INTRODUCCIÓN A SISTEMAS DE CONTROL

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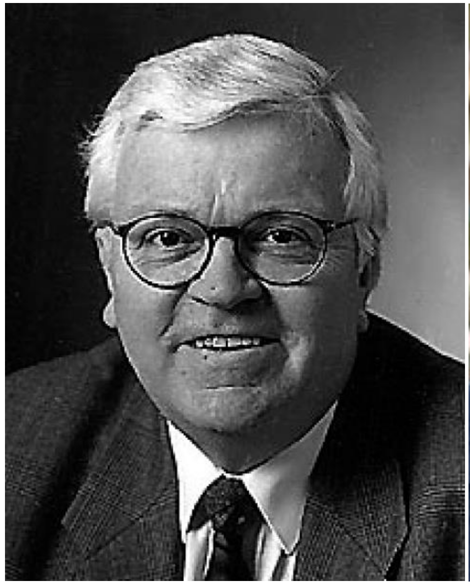
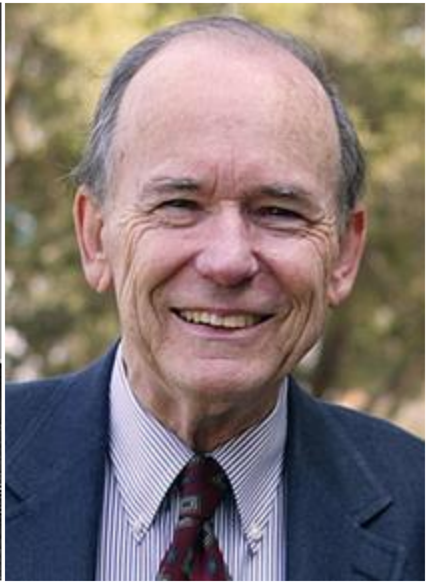
CONTROLABILIDAD, OBSERVABILIDAD Y CONTROL EN ESPACIO DE ESTADOS

CONTROLABILIDAD, OBSERVABILIDAD Y CONTROL EN ESPACIO DE ESTADOS

- Jürgen Ackermann y David Luenberger

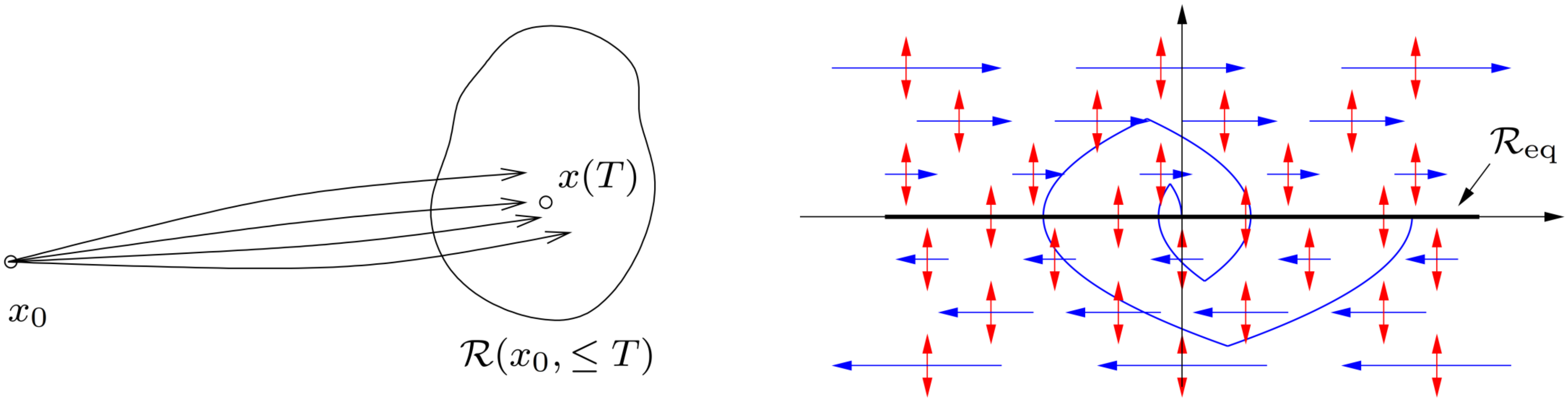



TEMARIO

- Realimentación de Estados
- Caley-Hamilton
- Matriz de Controlabilidad
- Fórmula de Ackerman

ALCANZABILIDAD (REACHABILITY)

Definition 7.1 (Reachability)**.** A linear system is *reachable* if for any $x_0, x_f \in \mathbb{R}^n$ there exists a $T > 0$ and $u\colon [0, T] \to \mathbb{R}$ such that if $x(0) = x_0$ then the corresponding solution satisfies $x(T) = x_f$.



(a) Reachable set (b) Reachability through control

ALCANZABILIDAD (REACHABILITY)

$$\frac{dx}{dt} = Ax + Bu \qquad x(t) = \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau = \int_0^t e^{A\tau} Bu(t-\tau)d\tau$$

$$e^{A\tau} = I\alpha_0(\tau) + A\alpha_1(\tau) + \cdots + A^{n-1}\alpha_{n-1}(\tau)$$

$$x(t) = B\int_0^t \alpha_0(\tau)u(t-\tau)\,d\tau + AB\int_0^t \alpha_1(\tau)u(t-\tau)\,d\tau + \cdots + A^{n-1}B\int_0^t \alpha_{n-1}(\tau)u(t-\tau)\,d\tau$$

$$x(t) = [B \quad AB \quad \cdots \quad A^{n-1}B]\begin{bmatrix} \int_0^t \alpha_0(\tau)u(t-\tau)d\tau \\ \int_0^t \alpha_1(\tau)u(t-\tau)d\tau \\ \vdots \\ \int_0^t \alpha_{n-1}(\tau)u(t-\tau)d\tau \end{bmatrix}$$

ALCANZABILIDAD (REACHABILITY)

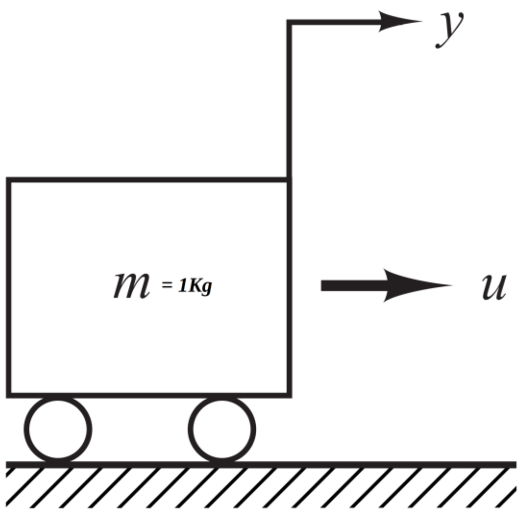
Theorem 7.1 (Reachability rank condition). *A linear system of the form* (7.1) *is reachable if and only if the reachability matrix* W_r *is invertible (full column rank).*

SISTEMA CONTROLABLE: EJEMPLO

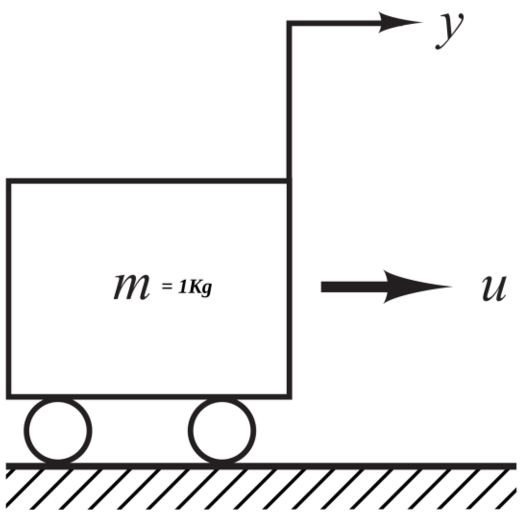
$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = 0$$



SISTEMA CONTROLABLE: EJEMPLO



$$\dot{x} = Ax + Bu \qquad u = -Kx$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_1 \end{bmatrix} \qquad \tilde{A} = A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

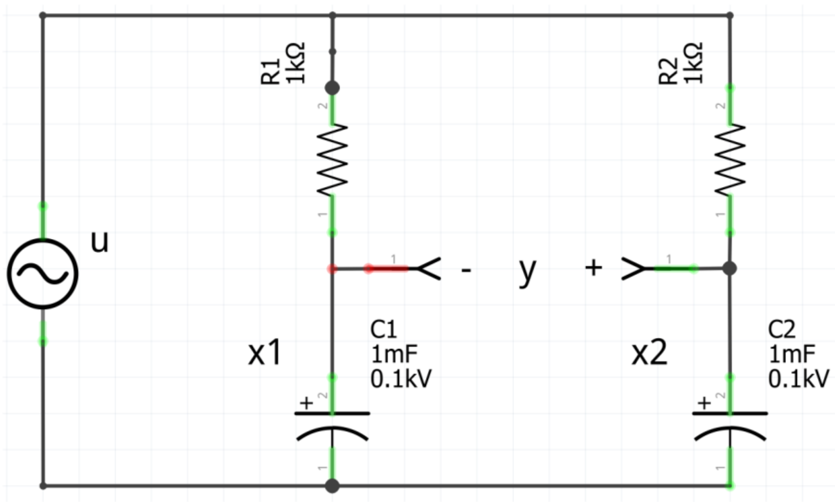
Siendo el polinomio característico de $\tilde{A}$ igual a $|sI - \tilde{A}| = s^2 + k_2 s + k_1$ queda en evidencia que los autovalores de lazo cerrado pueden ubicarse en una locación arbitraria cambiando el valor de $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$.

EJEMPLO:
SISTEMA NO CONTROLABLE

$$\dot{x} = Ax + Bu \qquad y = Cx + Du$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 1 \end{bmatrix} \qquad D = 0$$



SISTEMA NO CONTROLABLE: CAMBIO DE COORDENADAS

$$z = Tx = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad \dot{z} = T\dot{x} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} \qquad T = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$x = T^{-1}z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \dot{x} = T^{-1}\dot{z} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \qquad T^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$z_1 = x_1 \qquad z_2 = x_2 - x_1$$

$$x_1 = z_1 \qquad x_2 = z_1 + z_2$$

SISTEMA NO CONTROLABLE: EJEMPLO

$$\dot{z} = T\dot{x}$$

$$\dot{z} = T\left(Ax + Bu\right)$$

$$\dot{z} = T\left(AT^{-1}z + Bu\right)$$

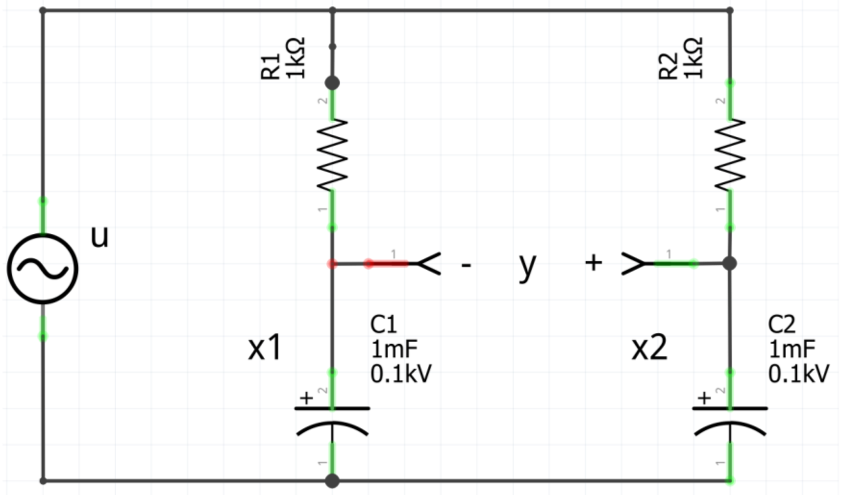
$$\dot{z} = \underbrace{TAT^{-1}}_{\bar{A}}z + \underbrace{TB}_{\bar{B}}u$$

$$\dot{z} = \bar{A}z + \bar{B}u$$

$$y = Cx + Du$$

$$y = \overbrace{CT^{-1}}^{\bar{C}}z + Du$$

$$y = \bar{C}z + Du$$

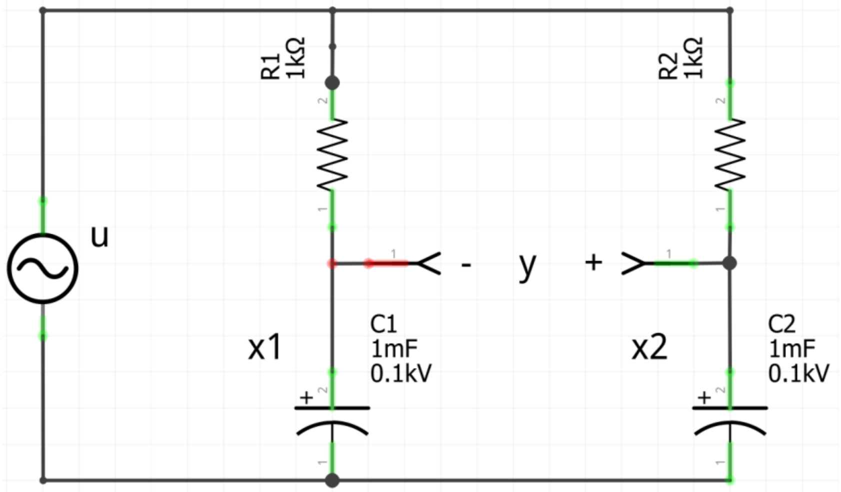


SISTEMA NO CONTROLABLE: EJEMPLO

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

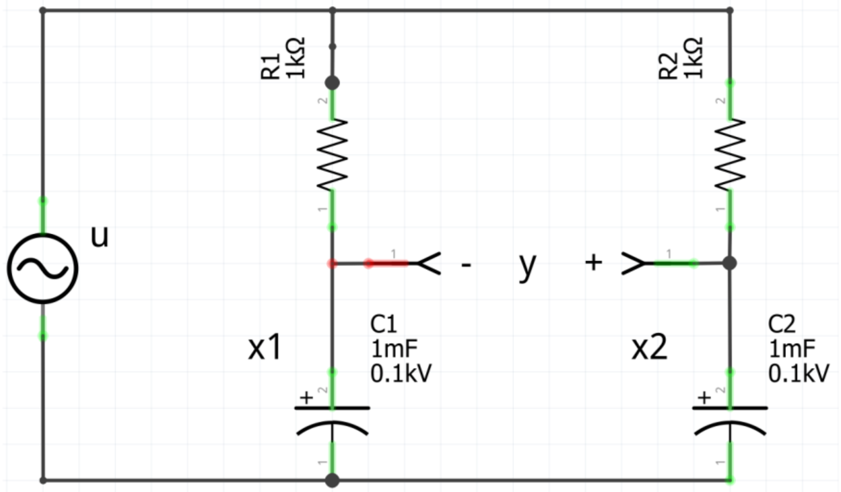
$$\bar{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$



SISTEMA NO CONTROLABLE: EJEMPLO

$$u = -Kx = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -k_1 x_1 - k_2 x_2$$

$$u = -\underbrace{KT^{-1}}_{\bar{K}} z = -\underbrace{\begin{bmatrix} \bar{k}_1 & \bar{k}_2 \end{bmatrix}}_{\bar{K}} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = -\bar{k}_1 z_1 - \bar{k}_2 z_2$$



SISTEMA NO CONTROLABLE: AUTOVALORES DE LAZO CERRADO EN "x"

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$K = \begin{bmatrix} k_1 & k_1 \end{bmatrix} \qquad \tilde{A} = A - BK = \begin{bmatrix} -1 - k_1 & -k_2 \\ -k_1 & -1 - k_2 \end{bmatrix}$$

SISTEMA NO CONTROLABLE: AUTOVALORES DE LAZO CERRADO EN "z"

$$\bar{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

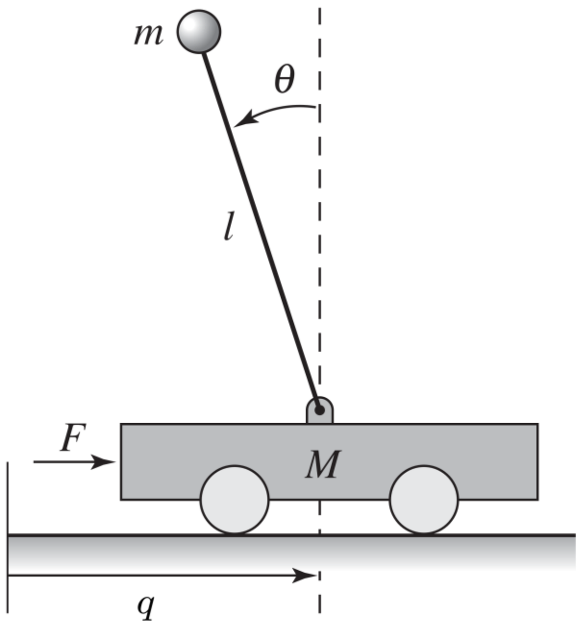
$$\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

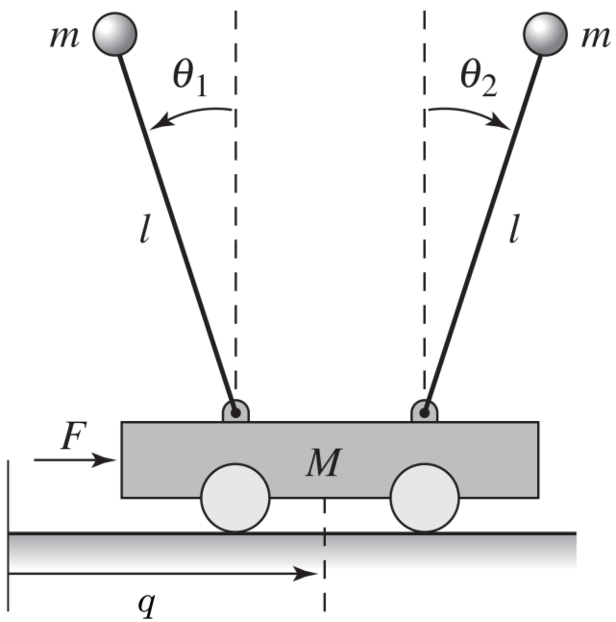
$$\bar{K} = \begin{bmatrix} \bar{k}_1 & \bar{k}_1 \end{bmatrix}$$

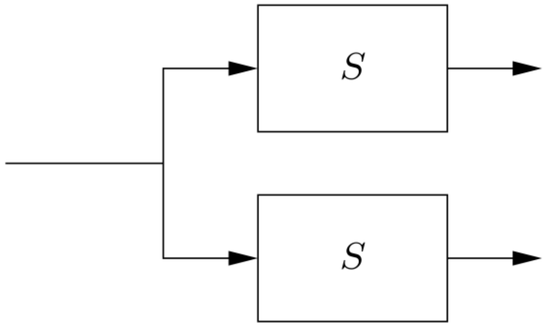
$$\widehat{\tilde{A}} = T\tilde{A}T^{-1} = \bar{A} - \bar{B}\bar{K} = \begin{bmatrix} -1-\bar{k}_1 & -\bar{k}_2 \\ 0 & -1 \end{bmatrix}$$

$$|sI - A| = \left(s + \left(1 + \bar{k}_1\right)\right)(s+1)$$

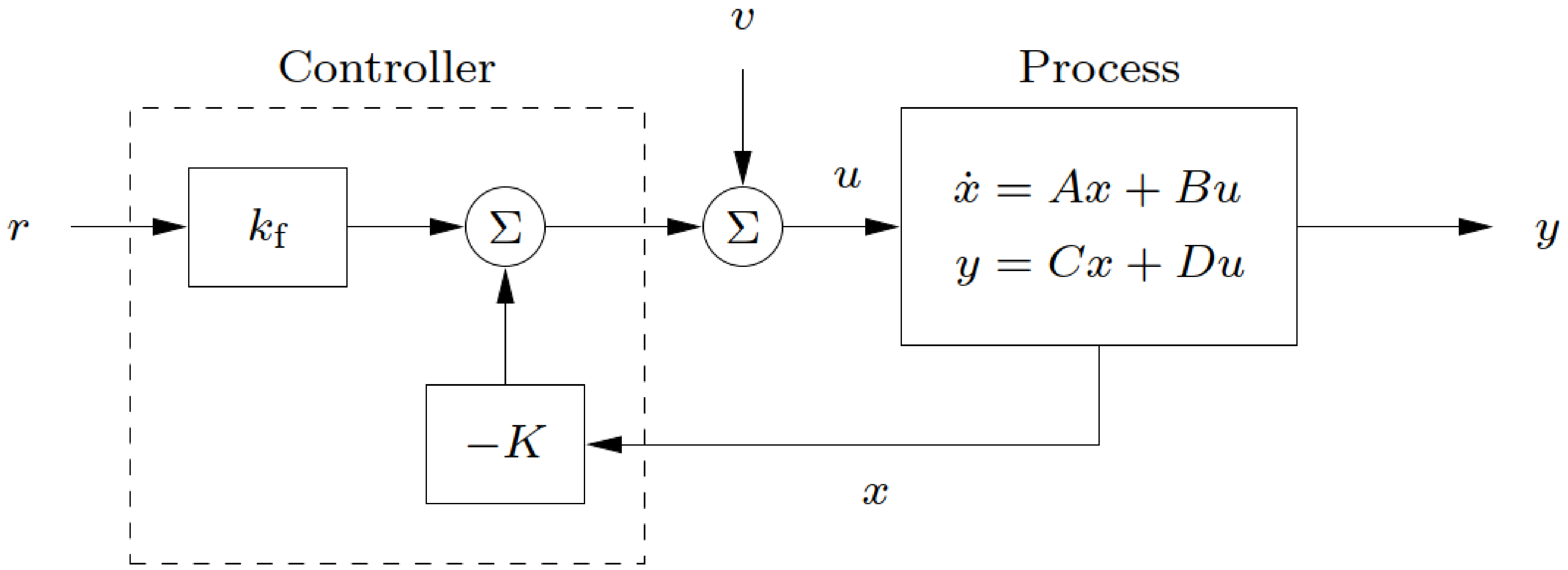
Otro ejemplo de sistema no controlable







FÓRMULA DE ACKERMANN



$$\frac{dx}{dt} = (A - BK)x + Bk_f r$$

ASIGNACIÓN DE AUTOVALORES (POLE PLACEMENT)

$\dot{x} = Ax + Bu$	$y = Cx + Du$	$u = -Kx + k_f r$

$$\dot{x} = (A - BK)x + Bk_f r$$

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1} s + p_n$$

$x_e = -[(A - BK)^{-1}]Bk_f r$	$y_e = Cx_e + Du_e$

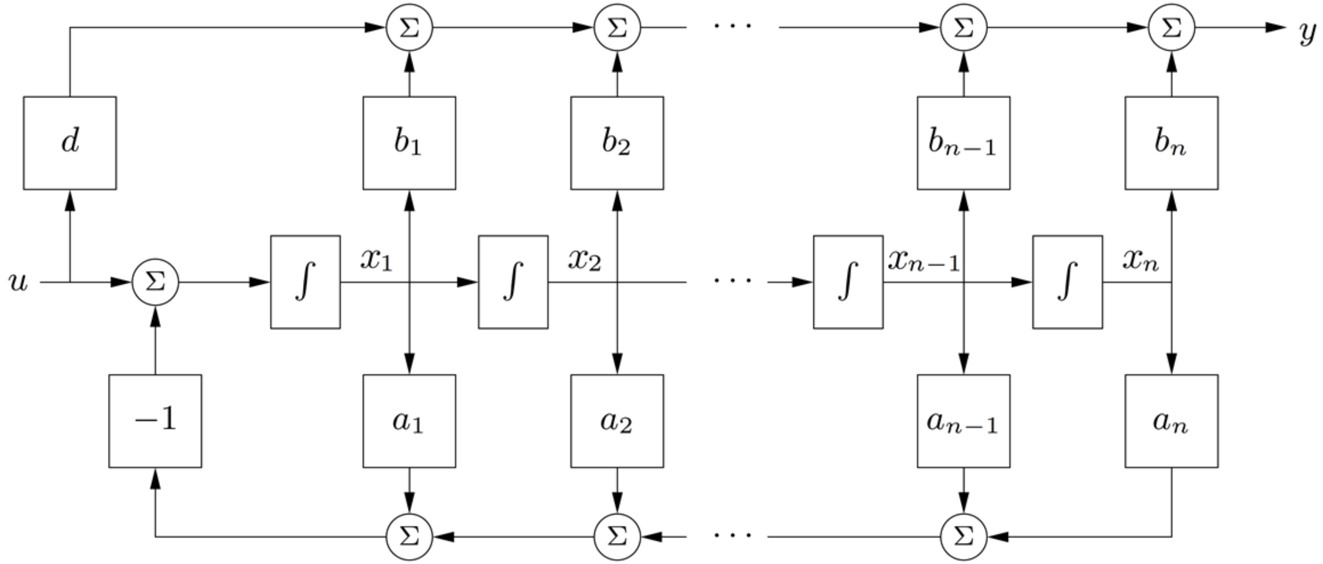
$$k_f = -\frac{1}{C[(A - BK)^{-1}]B}$$

FORMA CANÓNICA CONTROLABLE (O "REACHABLE" - ALCANZABLE)

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & & & \\ & 1 & 0 & \multicolumn{2}{c}{0} \\ & 0 & \ddots & \ddots & \\ & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & b_3 & \dots & b_n \end{pmatrix} z + du.$$

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$



Forma Canónica Controlable

$$\tilde{A} = TAT^{-1}, \qquad \tilde{B} = TB \qquad\qquad z = Tx$$

$$\tilde{A}\tilde{B} = TAT^{-1}TB = TAB,$$

$$\tilde{A}^2\tilde{B} = (TAT^{-1})^2TB = TAT^{-1}TAT^{-1}TB = TA^2B$$

$$\vdots$$

$$\tilde{A}^n\tilde{B} = TA^nB$$

$$\tilde{W}_\mathrm{r} = T\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix} = TW_\mathrm{r}$$

Forma Canónica Controlable

$$\tilde{W}_{\mathrm{r}} = \begin{pmatrix} \tilde{B} & \tilde{A}\tilde{B} & \dots & \tilde{A}^{n-1}\tilde{B} \end{pmatrix} = \begin{pmatrix} 1 & -a_1 & a_1^2 - a_2 & & \\ 0 & 1 & -a_1 & & * \\ & & \ddots & \ddots & \\ & 0 & & 1 & -a_1 \\ & & & & 1 \end{pmatrix}$$

$$\tilde{W}_r = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1}$$

$$T = \tilde{W}_{\mathrm{r}} W_{\mathrm{r}}^{-1}$$

Asignación de autovalores (*pole placement*)

$$\frac{dz}{dt} = \tilde{A}z + \tilde{B}u = \begin{pmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_n \\ 1 & 0 & & & \\ & 1 & 0 & & 0 \\ & & \ddots & \ddots & \\ & 0 & & & \\ & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \tilde{C}z = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} z.$$

$$u = -\tilde{K}z + k_\mathrm{f} r = -\tilde{k}_1 z_1 - \tilde{k}_2 z_2 - \cdots - \tilde{k}_n z_n + k_\mathrm{f} r$$

Asignación de autovalores (*pole placement*)

$$u = -\tilde{K}z + k_\mathrm{f} r = -\tilde{k}_1 z_1 - \tilde{k}_2 z_2 - \cdots - \tilde{k}_n z_n + k_\mathrm{f} r$$

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 - \tilde{k}_1 & -a_2 - \tilde{k}_2 & -a_3 - \tilde{k}_3 & \dots & -a_n - \tilde{k}_n \\ 1 & 0 & & & \\ & 1 & 0 & 0 & \\ & & \ddots & \ddots & \\ & 0 & & & \\ & & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} k_\mathrm{f} \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} r$$

$$y = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} z.$$

Asignación de autovalores (*pole placement*)

$$s^n + (a_1 + \tilde{k}_1)s^{n-1} + (a_2 + \tilde{k}_2)s^{n-2} + \cdots + (a_{n-1} + \tilde{k}_{n-1})s + a_n + \tilde{k}_n.$$

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n,$$

$$\tilde{k}_1 = p_1 - a_1, \qquad \tilde{k}_2 = p_2 - a_2, \qquad \ldots \qquad \tilde{k}_n = p_n - a_n.$$

$$\tilde{K} = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix}.$$

$$k_\mathrm{f} = \frac{a_n + \tilde{k}_n}{b_n} = \frac{p_n}{b_n}$$

Asignación de autovalores (*pole placement*)

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + Du$$

Theorem 7.3 (Eigenvalue assignment by state feedback). *Consider the system given by equation (7.20), with one input and one output. Let* $\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$ *be the characteristic polynomial of* A. *If the system is reachable, then there exists a control law*

$$u = -Kx + k_f r$$

that gives a closed loop system with the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_{n-1}s + p_n$$

and unity zero frequency gain between r *and* y. *The feedback gain is given by*

$$K = \tilde{K}T = \begin{pmatrix} p_1 - a_1 & p_2 - a_2 & \cdots & p_n - a_n \end{pmatrix} \tilde{W}_r W_r^{-1}, \tag{7.21}$$

where a_i *are the coefficients of the characteristic polynomial of the matrix* A *and the matrices* W_r *and* $\tilde{W}_r$ *are given by*

$$W_r = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}, \qquad \tilde{W}_r = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ & 1 & a_1 & \cdots & a_{n-2} \\ & & \ddots & \ddots & \vdots \\ & 0 & & 1 & a_1 \\ & & & & 1 \end{pmatrix}^{-1}.$$

The feedforward gain is given by

$$k_f = -1/\big(C(A - BK)^{-1}B\big).$$

Asignación de autovalores (*pole placement*)

$$W_r = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}, \qquad \tilde{W}_r = \begin{pmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_1 \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}^{-1}.$$

The reference gain is given by

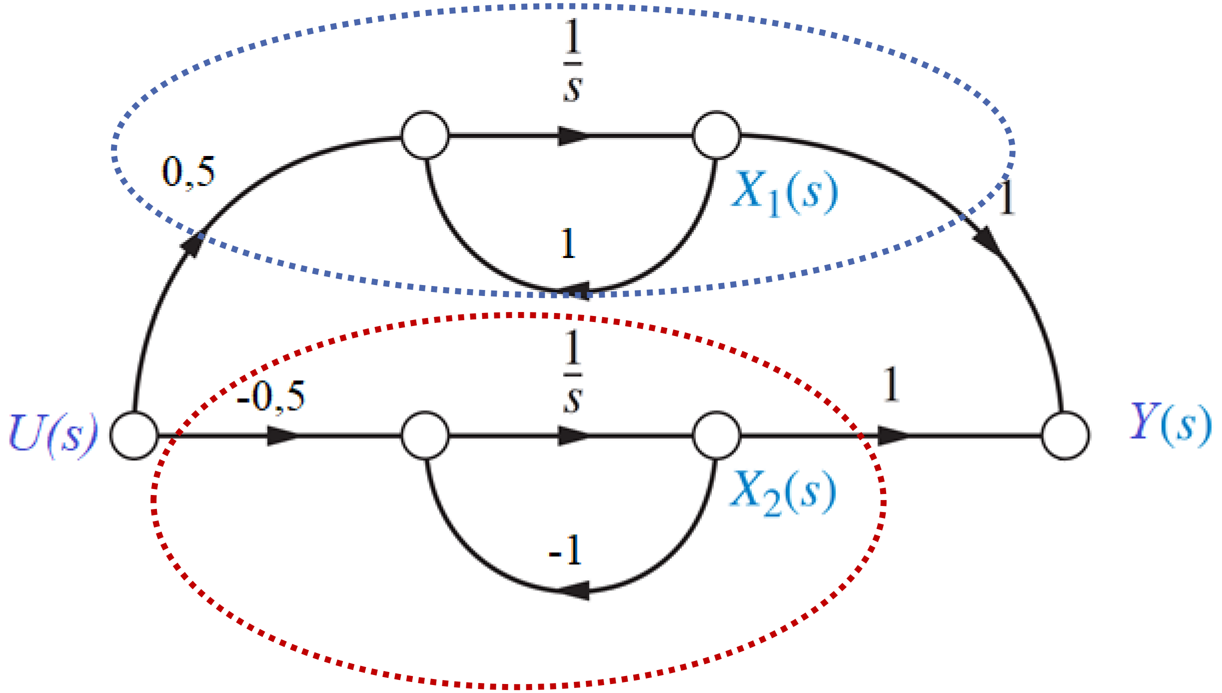
$$k_r = -1/\left(C(A-BK)^{-1}B\right).$$

FÓRMULA DE ACKERMANN: EJEMPLO

$$P(s) = \frac{1}{(s+1)(s-1)} = \frac{0{,}5}{s-1} + \frac{-0{,}5}{s+1}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0{,}5 \\ -0{,}5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = [0]$$



FÓRMULA DE ACKERMANN: EJEMPLO

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 0,5 \\ -0,5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = [0]$$

$$\det(sI - A) = s^2 - 1 = s^2 + a_1 s + a_2,$$

$$a_1 = 0, a_2 = 1$$

$$\det(sI - (A - BK)) = (s + 4)^2 = s^2 + p_1 s + p_2$$

$$p_1 = 8, p_2 = 16$$

$$W_r = \begin{bmatrix} B & AB \end{bmatrix}, \widetilde{W_r} = \begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$W_r = \begin{bmatrix} 0,5 & 0,5 \\ -0,5 & 0,5 \end{bmatrix}, \widetilde{W_r} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$K = \begin{bmatrix} p_1 - a_1 & p_2 - a_2 \end{bmatrix} \widetilde{W}_r W_r^{-1}$$

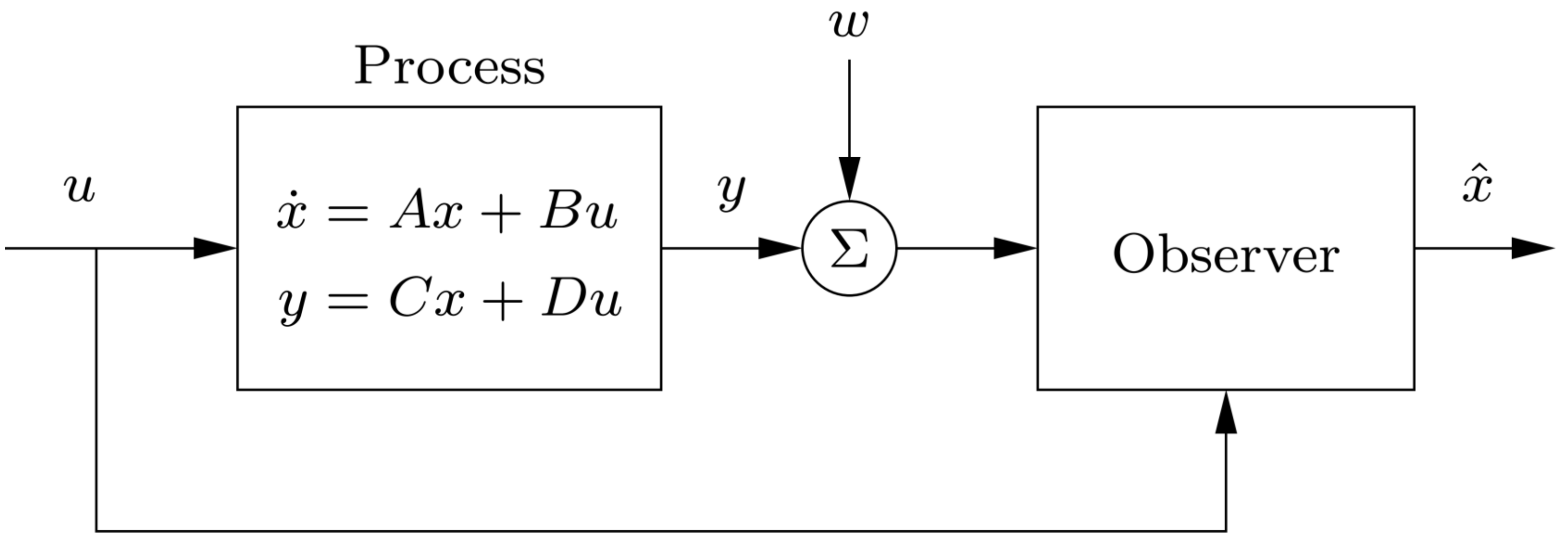
$$K = \begin{bmatrix} 8 - 0 & 16 - 1 \end{bmatrix} \widetilde{W}_r W_r^{-1}$$

FÓRMULA DE ACKERMANN: CÓDIGO

```
%% Sistema 1:

s=tf('s');

P=zpk(1/((s+1)*(s-1)))
[num,den]=tfdata(P,'v')
% polinomio de lazo abierto
a1=den(2);
a2=den(3);
% p(s) polinomio de lazo cerrado
p1=8;
p2=16;

[RR,PP,KK]=residue(num,den)
```

```
A=[1 0;0 -1];
B=[0.5;-0.5];
C=[1 1]
D=0;

Wrm=inv([1 a1;0 1]);
Wr=[B A*B]
K=[p1-a1 p2-a2]*Wrm*inv(Wr)

% corroborar mediante:
% K=acker(A,B,[-4 -4])
% eig(A-B*K)
```

OBSERVABILIDAD



Definition 8.1 (Observability). A linear system is *observable* if for every $T > 0$ it is possible to determine the state of the system $x(T)$ through measurements of $y(t)$ and $u(t)$ on the interval $[0, T]$.

Definición alternativa: El sistema (A,C) es no observable, si existe alguna condición inicial no nula que produzca salida nula.

OBSERVABILIDAD

Theorem 8.1 (Observability rank condition). *A linear system of the form (8.1) is observable if and only if the observability matrix W_o is full row rank.*

Proof. The sufficiency of the observability rank condition follows from the previous analysis. To prove necessity, suppose that the system is observable but W_o is not full row rank. Let $v \in \mathbb{R}^n$, $v \neq 0$, be a vector in the null space of W_o, so that $W_o v = 0$. (Such a v exists using the fact that the row and column rank of a matrix are always equal.) If we let $x(0) = v$ be the initial condition for the system and choose $u = 0$, then the output is given by $y(t) = Ce^{At}v$. Since e^{At} can be written as a power series in A and since A^n and higher powers can be rewritten in terms of lower powers of A (by the Cayley–Hamilton theorem), it follows that $y(t)$ will be identically zero (the reader should fill in the missing steps). However, if both the input and output of the system are zero, then a valid estimate of the state is $\hat{x} = 0$ for all time, which is clearly incorrect since $x(0) = v \neq 0$. Hence by contradiction we must have that W_o is full row rank if the system is observable. $\square$

OBSERVABILIDAD

$y(t) = Ce^{At}x(0)$	$y(t) \in \mathbb{R}^m$

$$y(t) = C\left[\sum_{k=0}^{n-1} \alpha_k(t)A^k\right]x(0)$$

$$y(t) = [I_m \quad I_m \quad \cdots \quad I_m]\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}x(0)$$

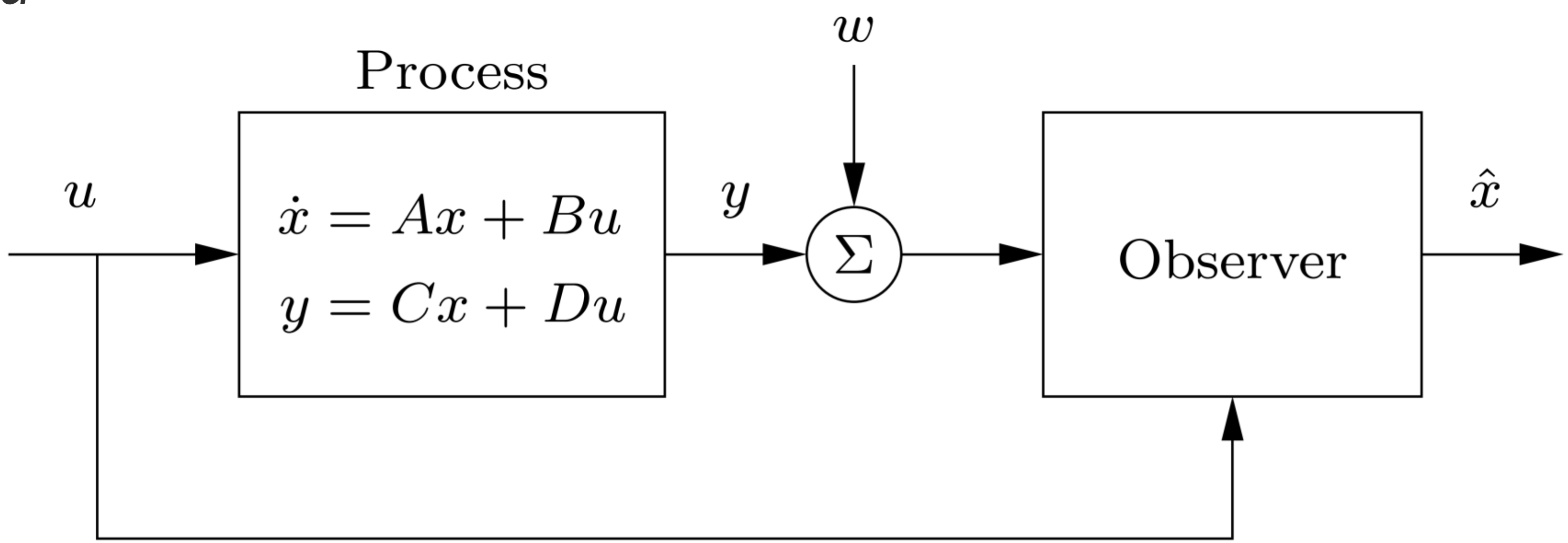
- Si la matriz de observabilidad no es de rango completo, va a existir una condición inicial $\boldsymbol{x(0)}$ que podrá darme salida nula.
- Así no se puede saber qué está pasando con el vector de estados.

Observador

- *Es un filtro.*
- *Toma la salida y la entrada y tiene una variable de estado interna propia.*
- *Se pretende que esa variable de estado interna del observador sea una estimación convergente de la variable de estado de la planta.*

$$\frac{d\hat{x}}{dt} = F\hat{x} + Gu + Hy,$$

$$\hat{x} \in \mathbb{R}^n \qquad \hat{x}(t) \to x(t) \text{ as } t \to \infty$$



Observador a lazo abierto

- *Prescinde de la “y”.*
- *Hace una “simulación” de la planta.*
- *No hay garantía de convergencia.*
- *El error depende de la dinámica de la planta.*
- *No funciona……*

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu \qquad \tilde{x} = x - \hat{x}$$

$$\frac{d\tilde{x}}{dt} = A\tilde{x}$$

Observador

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx$$

- *Agrego el factor de corrección:*

$$L(y - \hat{y})$$

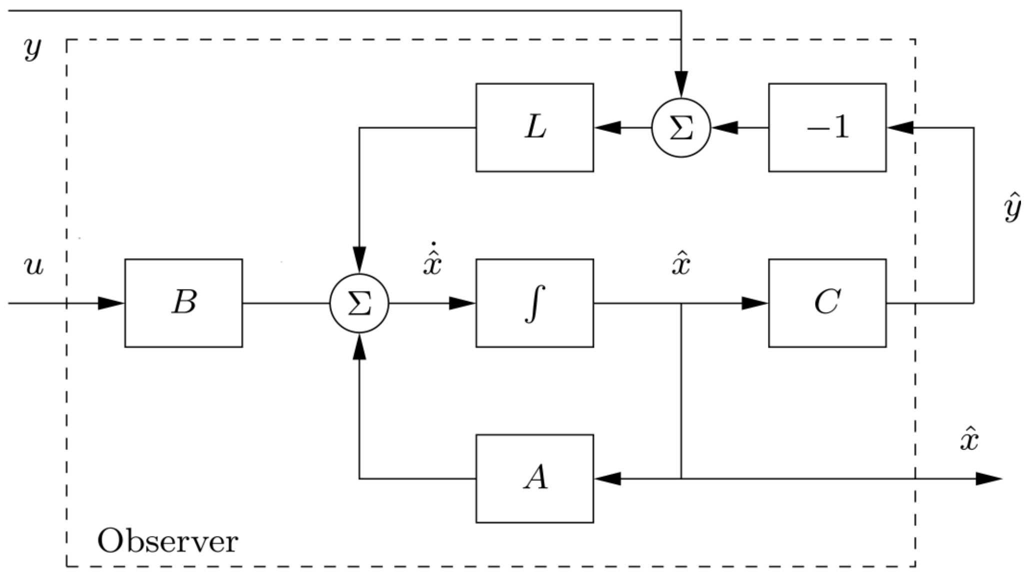
$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \tilde{x} = x - \hat{x}$$

- *La dinámica del error puede hacerse convergente si pueden ubicarse en el semiplano izquierdo los autovalores de*

$$A - LC$$

$$\frac{d\tilde{x}}{dt} = (A - LC)\tilde{x}$$

- *Es un problema algebraicamente equivalente al del cálculo de la realimentación de estados.*

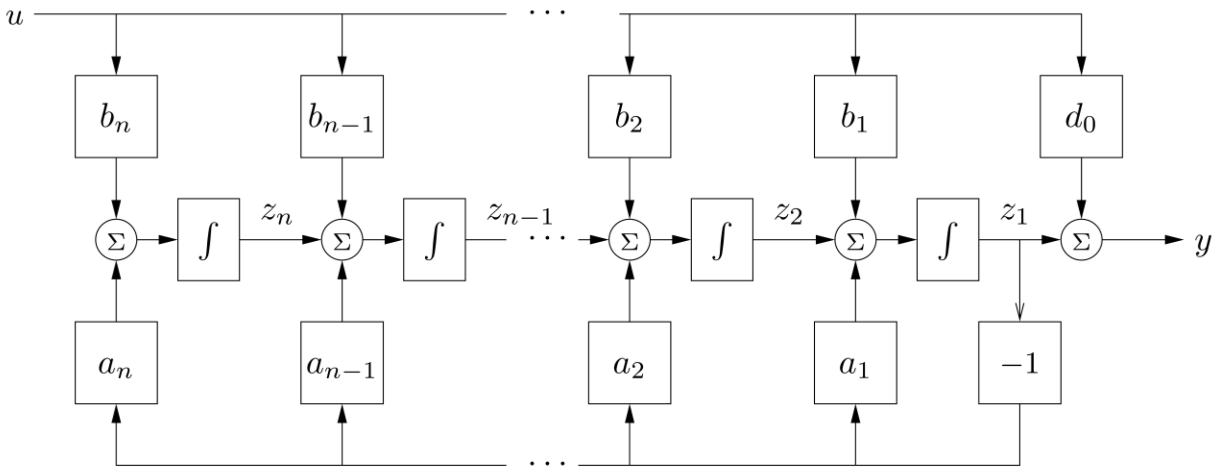


OBSERVADOR: FORMA CANÓNICA OBSERVABLE

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

$$W_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

OBSERVADOR: FORMA CANÓNICA OBSERVABLE



$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -a_n & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} z + d_0\, u$$

$$\lambda(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

OBSERVADOR: FORMA CANÓNICA OBSERVABLE

$$\frac{dz}{dt} = \tilde{A}z + \tilde{B}u \qquad y = \tilde{C}z + d_o u$$

$$\tilde{W}_o = \begin{bmatrix} \tilde{C} \\ \tilde{C}\tilde{A} \\ \vdots \\ \tilde{C}\tilde{A}^{n-1} \end{bmatrix}$$

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -a_n & & 0 & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} z + d_0\, u$$

$$\tilde{W}_\mathrm{o} = \begin{pmatrix} 1 & & & 0 \\ -a_1 & 1 & & \\ -a_1^2 - a_2 & -a_1 & 1 & \\ \vdots & \vdots & & \ddots \\ * & * & \cdots & 1 \end{pmatrix}$$

$$\tilde{W}_\mathrm{o}^{-1} = \begin{pmatrix} 1 & & & & 0 \\ a_1 & 1 & & & \\ a_2 & a_1 & 1 & & \\ & & & \ddots & \\ \vdots & \vdots & & & 1 \\ a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \end{pmatrix}$$

Observador

and the matrices W_o and $\widetilde{W}_o$ given by

$$W_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}, \qquad \widetilde{W}_o = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & 1 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & & 1 & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & 1 \end{pmatrix}^{-1}.$$

The resulting observer error $\tilde{x} = x - \hat{x}$ is governed by a differential equation having the characteristic polynomial

$$p(s) = s^n + p_1 s^{n-1} + \cdots + p_n.$$

Observador + Realimentación de estados observados

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \tilde{x} = x - \hat{x}$$

$$\frac{d\tilde{x}}{dt} = Ax - A\hat{x} - L(Cx - C\hat{x}) = A\tilde{x} - LC\tilde{x} = (A - LC)\tilde{x}$$

$$u = -K\hat{x} + k_\mathrm{f} r$$

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BK\hat{x} + Bk_\mathrm{f} r = Ax - BK(x - \tilde{x}) + Bk_\mathrm{f} r \\ &= (A - BK)x + BK\tilde{x} + Bk_\mathrm{f} r\end{aligned}$$

Observador + Realimentación de estados observados

$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bk_{\mathrm{f}} \\ 0 \end{pmatrix} r$$

$$\lambda(s) = \det\left(sI - A + BK\right)\det\left(sI - A + LC\right)$$

Observador + Realimentación de estados observados

Theorem 8.3 (Eigenvalue assignment by output feedback). *Consider the system*

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

The controller described by

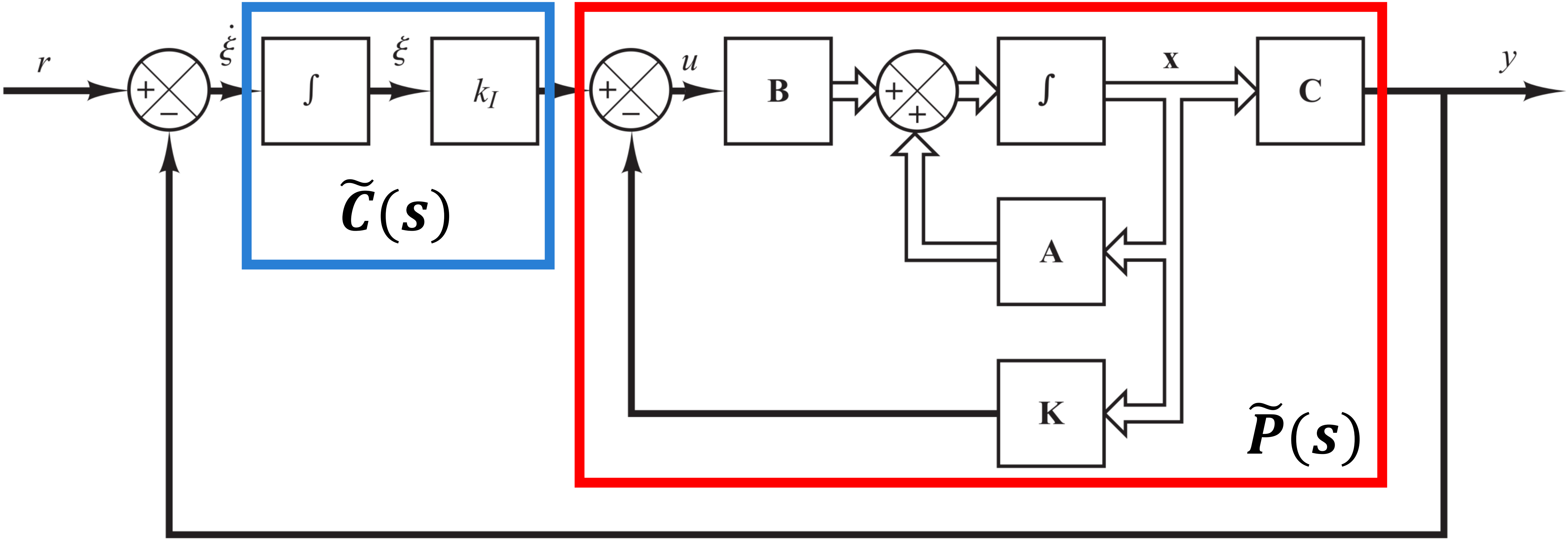
$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + Bu + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + Bk_f r + Ly, \\ u &= -K\hat{x} + k_f r \end{aligned}$$

gives a closed loop system with the characteristic polynomial

$$\lambda(s) = \det(sI - A + BK)\det(sI - A + LC).$$

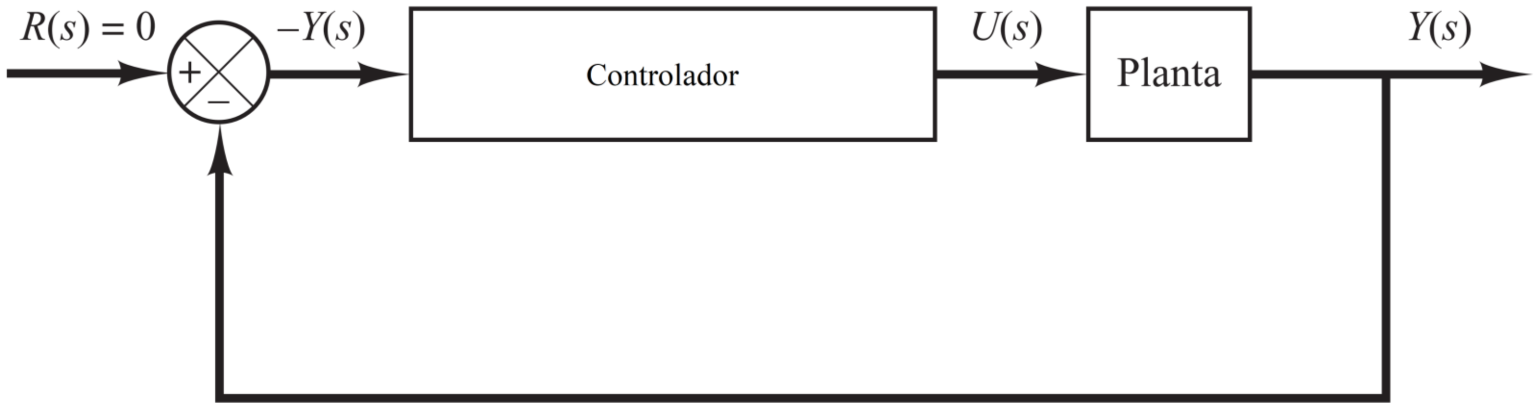
This polynomial can be assigned arbitrary roots if the system is reachable and observable.

ACCIÓN INTEGRAL EN ESPACIO DE ESTADOS

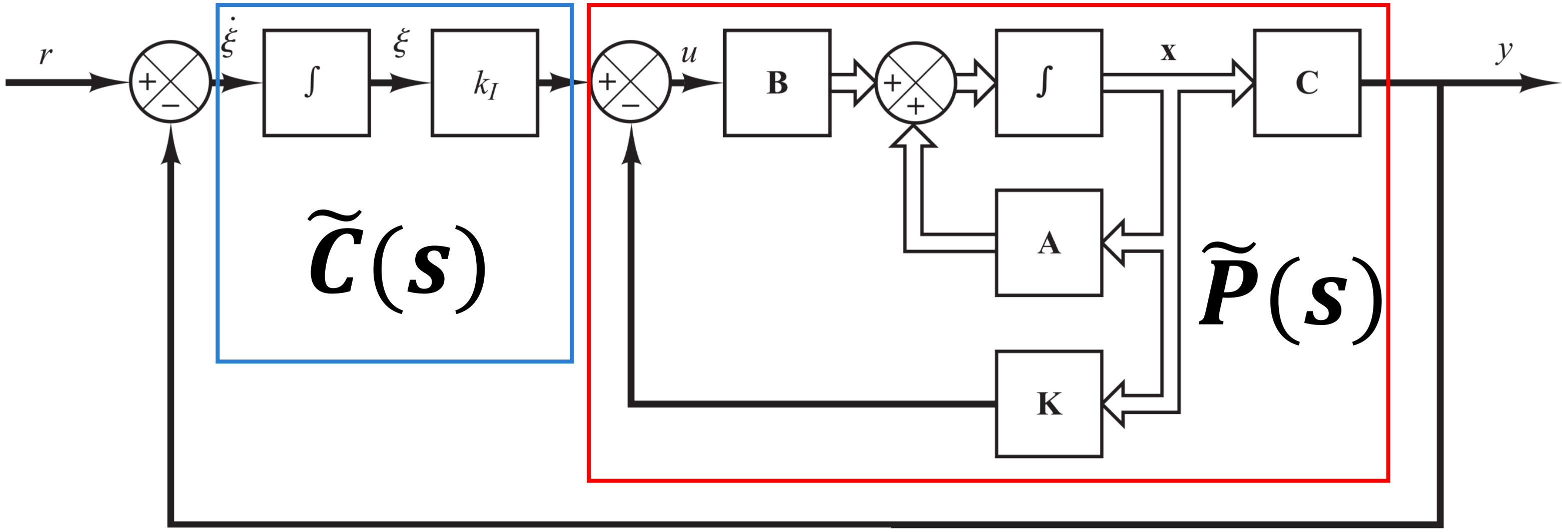


CONTROLADOR COMPLETO

Notar que así formulado, este controlador no se propone otra cosa que moldear la respuesta transitoria y carece de acción integral.



ACCIÓN INTEGRAL EN ESPACIO DE ESTADOS



$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\dot{\xi} = r - y = r - Cx - Du$$

$$u = -Kx + k_I\xi$$

$$\dot{x} = Ax + B(-Kx + k_I\xi)$$

$$y = Cx + D(-Kx + k_I\xi)$$

$$\dot{\xi} = r - y = r - Cx - D(-Kx + k_I\xi)$$

$$\dot{x} = Ax + B(-Kx + k_I\xi)$$

$$y = Cx + D(-Kx + k_I\xi)$$

$$\dot{\xi} = r - y = r - Cx - D(-Kx + k_I\xi)$$

$$\dot{x} = (A - BK)x + Bk_I\xi$$

$$y = (C - DK)x + Dk_I\xi$$

$$\dot{\xi} = r - y = r - (C - DK)x - Dk_I\xi$$

$$\dot{x} = (A - BK)\,x + Bk_I\xi \qquad y = (C - DK)\,x + Dk_I\xi$$

$$\dot{\xi} = r - y = r - (C - DK)\,x - Dk_I\xi$$

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A - BK & Bk_I \\ -(C - DK) & -Dk_I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} r$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix}}_{\dot{x}_a} = \underbrace{\left(\underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{A_a} - \underbrace{\begin{bmatrix} B \\ -D \end{bmatrix}}_{B_a} \underbrace{\begin{bmatrix} K & -k_I \end{bmatrix}}_{K_a} \right)}_{\tilde{A}_a} \underbrace{\begin{bmatrix} x \\ \xi \end{bmatrix}}_{x_a} + \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A - BK & Bk_I \\ -(C - DK) & -Dk_I \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} r$$

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix}}_{\dot{x}_a} = \underbrace{\left(\underbrace{\begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}}_{A_a} - \underbrace{\begin{bmatrix} B \\ -D \end{bmatrix}}_{B_a} \underbrace{\begin{bmatrix} K & -k_I \end{bmatrix}}_{K_a} \right)}_{\tilde{A}_a} \underbrace{\begin{bmatrix} x \\ \xi \end{bmatrix}}_{x_a} + \begin{bmatrix} 0_{n\times 1} \\ 1 \end{bmatrix} r$$

```
s=tf('s')

G=ss(zpk((s-1)*(s-3)/((s-5)*(s-4))))

Aa=[G.a  zeros(order(G),1);-G.c  0]

Ba=[G.b;-G.d]

Ka=acker(Aa,Ba,[-2  -2  -2])

K=Ka(1:order(G));

kI=-Ka(end);

L=acker(G.a',G.c',[-4  -4])'
```

Para una transferencia

$$G(s) = \frac{(s-1)(s-3)}{(s-5)(s-4)}$$

RESPUESTA DE SEGUNDO ORDEN

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u, \qquad y = q.$$

$$x = (q, \dot{q}/\omega_0)$$

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

$$\lambda = -\zeta\omega_0 \pm \omega_0\sqrt{(\zeta^2 - 1)}$$

$$\alpha = \omega_0(\zeta + \sqrt{\zeta^2 - 1}) \quad \beta = \omega_0(\zeta - \sqrt{\zeta^2 - 1})$$

$$\boxed{y(t) = \frac{\beta x_{10} + x_{20}}{\beta - \alpha} e^{-\alpha t} - \frac{\alpha x_{10} + x_{20}}{\beta - \alpha} e^{-\beta t}}$$

$$\boxed{y(t) = e^{-\zeta\omega_0 t}\big(x_{10} + (x_{20} + \zeta\omega_0 x_{10})t\big)}$$

RESPUESTA DE SEGUNDO ORDEN

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u, \qquad y = q.$$

$$x = (q, \dot{q}/\omega_0)$$

$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

$$\lambda = -\zeta\omega_0 \pm \omega_0\sqrt{(\zeta^2 - 1)}$$

$$\omega_\mathrm{d} = \omega_0\sqrt{1 - \zeta^2}$$

$$\boxed{y(t) = e^{-\zeta\omega_0 t}\left(x_{10}\cos\omega_\mathrm{d} t + \left(\frac{\zeta\omega_0}{\omega_\mathrm{d}}x_{10} + \frac{1}{\omega_\mathrm{d}}x_{20}\right)\sin\omega_\mathrm{d} t\right)}$$

RESPUESTA DE SEGUNDO ORDEN

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = k\omega_0^2 u, \qquad y = q.$$

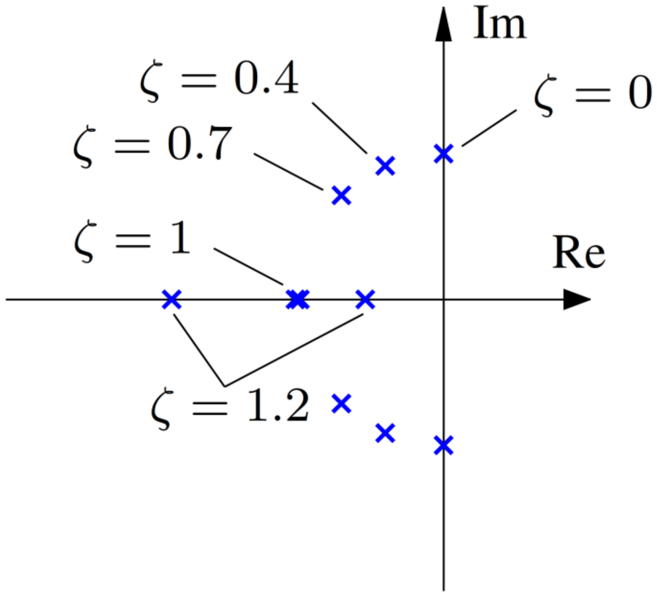
$$\frac{dx}{dt} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & -2\zeta\omega_0 \end{pmatrix} x + \begin{pmatrix} 0 \\ k\omega_0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

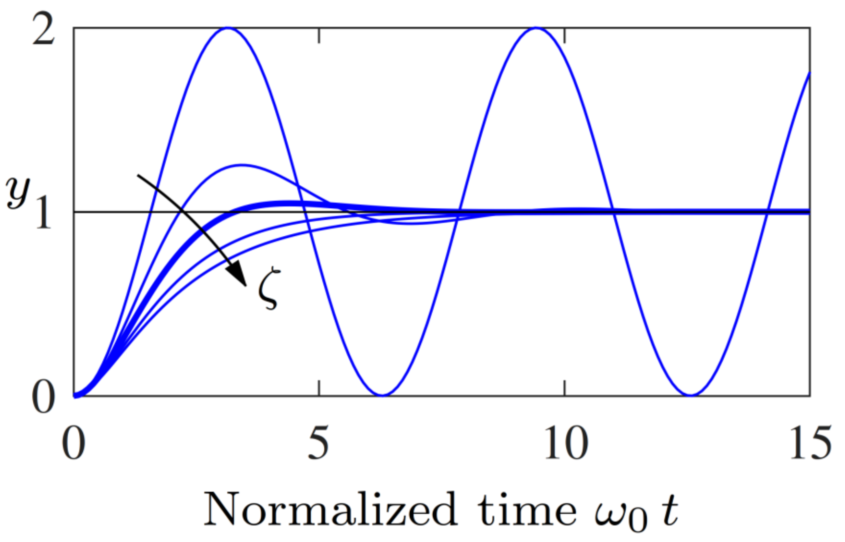
$$\boxed{y(t) = k\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t} \sin(\omega_d t + \varphi)\right)}$$

$$\lambda = -\zeta\omega_0 \pm \omega_0\sqrt{(\zeta^2 - 1)}$$

$$\omega_d = \omega_0\sqrt{1-\zeta^2}$$

$$x = (q, \dot{q}/\omega_0)$$





RESPUESTA DE SEGUNDO ORDEN

Table 7.1: Properties of the step response for a second-order system with $0 < \zeta \leq 1$.

Property	Value	$\zeta = 0.5$	$\zeta = 1/\sqrt{2}$	$\zeta = 1$
Steady-state value	k	k	k	k
Rise time (inverse slope)	$T_r = e^{\varphi/\tan\varphi}/\omega_0$	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
Overshoot	$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$	16%	4%	0%
Settling time (2%)	$T_s \approx 4/\zeta\omega_0$	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$

$$\boxed{y(t) = k\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_0 t}\sin(\omega_d t + \varphi)\right)}$$