FACULTAD
DE INGENIERIA
Universidad de Buenos Aires

Introducción a Sistemas de Control

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Estabilidad de Sistemas Lineales

$\dot{x} = Ax$	$z = Tx$
$\dot{z} = T\dot{x} = TAx = (TAT^{-1})z = \Lambda z$	Con "Λ" diagonal.
$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$	$(\Lambda t)^k = \begin{bmatrix} (\lambda_1 t)^k & & & 0 \\ & (\lambda_2 t)^k & & \\ & & \ddots & \\ 0 & & & (\lambda_n t)^k \end{bmatrix}$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix}$$

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Theorem 6.2 (Jordan decomposition). *Any matrix $A \in \mathbb{R}^{n\times n}$ can be transformed into Jordan form with the eigenvalues of A determining λ_i in the Jordan form.*

$$J = \begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{pmatrix} \qquad J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$

Each matrix J_i is called a *Jordan block.*

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$\dot{x} = Ax$	$z = Tx$
$\dot{z} = T\dot{x} = TAx = (TAT^{-1})z = Jz$	Con "J" de Jordan.
$J = \begin{bmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_k \end{bmatrix}$	$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 \\ 0 & \lambda_i & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}$

$$e^{Jt} = \begin{pmatrix} e^{J_1 t} & & & 0 \\ & e^{J_2 t} & & \\ & & \ddots & \\ 0 & & & e^{J_k t} \end{pmatrix}$$

Bloque de Jordan y Matriz Nilpotente

$$J_i = \lambda_i I + N$$

An interesting and useful feature of N is that its powers are easily computed. In particular its square is

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 1 \\ & & & 0 & 0 \\ 0 & & & & 0 \end{bmatrix}$$

and the successive powers of N have analogous structure, with the diagonal of ones shifting upwards, until eventually we find $N^n = 0$. That is N is *nilpotent* of order n.

State Space Systems: Autonomous system and Matrix Exp.

$$N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

State Space Systems: Autonomous system and Matrix Exp.

$$e^{J_i t} = e^{(\lambda_i I + N)t} = e^{\lambda_i I t} e^{Nt} = e^{\lambda_i t} e^{Nt}$$

$$e^{Nt} = \left(I + Nt + \frac{1}{2!} N^2 t^2 + \cdots + \frac{1}{(n_i - 1)!} N^{(n_i - 1)} t^{(n_i - 1)} \right)$$

$$e^{J_i t} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & \mathrm{t} \\ & 0 & & & 1 \end{pmatrix} e^{\lambda_i t}$$

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Theorem 5.1 (Stability of a linear system). *The system*

$$\frac{dx}{dt} = Ax$$

is asymptotically stable if and only if all eigenvalues of A have a strictly negative real part and is unstable if any eigenvalue of A has a strictly positive real part.

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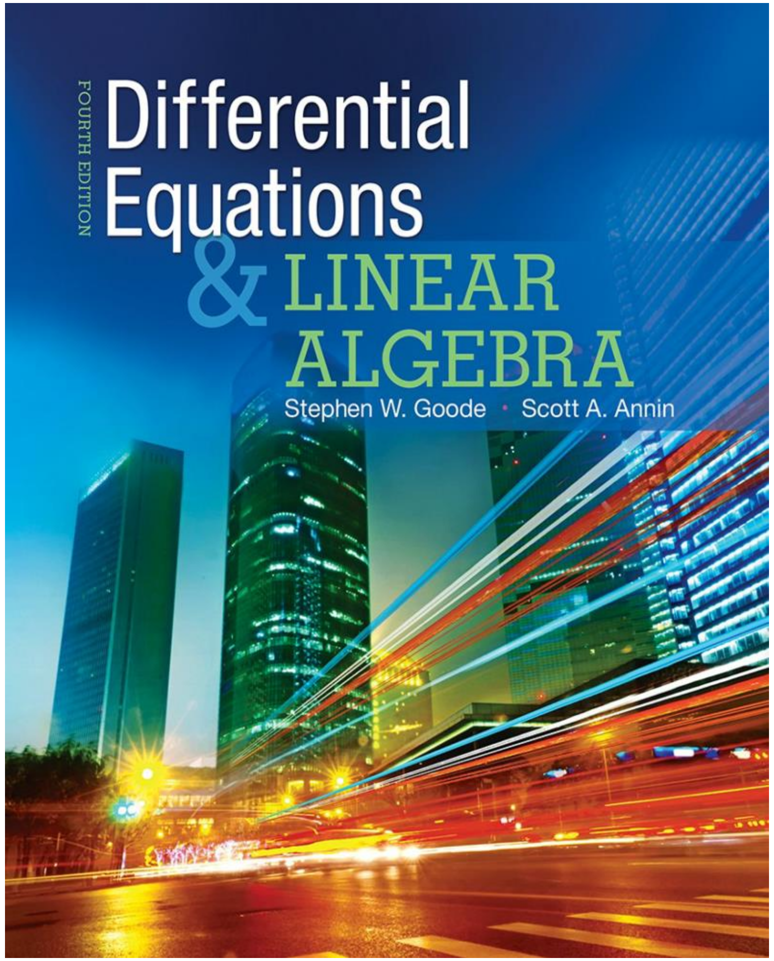
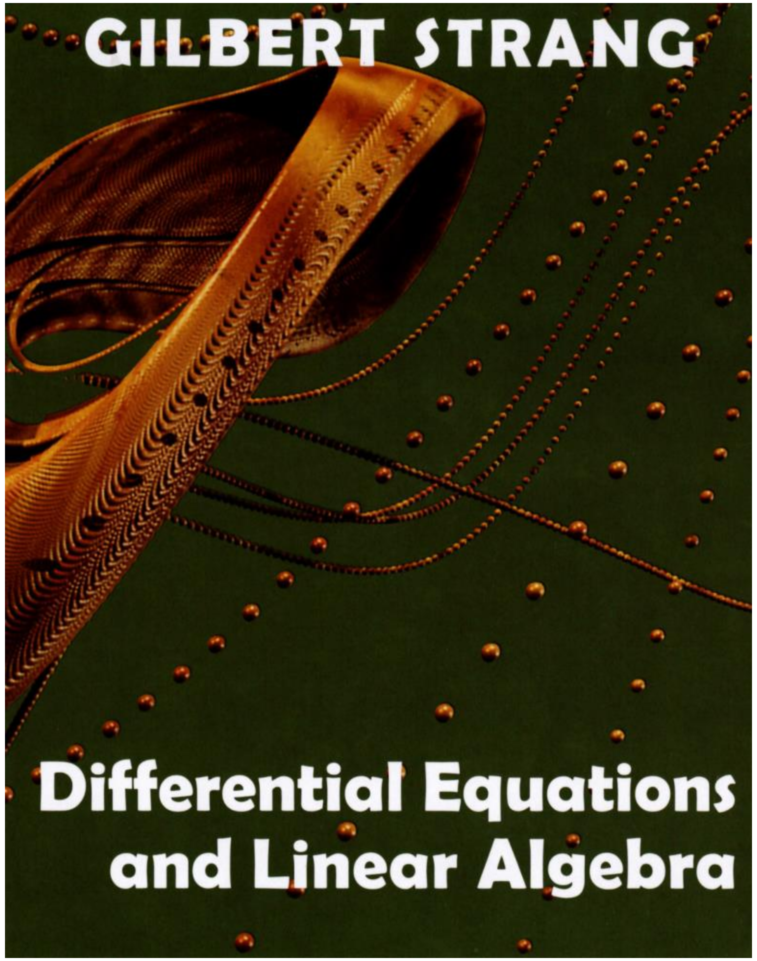
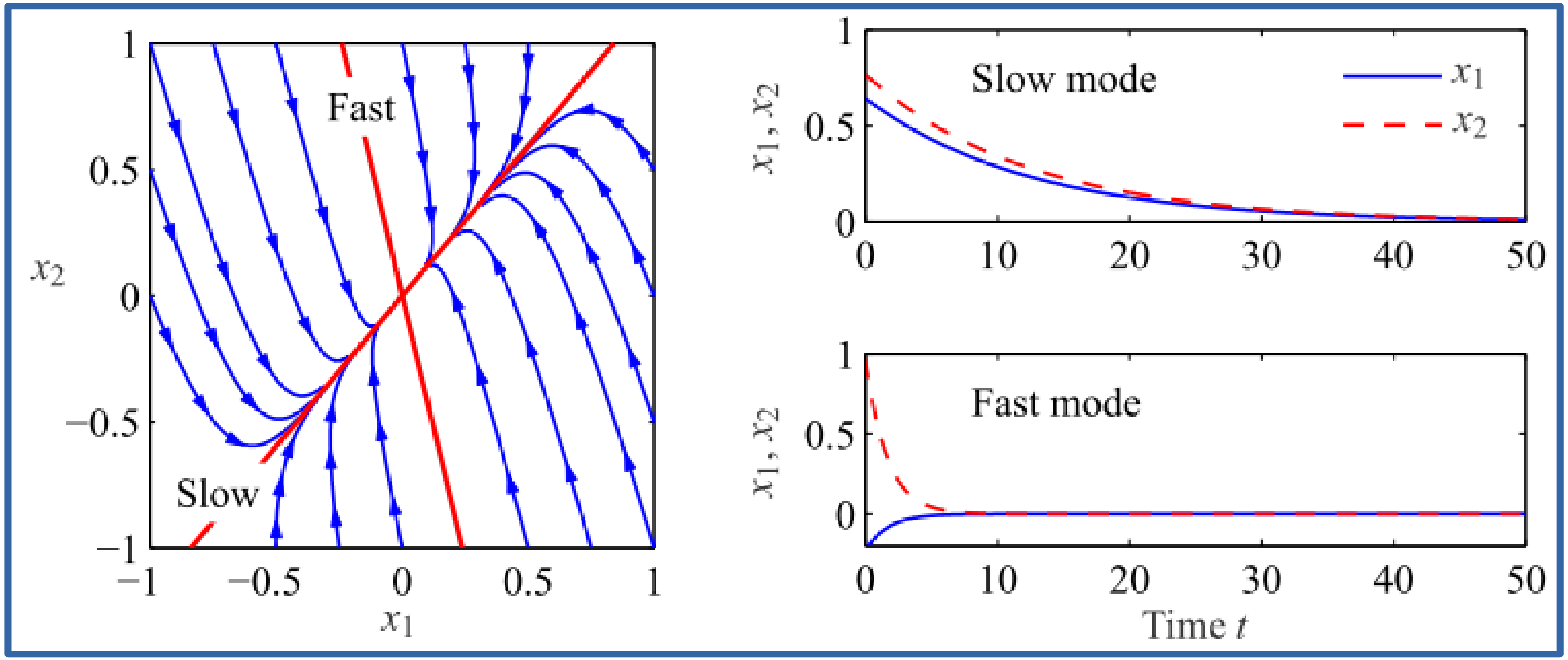
https://www.math.upenn.edu/~moose/240S2013/slides7-31.pdf





Retratos de Fase de Sistemas Lineales



$$\boxed{Av = \lambda v.}$$

$$\boxed{e^{At}v = \left(I + At + \frac{1}{2}A^2t^2 + \cdots\right)v = v + \lambda t v + \frac{\lambda^2 t^2}{2}v + \cdots = e^{\lambda t}v.}$$

Solución Forzada

$$\frac{dx}{dt} = Ax + Bu$$

Regla de Leibniz (Teorema fundamental del cálculo):

$$\boxed{\frac{d}{dt}\int_{f(t)}^{g(t)} H(t,\tau)d\tau = H(t,g(t))\dot{g}(t) - H(t,f(t))\dot{f}(t) + \int_{f(t)}^{g(t)} \frac{\partial}{\partial t}H(t,\tau)d\tau}$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$\frac{dx}{dt} = Ae^{At}x(0) + \int_0^t Ae^{A(t-\tau)}Bu(\tau)d\tau + Bu(t)$$