How is differential geometry used in the research of parametric curves and surfaces of constant curvature?

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1 Introduction

Research question : **How is differential geometry used in the research of parametric curves and surfaces of constant curvature?**

Following the introduction of calculus in the 17^{th} century, mathematicians can now study complex curves and surfaces using techniques such as integration. The arc length of a curve, an ancient problem of geometry, can consequently be calculated (Henderson). As a branch of differential geometry, I was introduced to the mathematical significance of curvature in the book "Math in 100 Key Breakthroughs", where I was intrigued by the technique used to quantify this subjective quality using intricate calculus and linear algebra. From as small as soap films to the grand-scale distortion of space-time, curvature is used to precisely describe those natural phenomena. As for human applications, when the mean curvature of a surface is zero for all points, the surface is considered to be minimal, a useful characteristic in material science or biology.

Curvature is the quantitative measurement of departure from straightness (Oresme and Clagett). Hence, the curvature of a parametric curve is its departure from a straight line and the curvature of a surface is how it deviates from a flat plane. This essay will focus on *Gaussian curvature* and *mean curvature* due to their relationship with minimal surface. The former is an intrinsic measure - the curvature at a point does not change if the surface is deformed. The latter is extrinsic - it describes the exact curvature of how the surface embeds in space.

2 Curvature of parametric curves

This section will discuss the curvature of parametric curves by first introducing them with a qualitative explanation of curvature and osculating circles. A proof of the explicit formula of curvature will be devised and few sample calculations will be made along with the derivation for curves of constant curvature. Note that all discussions regarding parametric curves are limited to $\mathbb{R}^2$ and $\mathbb{R}^3$

2.1 Parametric curves and the first formula for curvature

Let $\vec{\gamma}(t)$ denote the position vector of a curve at a specific time t. Note that a parametric curve in $\mathbb{R}^2$ can be written in the form of

$$\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

where $x(t)$ and $y(t)$ are functions for the x and y coordinates of the point in $\mathbb{R}^2$ Cartesian plane (Tapp 2). It follows trivially that the tangent vector is,

$$\vec{\gamma}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

It is important to note that $\vec{\gamma}'(t)$ could also be considered as the velocity vector which traces out the curve. When studying curvature, parametric equations are usually preferred over explicit functions because they can have many-to-one properties, thus allowing us to study more complicated curves and shapes. The equation below is the mathematical definition for the curvature of parametric curves .

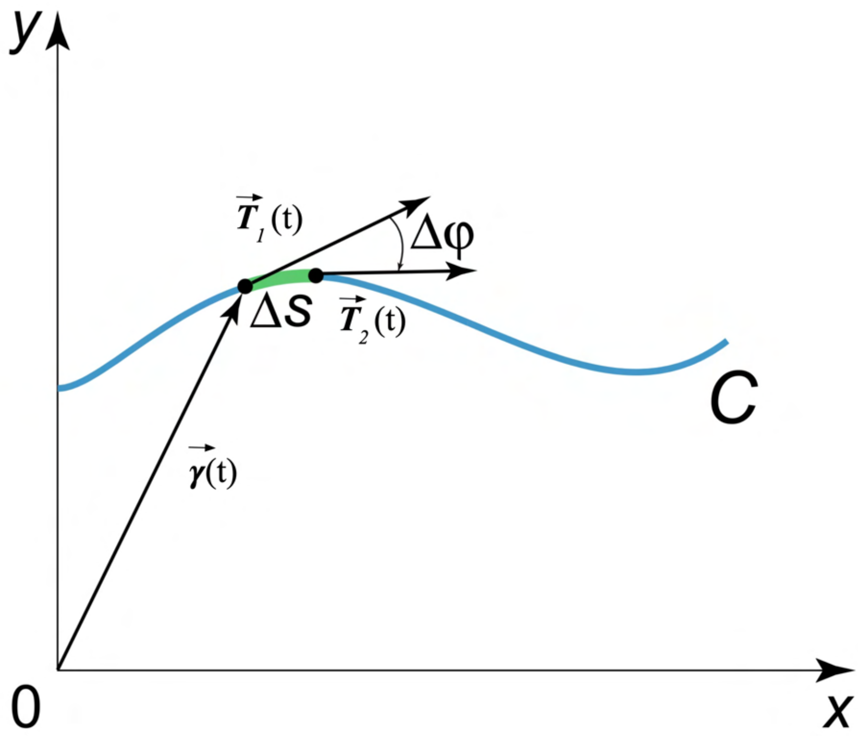
$$\kappa(t) \equiv \left| \frac{d\vec{T}}{ds} \right| \tag{2.1.1}$$

where s denotes arc-length and $\vec{T}(t)$ is defined as the *unit tangent vector* of a point, which can be explicitly written as

$$\vec{T}(t) = \frac{\vec{\gamma}'(t)}{|\vec{\gamma}'(t)|} \tag{2.1.2}$$

(2.1.1) : Weisstein. “Curvature”
(2.1.2) : Weisstein. “Curvature”

Figure 1: (Svirin). Visualisation of curvature on parametric curves. Adapted by candidate.



As shown in Figure 1, since the magnitude of $\vec{T}$ is always 1, the change in tangent vector, $d\vec{T}$, represents only the change in the direction of $\vec{T}$, i.e. $\Delta\varphi$. Equation (2.1.1) thus suggests that curvature is the magnitude of the change in direction of the unit tangent vector if we move an infinitesimally small increment along the curve, $\Delta s \to ds$. Intuitively, if the change in direction is large, the curve bends more at that point thus departing more from a straight line. Hence, the value of curvature should be large.

2.2 Osculating circle

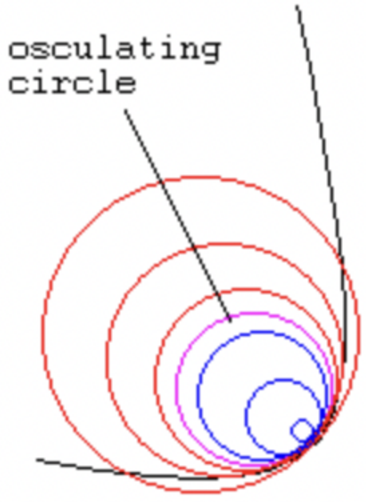
Another definition for the curvature of parametric curves is

$$\kappa \equiv \frac{1}{r} \tag{2.2.1}$$

where r is the *radius of curvature* - the radius of the *osculating circle* at that point on the curve. Since the word "osculate" means kissing in Latin, an *osculating circle* "kisses" or tangent to the point as shown in Figure 2

(2.2.1) : Tapp 30

Figure 2: (The History). Osculating circles on curves.



However, it is a sense "double tangent" to the point compared to other tangent circles. Hence, it provides a better local approximation to the curve. Therefore, the curvature on any straight line should be zero because the osculating circle is infinitely large $\therefore r \to \infty$, $\kappa \to 0$. This is conceptually true because curvature is the departure from straight line.

Moreover, Equation (2.2.1) also suggests that any circle should have a constant curvature because the osculating circle is the circle itself. This conjecture will be proven in Section 2.4.1 and Section 2.5.

Further note that definition (2.1.1) and definition (2.2.1) are equivalents (Jyoti Nivas College).

2.3 Proof of general formula

As defined previously, the curvature of a curve at a point is the magnitude of the rate of change of unit tangent vector w.r.t. to arc length. In this section, we will derive a formula for curvature in terms of $\gamma'(t)$ and $\gamma''(t)$. Using the chain rule and the identity $|\frac{ds}{dt}| \equiv |\frac{d\vec{\gamma}}{dt}|$ from Equation (3.1.2),

$$\kappa(t) = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{d\vec{\gamma}}{dt}}\right| = \frac{|\vec{T}'(t)|}{|\vec{\gamma}'(t)|}$$

Let us now derive a formula for $\vec{T}'(t)$.

Proof. By definition of unit tangent vector,

$$\vec{T}(t) = \frac{\vec{\gamma}'(t)}{|\vec{\gamma}'(t)|}$$

$$\Leftrightarrow \vec{\gamma}'(t) = \vec{T}(t)|\vec{\gamma}'(t)|$$

$$\Rightarrow \frac{d}{dt}(\vec{\gamma}'(t)) = \vec{\gamma}''(t) = \vec{T}'(t)|\vec{\gamma}'(t)| + \vec{T}(t)|\vec{\gamma}''(t)|$$

Now, by cross product $\vec{\gamma}'(t)$ with $\vec{\gamma}''(t)$,

$$\begin{aligned}\vec{\gamma}' \times \vec{\gamma}'' &= (\vec{T}|\vec{\gamma}'|) \times (\vec{T}'|\vec{\gamma}'| + \vec{T}|\vec{\gamma}''|)\\ &= (\vec{T} \times \vec{T}')|\vec{\gamma}'|^2 + |\vec{\gamma}'||\vec{\gamma}''|(\vec{T} \times \vec{T})\end{aligned}$$

Since $\vec{T}$ coincides thus parallel with $\vec{T}$, $(\vec{T} \times \vec{T}) = \vec{0}$. Hence,

$$\vec{\gamma}' \times \vec{\gamma}'' = (\vec{T} \times \vec{T}')|\vec{\gamma}'|^2$$

$$\Leftrightarrow |\vec{\gamma}' \times \vec{\gamma}''| = |\vec{\gamma}'|^2|\vec{T} \times \vec{T}'|$$

Lemma 2.1. Proposition : $\vec{T} \perp \vec{T}'$

Since $|\vec{T}| = 1$, by squaring both sides,

$$|\vec{T}|^2 = 1.$$

It follows that

$$\vec{T}\cdot\vec{T}=1$$

Differentiating both sides using product rule gives,

$$\vec{T}'\cdot\vec{T}+\vec{T}\cdot\vec{T}'=0$$

Since dot product is commutative,

$$2(\vec{T}'\cdot\vec{T})=0$$

$$\Leftrightarrow \vec{T}'\cdot\vec{T}=0$$

$\vec{T}'$ is therefore perpendicular to $\vec{T}$. $\square$

Now, Using *Lemma* 2.1, the magnitude of the vector $\vec{\gamma}'\times\vec{\gamma}''$ can be written as

$$|\vec{\gamma}'\times\vec{\gamma}''|=|\vec{\gamma}'|^2|\vec{T}||\vec{T}'|\sin\left(\frac{\pi}{2}\right)$$

Using the fact that $|\vec{T}|=\sin(\frac{\pi}{2})=1$,

$$|\vec{\gamma}'\times\vec{\gamma}''|=|\vec{\gamma}'|^2|\vec{T}'|$$

$$\Leftrightarrow |\vec{T}'|=\frac{|\vec{\gamma}'\times\vec{\gamma}''|}{|\vec{\gamma}'|^2}$$

$$\therefore \kappa(t)=\frac{|\vec{T}'|}{|\vec{\gamma}'|}=\frac{|\vec{\gamma}'\times\vec{\gamma}''|}{|\vec{\gamma}'|^3}$$

$\square$

However, this general formula is still in vector notation. To facilitate substitution, let us now derive an explicit formula for curvature in terms of $x(t)$ and $y(t)$ and their derivatives w.r.t. t.

Let us consider a continuous curve

$$\vec{\gamma}(t)=\begin{pmatrix}x(t)\\y(t)\end{pmatrix}.$$

It follows that

$$\vec{\gamma}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \text{ and } \vec{\gamma}''(t) = \begin{pmatrix} x''(t) \\ y''(t) \end{pmatrix}$$

Hence, using the cross product formula, since both vectors are two dimensional,

$$|\vec{\gamma}' \times \vec{\gamma}''| = |x'y'' - y'x''|$$

Let us now calculate the denominator,

$$\begin{aligned} |\vec{\gamma}'|^3 &= \left(\sqrt{x'(t)^2 + y'(t)^2}\right)^3 \\ &= \left((x')^2 + (y')^2\right)^{\frac{3}{2}} \end{aligned}$$

Hence, the explicit formula for the curvature of a 2D curve is

$$\kappa(t) = \frac{|x'y'' - y'x''|}{\left((x')^2 + (y')^2\right)^{\frac{3}{2}}} \tag{2.3.1}$$

Note that the absolute value sign on the numerator is necessary because the magnitude of a vector is always positive and also because $\kappa = \frac{1}{r}$ and the radius of curvature cannot be negative.

Moreover, if the curve is a well defined function, i.e. $y(t) = f(t)$, $\vec{\gamma}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$. It follows that $\vec{\gamma}'(t) = \begin{pmatrix} 1 \\ f'(t) \end{pmatrix}$ and $\vec{\gamma}''(t) = \begin{pmatrix} 0 \\ f''(t) \end{pmatrix}$. Therefore, the explicit formula for the curvature of a 2D function is

$$\kappa(t) = \frac{|y''|}{\left(1 + (y')^2\right)^{\frac{3}{2}}} \tag{2.3.2}$$

2.4 Sample calculations

In this section, we will employ the formulas derived from the previous section to solve for the curvature of few common curves.

2.4.1 Circle of radius $R \in \mathbb{R}^+$

Let

$$\vec{\gamma}(t) = \begin{pmatrix} R \times \cos(t) \\ R \times \sin(t) \end{pmatrix}$$

It follows trivially that

$$x'(t) = -R \times \sin(t) \qquad x''(t) = -R \times \cos(t)$$
$$y'(t) = R \times \cos(t) \qquad y''(t) = -R \times \sin(t)$$

Hence, using Equation 2.3.1,

$$\begin{aligned}\kappa(t) &= \frac{|R^2 \times \sin^2(t) + R^2 \times \cos^2(t)|}{\left(R^2 \times \sin^2(t) + R^2 \times \cos^2(t)\right)^{\frac{3}{2}}} \\ &= \frac{|R^2 \times (\sin^2(t) + \cos^2(t))|}{\left(R^2 \times (\sin^2(t) + \cos^2(t))\right)^{\frac{3}{2}}} \\ &= \frac{R^2}{R^3} = \frac{1}{R}\end{aligned}$$

Note that since $\kappa(t)$ is independent of t, circle is a *curve of constant curvature*. Intuitively, the radius of the osculating circle is the same for all points because the osculating circle is the circle itself. This is because $\kappa(t) = \frac{1}{r} = \frac{1}{R}$ implies that $r = R$. Hence, the radius of curvature is equal to the radius of the circle.

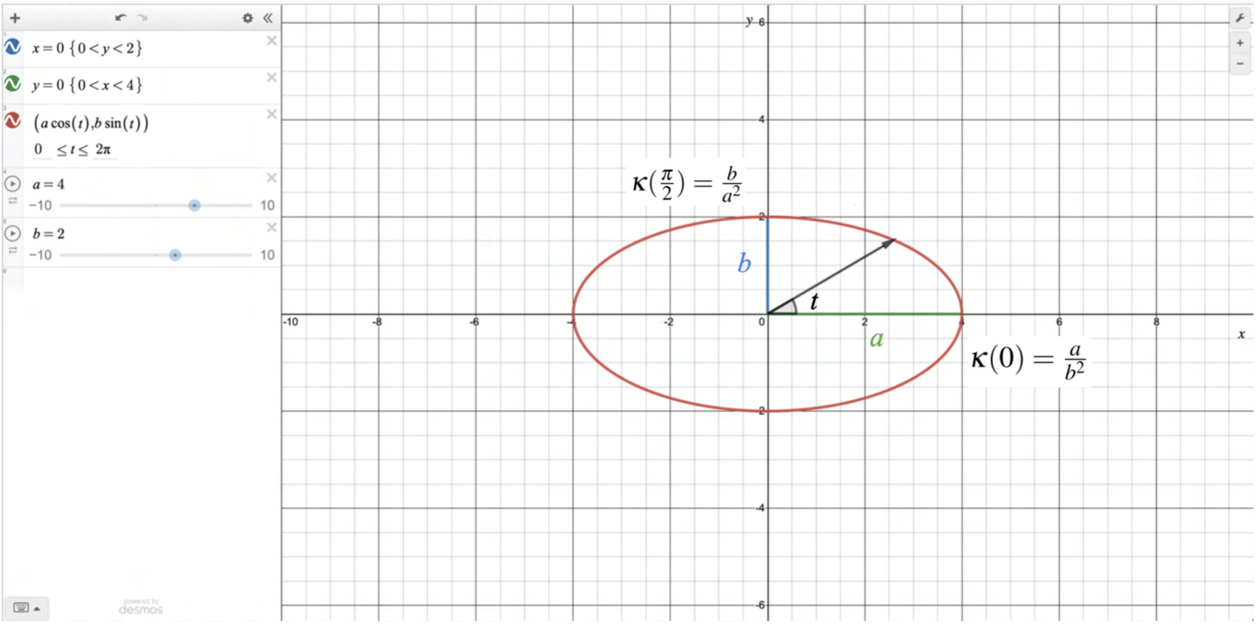
2.4.2 Ellipse

Let

$$\vec{\gamma}(t) = \begin{pmatrix} a \times \cos(t) \\ b \times \sin(t) \end{pmatrix} \tag{2.4.1}$$

where $a \in \mathbb{R}^+$ is the length of the *semi-major axis* of an ellipse and $b \in \mathbb{R}^+$ measures the *semi-minor axis* as shown in Figure 3.

Figure 3: A general ellipse, Figure made by candidate using Desmos



Hence,

$$x'(t) = -a \times \sin(t)$$

$$y'(t) = b \times \cos(t)$$

and

$$x''(t) = -a \times \cos(t)$$

$$y''(t) = -b \times \sin(t)$$

(2.4.1) : Math Open Reference

Similarly,

$$\begin{aligned}\kappa(t) &= \frac{|ab \times \sin^2(t) + ab \times \cos^2(t)|}{\left(a^2 \times \sin^2(t) + b^2 \times \cos^2(t)\right)^{\frac{3}{2}}} \\ &= \frac{ab}{\left(a^2 \times \sin^2(t) + b^2 \times \cos^2(t)\right)^{\frac{3}{2}}}\end{aligned}$$

It is important to note that in this case, the equation of curvature cannot be simplified any further. Moreover, the curvature along the ellipse varies with respect to t except for the special case where $a = b$ and the ellipse is actually a circle. Now, by substituting an angle value into t, the curvature at that specific point can be found. For example, at $t = 0$, $\kappa(0) = \frac{a}{b^2}$ and at $t = \frac{\pi}{2}$, $\kappa(\frac{\pi}{2}) = \frac{b}{a^2}$.

2.4.3 Helix

Since a Helix is a three-dimensional curve, the explicit formula for curvature must be different.

Let $\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ denote a continuous 3D curve.

$$\begin{aligned}|\vec{\gamma}'|^3 &= \left(\sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\right)^3 \\ &= \left((x')^2 + (y')^2 + (z')^2\right)^{\frac{3}{2}}\end{aligned}$$

and using the formula of cross product,

$$\vec{\gamma}' \times \vec{\gamma}'' = \begin{pmatrix} y'z'' - y''z' \\ \\ x''z' - x'z'' \\ \\ x'y'' - x''y' \end{pmatrix}$$

$$\Leftrightarrow |\vec{\gamma}' \times \vec{\gamma}''| = \sqrt{(y'z'' - y''z')^2 + (x''z' - x'z'')^2 + (x'y'' - x''y')^2}$$

Hence, the explicit formula for the curvature of a 3D curve is

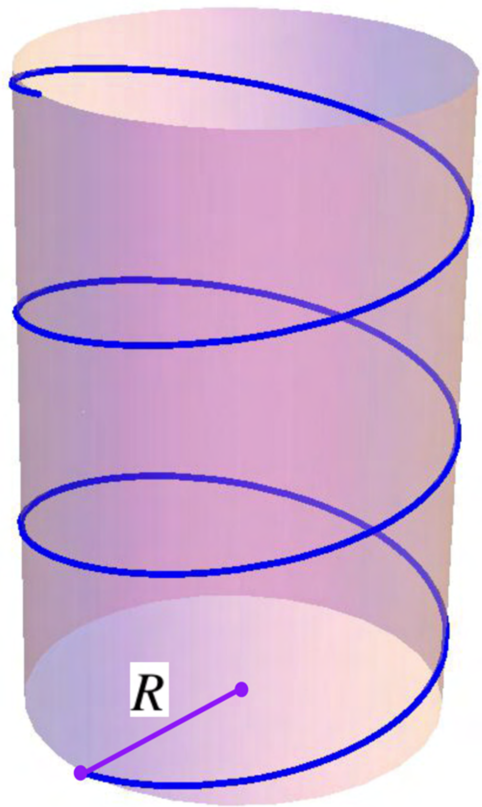
$$\kappa(t) = \frac{\sqrt{(y'z'' - y''z')^2 + (x''z' - x'z'')^2 + (x'y'' - x''y')^2}}{\left((x')^2 + (y')^2 + (z')^2\right)^{\frac{3}{2}}} \tag{2.4.2}$$

Let us now calculate the curvature of a generalised helix exemplified by Figure 4. It is worth noting that the parametric equation of a helix is very similar to the parametric equation of a circle and can be deduced trivially.

$$\vec{\gamma}(t) = \begin{pmatrix} R \times \cos(t) \\ R \times \sin(t) \\ a \times t \end{pmatrix}$$

where $a \in \mathbb{R}\backslash\{0\}$ is a parameter that can be considered as the speed at which the helix travels along the z-axis in circular motion.

Figure 4: (Lăzureanu). A general Helix with radius R. Image adapted by candidate.



Hence

$$\begin{aligned}
x'(t) &= -R \times \sin(t) \\
y'(t) &= R \times \cos(t) \\
z'(t) &= a
\end{aligned}$$

and

$$\begin{aligned}
x''(t) &= -R \times \cos(t) \\
y''(t) &= -R \times \sin(t) \\
z''(t) &= 0
\end{aligned}$$

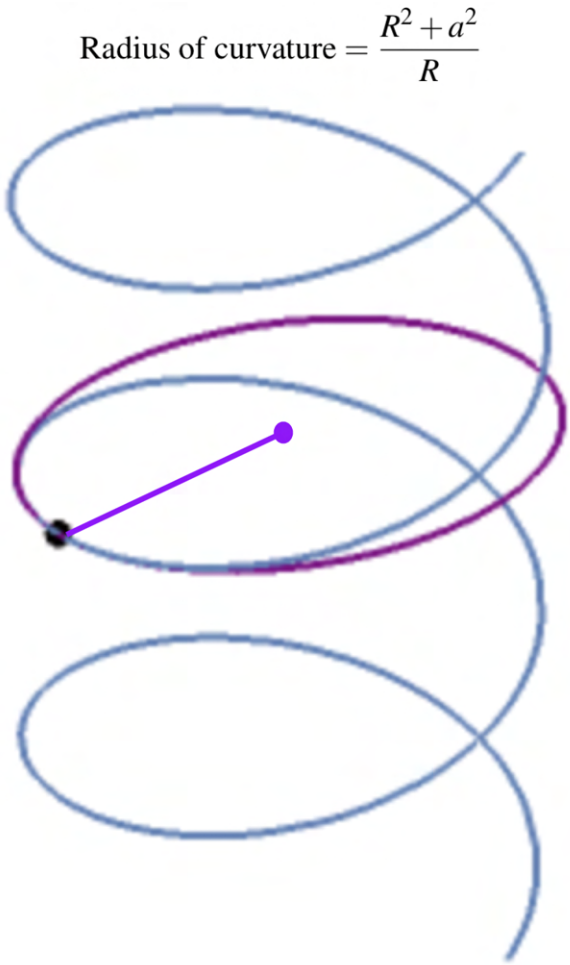
Now, by substituting $z''(t) = 0$ into Equation (2.4.2),

$$\begin{aligned}
\kappa(t) &= \frac{\sqrt{(y''z')^2 + (x''z')^2 + (x'y'' - x''y')^2}}{\left((x')^2 + (y')^2 + (z')^2\right)^{\frac{3}{2}}} \\
&= \frac{\sqrt{(a^2R^2\sin^2(t) + a^2R^2\cos^2(t) + (R^2\sin^2(t) + R^2\cos^2(t))^2}}{\left(R^2\sin^2(t) + R^2\cos^2(t) + a^2\right)^{\frac{3}{2}}}
\end{aligned}$$

and using the Pythagorean identity to simplify the numerator and denominator,

$$\begin{aligned}
\kappa(t) &= \frac{\sqrt{a^2R^2 + R^4}}{(R^2 + a^2)^{\frac{3}{2}}} \\
&= \frac{R \times \sqrt{a^2 + R^2}}{(R^2 + a^2)^{\frac{3}{2}}} \\
&= \frac{R}{R^2 + a^2}
\end{aligned}$$

Figure 5: (Abby). Osculating circle on Helix. Image made using Wolfram demonstration project.



It is again interesting to note that the helix has a constant curvature. Despite its similarity with a circle, the curvature of a helix is smaller than the curvature of a circle with the same radius. Intuitively, this is because the circle is essentially stretched vertically into a helix, the radius of the osculating circle is thus larger than R. Furthermore, a larger value of a denotes a larger vertical stretch, thereby smaller curvature.

2.5 Curves of constant curvature

This section will now prove that in the planar case, only lines and circles have constant curvature (Svirin). Using equation (2.3.2), let us now find the function, $y = f(x)$ which has constant curvature. Let $\kappa(t) = \frac{1}{r}$, where r is a constant representing the radius of curvature.

Proof.

$$\frac{1}{r} = \frac{|y''|}{\left(1+(y')^2\right)^{\frac{3}{2}}}$$

First note that there is a trivial solution to the problem, the general equation of a line, $y = mx + c$. This is because $y' = m$ and therefore $y'' = 0$. Hence, for the non-trivial case, we will need to solve this non-linear second-order differential equation. By first removing the absolute value sign from $|y''|$ and then integrate both sides w.r.t. x, we obtain

$$\pm \int \frac{dx}{r} = \int \frac{y''}{\left(1+(y')^2\right)^{\frac{3}{2}}} dx$$

The problem can be simplified further by substituting $y'(x) = u(x)$ to reduce the order of the problem. Using integration by substitution, since $\frac{du}{dx} = u'$,

$$\begin{aligned} \pm \int \frac{1}{r} dx &= \int \frac{u'}{(1+u^2)^{\frac{3}{2}}} \times \frac{du}{u'} \\ &= \int \frac{du}{(1+u^2)^{\frac{3}{2}}} \end{aligned}$$

Since the denominator of Right Hand Side (RHS) is $(1+u^2)^{\frac{3}{2}}$, trigonometry substitution is best suited to simplify RHS because $1 + \tan^2 x = \sec^2 x$ and $\frac{d}{dx}\big(\arctan(x)\big) = \frac{1}{1+x^2}$.

$$\begin{aligned} &\text{Let } u = \tan(z) \\ &\Rightarrow \frac{dz}{du} = \frac{1}{1+u^2} \end{aligned}$$

Hence, using integration by substitution and the trigonometric identity above,

$$
\begin{aligned}
\therefore RHS &= \int \frac{du}{(1+u^2)^{\frac{3}{2}}} \\
&= \int \frac{1}{(1+u^2)^{\frac{1}{2}}} \times \frac{1}{1+u^2} \times du \\
&= \int \frac{1}{(1+\tan^2 z)^{\frac{1}{2}}} \times \frac{dz}{du} \times du \\
&= \int \frac{dz}{(\sec^2 z)^{\frac{1}{2}}} \\
&= \int \cos(z) dz = \sin(z) + c
\end{aligned}
$$

where c is an integration constant. Hence, integrating LHS gives

$$\therefore \pm \frac{1}{R}(x + C_1) = \sin(z) \tag{2.5.1}$$

where C_1 is another integration constant taken into account c.

Lemma 2.2 ($\sin\theta \equiv \sqrt{\frac{\tan^2\theta}{\tan^2\theta+1}}$). Let us now prove this trigonometric identity so that we can substitute z with u and subsequently with y back into Equation (2.5.1)

Proof.

$$
\begin{aligned}
\sin\theta &= \sqrt{\frac{\tan^2\theta}{\tan^2\theta + 1}} \\
&= \sqrt{\frac{\tan^2\theta}{\sec^2\theta}} \\
&= \frac{\sin\theta}{\cos\theta} \times \cos\theta \\
&= \sin\theta
\end{aligned}
$$

□

Using *Lemma* 2.2. and the substitution $u = \tan z$ previously,

$$\sin(z) \equiv \sqrt{\frac{\tan^2 z}{\tan^2 z + 1}} \Leftrightarrow \sqrt{\frac{u^2}{u^2+1}} = \pm\frac{1}{R}(x+C_1)$$

We could eliminate the $\pm$ sign by squaring both sides,

$$\therefore \frac{u^2}{u^2+1} = \frac{1}{R^2}(x+C_1)^2$$

$$\Leftrightarrow R^2u^2 = (x+C_1)^2u^2 + (x+C_1)^2$$

$$\Leftrightarrow \Big(R^2 - (x+C_1)^2\Big)u^2 = (x+C_1)^2$$

$$\Leftrightarrow u^2 = \frac{(x+C_1)^2}{R^2-(x+C_1)^2}$$

$$\therefore u = \pm\sqrt{\frac{(x+C_1)^2}{R^2-(x+C_1)^2}}$$

substituting back $u = y'$ gives

$$y' = \pm\sqrt{\frac{(x+C_1)^2}{R^2-(x+C_1)^2}}$$

$$\Rightarrow y = \pm\int\sqrt{\frac{(x+C_1)^2}{R^2-(x+C_1)^2}}\,dx$$

Let us now substitute

$$x + C_1 = R\sin(t)$$

$$\Rightarrow \frac{dx}{dt} = R\cos(t)$$

This substitution is used because the denominator would simplify to $R^2(1-\sin^2 t) = R^2\cos^2 t$.

$$
\begin{aligned}
\therefore y &= \pm \int \sqrt{\frac{R^2 \sin^2 t}{R^2 - R^2 \times \sin^2 t}} \times R\cos(t) \times \frac{dt}{dx} \times dx \\
&= \pm \int \sqrt{\frac{R^2 \sin^2 t}{R^2 \cos^2 t}} \times R\cos(t) \times dt \\
&= \pm R \int \sin(t)\, dt \\
&= \mp R\cos(t) - C_2 \\
\Leftrightarrow y + C_2 &= \mp R\cos(t)
\end{aligned}
$$

where C_2 is another integration constant. Hence, by squaring the previous substitution $x+C_1 = R\sin(t)$ and $y+C_2 = \mp R\cos(t)$, we obtain the following two equations.

$$
\begin{cases}
(x+C_1)^2 = R^2 \sin^2(t) \\
(y+C_2)^2 = R^2 \cos^2(t)
\end{cases}
$$

Hence using Pythagorean identity,

$$
\underbrace{(x+C_1)^2}_{R^2\sin^2(t)} + \underbrace{(y+C_2)^2}_{R^2\cos^2(t)} = R^2
$$

Note that the equation above is the general formula for a circle. Therefore, the other planar curve with constant curvature is a circle with radius R whose centre locates at $(-C_1, -C_2)$. Thereby **all circles** have constant curvature. $\square$

Theorem 2.3. Only circles and straight lines are **curves of constant curvature** in $\mathbb{R}^2$.

3 Curvature of parametric surfaces

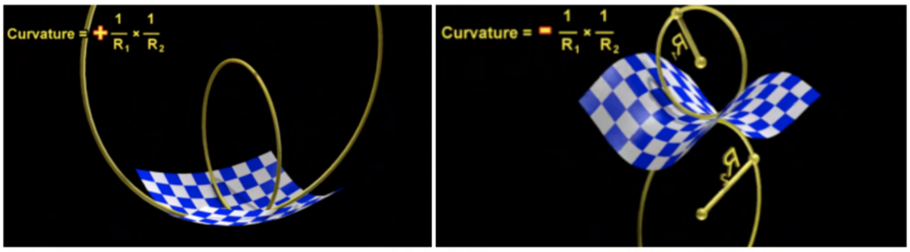
Two types of surface curvature will be discussed in this section. *Mean Curvature*, denoted by H, extrinsically measures the curvature of the surface at a local point whereas *Gaussian Curvature*, represented by K is an intrinsic measurement. They are defined as

$$H = \frac{k_1 + k_2}{2} = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2}) \tag{3.0.1}$$

$$K = k_1 k_2 = \frac{1}{R_1} \times \frac{1}{R_2} \tag{3.0.2}$$

where k_1 and k_2 are the *Principal Curvatures* at a point on a surface and $k_1 \equiv \frac{1}{R_1}$, $k_2 \equiv \frac{1}{R_2}$. It is worth noting that, Equation (2.2.1), the relationship between curvature and radius of curvature of parametric curves also holds for principal curvatures.

Figure 6: (Bio). Osculating circles on surfaces.



Hence, as shown in Figure 6, if both osculating circles are on the same side (in another word, the surface curves away from the point in the same direction), the surface has positive Gaussian curvature at the point. The inverse of the statement holds true as well. Now note that the value of k_1, k_2 are the solutions of the following equation,

$$\det \begin{pmatrix} L - kE & M - kF \\ M - kF & N - kG \end{pmatrix} = 0 \tag{3.0.3}$$

where E, F, G are the coefficients of **first fundamental form** and L, M, N are the coefficients of **second fundamental form** (E.V.Shikin). Before going into greater details regarding those coefficients in Section 3.1, let us first expand equation (3.0.3). It is thereby possible to rewrite

(3.0.1), (3.0.2) : E.V.Shikin, “Principal curvature.” Encyclopedia of Mathematics
(3.0.3) : E.V.Shikin, “Principal curvature.” Encyclopedia of Mathematics

Equation (3.0.1) and (3.0.2) in polynomial form using the coefficients.

$$\begin{aligned}
&(L-kE)\times(N-kG)-(M-kF)^2=0\\
\Leftrightarrow\ &EGk^2+LN-(LG+EN)k-F^2k^2-M^2+2MFk=0\\
\Leftrightarrow\ &\underbrace{(EG-F^2)}_{a}k^2+\underbrace{(2MF-LG-EN)}_{b}k+\underbrace{(LN-M^2)}_{c}=0
\end{aligned}$$

Since K is the product of the roots of this quadratic equation and H is the sum of its roots divided by two, we can fore use Vieta's formula to solve for H and K.

$$K=k_1k_2=\frac{c}{a}=\frac{LN-M^2}{EG-F^2} \tag{3.0.4}$$

$$H=\frac{k_1+k_2}{2}=\frac{-b}{2a}=\frac{LG+EN-2MF}{2(EG-F^2)}. \tag{3.0.5}$$

Conversely, if we have the value of H and K, we can compute the principal curvature from the following quadratic equation also using Vieta's formula.

$$k^2-2\times\left(\frac{k_1+k_2}{2}\right)k+(k_1k_2)=0$$

$$\Leftrightarrow k^2-2Hk+K=0$$

$$\Leftrightarrow k_1=H+\sqrt{H^2-K} \tag{3.0.6}$$

$$\Leftrightarrow k_2=H-\sqrt{H^2-K} \tag{3.0.7}$$

After learning about the equations which define curvature, let us now consider a parametric surface in three-dimensional Euclidean space.

$$\vec{\sigma}(u,v)=\begin{pmatrix}x(u,v)\\ y(u,v)\\ z(u,v)\end{pmatrix}$$

Note that parametric surfaces have a similar form as parametric curves in $\mathbb{R}^3$. However, there are two parameters exemplified by u and v whereas parametric curves only have one.

Moreover, since there are two variables, multi-variable calculus is needed to analyse a surface. The tool of **partial derivatives** is especially important. When taking the partial derivative of a multi-variable function w.r.t. one of its variables, we simply treat other variables as constants. For example, $f(x,y) = x^2 + y^2$ is the explicit function which plots out an *Elliptic Paraboloid* in Section 3.2.1. The first and second partial derivatives of $f(x,y)$ both w.r.t. x are as follows,

$$\frac{\partial f(x,y)}{\partial x} = f_x = 2x.$$
$$\frac{\partial f(x,y)}{\partial x \partial x} = f_{xx} = 2$$

3.1 The First and Second fundamental forms

The First fundamental form describes the metric properties of a surface. i.e. Properties associated with distances, exemplified by surface area and arc length. ds, an infinitesimally small displacement along the surface $\vec{\sigma}(u,v)$ can be written in terms of the coefficients of the first fundamental form .

$$I = ds^2 = Edu^2 + 2F\,dudv + Gdv^2. \tag{3.1.1}$$

This basic understanding of multivariable calculus allows us to define the coefficients of first fundamental form (MIT. "3.2 First Fundamental Form I (Metric)").

$$\begin{cases} E = \vec{\sigma}_u \cdot \vec{\sigma}_u \\ F = \vec{\sigma}_u \cdot \vec{\sigma}_v \\ G = \vec{\sigma}_v \cdot \vec{\sigma}_v \end{cases}$$

where $\vec{\sigma}_u$ and $\vec{\sigma}_v$ are the **partial derivatives** of the surface $\vec{\sigma}(u,v)$ w.r.t. u and v. In its explicit form,

$$\frac{\partial \vec{\sigma}(u,v)}{\partial u} = \vec{\sigma}_u = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} = \begin{pmatrix} x_u \\ y_u \\ z_u \end{pmatrix}$$

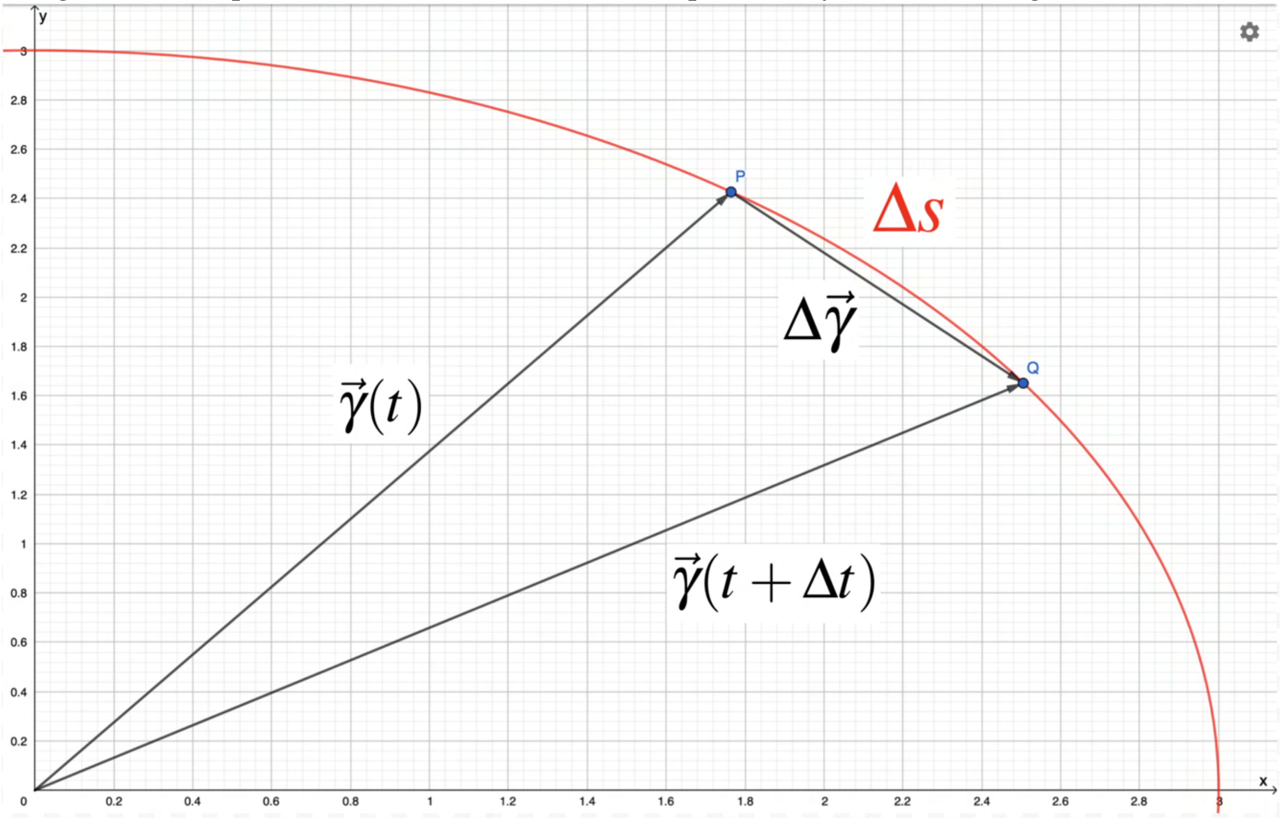
(3.1.1) : MIT. "3.2 First Fundamental Form I (Metric)"

Similarly,

$$\vec{\sigma}_v = \begin{pmatrix} x_v \\ y_v \\ z_v \end{pmatrix}$$

Let us now prove Equation 3.1.1 and its coefficients by first considering a parametric curve $\vec{\gamma}(t)$.

Figure 7: Principal of First Fundamental Form. Graph made by candidate using GeoGebra



Proof. First note that for a small increment along the curve, the corresponding change in arc length can be intuitively approximated as

$$\Delta s \approx |\Delta\vec{\gamma}|$$

which is equivalent to

$$\Delta s \approx |\vec{\gamma}(t+\Delta t) - \vec{\gamma}(t)|$$

Let us now apply Taylor's expansion about the point t,

$$\begin{aligned}\Delta s &\approx \left|\frac{d\vec{\gamma}}{dt}(t+\Delta t) - \frac{d\vec{\gamma}}{dt}(t)\right| \\ &\approx \left|\frac{d\vec{\gamma}}{dt}\right|\Delta t\end{aligned}$$

As $\Delta t \to 0$, $\Delta s \to ds$ becomes the *differential arc length*.

$$\therefore ds = \left|\frac{d\vec{\gamma}}{dt}\right| dt \tag{3.1.2}$$

Using the equation derived above, let us substitute $\vec{\sigma}(u,v) = \vec{\gamma}(t)$ to derive the general case for a parametric surface.

$$\therefore ds = \left|\frac{d\vec{\sigma}}{dt}\right| dt$$

In order to simplify this further and write it in the form of its coefficients, let us first define the chain rule for functions of two variables .

Definition 3.1. Let $z = f(x,y)$ where $x = g(t)$ and $y = h(t)$.

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{3.1.3}$$

$$\therefore ds = \left|\vec{\sigma}_u \frac{du}{dt} + \vec{\sigma}_v \frac{dv}{dt}\right| dt$$

Using the square root identity $|x| = \sqrt{x^2}$,

$$\begin{aligned}ds &= \sqrt{\left(\vec{\sigma}_u \frac{du}{dt} + \vec{\sigma}_v \frac{dv}{dt}\right)^2} dt \\ &= \sqrt{(\vec{\sigma}_u \cdot \vec{\sigma}_u)\left(\frac{du}{dt}\right)^2 + (\vec{\sigma}_u \cdot \vec{\sigma}_v)\left(\frac{du}{dt} \times \frac{dv}{dt}\right) + (\vec{\sigma}_v \cdot \vec{\sigma}_v)\left(\frac{dv}{dt}\right)^2} dt \\ &= \sqrt{\underbrace{(\vec{\sigma}_u \cdot \vec{\sigma}_u)}_{E} du^2 + \underbrace{(\vec{\sigma}_u \cdot \vec{\sigma}_v)}_{F} dudv + \underbrace{(\vec{\sigma}_v \cdot \vec{\sigma}_v)}_{G} dv^2}\end{aligned}$$

□

(3.1.3) : Dawkins, "Section 2-6 : Chain Rule." Paul's online note

Hence, one can for example calculate the arc length of a parametric curve on a surface using those coefficients.

$$ds = \sqrt{E\left(\frac{du}{dt}\right)^2 + F\left(\frac{du}{dt} \times \frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}dt$$
$$\Leftrightarrow s = \int_{t_2}^{t_1} \sqrt{E\left(\frac{du}{dt}\right)^2 + F\left(\frac{du}{dt} \times \frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2}dt$$

where t_1, t_2 are the upper and lower bounds of the curve respectively.

3.1.1 Second fundamental form

The second fundamental form on the other hand directly describes the local deviation of the surface from its tangent plane. Therefore, it is an external measure of curvature (Powell). Its coefficients can be defined in a similar manner.

$$II = Ldu^2 + 2Mdudv + Ndv^2 \tag{3.1.4}$$

where its coefficients can be calculated as follows (MIT. "3.3 Second Fundamental Form II (Curvature)").

$$\begin{cases} L = \vec{\sigma}_{uu} \cdot \hat{\mathbf{n}} \\ M = \vec{\sigma}_{uv} \cdot \hat{\mathbf{n}} \\ N = \vec{\sigma}_{vv} \cdot \hat{\mathbf{n}} \end{cases}$$

where $\hat{\mathbf{n}}$ is the *unit normal vector* at the point on the surface . It can be computed from

$$\hat{\mathbf{n}} = \frac{\vec{\sigma}_u \times \vec{\sigma}_v}{|\vec{\sigma}_u \times \vec{\sigma}_v|} \tag{3.1.5}$$

Let us now prove the second fundamental form of the surface (Powell).

(3.1.4) : MIT. "3.3 Second Fundamental Form II (Curvature)"
(3.1.5) : Weisstein, "Normal Vector"

Proof. In this proof, it is needed to extend the definition for Taylor's expansion for Functions of two variables.

Definition 3.2. Let $f(x,y)$ denote a two variable function whose partial derivatives are well defined. Around a point (a,b), the first two terms of the Taylor's series are as follows (Seeburger).

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \frac{f_{xx}(a,b)}{2}(x-a)^2 + \frac{f_{yy}(a,b)}{2}(y-b)^2 + f_{xy}(x-a)(y-b)$$

Hence, to quantify the deviation from the tangent plane, we need to calculate

$$2 \times II = \Big(\underbrace{\vec{\sigma}(u+\delta u, v+\delta v) - \vec{\sigma}(u,v)}_{\text{displacement from the tangent point}}\Big) \cdot \hat{\mathbf{n}} \tag{3.1.6}$$

Hence, by dot product the displacement with the unit normal vector, the second fundamental form describes how close to being parallel is the displacement vector to the normal vector, in another word, how close to being perpendicular is the displacement of the surface to the tangent plane.

Using $\vec{a} \cdot \vec{b} = |a||b|\cos\theta$, if the curvature is small, the displacement vector would be less close to being perpendicular to the tangent plane. Hence, the angle between the displacement vector and the unit normal vector should be large and close to $\frac{\pi}{2}$. Therefore, the value of the second fundamental form should also be small.

Let us first use Definition 3.2 to approximate $\vec{\sigma}(u+\delta u, v+\delta v)$ around the point (u,v).

$$\vec{\sigma}(u+\delta u, v+\delta v) = \vec{\sigma} + \vec{\sigma}_u \times (u-u-\delta u) + \vec{\sigma}_v \times (v-v-\delta v) + \frac{\vec{\sigma}_{uu}}{2}(u-u-\delta u)^2 + \frac{\vec{\sigma}_{uv}}{2}(u-u-\delta u)(v-v-\delta v) + \frac{\vec{\sigma}_{vv}}{2}(v-v-\delta v)^2 + \text{Higher-Order Terms}$$

Which can be simplified to

$$\vec{\sigma}(u+\delta u, v+\delta v) = \vec{\sigma} {\color{red}- \vec{\sigma}_u \delta u - \vec{\sigma}_v \delta v} + \frac{\vec{\sigma}_{uu}}{2}\delta u^2 + \vec{\sigma}_{uv}\delta u \delta v + \frac{\vec{\sigma}_{vv}}{2}\delta v^2 + \ldots$$

Note that since $\hat{\mathbf{n}} = \frac{\vec{\sigma}_u \times \vec{\sigma}_v}{|\vec{\sigma}_u \times \vec{\sigma}_v|}$, $\hat{\mathbf{n}}$ is orthogonal to both $\vec{\sigma}_u$ and $\vec{\sigma}_v$. Therefore, when dot product the terms $(-\vec{\sigma}_u \delta u - \vec{\sigma}_v \delta v)$ with $\hat{\mathbf{n}}$, we obtain 0 because the value of θ is $\frac{\pi}{2}$. Hence,

$$\begin{aligned}\Big(\vec{\sigma}(u+\delta u, v+\delta v) - \vec{\sigma}(u,v)\Big) \cdot \hat{\mathbf{n}} &\approx \frac{1}{2}\Big(\vec{\sigma}_{uu}\delta u^2 + 2\vec{\sigma}_{uv}\delta u \delta v + \vec{\sigma}_{vv}\delta v^2\Big) \cdot \hat{\mathbf{n}} \\ &\approx \frac{1}{2}\Big(\underbrace{(\vec{\sigma}_{uu} \cdot \hat{\mathbf{n}})\delta u^2 + 2(\vec{\sigma}_{uv} \cdot \hat{\mathbf{n}})\delta u \delta v + (\vec{\sigma}_{vv} \cdot \hat{\mathbf{n}})\delta v^2}_{II}\Big)\end{aligned}$$

We can therefore derive the coefficients of second fundamental form because higher-order terms can be ignored since δu^3 and δv^3 are sufficiently small.

$$II = \underbrace{(\vec{\sigma}_{uu} \cdot \hat{\mathbf{n}})}_{L}\delta u^2 + 2\underbrace{(\vec{\sigma}_{uv} \cdot \hat{\mathbf{n}})}_{M}\delta u \delta v + \underbrace{(\vec{\sigma}_{vv} \cdot \hat{\mathbf{n}})}_{N}\delta v^2 \tag{3.1.7}$$

□

3.2 Curvature of paraboloids

Before discussing the surfaces of constant curvature, sample calculations for the curvature of the family of paraboloids will be made in this section to illustrate some essential concepts. Furthermore, note that Elliptic paraboloids have **positive** Gaussian curvature everywhere whereas hyperbolic paraboloids have **negative** Gaussian curvature everywhere and the parabolic cylinder has $K = 0$ at all points.

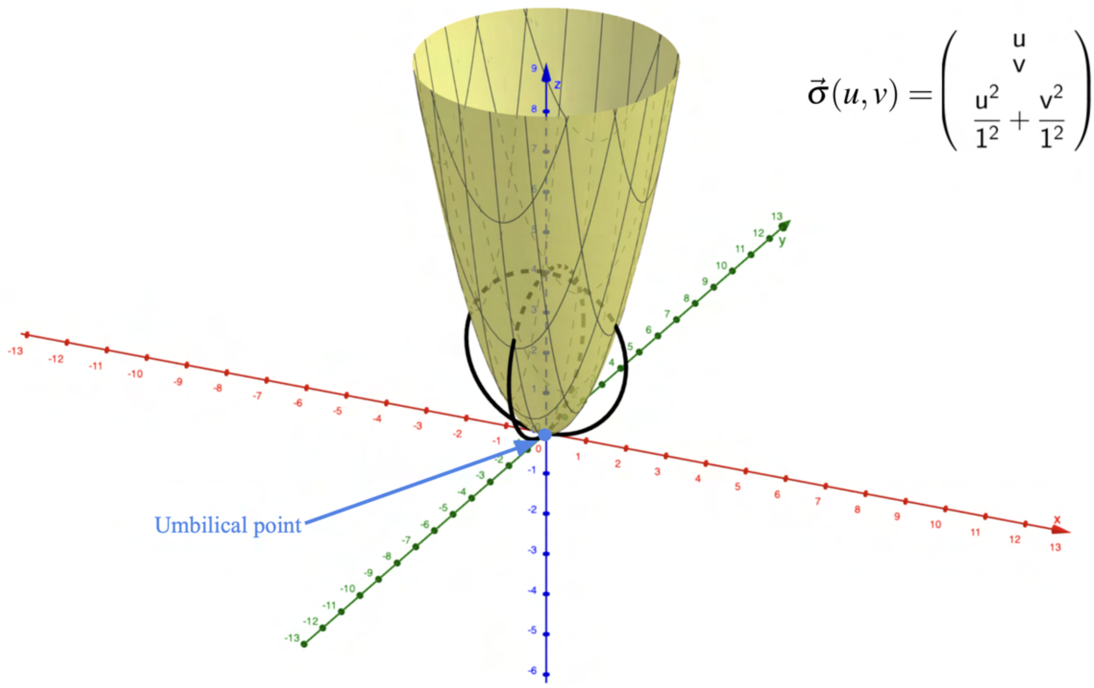
3.2.1 Elliptic paraboloid

Given previously that the explicit formula for an elliptic parabolid is $f(x,y) = x^2 + y^2$, it follows naturally that the general parametric equation for an Elliptic paraboloid is

$$\vec{\sigma}(x,y) = \begin{pmatrix} x \\ y \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \end{pmatrix} \tag{3.2.1}$$

where $a, b \in \mathbb{R}\backslash\{0\}$ are the scaling constants of the surface.

Figure 8: Elliptic paraboloid, Figure made by candidate using GeoGebra



(3.2.1) : "Paraboloid", Wikipedia

Its partial derivative is as follows,

$$\vec{\sigma}_x = \begin{pmatrix} 1 \\ 0 \\ \frac{2x}{a^2} \end{pmatrix} \quad \vec{\sigma}_y = \begin{pmatrix} 0 \\ 1 \\ \frac{2y}{b^2} \end{pmatrix}$$

We can therefore first compute the coefficients of first fundamental form.

$$\begin{cases} E = 1 + \dfrac{4x^2}{a^4} \\ F = \dfrac{4xy}{a^2b^2} \\ G = 1 + \dfrac{4y^2}{b^4} \end{cases}$$

To calculate L, M, N, we first need to compute the normal vector, $\vec{n}$.

$$\vec{n} = \vec{\sigma}_x \times \vec{\sigma}_y = \begin{pmatrix} 1 \\ 0 \\ \frac{2x}{a^2} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{2y}{b^2} \end{pmatrix} = \det \begin{pmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & \frac{2x}{a^2} \\ 0 & 1 & \frac{2y}{b^2} \end{pmatrix}$$

Hence, by taking the discriminant of this three by three matrix,

$$\therefore \vec{n} = \begin{pmatrix} -\frac{2x}{a^2} \\ -\frac{2y}{b^2} \\ 1 \end{pmatrix}$$

$$\Leftrightarrow |\vec{n}| = \sqrt{1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}}$$

$$\Leftrightarrow \hat{\mathbf{n}} = \frac{1}{\sqrt{1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}}} \times \begin{pmatrix} -\frac{2x}{a^2} \\ -\frac{2y}{b^2} \\ 1 \end{pmatrix}$$

Let us now compute the second partial derivatives of $\vec{\sigma}(x,y)$.

$$\vec{\sigma}_{xx} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{a^2} \end{pmatrix}; \ \vec{\sigma}_{yy} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{b^2} \end{pmatrix}; \ \vec{\sigma}_{xy} = \frac{\partial \vec{\sigma}(x,y)}{\partial x \partial y} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The coefficients of second fundamental form are therefore

$$\begin{cases} L = \dfrac{2}{a^2} \times \dfrac{1}{|\vec{n}|} \\ M = 0 \\ N = \dfrac{2}{b^2} \times \dfrac{1}{|\vec{n}|} \end{cases}$$

Hence, by substituting the value of E,F,G and L,M,N into Equation (3.0.5) and (3.0.4), we can obtain the curvature of the surface at any point.

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} \\ &= \frac{\frac{1}{|\vec{n}|^2} \times \frac{4}{a^2b^2}}{\left(1 + \frac{4x^2}{a^4}\right)\left(1 + \frac{4y^2}{b^4}\right) - \frac{16x^2y^2}{a^4b^4}} \\ &= \frac{\frac{1}{|\vec{n}|^2} \times \frac{4}{a^2b^2}}{1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4} + \frac{16x^2y^2}{a^4b^4} - \frac{16x^2y^2}{a^4b^4}} \\ &= \frac{\frac{1}{|\vec{n}|^2} \times \frac{4}{a^2b^2}}{|\vec{n}|^2} = \frac{4}{a^2b^2|\vec{n}|^4} \\ &= \frac{4}{a^2b^2\left(1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^2} \end{aligned}$$

$$\begin{aligned} H &= \frac{LG + EN - 2MF}{2(EG - F^2)} \\ &= \frac{\dfrac{2 \times \left(1 + \frac{4y^2}{b^4}\right)}{a^2 \times |\vec{n}|} + \dfrac{2 \times \left(1 + \frac{4x^2}{a^4}\right)}{b^2 \times |\vec{n}|}}{2 \times |\vec{n}|^2} \\ &= \frac{b^2 + \dfrac{4y^2}{b^2} + a^2 + \dfrac{4x^2}{a^2}}{a^2b^2|\vec{n}|^3} \\ &= \frac{b^2 + \dfrac{4y^2}{b^2} + a^2 + \dfrac{4x^2}{a^2}}{a^2b^2\sqrt{\left(1 + \frac{4x^2}{a^4} + \frac{4y^2}{b^4}\right)^3}} \end{aligned}$$

It is interesting to note that for a special case of elliptic paraboloid where $a = b$, there exists an *umbilical point* at the minimum point on the surface as illustrated in Figure 8.

Definition 3.3. An *Umbilical point* on the surface is **locally spherical**, meaning that $k_1 = k_2$ at that point (Weisstein, ”Umbilical Point”). This is because, for any point on a sphere, $k_1 = k_2$.

Proof. By substituting $x = y = 0$ and $b = a$, we obtain $K = \dfrac{4}{a^4}$ and $H = \dfrac{2}{a^2}$. Hence,

$$k_1 = k_2 = \frac{2}{a^2}$$

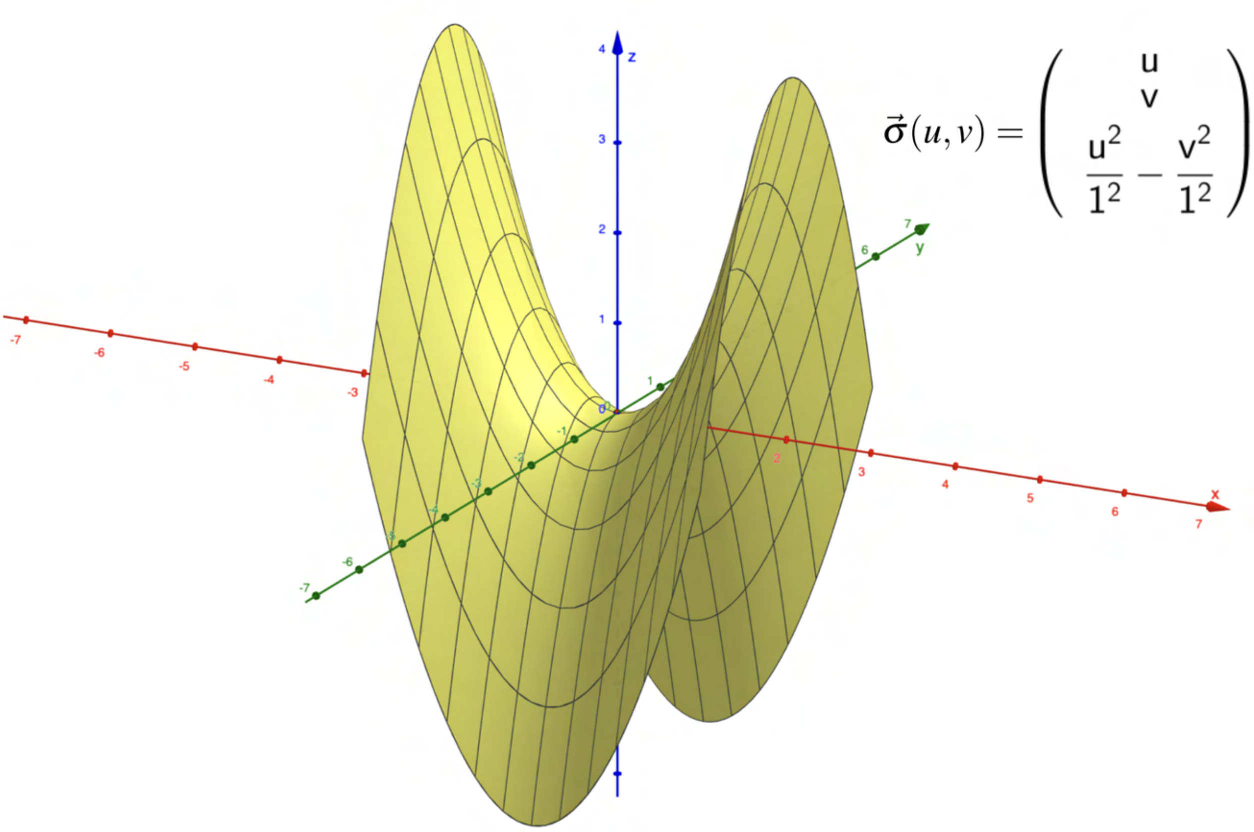
□

3.2.2 Hyperbolic paraboloid

Another type of paraboloid is the *Hyperbolic paraboloid*, which looks like a saddle in Figure 9. It has the general equation of

$$\vec{\sigma}(x,y) = \begin{pmatrix} x \\ y \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} \end{pmatrix} \tag{3.2.2}$$

Figure 9: Hyperbolic paraboloid, Figure made by candidate using GeoGebra



(3.2.2) : Weisstein, "Hyperbolic Paraboloid"

Using the same technique, the Gaussian and mean curvature of a general hyperbolic paraboloid is

$$K = \frac{-4}{a^2b^2\left(1+\frac{4x^2}{a^4}+\frac{4y^2}{b^4}\right)^2}$$

$$H = \frac{b^2+\frac{4y^2}{b^2}-a^2-\frac{4x^2}{a^2}}{a^2b^2\sqrt{\left(1+\frac{4x^2}{a^4}+\frac{4y^2}{b^4}\right)^3}}$$

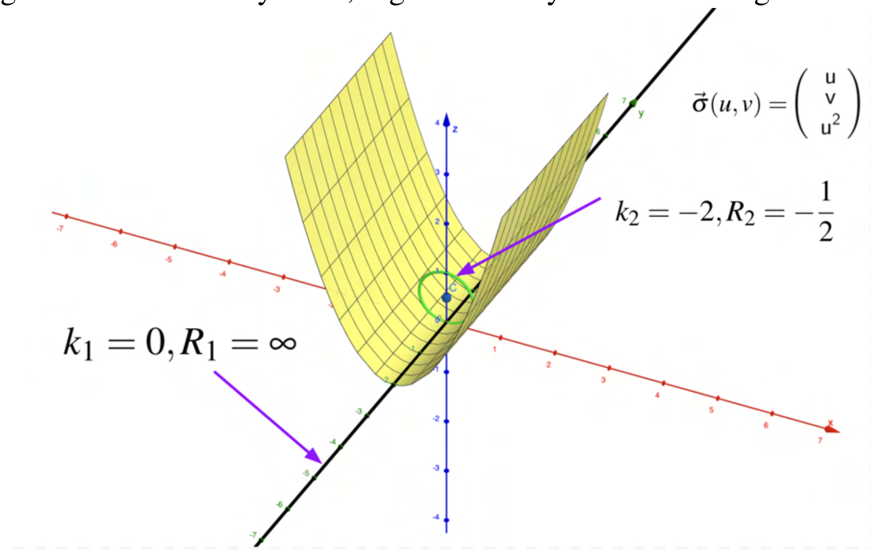
3.2.3 Parabolic cylinder

A parabolic cylinder is defined as

$$\vec{\sigma}(x,y) = \begin{pmatrix} x \\ a\times x^2 \\ y \end{pmatrix} \tag{3.2.3}$$

where $a \in \mathbb{R}\backslash\{0\}$ is a scaling constant.

Figure 10: Parabolic cylinder, Figure made by candidate using GeoGebra



Note that if we ignore the z component of the vector, the x and y components together are the parametric curve of a general parabola. Hence by simply adding z component, we derive the surface in Figure 10. Now, using the same technique,

(3.2.3) : Rejbrand, "Parabolic Cylinder"

$$\begin{cases} E = 1 + 4a^2x^2 \\ F = 0 \\ G = 1 \end{cases} \qquad \begin{cases} L = \frac{-2a}{\sqrt{4a^2x^2+1}} \\ M = 0 \\ N = 0 \end{cases}$$

note that for Gaussian curvature, K, since $LN - M^2 = 0$, the surface has **constant Gaussian curvature of 0.** This is because Gaussian curvature is an intrinsic measure and a flat plane can be deformed into a parabolic cylinder, meaning that its Gaussian curvature would be zero at all points, just like a flat plane. However, this does not imply that the surface has zero mean curvature. In fact,

$$H = \frac{L \times 1 + 0 - 2 \times 0}{2(E \times 1 - 0^2)} = \frac{L}{2E} = \frac{-a}{(4a^2x^2+1)^{\frac{3}{2}}}$$

3.3 Surfaces of constant curvature

An important class of surfaces that will be discussed in this paper is surfaces of revolution. Note that any such surface takes the form of

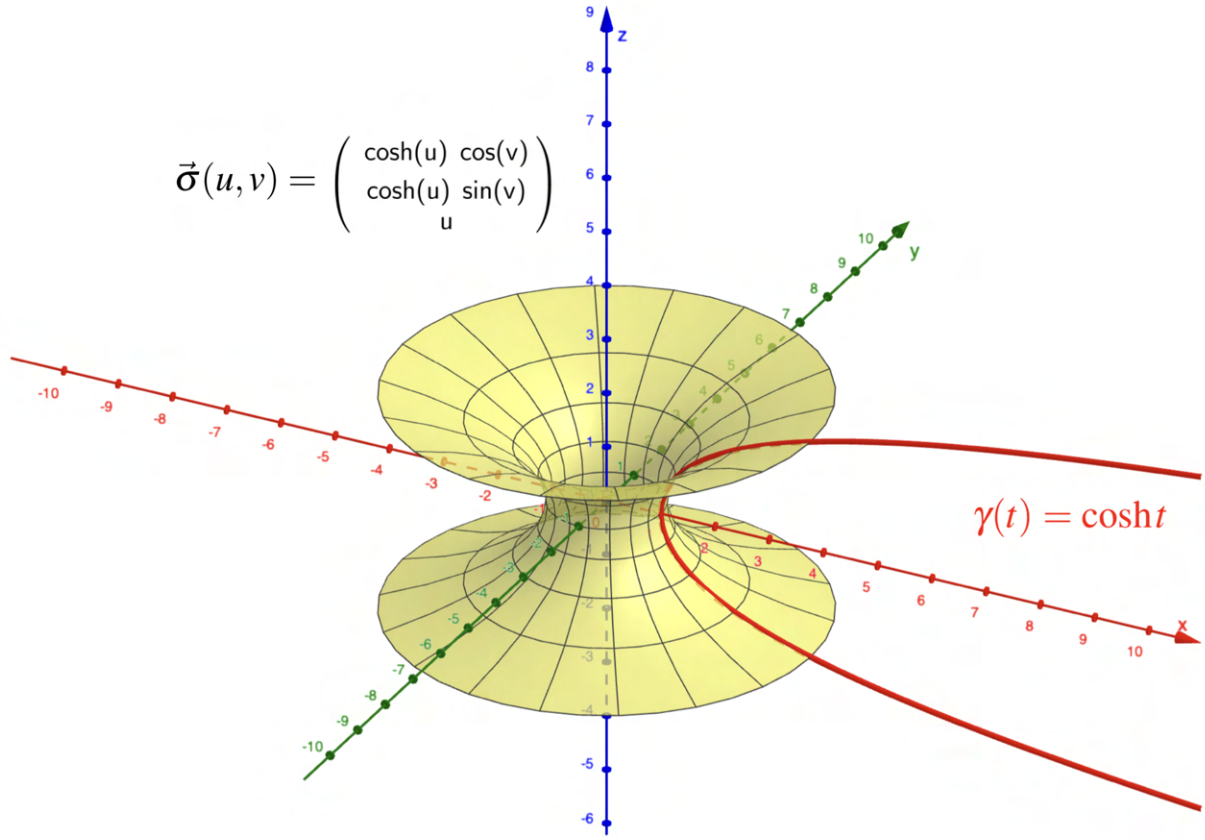
$$\vec{\sigma}(\theta, t) = \begin{pmatrix} x(t) \times \cos(\theta) \\ x(t) \times \sin(\theta) \\ z(t) \end{pmatrix} \tag{3.3.1}$$

where the parametric curve, $\vec{\gamma}(t)$, which is rotated around the z-axis has the equation

$$\vec{\gamma}(t) = \begin{pmatrix} x(t) \\ z(t) \end{pmatrix}$$

This is simply because $\vec{\varepsilon}(t) = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ is the parametric equation of the circle, thereby allowing the revolution of the curve. An example of such a surface is *Catenoid*, which will be discussed in Section 3.4.2.

Figure 11: Catenoid and Catenary, Figure made by candidate using GeoGebra



3.3.1 Sphere

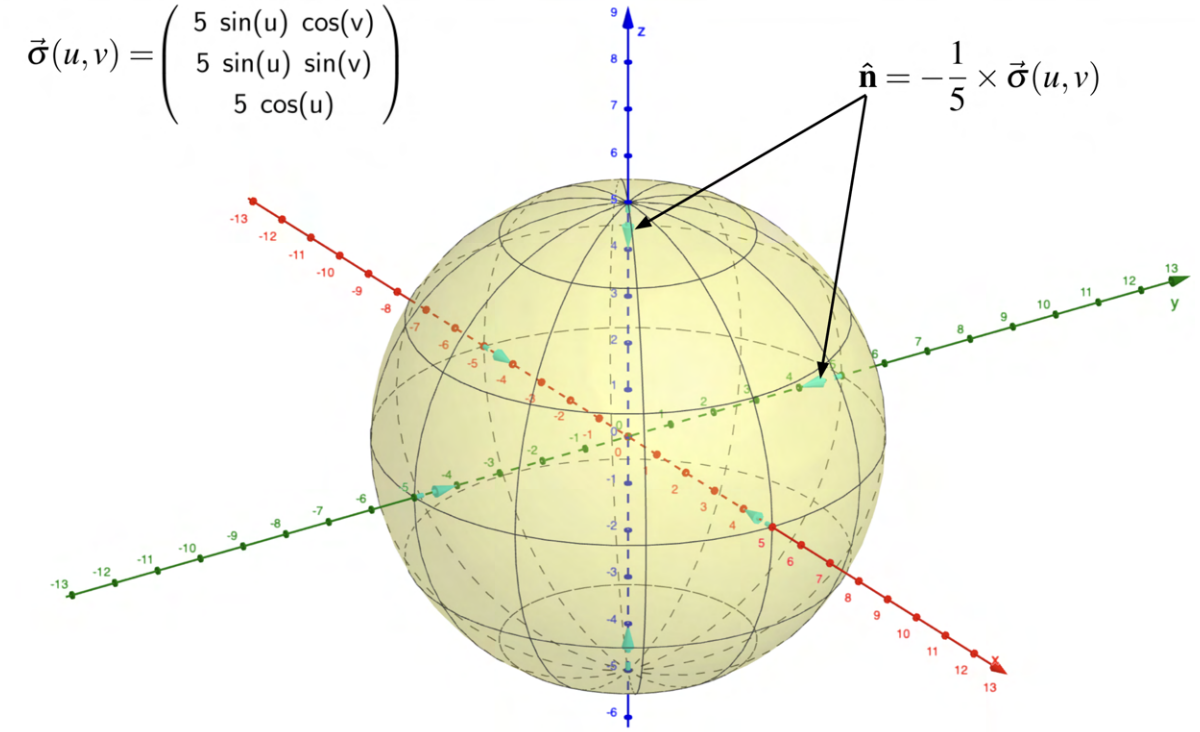
Hence, for a sphere,

$$\vec{\gamma}(t) = \begin{pmatrix} r \times \cos(t) \\ r \times \sin(t) \end{pmatrix} \Rightarrow \vec{\sigma}(\theta, t) = \begin{pmatrix} r \times \cos(t) \times \cos(\theta) \\ r \times \cos(t) \times \sin(\theta) \\ r \times \sin(t) \end{pmatrix}$$

where $r \in \mathbb{R}^+$ is the radius of the sphere. By substituting $\phi = \frac{\pi}{2} - t$, we can obtain the conventional form of the parametric equation (Weisstein, "Sphere").

$$\therefore \vec{\sigma}(\theta, \phi) = \begin{pmatrix} r \times \sin(\phi) \times \cos(\theta) \\ r \times \sin(\phi) \times \sin(\theta) \\ r \times \cos(\phi) \end{pmatrix}$$

Figure 12: Sphere, Figure made by candidate using GeoGebra



By taking the partial derivative w.r.t. θ and ϕ, we obtain

$$\vec{\sigma}_\theta = \begin{pmatrix} -r \times \sin(\phi) \times \sin(\theta) \\ r \times \sin(\phi) \times \cos(\theta) \\ 0 \end{pmatrix} ; \ \vec{\sigma}_\phi = \begin{pmatrix} r \times \cos(\phi) \times \cos(\theta) \\ r \times \cos(\phi) \times \sin(\theta) \\ -r \times \sin(\phi) \end{pmatrix}$$

We can therefore first compute the coefficients of first fundamental form.

$$\begin{cases} E = r^2 \sin^2(\theta) \sin^2(\phi) + r^2 \cos^2(\theta) \sin^2(\phi) = r^2 \sin^2(\phi) \\ F = r^2 \sin(\theta) \cos(\theta) \sin(\phi) \cos(\phi) - r^2 \sin(\theta) \cos(\theta) \sin(\phi) \cos(\phi) = 0 \\ G = r^2 \cos^2(\phi) \times (\sin^2(\theta) + \cos^2(\theta)) + r^2 \sin^2(\phi) = r^2 \end{cases}$$

Let us now compute the normal vector on a sphere,

$$\vec{n} = \vec{\sigma}_\theta \times \vec{\sigma}_\phi = \begin{pmatrix} -r^2 \sin^2(\phi) \cos(\theta) \\ -r^2 \sin^2(\phi) \sin(\theta) \\ -r^2 \sin^2(\theta) \cos(\phi) \sin(\phi) - r^2 \cos^2(\theta) \cos(\phi) \sin(\phi) \end{pmatrix}$$

Through simplification and factorisation,

$$\therefore \vec{n} = -r^2 \sin(\phi) \times \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}$$

Moreover, note that the unit vector $\hat{\mathbf{n}}$ are parallel but opposite to the direction of the position vector $\vec{\sigma}(\theta, \phi)$.

$$\begin{aligned} |\vec{n}| &= r^2 \sin(\phi)\sqrt{\sin^2(\phi) \times (\sin^2(\theta) + \cos^2(\theta)) + \cos^2(\phi)} \\ &= r^2 \sin(\phi) \\ \therefore \hat{\mathbf{n}} &= \begin{pmatrix} -\sin(\phi)\cos(\theta) \\ -\sin(\phi)\sin(\theta) \\ -\cos(\phi) \end{pmatrix} = -\frac{1}{r} \times \vec{\sigma}(\theta, \phi) \end{aligned}$$

Therefore, all unit normal vectors are pointing towards the origin of the sphere as shown in Figure 12. Let us now consider the second partial derivatives,

$$\vec{\sigma}_{\theta\theta} = \begin{pmatrix} -r \times \sin(\phi) \times \cos(\theta) \\ -r \times \sin(\phi) \times \sin(\theta) \\ 0 \end{pmatrix}; \; \vec{\sigma}_{\phi\phi} = \begin{pmatrix} -r \times \sin(\phi) \times \cos(\theta) \\ -r \times \sin(\phi) \times \sin(\theta) \\ -r \times \cos(\phi) \end{pmatrix}; \; \vec{\sigma}_{\theta\phi} = \begin{pmatrix} -r \times \cos(\phi) \times \sin(\theta) \\ r \times \cos(\phi) \times \cos(\theta) \\ 0 \end{pmatrix}$$

We can therefore compute the coefficients of second fundamental form.

$$\begin{cases} L = r\cos^2(\theta)\sin^2(\phi) + r\sin^2(\theta)\sin^2(\phi) = r\sin^2(\phi) \\ M = 0 \\ N = r\cos^2(\theta)\sin^2(\phi) + r\sin^2(\theta)\sin^2(\phi) + r\cos^2(\phi) = r \end{cases}$$

Using equation (3.0.3),

$$
\begin{aligned}
(L-KE)\times(N-kG)&=0\\
\Leftrightarrow (r\sin^2(\phi)-r^2\sin^2(\phi)k)(r-r^2k)&=0\\
\Leftrightarrow r^2\sin^2(\phi)(1-rk)=0\Leftrightarrow k_1=k_2&=\frac{1}{r}
\end{aligned}
$$

Hence, all points on the surface are **umbilical** and both Gaussian and mean curvature are constant and positive throughout the surface.

$$
\begin{aligned}
K&=\frac{1}{r^2}\\
H&=\frac{1}{r}
\end{aligned}
$$

3.3.2 General formula for surfaces of revolution

Let us now derive a general formula for the curvature of surfaces of revolution (Weisstein "Surface of Revolution"). Using equation (3.3.1),

$$
\vec{\sigma}_\theta=\begin{pmatrix}-x\sin(\theta)\\ x\cos(\theta)\\ 0\end{pmatrix};\ \vec{\sigma}_t=\begin{pmatrix}x'\cos(\theta)\\ x'\sin(\theta)\\ z'\end{pmatrix}
$$

Hence,

$$
\begin{cases}E=x^2\\ F=0\\ G=(x')^2+(z')^2\end{cases}
$$

Let us now compute the unit normal vector for this general case,

$$
\therefore \hat{\mathbf{n}}=\frac{\vec{\sigma}_\theta\times\vec{\sigma}_t}{|\vec{\sigma}_\theta\times\vec{\sigma}_t|}=\frac{1}{\sqrt{(x')^2+(z')^2}}\times\begin{pmatrix}z'\cos(\theta)\\ z'\sin(\theta)\\ -x'\end{pmatrix}
$$

Now by taking the second partial derivatives,

$$\vec{\sigma}_{\theta\theta} = \begin{pmatrix} -x\cos(\theta) \\ -x\sin(\theta) \\ 0 \end{pmatrix} ; \ \vec{\sigma}_{tt} = \begin{pmatrix} x''\cos(\theta) \\ x''\sin(\theta) \\ z'' \end{pmatrix} ; \ \vec{\sigma}_{xy} = \begin{pmatrix} -x'\sin(\theta) \\ x'\cos(\theta) \\ 0 \end{pmatrix}$$

which gives

$$\begin{cases} L = -\dfrac{xz'}{\sqrt{(x')^2+(z')^2}} \\ M = 0 \\ N = \dfrac{x''z' - x'z''}{\sqrt{(x')^2+(z')^2}} \end{cases}$$

Therefore,

$$K = \frac{LN}{EG} = \frac{-z'(x''z' - x'z'')}{x \times \left((x')^2+(z')^2\right)^2} \tag{3.3.2}$$

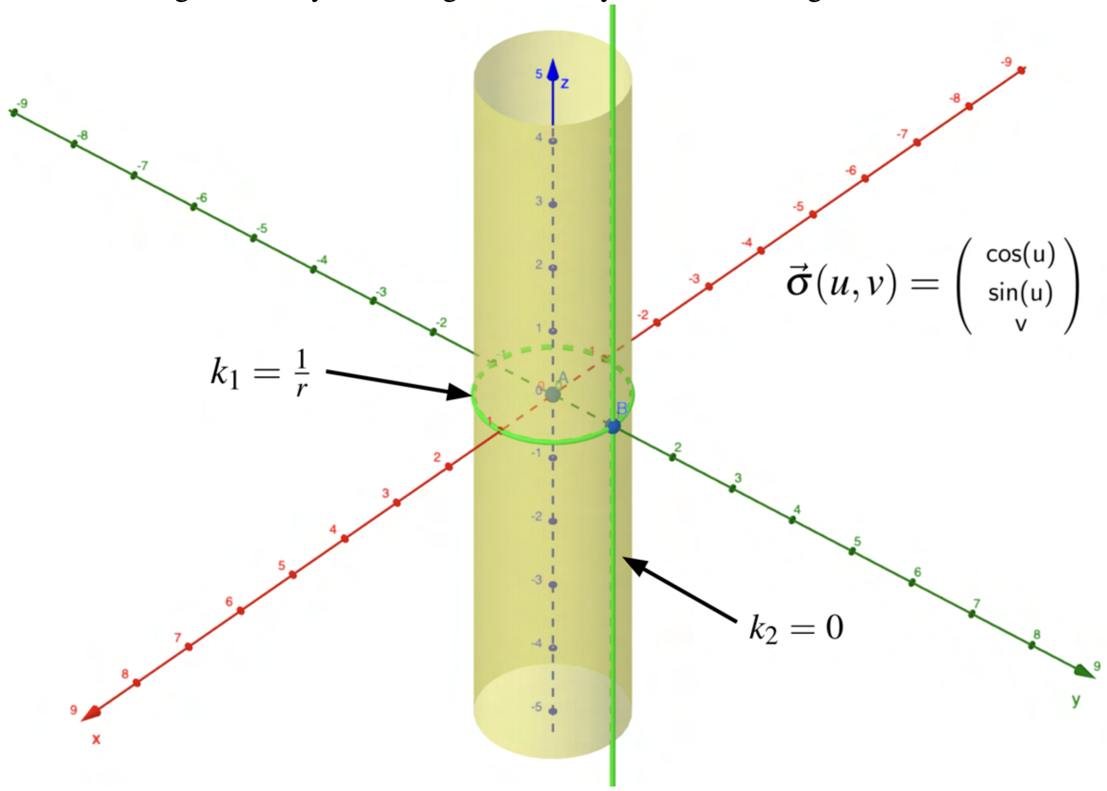
The explicit form of the mean curvature can be derived in a similar manner

$$H = \frac{LG+NE}{2EG} = \frac{x \times (x''z' - x'z'') - z' \times ((x')^2+(z')^2)}{2x \times \left((x')^2+(z')^2\right)^{\frac{3}{2}}} \tag{3.3.3}$$

3.3.3 Constant-Mean-Curvature surface (CMC)

CMC are surfaces with constant mean curvature everywhere. Note that Minimal surface, an important class of *CMC* with $H = 0$ will be discussed in section 3.4. As proven previously, sphere is a *CMC* because it has constant mean curvature of $H = \frac{1}{r}$. Another surface with non-zero mean curvature is cylinder.

Figure 13: Cylinder, Figure made by candidate using GeoGebra



Proof. Let us consider a cylinder that is infinitely long so that we do not need to consider the curvatures at the two ends. Let us chose an arbitrary point B on the surface and consider the osculating circles in two *principal directions*. Along the direction parallel to the cylinder's axis, the minimum principal curvature, $k_2 = 0$ at all points because it is a straight line. As for k_1, let us consider the axis which is perpendicular to the cylinder's axis, which shows us the cross-section of the cylinder, represented by a circle. Hence, $k_1 = \frac{1}{r}$.

$$\therefore H = \frac{1}{2r}$$

A more rigorous proof is as follows,

$$\vec{\sigma}(\theta,t) = \begin{pmatrix} r \times \cos(\theta) \\ r \times \sin(\theta) \\ t \end{pmatrix}$$

where $r \in \mathbb{R}\backslash\{0\}$ is the radius of the cylinder. By following the previous procedure,

$$\begin{cases} E = 1 \\ F = 0 \\ G = r^2 \end{cases} \Leftrightarrow \hat{\mathbf{n}} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} L = 0 \\ M = 0 \\ N = r \end{cases}$$

Hence, using equation (3.3.3),

$$H = \frac{r}{2r^2} = \frac{1}{2r}$$

□

However, it is worth noting that in many cases, *CMC* does not have constant Gaussian curvature, except for sphere and the trivial case for any planes, where $H = K = 0$ because they have no deviation from a plane.

3.4 Minimal surface

Definition 3.4 (Minimal surface)**.** For a given boundary condition, a *minimal surface* minimises its surface area (Weisstein "Minimal Surface"). For example, let us consider a sealed cylinder and a sphere both with radius $r \in \mathbb{R}^+$ and volume $100m^3$. It follows trivially that the surface area of the sphere is approximately $479m^2$ and the surface area of the cylinder is around $272m^2$. Hence, we can conclude that a cylinder minimises its surface area more than a sphere given a boundary condition. However, a true *minimal surface* would minimise this area further. It is mathematically defined to have $H = 0$ everywhere (Weisstein "Minimal Surface").

3.4.1 Hyperbolic functions

One of the classical minimal surfaces is *Catenoid* and its transformed surface, *Helicoid*. However, before studying those surfaces in detail, some knowledge and understanding of hyperbolic functions are needed (Weisstein "Hyperbolic Functions"). Let us first define some hyperbolic functions. Note that similar to the complex definition of sine and cosine function,

$$-i\sin iz = \sinh z \equiv \frac{e^z - e^{-z}}{2} \tag{3.4.1}$$
$$\cos iz = \cosh z \equiv \frac{e^z + e^{-z}}{2} \tag{3.4.2}$$
$$\tanh z \equiv \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}} \tag{3.4.3}$$

An important identity can be deduced from this definition,

Lemma 3.1. *Proof.*

$$\begin{aligned} \cos^2 iz - (-i\sin iz)^2 &= \cos^2 iz + \sin^2 iz = 1 \\ &\Leftrightarrow \cosh^2 z - \sinh^2 z = 1 \end{aligned} \tag{3.4.4}$$

□

For Section 3.5, an understanding of reciprocal hyperbolic functions is also needed. Similar to trigonometric equations,

$$\text{cosech} z \equiv \frac{1}{\sinh z} \tag{3.4.5}$$
$$\text{sech} z \equiv \frac{1}{\cosh z} \tag{3.4.6}$$

Hence, another essential identity can also be derived

Lemma 3.2. *Proof.*

$$\begin{aligned} \tanh^2 z + \text{sech}^2 z &= \frac{\sinh^2 z}{\cosh^2 z} + \frac{1}{\cosh^2 z} \\ &= \frac{\sinh^2 z + 1}{\cosh^2 z} \end{aligned}$$

Using *Lemma* (3.1),

$$\tanh^2 z + \mathrm{sech}^2 z = \frac{\cosh^2 z}{\cosh^2 z} = 1 \tag{3.4.7}$$

□

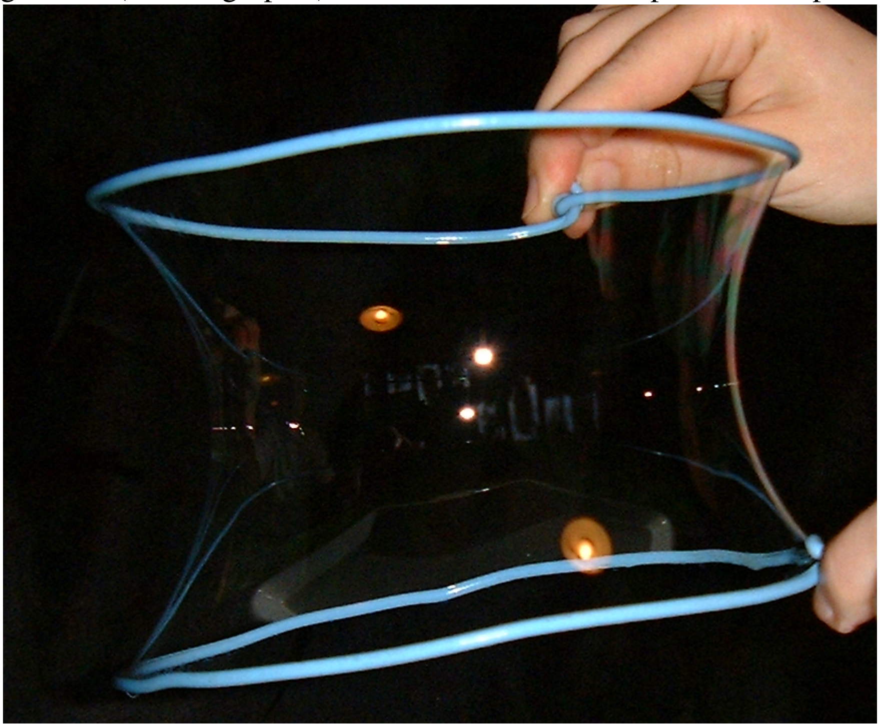
Let us now consider the derivatives of those hyperbolic functions.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}(\sinh z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^z - e^{-z}}{2}\right) = \frac{e^z + e^{-z}}{2} = \cosh z \\
\frac{\mathrm{d}}{\mathrm{d}z}(\cosh z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^z - e^{-z}}{2} = \sinh z \\
\frac{\mathrm{d}}{\mathrm{d}z}(\tanh z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\sinh z}{\cosh z}\right) = \frac{\cosh^2 z - \sinh^2 z}{\cosh^2 z} = \mathrm{sech}^2 z \\
\frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{sech} z) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1}{\cosh z}\right) = -\frac{\sinh z}{\cosh^2 z} = -\tanh z\ \mathrm{sech} z
\end{aligned}$$

3.4.2 Catenoid

Catenoid has a profound connection with nature. For example, it is the shape of the soap film confined by two circular rings such as in Figure 14. Moreover, it is the surface of revolution of catenary, the shape of a flexible hanging rope.

Figure 14: (Blinking Spirit). Catenoid in nature : shape of the soap film.



Proof. Let us now consider the parametric equation for a Catenoid, note that it is the surface of revolution created by rotating the function $\gamma(t) = \cosh t$. An example of such a surface can be seen in Figure 11

$$\vec{\sigma}(\theta,t) = \begin{pmatrix} a\cosh t \times \cos(\theta) \\ a\cosh t \times \sin(\theta) \\ a \times t \end{pmatrix} \tag{3.4.8}$$

where $a \in \mathbb{R}\backslash\{0\}$ is the scaling parameter . Using the general equations derived in Section 3.3.2, we obtain

$$\begin{cases} E = a^2\cosh^2 t \\ F = 0 \\ G = a^2\sinh^2 t + a^2 = a^2\cosh^2 t \ \text{ using Lemma 3.1} \end{cases}$$

Since $E = G$ and $F = 0$, The formula for mean curvature can be reduced to

$$H = \frac{E \times (N+L)}{EG} = \frac{N+L}{G} = \frac{(\vec{\sigma}_{tt} + \vec{\sigma}_{\theta\theta}) \cdot \hat{\mathbf{n}}}{G}$$

Now note that since $\dfrac{\mathrm{d}^2}{\mathrm{d}z^2}(\cosh z) = \cosh z \Leftrightarrow x'' = x$ and $z'' = 0$,

$$\therefore \vec{\sigma}_{tt} = -\vec{\sigma}_{\theta\theta}$$

Hence,

$$H = \frac{\vec{0} \cdot \hat{\mathbf{n}}}{G} = 0$$

Catenoid is therefore a minimal surface because $H = 0, \forall(\theta,t) \in [0,2\pi] \times \mathbb{R}$. However, the Gaussian curvature of Catenoid is neither zero nor constant.

$$k_1, k_2 = \pm\frac{1}{a\cosh^2 t} \Leftrightarrow K = \frac{1}{a^2\cosh^4 t}$$

□

(3.4.8) : Tapp 233

3.4.3 Helicoid

Note that it is possible to continuously deform a Catenoid into a Helicoid using the following parametrization of deformation. However, to do so, it is needed to introduce a new deformation parameter, ψ, which rotates parts of the Catenoid upwards, creating a Helicoid (Figure 15).

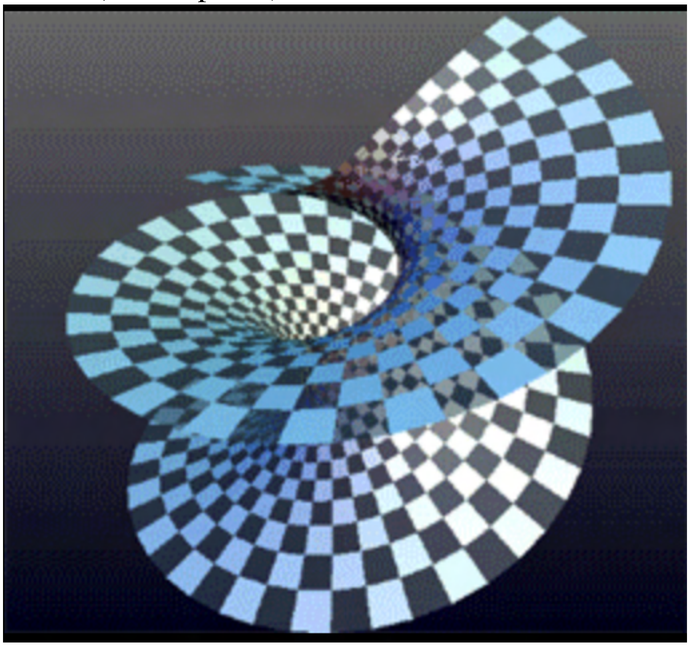
$$x(\theta,t) = \cos\psi \sinh t \sin\theta + \sin\psi \cosh t \cos\theta$$

$$y(\theta,t) = -\cos\psi \cosh t \sin\theta + \sin\psi \cosh t \sin\theta$$

$$z(\theta,t) = \theta\cos\psi + t\sin\psi$$

However, its proof is beyond the scope of this paper and insignificant to the research question thereby will not be included. Let us now prove that Helicoid is minimal.

Figure 15: (Wickerprints). Cetanoid deformation to Helicoid.



Proof. Let us first consider the general parametric formula of a Helicoid

$$\vec{\sigma}(\theta,t) = \begin{pmatrix} a\sinh t \times \cos(\theta) \\ a\sinh t \times \sin(\theta) \\ a \times \theta \end{pmatrix} \tag{3.4.9}$$

(3.4.9) : Tapp 233

By taking its partial derivatives, we obtain

$$E = G = a^2 \cosh^2 t$$

$$F = 0$$

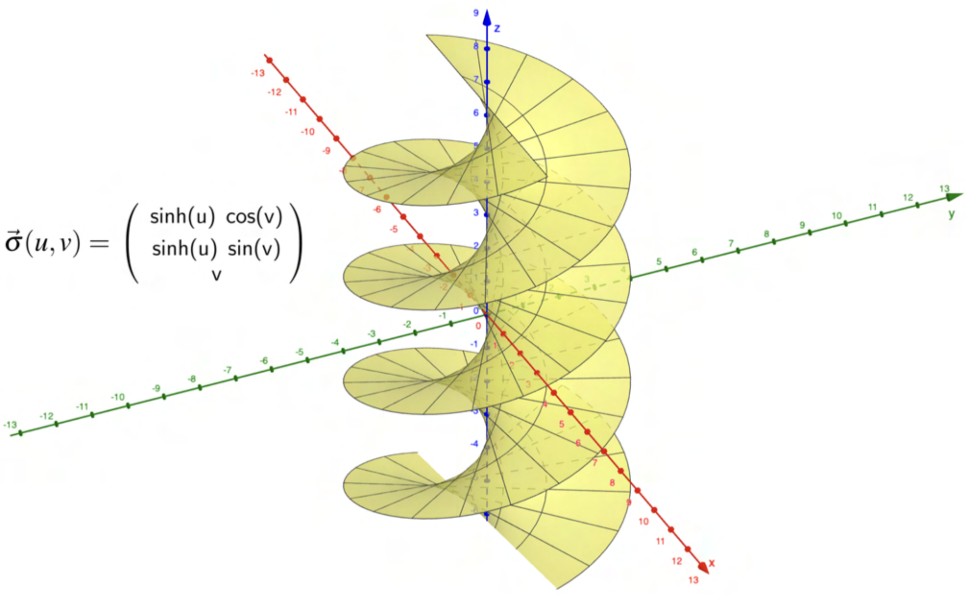
Since

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}(\sinh t) = \sinh t \Leftrightarrow x'' = x$$

$$\vec{\sigma}_{tt} = -\vec{\sigma}_{\theta\theta}$$

Helicoid is also a minimal surface. $\square$

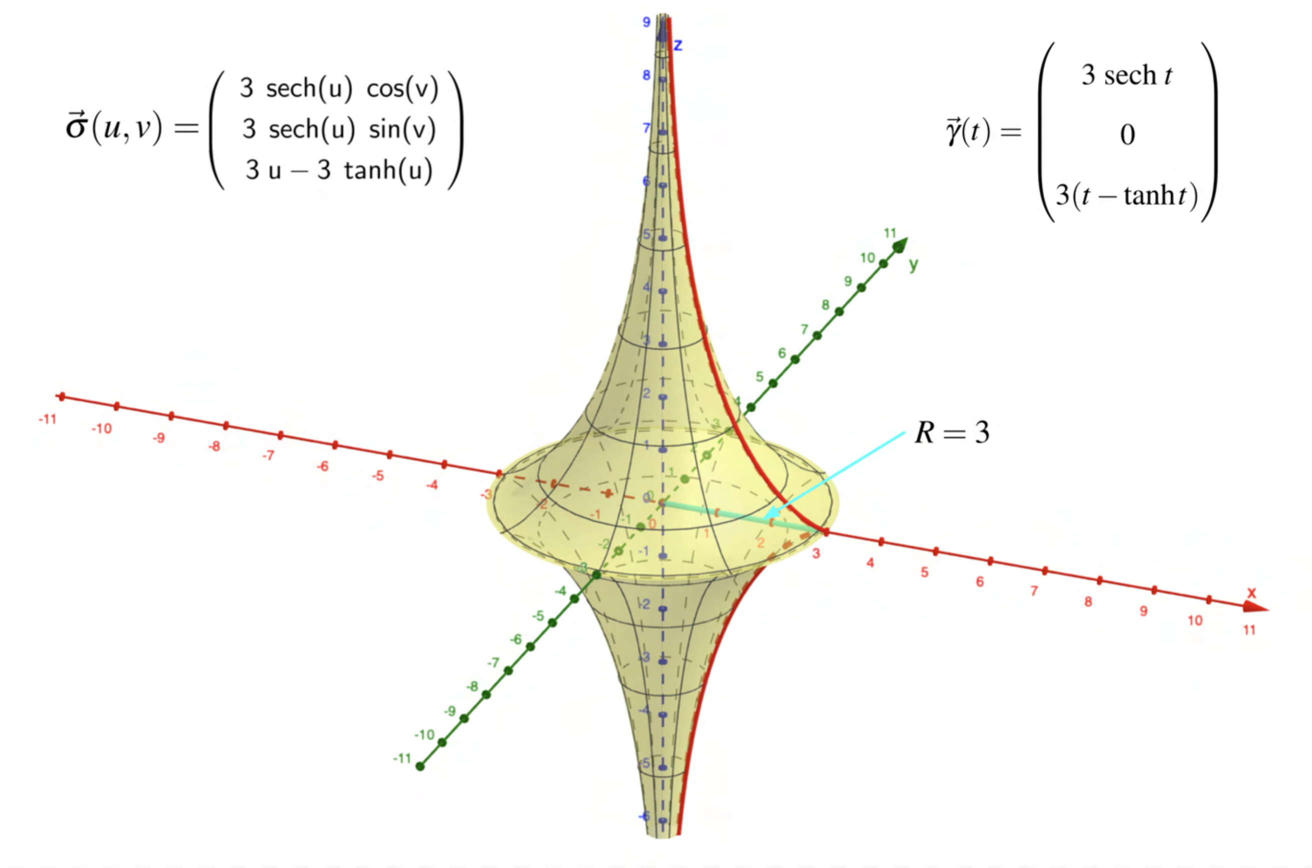
Figure 16: Helicoid, Figure made by candidate using GeoGebra



3.5 Surface of constant Gaussian curvature

In this section, the curvature of *pseudosphere* will be explored. It is a surface of revolution which has a constant negative Gaussian curvature produced by rotating a *tractrix* around its vertical asymptote (Weisstein "Pseudosphere"). It is called a pseudosphere because it has $K = -\frac{1}{R^2}$ everywhere on the surface. Let us now formally prove its constant Gaussian curvature.

Figure 17: Pseudosphere with $R = 3$, Figure made by candidate using GeoGebra



Proof.

$$\vec{\sigma}(\theta,t) = \begin{pmatrix} R\ \mathrm{sech}\, t \times \cos(\theta) \\ R\ \mathrm{sech}\, t \times \sin(\theta) \\ R(t - \tanh t) \end{pmatrix} \tag{3.5.1}$$

Before applying the explicit formulas derived in section 3.3.2, we first need to compute the derivatives of $x(t)$ and $z(t)$ w.r.t. t.

$$\begin{cases} x = R\mathrm{sech}\, t \\ x' = R(-\tanh t\ \mathrm{sech}\, t) \\ x'' = R(-\mathrm{sech}^3 t + \tanh^2 t\ \mathrm{sech}\, t) \end{cases} \qquad \begin{cases} z = R(t - \tanh t) \\ z' = R(1 - \mathrm{sech}^2 t) = R\tanh^2 t \\ z'' = 2R\mathrm{sech}^2 t \tanh t \end{cases}$$

Since direct substitution is too complicated, we can first compute the coefficients of first and

(3.5.1) : Weisstein "Pseudosphere"

second fundamental forms.

$$\begin{cases} E = R^2 \text{sech}^2 t \\ F = 0 \\ G = R^2 \tanh^4 t + R^2 (\tanh^2 t \ \text{sech}^2 t) = R^2 \tanh^2 t \underbrace{(\tanh^2 t + \ \text{sech}^2 t)}_{=1} = R^2 \tanh^2 t \end{cases}$$

Since

$$\sqrt{G} = \sqrt{(x')^2 + (z')^2} = R \tanh t,$$

Expressions for L and N can therefore be written as

$$\begin{aligned} L &= -\frac{R^2 \ \text{sech}\ t \tanh^2 t}{R \tanh t} \\ &= -R \ \text{sech}\ t \tanh t \\ N &= \frac{R^2 (-\text{sech}^3 t + \tanh^2 t \ \text{sech}\ t) \tanh^2 t + R^2 \tanh^2 t \ \text{sech}^3 t}{R \tanh t} \\ &= R \tanh t \ \text{sech}\ t \underbrace{(\tanh^2 t - \text{sech}^2 t + 2\text{sech}^2 t)}_{=1} \\ &= R \tanh t \ \text{sech}\ t \end{aligned}$$

Hence,

$$K = \frac{LN}{EG} = -\frac{R^2 \text{sech}^2 t \tanh^2 t}{R^4 \text{sech}^2 t \tanh^2 t} = -\frac{1}{R^2}$$

□

4 Conclusion

In conclusion, **this paper developed complete and thorough methods of researching and proving parametric curves and surfaces of constant curvature with the help of specific examples and figures.** Starting with the fundamentals in differential geometry exemplified by osculating circles; first and second fundamental forms, etc. Explicit formulas are derived for general cases and the significance of the results was discussed. An emphasis on surfaces of revolution and minimal surface was made due to their importance in real-world applications.

However, some concepts such as the mathematical proof for the definition of minimal surfaces were not included in this paper because the technical preliminary required to understand the proof is beyond the scope of this paper. Methods of solving differential equations such as (3.3.2) and (3.3.3) to derive the equations of the curves which produces surfaces of revolution with constant curvature could also be further developed. Moreover, other CMC exemplified by unduloid and nodoid are not discussed because the functions which define those surfaces are too advanced.

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