FACULTAD
DE INGENIERIA
Universidad de Buenos Aires

Introducción a Sistemas de Control

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ESTABILIDAD INTERNA

Interconexión de Sistemas

$$\dot{x}_1 = A_1 x_1 + B_1 u$$

$$y_1 = C_1 x_1 + D_1 u$$

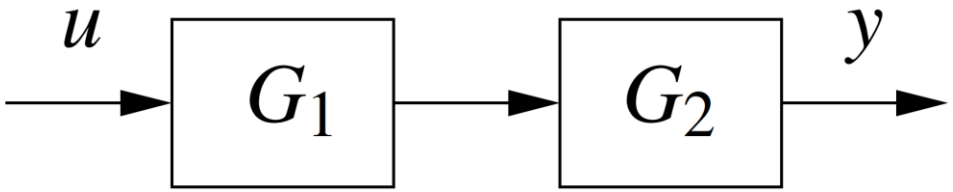
$$\dot{x}_2 = A_2 x_2 + B_2 y_1$$

$$y = C_2 x_2 + D_2 y_1$$

$$G_1(s) = C_1 \left(sI - A_1\right)^{-1} B_1 + D_1$$

$$G_2(s) = C_2 \left(sI - A_2\right)^{-1} B_2 + D_2$$

Interconexión Serie



$$\dot{x}_1 = A_1 x_1 + B_1 u$$

$$\dot{x}_2 = A_2 x_2 + B_2 (C_1 x_1 + D_1 u)$$

$$y = C_2 x_2 + D_2 (C_1 x_1 + D_1 u)$$

$$\dot{x}_1 = A_1 x_1 + B_1 u$$

$$\dot{x}_2 = B_1 C_1 x_1 + A_2 x_2 + B_2 D_1 u$$

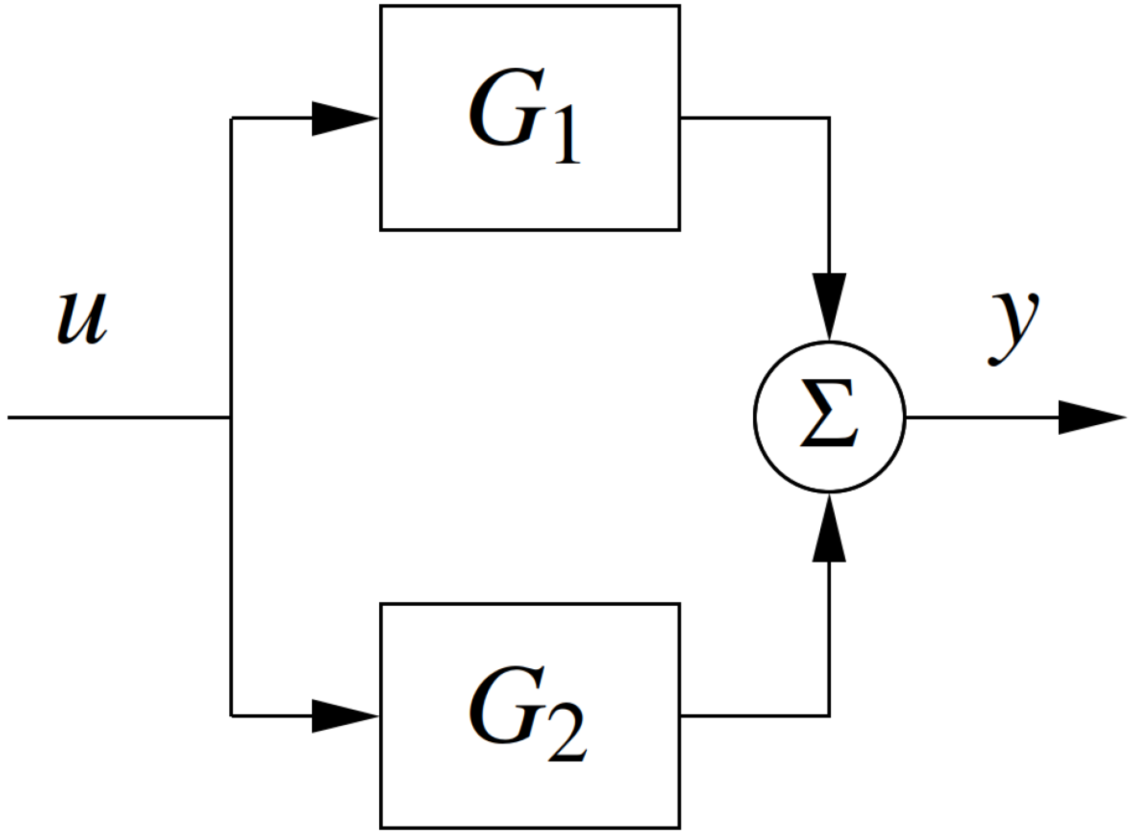
$$y = C_2 x_2 + D_2 C_1 x_1 + D_2 D_1 u$$

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{x}} = \overbrace{\begin{bmatrix} A_1 & 0 \\ B_1 C_1 & A_2 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{x} + \overbrace{\begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix}}^{B} u$$

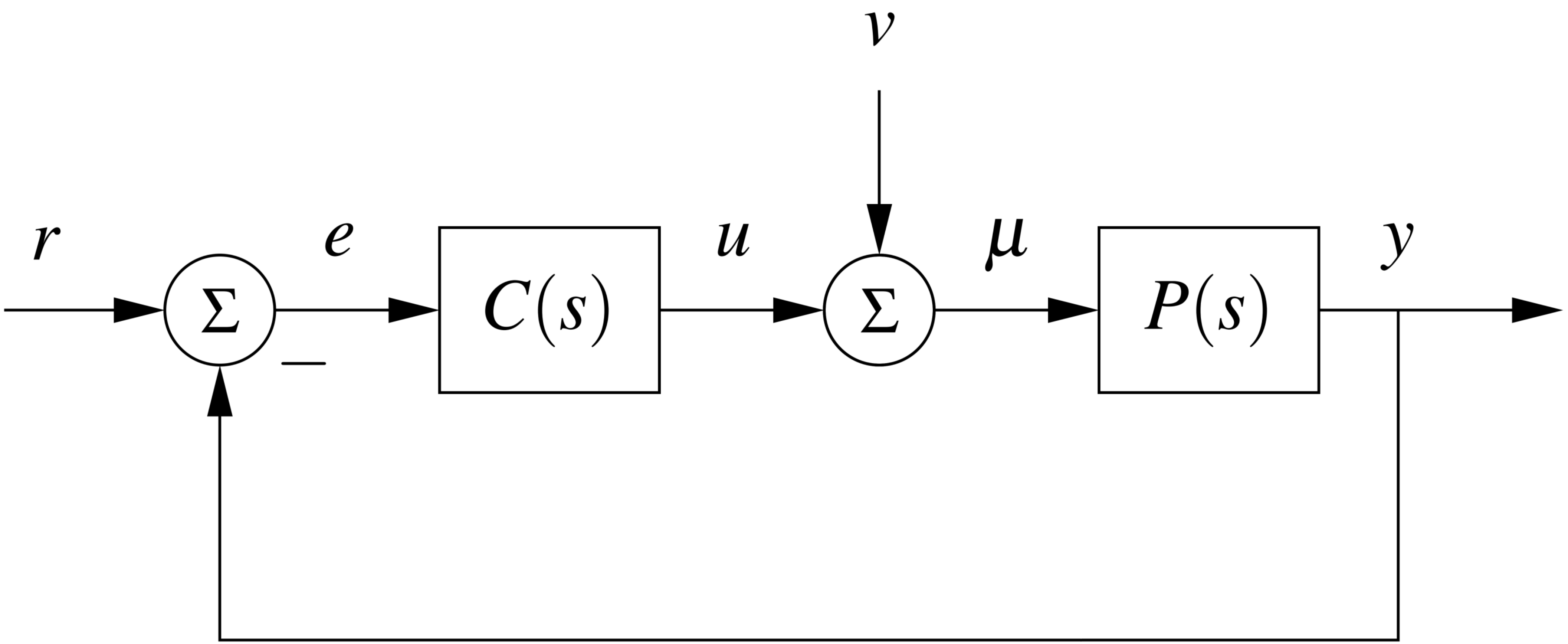
$$y = \underbrace{\begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{D_2 D_1}_{D} u.$$

INTERCONEXIÓN PARALELO

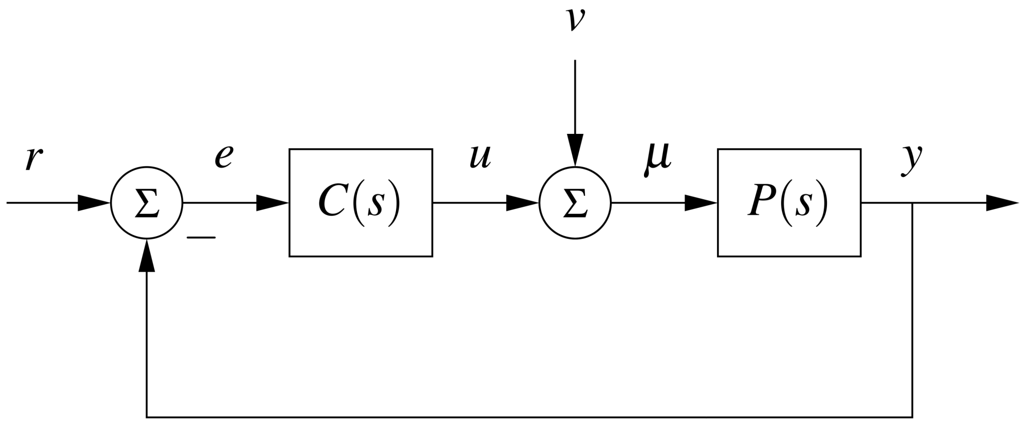
- Completar como ejercicio



Interconexión Feedback



Interconexión Feedback

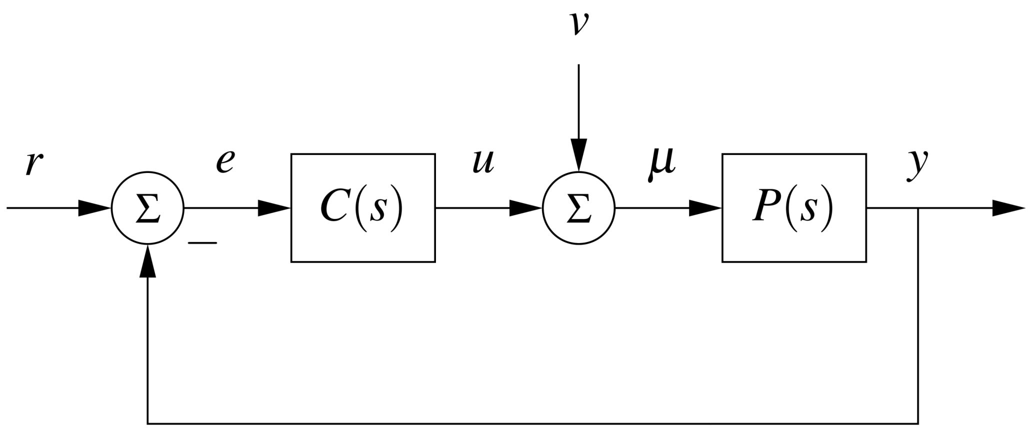


$$\dot{x} = Ax + B\mu \qquad \dot{x}_c = A_c x_c + B_c e$$

$$y = Cx + D\mu \qquad u = C_c x_c + D_c e$$

$$e = r - y \qquad \mu = v + u$$

Interconexión Feedback



$$\dot{x}_{cl} = A_{cl} x_{cl} + B_{cl} \begin{bmatrix} r \\ v \end{bmatrix}$$

$$\begin{bmatrix} e \\ \mu \end{bmatrix} = C_{cl} x_{cl} + D_{cl} \begin{bmatrix} r \\ v \end{bmatrix}$$

$$x_{cl} = \begin{bmatrix} x \\ x_c \end{bmatrix}$$

Interconexión Feedback

$$\dot{x} = Ax + B\mu$$

$$\dot{x}_c = A_c x_c + B_c e$$

$$y = Cx$$

$$u = C_c x_c$$

$$e = r - y$$

$$\mu = v + u$$

$$\dot{x} = Ax + B\,(v + u)$$

$$\dot{x}_c = A_c x_c + B_c \left(r - y\right)$$

$$\dot{x} = Ax + B\,(v + C_c x_c)$$

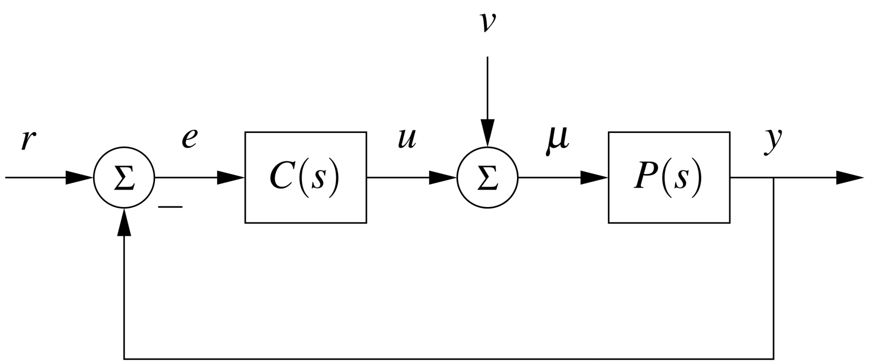
$$\dot{x}_c = A_c x_c + B_c\,(r - Cx)$$

Interconexión Feedback

$$\overbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix}}^{\dot{x}_{cl}} = \overbrace{\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}}^{A_{cl}} \overbrace{\begin{bmatrix} x \\ x_c \end{bmatrix}}^{x_{cl}} + \overbrace{\begin{bmatrix} 0 & B \\ B_c & 0 \end{bmatrix}}^{B_{cl}} \begin{bmatrix} r \\ v \end{bmatrix}$$

$$\begin{bmatrix} e \\ \mu \end{bmatrix} = \underbrace{\begin{bmatrix} -C & 0 \\ 0 & C_c \end{bmatrix}}_{C_{cl}} \underbrace{\begin{bmatrix} x \\ x_c \end{bmatrix}}_{x_{cl}} + \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}}_{D_{cl}} \begin{bmatrix} r \\ v \end{bmatrix}$$

Interconexión Feedback



$$\dot{x} = Ax + B\mu \qquad \dot{x}_c = A_c x_c + B_c e$$

$$y = Cx + D\mu \qquad u = C_c x_c + D_c e$$

$$e = r - y \qquad \mu = v + u$$

Interconexión Feedback

$$y = Cx + D(v+u) \qquad u = C_c x_c + D_c(r-y)$$

$$y = Cx + D\left(v + C_c x_c + D_c(r-y)\right) \qquad u = C_c x_c + D_c(r - [Cx + D(v+u)])$$

$$\underbrace{(I + DD_c)}_{\Phi_o} y = Cx + D(v + C_c x_c + D_c r) \qquad \underbrace{(I + D_c D)}_{\Phi_i} u = C_c x_c + D_c(r - [Cx + Dv])$$

$$y = \Phi_o^{-1}[Cx + D(v + C_c x_c + D_c r)] \qquad u = \Phi_i^{-1}[C_c x_c + D_c(r - [Cx + Dv])]$$

Interconexión Feedback

$$\dot{x} = Ax + B(v + u) \qquad \dot{x}_c = A_c x_c + B_c \left(r - y\right)$$

$$\dot{x} = Ax + B\left(v + \overbrace{\left\{\Phi_i^{-1}\left[C_c x_c + D_c\left(r - [Cx + Dv]\right)\right]\right\}}^{u}\right)$$

$$\dot{x}_c = A_c x_c + B_c\left(r - \underbrace{\left\{\Phi_o^{-1}\left[Cx + D\left(v + C_c x_c + D_c r\right)\right]\right\}}_{y}\right)$$

Interconexión Feedback

$$\dot{x} = Ax + B\Phi_i^{-1}C_c x_c + B\Phi_i^{-1}D_c r - B\Phi_i^{-1}D_c C x - B\Phi_i^{-1}D_c D v + B v$$

$$\dot{x}_c = A_c x_c + B_c r - B_c\Phi_o^{-1}C x - B_c\Phi_o^{-1}D C_c x_c - B_c\Phi_o^{-1}D D_c r - B_c\Phi_o^{-1}D v$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix} = \begin{bmatrix} (A - B\Phi_i^{-1}D_c C) & B\Phi_i^{-1}C_c \\ -B_c\Phi_o^{-1}C & (A_c - B_c\Phi_o^{-1}DC_c) \end{bmatrix} \begin{bmatrix} x \\ x_c \end{bmatrix} + \begin{bmatrix} B\Phi_i^{-1}D_c & B\Phi_i^{-1} \\ B_c\Phi_o^{-1} & -B_c\Phi_o^{-1}D \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix}$$

Interconexión Feedback

$$e = r - y = r - \Phi_o^{-1}\left[Cx + D\left(v + C_c x_c + D_c r\right)\right]$$

$$\mu = v + u = v + \Phi_i^{-1}\left[C_c x_c + D_c\left(r - [Cx + Dv]\right)\right]$$

$$e = -\Phi_o^{-1}Cx - \Phi_o^{-1}DC_c x_c + (I - \Phi_o^{-1}DD_c)r - \Phi_o^{-1}Dv$$

$$\mu = -\Phi_i^{-1}D_c Cx + \Phi_i^{-1}C_c x_c + \Phi_i^{-1}D_c r + (I + \Phi_i^{-1}D_c D)v$$

Interconexión en Transferencia

$$C(s) = \frac{n_c(s)}{d_c(s)}$$

$$P(s) = \frac{n_p(s)}{d_p(s)}$$

$$\lambda(s) = n_c(s)n_p(s) + d_c(s)d_p(s)$$

$$S(s) = \frac{1}{1 + P(s)C(s)} = \frac{d_p d_c}{\lambda(s)},$$

$$T(s) = 1 - S(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{n_p n_c}{\lambda(s)}$$

$$PS(s) = \frac{1}{1 + P(s)C(s)} = \frac{n_p d_c}{\lambda(s)}$$

$$CS(s) = \frac{1}{1 + P(s)C(s)} = \frac{d_p n_c}{\lambda(s)}$$

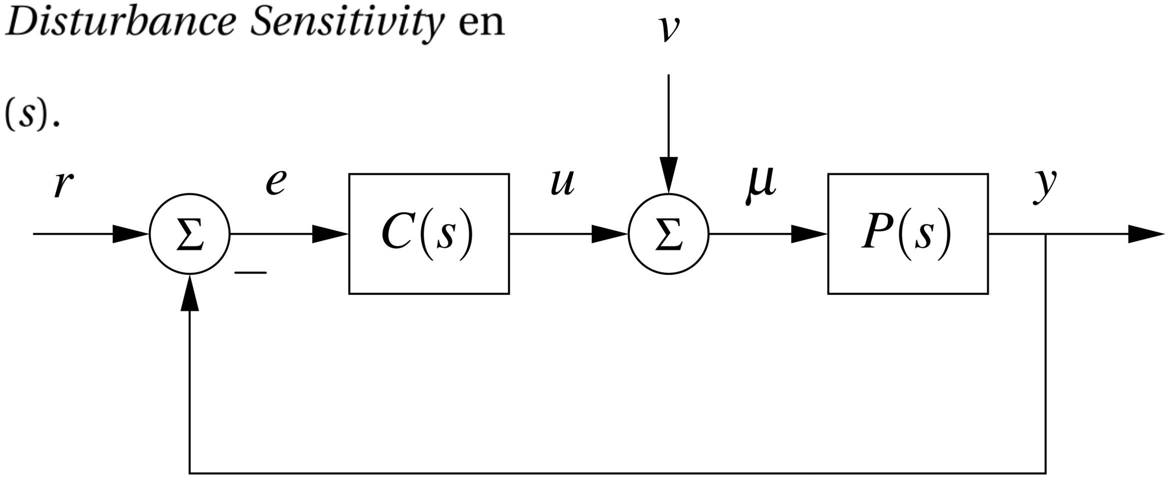
Funciones de sensibilidad

$S(s)$ Función de **Sensibilidad**, propiamente dicha. Es la transferencia de $R(s)$ a $E(s)$, y también de $V(s)$ a $\mu(s)$. También es la transferencia de $\eta(s)$ a $E(s)$.

$T(s)$ Función de **Sensibilidad Complementaria**. Es la transferencia de $R(s)$ a $Y(s)$.

$CS(s)$ **Sensibilidad al ruido de medición** o sensibilidad de salida (*noise Sensitivity* en Åström y Murray 2020) o Sensibilidad de la Acción de Control (*Control Sensitivity* en Goodwin, Graebe y Salgado 2000). Es la transferencia de $R(s)$ a $U(s)$ y también de $\eta(s)$ a $U(s)$

$PS(s)$ Sensibilidad de carga o Sensibilidad de entrada (*Input (o Load) Sensitivity* en Åström y Murray 2020) o **Sensibilidad a la Perturbación de Entrada** (*Input Disturbance Sensitivity* en Goodwin, Graebe y Salgado 2000). Es la transferencia de $V(s)$ a $Y(s)$.



Cancelación Polo/Cero

$$P(s) = \frac{n_p(s)}{d_p(s)} = \frac{\bar{n}_p(s)(s-z)}{\bar{d}_p(s)(s-p)}$$

$$C(s) = \frac{n_c(s)}{d_c(s)} = \frac{\bar{n}_c(s)(s-p)}{\bar{d}_c(s)(s-z)}$$

$$\lambda(s) = n_c(s)\, n_p(s) + d_c(s)\, d_p(s)$$

$$\bar{\lambda}(s) = \bar{n}_c(s)\, \bar{n}_p(s) + \bar{d}_c(s)\, \bar{d}_p(s)$$

$$\lambda(s) = (s-z)(s-p)\, \bar{\lambda}(s)$$

Estabilidad interna

El sistema a lazo cerrado es internamente estable si la matriz A_{cl} tiene sus autovalores con parte real negativa.

Equivalentemente, si las transferencias $S(s), PS(s)$ y $CS(s)$ son BIBO estables, i.e., si tienen todos sus polos tiene parte real negativa, el sistema es internamente estable.

$$\overbrace{\begin{bmatrix} \dot{x} \\ \dot{x}_c \end{bmatrix}}^{\dot{x}_{cl}} = \overbrace{\begin{bmatrix} A & BC_c \\ B_cC & A_c \end{bmatrix}}^{A_{cl}} \overbrace{\begin{bmatrix} x \\ x_c \end{bmatrix}}^{x_{cl}} + \overbrace{\begin{bmatrix} 0 & B \\ B_c & 0 \end{bmatrix}}^{B_{cl}} \begin{bmatrix} r \\ v \end{bmatrix}$$

$$\begin{bmatrix} e \\ \mu \end{bmatrix} = \underbrace{\begin{bmatrix} -C & 0 \\ 0 & C_c \end{bmatrix}}_{C_{cl}} \underbrace{\begin{bmatrix} x \\ x_c \end{bmatrix}}_{x_{cl}} + \underbrace{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}}_{D_{cl}} \begin{bmatrix} r \\ v \end{bmatrix}$$

$$\dot{x}_{cl} = A_{cl}x_{cl} + B_{cl}\begin{bmatrix} r \\ v \end{bmatrix}$$

$$\begin{bmatrix} e \\ \mu \end{bmatrix} = C_{cl}x_{cl} + D_{cl}\begin{bmatrix} r \\ v \end{bmatrix}$$

$$\begin{bmatrix} E(s) \\ \mu(s) \end{bmatrix} = \begin{bmatrix} S(s) & PS(s) \\ CS(s) & S(s) \end{bmatrix} \begin{bmatrix} R(s) \\ V(s) \end{bmatrix}$$

Estabilidad interna

$$P(s) = \frac{s-2}{(s+1)(s-1)} = \frac{s-2}{s^2-1} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

$$C(s) = \frac{1}{s}\frac{(s+1)(s-1)}{s-2} = \frac{s^2-1}{s^2-2s} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2}$$

$$\dot{x} = Ax + Bu \qquad y = Cx$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 \end{bmatrix} \qquad D = 0$$

$$= \begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \qquad = b_0$$

$$\dot{x}_c = A_c x_c + B_c e \qquad u = C_c x_c + D_c e$$

$$A_c = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \qquad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$$

$$C_c = \begin{bmatrix} -1 & 2 \end{bmatrix} \qquad D_c = 1$$

$$= \begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \qquad = b_0$$

Estabilidad interna

$$A_{cl} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 2 \end{bmatrix} \qquad B_{cl} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 2 & -1 & -1 & 2 \end{bmatrix} \qquad D_{cl} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} R(s) \\ V(s) \end{bmatrix} \mapsto \begin{bmatrix} E(s) \\ \mu(s) \end{bmatrix} = \begin{bmatrix} S(s) & PS(s) \\ CS(s) & S(s) \end{bmatrix} = \begin{bmatrix} \frac{s}{s+1} & \frac{-s(s-2)}{(s+1)^2(s-1)} \\ \frac{s-1}{s-2} & \frac{s}{(s+1)} \end{bmatrix}$$