CONICET

FACULTAD DE INGENIERIA
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INTRODUCCIÓN A SISTEMAS DE CONTROL

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ELEMENTOS BÁSICOS DE CONTROL EN TIEMPO DISCRETO

SISTEMAS EN TIEMPO DISCRETO

$$y(k) \longleftarrow \boxed{P} \longleftarrow u(k)$$

Input and output signals are sequences:

$$u = \{u(0), u(1), u(2), \ldots, u(k), \ldots\} \qquad \text{and} \qquad y = \{y(0), y(1), y(2), \ldots, y(k), \ldots\}.$$

Causal LTI/LSI models can be described by difference equations:

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) \ \ldots \ - a_n y(k-n) \ + b_0 u(k) + b_1 u(k-1) \ \ldots \ b_m u(k-m).$$

Note that the current output, $y(k)$, depends only on current and past inputs, $u(k)$, $u(k-1)$, $\ldots$, and past outputs, $y(k-1)$, $\ldots$.

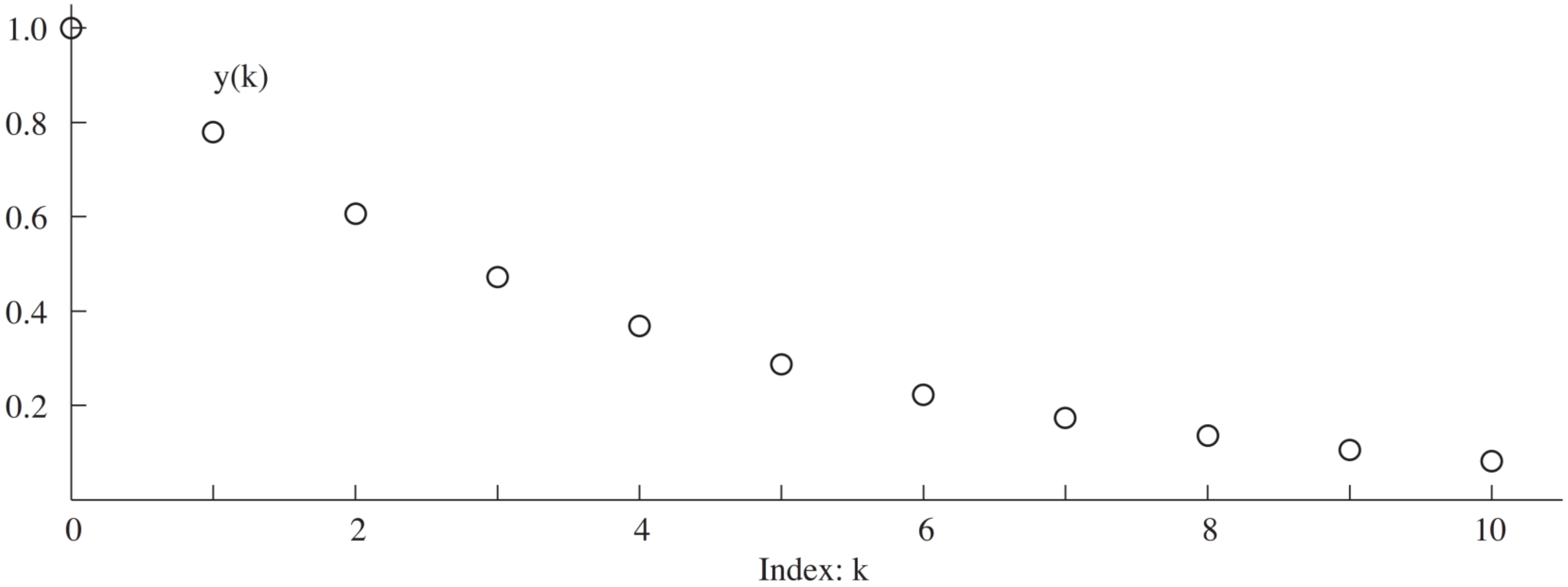
TRANSFORMADA "Z"

Recall the Z-Transform definition:

$$y(z) := \sum_{k=-\infty}^{\infty} y(k) z^{-k},$$

where $z \in \mathcal{C}$. This has an associated region of convergence: $r_0 < |z| < R_0$.

Example: $y(k) = \begin{cases} 0 & \text{for } k < 0, \\ \mathrm{e}^{-akT} & \text{for } k \geq 0 \end{cases}$



$$\begin{aligned} a &= 0.25, \\ T &= 1 \end{aligned}$$

$$
\begin{aligned}
y(z) &= \sum_{k=0}^{\infty} \mathrm{e}^{-akT} z^{-k} \quad = \quad \sum_{k=0}^{\infty} \left(\mathrm{e}^{-aT} z^{-1}\right)^k \\
&= \frac{1}{1-\mathrm{e}^{-aT} z^{-1}}, \qquad \text{for } \mathrm{e}^{-aT} < |z| < \infty, \\
&= \frac{z}{z-\mathrm{e}^{-aT}}, \qquad \text{for } \mathrm{e}^{-aT} < |z| < \infty.
\end{aligned}
$$

The region of convergence is needed to reconstruct the signal. (Exercise: reconstruct this Z-tranform for $|z| < \mathrm{e}^{-aT}$)

Shifted sequences

$$
\begin{aligned}
\mathcal{Z}\{y(k-l)\} &= \sum_{k=-\infty}^{\infty} y(k-l) z^{-k}, \\
&= \sum_{i=-\infty}^{\infty} y(i) z^{-i} z^{-l}, \text{by substituting } i = k-l, \\
&= z^{-l} \sum_{i=\infty}^{\infty} y(i) z^{-i} \\
&= z^{-l} y(z).
\end{aligned}
$$

TRANSFORMADA "Z"

TRANSFERENCIAS EN "Z"

Applying Z-transforms to the difference equations gives:

$$
\begin{aligned}
y(z) &= \sum_{k=-\infty}^{\infty} y(k) z^{-k} \\
&= \sum_{k=-\infty}^{\infty} \left(-a_1 y(k-1) \ \ldots - a_n y(k-n) + b_0 u(k) \ \ldots + b_m u(k-m)\right) z^{-k}, \\
&= -a_1 \sum_{k=-\infty}^{\infty} y(k-1) z^{-k} \ \ldots - a_n \sum_{k=-\infty}^{\infty} y(k-n) z^{-k} + b_0 \sum_{k=-\infty}^{\infty} u(k) z^{-k} \ \ldots + b_m \sum_{k=-\infty}^{\infty} u(k-m) z^{-k}, \\
&= -a_1 z^{-1} y(z) \ \ldots - a_n z^{-n} y(z) + b_0 u(z) \ \ldots \ b_m z^{-m} u(z),
\end{aligned}
$$

Rearranging gives,

$$
\left(1 + a_1 z^{-1} + \cdots + a_n z^{-n}\right) y(z) = \left(b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}\right) u(z).
$$

From this we get the transfer function,

$$
P(z) = \frac{y(z)}{u(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots + a_n z^{-n}} = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + \cdots + a_n} = \frac{b(z)}{a(z)}.
$$

POLOS Y CEROS EN "Z"

$$P(z) = \frac{b(z)}{a(z)}$$

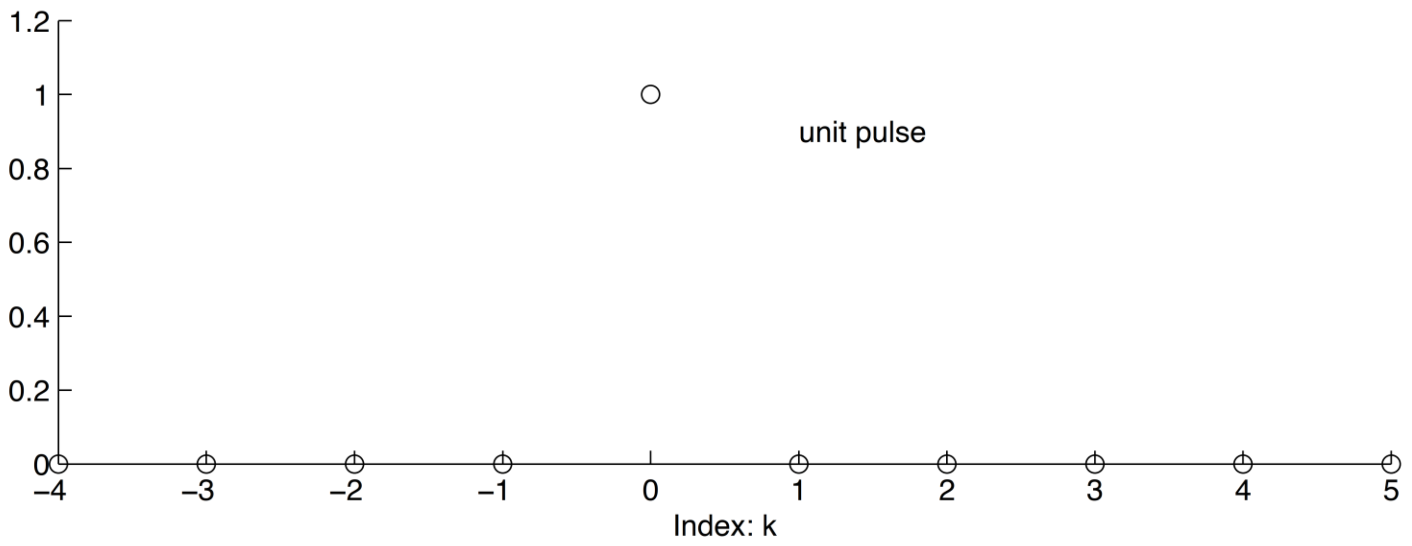
The pole and zero positions determine the typical plant responses.

The poles are the roots of $a(z) = 0$.

The zeros are the roots of $b(z) = 0$.

Unit pulse response: $u(k) = \begin{cases} 0 & \text{for } k < 0, \\ 1 & \text{for } k = 0, \\ 0 & \text{for } k > 0 \end{cases}$

This is used to characterise LTI discrete-time systems (cf. impulse response)



Note that if $u(k)$ is the unit pulse then,

$$u(z) = \sum_{k=-\infty}^{\infty} u(k) z^k = 1 z^0 = 1.$$

RESPUESTA A LA "δ" DE KRONECKER

So the system pulse response is given by,

$$y(k) = \mathcal{Z}^{-1}\{y(z)\} = \mathcal{Z}^{-1}\{P(z)u(z)\} = \mathcal{Z}^{-1}\{P(z)\},$$

which is the inverse Z-transform of the transfer function. (cf. continuous-time).

First order example:

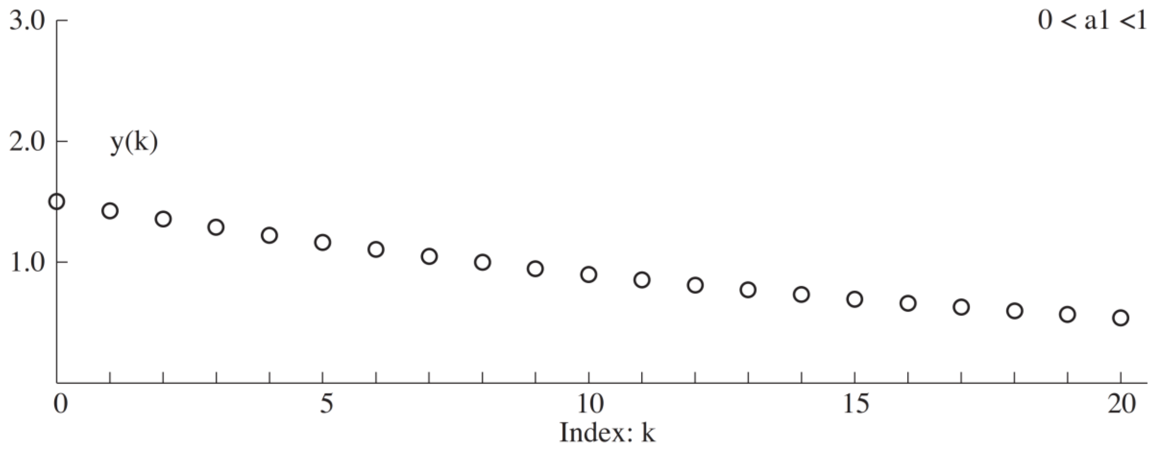
$$P(z) = \frac{b_1 z}{z - a_1}, \qquad \text{and so the pulse response is} \qquad y(k) = \mathcal{Z}^{-1}\{P(z)\} = b_1\, a_1^k.$$

We will use this example to look at the various behaviors possible.

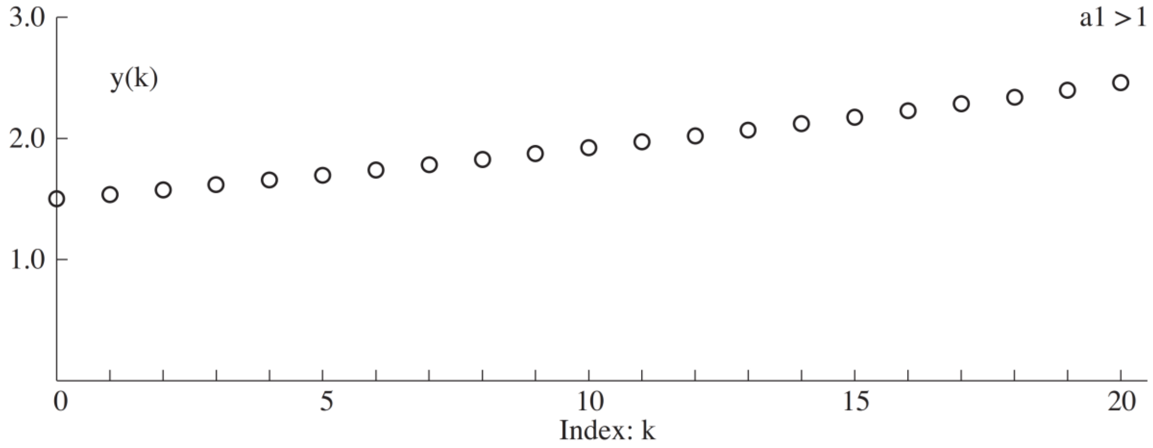
RESPUESTA A LA "δ" DE KRONECKER

The examples are illustrated for $b_1 = 1.5$.

$0 < a_1 < 1$
(inside the unit circle)

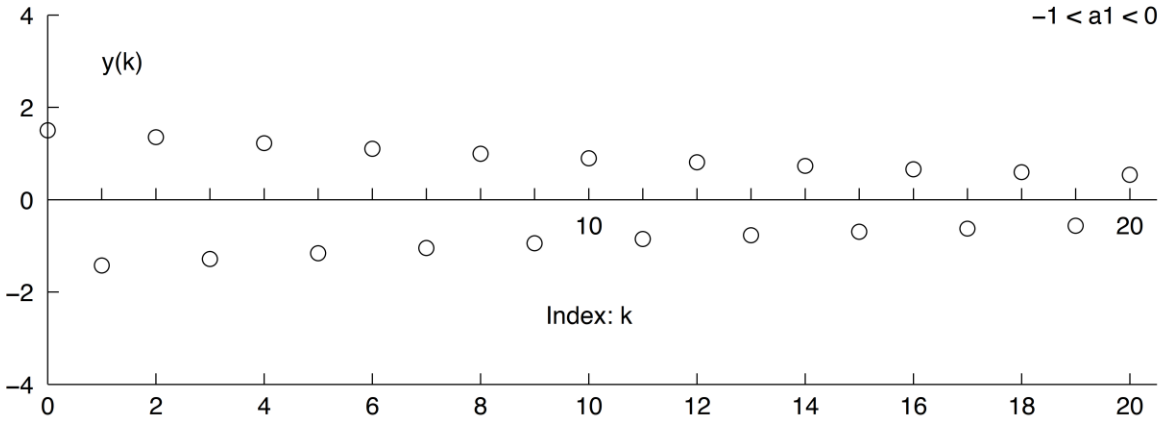


$a_1 > 1$
(outside the unit circle)

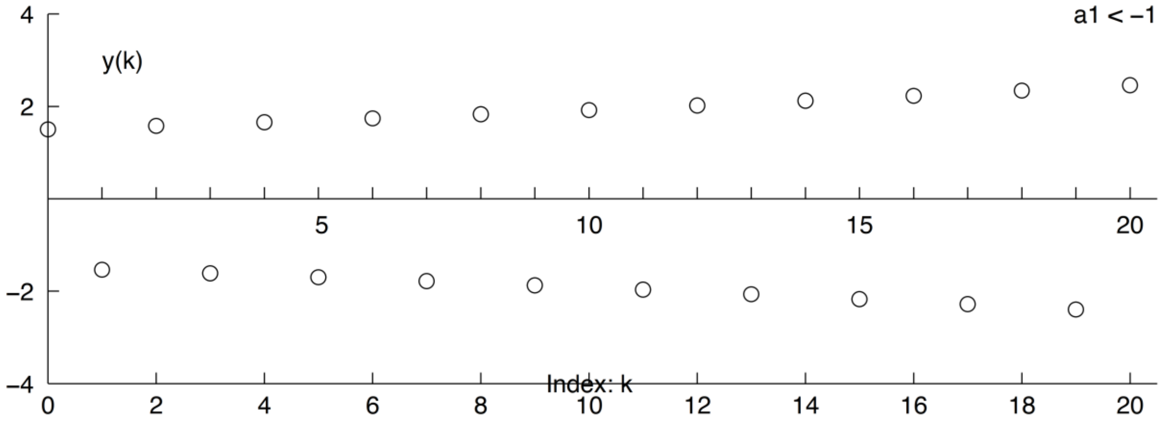


RESPUESTA A LA "δ" DE KRONECKER

$-1 < a_1 < 0$
(inside the unit circle)

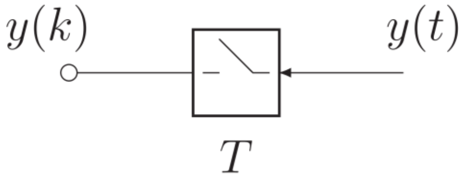


$a_1 < -1$
(outside the unit circle)



We can see that for $|a_1| < 1$ the responses decay. If $|a_1| > 1$ the responses blow up. If a_1 is negative the responses alternate in sign.

SAMPLING



Example (single pole signal)

Consider, $\quad y(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad$ with $\quad a > 0.$

Laplace transform: $\quad y(s) = \dfrac{1}{s+a}.$

Sampled signal: $\quad y(k) = y(t)\Big|_{t=kT} = e^{-akT} = \left(e^{-aT}\right)^k.$

Z-transform, $\quad y(z) = \dfrac{z}{z - e^{-aT}}.$

The s-plane pole is at $s_1 = -a$, and the corresponding z-plane pole is at $z_1 = e^{-aT}$.

Example: (second order)

Now consider a damped sinusoidal signal, $y(t) = \mathrm{e}^{-\alpha t}\sin(\beta t), \quad t \geq 0$, with $\alpha > 0$.

Laplace transform: $y(s) = \dfrac{\beta}{(s+\alpha)^2 + \beta^2}$, Poles: $s_{1,2} = -\alpha \pm j\beta$.

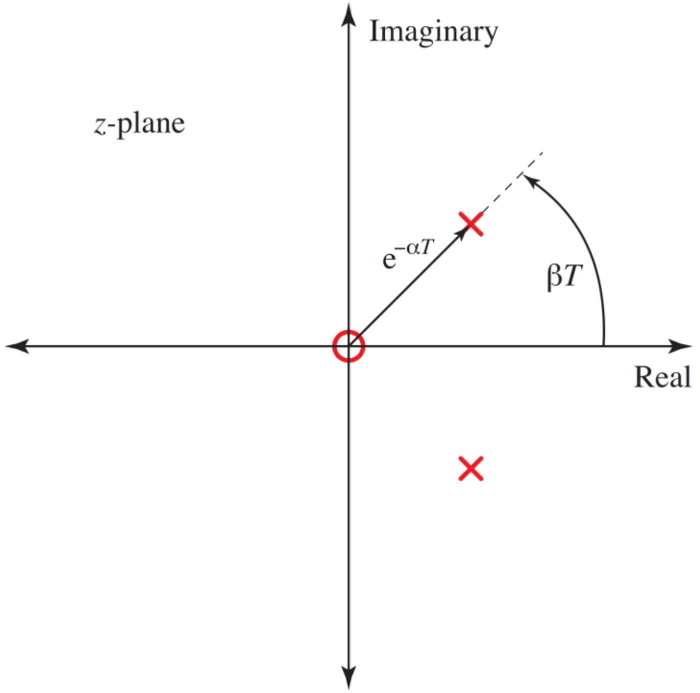
SAMPLING

Sampled signal: $y(k) = \mathrm{e}^{-\alpha kT}\sin(\beta kT), \quad k \geq 0$.

Z-transform: $y(z) = \dfrac{z^{-1}\mathrm{e}^{-\alpha T}\sin(\beta T)}{1 - z^{-1}2\mathrm{e}^{-\alpha T}\cos(\beta T) + z^{-2}\mathrm{e}^{-2\alpha T}}$.

Z domain poles given by: $z^2 - 2\mathrm{e}^{-\alpha T}\cos(\beta T)z + \mathrm{e}^{-2\alpha T} = 0$.

$$\begin{aligned}
z_{1,2} &= \mathrm{e}^{-\alpha T}\cos(\beta T) \pm \sqrt{\mathrm{e}^{2\alpha T}\cos^2(\beta T) - \mathrm{e}^{-2\alpha T}} \\
&= \mathrm{e}^{-\alpha T}\left(\cos(\beta T) \pm j\sqrt{1-\cos^2(\beta T)}\right) \\
&= \mathrm{e}^{-\alpha T}(\cos(\beta T) \pm j\sin(\beta T)) \\
&= \mathrm{e}^{-\alpha T}\mathrm{e}^{\pm j\beta T} \\
&= \mathrm{e}^{(-\alpha \pm j\beta)T}.
\end{aligned}$$



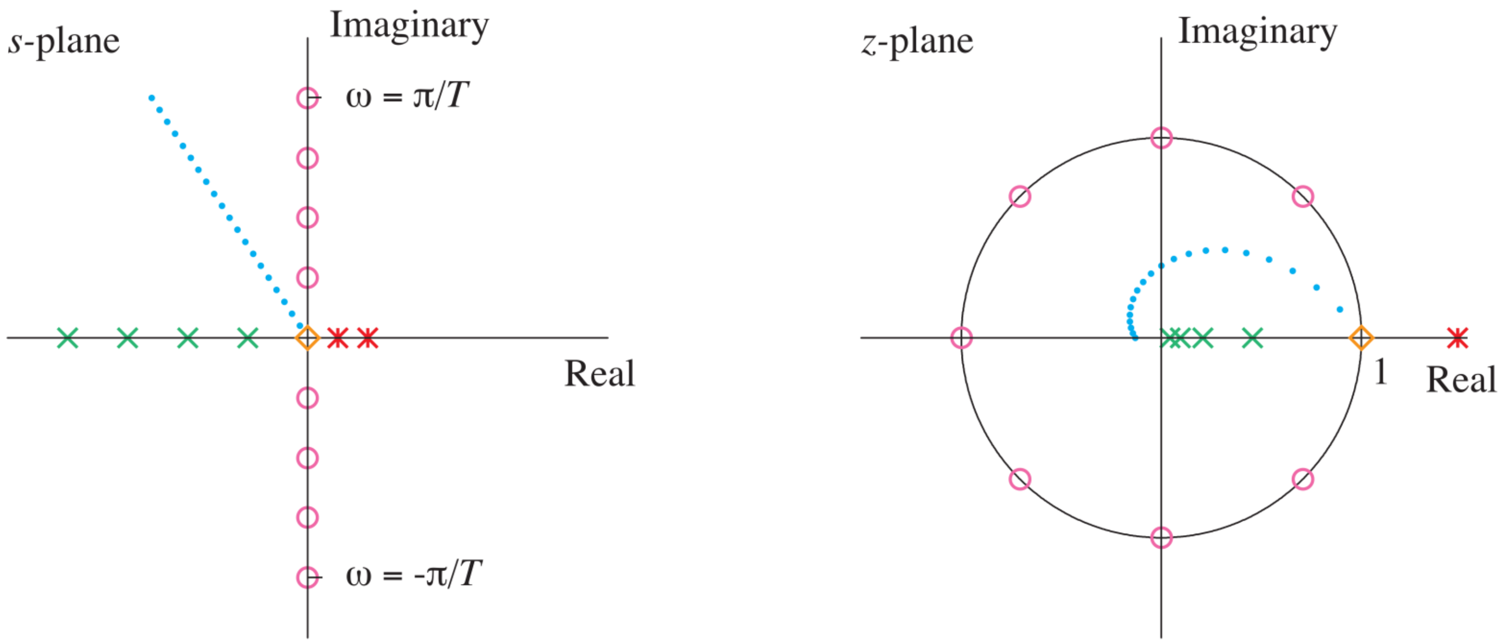
SAMPLING

General case:

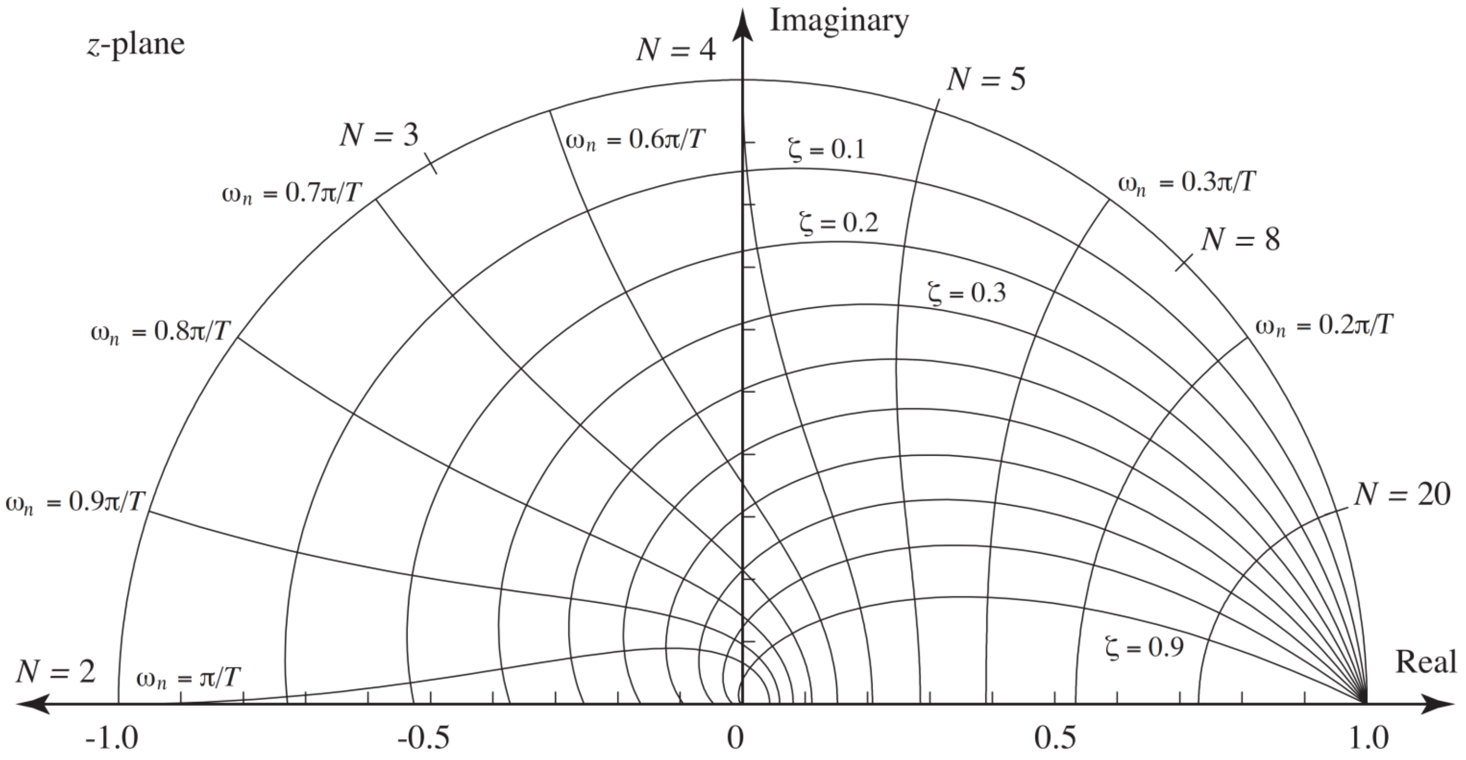
Sampling maps the s-domain poles to the z-domain via: $z_i = \mathrm{e}^{s_i T}$.

Stable continuous-time signals $(\mathrm{Re}\{s_i\} < 0)$ map to stable discrete-time signals $(|z_i| < 1)$.

Pole locations under sampling:



Sampled pole locations: (in detail)



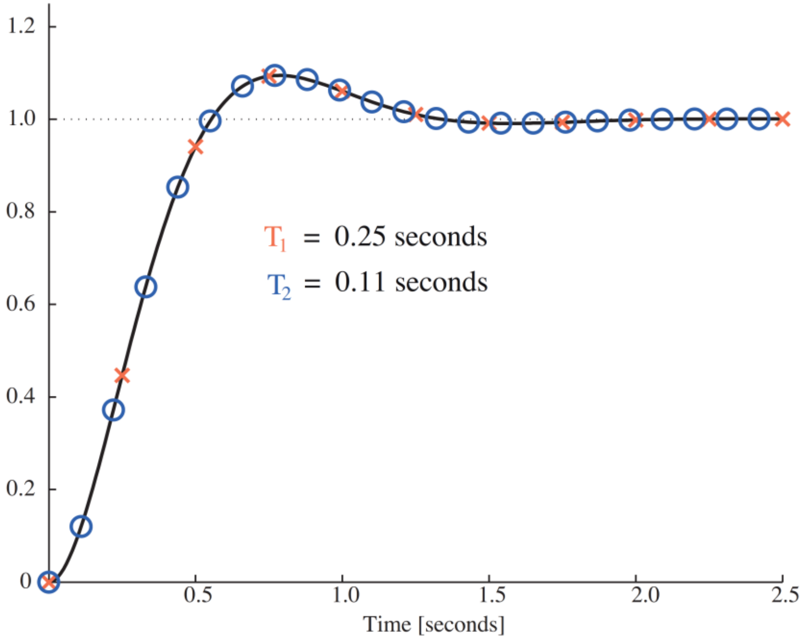
SAMPLING

Changing the sampling frequency. (recall that $z_i = e^{s_i T}$.)

Decreasing T: decrease decay rate ($r \to 1$)
decrease oscillation frequency ($\theta \to 0$)
poles track constant damping curves towards 1

This is simply because there are more samples taken in the same time period.

Sample rate effects: $z_i = e^{s_i T}$, changing T changes the pole positions.



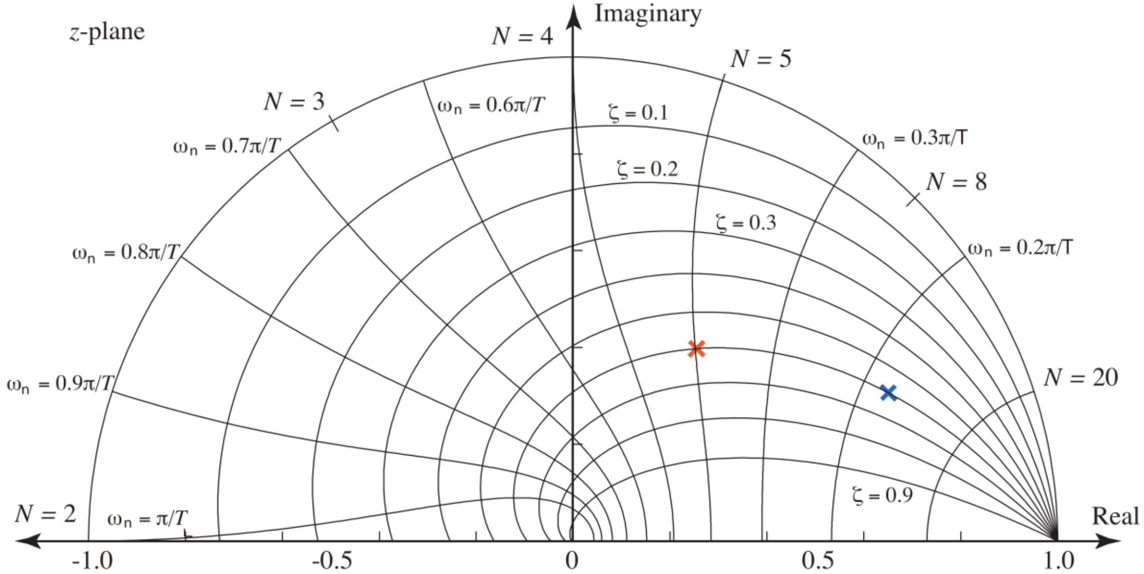
Continuous closed-loop system step response:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\zeta = 0.6,$

$\omega_n = 5$ rad./sec.,
$s_{1,2} = -3 \pm 4i.$

SAMPLING



Discrete pole positions:

Period T_1: $z_{1,2} = 0.255 \pm 0.398i$
Period T_2: $z_{1,2} = 0.650 \pm 0.306i$

ALIASING

Aliasing: What happens to signals of high frequencies ($\omega > \pi/T$) ?

As $z_i = \mathrm{e}^{s_i T}$, sinusoids of frequencies from -π/T to π/T radians/second are mapped onto the unit disk by sampling.

Consider $y(t) = \sin \omega_1 t$, which has Laplace transform: $y(s) = \dfrac{\omega_1}{s^2 + \omega_1^2}$.

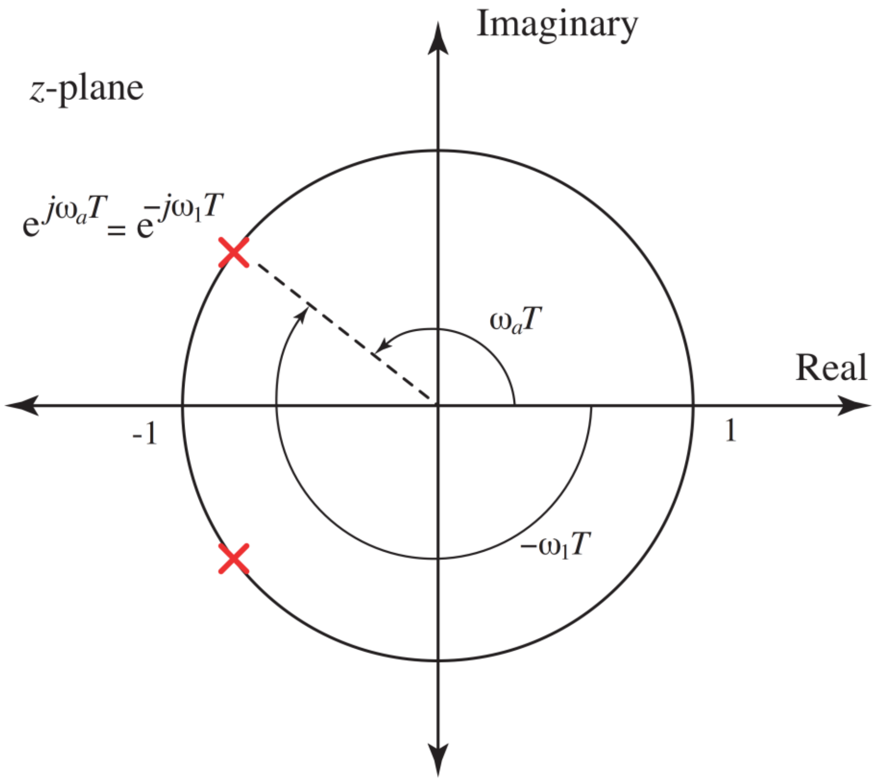
Poles are $s_{1,2} = \pm j\omega_1$.

Sample at period T: $y(k) = \sin \omega_1 kT$,

Z-transform: $y(z) = \dfrac{z \sin w_1 T}{z^2 - 2\cos \omega_1 T z + 1}$.

Poles of $y(z)$ are $z_{1,2} = \mathrm{e}^{\pm j\omega_1 T}$.

Slow sampling, $T > \pi/\omega_1$, implies that $\omega_1 T > \pi$.
The pole angle is greater than π.



ALIASING

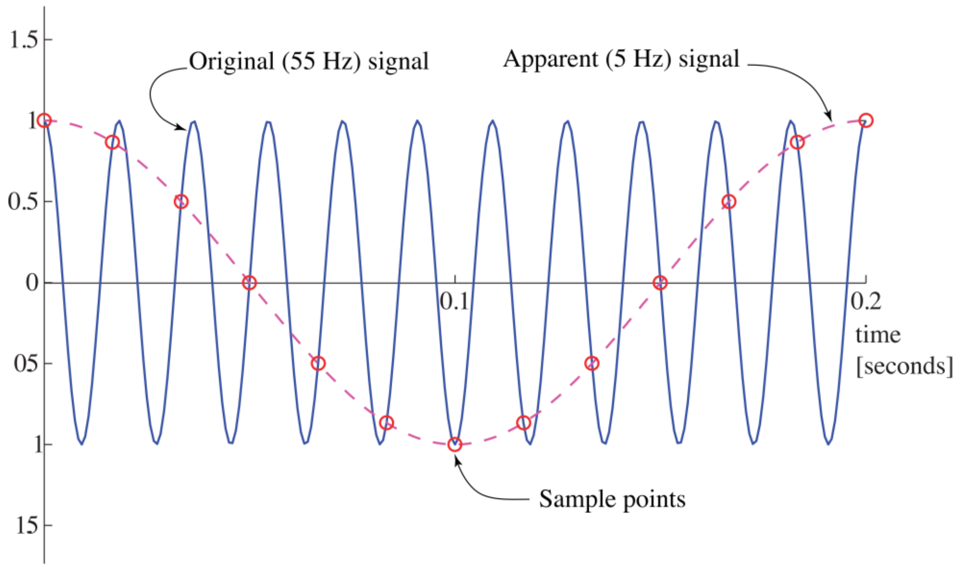
Having $w_1 T > \pi$, means that,

$$e^{-j\omega_1 T} = e^{j(2\pi - \omega_1 T)}, \quad \text{and} \quad e^{j\omega_1 T} = e^{-j(2\pi - \omega_1 T)}.$$

Now, if $(2\pi - \omega_1 T)$ lies in the range 0 to π radians, the pole pattern is identical to that of a sinusoid of a lower frequency, ω_a, where $\quad \omega_a T = 2\pi - \omega_1 T$.

Equivalently, the apparent frequency is, $\quad \omega_a = \dfrac{2\pi}{T} - \omega_1. \quad$ (sampling freq: $2\pi/T$ rad/sec).

Example: 55 Hz signal sampled at 60 Hz



Example:

55 Hz Signal: $y(t) = \cos(2\pi 55 t)$

Sampling frequency: $1/T = 60$ Hz

Then $y(k) = \cos(2\pi 55 t)|_{t=kT} = \cos(2\pi 5 t)|_{t=kT}$

Indistinguishable from a sampled 5 Hz signal!

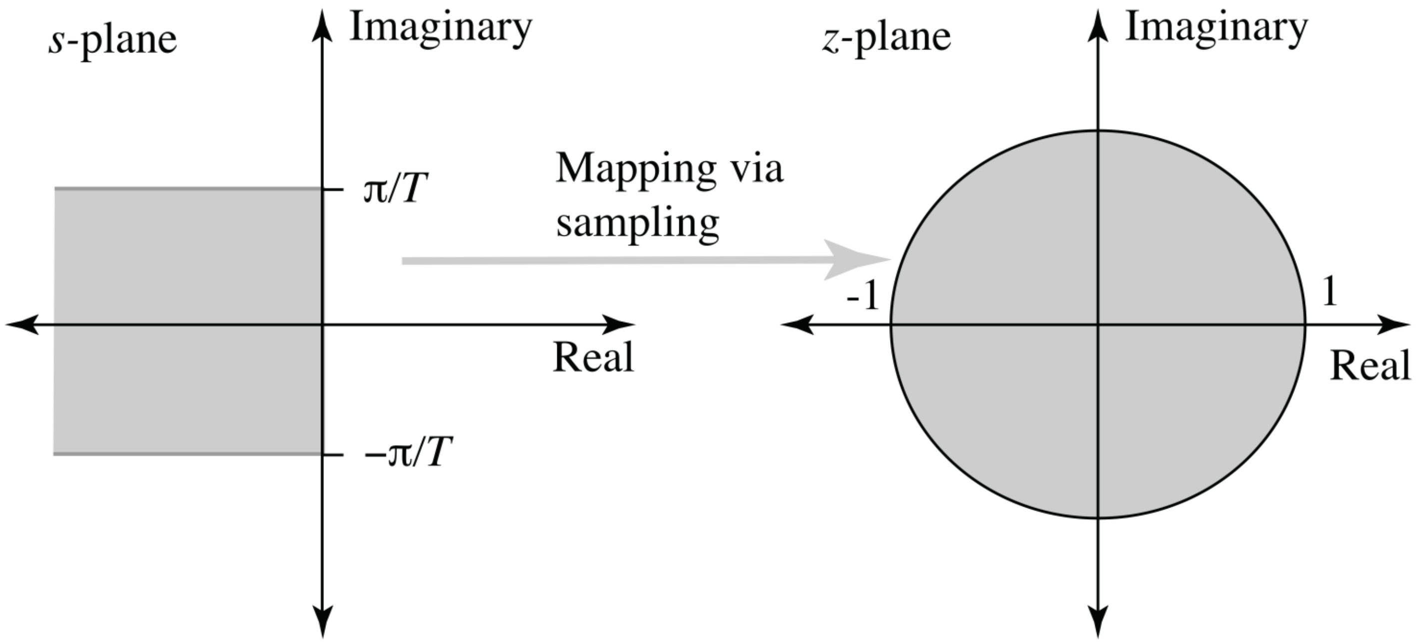
Sampled signals

The unit disk can only represent signals of frequency up to 1/2 the sampling frequency. (**Nyquist frequency**).

Sampling operation maps signal poles via: $z_i = \mathrm{e}^{s_i T}$.

Maps the horizontal strip from $-j\pi/T$ to $j\pi/T$ onto the whole z-plane.

ALIASING

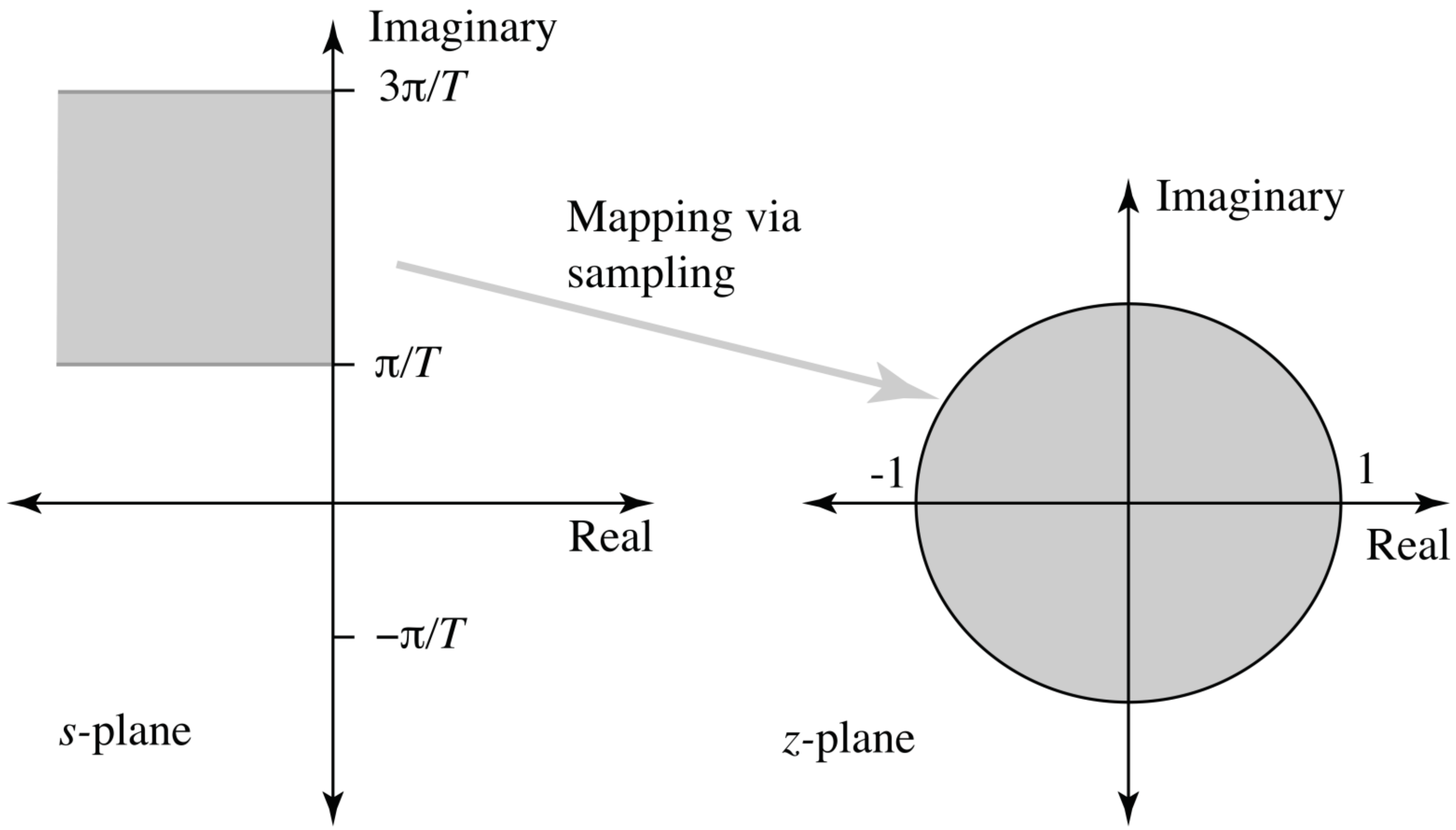


And $\mathrm{Re}\{s\} < 0$ in this strip maps to the inside of the unit disk.

Aliasing: (ambiguous mapping of higher frequency signals)

Sampling also maps the next strip (from $j\pi/T$ to $j3\pi/T$) onto the whole z-plane and adds it into the result.

ALIASING



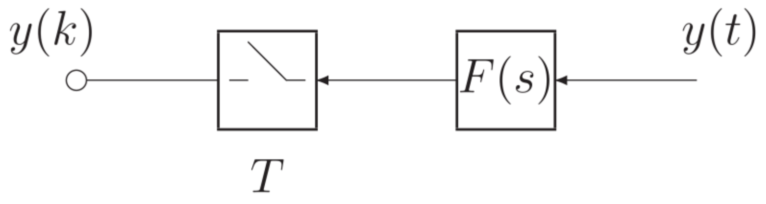
Also true for all (infinite) $2\pi/T$ wide strips above and below the lowest frequency strip.

Consequences of aliasing:

- Ambiguity. Our computer/controller cannot distinguish between frequencies inside the $-\pi/T$ to π/T range and those outside of it.
  - Controller will respond incorrectly to an aliased signal (e.g. disturbance or error).
  - An aliased signal cannot be reconstructed (signal processing).

ALIASING

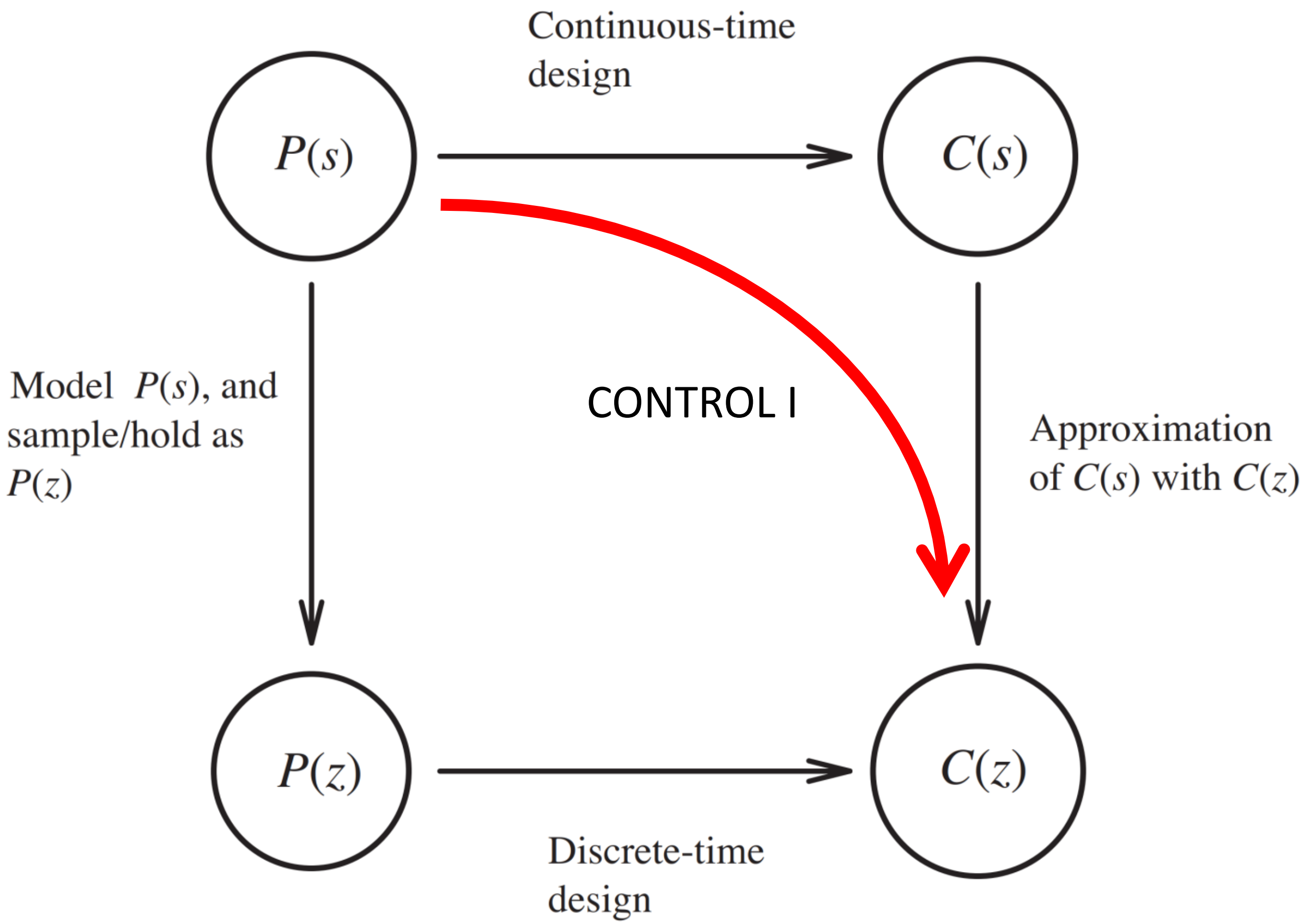
Amelioration of the problem:



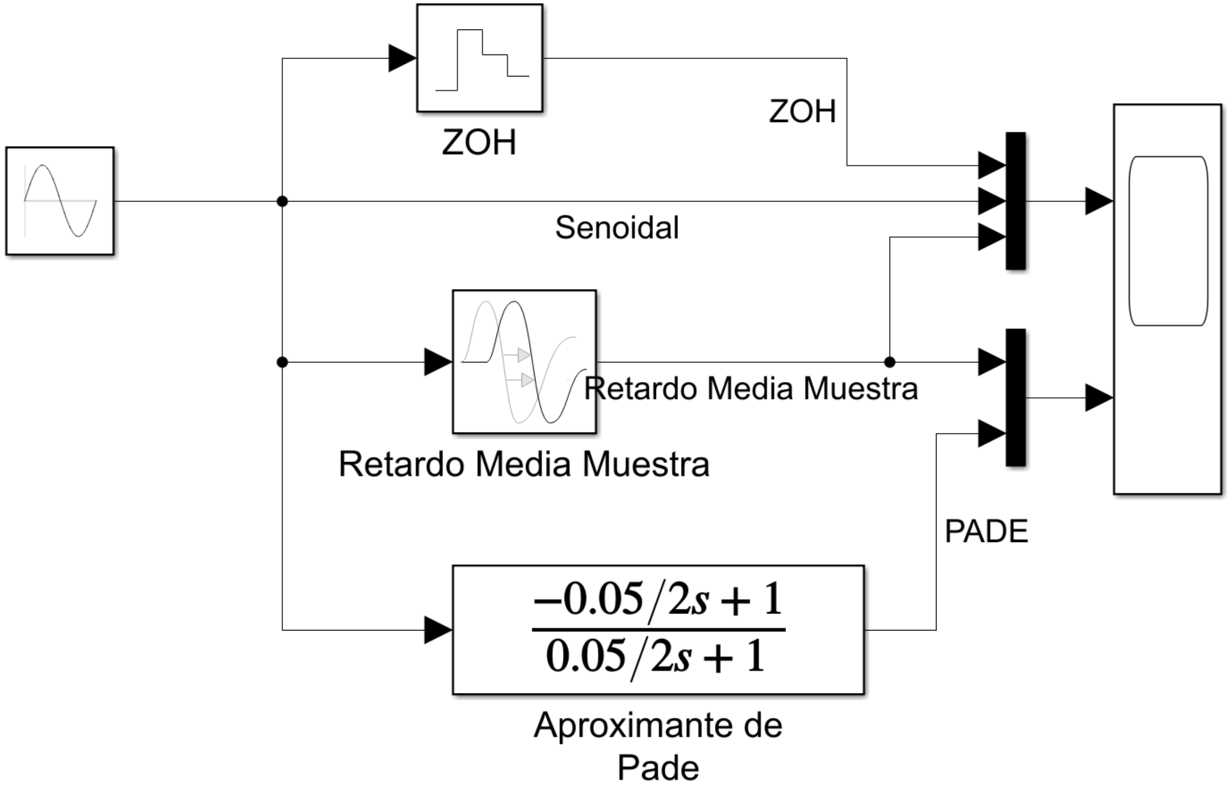
- Anti-aliasing filter. Low pass, rejecting $|\omega| > \pi/T$.
  - High frequency signals no longer enter loop erroneously.
  - High frequency disturbances/errors are “invisible.”
  - Filter adds phase lag to the loop. (**Potentially destabilizing!**)

APROXIMAR O HACER ZERO-ORDER-HOLD

- Se puede diseñar en continuo y aproximar por un C(z) discreto.
- O discretizar la planta y diseñar en discreto.



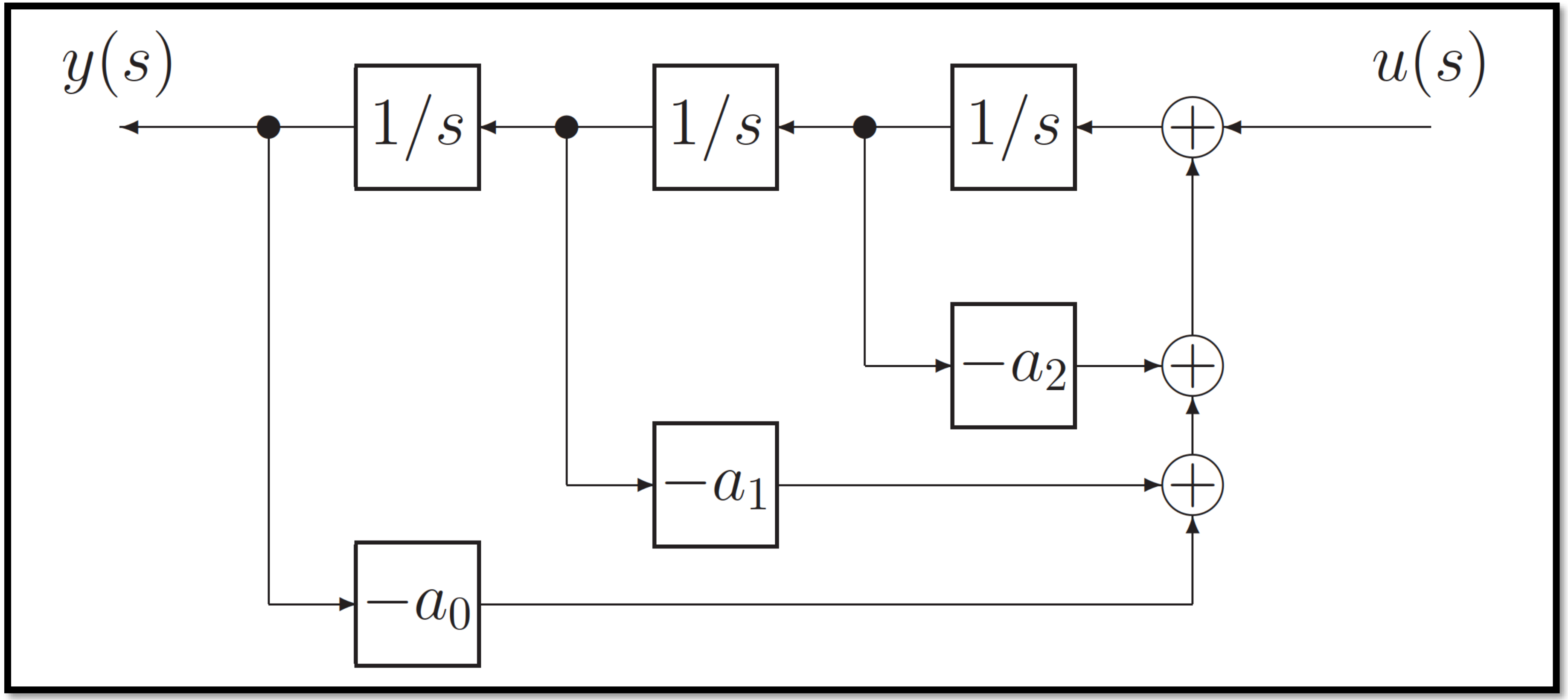
¿CÓMO TRATAR EL ZOH EN TIEMPO CONTINUO?



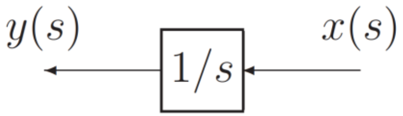
En este ejemplo se ilustra como qué transformación no lineal realiza el Zero Order Hold (ZOH) sobre una senoidal y cómo aproximarla quedándonos con el primer armónico de la señal de salida del ZOH. Esto da como transformación LTI que captura en tiempo continuo el efecto de implementar un controlador en tiempo discreto, un retardo de medio intervalo de muestreo. Este “retardo de media muestra” es a su vez aproximado por un aproximante de Padé en este caso de primer orden, aunque pueden ponerse aproximantes de órdenes superiores.

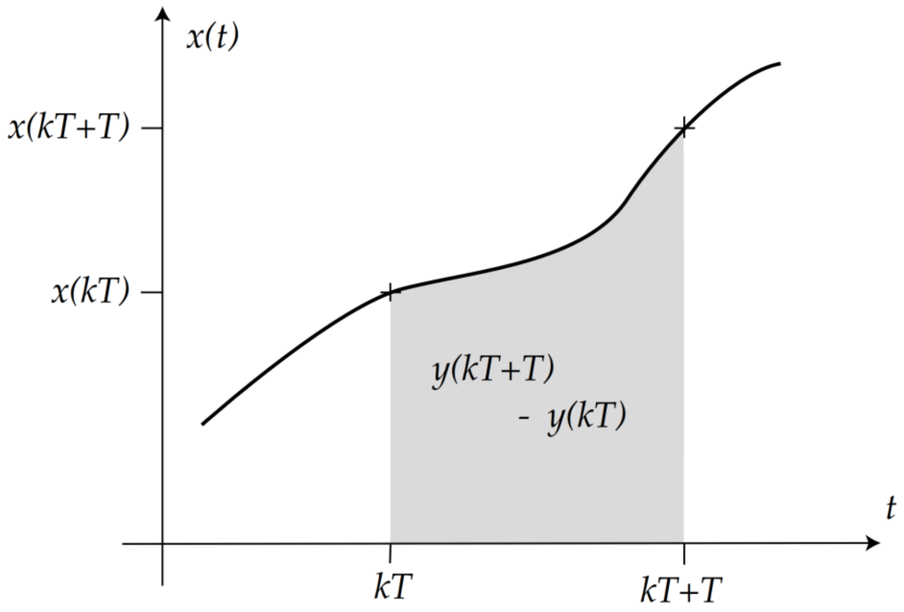
APROXIMACIONES DEL INTEGRADOR:

$$C(s) = \frac{y(s)}{u(s)} = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}$$



Integration:





$$y(t) = y(0) + \int_0^t x(\tau)\, d\tau,$$

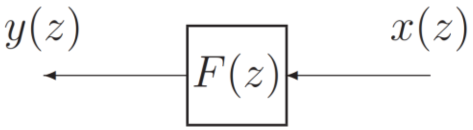
The output, $y(t)$, over a single sample period of T seconds, is given by

$$y(kT + T) = y(kT) + \int_{kT}^{kT+T} x(\tau)\, d\tau.$$

Objective:

Find a discrete-time approximation, $F(z)$, to the input-output relationship of the integrator.

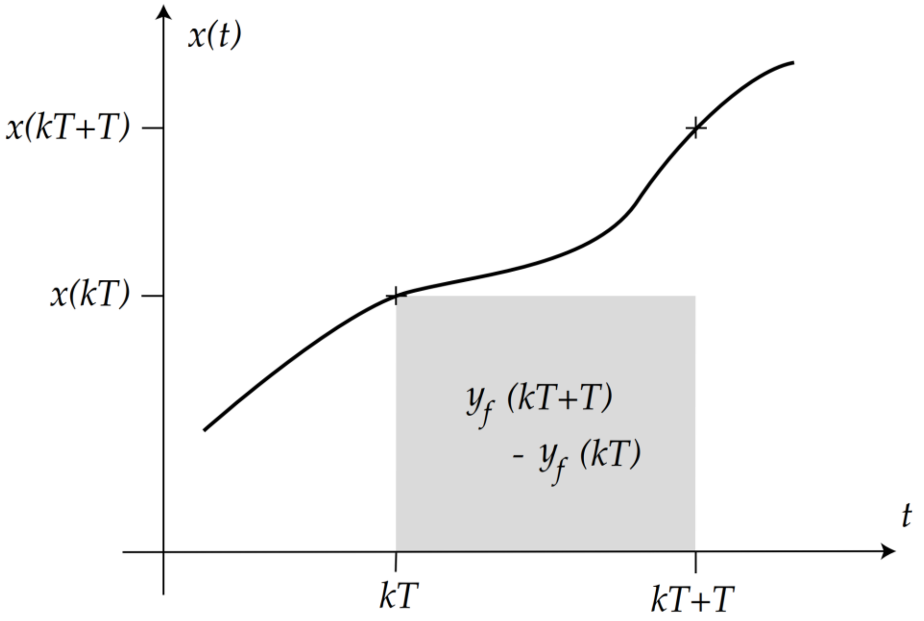
Find $F(z) \approx 1/s$, then, $s \approx F^{-1}(z)$,



and $C(z) = C(s) \,|_{s=F^{-1}(z)}$.

Forward difference approximation:

$$y_f(kT+T) = y_f(kT) + Tx(kT).$$



By taking z-transforms,

$$zy_f(z) = y_f(z) + Tx(z),$$

or,

$$\frac{y_f(z)}{x(z)} = \frac{T}{z-1}.$$

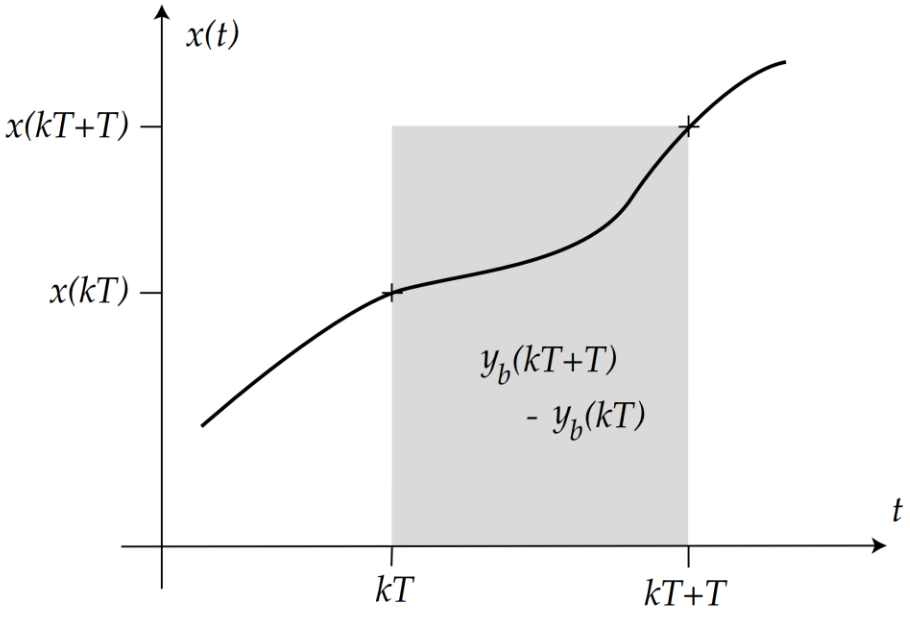
So, the approximation is: $\quad \dfrac{1}{s} \approx \dfrac{T}{z-1}.$

This is equivalent to the substitution: $\quad s = \dfrac{z-1}{T}.$

This approximation is also known as an Euler approximation.

Backward difference approximation:

$$y_b(kT+T) = y_b(kT) + Tx(kT+T).$$



In the z-domain this gives,

$$zy_b(z) = y_b(z) + zTx(z),$$

or, equivalently,

$$\frac{y_b(z)}{x(z)} = \frac{Tz}{z-1}.$$

So the approximation is:

$$\frac{1}{s} \approx \frac{Tz}{z-1},$$

which is equivalent to the substitution: $\quad s = \dfrac{z-1}{Tz}.$

Trapezoidal approximation:

$$y_{bl}(kT + T) = y_{bl}(kT) + Tx(kT) + (x(kT + T) - x(kT))T/2.$$

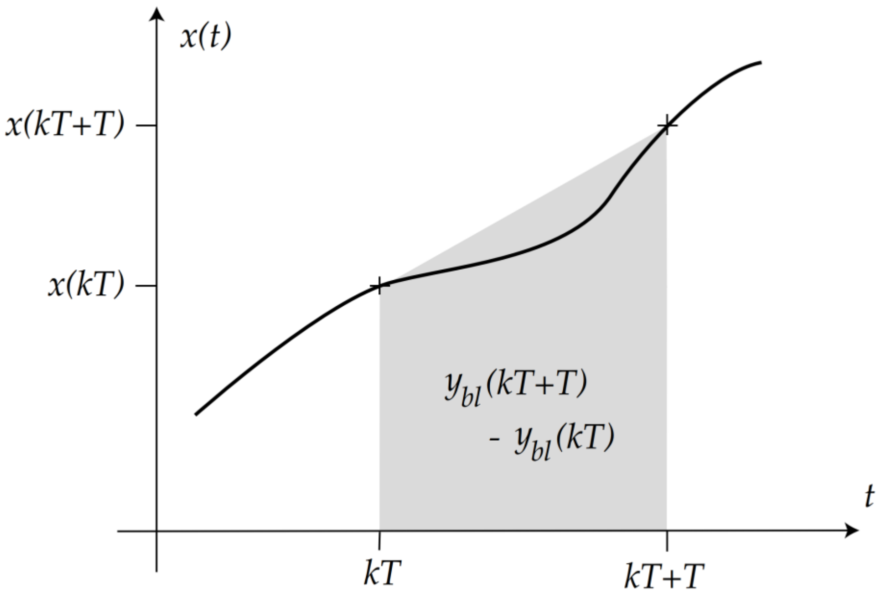
Taking z-transforms,

$$zy_{bl}(z) = y_{bl}(z) + Tx(z) + \frac{T}{2}(z - 1)x(z),$$

which gives,

$$\frac{y_{bl}(z)}{x(z)} = \frac{T}{2}\frac{z + 1}{z - 1}.$$

So the approximation is: $\quad \dfrac{1}{s} \approx \dfrac{T}{2}\dfrac{z+1}{z-1}.$



The substitution is therefore, $\quad s = \dfrac{2}{T}\dfrac{z-1}{z+1}.$

This approximation is also known as:

- Bilinear approximation (based on the mathematical form).
- Tustin approximation (from the British engineer who first used it for this purpose).

Properties:

Controller order:

The forward, backward and trapezoidal approximations all preserve the order of the controller.

If $C(s)$ is an nth order transfer function, the $C(z)$ is also nth order with any of these approximations.

It is possible to derive higher order approximations to integration (quadratic or higher order polynomial fits). These will make the order of $C(z)$ greater than $C(s)$.

Stability:

Two issues:

- **Controller stability:** If $C(s)$ is stable, is $C(z)$ stable?
- **Closed-loop stability:** If $\dfrac{1}{1 + P(s)C(s)}$ is stable, is $\dfrac{1}{1 + P(z)C(z)}$ stable?

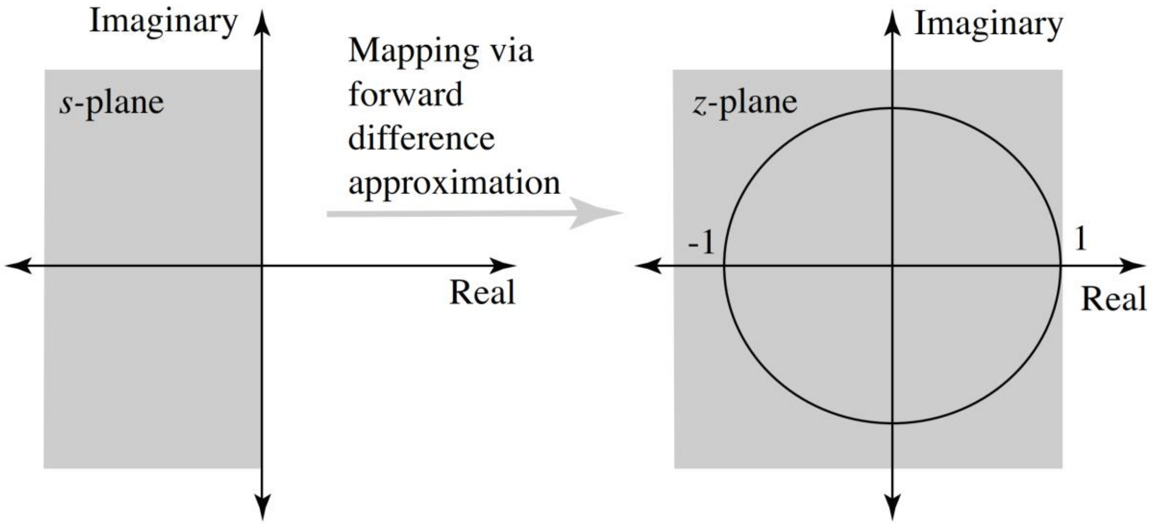
To investigate controller stability we have to look more closely at how the approximations map the s-plane to the z-plane.

Controller stability:

Forward difference/Euler approximation:

$$s = \frac{z-1}{T}.$$

This maps the left half s-plane onto the region shown.



This maps to more than just the unit disk.

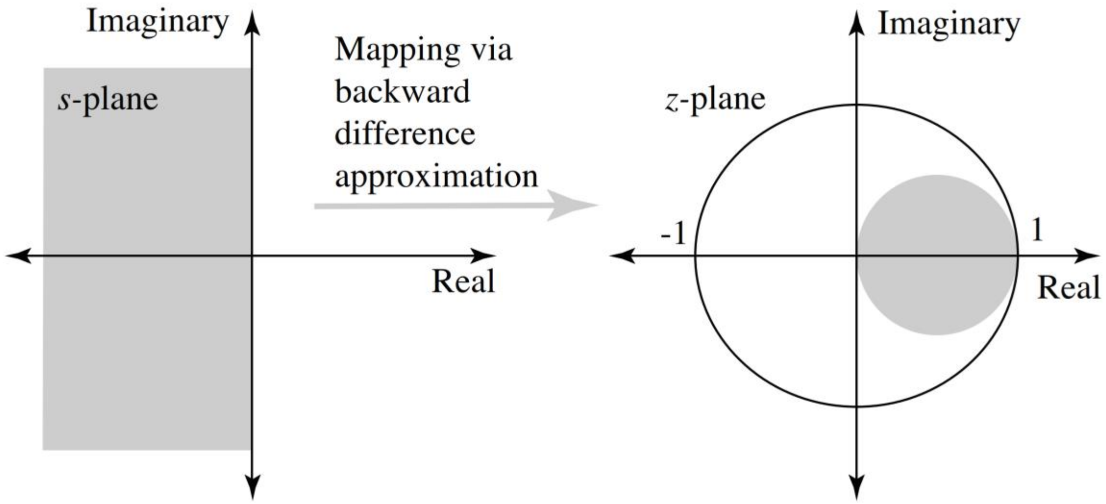
Controllers, $C(s)$, with high frequency or lightly damped poles will give **unstable** $C(z)$.

Controller stability:

Backward difference approximation:

$$s = \frac{z-1}{Tz},$$

This maps the left half s-plane onto the region shown.



This maps to the inside of the unit disk. So stable $C(s)$ imples **stable** $C(z)$.

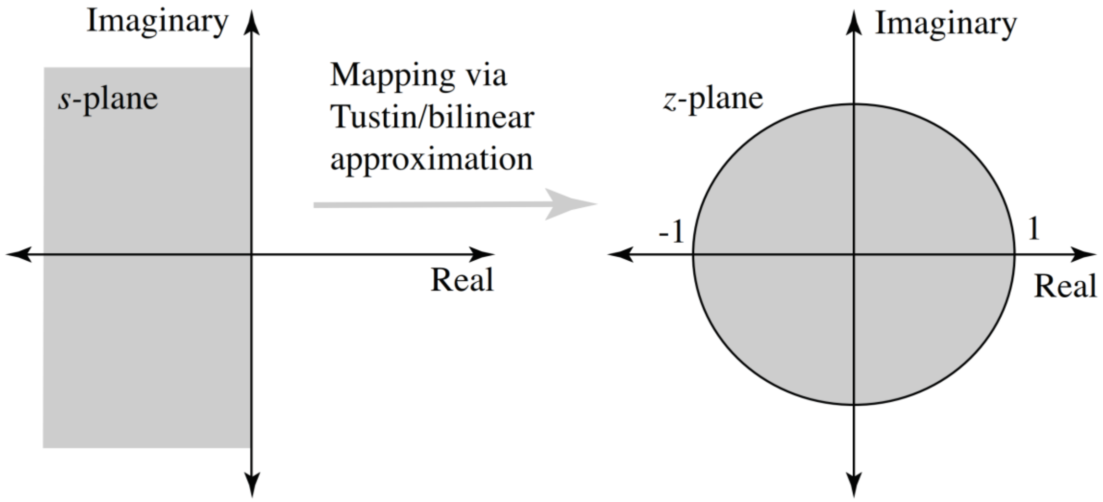
$C(z)$ cannot have lightly damped poles, even if $C(s)$ had lightly damped poles.

Controller stability:

Trapezoidal/Bilinear/Tustin approximation:

$$s = \frac{2}{T}\frac{z-1}{z+1},$$

This maps the left half s-plane onto the region shown.



This maps to the entire right-half plane exactly onto the unit disk.

So $C(s)$ is stable $\iff$ $C(z)$ is **stable**.

This is why this approximation is the most commonly used.

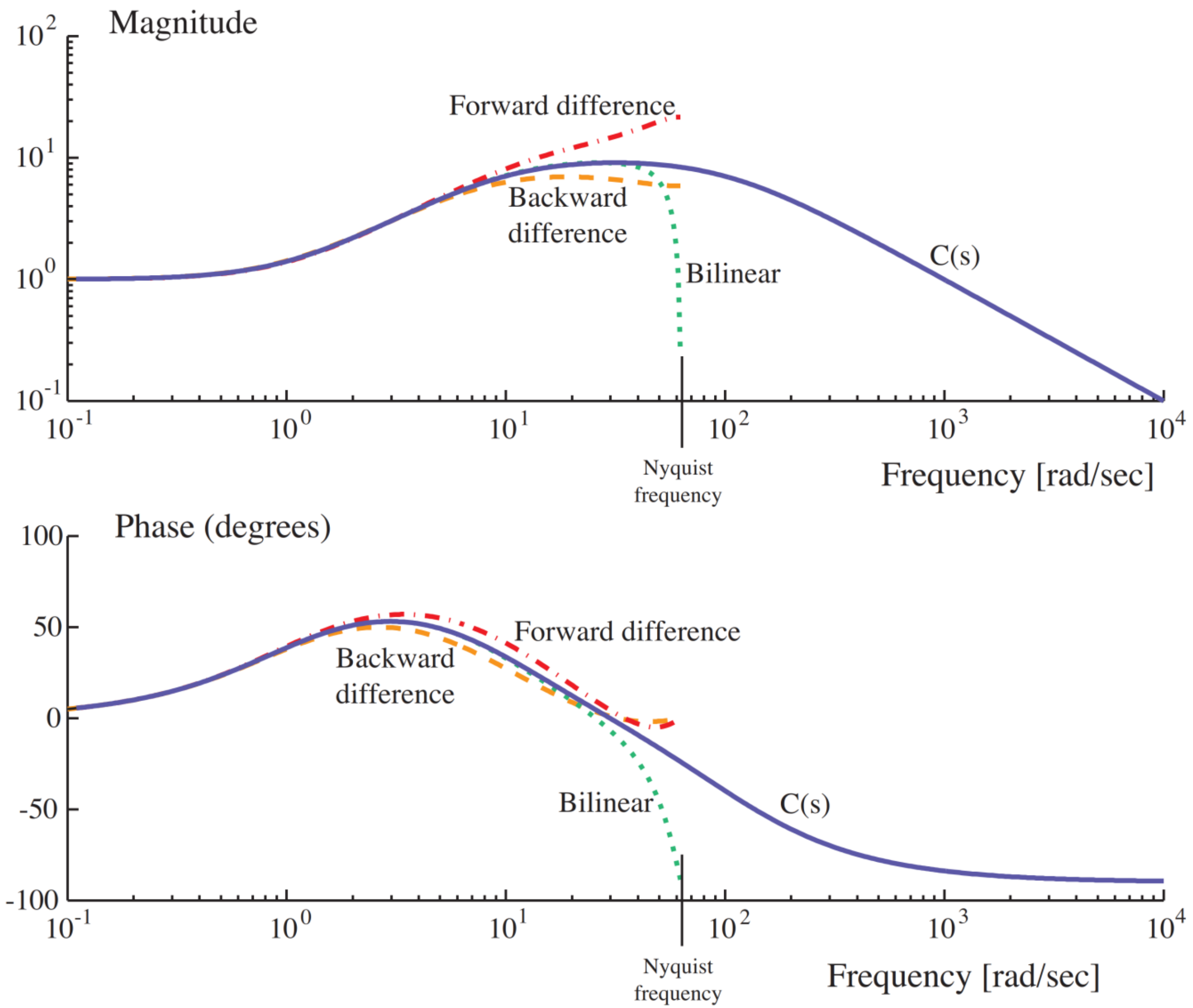
A Comparison

Consider the controller: $$C(s) = \frac{(s+1)}{(0.1s+1)(0.01s+1)}.$$

A lead-lag controller producing the maximum phase lead around 30 rad/sec. ($\approx$ 4.8 Hz).

Using a sample period of $T = 0.05$ second gives a Nyquist frequency of 10 Hz.

A Comparison: All approximations have significant errors close to the Nyquist frequency.



Frequency distortion: Bilinear approximation

Bilinear approximation maps **all** continuous frequencies (ω) from 0 to $j\infty$ to discrete frequencies ($\mathrm{e}^{j\Omega T}$) with Ω from 0 to π/T. In particular, $s = j\infty$ maps to $z = \mathrm{e}^{j\pi} = -1$.

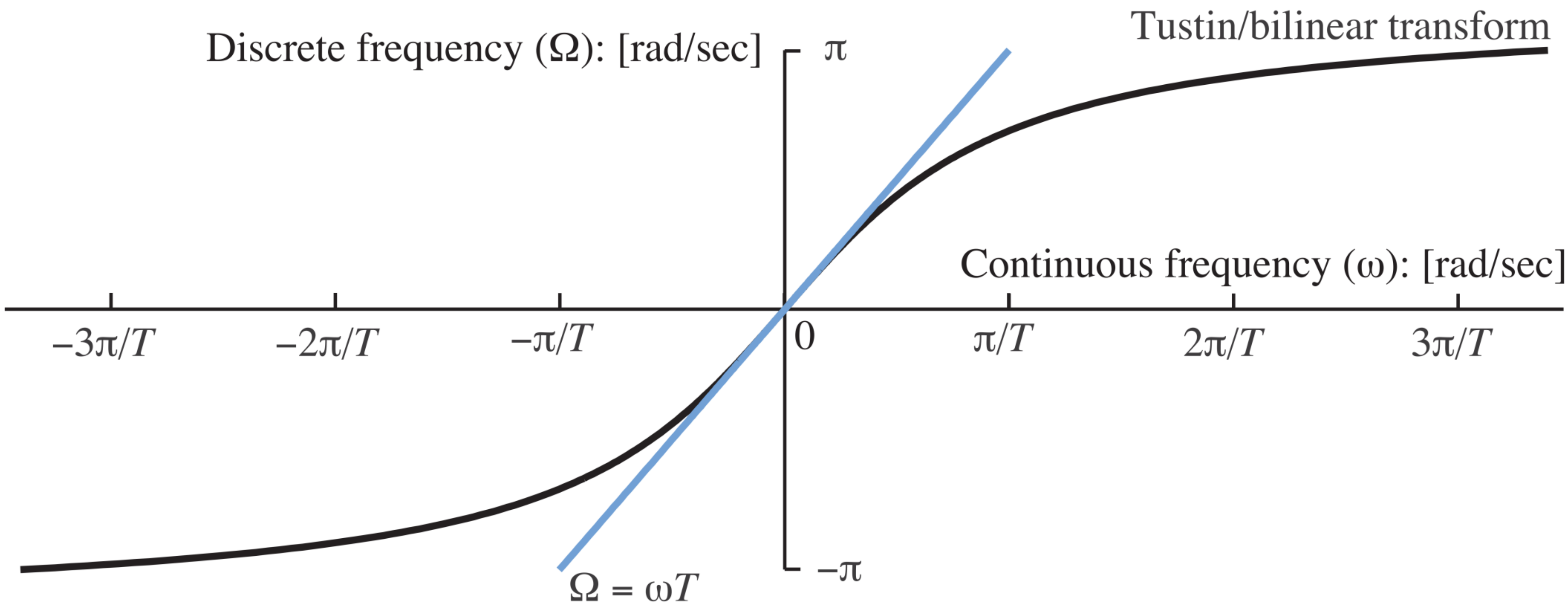
Sampling would map frequencies via $\omega = \Omega$, so $z = -1$ would correspond to a continuous frequency $\omega = j\pi/T$.

Substituting $s = j\omega$ and $z = \mathrm{e}^{j\Omega T}$ into $s = \frac{2}{T}\frac{z-1}{z+1}$, gives,

$$
\begin{aligned}
j\omega &= \frac{2}{T}\frac{(1-\mathrm{e}^{-j\Omega T})}{1+\mathrm{e}^{-j\Omega T}} \\
&= \frac{2}{T}\frac{j\sin(\Omega T/2)}{\cos(\Omega T/2)} \\
&= \frac{2}{T} j\tan(\Omega T/2),
\end{aligned}
$$

which implies that the distortion is given by $\quad \Omega = \frac{2}{T}\tan^{-1}(\omega T/2).$

Frequency distortion (Bilinear approximation) $\Omega = \dfrac{2}{T}\tan^{-1}(\omega T/2)$.



The line $\Omega = \omega T$ is the equivalent sampled frequency mapping.

Reducing the distortion: prewarping

The transformation $s = \dfrac{\alpha(z-1)}{(z+1)}$, maps $\mathrm{Re}\{s\} < 0$ to $|z| < 1$.

α is a degree of freedom that can be exploited to modify the frequency distortion.

Prewarping:

Select α to make $C(j\omega_0) = C_z\left(\mathrm{e}^{j\omega_0 T}\right)$.

This makes $C(s) = C_z(z)$ at DC and at $s = j\omega_0$ (ω_0 is the *prewarping frequency*).

To solve for α,

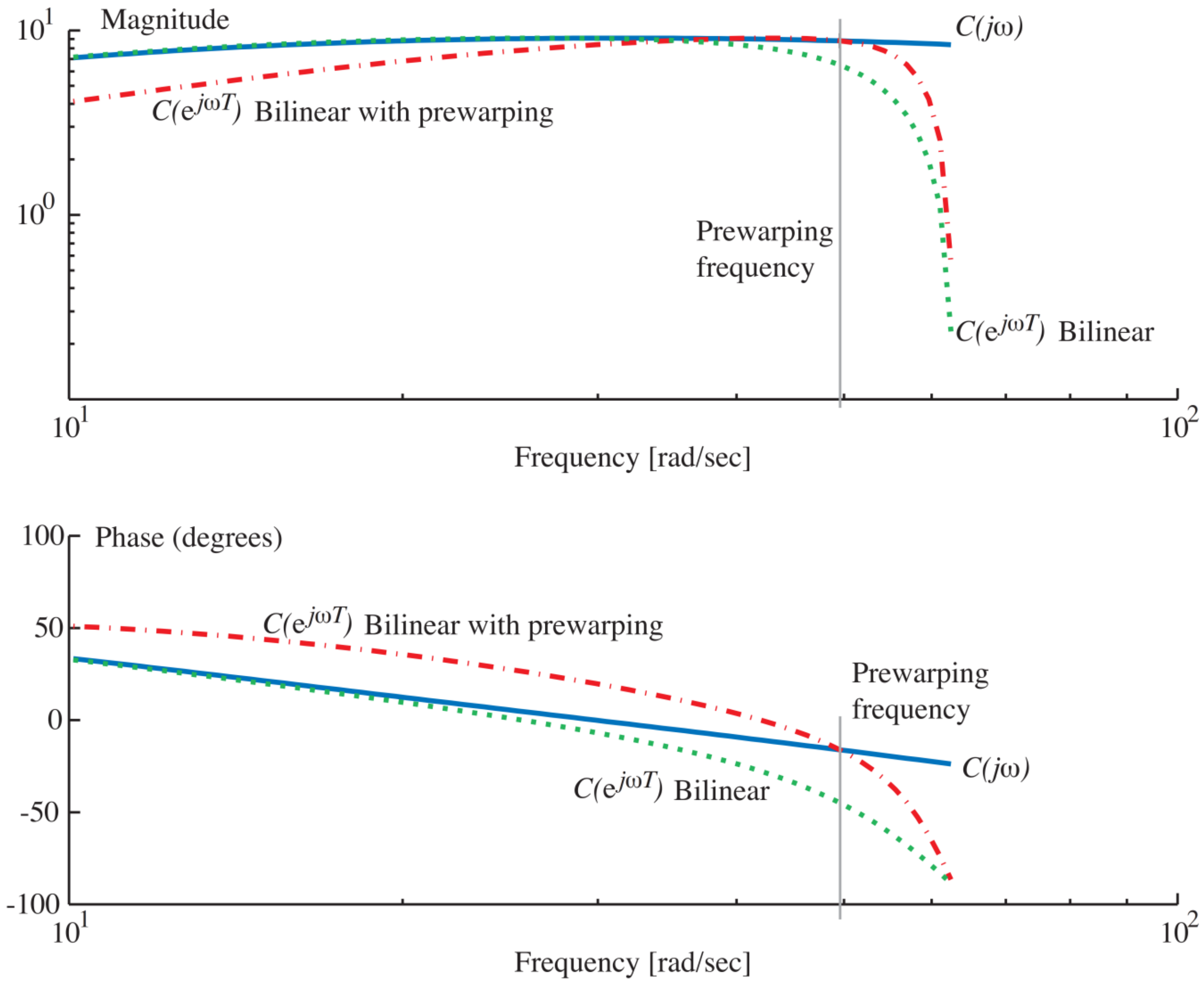
$$j\omega_0 \quad = \quad \frac{\alpha(\mathrm{e}^{j\omega_0 T} - 1)}{(\mathrm{e}^{j\omega_0 T} + 1)} \quad = \quad j\alpha \tan(\omega_0 T/2),$$

which implies that

$$\alpha = \frac{\omega_0}{\tan(\omega_0 T/2)}.$$

Example revisited Choose a prewarping frequency: $\omega_0 = 50$ rad/sec.

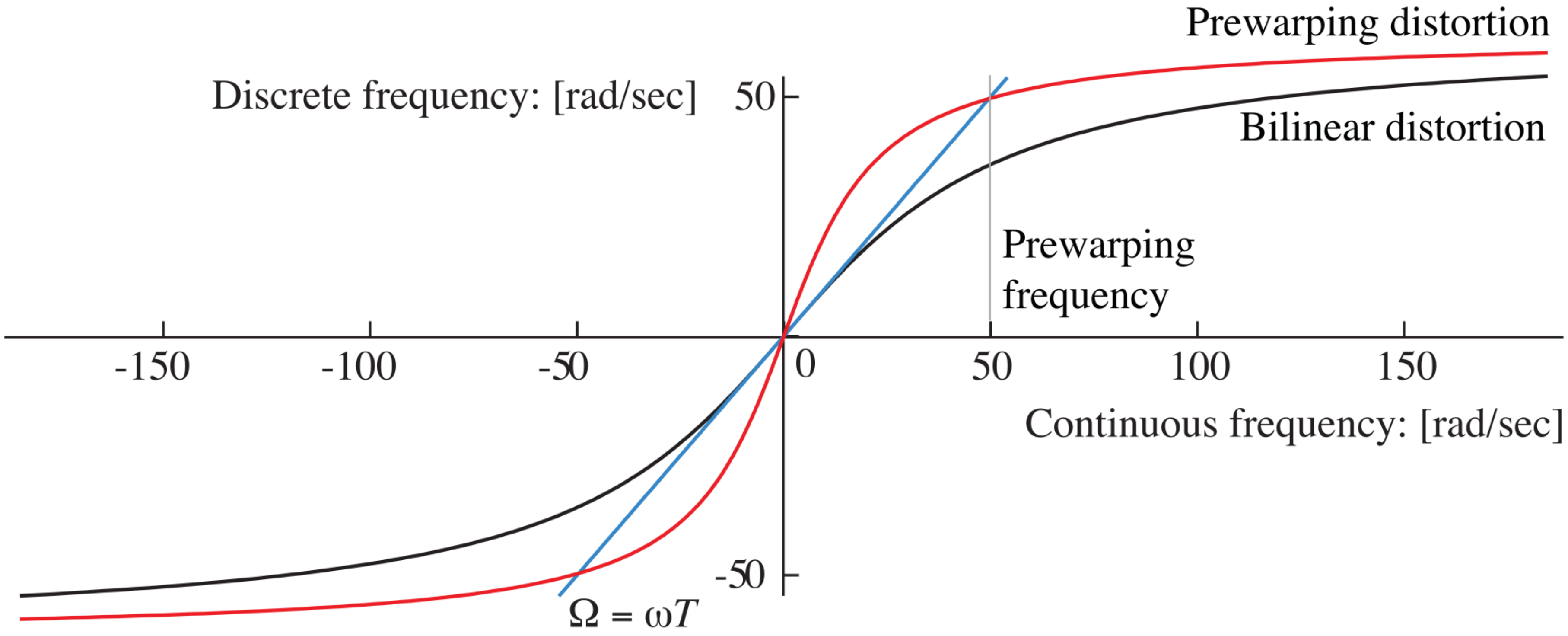
Prewarped bilinear/Tustin: $C_z(z) = C(s)\,|_{s=\alpha\frac{z-1}{z+1}}$ which gives $C(j50) = C_z(e^{j50T})$.



Example revisited

Frequency distortion (Bilinear): $\Omega = \dfrac{2}{T}\tan^{-1}(\omega T/2)$.

Frequency distortion (Bilinear with prewarping): $\Omega = \dfrac{2}{T}\tan^{-1}(\omega/\alpha)$



Choosing a prewarping frequency

The prewarping frequency must be in the range: $0 < \omega_0 < \pi/T$.

- $\alpha = 2/T$ (standard bilinear) corresponds to $\omega_0 = 0$.
- $\omega_0 = \pi/T$ is impossible.

Possible choices for ω_0:

- The cross-over frequency (which will help preserve the phase margin).
- The frequency of a critical notch.
- The frequency of a critical oscillatory mode.

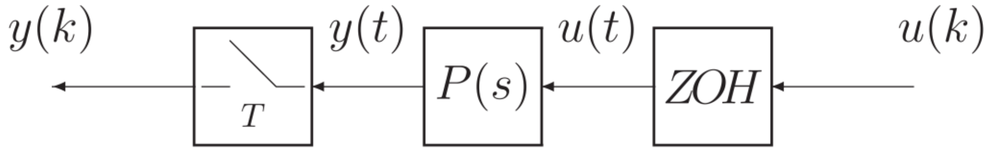
The best choice depends on the most important features in your control design.

Remember: $C(s)$ stable implies $C(z)$ stable, but you **must** check that $\dfrac{1}{1+P(z)C(z)}$ is stable!

Continuous to discrete transforms in state-space

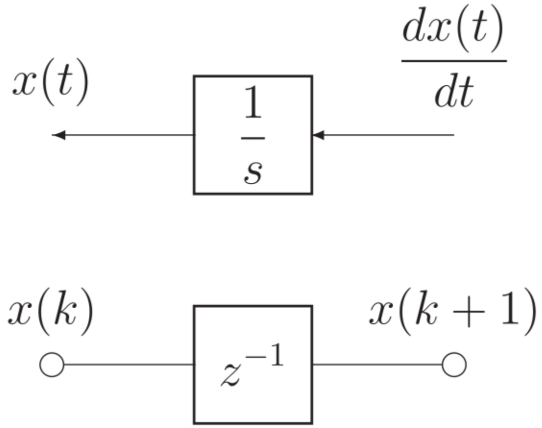
We have several ways of calculating a discrete-time transfer function from a continuous-time one, depending on the application.

ZOH Equivalence



$$P(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{P(s)}{s}\right\}.$$

Controller approximation

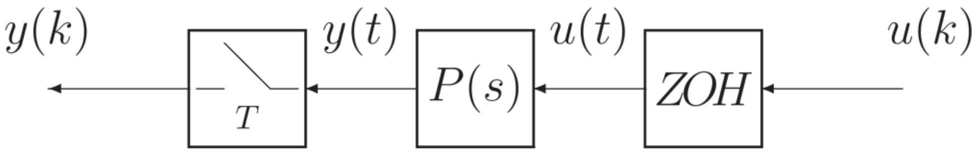


Forward difference: $C(z) = C(s)\,\big|_{s=\frac{z-1}{T}}$

Backward difference: $C(z) = C(s)\,\big|_{s=\frac{z-1}{Tz}}$

Tustin/bilinear: $C(z) = C(s)\,\big|_{s=\frac{2}{T}\frac{(z-1)}{(z+1)}}$

ZOH Equivalence



$$P(s) \quad \begin{cases} \dfrac{dx(t)}{dt} &= A\,x(t) + B\,u(t) \\ y(t) &= C\,x(t) + D\,u(t). \end{cases}$$

We would like to get a description of the form,

$$P(z) \quad \begin{cases} x(k+1) &= A_d\,x(k) + B_d\,u(k) \\ y(k) &= C_d\,x(k) + D_d\,u(k). \end{cases}$$

Approach: Solve the state equation over one sample period.

$$x(t) = \mathrm{e}^{At}\,x(0) + \int_0^t \mathrm{e}^{A(t-\tau)}\,Bu(\tau)\,d\tau,$$

And over a single sample period (kT to $kT+T$) this is,

$$x(kT+T) = \mathrm{e}^{AT}\,x(kT) + \int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\,Bu(\tau)\,d\tau,$$

Key observation

The integration involves $u(\tau)$ from $\tau = kT$ to $\tau = kT + T$.

But $u(\tau)$ is **constant** over this time period. It is the output of a *ZOH*.

So, $u(\tau) = u(k)$ for $kT \leq \tau < kT + T$.

Therefore,

$$x(kT+T) \;=\; \mathrm{e}^{AT}\, x(kT) \;+\; \left[\int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\, B\, d\tau,\right] u(k).$$

By our sampling definitions, $x(t)\mid_{t=kT} = x(k)$, so

$$\begin{aligned} x(k+1) \;&=\; \mathrm{e}^{AT}\, x(k) \;+\; \left[\int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\, B\, d\tau,\right] u(k). \\ &=\; A_d\, x(k) \;+\; B_d\, u(k), \end{aligned}$$

$$\text{where,}\quad A_d = \mathrm{e}^{AT}, \quad \text{and} \quad B_d = \int_{kT}^{kT+T} \mathrm{e}^{A(kT+T-\tau)}\, B\, d\tau.$$

To simplify the B_d integral define $\eta = kT + T - \tau$ to get,

$$B_d = \int_0^T \mathrm{e}^{A\eta}\, B\, d\eta.$$

ZOH equivalent

So far we have calculated $x(k+1)$ as a linear function of $x(k)$ and $u(k)$. What about $y(k)$?

$$y(kT) \;=\; C\,x(kT) \;+\; D\,u(kT).$$

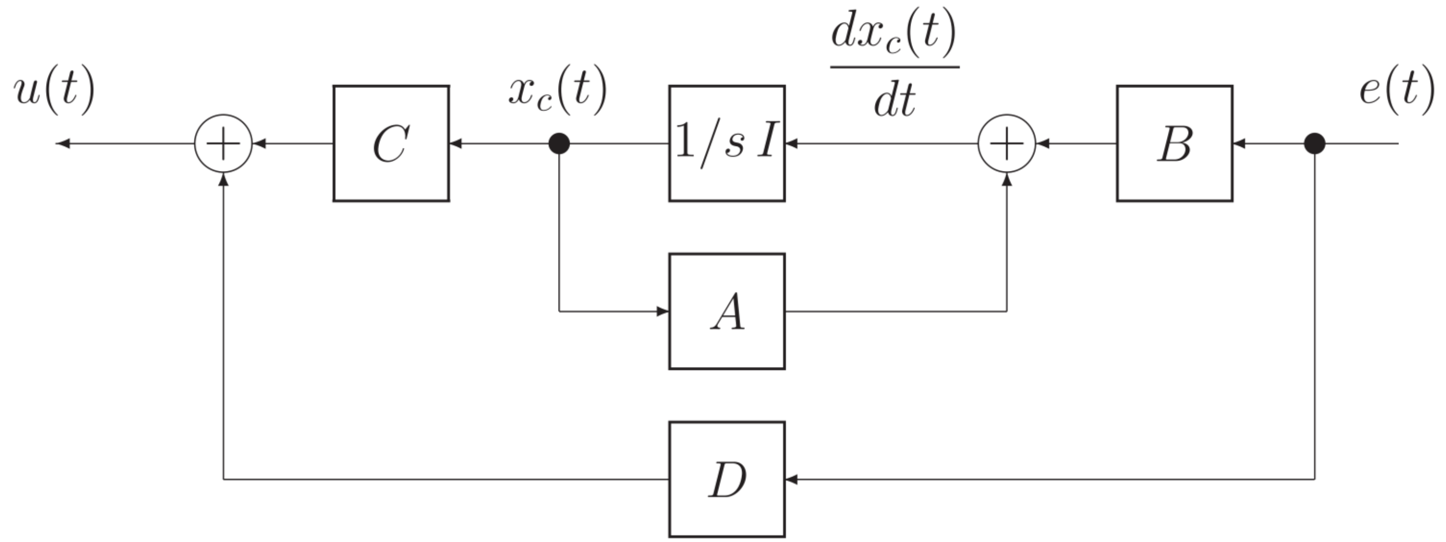
By definition, $y(k) = y(kT)$, and as $u(t)$ is constant over the sample period, $u(k) = u(kT)$.

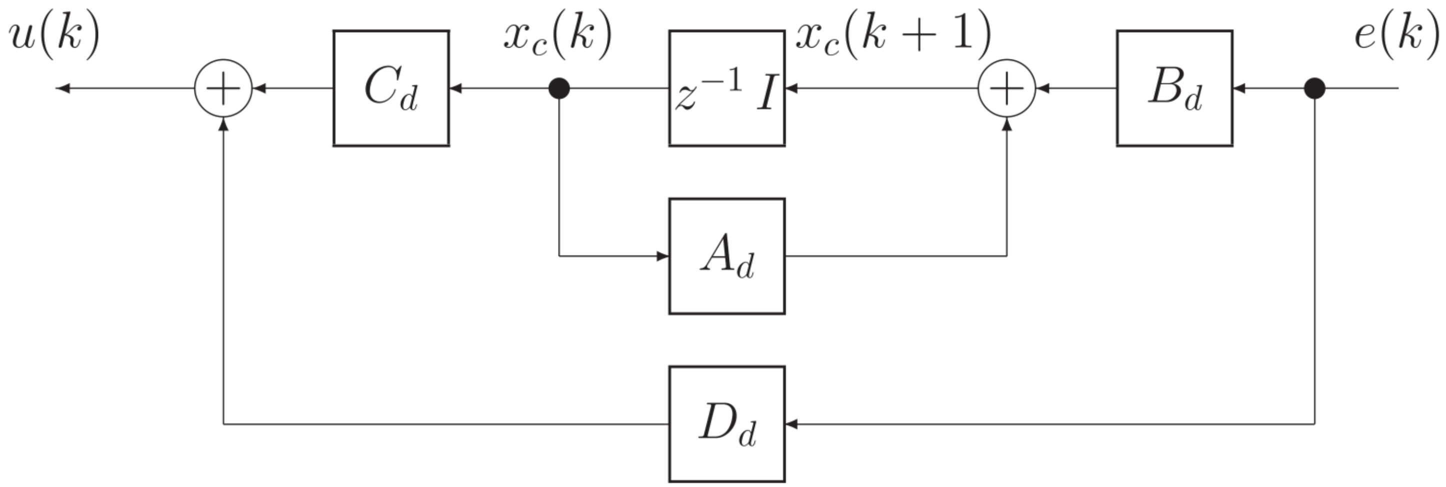
$$y(k) \;=\; C\,x(k) \;+\; D\,u(k).$$

Clearly then, $C_d = C$ and $D_d = D$.

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \quad \overset{\text{ZOH}}{\Longrightarrow} \quad \left[\begin{array}{c|c} \mathrm{e}^{AT} & \displaystyle\int_0^T \mathrm{e}^{A\eta}\, B\, d\eta \\ \hline C & D \end{array}\right]$$

A_d and B_d are calculated via MATLAB commands `c2d` or `zohequiv`.


$u(t)$
C
$x_c(t)$
$1/s\, I$
$\frac{dx_c(t)}{dt}$
B
$e(t)$
A
D


$u(k)$
C_d
$x_c(k)$
$z^{-1}\, I$
$x_c(k+1)$
B_d
$e(k)$
A_d
D_d

Controller (continuous-time)

$$u(s) \longleftarrow \boxed{C(s)} \longleftarrow e(s)$$

$$\begin{aligned} \frac{dx_c(t)}{dt} &= A\,x_c(t) + B\,e(t) \\ u(t) &= C\,x_c(t) + D\,e(t). \end{aligned}$$

Forward difference approximation $\quad \dfrac{1}{s} \approx \dfrac{T}{z-1} \quad$ or $\quad s \approx \dfrac{z-1}{T}.$

Take a Laplace transform of the controller equations,

$$\begin{aligned} s\,x_c(s) &= A\,x_c(s) + B\,e(s) \\ u(s) &= C\,x_c(s) + D\,e(s), \end{aligned}$$

and substitute, $s = \dfrac{z-1}{T}$, $\quad x(s) = x(z)$, $\quad e(s) = e(z)$ and $\quad u(s) = u(z)$.

This effectively replaces the $1/s$ block with a forward difference approximation to integration, and relabels all of the signals in the diagram as discrete-time signals.

The result of these substitutions is,

$$\begin{aligned} \frac{z-1}{T}\, x_c(z) &= A\, x_c(z) + B\, e(z) \\ u(z) &= C\, x_c(z) + D\, e(z). \end{aligned}$$

This is easily rearranged to get,

$$\begin{aligned} (z-1)\, x_c(z) &= AT\, x_c(z) + BT\, e(z) \\ u(z) &= C\, x_c(z) + D\, e(z), \end{aligned} \quad \text{or,} \quad \begin{aligned} z\, x_c(z) &= (I + AT)\, x_c(z) + BT\, e(z) \\ u(z) &= C\, x_c(z) + D\, e(z). \end{aligned}$$

This is now in discrete-time state-space form,

$$\begin{aligned} x_c(k+1) &= (I + AT)\, x_c(k) + BT\, e(k) \\ u(k) &= C\, x_c(k) + D\, e(k). \end{aligned}$$

Clearly then,

$$A_d = I + AT, \quad B_d = BT, \quad C_d = C \quad \text{and } D_d = D.$$

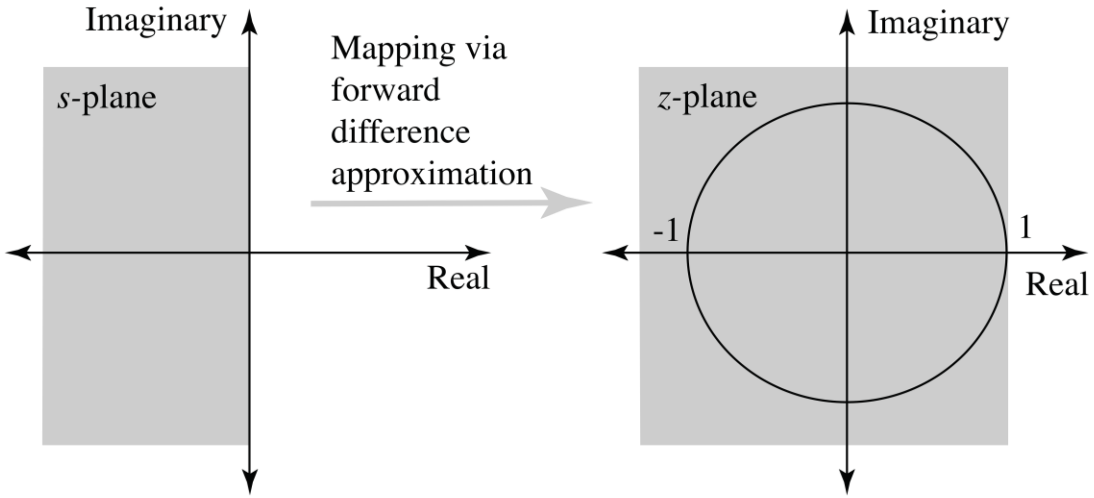
Poles of $C(z)$? Compare the eigenvalues of A_d to A.

$$A_d = I + AT$$

Exercises: Use the determinant definition of eigenvalues to show:

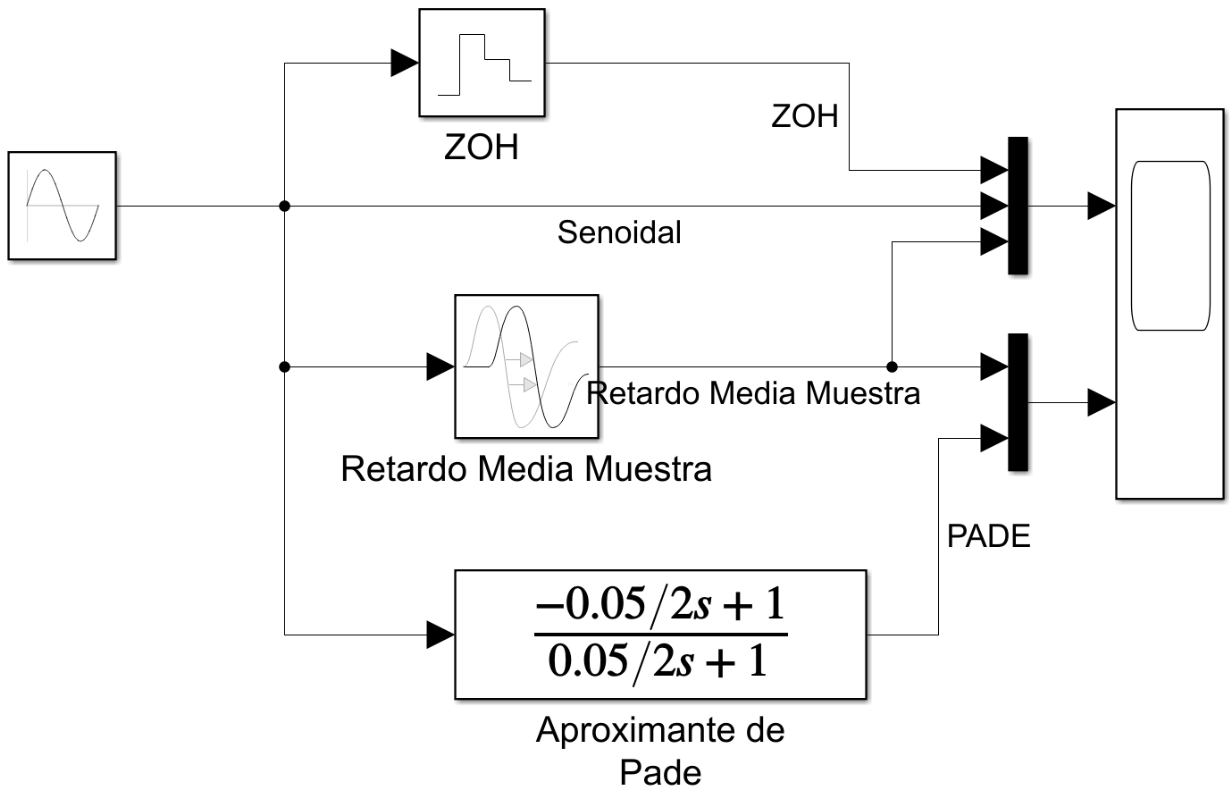
1. multiplying a matrix by a scalar multiplies all of the eigenvalues by the scalar;
2. adding the identity to a matrix adds 1 to all of the eigenvalues.

Eigenvalue/pole mapping result:



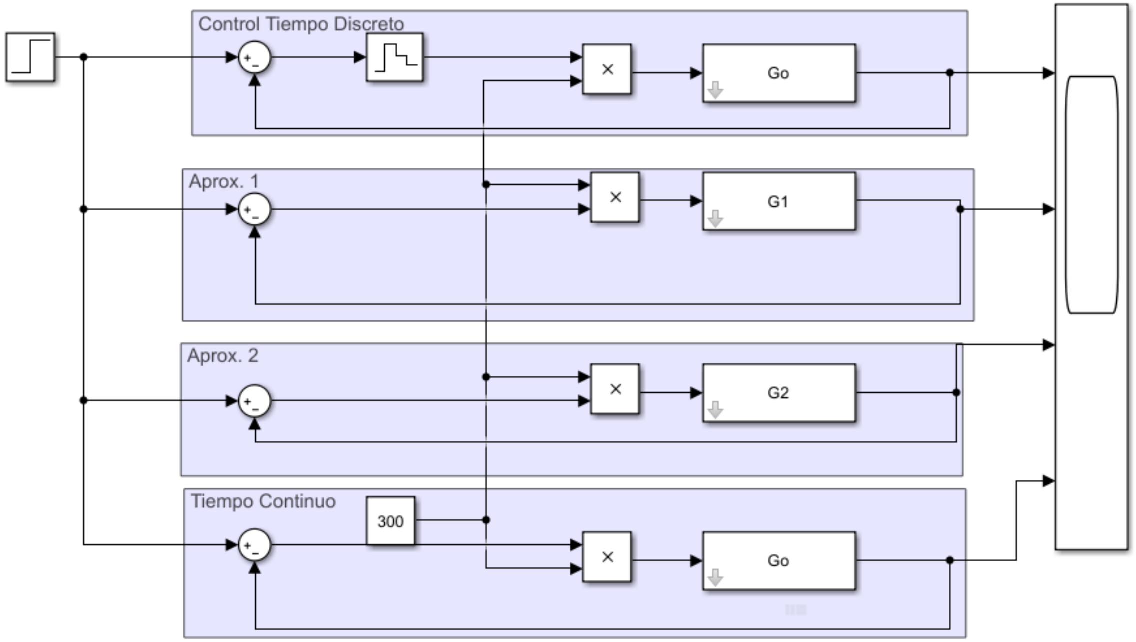
Exercise: Repeat this for the backward difference and bilinear approximations.

EJEMPLO: CONTROL PROPORCIONAL EN TIEMPO DISCRETO PARA UN SISTEMA DE PRIMER ORDEN.

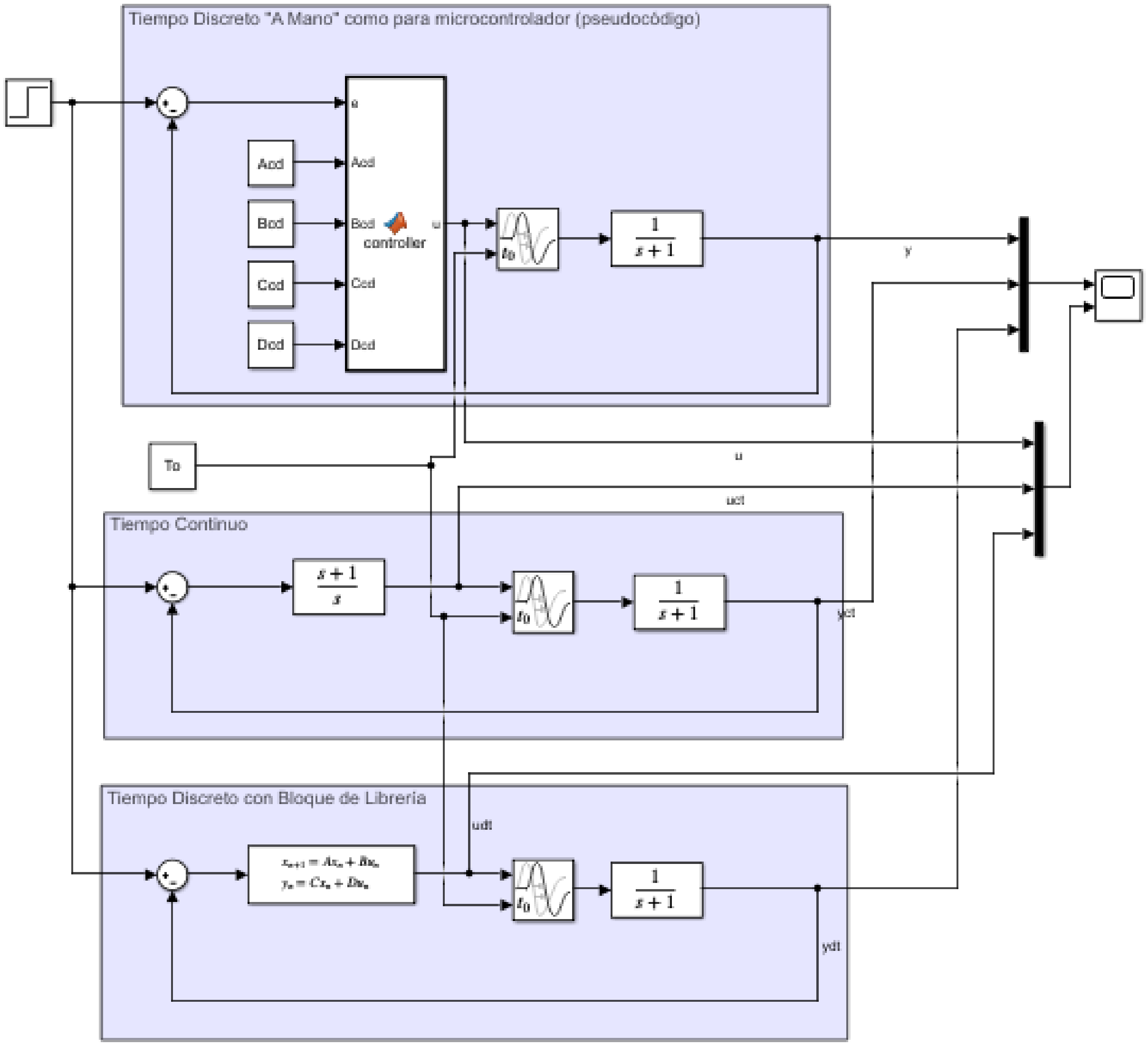


En este ejemplo se ilustra como qué transformación no lineal realiza el Zero Order Hold (ZOH) sobre una senoidal y cómo aproximarla quedándonos con el primer armónico de la señal de salida del ZOH. Esto da como transformación LTI que captura en tiempo continuo el efecto de implementar un controlador en tiempo discreto, un retardo de medio intervalo de muestreo. Este “retardo de media muestra” es a su vez aproximado por un aproximante de Padé en este caso de primer orden, aunque pueden ponerse aproximantes de órdenes superiores.

EJEMPLO: PRIMER ARMÓNICO DE SALIDA DEL ZOH



En este ejemplo se ilustra como un simple lazo de primer orden con acción proporcional, puede volverse inestable con un controlador implementado en tiempo discreto lo cual se puede predecir a través del método visto en clase a través del cual el efecto de la digitalización se captura como modelo LTI a través de un retardo de media muestra a su vez aproximado por un Pade de órdenes 1,2,3 y 4.



PROPORCIONAL EN TIEMPO DISCRETO PARA UN SISTEMA DE PRIMER ORDEN.

En este ejemplo se ilustra como un simple lazo de primer orden con retardo y control PI, como se aproxima el control de tiempo continuo por uno de tiempo discreto en espacio de estados.

Además, el bloque superior muestra el pseudocódigo para realizar la implementación digital en un Sistema digital como un microcontrolador.