Introduction and Application of Fourier Analysis

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1 Introduction

Fourier analysis is usually taught to engineering or mathematics undergraduate students in their 3^{rd} or 4^{th} year. It is the cornerstone of modern signal processing due to its ability to transform the time domain of a signal into its frequency domain.

Furthermore, Fourier analysis can be used (and was initially used) to solve partial differential equations, which predict a range of physical and other problems involving functions of several variables, such as fluid flow, heat and sound propagation etc. (Fourier analysis)

However, we are all born knowing how to do this complicated calculation. In fact, our ears and brain perform this intricate task continuously and subconsciously. However, this mathematical marvel and one of the most elegant equations in the history is not discovered until the late 18 Century by the French mathematician Joseph Fourier. The profound interrelationship between time and frequency domains is succinctly summarized by the two equations below, where ν denotes Frequency and t denotes Time.

$$F(\nu) = \int_{-\infty}^{\infty} f(t)e^{-2\pi it\nu}\, dt \tag{1}$$

$$f(t) = \int_{-\infty}^{\infty} F(\nu)e^{2\pi it\nu}\, d\nu \tag{2}$$

2 Technical Preliminaries

2.1 Integration by parts

Using Product Rule:

$$\frac{d}{dx}u(x) \cdot v(x) = u'(x) \cdot v(x) + v'(x) \cdot u(x)$$

Integrating both sides and split the terms give:

$$u(x) \cdot v(x) = \int u'(x) \cdot v(x)\, dx + \int v'(x) \cdot u(x), dx$$

$\therefore$

$$\int u'(x) \cdot v(x)\, dx = u(x) \cdot v(x) - \int v'(x) \cdot u(x), dx \tag{3}$$

$\square$

Tip: When performing integration by parts, let the part that is easier to integrate be $u'(x)$ and let the other part be $v(x)$.

2.2 Euler's equation

$$e^{iat} = \cos(at) + i\sin(at) \tag{4}$$

The MacLaurin series:

$$\begin{aligned}
\sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots
\end{aligned}$$

Substitute $z = ix$ in the last series:

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots \\
&= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \cdots \\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) \\
&= \cos x + i \sin x
\end{aligned}$$

$\square$

2.3 Deric-Delta function

In order to understand Fourier transforms and solve a few simple ones by hand, it is crucial to understand and fluently implement the Dirac Delta function. To see why, equation (1) is the function which transforms a signal in the time domain to a function of frequency domain.

It turns out to be extremely complicated to integrate over an infinite interval (an improper integral), especially when it involves finding the limit of complex exponents. Therefore, we use several properties of the delta function to simplify this process. The delta function is defined as a “function equal to zero everywhere except for zero and whose integral over the entire real line is equal to one” (Dirac delta), which gives it this important property.

$$\int_{-\infty}^{\infty} \delta(t-a) g(t) dt = g(a) \tag{5}$$

It is also important to note that the function can be expressed as the equation below because it greatly facilitates the substitution.

$$\delta(\nu - a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it(\nu - a)}\, dt \tag{6}$$

Now consider the classic example of Fourier transform below to calculate the frequencies of a generalized cosine function (ibmathsresources).

$$Let f(t) = cos(at), \text{where a} \in \mathbb{R}$$

Note that :

$$\begin{aligned}
e^{iat} &= \cos(at) + i\sin(at) \\
e^{-iat} &= \cos(at) - i\sin(at) \\
2\cos(at) &= e^{iat} + e^{-iat} \\
f(t) &= \frac{e^{iat} + e^{-iat}}{2}
\end{aligned}$$

Hence, using equation (1),

$$\begin{aligned}
F(\nu) &= 0.5\int_{-\infty}^{\infty} e^{iat-2\pi it\nu} + e^{-iat-2\pi it\nu}\, dt \\
&= 0.5\int_{-\infty}^{\infty} e^{iat-2\pi it\nu}\, dt + 0.5\int_{-\infty}^{\infty} e^{-iat-2\pi it\nu}\, dt (*)
\end{aligned}$$

Now, using equation (4), :

$$\begin{aligned}
\delta(a - 2\pi\nu) &= \frac{1}{2\pi}e^{it(a-2\pi\nu)} \\
\delta(-a - 2\pi\nu) &= \frac{1}{2\pi}e^{it(-a-2\pi\nu)}
\end{aligned}$$

Therefore, from (*),

$$F(\nu) = \pi(\delta(a - 2\pi\nu) + \delta(-a - 2\pi\nu))$$

$\therefore$

$$F(\nu) = 0$$

Except when the domain of the delta function = 0.
i.e.

$$a - 2\pi\nu = 0$$

or

$$-a - 2\pi\nu = 0$$

Thus,

$$\nu = \pm\frac{a}{2\pi}.$$

Note that using the definition of delta function, for other values of the domain,

$$F(\nu) = \pi(0 + 0) = 0$$

3 Fourier series

3.1 Real Fourier series and examples

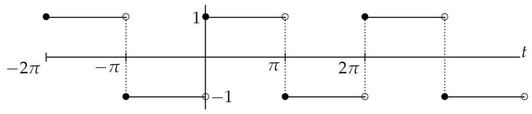
It is worth noticing that the entire Fourier transform is built upon the basic of the harmonic motion described by Fourier series. Fourier theorem states that a periodic and continuous function $f(t)$ can be expressed using an infinite series of sine and cosine terms. The Fourier series (equation (7)), would eventually converge to $f(t)$.

3.1.1 Square Wave Example

Compute the Fourier series of $f(t)$, where $f(t)$ is the square wave with period 2π. Which is defined over one period by

$$f(t) = \begin{cases} -1, & -\pi \le t < 0 \\ 1, & 0 \le t < \pi \end{cases}$$

Figure 1: The graph of the function over periods (MIT)



Solution: computing the Fourier coefficients given with Fourier's theorem:

$$f\ (t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nt) + b_n \sin nt) \tag{7}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt$$

We need to split the integrals into two corresponding pieces to where $f(t)$ is $+1$ and where it is -1, so that we may apply these formulas to this square wave function. i.e.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos\ nt\, dt = \frac{1}{\pi}\int_{-\pi}^{0} (-1)\cdot \cos\ nt\, dt + \frac{1}{\pi}\int_{0}^{\pi} (1)\cdot \cos\ nt\, dt$$

therefore, for n $\neq 0$:

$$a_n = -\frac{\sin nt}{n\pi}\bigg|_{-\pi}^{0} + \frac{\sin nt}{n\pi}\bigg|_{0}^{\pi} = 0,$$

as for $n = 0$, similarly we have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dt + \frac{1}{\pi}\int_{0}^{\pi}(1)\,dt$$
$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = 0.$$

Likewise

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cdot\sin nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\cdot\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(1)\cdot\sin nt\,dt \\ &= \frac{\cos nt}{n\pi}\bigg|_{-\pi}^{0} - \frac{\cos nt}{n\pi}\bigg|_{0}^{\pi} = \frac{1-\cos(-n\pi)}{n\pi} - \frac{\cos n\pi - 1}{n\pi} \\ &= \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}(1-(-1)^n) = \begin{cases} 0, & \text{for n is even} \\ \frac{4}{n\pi}, & \text{for n is odd} \end{cases} \end{aligned}$$

Therefore the Fourier series for $f(t)$ is

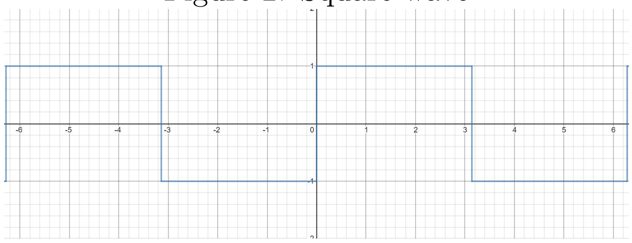
$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt = \frac{4}{\pi}\,(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots).$$

□

3.1.2 Using sine waves to output square waves

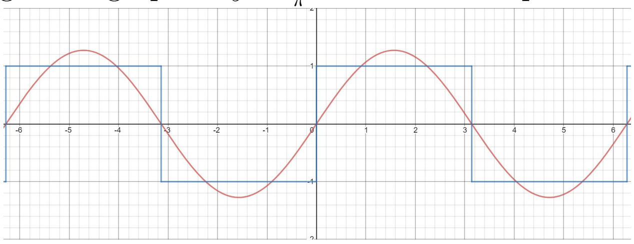
According to the Fourier theorem, the combination of sine and cosine waves can compose other periodic functions. We therefore make square waves using sine waves.
Our target square wave looks like this:

Figure 2: Square wave



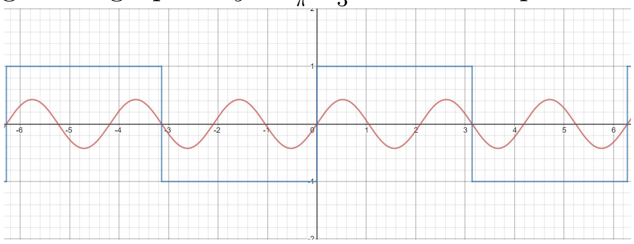
If we input the function $y = \frac{4}{\pi} \sin x$, then we have:

Figure 3: graph of $y = \frac{4}{\pi} \sin x$ and the square wave



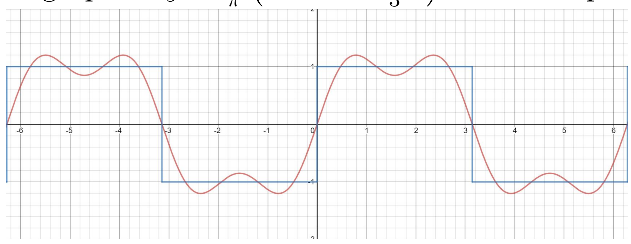
Input $y = \frac{4}{\pi} \cdot \frac{\sin 3x}{3}$, we have

Figure 4: graph of $y = \frac{4}{\pi} \frac{\sin 3x}{3}$ and the square wave



Now we add the two functions above to get:

Figure 5: graph of $y = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} \right)$ and the square wave



We can clearly see that the function $y = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} \right)$ is somewhat approaching our target square wave. If we continue to add more sine waves to the function:

Figure 6: graph of $y = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7} + \frac{\sin 9x}{9} \right)$ and the square wave

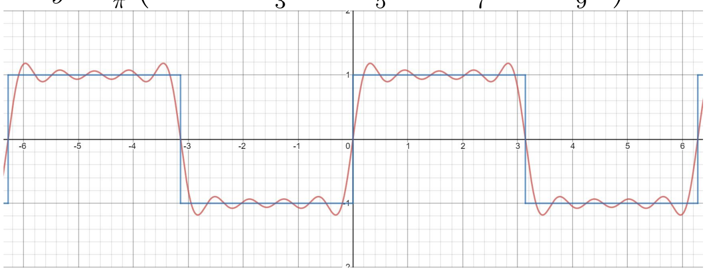
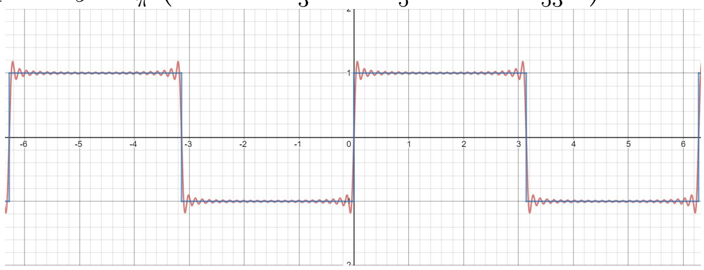


Figure 7: graph of $y = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \cdots + \frac{\sin 53x}{53}\right)$ and the square wave



Therefore, if an infinite series of sine waves were to be added in this pattern, the limit of which would tend to the form of a square wave. Therefore, we can deduce that:

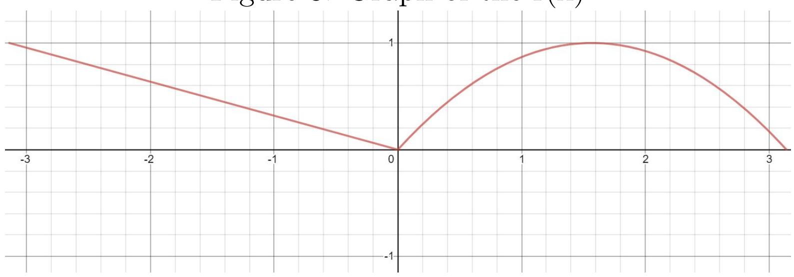
$$\text{A square wave} = \frac{4}{\pi}\left(\sin x + \frac{\sin 5x}{3} + \frac{\sin 5x}{5} + \cdots\right)$$

And this is the fundamental idea of Fourier series.

3.1.3 Linear and parabola example

$$r(x) = \begin{cases} -\frac{1}{\pi}x, & -\pi \leq x \leq 0 \\ -\frac{4}{\pi^2} \cdot (x^2 - \pi x), & 0 < x \leq \pi \end{cases}$$

Figure 8: Graph of the r(x)



$$\begin{aligned}
a_0 &= \frac{1}{\pi}\left(\int_{\pi}^{0} -\frac{x}{\pi}dx - \frac{4}{\pi^2}\int_0^{\pi} x^2 - \pi x\,dx\right) \\
&= \frac{1}{\pi}\left[\left(-\frac{x^2}{2\pi}\Big|_{-\pi}^{0}\right) - \frac{4}{\pi^2}\cdot\left(\frac{x^3}{3} - \frac{\pi}{2}x^2\Big|_0^{\pi}\right)\right] \\
&= \frac{1}{\pi}\left[\left(0 + \frac{\pi}{2}\right) - \frac{4}{\pi^2}\left(\frac{\pi^3}{3} - \frac{\pi^3}{2} - 0\right)\right] \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{4}{\pi^2}\cdot - \frac{\pi^2}{6}\right) \\
&= \frac{1}{\pi}\left(\frac{\pi}{2} + \frac{2\pi}{3}\right) = \frac{1}{\pi}\cdot\frac{7\pi}{6} = \frac{7}{6}
\end{aligned}$$

$$① \int x \cdot \cos x dx = \frac{x}{n}\sin nx - \frac{1}{n}\int \sin nx\,dx$$

$$= \frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx + c$$

where we have $u = x \quad u' = 1 \quad v' = \cos nx \quad v = \frac{1}{n}\sin nx.$

$$② \int x \cdot \sin nx\,dx = -\frac{x}{n}\cos nx + \frac{1}{n}\int \cos nx dx$$

$$= -\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx + c$$

where we have $u = x \quad u' = 2x \quad v' = \cos nx \quad v = -\frac{1}{n}\cos nx.$

$$③ \int x^2\cos nx\,dx = \frac{x^2}{n}\sin nx - \frac{2}{n}\int x\sin nx\,dx$$

$$= \frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx + c$$

where we have $u = x^2 \quad u' = 2x \quad v' = \cos nx \quad v = \frac{1}{n}\sin nx.$

$$④ \int x^2\sin nx\,dx = -\frac{x^2}{n}\cos nx + \frac{2}{n}\int x\cos nx\,dx$$

$$= -\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx + c$$

where we have $u = x^2 \quad u' = 2x \quad v' = \sin nx \quad v = -\frac{1}{n}\cos nx.$

$$\begin{aligned}
\therefore\ a_n &= \frac{1}{\pi}\left(-\frac{1}{\pi}\int_{-\pi}^{0} x\cdot\cos nx\,dx - \frac{4}{\pi^2}\int_0^{\pi} x^2\cdot\cos nx\,dx + \frac{4}{\pi}\int_0^{\pi} x\cdot\cos nx\,dx\right)\\
&= -\frac{1}{\pi^2}\left[\left(\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\Bigg|_{-\pi}^{0}\right] - \frac{4}{\pi^3}\left[\left(\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right)\Bigg|_{0}^{\pi}\right]+\\
&\quad \frac{4}{\pi^2}\left[\left(\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right)\Bigg|_{0}^{\pi}\right]\\
&= -\frac{1}{\pi^2}\left(\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right) - \frac{4}{\pi^3}\left(\frac{2\pi}{n^2}(-1)^n\right) + \frac{4}{n^2}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\\
&= -\frac{1}{n^2\pi^2}\left(1-(-1)^n + 8(-1)^n - 4(-1)^n + 4\right)\\
&= -\frac{1}{n^2\pi^2}\left(5 + 3\cdot(-1)^n\right),
\end{aligned}$$

And

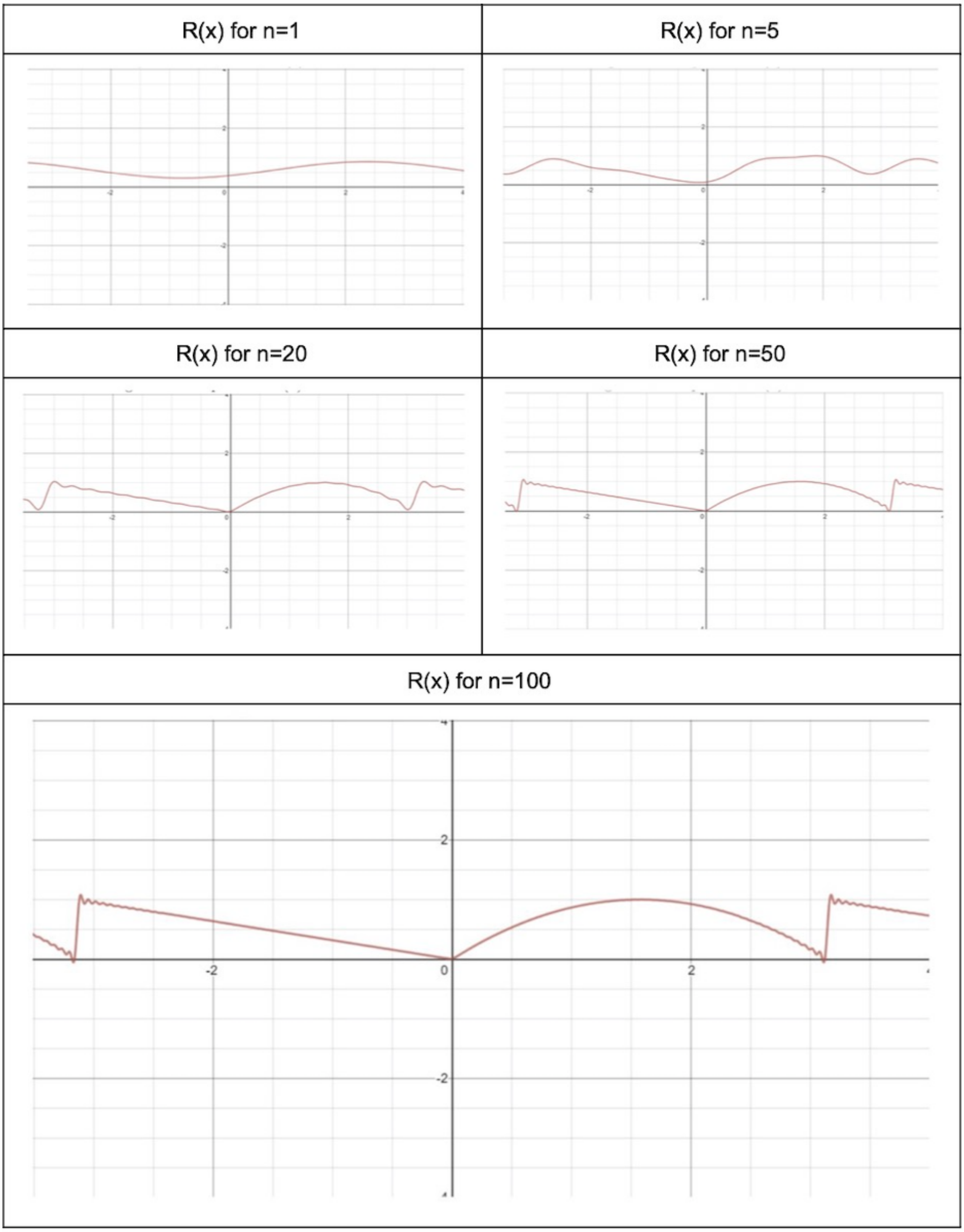
$$\begin{aligned}
b_n &= \frac{1}{\pi}\left(-\frac{1}{\pi}\int_{-\pi}^{0} x\cdot\sin nx\,dx - \frac{4}{\pi^2}\int_0^{\pi} x^2\cdot\sin nx\,dx + \frac{4}{\pi}\int_0^{\pi} x\cdot\sin nx\,dx\right)\\
&= -\frac{1}{\pi^2}\left[\left(-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right)\Bigg|_{-\pi}^{0}\right] - \frac{4}{\pi^3}\left[\left(-\frac{x^2}{n}\cos nx + \frac{2x}{n^2}\sin nx + \frac{2}{n^3}\cos nx\right)\Bigg|_{0}^{\pi}\right]+\\
&\quad \frac{4}{\pi^2}\left[\left(-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right)\Bigg|_{0}^{\pi}\right]\\
&= -\frac{1}{n^2}\left(0 - \frac{\pi}{n}(-1)^n\right) - \frac{4}{\pi^3}\left(-\frac{\pi^2}{n}(-1)^n + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}\right) + \frac{4}{\pi^2}\left(-\frac{\pi}{n}(-1)^n\right)\\
&= \frac{1}{n\pi}(-1)^n + \frac{4}{n\pi}(-1)^n - \frac{8}{n^3\pi^3}(-1)^n + \frac{8}{n^3\pi^3} - \frac{4}{n\pi}(-1)^n\\
&= \frac{1}{n^3\pi^3}(n^2\pi^2(-1)^n - 8(-1)^n + 8)\\
&= \frac{(-1)^n}{n\pi} - \frac{8(-1)^n}{n^3\pi^3} + \frac{8}{n^3\pi^3}.
\end{aligned}$$

Hence

$$a_n = -\frac{1}{\pi^2 n^2}(5+3(-1)^n);\ b_n = \frac{(-1)^n}{\pi n} - \frac{8(-1)^n}{\pi^3 n^3} + \frac{8}{\pi^3 n^3}.$$

$$\therefore r(x) = \frac{7}{12} + \sum_{n=1}^{\infty}\left[-\frac{1}{\pi^2 n^2}(5+3(-1)^n)\cos(nx) + \left(\frac{(-1)^n}{\pi n} - \frac{8(-1)^n}{\pi^3 n^3} + \frac{8}{\pi^3 n^3}\right)\sin(nx)\right].$$

Figure 9: Representation of r(x) through Fourier series



3.2 Deduction of Fourier transform formula

As demonstrated earlier, we have the Real Fourier series:

$$f\left(t\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nt+b_n\sin nt\right)$$

Let $\omega=2\pi\nu$ denote angular frequency and since

$$\begin{aligned}\cos(\omega\theta)&=\frac{e^{i\omega\theta}+e^{-i\omega\theta}}{2}\\ \sin(\omega\theta)&=\frac{e^{i\omega\theta}-e^{-i\omega\theta}}{2i}\end{aligned}$$

$$\begin{aligned}\therefore f\left(t\right)&=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos n\omega t+b_n\sin n\omega t\right)\\ &=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\frac{e^{in\omega t}+e^{-in\omega t}}{2}+b_n\frac{e^{in\omega t}-e^{-in\omega t}}{2i}\right)\\ &=\frac{a_0}{2}+\sum_{n=1}^{\infty}\frac{a_n-ib_n}{2}e^{in\omega t}+\sum_{n=1}^{\infty}\frac{a_n+ib_n}{2}e^{-in\omega t}\end{aligned}$$

Note that

$$\sum_{n=1}^{\infty}\frac{a_n+ib_n}{2}e^{-in\omega t}\equiv\sum_{n=-\infty}^{-1}\frac{a_n-ib_n}{2}e^{in\omega t}$$

Let

$$f(t)=\sum_{n=-\infty}^{\infty}c_n\cdot e^{in\omega t}$$

Where c_n is

$$c_0=\frac{a_0}{2},\ c_n=\frac{a_n-ib_n}{2},\ c_{-n}=\frac{a_n+ib_n}{2}.$$

Now, by substituting a_n and b_n,

$$\begin{aligned}c_n&=\frac{1}{2}\left(\frac{1}{L}\int_{L}^{-L}f(t)cos(n\omega t)dt-\frac{i}{L}\int_{L}^{-L}f(t)sin(n\omega t)dt\right)\\ &=\frac{1}{2L}\left(\int_{L}^{-L}f(t)\left(cos(n\omega t)-isin(n\omega t)dt\right)\right)\\ &=\frac{1}{2L}\cdot\int_{L}^{-L}e^{-in2\pi\nu t}dt\end{aligned}$$

Using

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2L} \cdot \int_{-L}^{L} f(t)e^{-2\pi in\nu t} \cdot dt \cdot e^{2\pi in\nu t}, \text{ where } \nu \text{ denotes frequency}$$

$$\because \frac{1}{2L} = \frac{1}{period} = \Delta\nu$$

$$\therefore f(t) = \sum_{-\infty}^{\infty} \int_{-L}^{L} f(t) \cdot e^{-2\pi in\nu t} dt \cdot e^{2\pi in\nu t} \cdot \Delta\nu$$

To derive the Fourier transform's formula, we take the limit of the half period as L$\rightarrow \infty$.

$$\therefore f(t) = \lim_{L\rightarrow\infty} \left(\sum_{-\infty}^{\infty} \int_{-L}^{L} f(t) \cdot e^{-2\pi in\nu t} dt \cdot e^{2\pi in\nu t} \cdot \Delta\nu \right).$$

As $L \rightarrow \infty$,

$$\Delta\nu \rightarrow d\nu, \quad n\nu \rightarrow \nu, \quad \sum_{-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}.$$

Thus,

$$f(t) = \int_{-\infty}^{\infty} \left[\left(\int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i\nu t} dt \right) \cdot e^{2\pi i\nu t} \right] d\nu$$

Hence,

$$f(t) = \int_{-\infty}^{\infty} F(\nu) e^{2\pi it\nu} \, d\nu$$

Where

$$F(\nu) = \int_{-\infty}^{\infty} f(t) e^{-2\pi it\nu} \, dt$$

$\square$

4 Examples of Fourier transform

4.1 $f(t) = e^{-a|t|}$ where $a > 0$ and $t \in \mathbb{R}$

Using equation (1) and let the angular frequency, $\omega = 2\pi\nu$,

$$\begin{aligned}
F(\nu) &= \int_{-\infty}^{\infty} e^{-a|t|} \cdot e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{at} \cdot e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-at} \cdot e^{-i\omega t}\, dt \\
&= \int_{-\infty}^{0} e^{t(a-i\omega)}\, dt + \int_{0}^{\infty} e^{t(-a-i\omega)}\, dt \\
&= \frac{1}{a-i\omega} \cdot e^{t(a-i\omega)}\Big|_{-\infty}^{0} - \frac{1}{a+i\omega} \cdot e^{-t(a+i\omega)}\Big|_{0}^{\infty} (*)
\end{aligned}$$

Note that $e^0 = 1$ and $\lim_{t\to-\infty} e^t = 0$. So from (*),

$$\begin{aligned}
F(\nu) &= \frac{1}{a-i\omega} \cdot (1-0) - \frac{1}{a+i\omega} \cdot (0-1) \\
&= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} \\
&= \frac{2a}{a^2+\omega^2} = \frac{2a}{a^2+4\pi^2\nu^2}
\end{aligned}$$

To find the maximum frequency, using quotient rule to differentiate $F(\omega)$ with respect to ω.

$$F'(\omega) = \frac{-2a \cdot 2\omega}{(a^2+\omega^2)^2} = \frac{-4a\omega}{(a^2+\omega^2)^2}$$

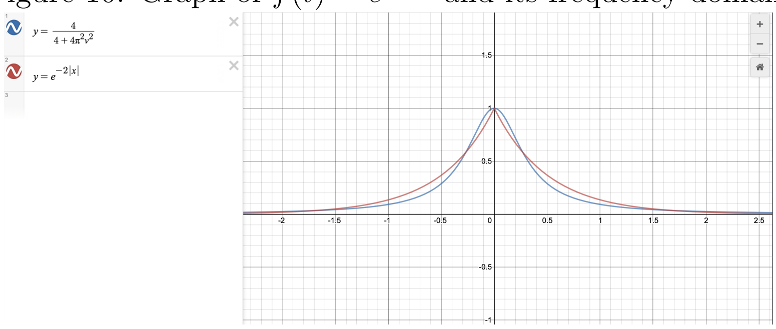
$\therefore$ stationary point at $\omega = 0$

$$F(0) = \frac{2a}{a^2} = \frac{2}{a}$$

$\therefore$ max frequency at 2/a

$\square$

Figure 10: Graph of $f(t) = e^{-2|t|}$ and its frequency domain



4.2 $f(t) = \sin(at)$, where $a > 0$ and $t \in \mathbb{R}$

Now note that:

$$\sin(t) = \frac{e^{iat} - e^{-iat}}{2i}$$

Hence,

$$\begin{aligned}
F(\nu) &= \int_{-\infty}^{\infty} \frac{e^{iat} - e^{-iat}}{2i} \cdot e^{-2\pi i t\nu}\, dt \\
&= \frac{1}{2i} \int_{-\infty}^{\infty} \left(e^{iat-2\pi it\nu} - e^{-iat-2\pi it\nu}\right)\, dt \\
&= \frac{1}{2i} \int_{-\infty}^{\infty} e^{it(a-2\pi\nu)}\, dt - \frac{1}{2i} \int_{-\infty}^{\infty} e^{it(-a-2\pi\nu)}\, dt
\end{aligned}$$

Using equation (4),

$$F(\nu) = \frac{\pi}{i} \cdot \delta(a - 2\pi\nu) - \frac{\pi}{i} \cdot \delta(-a - 2\pi\nu)$$

Rationalise the denominator and factorise gives:

$$F(\nu) = -\pi i(\delta(a - 2\pi\nu) + \delta(-a - 2\pi\nu))$$

$\therefore$

$$F(\nu) = 0$$

Except when the domain of the delta function $= 0$.
i.e.

$$a - 2\pi\nu = 0$$

or

$$-a - 2\pi\nu = 0$$

Thus,

$$\nu = \pm\frac{a}{2\pi}$$

To find out the prevalence of the respective frequencies,
for

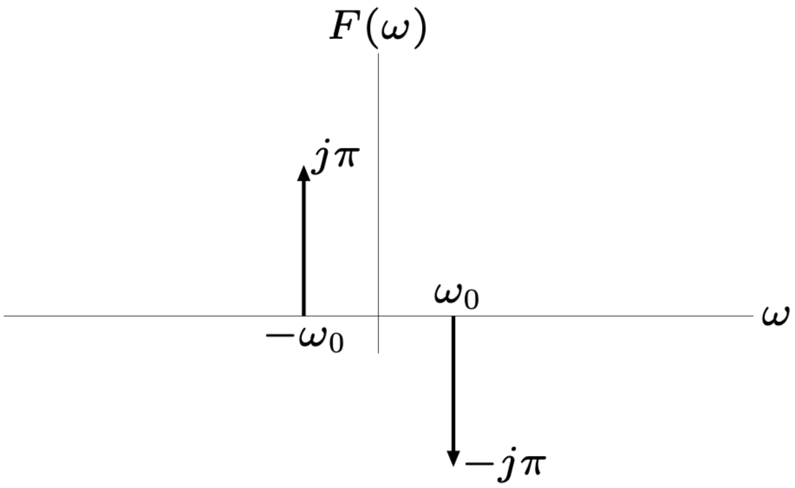
$$\nu = \frac{a}{2\pi}, \quad F(\nu) = -\pi i(1 - 0) = -\pi i$$

for

$$\nu = -\frac{a}{2\pi}, \quad F(\nu) = -\pi i(0 - 1) = \pi i.$$

$\square$

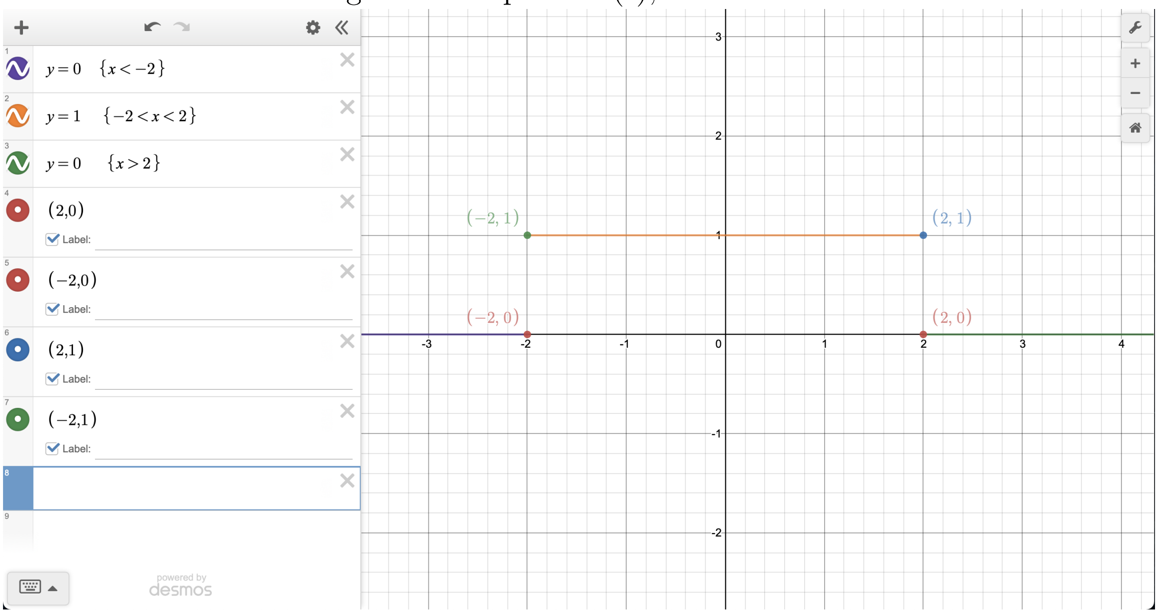
Figure 11: Impulse diagram of $\sin(at)$ (Stanford)



4.3 Rectangular pulse

$$f(t) = \begin{cases} 1, & -T \leq t \leq T \\ 0, & |t| > T \end{cases}$$

Figure 12: A plot of f(t), where T = 2



Using equation (1) and split the terms,

$$
\begin{aligned}
F(\omega) &= \int_{-T}^{-\infty} 0 \cdot e^{-i\omega t}\ dt + \int_{T}^{-T} 1 \cdot e^{-i\omega t}\ dt + \int_{\infty}^{T} 0 \cdot e^{-i\omega t}\ dt \\
&= \int_{T}^{-T} e^{-i\omega t}\ dt \\
&= -\frac{1}{i\omega} \cdot e^{-i\omega t}\Big|_{-T}^{T} \\
&= -\frac{1}{i\omega} \cdot (e^{-i\omega T} - e^{i\omega T})
\end{aligned}
$$

Since

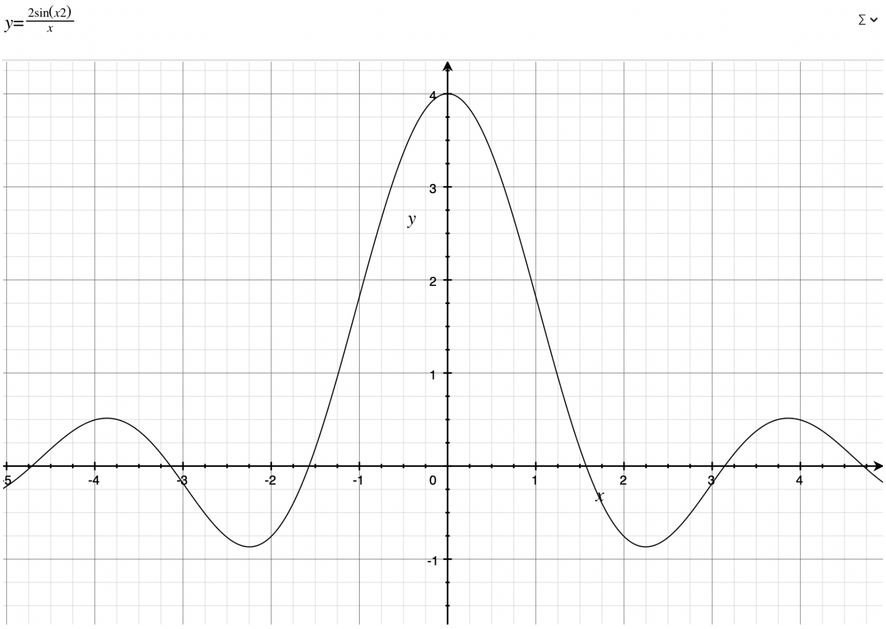
$$
\begin{aligned}
sin(\omega T) &= \frac{e^{-i\omega T} - e^{i\omega T}}{2i} \\
-2i\sin(\omega T) &= e^{-i\omega T} - e^{i\omega T}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
F(\omega) &= -\frac{1}{i\omega} \cdot -2i\sin(\omega T) \\
&= \frac{2\sin(\omega T)}{\omega}
\end{aligned}
$$

□

Figure 13: A plot of $F(\omega) = \frac{2\sin(\omega T)}{\omega}$, where T = 2



5 Various applications of Fourier Transform

5.1 Earthquake

Even if an earthquake does not occur, the ground is still constantly in vibration which can be detected by a seismograph. Fourier Transform helps in analyzing the composition and oscillation characteristics in a seismograph, it is therefore a crucial tool in seismology.

When the sampling rate in a seismograph is high enough to be detected as a signal, the time signal is then transformed into a frequency signal that demonstrates the exact frequencies of which the signal is composed of, with the help of Fourier transform. The finite time sequence of recorded data in seismographs can be broken down into a series of harmonic oscillations.

Transitory signals such as earthquake seismographs have more complicated wave forms, its waveform is determined by

- The earthquake source
- The structure of the Earth
- The properties of the seismograph

We can strictly speaking consider Seismic noise to be deterministic albeit we don't have a set of accurate parameters to fully predict or interpret the waveform. Therefore, due to the huge number of variables influencing the signal, we can generally say that seismic waves are stochastic. Hence, we can use Fourier transform to compute the frequency spectrum of the noise. (Bormann)

Definition 5.1 (Stochastic)**.** Stochastic models possess some inherent randomness. The same set of parameter values and initial conditions will lead to an ensemble of different outputs.

Definition 5.2 (Deterministic)**.** In deterministic models, the output of the model is fully determined by the parameter values and the initial conditions (”Deterministic vs. stochastic”).

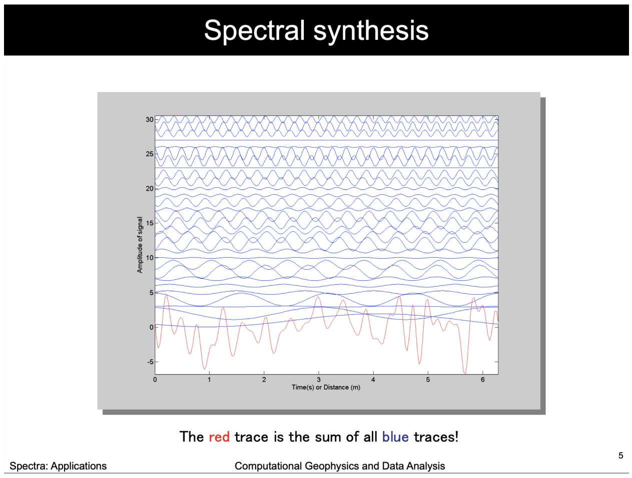


Figure 14: Analysis of a seismograph (Fourier Transform: Applications in Seismology.)

5.2 Computed tomography (CT)

CT Medical Reconstruction Energy: sending electromagnetic waves through an object, the intensity of these waves is detected on a sensor located below the object. CT scanner: rotating an X-ray machine around the object, soft tissues would lead to greater intensity routines, harder tissues would lead to lower intensity

CT scan has the ability of gathering compiling 360 degrees views of a target area (different to X-rays) Fourier transform allows computers to turn the list of light intensities into a 3D image

CT scans take projections from many different angles and produce 2D slices (like pictures taken by a camera), Fourier transform allows to take measurements as a function of angles and invert the data into pixels.

E.g. f(x, y) is the original image, the projection is taken of the image at angle theta, the projection is denoted P theta (r).

- A 1D Fourier transform of the image will produce a slice image of the object would graph an one-one mapping
- A 2D Fourier transform of the image will produce in the frequency domain a many-many (circular) graph

Fourier central slice theorem - 1D Fourier transform of the projection of the image is identical as the values of the 2D Fourier transform of the object along a line drawn through the centre of the 2D Fourier transform plane

Fourier transform, in addition to these, also allows taking these image slices to build up a 3D image of the object that the CT scan measures. (Todorovic)

5.3 Image compression

Background: Image is stored as binary data in computer memory where it is organized into a giant matrix full of zeros and ones. When we digitally modify an image, we are simply operating with the numbers within. When we take a coloured photo, the camera captures different light intensities of Red, Green and Blue (RGB) lights. Therefore each pixel is represented as a 3 by 1 matrix. It is practical to compress images since it is not necessary to have that much information stored for the majority of its applications. (Al-Azawi)

JPEG compression: In summary, the algorithm divide the image up into 8*8 (since 8 bytes=1bit) or 16*16 blocks. We can then think of each row of the image as a signal (a wave) and the value of each entry as the position on the wave. We can therefore use Fourier transform to convert spatial signals into spatial frequencies.†

Because human eyes cannot perceive all the minute changes in colours, we can set a limit for the spatial frequencies where we can discard all the regions of pixels that are above this requirement (replace them by 0). This is because higher frequencies denote sharper changes of colours in the region. Hence, we can reduce the size of the image without having noticeable changes. Lower the limit, more compression, worse quality. (Al-Azawi)

† I have to admit that Discrete Cosine Transform (DSC) is more commonly used in JPEG compression as it provides a more accurate compression result.

5.4 Noise cancelling

5.4.1 Mechanism of noise cancelling headphone

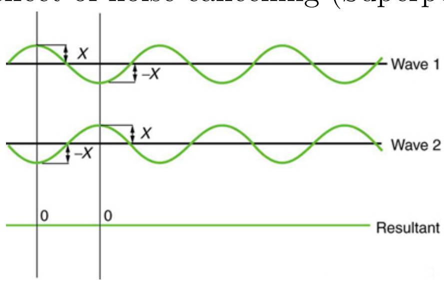
We first use Fourier transform to map the sound wave from its time domain to its frequency domain to find out its composite frequencies. Then we can set a limit for the frequencies that are either below or above the range of ideal sound, such that we may assume them to be background white noise. Afterwards, we can use the speaker to generate identical frequencies of the white noise with a phase change of π. Thereby, the sound waves of the background noises experience destructive interference and hence achieving a cancellation. Note that the majority of noise cancelling headphones apply the same mechanism. Typically, they have an external microphone to pick up the background noises, simultaneous there is also a small speaker attached to generate frequencies of the background noises with a π phase shift to rule them out.

The aim for carrying out this experiment is to verify the shape of the different waves and hence draw a comparison with the approximate analytical result through using Fourier series. In addition, we are also looking into the effect of noise cancelling of two speakers through testing the superposition between the two.

5.4.2 Theory

As there will always be interference between the pure constructive wave (such as wave 1 in the diagram below) and the pure destructive wave (wave 2 in the diagram below), the two waves are required to produce a zero amplitude, in which case they are identical and precisely aligned such that we have a complete cancellation. In practice, we can phase change the constructive wave by π (Fig. 15). However, it would be challenging for us to discover the actual superposition as it can vary from place and time, it could be even harder to be discovered considering the fact that the waves generated from the speakers will not be perfectly identical. Therefore, it would be more reasonable for us to record our stats along the centre line of symmetry between the speakers, because the energy generated from the two speakers will be approximately the same along this line, therefore we could potentially discover the optimal position of noise cancellation.

Figure 15: Theoretical effect of noise cancelling (Superposition and Interference)

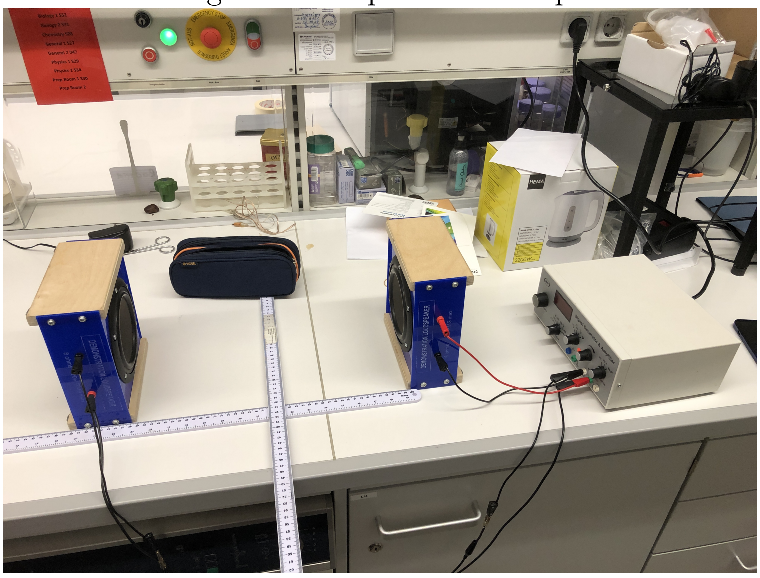


5.4.3 Methodology

We used a signal generator connected to a single speaker first to produce pure sound waves. The generator can produce pure sin waves, square waves and triangle waves of different frequencies and amplitudes by vibrating the central diaphragm of the speaker. We used applications on our phone : SoundSpectrumAnalysis and Phyphox to plot out the waveform and the frequency spectrum of the sound through Fast Fourier Transform. We have tested the frequency spectrum and wave form of all three types of waves at 200, 400, 500, 700 and 1000. Only the results of the 700 Hz trials are displayed for the purpose of controlling our variables.

In the second part of our experiment, we connected another speaker to the generator with reversed input-output cables to generate sound waves of phase difference π. Then we separate the two speakers by a certain distance, place a ruler on the line of symmetry and use the applications to analyse the superposition along the axis of symmetry.

Figure 16: Experiment setup



5.4.4 Results and Interpretation

From Table 1, the 700 Hz pure sin wave consists of predominantly 1 frequency at around 725 Hz. This shows the device we used to produce this sound wave is not as accurate. Some frequencies lower than 200 Hz are also recorded because they are simply background noises. It is also worth noting that not only the diaphragm is vibrating, e.g. the device, the table, the cables, all oscillates at similar frequencies because they are all connected. Therefore we can observe some frequencies in the sin wave spectrum that are closer to the main frequency.

In Table 1, column 3, we can observe that the average period of the wave is 1.4ms. Using $T = \frac{1}{\nu}$, the base frequency of the signals should be approximately 714 Hz. In addition, we can observe that the wave forms of the results corresponds precisely with theoretical shapes of the graph albeit the square waves cannot be represented perfectly as the algorithm that generates such waves does not apply the complete or the correct Fourier transform of the wave. It is also worth noting that the frequency spectrum of the square wave does not correspond with the analytical result because the algorithm of the signal generator is not precise enough. From our results, We have observed frequencies of even multiples of the 714Hz, while theoretically a Fourier transform only generates frequencies of odd multiples as we have proven in section 3.1.1. This discrepancy between our results is also because of the inaccuracies within the equipment. Therefore, it raises another important question: If the frequencies of the even multiples are neglected and then we plot its waveform, will it produce a better fit to the theoretical form?

Figure 17: Table 1: Results from 1 single speaker

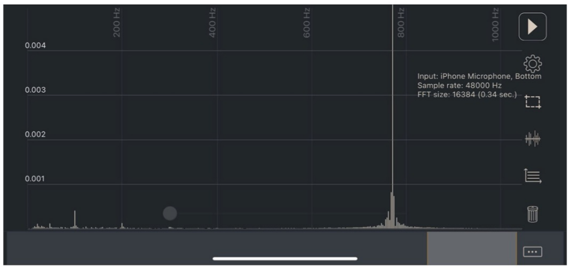
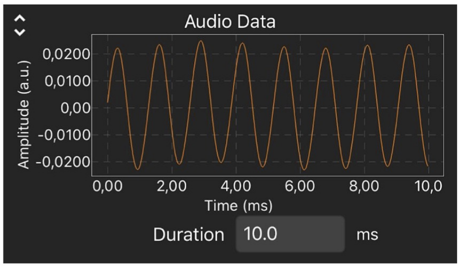
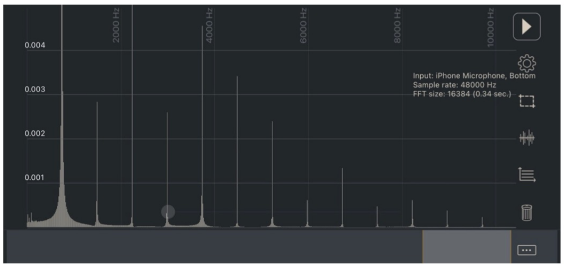
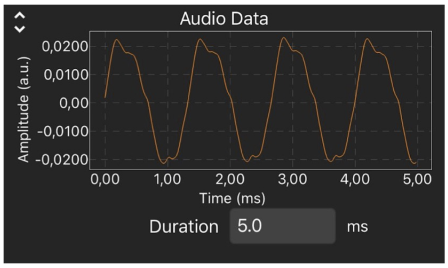
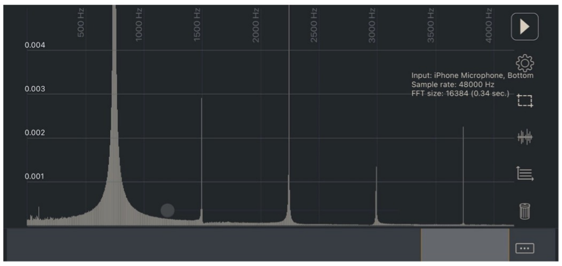
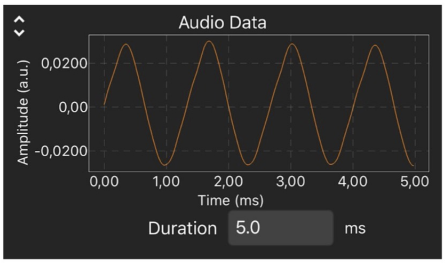
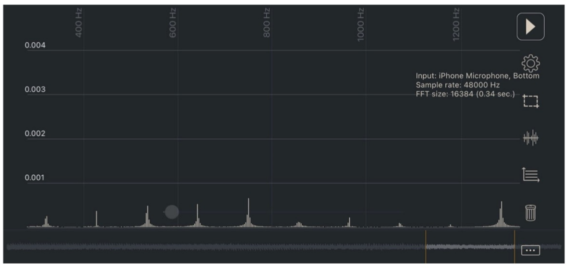
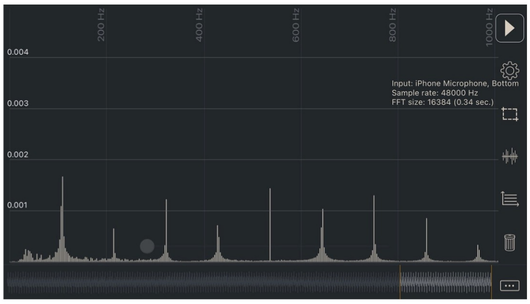
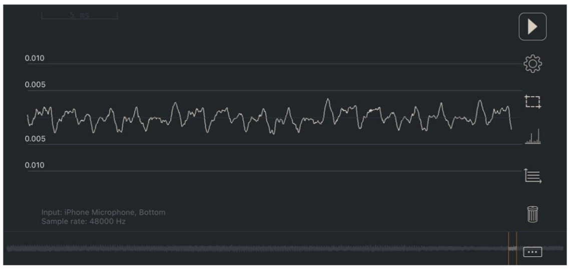
<table>
<tr><th>Frequency spectrum</th><th></th><th>Wave form</th></tr>
<tr><td>
</td><td>Sin wave</td><td>
</td></tr>
<tr><td>
</td><td>Square wave</td><td>
</td></tr>
<tr><td>
</td><td>Triangle wave</td><td>
</td></tr>
</table>

Figure 18: Table 2: Noise cancelling

Two speakers with π phase difference placed 62cm apart	No noise cancelling, square wave of 100 Hz from 1 speaker
	
	

We have managed to find a few positions of destructive interference (quiet spots) along the axis on symmetry (Table 2). The amplitude of the noise is visibly reduced, (bottom left, A<0.005) with no visible peaks and troughs compared to the Figure on its right. The intensities of the frequencies also all fell below 0.001 in the frequency spectrum. Therefore, the theoretical destructive interference between the two waves is proven. However it is impossible to completely achieve absolute noise cancelling with this experimental setup. Reasons being the inaccuracies within the devices, applications and background noises.

The extrapolation process is shown in Figure 19. A curve line of best fit is used because the Fourier coefficients of the square wave is inversely proportional to its multiplier of the base frequency. Then by graphing the function using the parameters given by the spectrum, we have produced the waves in Table 3. Within the range of $-0.005 \leq y \leq 0.005$, we can observe that the lines are almost perfectly vertical, hence the correct shape. However, the peaks and troughs are still uneven. The graphs still do not look like a square waves because : 1) Sin waves with higher frequencies are not used, only the first 6 terms of the Fourier series is used 2) The Fourier coefficients are very rough estimates from the spectrum 3) a_0 cannot be determined from the spectrum and the first two sinusoids that have the greatest impact on the wave form are extrapolated from the data.

Figure 19: Extrapolation of the frequency spectrum of 700 Hz square wave

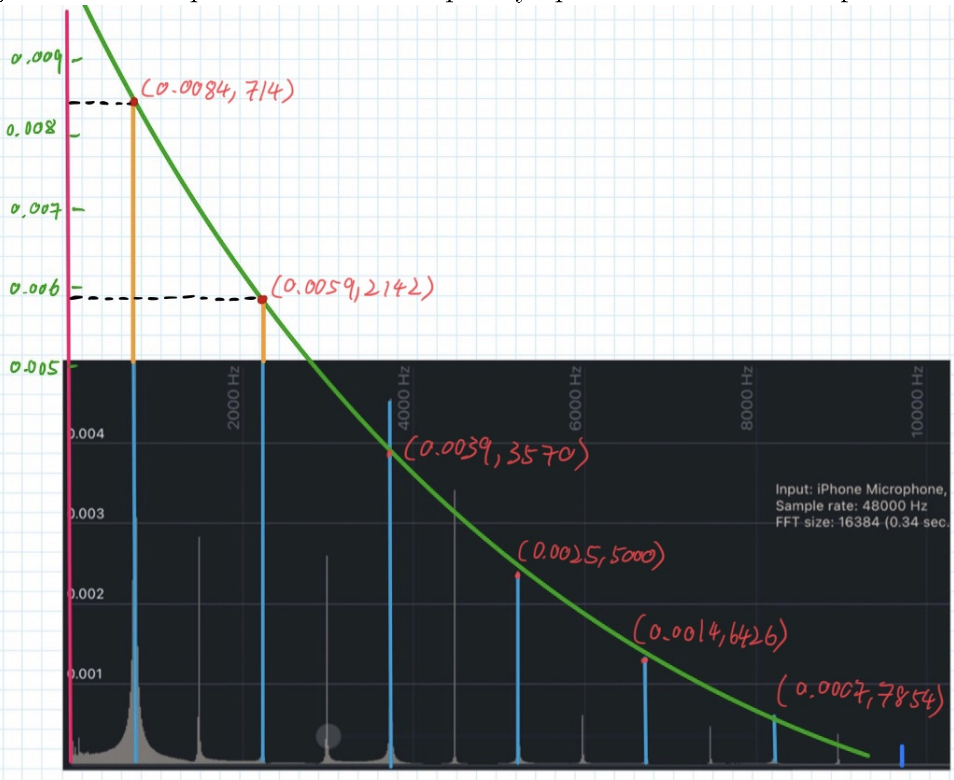
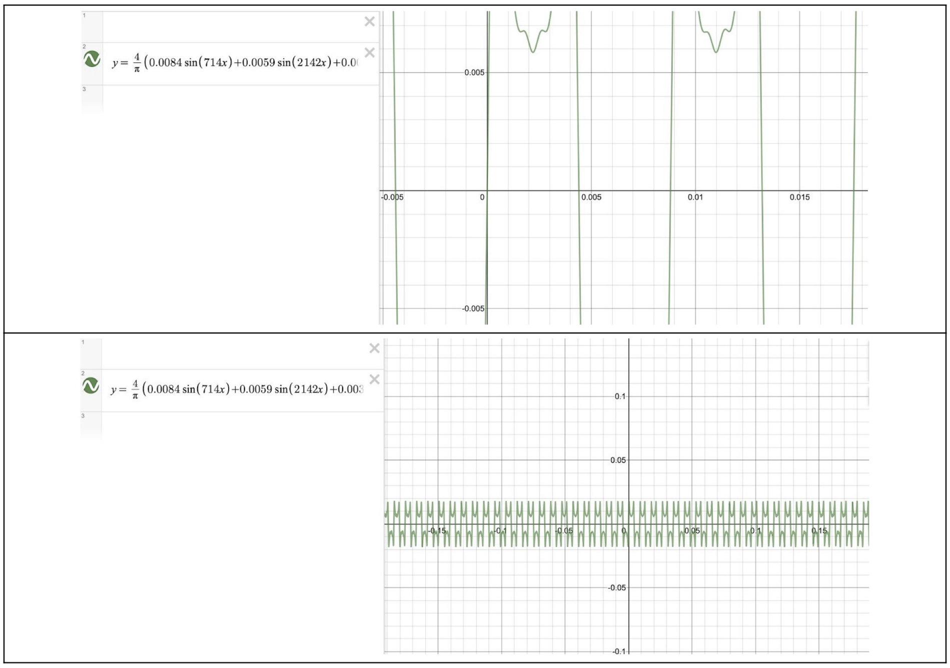
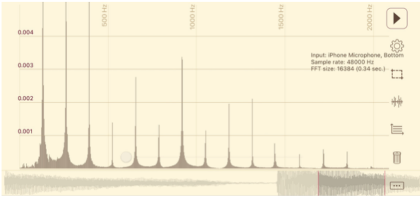
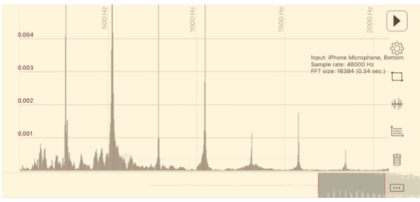
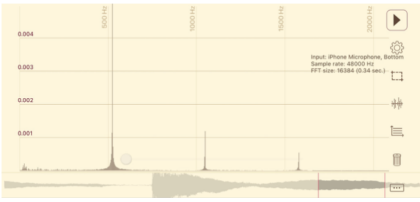


Figure 20: Table 3: Graphing the possible correction



5.5 Sound Spectrum Analysis of the C note

Figure 21: Table 4 : Sound frequency spectrum of the note C of different octaves made using Fast Fourier Transform

	C3 - Piano
	C4 - Piano
	C5 - Piano

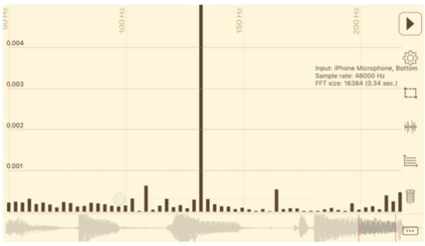
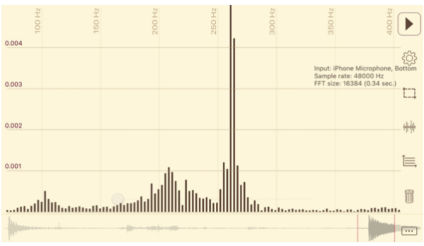
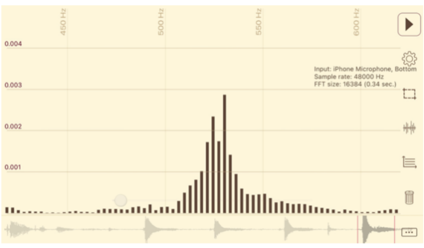
	C3 - Guitar
	C4 - Guitar
	C5 - Guitar

Figure 22: Theoretical predominant frequency of each note.

C3	130.81 Hz
C4	261.63 Hz
C5	523.25 Hz

(B. H. Suits)

Another application of the Fourier Transform that we have explored, in fact, is vastly different from the field of physics and engineering - the arts. We listen to all different types of music everyday, that it becomes natural for us to be aware of the difference in tones. A musical tone is typically characterized by its pitch, intensity and timbre, while different notes are associated with different frequencies. We therefore explored the correlation in frequency among some C notes on different octaves. Nonetheless, our instruments of investigation are limited to the piano and the guitar due to the pandemic.

It is worth noting that in Figure 22, the theoretical predominant frequency of C4 doubled the frequency of C3 because it is one octave higher. This trend can also be seen in Figure 21, Table 4, the frequency spectrum of C3, C4 and C5 played by the piano clearly correspond to their predominant frequencies and their harmonics. The peaks in the frequency spectrum of those notes are spaced equally, further proving that they are harmonic to each other. On the other hand, the music notes played by the guitar string evident have a worse timbre as suggested by the frequency intensity found near the predominant frequency. This could also be due to the tuning of the guitar string.

6 Conclusion

In conclusion, this paper has covered: Fourier series; Introduction to Fourier Transform; its application in seismic wave analysis, computed tomography, image compression as well as the timbre of musical tone. The study has verified through experiment the existence of destructive interference between two sound waves using FFT. The waves forms of sin, triangle and square waves are also analysed and improved. However, we can still further investigate:

- Derivation of Real Fourier Series.
- Properties of Fourier transform.
- Inverse Fourier transform.
- The use of Fourier transform to solve partial differential equations.
- Understand Generalizations of Fourier transform, notably Fast Fourier Transform (FFT) and Discrete Fourier transform.

For the experimental section, we can further explore :

- Is there more than 1 superposition?
- Use pure sine waves to obtain a better noise cancelling result.
- What's the relationship between the distance of superposition along the axis of symmetry and the separation between two speakers?
- Using multiple speakers to generate pure sine waves which correspond to the theoretical frequency spectrum that we have extrapolated.
- Write an algorithm to identify the tone quality of different instruments. Hence it will be able to distinguish the different instruments played in a song.

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