FACULTAD DE INGENIERIA
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Introducción a Sistemas de Control

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REALIZACIONES DE TRANSFERENCIAS SISO

Transferencias SISO (según el cap. 3 del libro):

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = b_0 \frac{d^m u}{dt^m} + b_1 \frac{d^{m-1} u}{dt^{m-1}} + \cdots + b_m u$$

- Notar que así definida no se especifica si $G(s)$ es propia.

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_n$$

$$b(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m$$

- Supondremos que $m \leq n$.

$$G(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b(s)}{a(s)}$$

Transferencias SISO: estrictamente propia

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = b_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + b_{n-1} \frac{du}{dt} + b_n u$$

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

$$b(s) = b_1 s^{n-1} + \cdots + b_{n-1} s + b_n$$

$$G(s) = \frac{b(s)}{a(s)}$$

Transferencias SISO: bipropia

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_{n-1} \frac{dy}{dt} + a_n y = \beta_0 \frac{d^n u}{dt^n} + \beta_1 \frac{d^{n-1} u}{dt^{n-1}} + \cdots + \beta_{n-1} \frac{du}{dt} + \beta_n u$$

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$
$$b(s) = b_1 s^{n-1} + \cdots + b_{n-1} s + b_n$$
$$\beta(s) = \beta_0 s^n + \beta_1 s^{n-1} + \cdots + \beta_{n-1} s + \beta_n$$

$$G(s) = \frac{\beta(s)}{a(s)} = \frac{b(s)}{a(s)} + d$$

Transferencias SISO: bipropia

$$b(s) = b_1 s^{n-1} + \cdots + b_{n-1}s + b_n$$

$$\beta(s) = \beta_0 s^n + \beta_1 s^{n-1} + \cdots + \beta_{n-1}s + \beta_n$$

$$\beta(s) = b(s) + d.\,a(s)$$

Ejercicios. Dada:

$$G(s) = \frac{32s^2 + 8s + 49}{s^2 + 2s + 1}$$

Encontrar “$b(s) = b_1 s + b_2$” y “d”.

¿Qué transferencias se pueden representar en espacio de estados? SOLO LAS **PROPIAS**

$$G(s) = C(sI - A)^{-1}B + D = C(sI - A)^{-1}B + d$$

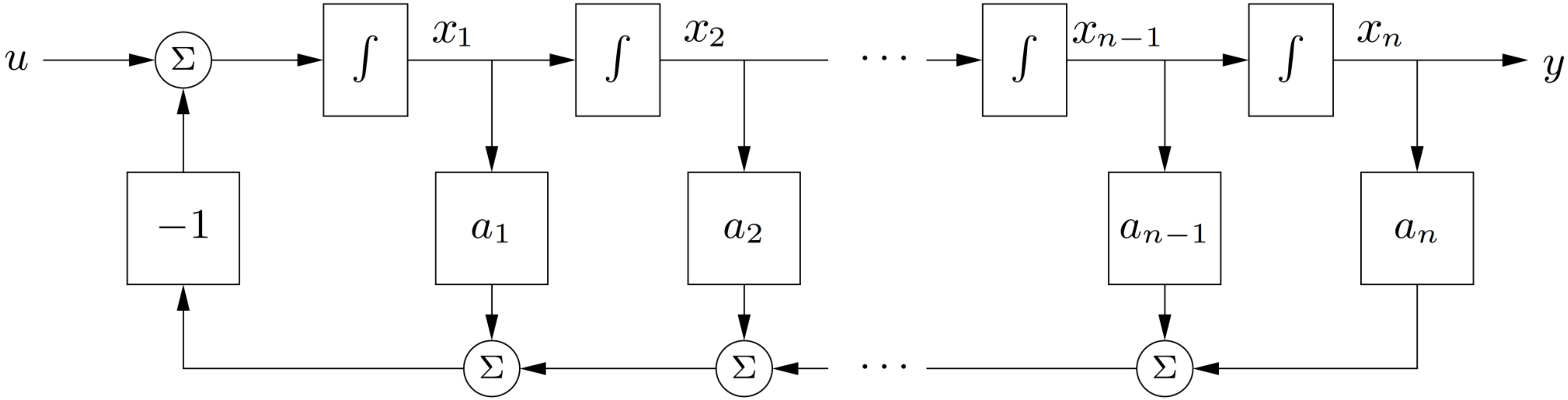
$$(sI - A)^{-1} = \frac{\left(Adj(sI - A)\right)^T}{|sI - A|}$$

- $Adj(sI - A)$: sus elementos son polinomios de orden $\leq n - 1$.
- $C(sI - A)^{-1}B$ es la parte estrictamente propia.
- Si $D \neq 0$ la transferencia será bipropia, pero nunca impropia.

Grado Relativo "n"

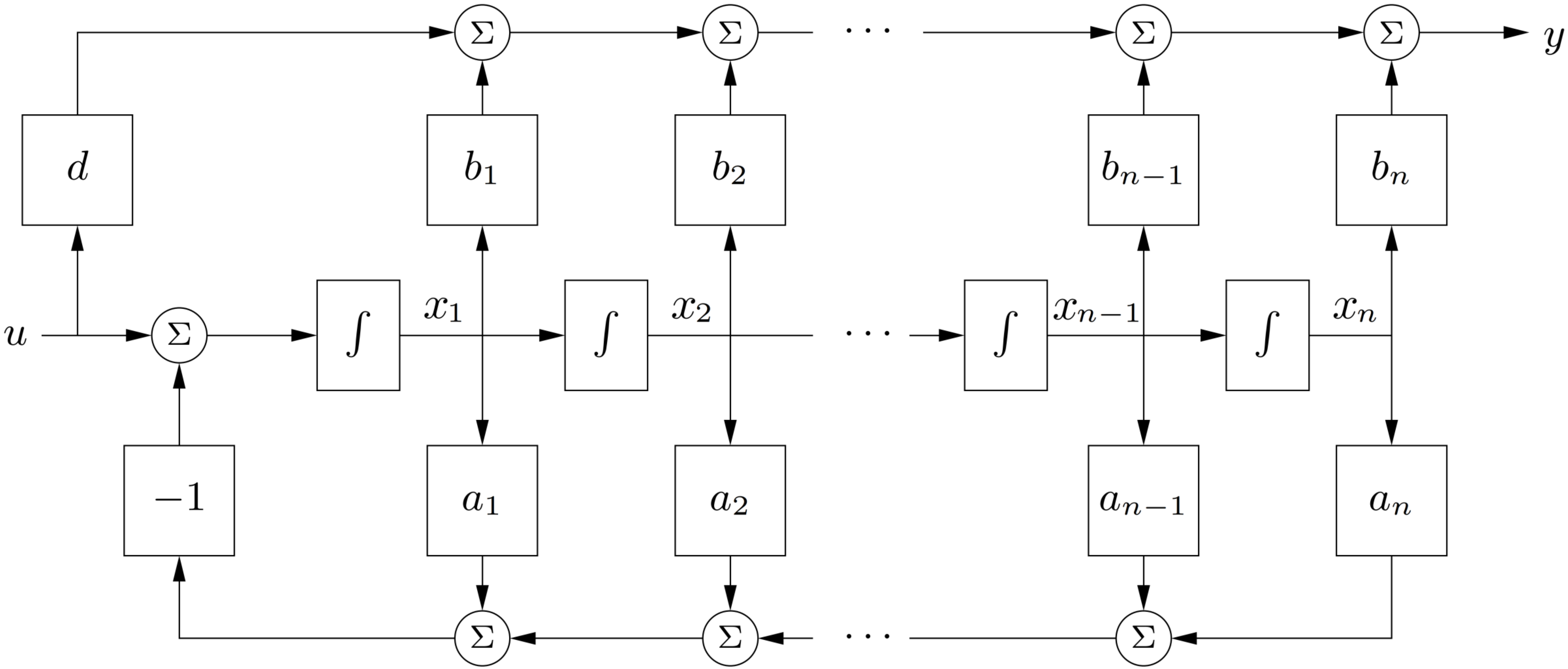
$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_n y = u$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} d^{n-1}y/dt^{n-1} \\ d^{n-2}y/dt^{n-2} \\ \vdots \\ dy/dt \\ y \end{pmatrix} \qquad \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -a_1 x_1 - \cdots - a_n x_n \\ x_1 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{pmatrix} + \begin{pmatrix} u \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$



$$y = b_1 x_1 + b_2 x_2 + \cdots + b_n x_n + du$$

Caso General



$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} x + du.$$

Caso General

Forma de Jordan vía Fracciones Simples

Repaso de fracciones simples:

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}, \qquad \text{para } m < n$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \cdots + \frac{a_n}{s+p_n}$$

$$\begin{aligned}\left[(s+p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} &= \left[\frac{a_1}{s+p_1}(s+p_k) + \frac{a_2}{s+p_2}(s+p_k)\right. \\ &\qquad \left. + \cdots + \frac{a_k}{s+p_k}(s+p_k) + \cdots + \frac{a_n}{s+p_n}(s+p_k)\right]_{s=-p_k} \\ &= a_k\end{aligned}$$

Forma de Jordan vía Fracciones Simples

Repaso de fracciones simples:

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

$$(s+1)^3 \frac{B(s)}{A(s)} = b_1(s+1)^2 + b_2(s+1) + b_3$$

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s+1)$$

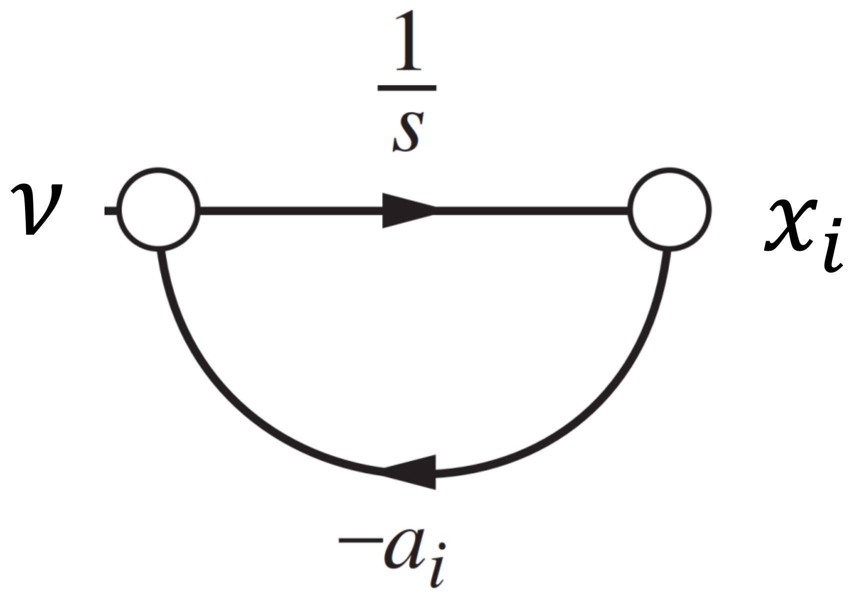
$$\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

$$\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2b_1$$

En matlab se puede usar la función “residue” para calcular los coeficientes de la descomposición.

Forma de Jordan vía Fracciones Simples



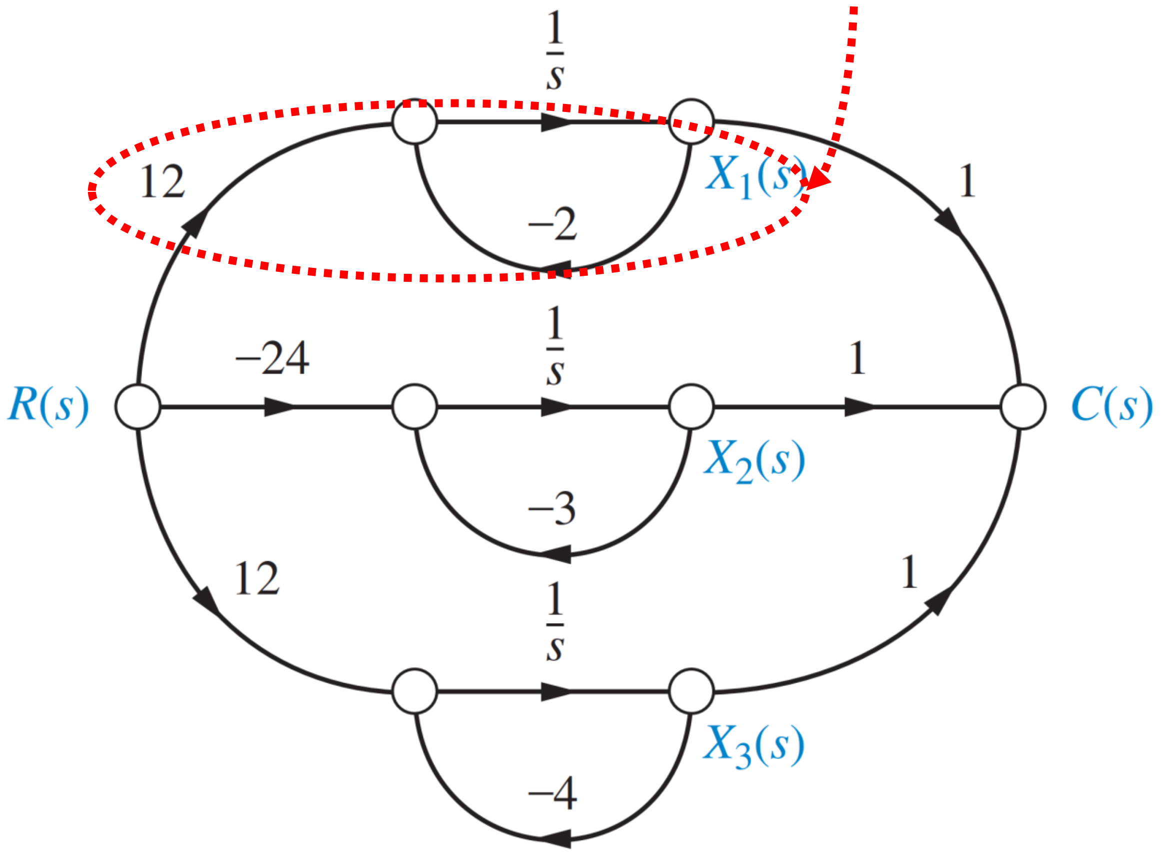
$$\frac{X_i(s)}{\mathrm{N}(s)} = \frac{1}{s + a_i}$$

$$\mathrm{N}(s) = \mathcal{L}\{\nu\}$$

$$\dot{x}_i = -a_i x_i + \nu$$

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

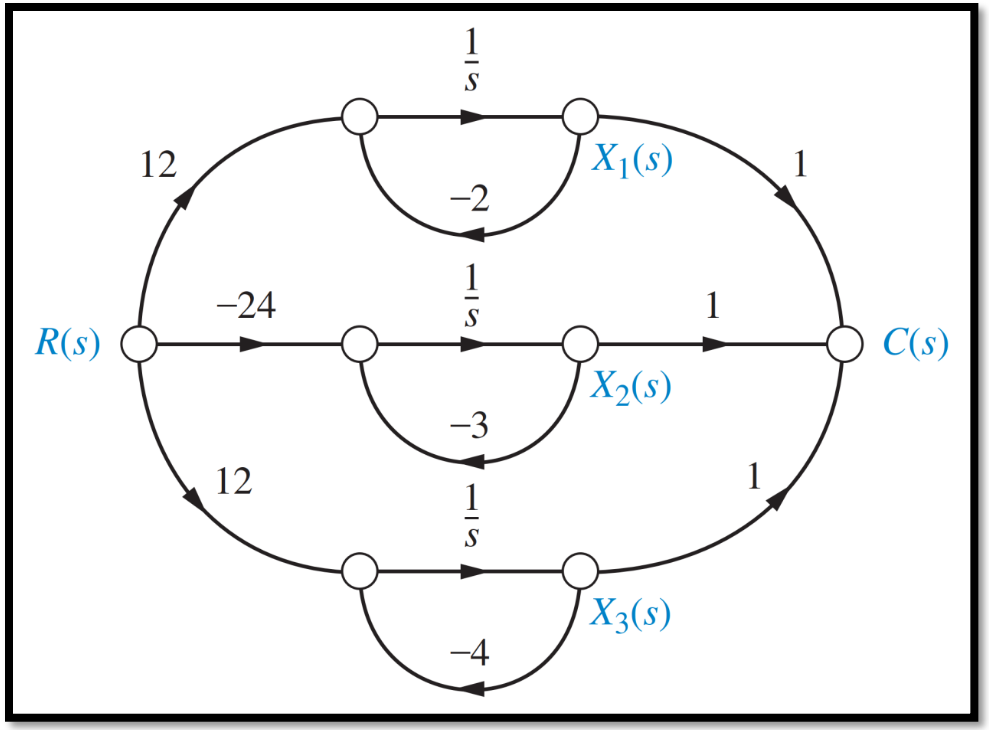
Forma de Jordan vía Fracciones Simples



$$\dot{x}_1 = -2x_1 \qquad\qquad\qquad +12r$$

$$\dot{x}_2 = \qquad -3x_2 \qquad -24r$$

$$\dot{x}_3 = \qquad\qquad\qquad -4x_3 +12r$$



$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

$$y = c(t) = x_1 + x_2 + x_3$$

$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\mathbf{x}$$

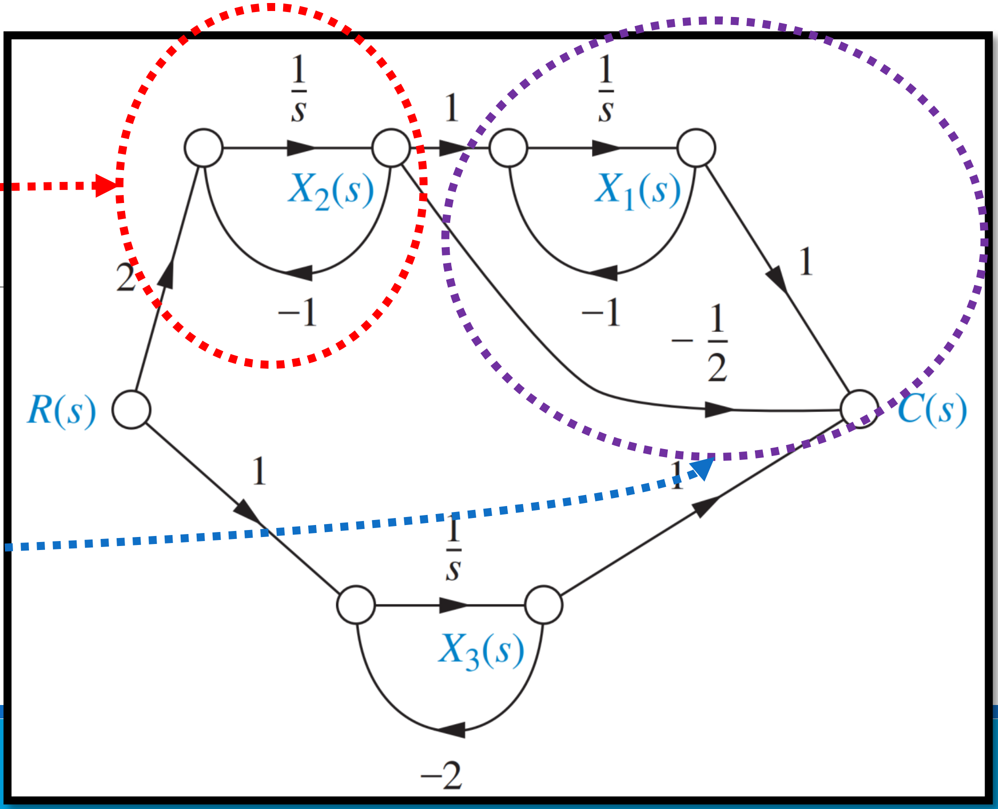
Con polos múltiples

$$\boxed{\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)}}$$

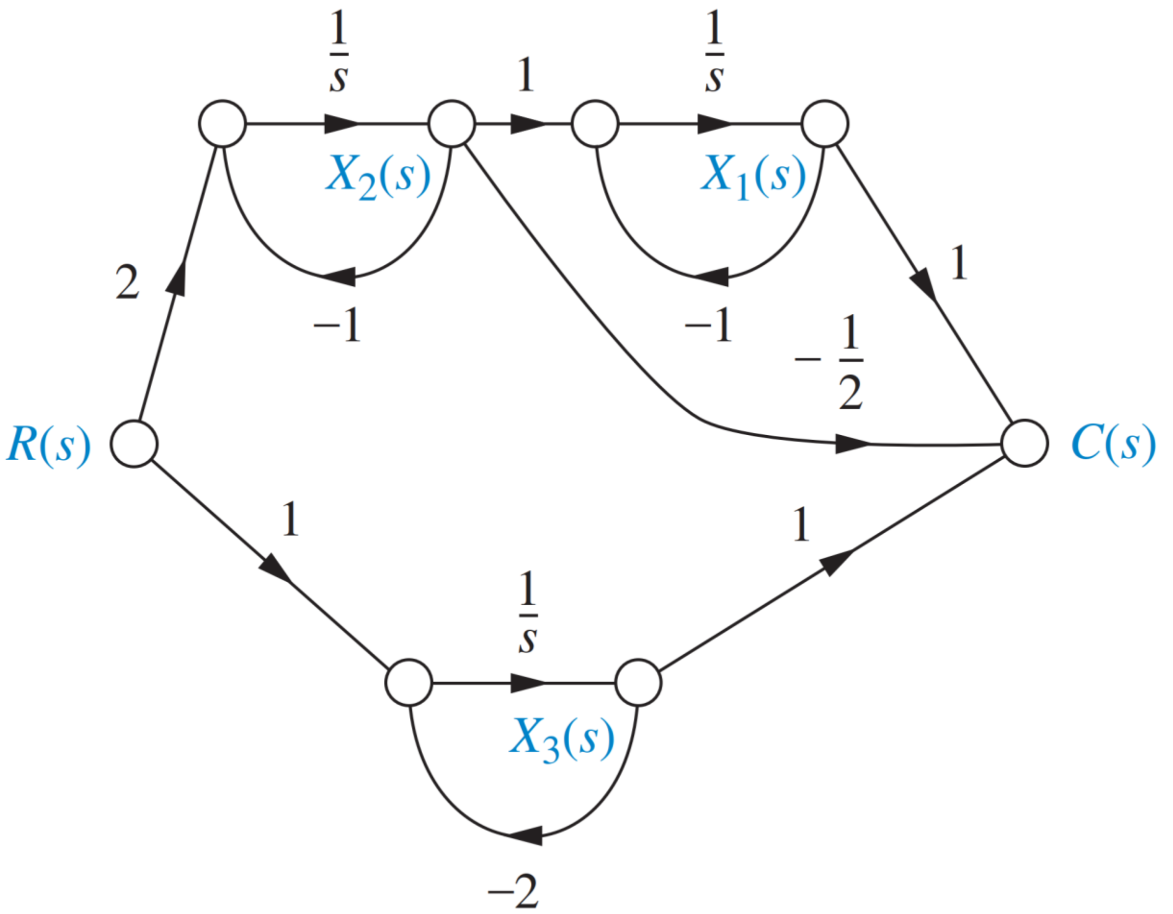
$$\boxed{\frac{C(s)}{R(s)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}}$$

$$\frac{2}{(s+1)^2} - \frac{1}{(s+1)} =$$

$$\frac{2}{(s+1)}\left(\frac{1}{(s+1)} + -\frac{1}{2}\right)$$



$$
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= -x_2 + 2r \\
\dot{x}_3 &= -2x_3 + r \\
y = c(t) &= x_1 - \frac{1}{2}x_2 + x_3
\end{aligned}
$$



$$
\dot{\mathbf{x}} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r
$$

$$
y = \begin{bmatrix} 1 & -\frac{1}{2} & 1 \end{bmatrix}\mathbf{x}
$$

Forma canónica del observador

$$\boxed{\frac{C(s)}{R(s)} = \frac{\dfrac{1}{s} + \dfrac{7}{s^2} + \dfrac{2}{s^3}}{1 + \dfrac{9}{s} + \dfrac{26}{s^2} + \dfrac{24}{s^3}}}$$

$$\frac{s^2 + 7s + 2}{s^3 + 9s^2 + 26s + 24}$$

$$\boxed{\left[\frac{1}{s} + \frac{7}{s^2} + \frac{2}{s^3}\right] R(s) = \left[1 + \frac{9}{s} + \frac{26}{s^2} + \frac{24}{s^3}\right] C(s)}$$

Forma canónica del observador

$$\left[\frac{1}{s}+\frac{7}{s^2}+\frac{2}{s^3}\right]R(s)=\left[1+\frac{9}{s}+\frac{26}{s^2}+\frac{24}{s^3}\right]C(s)$$

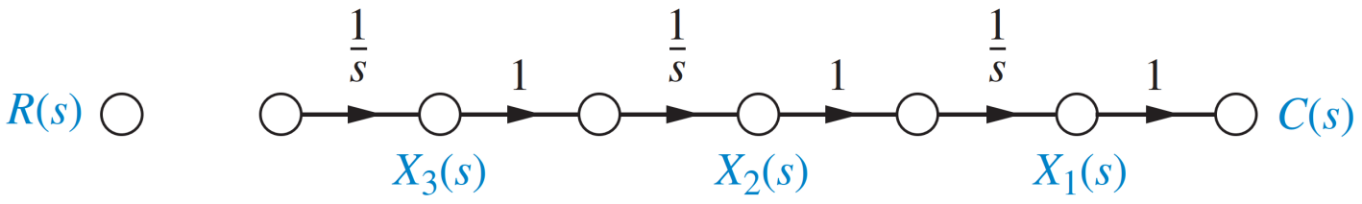
$$C(s)=\frac{1}{s}[R(s)-9C(s)]+\frac{1}{s^2}[7R(s)-26C(s)]+\frac{1}{s^3}[2R(s)-24C(s)]$$

Forma canónica del observador

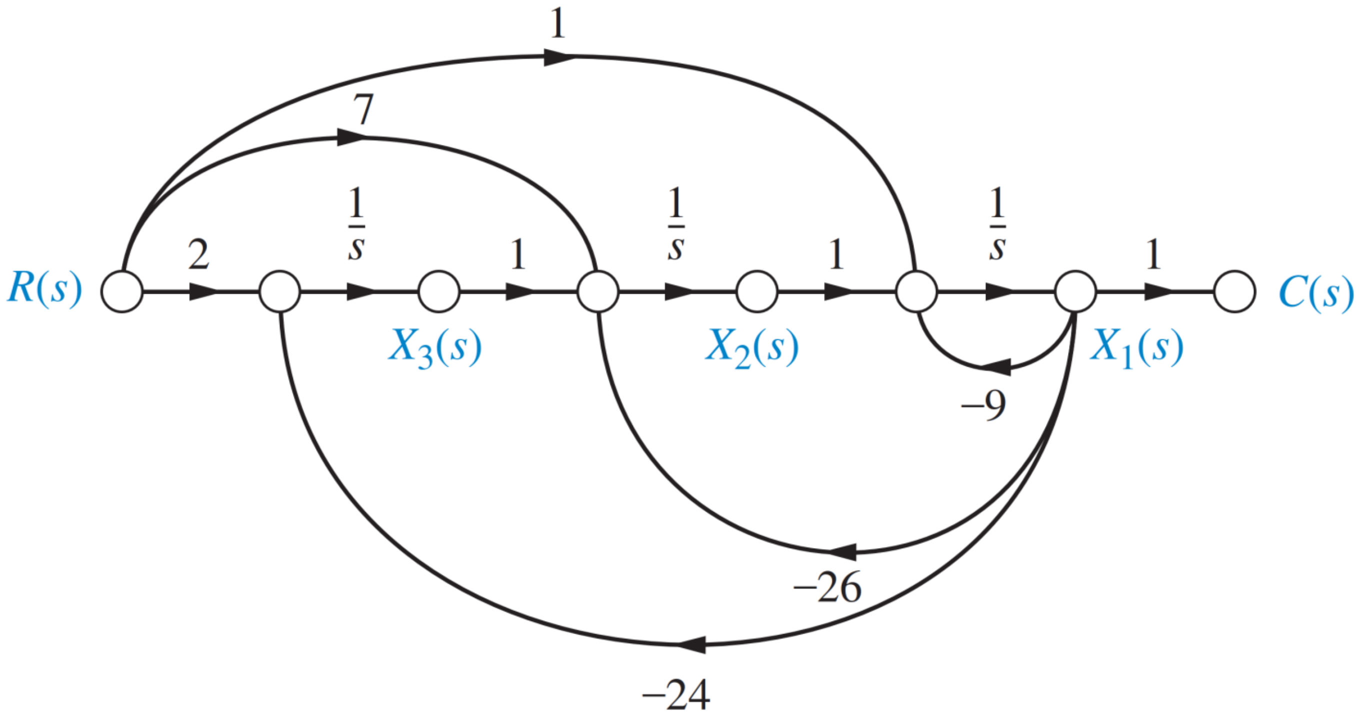
$$\boxed{C(s) = \frac{1}{s}[R(s) - 9C(s)] + \frac{1}{s^2}[7R(s) - 26C(s)] + \frac{1}{s^3}[2R(s) - 24C(s)]}$$

$$\boxed{C(s) = \frac{1}{s}\left[[R(s) - 9C(s)] + \frac{1}{s}\left([7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)]\right)\right]}$$

$$C(s) = \frac{1}{s}\left[[R(s) - 9C(s)] + \frac{1}{s}\left([7R(s) - 26C(s)] + \frac{1}{s}[2R(s) - 24C(s)]\right)\right]$$



(a)



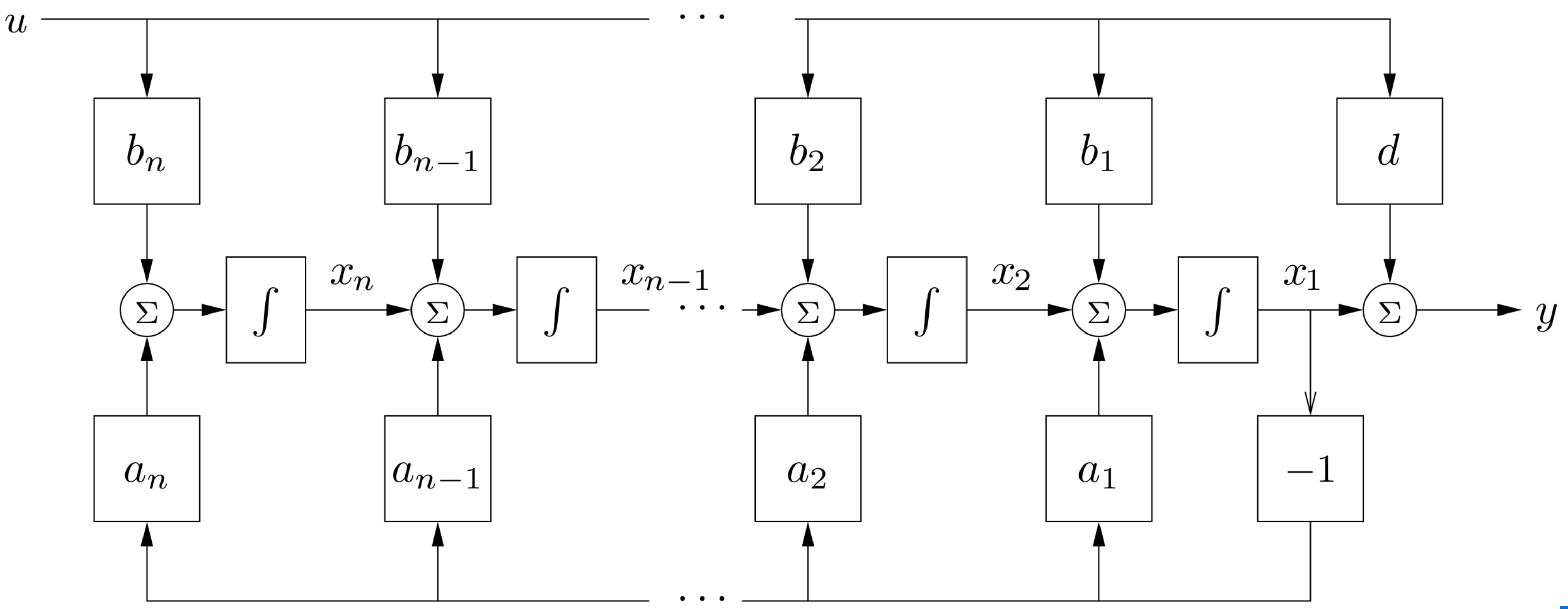
$$C(s)=\frac{1}{s}\left[[R(s)-9C(s)]+\frac{1}{s}\left([7R(s)-26C(s)]+\frac{1}{s}[2R(s)-24C(s)]\right)\right]$$

$$\begin{aligned}
\dot{x}_1 &= -9x_1 + x_2 && + r \\
\dot{x}_2 &= -26x_1 && + x_3 + 7r \\
\dot{x}_3 &= -24x_1 && + 2r
\end{aligned}$$

$$\dot{\mathbf{x}}=\begin{bmatrix}-9 & 1 & 0\\ -26 & 0 & 1\\ -24 & 0 & 0\end{bmatrix}\mathbf{x}+\begin{bmatrix}1\\7\\2\end{bmatrix}r$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{x}$$

Forma canónica del observador



Forma canónica del observador

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -a_n & & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} x + du$$

PHASE VARIABLE FORM

CONTROLLER CANONICAL

CONTROLLABLE O
REACHABLE CANONICAL

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} x + du.$$

OBSERVER
CANONICAL

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 & 1 & & 0 \\ -a_2 & 0 & \ddots & \\ \vdots & & \ddots & 1 \\ -a_n & & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} x + du$$

Conclusiones

- En todos los libros (Ogata, NISE, Astrom y Murray) están desarrolladas las fórmulas de las realizaciones de las funciones de transferencias SISO.

- Pueden diferir en si x_1 de uno es x_n en el otro, o si una la llama "*Controlable Canonical*" "*Controller Canonical*", o bien "*Observer Canonical*" vs. "*Observable Canonical*".

- Cada forma canónica tiene un atractivo, y son todas similares – en sentido matemático – se usan para ver distintas propiedades o para implementar numéricamente la programación de un sistema en espacio de estados.