CHAPTER 9

Random Processes

In certain random experiments, the outcome is a function of time or space. For example, in speech recognition systems, decisions are made on the basis of a voltage waveform corresponding to a speech utterance. In an image processing system, the intensity and color of the image varies over a rectangular region. In a peer-to-peer network, the number of peers in the system varies with time. In some situations, two or more functions of time may be of interest. For example, the temperature in a certain city and the demand placed on the local electric power utility vary together in time.

The random time functions in the above examples can be viewed as numerical quantities that evolve randomly in time or space. Thus what we really have is a family of random variables indexed by the time or space variable. In this chapter we begin the study of random processes. We will proceed as follows:

- In Section 9.1 we introduce the notion of a *random process* (or *stochastic process*), which is defined as an *indexed family of random variables*.
- We are interested in specifying the joint behavior of the random variables within a family (i.e., the temperature at two time instants). In Section 9.2 we see that this is done by specifying joint distribution functions, as well as mean and covariance functions.
- In Sections 9.3 to 9.5 we present examples of stochastic processes and show how models of complex processes can be developed from a few simple models.
- In Section 9.6 we introduce the class of stationary random processes that can be viewed as random processes in "steady state."
- In Section 9.7 we investigate the continuity properties of random processes and define their derivatives and integrals.
- In Section 9.8 we examine the properties of time averages of random processes and the problem of estimating the parameters of a random process.
- In Section 9.9 we describe methods for representing random processes by Fourier series and by the Karhunen-Loeve expansion. ???
- Finally, in Section 9.10 we present methods for generating random processes.

9.1 DEFINITION OF A RANDOM PROCESS

Realizaciones

Consider a random experiment specified by the outcomes ζ from some sample space S, by the events defined on S, and by the probabilities on these events. Suppose that to every outcome $\zeta \in S$, we assign a function of time according to some rule:

$$X(t, \zeta) \qquad t \in I.$$

En criollo, el resultado de mi experimento ahora es una función. O también, para cada tiempo distinto tengo una variable aleatoria distinta.

The graph of the function $X(t, \zeta)$ versus t, for ζ fixed, is called a **realization**, **sample path**, or **sample function** of the random process. Thus we can view the outcome of the random experiment as producing an entire function of time as shown in Fig. 9.1. On the other hand, if we fix a time t_k from the index set I, then $X(t_k, \zeta)$ is a random variable (see Fig. 9.1) since we are mapping ζ onto a real number. Thus we have created a family (or ensemble) of random variables indexed by the parameter t, $\{X(t, \zeta), t \in I\}$. This family is called a **random process**. We also refer to random processes as **stochastic processes**. We usually suppress the ζ and use $X(t)$ to denote a random process.

A stochastic process is said to be **discrete-time** if the index set I is a countable set (i.e., the set of integers or the set of nonnegative integers). When dealing with discrete-time processes, we usually use n to denote the time index and X_n to denote the random process. A **continuous-time** stochastic process is one in which I is continuous (i.e., the real line or the nonnegative real line).

The following example shows how we can imagine a stochastic process as resulting from nature selecting ζ at the beginning of time and gradually revealing it in time through $X(t, \zeta)$.

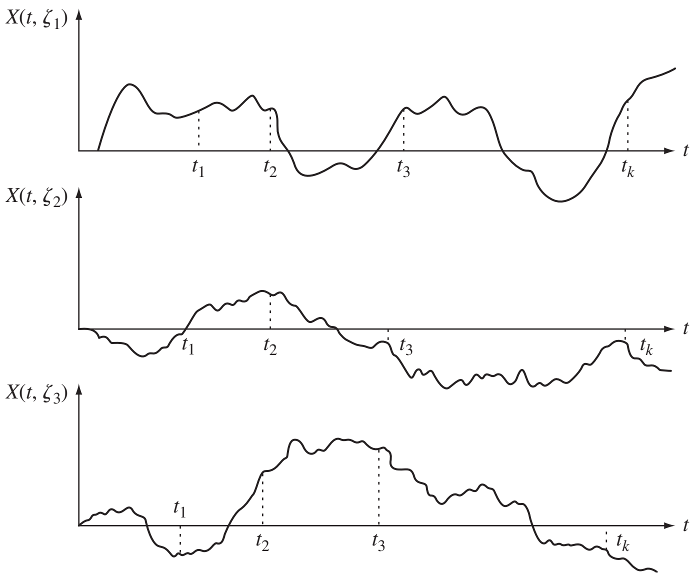

FIGURE 9.1
Several realizations of a random process.

Example 9.1 Random Binary Sequence

Let ζ be a number selected at random from the interval $S = [0, 1]$, and let $b_1 b_2 \ldots$ be the binary expansion of ζ:

¿Cuál es más difícil de escribir?

¿ζ ó ξ?

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i} \qquad \text{where } b_i \in \{0, 1\}.$$

$\zeta = 0{,}011011000010$

Define the discrete-time random process $X(n, \zeta)$ by

$$X(n, \zeta) = b_n \qquad n = 1, 2, \ldots .$$

The resulting process is sequence of binary numbers, with $X(n, \zeta)$ equal to the nth number in the binary expansion of ζ.

Example 9.2 Random Sinusoids

Let ζ be selected at random from the interval $[-1, 1]$. Define the continuous-time random process $X(t, \zeta)$ by

$$X(t, \zeta) = \zeta \cos(2\pi t) \qquad -\infty < t < \infty.$$

The realizations of this random process are sinusoids with amplitude ζ, as shown in Fig. 9.2(a).

Let ζ be selected at random from the interval $(-\pi, \pi)$ and let $Y(t, \zeta) = \cos(2\pi t + \zeta)$. The realizations of $Y(t, \zeta)$ are phase-shifted versions of $\cos 2\pi t$ as shown in Fig 9.2(b).

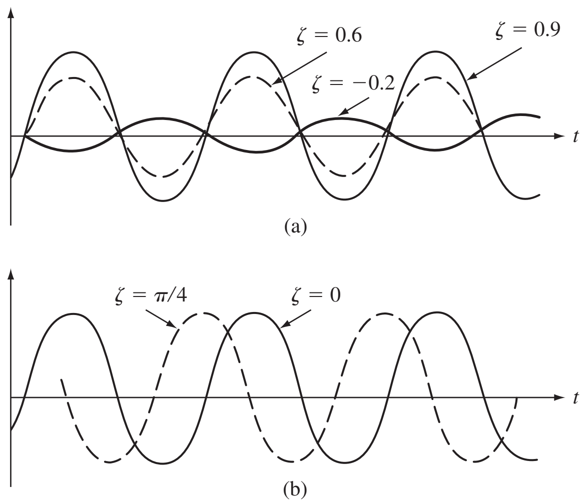

FIGURE 9.2
(a) Sinusoid with random amplitude, (b) Sinusoid with random phase.

The randomness in ζ induces randomness in the observed function $X(t, \zeta)$. In principle, one can deduce the probability of events involving a stochastic process at various instants of time from probabilities involving ζ by using the equivalent-event method introduced in Chapter 4.

Example 9.3

Find the following probabilities for the random process introduced in Example 9.1: $P[X(1, \zeta) = 0]$ and $P[X(1, \zeta) = 0 \text{ and } X(2, \zeta) = 1]$.

The probabilities are obtained by finding the equivalent events in terms of ζ:

Bué, medio rápido

$$P[X(1, \zeta) = 0] = P\left[0 \le \zeta < \frac{1}{2}\right] = \frac{1}{2}$$

$$P[X(1, \zeta) = 0 \text{ and } X(2, \zeta) = 1] = P\left[\frac{1}{4} \le \zeta < \frac{1}{2}\right] = \frac{1}{4},$$

since all points in the interval $[0 \le \zeta \le 1]$ begin with $b_1 = 0$ and all points in $[1/4, 1/2)$ begin with $b_1 = 0$ and $b_2 = 1$. Clearly, any sequence of k bits has a corresponding subinterval of length (and hence probability) 2^{-k}.

Example 9.4

Find the pdf of $X_0 = X(t_0, \zeta)$ and $Y(t_0, \zeta)$ in Example 9.2.

If t_0 is such that $\cos(2\pi t_0) = 0$, then $X(t_0, \zeta) = 0$ for all ζ and the pdf of $X(t_0)$ is a delta function of unit weight at $x = 0$. Otherwise, $X(t_0, \zeta)$ is uniformly distributed in the interval $(-\cos 2\pi t_0, \cos 2\pi t_0)$ since ζ is uniformly distributed in $[-1, 1]$ (see Fig. 9.3a). Note that the pdf of $X(t_0, \zeta)$ depends on t_0.

The approach used in Example 4.36 can be used to show that $Y(t_0, \zeta)$ has an arcsine distribution:

$$f_Y(y) = \frac{1}{\pi\sqrt{1 - y^2}}, \qquad |y| < 1$$

(see Fig. 9.3b). Note that the pdf of $Y(t_0, \zeta)$ does not depend on t_0.

Figure 9.3(c) shows a histogram of 1000 samples of the amplitudes $X(t_0, \zeta)$ at $t_0 = 0$, which can be seen to be approximately uniformly distributed in $[-1, 1]$. Figure 9.3(d) shows the histogram for the samples of the sinusoid with random phase. Clearly there is agreement with the arcsine pdf.

In general, the sample paths of a stochastic process can be quite complicated and cannot be described by simple formulas. In addition, it is usually not possible to identify an underlying probability space for the family of observed functions of time. Thus the equivalent-event approach for computing the probability of events involving $X(t, \zeta)$ in terms of the probabilities of events involving ζ does not prove useful in

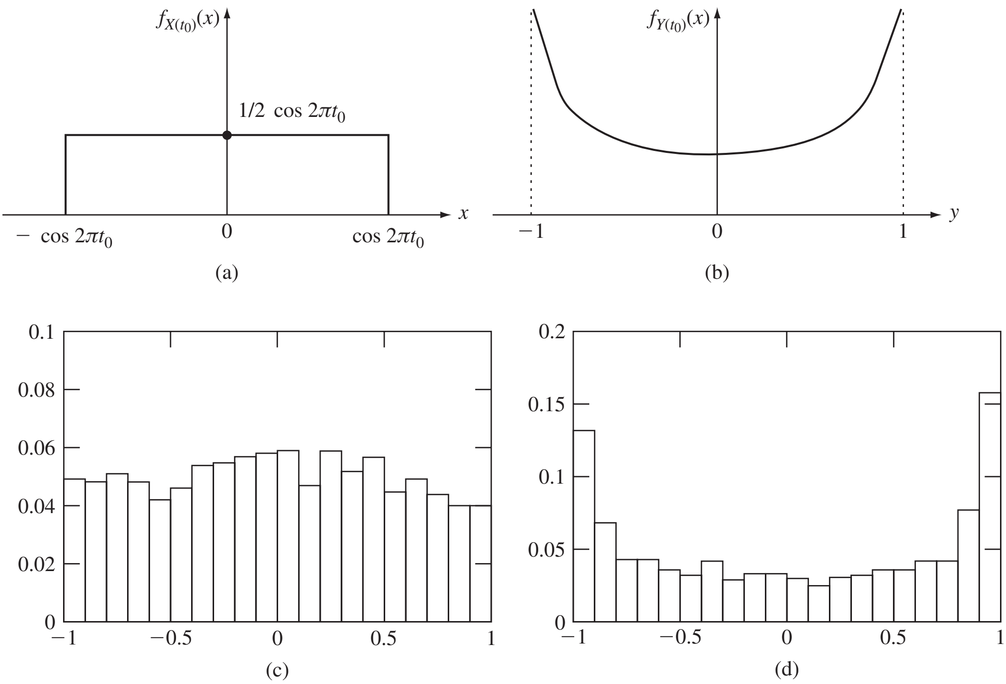

FIGURE 9.3
(a) pdf of sinusoid with random amplitude. (b) pdf of sinusoid with random phase. (c) Histogram of samples from uniform amplitude sinusoid at $t = 0$. (d) Histogram of samples from random phase sinusoid at $t = 0$.

practice. In the next section we show an alternative method for specifying the probabilities of events involving a stochastic process.

9.2 SPECIFYING A RANDOM PROCESS

There are many questions regarding random processes that cannot be answered with just knowledge of the distribution at a single time instant. For example, we may be interested in the temperature at a given locale at two different times. This requires the following information:

$$P[x_1 < X(t_1) \leq x_1, x_2 < X(t_2) \leq x_2].$$

In another example, the speech compression system in a cellular phone predicts the value of the speech signal at the next sampling time based on the previous k samples. Thus we may be interested in the following probability:

$$P[a < X(t_{k+1}) \leq b \,|\, X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_k) = x_k].$$

It is clear that a general description of a random process should provide probabilities for *vectors* of samples of the process. → ¿Y si el indice es continuo?

9.2.1 Joint Distributions of Time Samples

Let $X_1, X_2, \ldots, X_k$ be the k random variables obtained by sampling the random process $X(t, \zeta)$ at the times $t_1, t_2, \ldots, t_k$:

$$X_1 = X(t_1, \zeta), X_2 = X(t_2, \zeta,), \ldots, X_k = X(t_k, \zeta),$$

as shown in Fig. 9.1. The joint behavior of the random process at these k time instants is specified by the joint cumulative distribution of the *vector* random variable $X_1, X_2, \ldots, X_k$. The probabilities of any event involving the random process at all or some of these time instants can be computed from this cdf using the methods developed for vector random variables in Chapter 6. Thus, *a stochastic process is specified by the collection of kth-order joint cumulative distribution functions*:

$$F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = P[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_k) \le x_k], \quad (9.1)$$

for any k and any choice of sampling instants $t_1, \ldots, t_k$. Note that the collection of cdf's must be consistent in the sense that lower-order cdf's are obtained as marginals of higher-order cdf's. If the stochastic process is *continuous-valued*, then a collection of probability density functions can be used instead:

$$\begin{aligned} f_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k)\, dx_1 \ldots dx_n \\ = P\{x_1 < X(t_1) \le x_1 + dx_1, \ldots, x_k < X(t_k) \le x_k + dx_k]. \quad (9.2) \end{aligned}$$

If the stochastic process is *discrete-valued*, then a collection of probability mass functions can be used to specify the stochastic process:

$$p_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) = P[X(t_1) = x_1, X(t_2) = x_2, \ldots, X(t_k) = x_k] \quad (9.3)$$

↳ creo q' acá venían

for any k and any choice of sampling instants $n_1, \ldots, n_k$.

At first glance it does not appear that we have made much progress in specifying random processes because we are now confronted with the task of specifying a vast collection of joint cdf's! However, this approach works because most useful models of stochastic processes are obtained by elaborating on a few simple models, so the methods developed in Chapters 5 and 6 of this book can be used to derive the required cdf's. The following examples give a preview of how we construct complex models from simple models. We develop these important examples more fully in Sections 9.3 to 9.5.

Example 9.5 iid Bernoulli Random Variables

Let X_n be a sequence of independent, identically distributed Bernoulli random variables with $p = 1/2$. The joint pmf for any k time samples is then

$$P[X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k] = P[X_1 = x_1] \ldots P[X_k = x_k] = \left(\frac{1}{2}\right)^k$$

where $x_i \in \{0, 1\}$ for all i. This binary random process is equivalent to the one discussed in Example 9.1.

Example 9.6 iid Gaussian Random Variables

Let X_n be a sequence of independent, identically distributed Gaussian random variables with zero mean and variance σ_X^2. The joint pdf for any k time samples is then

$$f_{X_1,X_2,\ldots,X_k}(x_1, x_2, \ldots, x_k) = \frac{1}{(2\pi\sigma^2)^{k/2}} e^{-(x_1^2+x_2^2+\cdots+x_k^2)/2\sigma^2}.$$

The following two examples show how more complex and interesting processes can be built from iid sequences.

Example 9.7 Binomial Counting Process

Let X_n be a sequence of independent, identically distributed Bernoulli random variables with $p = 1/2$. Let S_n be the number of 1's in the first n trials:

RANDOM WALK

$$S_n = X_1 + X_2 + \cdots + X_n \quad \text{for} \quad n = 0, 1, \ldots.$$

S_n is an integer-valued nondecreasing function of n that grows by unit steps after a random number of time instants. From previous chapters we know that S_n is a binomial random variable with parameters n and $p = 1/2$. In the next section we show how to find the joint pmf's of S_n using conditional probabilities.

Example 9.8 Filtered Noisy Signal

Let X_j be a sequence of independent, identically distributed observations of a signal voltage μ corrupted by zero-mean Gaussian noise N_j with variance σ^2:

$$X_j = \mu + N_j \quad \text{for} \quad j = 0, 1, \ldots.$$

Consider the signal that results from averaging the sequence of observations:

MOY BUENO

$$S_n = (X_1 + X_2 + \cdots + X_n)/n \quad \text{for} \quad n = 0, 1, \ldots.$$

From previous chapters we know that S_n is the sample mean of an iid sequence of Gaussian random variables. We know that S_n itself is a Gaussian random variable with mean μ and variance σ^2/n, and so it tends towards the value μ as n increases. In a later section, we show that S_n is an example from the class of Gaussian random processes.

9.2.2 The Mean, Autocorrelation, and Autocovariance Functions

The moments of time samples of a random process can be used to partially specify the random process because they summarize the information contained in the joint cdf's.

The **mean function** $m_X(t)$ and the **variance function VAR[$X(t)$]** of a continuous-time random process $X(t)$ are defined by

La evolución de la media y la varianza en función del tiempo

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\, dx, \tag{9.4}$$

and

$$\text{VAR}[X(t)] = \int_{-\infty}^{\infty} (x - m_X(t))^2 f_{X(t)}(x)\, dx, \tag{9.5}$$

where $f_{X(t)}(x)$ is the pdf of $X(t)$. Note that $m_X(t)$ and VAR[$X(t)$] are *deterministic* functions of time. Trends in the behavior of $X(t)$ are reflected in the variation of $m_X(t)$ with time. The variance gives an indication of the spread in the values taken on by $X(t)$ at different time instants.

The **autocorrelation** $R_X(t_1, t_2)$ of a random process $X(t)$ is defined as the joint moment of $X(t_1)$ and $X(t_2)$:

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),X(t_2)}(x, y)\, dx\, dy, \tag{9.6}$$

where $f_{X(t_1),X(t_2)}(x, y)$ is the second-order pdf of $X(t)$. In general, the autocorrelation is a function of t_1 and t_2. Note that $R_X(t, t) = E[X^2(t)]$.

The **autocovariance** $C_X(t_1, t_2)$ of a random process $X(t)$ is defined as the covariance of $X(t_1)$ and $X(t_2)$:

$$C_X(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]. \tag{9.7}$$

From Eq. (5.30), the autocovariance can be expressed in terms of the autocorrelation and the means:

Como fue siempre

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1)m_X(t_2). \tag{9.8}$$

Note that the variance of $X(t)$ can be obtained from $C_X(t_1, t_2)$:

Tal vez convenga distinguir

$$\text{VAR}[X(t)] = E[(X(t) - m_X(t))^2] = C_X(t, t). \tag{9.9}$$

The *correlation coefficient* of $X(t)$ is defined as the correlation coefficient of $X(t_1)$ and $X(t_2)$ (see Eq. 5.31):

$X(t_1) = X$

$X(t_2) = Y \Rightarrow C(t_1, t_2) = \text{cov}(X, Y)$

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)}\sqrt{C_X(t_2, t_2)}}. \tag{9.10}$$

From Eq. (5.32) we have that $|\rho_X(t_1, t_2)| \leq 1$. Recall that the correlation coefficient is a measure of the extent to which a random variable can be predicted as a linear function of another. In Chapter 10, we will see that the autocovariance function and the autocorrelation function play a critical role in the design of linear methods for analyzing and processing random signals.

AUTOcorrelación y AUTOcovarianza xq estoy comparando EL MISMO proceso, solo que en índices diferentes

4 personajes: 1) MEDIA(x), 2) VAR(x) → 4) AUTOCOV(t_1, t_2)
3) AUTOCORR(t_1, t_2), 4)

Por ahora estos son mis momentos

The mean, variance, autocorrelation, and autocovariance functions for discrete-time random processes are defined in the same manner as above. We use a slightly different notation for the time index. The **mean and variance** of a *discrete-time* random process X_n are defined as:

$$m_X(n) = E[X_n] \quad \text{and} \quad \text{VAR}[X_n] = E[(X_n - m_X(n))^2]. \tag{9.11}$$

The **autocorrelation and autocovariance functions** of a discrete-time random process X_n are defined as follows:

$$R_X(n_1, n_2) = E[X(n_1)X(n_2)] \tag{9.12}$$

and

¿Y la varianza?

$$\begin{aligned} C_X(n_1, n_2) &= E[\{X(n_1) - m_X(n_1)\}\{X(n_2) - m_X(n_2)\}] \\ &= R_X(n_1, n_2) - m_X(n_1)m_X(n_2). \end{aligned} \tag{9.13}$$

Before proceeding to examples, we reiterate that the mean, autocorrelation, and autocovariance functions are only partial descriptions of a random process. Thus we will see later in the chapter that it is possible for two quite different random processes to have the same mean, autocorrelation, and autocovariance functions.

Example 9.9 Sinusoid with Random Amplitude

Let $X(t) = A \cos 2\pi t$, where A is some random variable (see Fig. 9.2a). The mean of $X(t)$ is found using Eq. (4.30):

$$m_X(t) = E[A \cos 2\pi t] = E[A] \cos 2\pi t.$$

Note that the mean varies with t. In particular, note that the process is always zero for values of t where $\cos 2\pi t = 0$.

The autocorrelation is

$$\begin{aligned} R_X(t_1, t_2) &= E[A \cos 2\pi t_1 \, A \cos 2\pi t_2] \\ &= E[A^2] \cos 2\pi t_1 \cos 2\pi t_2, \end{aligned}$$

and the autocovariance is then

OBS: podés elegir t de manera tal que X(t1) y X(t2) queden descorrelacionados (y si fueran gaussianas podrías decir que son indep.)

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2) \\ &= \{E[A^2] - E[A]^2\} \cos 2\pi t_1 \cos 2\pi t_2 \\ &= \text{VAR}[A] \cos 2\pi t_1 \cos 2\pi t_2. \end{aligned}$$

Example 9.10 Sinusoid with Random Phase

Let $X(t) = \cos(\omega t + \Theta)$, where Θ is uniformly distributed in the interval $(-\pi, \pi)$ (see Fig. 9.2b). The mean of $X(t)$ is found using Eq. (4.30):

$$m_X(t) = E[\cos(\omega t + \Theta)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\, d\theta = 0.$$ ∀t

The autocorrelation and autocovariance are then

$$C_X(t_1, t_2) = R_X(t_1, t_2) = E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)]$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\{\cos(\omega(t_1 - t_2) + \cos(\omega(t_1 + t_2) + 2\theta)\}\, d\theta$$

Si tomo muestras separadas por algún múltiplo de medio período, tengo descorrelación

$$= \frac{1}{2}\cos(\omega(t_1 - t_2)),$$

WTF

where we used the identity $\cos(a)\cos(b) = 1/2\cos(a + b) + 1/2\cos(a - b)$. Note that $m_X(t)$ is a constant and that $C_X(t_1, t_2)$ depends only on $|t_1 - t_2|$. Note as well that the samples at time t_1 and t_2 are uncorrelated if $\omega(t_1 - t_2) = k\pi$ where k is any integer.

9.2.3 Multiple Random Processes

In most situations we deal with more than one random process at a time. For example, we may be interested in the temperatures at city a, $X(t)$, and city b, $Y(t)$. Another very common example involves a random process $X(t)$ that is the "input" to a system and another random process $Y(t)$ that is the "output" of the system. Naturally, we are interested in the interplay between $X(t)$ and $Y(t)$.

The joint behavior of two or more random processes is specified by the collection of joint distributions for all possible choices of time samples of the processes. Thus for a pair of continuous-valued random processes $X(t)$ and $Y(t)$ we must specify all possible joint density functions of $X(t_1), \ldots, X(t_k)$ and $Y(t'_1), \ldots, Y(t'_j)$ for all k, j, and all choices of $t_1, \ldots, t_k$ and $t'_1, \ldots, t'_j$. For example, the simplest joint pdf would be:

Se re pudrió

$$f_{X(t_1),Y(t_2)}(x, y)\, dx\, dy = P\{x < X(t_1) \le x + dx, y < Y(t_2) \le y + dy\}.$$

Note that the time indices of $X(t)$ and $Y(t)$ need not be the same. For example, we may be interested in the input at time t_1 and the output at a later time t_2.

The random processes $X(t)$ and $Y(t)$ are said to be **independent random processes** if the vector random variables $\mathbf{X} = (X(t_1), \ldots, X(t_k))$ and $\mathbf{Y} = (Y(t'_1), \ldots, Y(t'_j))$ are independent for *all* k,j, and *all* choices of $t_1, \ldots, t_k$ and $t'_1, \ldots, t'_j$:

Definición de independencia entre procesos aleatorios

$$F_{\mathbf{X},\mathbf{Y}}(x_1, \ldots, x_k, y_1, \ldots, y_j) = F_{\mathbf{X}}(X_1, \ldots, X_k)\, F_{\mathbf{Y}}(y_1, \ldots, y_j).$$

The **cross-correlation** $\boldsymbol{R_{X,Y}(t_1, t_2)}$ of $X(t)$ and $Y(t)$ is defined by

ortogonalidad entre procesos

$$R_{X,Y}(t_1, t_2) = E[X(t_1)Y(t_2)]. \tag{9.14}$$

The processes $X(t)$ and $Y(t)$ are said to be **orthogonal random processes** if

$$R_{X,Y}(t_1, t_2) = 0 \quad \text{for all } t_1 \text{ and } t_2. \tag{9.15}$$

No tienen nada q' ver uno con el otro

The **cross-covariance** $C_{X,Y}(t_1, t_2)$ of $X(t)$ and $Y(t)$ is defined by

tiempo de x

tiempo de y

$$\begin{aligned} C_{X,Y}(t_1, t_2) &= E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_X(t_2)\}] \\ &= R_{X,Y}(t_1, t_2) - m_X(t_1)m_X(t_2). \end{aligned} \tag{9.16}$$

The processes $X(t)$ and $Y(t)$ are said to be **uncorrelated random processes** if

$$C_{X,Y}(t_1, t_2) = 0 \quad \text{for all } t_1 \text{ and } t_2. \tag{9.17}$$

Example 9.11

Let $X(t) = \cos(\omega t + \Theta)$ and $Y(t) = \sin(\omega t + \Theta)$, where Θ is a random variable uniformly distributed in $[-\pi, \pi]$. Find the cross-covariance of $X(t)$ and $Y(t)$.

From Example 9.10 we know that $X(t)$ and $Y(t)$ are zero mean. From Eq. (9.16), the cross-covariance is then equal to the cross-correlation:

$$\begin{aligned} C_{X,Y}(t_1, t_2) = R_{X,Y}(t_1, t_2) &= E[\cos(\omega t_1 + \Theta)\sin(\omega t_2 + \Theta)] \\ &= E\left[-\frac{1}{2}\sin(\omega(t_1 - t_2)) + \frac{1}{2}\sin(\omega(t_1 + t_2) + 2\Theta)\right] \\ &= -\frac{1}{2}\sin(\omega(t_1 - t_2)), \end{aligned}$$

since $E[\sin(\omega(t_1 + t_2) + 2\Theta)] = 0$. $X(t)$ and $Y(t)$ are not uncorrelated random processes because the cross-covariance is not equal to zero for all choices of time samples. Note, however, that $X(t_1)$ and $Y(t_2)$ are *uncorrelated random variables* for t_1 and t_2 such that $\omega(t_1 - t_2) = k\pi$ where k is any integer.

Example 9.12 Signal Plus Noise

Suppose process $Y(t)$ consists of a desired signal $X(t)$ plus noise $N(t)$:

$$Y(t) = X(t) + N(t).$$

Find the cross-correlation between the observed signal and the desired signal assuming that $X(t)$ and $N(t)$ are independent random processes.

From Eq. (8.14), we have

No lo dijimos, pero si los procesos son indep, la media del producto es el producto de las medias (la derecha por la izquierda)

$$\begin{aligned} R_{XY}(t_1, t_2) &= E[X(t_1)Y(t_2)] \\ &= E[X(t_1)\{X(t_2) + N(t_2)\}] \\ &= R_X(t_1, t_2) + E[X(t_1)]E[N(t_2)] \\ &= R_X(t_1, t_2) + m_X(t_l)m_N(t_2), \end{aligned}$$

where the third equality followed from the fact that $X(t)$ and $N(t)$ are independent.

¿Y cuál es la diferencia entre dos procesos ortogonales y dos procesos descorrelacionados? Si 2 procesos son decorrelacionados, su correlación cruzada es el producto de sus medias. Si dos procesos son ortogonales, su covarianza cruzada es menos el producto de sus medias. Si son ortogonales Y descorrelacionados, el producto de sus medias debe ser cero, o sea que por lo menos una es cero. Y con todo esto qué?

9.3 DISCRETE-TIME PROCESSES: SUM PROCESS, BINOMIAL COUNTING PROCESS, AND RANDOM WALK

In this section we introduce several important discrete-time random processes. We begin with the simplest class of random processes—independent, identically distributed sequences—and then consider the sum process that results from adding an iid sequence. We show that the sum process satisfies the independent increments property as well as the Markov property. Both of these properties greatly facilitate the calculation of joint probabilities. We also introduce the binomial counting process and the random walk process as special cases of sum processes.

9.3.1 iid Random Process

Let X_n be a discrete-time random process consisting of a sequence of independent, identically distributed (iid) random variables with common cdf $F_X(x)$, mean m, and variance σ^2. The sequence X_n is called the **iid random process**.

The joint cdf for any time instants $n_1, \ldots, n_k$ is given by

$$\begin{aligned} F_{X_1,\ldots,X_k}(x_1, x_2, \ldots, x_k) &= P[X_1 \le x_1, X_2 \le x_2, \ldots, X_k \le x_k] \\ &= F_X(x_1)F_X(x_2)\ldots F_X(x_k), \end{aligned} \tag{9.18}$$

where, for simplicity, X_k denotes X_{n_k}. Equation (9.18) implies that if X_n is discrete-valued, the joint pmf factors into the product of individual pmf's, and if X_n is continuous-valued, the joint pdf factors into the product of the individual pdf's.

The *mean of an iid process* is obtained from Eq. (9.4):

$$m_X(n) = E[X_n] = m \qquad \text{for all } n. \tag{9.19}$$

Thus, the mean is constant.

The autocovariance function is obtained from Eq. (9.6) as follows. If $n_1 \ne n_2$, then

$$\begin{aligned} C_X(n_1, n_2) &= E[(X_{n_1} - m)(X_{n_2} - m)] \\ &= E[(X_{n_1} - m)]E[(X_{n_2} - m)] = 0, \end{aligned}$$

since X_{n_1} and X_{n_2} are independent random variables. If $n_1 = n_2 = n$, then

$$C_X(n_1, n_2) = E[(X_n - m)^2] = \sigma^2.$$

We can express the *autocovariance of the iid process* in compact form as follows:

$$C_X(n_1, n_2) = \sigma^2 \delta_{n_1 n_2}, \tag{9.20}$$

where $\delta_{n_1 n_2} = 1$ if $n_1 = n_2$, and 0 otherwise. Therefore the autocovariance function is zero everywhere except for $n_1 = n_2$. The *autocorrelation function of the iid process* is found from Eq. (9.7):

$$R_X(n_1, n_2) = C_X(n_1, n_2) + m^2. \tag{9.21}$$

Kronecker y la puta que te parió

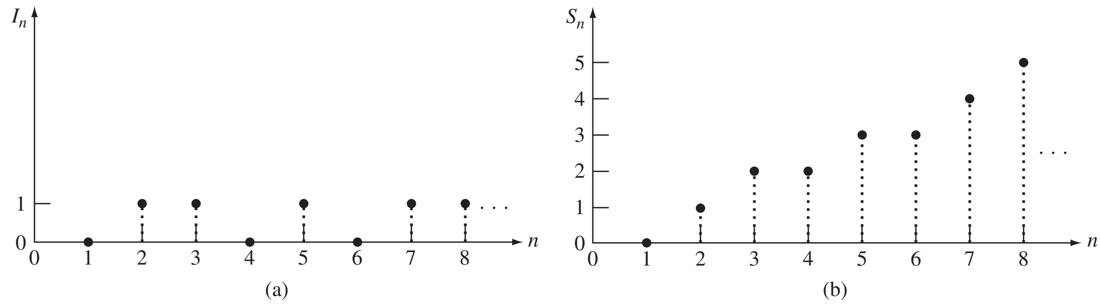

FIGURE 9.4
(a) Realization of a Bernoulli process. $I_n = 1$ indicates that a light bulb fails and is replaced on day n. (b) Realization of a binomial process. S_n denotes the number of light bulbs that have failed up to time n.

Example 9.13 Bernoulli Random Process

Let I_n be a sequence of independent Bernoulli random variables. I_n is then an iid random process taking on values from the set $\{0, 1\}$. A realization of such a process is shown in Fig. 9.4(a). For example, I_n could be an indicator function for the event "a light bulb fails and is replaced on day n."

Since I_n is a Bernoulli random variable, it has mean and variance

$$m_I(n) = p \qquad \text{VAR}[I_n] = p(1 - p).$$

The independence of the I_n's makes probabilities easy to compute. For example, the probability that the first four bits in the sequence are 1001 is

$$\begin{aligned} &P[I_1 = 1, I_2 = 0, I_3 = 0, I_4 = 1] \\ &\quad = P[I_1 = 1]P[I_2 = 0]P[I_3 = 0]P[I_4 = 1] \\ &\quad = p^2(1 - p)^2. \end{aligned}$$

$p^k(1-p)^m$

Donde k es la cantidad de veces que el evento ocurre, y m la cantidad de veces que ocurre lo contrario

Similarly, the probability that the second bit is 0 and the seventh is 1 is

$$P[I_2 = 0, I_7 = 1] = P[I_2 = 0]P[I_7 = 1] = p(1 - p).$$

Example 9.14 Random Step Process

An up-down counter is driven by +1 or −1 pulses. Let the input to the counter be given by $D_n = 2I_n - 1$, where I_n is the Bernoulli random process, then

$$D_n = \begin{cases} +1 & \text{if } I_n = 1 \\ -1 & \text{if } I_n = 0. \end{cases}$$

For example, D_n might represent the change in position of a particle that moves along a straight line in jumps of ± 1 every time unit. A realization of D_n is shown in Fig. 9.5(a).

¿Experimento de Galton?

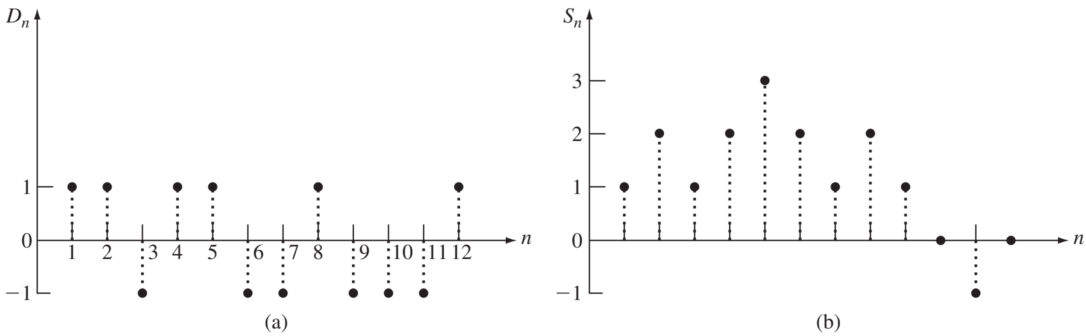

FIGURE 9.5
(a) Realization of a random step process. $D_n = 1$ implies that the particle moves one step to the right at time n. (b) Realization of a random walk process. S_n denotes the position of a particle at time n.

The mean of D_n is

$$m_D(n) = E[D_n] = E[2I_n - 1] = 2E[I_n] - 1 = 2p - 1.$$

The variance of D_n is found from Eqs. (4.37) and (4.38):

$$\text{VAR}[D_n] = \text{VAR}[2I_n - 1] = 2^2\,\text{VAR}[I_n] = 4p(1 - p).$$

The probabilities of events involving D_n are computed as in Example 9.13.

9.3.2 Independent Increments and Markov Properties of Random Processes

Before proceeding to build random processes from iid processes, we present two very useful properties of random processes. Let $X(t)$ be a random process and consider two time instants, $t_1 < t_2$. The increment of the random process in the interval $t_1 < t \le t_2$ is defined as $X(t_2) - X(t_1)$. *A random process $X(t)$ is said to have* ***independent increments*** *if the increments in disjoint intervals are independent random variables*, that is, for any k and any choice of sampling instants $t_1 < t_2 < \cdots < t_k$, the associated increments

Cuánto aumenta X(t) entre t4 y t3 es independiente de cuánto aumentó X(t) entre t2 y t1

$$X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_k) - X(t_{k-1})$$

are independent random variables. In the next subsection, we show that the joint pdf (pmf) of $X(t_1), X(t_2), \ldots, X(t_k)$ is given by the product of the pdf (pmf) of $X(t_1)$ and the marginal pdf's (pmf's) of the individual increments.

Another useful property of random processes that allows us to readily obtain the joint probabilities is the Markov property. *A random process $X(t)$ is said to be* ***Markov*** *if the future of the process given the present is independent of the past;* that is, for any k and any choice of sampling instants $t_1 < t_2 < \cdots < t_k$ and for any $x_1, x_2, \ldots, x_k$,

Presente

$$f_{X(t_k)}(x_k \,|\, \boxed{X(t_{k-1}) = x_{k-1}}, \boxed{\ldots, X(t_1) = x_1})$$
$$= f_{X(t_k)}(x_k \,|\, X(t_{k-1}) = x_{k-1}) \qquad (9.22)$$

futuro

Pasado

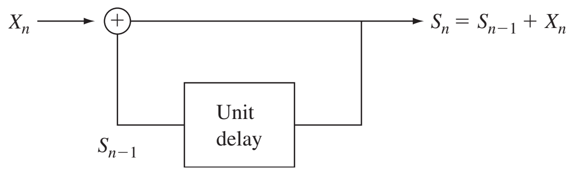

FIGURE 9.6
The sum process $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, can be generated in this way.

if $X(t)$ is continuous-valued, and

$$\begin{aligned} P[X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}, \ldots, X(t_1) = x_1] \\ = P[X(t_k) = x_k \mid X(t_{k-1}) = x_{k-1}] \end{aligned} \tag{9.23}$$

No parece importante

if $X(t)$ is discrete-valued. The expressions on the right-hand side of the above two equations are called the transition pdf and transition pmf, respectively. In the next sections we encounter several processes that satisfy the Markov property. Chapter 11 is entirely devoted to random processes that satisfy this property.

It is easy to show that a random process that has independent increments is also a Markov process. The converse is not true; that is, the Markov property does not imply independent increments. → Curioso, pero me fui de tema

9.3.3 Sum Processes: The Binomial Counting and Random Walk Processes

Many interesting random processes are obtained as the sum of a sequence of iid random variables, $X_1, X_2, \ldots$:

$$\begin{aligned} S_n &= X_1 + X_2 + \cdots + X_n \qquad n = 1, 2, \ldots \\ &= S_{n-1} + X_n, \end{aligned} \tag{9.24}$$

where $S_0 = 0$. We call S_n the **sum process**. The pdf or pmf of S_n is found using the convolution or characteristic-equation methods presented in Section 7.1. Note that S_n depends on the "past," $S_1, \ldots, S_{n-1}$, only through S_{n-1}, that is, S_n is independent of the past when S_{n-1} is known. This can be seen clearly from Fig. 9.6, which shows a recursive procedure for computing S_n in terms of S_{n-1} and the increment X_n. Thus *S_n is a Markov process*.

Example 9.15 Binomial Counting Process

Let the I_i be the sequence of independent Bernoulli random variables in Example 9.13, and let S_n be the corresponding sum process. S_n is then the *counting process* that gives the number of successes in the first n Bernoulli trials. The sample function for S_n corresponding to a particular sequence of I_i's is shown in Fig. 9.4(b). Note that the counting process can only increase over time. Note as well that the binomial process can increase by at most one unit at a time. If I_n indicates that a light bulb fails and is replaced on day n, then S_n denotes the number of light bulbs that have failed up to day n.

Since S_n is the sum of n independent Bernoulli random variables, S_n is a binomial random variable with parameters n and $p = P[I = 1]$:

$$P[S_n = j] = \binom{n}{j} p^j (1 - p)^{n-j} \qquad \text{for } 0 \le j \le n,$$

and zero otherwise. Thus S_n has mean np and variance $np(1 - p)$. Note that the mean and variance of this process grow linearly with time. This reflects the fact that as time progresses, that is, as n grows, the range of values that can be assumed by the process increases. If $p > 0$ then we also know that S_n has a tendency to grow steadily without bound over time.

The Markov property of the binomial counting process is easy to deduce. Given that the current value of the process at time $n - 1$ is $S_{n-1} = k$, the process at the next time instant will be k with probability $1 - p$ or $k + 1$ with probability p. Once we know the value of the process at time $n - 1$, the values of the random process prior to time $n - 1$ are irrelevant.

Example 9.16 One-Dimensional Random Walk

Let D_n be the iid process of ± 1 random variables in Example 9.14, and let S_n be the corresponding sum process. S_n can represent the position of a particle at time n. The random process S_n is an example of a **one-dimensional random walk**. A sample function of S_n is shown in Fig. 9.5(b). Unlike the binomial process, the random walk can increase or decrease over time. The random walk process changes by one unit at a time.

The pmf of S_n is found as follows. If there are k "+1"s in the first n trials, then there are $n - k$ "−1"s, and $S_n = k - (n - k) = 2k - n$. Conversely, $S_n = j$ if the number of +1's is $k = (j + n)/2$. If $(j + n)/2$ is not an integer, then S_n cannot equal j. Thus

KHÉ?

$$P[S_n = 2k - n] = \binom{n}{k} p^k (1 - p)^{n-k} \qquad \text{for } k \in \{0, 1, \ldots, n\}.$$

Since k is the number of successes in n Bernoulli trials, the mean of the random walk is:

$$E[S_n] = 2np - n = n(2p - 1).$$

As time progresses, the random walk can fluctuate over an increasingly broader range of positive and negative values. S_n has a tendency to either grow if $p > 1/2$, or to decrease if $p < 1/2$. The case $p = 1/2$ provides a precarious balance, and we will see later, in Chapter 12, very interesting dynamics. Figure 9.7(a) shows the first 100 steps from a sample function of the random walk with $p = 1/2$. Figure 9.7(b) shows four sample functions of the random walk process with $p = 1/2$ for 1000 steps. Figure 9.7(c) shows four sample functions in the asymmetric case where $p = 3/4$. Note the strong linear growth trend in the process.

The sum process S_n has **independent increments** in nonoverlapping time intervals. To see this consider two time intervals: $n_0 < n \le n_1$ and $n_2 < n \le n_3$, where $n_1 \le n_2$. The increments of S_n in these disjoint time intervals are given by

$$\begin{aligned} S_{n_1} - S_{n_0} &= X_{n_0+1} + \cdots + X_{n_1} \\ S_{n_3} - S_{n_2} &= X_{n_2+1} + \cdots + X_{n_3}. \end{aligned} \tag{9.25}$$

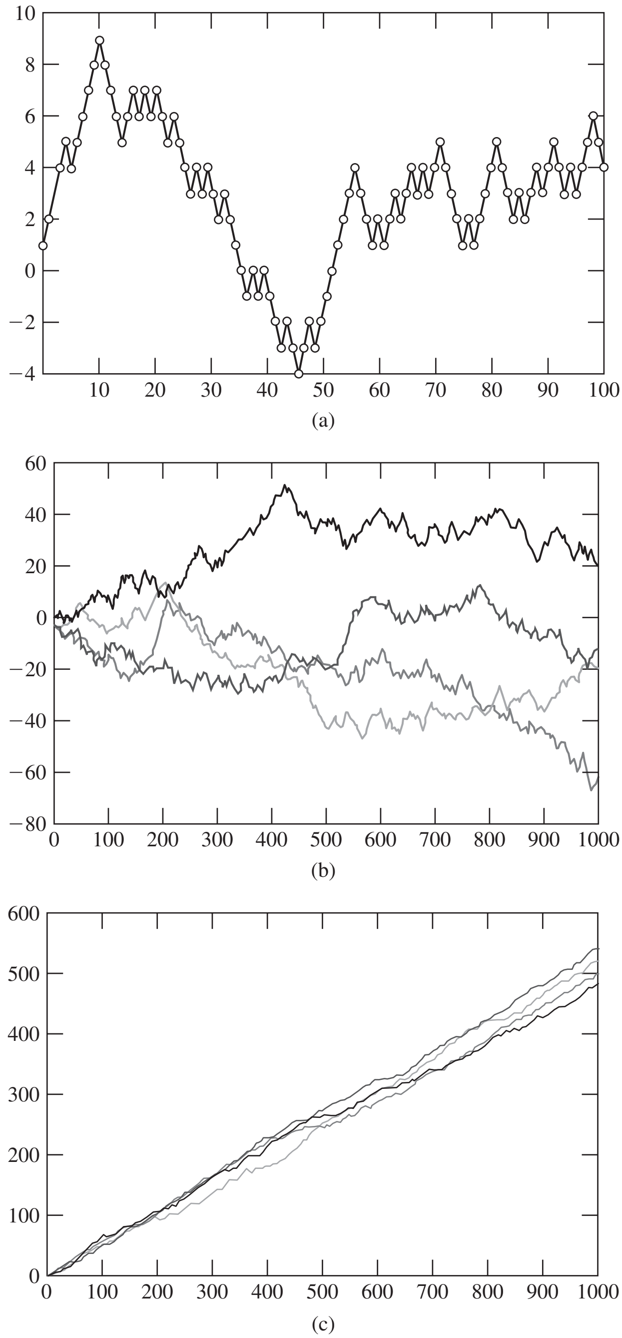

FIGURE 9.7
(a) Random walk process with $p = 1/2$. (b) Four sample functions of symmetric random walk process with $p = 1/2$. (c) Four sample functions of asymmetric random walk with $p = 3/4$.

The above increments do not have any of the X_n's in common, so the independence of the X_n's implies that the increments $(S_{n_1} - S_{n_0})$ and $(S_{n_3} - S_{n_2})$ are independent random variables.

For $n' > n$, the increment $S_{n'} - S_n$ is the sum of $n' - n$ iid random variables, so it has the same distribution as $S_{n'-n}$, the sum of the first $n' - n$ X's, that is,

$$P[S_{n'} - S_n = y] = P[S_{n'-n} = y]. \tag{9.26}$$

Thus increments in intervals of the same length have the same distribution regardless of when the interval begins. For this reason, we also say that S_n has **stationary increments**.

Example 9.17 Independent and Stationary Increments of Binomial Process and Random Walk

The independent and stationary increments property is particularly easy to see for the binomial process since the increments in an interval are the number of successes in the corresponding Bernoulli trials. The independent increment property follows from the fact that the numbers of successes in disjoint time intervals are independent. The stationary increments property follows from the fact that the pmf for the increment in a time interval is the binomial pmf with the corresponding number of trials.

The increment in a random walk process is determined by the same number of successes as a binomial process. It then follows that the random walk also has independent and stationary increments.

The independent and stationary increments property of the sum process S_n makes it easy to compute the joint pmf/pdf for any number of time instants. For simplicity, suppose that the X_n are integer-valued, so S_n is also integer-valued. We compute the joint pmf of S_n at times n_1, n_2, and n_3:

$$\begin{aligned} P[S_{n_1} &= y_1, S_{n_2} = y_2, S_{n_3} = y_3] \\ &= P[S_{n_1} = y_1, S_{n_2} - S_{n_1} = y_2 - y_1, S_{n_3} - S_{n_2} = y_3 - y_2], \end{aligned} \tag{9.27}$$

since the process is equal to y_1, y_2, and y_3 at times n_1, n_2, and n_3, if and only if it is equal to y_1 at time n_1, and the subsequent increments are $y_2 - y_1$, and $y_3 - y_2$. The independent increments property then implies that

$$\begin{aligned} P[S_{n_1} &= y_1, S_{n_2} = y_2, S_{n_3} = y_3] \\ &= P[S_{n_1} = y_1]P[S_{n_2} - S_{n_1} = y_2 - y_1]P[S_{n_3} - S_{n_2} = y_3 - y_2]. \end{aligned} \tag{9.28}$$

Finally, the stationary increments property implies that the *joint pmf of S_n is given by*:

$$\begin{aligned} P[S_{n_1} &= y_1, S_{n_2} = y_2, S_{n_3} = y_3] \\ &= P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1]P[S_{n_3-n_2} = y_3 - y_2]. \end{aligned}$$

Clearly, we can use this procedure to write the joint pmf of S_n at any time instants $n_1 < n_2 < \cdots < n_k$ in terms of the pmf at the initial time instant and the pmf's of the subsequent increments:

La propiedad de incrementos independientes en intervalos disjuntos simplifica todo con un poco de viveza

$$P[S_{n_1} = y_1, S_{n_2} = y_2, \ldots, S_{n_k} = y_k]$$
$$= P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1]\ldots P[S_{n_k-n_{k-1}} = y_k - y_{k-1}]. \tag{9.29}$$

If the X_n are continuous-valued random variables, then it can be shown that *the joint density of S_n at times $n_1, n_2, \ldots, n_k$* is:

$$f_{S_{n_1},S_{n_2},\ldots,S_{n_k}}(y_1, y_2, \ldots, y_k) = f_{S_{n_1}}(y_1)f_{S_{n_2-n_1}}(y_2 - y_1)\ldots f_{S_{n_k-n_{k-1}}}(y_k - y_{k-1}). \tag{9.30}$$

Example 9.18 Joint pmf of Binomial Counting Process

Find the joint pmf for the binomial counting process at times n_1 and n_2. Find the probability that $P[S_{n_1} = 0, S_{n_2} = n_2 - n_1]$, that is, the first n_1 trials are failures and the remaining trials are all successes.

Following the above approach we have

$$\begin{aligned} P[S_{n_1} = y_1, S_{n_2} = y_2] &= P[S_{n_1} = y_1]P[S_{n_2} - S_{n_1} = y_2 - y_1] \\ &= \binom{n_2 - n_1}{y_2 - y_1}p^{y_2-y_1}(1-p)^{n_2-n_1-y_2+y_1}\binom{n_1}{y_1}p^{y_1}(1-p)^{n_1-y_1} \\ &= \binom{n_2 - n_1}{y_2 - y_1}\binom{n_1}{y_1}p^{y_2}(1-p)^{n_2-y_2}. \end{aligned}$$

¿Y para qué damos tantas vueltas?

The requested probability is then:

$$P[S_{n_1} = 0, S_{n_2} = n_2 - n_1] = \binom{n_2 - n_1}{n_2 - n_1}\binom{n_1}{0}p^{n_2-n_1}(1-p)^{n_1} = p^{n_2-n_1}(1-p)^{n_1}$$

which is what we would obtain from a direct calculation for Bernoulli trials.

Example 9.19 Joint pdf of Sum of iid Gaussian Sequence

Let X_n be a sequence of iid Gaussian random variables with zero mean and variance σ^2. Find the joint pdf of the corresponding sum process at times n_1 and n_2.

From Example 7.3, we know that S_n is a Gaussian random variable with mean zero and variance $n\sigma^2$. The joint pdf of S_n at times n_j and n_2 is given by

$$\begin{aligned} f_{S_{n_1},S_{n_2}}(y_1, y_2) &= f_{S_{n_2-n_1}}(y_2 - y_1)f_{S_{n_1}}(y_1) \\ &= \frac{1}{\sqrt{2\pi(n_2 - n_1)\sigma^2}}e^{-(y_2-y_1)^2/[2(n_2-n_1)\sigma^2]}\frac{1}{\sqrt{2\pi n_1\sigma^2}}e^{-y_1^2/2n_1\sigma^2}. \end{aligned}$$

Since the sum process S_n is the sum of n iid random variables, it has mean and variance:

$$m_S(n) = E[S_n] = nE[X] = nm \tag{9.31}$$
$$\text{VAR}[S_n] = n\,\text{VAR}[X] = n\sigma^2. \tag{9.32}$$

OJO Con este HDP

The property of independent increments allows us to compute the autocovariance in an interesting way. Suppose $n \le k$ so $n = \min(n, k)$, then

$$\begin{aligned} C_S(n,k) &= E[(S_n - nm)(S_k - km)] \\ &= E[(S_n - nm)\{(S_n - nm) + (S_k - km) - (S_n - nm)\}] \\ &= E[(S_n - nm)^2] + E[(S_n - nm)(S_k - S_n - (k - n)m)]. \end{aligned}$$

Since S_n and the increment $S_k - S_n$ are independent,

$$\begin{aligned} C_S(n,k) &= E[(S_n - nm)^2] + \underbrace{E[(S_n - nm)]}_{=0}E[(S_k - S_n - (k - n)m)] \\ &= E[(S_n - nm)^2] \\ &= \text{VAR}[S_n] = n\sigma^2, \end{aligned}$$

since $E[S_n - nm] = 0$. Similarly, if $k = \min(n, k)$, we would have obtained $k\sigma^2$. Therefore the *autocovariance of the sum process* is

(asumiendo n<k) S en k NO es indep de S en n.

$$C_S(n,k) = \min(n,k)\sigma^2. \tag{9.33}$$

Mientras mayor sea Sn, mayor será Sk, de ahí que la autocovarianza del proceso crezca con n.

Example 9.20 Autocovariance of Random Walk

Find the autocovariance of the one-dimensional random walk.

From Example 9.14 and Eqs. (9.32) and (9.33), S_n has mean $n(2p - 1)$ and variance $4np(1 - p)$. Thus its autocovariance is given by

$$C_s(n,k) = \min(n,k)4p(1 - p).$$

TURBIO ES POCO

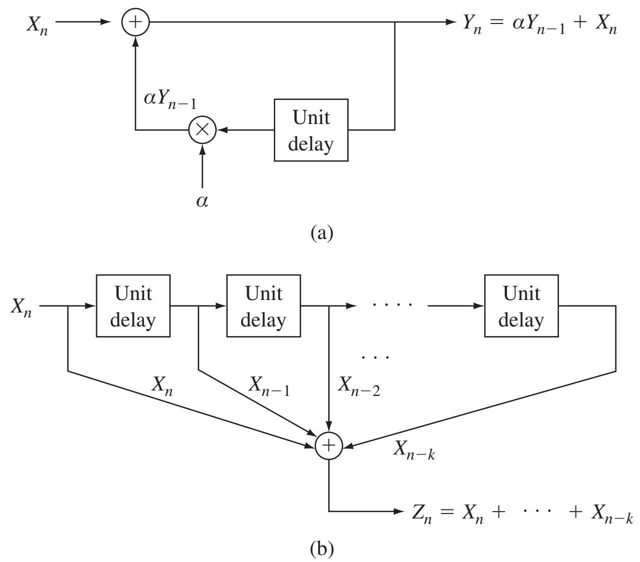

FIGURE 9.8
(a) First-order autoregressive process; (b) Moving average process.

The sum process can be generalized in a number of ways. For example, the recursive structure in Fig. 9.6 can be modified as shown in Fig. 9.8(a). We then obtain first-order *autoregressive random processes*, which are of interest in time series analysis and in digital signal processing. If instead we use the structure shown in Fig. 9.8(b), we obtain an example of a *moving average process.* We investigate these processes in Chapter 10.

9.4 POISSON AND ASSOCIATED RANDOM PROCESSES → 15/4 → hasta acá llegó el amor.

In this section we develop the Poisson random process, which plays an important role in models that involve counting of events and that find application in areas such as queueing systems and reliability analysis. We show how the continuous-time Poisson random process can be obtained as the limit of a discrete-time process. We also introduce several random processes that are derived from the Poisson process.

9.4.1 Poisson Process

Consider a situation in which events occur at random instants of time at an average rate of λ events per second. For example, an event could represent the arrival of a customer to a service station or the breakdown of a component in some system. Let $N(t)$ be the number of event occurrences in the time interval $[0, t]$. $N(t)$ is then a nondecreasing, integer-valued, continuous-time random process as shown in Fig. 9.9.

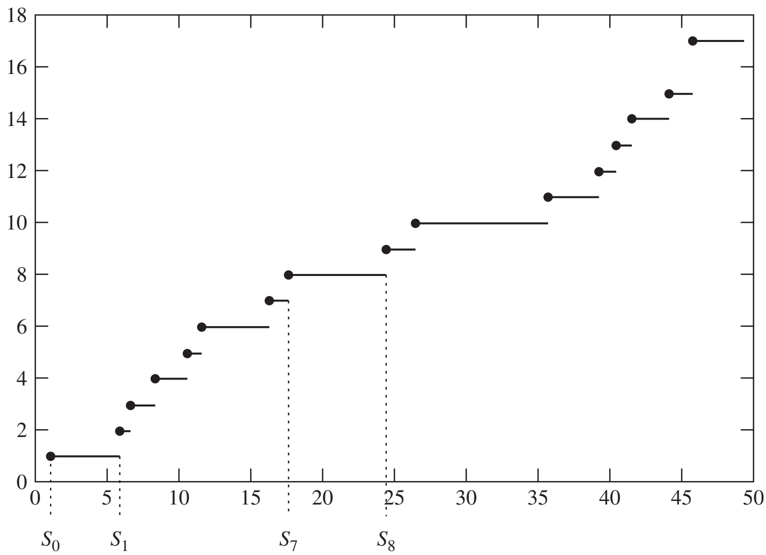

FIGURE 9.9
A sample path of the Poisson counting process. The event occurrence times are denoted by $S_1, S_2, \ldots$. The jth interevent time is denoted by $X_j = S_j - S_{j-1}$.

Suppose that the interval $[0, t]$ is divided into n subintervals of very short duration $\delta = t/n$. Assume that the following two conditions hold:

1. The probability of more than one event occurrence in a subinterval is negligible compared to the probability of observing one or zero events.
2. Whether or not an event occurs in a subinterval is independent of the outcomes in other subintervals.

The first assumption implies that the outcome in each subinterval can be viewed as a Bernoulli trial. The second assumption implies that these Bernoulli trials are independent. The two assumptions together imply that the counting process $N(t)$ *can be approximated by the binomial counting process* discussed in the previous section.

If the probability of an event occurrence in each subinterval is p, then the expected number of event occurrences in the interval $[0, t]$ is np. Since events occur at a rate of λ events per second, the average number of events in the interval $[0, t]$ is λt. Thus we must have that

$$\lambda t = np.$$

If we now let $n \to \infty$ (i.e., $\delta = t/n \to 0$) and $p \to 0$ while $np = \lambda t$ remains fixed, then from Eq. (3.40) the binomial distribution approaches a Poisson distribution with parameter λt. We therefore conclude that the number of event occurrences $N(t)$ in the interval $[0, t]$ has a Poisson distribution with mean λt:

$$P[N(t) = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad \text{for } k = 0, 1, \ldots. \tag{9.34a}$$

For this reason $N(t)$ is called the **Poisson process**. The mean function and the variance function of the Poisson process are given by:

$$m_N(t) = E[N(t) = k] = \lambda t \quad \text{and} \quad \text{VAR}[N(t)] = \lambda t. \tag{9.34b}$$

In Section 11.3 we rederive the Poisson process using results from Markov chain theory.

The process $N(t)$ inherits the property of independent and stationary increments from the underlying binomial process. First, *the distribution for the number of event occurrences in any interval of length t is given by Eq. (9.34a)*. Next, the independent and stationary increments property allows us to write the joint pmf for $N(t)$ at any number of points. For example, for $t_1 < t_2$,

$$\begin{aligned} P[N(t_1) = i, N(t_2) = j] &= P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] \\ &= P[N(t_1) = i]P[N(t_2 - t_1) = j - i] \\ &= \frac{(\lambda t_1)^i e^{-\lambda t_1}}{i!} \frac{(\lambda(t_2 - t_1))^j e^{-\lambda(t_2 - t_1)}}{(j - i)!}. \end{aligned} \tag{9.35a}$$

The independent increments property also allows us to calculate the autocovariance of $N(t)$. For $t_1 \le t_2$:

$$
\begin{aligned}
C_N(t_1, t_2) &= E[(N(t_1) - \lambda t_1)(N(t_2) - \lambda t_2)] \\
&= E[(N(t_1) - \lambda t_1)\{N(t_2) - N(t_1) - \lambda t_2 + \lambda t_1 + (N(t_1) - \lambda t_1)\}] \\
&= E[(N(t_1) - \lambda t_1)]E[(N(t_2) - N(t_1) - \lambda(t_2 - t_1)] + \text{VAR}[N(t_1)] \\
&= \text{VAR}[N(t_1)] = \lambda t_1. \qquad (9.35\text{b})
\end{aligned}
$$

Example 9.21

Inquiries arrive at a recorded message device according to a Poisson process of rate 15 inquiries per minute. Find the probability that in a 1-minute period, 3 inquiries arrive during the first 10 seconds and 2 inquiries arrive during the last 15 seconds.

The arrival rate in seconds is $\lambda = 15/60 = 1/4$ inquiries per second. Writing time in seconds, the probability of interest is

$$P[N(10) = 3 \text{ and } N(60) - N(45) = 2].$$

By applying first the independent increments property, and then the stationary increments property, we obtain

$$
\begin{aligned}
P[N(10) = 3 \text{ and } N(60) - N(45) = 2] \\
&= P[N(10) = 3]P[N(60) - N(45) = 2] \\
&= P[N(10) = 3]P[N(60 - 45) = 2] \\
&= \frac{(10/4)^3 e^{-10/4}}{3!} \frac{(15/4)^2 e^{-15/4}}{2!}.
\end{aligned}
$$

Consider the time T between event occurrences in a Poisson process. Again suppose that the time interval $[0, t]$ is divided into n subintervals of length $\delta = t/n$. The probability that the interevent time T exceeds t seconds is equivalent to no event occurring in t seconds (or in n Bernoulli trials):

$$
\begin{aligned}
P[T > t] &= P[\text{no events in } t \text{ seconds}] \\
&= (1 - p)^n \\
&= \left(1 - \frac{\lambda t}{n}\right)^n \\
&\rightarrow e^{-\lambda t} \qquad \text{as } n \rightarrow \infty. \qquad (9.36)
\end{aligned}
$$

Equation (9.36) implies that T is an exponential random variable with parameter λ. Since the times between event occurrences in the underlying binomial process are independent geometric random variables, it follows that the sequence of interevent times in a Poisson process is composed of independent random variables. We therefore conclude that the *interevent times in a Poisson process form an iid sequence of exponential random variables with mean* $1/\lambda$.

Another quantity of interest is the time S_n at which the nth event occurs in a Poisson process. Let T_j denote the iid exponential interarrival times, then

$$S_n = T_1 + T_2 + \cdots + T_n.$$

In Example 7.5, we saw that the sum of n iid exponential random variables has an Erlang distribution. Thus the pdf of S_n *is an Erlang random variable*:

$$f_{S_n}(y) = \frac{(\lambda y)^{n-1}}{(n-1)!}\lambda e^{-\lambda y} \qquad \text{for } y \geq 0. \tag{9.37}$$

Example 9.22

Find the mean and variance of the time until the tenth inquiry in Example 9.20.

The arrival rate is $\lambda = 1/4$ inquiries per second, so the interarrival times are exponential random variables with parameter λ. From Table 4.1, the mean and variance of exponential interarrival times then $1/\lambda$ and $1/\lambda^2$, respectively. The time of the tenth arrival is the sum of ten such iid random variables, thus

$$E[S_{10}] = 10E[T] = \frac{10}{\lambda} = 40 \text{ sec}$$
$$\text{VAR}[S_{10}] = 10\,\text{VAR}[T] = \frac{10}{\lambda^2} = 160 \text{ sec}^2.$$

In applications where the Poisson process models customer interarrival times, it is customary to say that arrivals occur "at random." We now explain what is meant by this statement. Suppose that we are *given* that only one arrival occurred in an interval $[0, t]$ and we let X be the arrival time of the single customer. For $0 < x < t$, $N(x)$ is the number of events up to time x, and $N(t) - N(x)$ is the increment in the interval $(x, t]$, then:

$$\begin{aligned}
P[X \leq x] &= P[N(x) = 1 \,|\, N(t) = 1] \\
&= \frac{P[N(x) = 1 \text{ and } N(t) = 1]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1 \text{ and } N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{P[N(x) = 1]P[N(t) - N(x) = 0]}{P[N(t) = 1]} \\
&= \frac{\lambda x e^{-\lambda x} e^{-\lambda(t-x)}}{\lambda t e^{-\lambda t}} \\
&= \frac{x}{t}.
\end{aligned} \tag{9.38}$$

Equation (9.38) implies that given that one arrival has occurred in the interval $[0, t]$, then the customer arrival time is uniformly distributed in the interval $[0, t]$. It is in this sense that customer arrival times occur "at random." It can be shown that *if the number of amvals in the interval* $[0, t]$ *is k, then the individual arrival times are distributed independently and uniformly in the interval.*

Example 9.23

Suppose two customers arrive at a shop during a two-minute period. Find the probability that both customers arrived during the first minute.

The arrival times of the customers are independent and uniformly distributed in the two-minute interval. Each customer arrives during the first minute with probability 1/2. Thus the probability that both arrive during the first minute is $(1/2)^2 = 1/4$. This answer can be verified by showing that $P[N(1) = 2 \mid N(2) = 2] = 1/4$.

9.4.2 Random Telegraph Signal and Other Processes Derived from the Poisson Process

Many processes are derived from the Poisson process. In this section, we present two examples of such random processes.

Example 9.24 Random Telegraph Signal

Consider a random process $X(t)$ that assumes the values ± 1. Suppose that $X(0) = +1$ or -1 with probability 1/2, and suppose that $X(t)$ changes polarity with each occurrence of an event in a Poisson process of rate α. Figure 9.10 shows a sample function of $X(t)$.

The pmf of $X(t)$ is given by

$$\begin{aligned} P[X(t) = \pm 1] = P[X(t) &= \pm 1 \mid X(0) = 1]P[X(0) = 1] \\ &+ P[X(t) = \pm 1 \mid X(0) = -1]P[X(0) = -1]. \end{aligned} \tag{9.39}$$

The conditional pmf's are found by noting that $X(t)$ will have the same polarity as $X(0)$ only when an even number of events occur in the interval $(0, t]$. Thus

$$\begin{aligned} P[X(t) = \pm 1 \mid X(0) = \pm 1] &= P[N(t) = \text{even integer}] \\ &= \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j}}{(2j)!} e^{-\alpha t} \\ &= e^{-\alpha t} \frac{1}{2}\{e^{\alpha t} + e^{-\alpha t}\} \\ &= \frac{1}{2}(1 + e^{-2\alpha t}). \end{aligned} \tag{9.40}$$

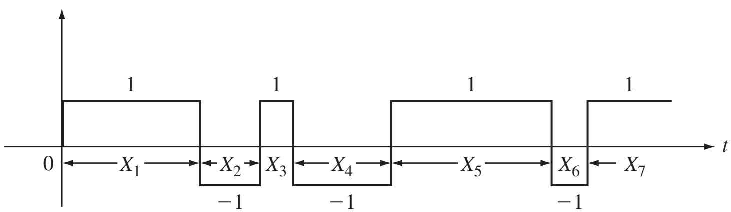

FIGURE 9.10
Sample path of a random telegraph signal. The times between transitions X_j are iid exponential random variables.

$X(t)$ and $X(0)$ will differ in sign if the number of events in t is odd:

$$P[X(t) = \pm 1 \,|\, X(0) = \mp 1] = \sum_{j=0}^{\infty} \frac{(\alpha t)^{2j+1}}{(2j+1)!} e^{-\alpha t}$$
$$= e^{-\alpha t} \frac{1}{2}\{e^{\alpha t} - e^{-\alpha t}\}$$
$$= \frac{1}{2}(1 - e^{-2\alpha t}). \tag{9.41}$$

We obtain the pmf for $X(t)$ by substituting into Eq. (9.40):

$$P[X(t) = 1] = \frac{1}{2}\frac{1}{2}\{1 + e^{-2\alpha t}\} + \frac{1}{2}\frac{1}{2}\{1 - e^{-2\alpha t}\} = \frac{1}{2}$$
$$P[X(t) = -1] = 1 - P[X(t) = 1] = \frac{1}{2}. \tag{9.42}$$

Thus the random telegraph signal is equally likely to be ± 1 at any time $t > 0$.

The mean and variance of $X(t)$ are

$$m_X(t) = 1P[X(t) = 1] + (-1)P[X(t) = -1] = 0$$
$$\text{VAR}[X(t)] = E[X(t)^2] = (1^2)P[X(t) = 1] + (-1)^2 P[X(t) = -1] = 1. \tag{9.43}$$

The autocovariance of $X(t)$ is found as follows:

$$C_X(t_1, t_2) = E[X(t_1)X(t_2)]$$
$$= 1P[X(t_1) = X(t_2)] + (-1)P[X(t_1) \neq X(t_2)]$$
$$= \frac{1}{2}\{1 + e^{-2\alpha|t_2 - t_1|}\} - \frac{1}{2}\{1 - e^{-2\alpha|t_2 - t_1|}\}$$
$$= e^{-2\alpha|t_2 - t_1|}. \tag{9.44}$$

Thus time samples of $X(t)$ become less and less correlated as the time between them increases.

The Poisson process and the random telegraph processes are examples of the continuous-time Markov chain processes that are discussed in Chapter 11.

Example 9.25 Filtered Poisson Impulse Train

The Poisson process is zero at $t = 0$ and increases by one unit at the random arrival times $S_j, j = 1, 2, \ldots$. Thus the Poisson process can be expressed as the sum of randomly shifted step functions:

$$N(t) = \sum_{i=1}^{\infty} u(t - S_i) \qquad N(0) = 0,$$

where the S_i are the arrival times.

Since the integral of a delta function $\delta(t - S)$ is a step function $u(t - S)$, we can view $N(t)$ as the result of integrating a train of delta functions that occur at times S_n, as shown in Fig. 9.11(a):

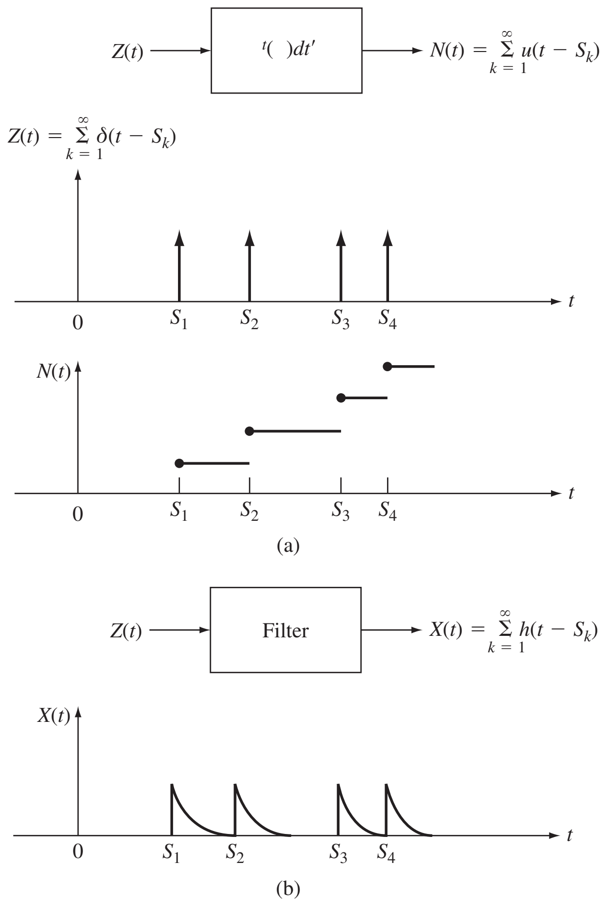

FIGURE 9.11
(a) Poisson process as integral of train of delta functions. (b) Filtered train of delta functions.

$$Z(t) = \sum_{i=1}^{\infty} \delta(t - S_i).$$

We can obtain other continuous-time processes by replacing the step function by another function $h(t)$,[1] as shown in Fig. 9.11(b):

$$X(t) = \sum_{i=1}^{\infty} h(t - S_i). \tag{9.45}$$

For example, $h(t)$ could represent the current pulse that results when a photoelectron hits a detector. $X(t)$ is then the total current flowing at time t. $X(t)$ is called a **shot noise** process.

[1]This is equivalent to passing $Z(t)$ through a linear system whose response to a delta function is $h(t)$.

The following example shows how the properties of the Poisson process can be used to evaluate averages involving the filtered process.

Example 9.26 Mean of Shot Noise Process

Find the expected value of the shot noise process $X(t)$.

We condition on $N(t)$, the number of impulses that have occurred up to time t:

$$E[X(t)] = E[E[X(t)|N(t)]].$$

Suppose $N(t) = k$, then

$$\begin{aligned} E[X(t)|N(t) = k] &= E\left[\sum_{j=1}^{k} h(t - S_j)\right] \\ &= \sum_{j=1}^{k} E[h(t - S_j)]. \end{aligned}$$

Since the arrival times, $S_1, \ldots, S_k$, when the impulses occurred are independent, uniformly distributed in the interval $[0, t]$,

$$E[h(t - S_j)] = \int_0^t h(t - s)\frac{ds}{t} = \frac{1}{t}\int_0^t h(u)\,du.$$

Thus

$$E[X(t)|N(t) = k] = \frac{k}{t}\int_0^t h(u)\,du,$$

and

$$E[X(t)|N(t)] = \frac{N(t)}{t}\int_0^t h(u)\,du.$$

Finally, we obtain

$$\begin{aligned} E[X(t)] &= E[E[X(t)|N(t)]] \\ &= \frac{E[N(t)]}{t}\int_0^t h(u)\,du \\ &= \lambda\int_0^t h(u)\,du, \end{aligned} \tag{9.46}$$

where we used the fact that $E[N(t)] = \lambda t$. Note that $E[X(t)]$ approaches a constant value as t becomes large if the above integral is finite.

9.5 GAUSSIAN RANDOM PROCESSES, WIENER PROCESS, AND BROWNIAN MOTION

In this section we continue the introduction of important random processes. First, we introduce the class of Gaussian random processes which find many important applications in electrical engineering. We then develop an example of a Gaussian random process: the Wiener random process which is used to model Brownian motion.

9.5.1 Gaussian Random Processes

A random process $X(t)$ is a **Gaussian random process** if the samples $X_1 = X(t_1)$, $X_2 = X(t_2), \ldots, X_k = X(t_k)$ are jointly Gaussian random variables for all k, and all choices of $t_1, \ldots, t_k$. This definition applies to both discrete-time and continuous-time processes. Recall from Eq. (6.42) that the joint pdf of jointly Gaussian random variables is determined by the vector of means and by the covariance matrix:

$$f_{X_1, X_2, \ldots, X_k}(x_1, x_2, \ldots, x_k) = \frac{e^{-1/2(\mathbf{x}-\mathbf{m})^{\mathrm{T}}K^{-1}(\mathbf{x}-\mathbf{m})}}{(2\pi)^{k/2}|K|^{1/2}}. \tag{9.47a}$$

In the case of Gaussian random processes, the mean vector and the covariance matrix are the values of the mean function and covariance function at the corresponding time instants:

$$\mathbf{m} = \begin{bmatrix} m_X(t_1) \\ \vdots \\ m_X(t_k) \end{bmatrix} \qquad K = \begin{bmatrix} C_X(t_1, t_1) & C_X(t_1, t_2) & \cdots & C_X(t_1, t_k) \\ C_X(t_2, t_1) & C_X(t_2, t_2) & \cdots & C_X(t_2, t_k) \\ \vdots & \vdots & & \vdots \\ C_X(t_k, t_1) & \cdots & & C_X(t_k, t_k) \end{bmatrix}. \tag{9.47b}$$

Gaussian random processes therefore have the very special property that their joint pdf's are completely specified by the mean function of the process $m_X(t)$ and by the covariance function $C_X(t_1, t_2)$. In Chapter 6 we saw that the linear transformations of jointly Gaussian random vectors result in jointly Gaussian random vectors as well. We will see in Chapter 10 that Gaussian random processes also have the property that the linear operations on a Gaussian process (e.g., a sum, derivative, or integral) results in another Gaussian random process. These two properties, combined with the fact that many signal and noise processes are accurately modeled as Gaussian, make Gaussian random processes the most useful model in signal processing.

Example 9.27 iid Discrete-Time Gaussian Random Process

Let the discrete-time random process X_n be a sequence of independent Gaussian random variables with mean m and variance σ^2. The covariance matrix for the times $n_1, \ldots, n_k$ is

$$\{C_X(n_1, n_2)\} = \{\sigma^2\,\delta_{ij}\} = \sigma^2 I,$$

where $\delta_{ij} = 1$ when $i = j$ and 0 otherwise, and I is the identity matrix. Thus the joint pdf for the vector $\mathbf{X}_n = (X_{n_1}, \ldots, X_{n_k})$ is

$$f_{\mathbf{X}_n}(x_1, x_2, \ldots, x_k) = \frac{1}{(2\pi\sigma^2)^{k/2}} \exp\left\{-\sum_{i=1}^{k}(x_i - m)^2/2\sigma^2\right\}.$$

The Gaussian iid random process has the property that the value at every time instant is independent of the value at all other time instants.

Example 9.28 Continuous-Time Gaussian Random Process

Let $X(t)$ be a continuous-time Gaussian random process with mean function and covariance function given by:

$$m_X(t) = 3t \qquad C_X(t_1, t_2) = 9e^{-2|t_1 - t_2|}.$$

Find $P[X(3) < 6]$ and $P[X(1) + X(2) > 2]$.

The sample $X(3)$ has a Gaussian pdf with mean $m_X(3) = 3(3) = 9$ and variance $\sigma_X^2(3) = C_X(3, 3) = 9e^{-2|3-3|} = 9$. To calculate $P[X(3) < 6]$ we put $X(3)$ in standard form:

$$P[X(3) < 6] = P\left[\frac{X(3) - 9}{\sqrt{9}} < \frac{6 - 9}{\sqrt{9}}\right] = 1 - Q(-1) = Q(1) = 0.16.$$

From Example 6.24 we know that the sum of two Gaussian random variables is also a Gaussian random variable with mean and variance given by Eq. (6.47). Therefore the mean and variance of $X(1) + X(2)$ are given by:

$$E[X(1) + X(2)] = m_X(1) + m_X(2) = 3 + 6 = 9$$

$$\begin{aligned} \text{VAR}[X(1) + X(2)] &= C_X(1, 1) + C_X(1, 2) + C_X(2, 1) + C_X(2, 2) \\ &= 9\{e^{-2|1-1|} + e^{-2|2-1|} + e^{-2|1-2|} + e^{-2|2-2|}\} \\ &= 9\{2 + 2e^{-2}\} = 20.43. \end{aligned}$$

To calculate $P[X(1) + X(2) > 2]$ we put $X(1) + X(2)$ in standard form:

$$P[X(1) + X(2) > 15] = P\left[\frac{X(1) + X(2) - 9}{\sqrt{20.43}} > \frac{15 - 9}{\sqrt{20.43}}\right] = Q(1.327) = 0.0922.$$

9.5.2 Wiener Process and Brownian Motion

We now construct a continuous-time Gaussian random process as a limit of a discrete-time process. Suppose that the symmetric random walk process (i.e., $p = 1/2$) of Example 9.16 takes steps of magnitude $\pm h$ every δ seconds. We obtain a continuous-time process by letting $X_\delta(t)$ be the accumulated sum of the random step process up to time t. $X_\delta(t)$ is a staircase function of time that takes jumps of $\pm h$ every δ seconds. At time t, the process will have taken $n = [t/\delta]$ jumps, so it is equal to

$$X_\delta(t) = h(D_1 + D_2 + \cdots + D_{[t/\delta]}) = hS_n. \tag{9.48}$$

The mean and variance of $X_\delta(t)$ are

$$E[X_\delta(t)] = hE[S_n] = 0$$
$$\text{VAR}[X_\delta(t)] = h^2 n\, \text{VAR}[D_n] = h^2 n,$$

where we used the fact that $\text{VAR}[D_n] = 4p(1 - p) = 1$ since $p = 1/2$.

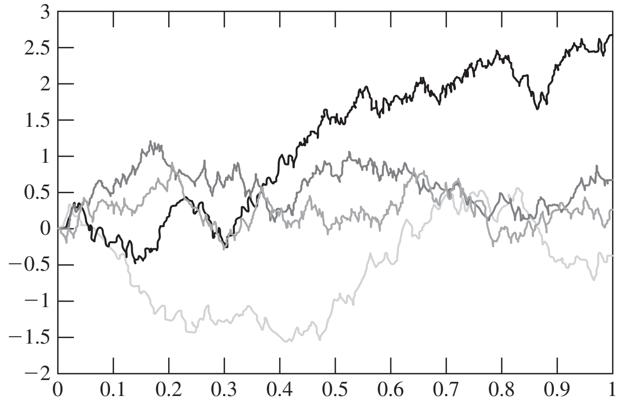

FIGURE 9.12
Four sample functions of the Wiener process.

Suppose that we take a limit where we simultaneously shrink the size of the jumps and the time between jumps. In particular let $\delta \to 0$ and $h \to 0$ with $h = \sqrt{\alpha\delta}$ and let $X(t)$ denote the resulting process.

$X(t)$ then has *mean* and *variance* given by

$$E[X(t)] = 0 \tag{9.49a}$$

$$\text{VAR}[X(t)] = (\sqrt{\alpha\delta})^2(t/\delta) = \alpha t. \tag{9.49b}$$

Thus we obtain a continuous-time process $X(t)$ that begins at the origin, has zero mean for all time, but has a variance that increases linearly with time. Figure 9.12 shows four sample functions of the process. Note the similarities in fluctuations to the realizations of a symmetric random walk in Fig. 9.7(b). $X(t)$ is called the **Wiener random process**. It is used to model *Brownian motion*, the motion of particles suspended in a fluid that move under the rapid and random impact of neighboring particles.

As $\delta \to 0$, Eq. (9.48) implies that $X(t)$ approaches the sum of an infinite number of random variables since $n = [t/\delta] \to \infty$:

$$X(t) = \lim_{\delta \to 0} hS_n = \lim_{n \to \infty} \sqrt{\alpha t}\frac{S_n}{\sqrt{n}}. \tag{9.50}$$

By the central limit theorem the pdf of $X(t)$ therefore approaches that of a Gaussian random variable with mean zero and variance αt:

$$f_{X(t)}(x) = \frac{1}{\sqrt{2\pi\alpha t}} e^{-x^2/2\alpha t}. \tag{9.51}$$

$X(t)$ inherits the property of independent and stationary increments from the random walk process from which it is derived. As a result, the *joint pdf of* $X(t)$ at several times $t_1, t_2, \ldots, t_k$ can be obtained by using Eq. (9.30):

$$f_{X(t_1),\ldots,X(t_k)}(x_1,\ldots,x_k) = f_{X(t_1)}(x_1)f_{X(t_2-t_1)}(x_2-x_1)\ldots f_{X(t_k-t_{k-1})}(x_k-x_{k-1})$$
$$= \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{\alpha t_1}+\frac{(x_2-x_1)^2}{\alpha(t_2-t_1)}+\cdots+\frac{(x_k-x_{k-1})^2}{\alpha(t_k-t_{k-1})}\right]\right\}}{\sqrt{(2\pi\alpha)^k t_1(t_2-t_1)\ldots(t_k-t_{k-1})}}. \quad (9.52)$$

The independent increments property and the same sequence of steps that led to Eq. (9.33) can be used to show that the *autocovariance of* $X(t)$ is given by

$$C_X(t_1, t_2) = \alpha \min(t_1, t_2) = \alpha\, t_1 \text{ for } t_1 < t_2. \quad (9.53)$$

By comparing Eq. (9.53) and Eq. (9.35b), we see that the Wiener process and the Poisson process have the same covariance function despite the fact that the two processes have very different sample functions. This underscores the fact that the mean and autocovariance functions are only partial descriptions of a random process.

Example 9.29

Show that the Wiener process is a Gaussian random process.

Equation (9.52) shows that the random variables $X(t_1), X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_k) - X(t_{k-1})$, are independent Gaussian random variables. The random variables $X(t_1), X(t_2), X(t_3), \ldots, X(t_k)$, can be obtained from the $X(t_1)$ and the increments by a linear transformation:

$$\begin{aligned} X(t_1) &= X(t_1) \\ X(t_2) &= X(t_1) + (X(t_2) - X(t_1)) \\ X(t_3) &= X(t_1) + (X(t_2) - X(t_1)) + (X(t_3) - X(t_2)) \\ &\vdots \\ X(t_k) &= X(t_1) + (X(t_2) - X(t_1)) + \cdots + (X(t_k) - X(t_{k-1})). \end{aligned} \quad (9.54)$$

It then follows (from Eq. 6.45) that $X(t_1), X(t_2), X(t_3), \ldots, X(t_k)$ are jointly Gaussian random variables, and that $X(t)$ is a Gaussian random process.

9.6 STATIONARY RANDOM PROCESSES

Many random processes have the property that the nature of the randomness in the process does not change with time. An observation of the process in the time interval (t_0, t_1) exhibits the same type of random behavior as an observation in some other time interval $(t_0 + \tau, t_1 + \tau)$. This leads us to postulate that the probabilities of samples of the process do not depend on the instant when we begin taking observations, that is, probabilities involving samples taken at times $t_1, \ldots, t_k$ will not differ from those taken at $t_1 + \tau, \ldots, t_k + \tau$.

Example 9.30 Stationarity and Transience

An urn has 6 white balls each with the label "0" and 5 white balls with the label "1". The following sequence of experiments is performed: A ball is selected and the number noted; the first time a white ball is selected it is not put back in the urn, but otherwise balls are always put back in the urn.

!!!

The random process that results from this sequence of experiments clearly has a transient phase and a stationary phase. The transient phase consists of a string of n consecutive 1's and it ends with the first occurrence of a "0". During the transient phase $P[I_n = 0] = 6/11$, and the mean duration of the transient phase is geometrically distributed with mean 11/6. After the first occurrence of a "0", the process enters a "stationary" phase where the process is a binary equiprobable iid sequence. The statistical behavior of the process does not change once the stationary phase is reached.

If we are dealing with random processes that began at $t = -\infty$, then the above condition can be stated precisely as follows. *A discrete-time or continuous-time random process X(t) is* ***stationary*** *if the joint distribution of any set of samples does not depend on the placement of the time origin.* This means that the joint cdf of $X(t_1), X(t_2), \ldots, X(t_k)$ is the same as that of $X(t_1 + \tau), X(t_2 + \tau), \ldots, X(t_k + \tau)$:

$$F_{X(t_1),\ldots,X(t_k)}(x_1, \ldots, x_k) = F_{X(t_1+\tau),\ldots,X(t_k+\tau)}(x_1, \ldots, x_k), \tag{9.55}$$

for all time shifts τ, all k, and all choices of sample times $t_1, \ldots, t_k$. If a process begins at some definite time (i.e., $n = 0$ or $t = 0$), then we say it is stationary if its joint distributions do not change under time shifts to the right.

Two processes $X(t)$ and $Y(t)$ are said to be **jointly stationary** if the joint cdf's of $X(t_1), \ldots, X(t_k)$ and $Y(t'_1), \ldots, Y(t'_j)$ do not depend on the placement of the time origin for all k and j and all choices of sampling times $t_1, \ldots, t_k$ and $t'_1, \ldots, t'_j$.

The *first-order cdf of a stationary random process must be independent of time*, since by Eq. (9.55),

$$F_{X(t)}(x) = F_{X(t+\tau)}(x) = F_X(x) \qquad \text{all } t, \tau. \tag{9.56}$$

This implies that the mean and variance of $X(t)$ are constant and independent of time:

$$m_X(t) = E[X(t)] = m \qquad \text{for all } t \tag{9.57}$$

$$\text{VAR}[X(t)] = E[(X(t) - m)^2] = \sigma^2 \qquad \text{for all } t. \tag{9.58}$$

The *second-order cdf of a stationary random process can depend only on the time difference between the samples* and not on the particular time of the samples, since by Eq. (9.55),

$$F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(0),X(t_2-t_1)}(x_1, x_2) \qquad \text{for all } t_1, t_2. \tag{9.59}$$

This implies that the autocorrelation and the autocovariance of $X(t)$ can depend only on $t_2 - t_1$:

$$R_X(t_1, t_2) = R_X(t_2 - t_1) \qquad \text{for all } t_1, t_2 \tag{9.60}$$

$$C_X(t_1, t_2) = C_X(t_2 - t_1) \qquad \text{for all } t_1, t_2. \tag{9.61}$$

Example 9.31 iid Random Process

Show that the iid random process is stationary.

The joint cdf for the samples at any k time instants, $t_1, \ldots, t_k$, is

$$F_{X(t_1),\dots,X(t_k)}(x_1, x_2, \dots, x_k) = F_X(x_1)F_X(x_2)\dots F_X(x_k)$$
$$= F_{X(t_1+\tau),\dots,X(t_k+\tau)}(x_1, \dots, x_k),$$

for all k, $t_1, \dots, t_k$. Thus Eq. (9.55) is satisfied, and so the iid random process is stationary.

Example 9.32

Is the sum process a discrete-time stationary process?

The sum process is defined by $S_n = X_1 + X_2 + \cdots + X_n$, where the X_i are an iid sequence. The process has mean and variance

$$m_S(n) = nm \qquad \text{VAR}[S_n] = n\sigma^2,$$

where m and σ^2 are the mean and variance of the X_n. It can be seen that the mean and variance are not constant but grow linearly with the time index n. Therefore the sum process cannot be a stationary process.

Example 9.33 Random Telegraph Signal

Show that the random telegraph signal discussed in Example 9.24 is a stationary random process when $P[X(0) = \pm 1] = 1/2$. Show that $X(t)$ settles into stationary behavior as $t \to \infty$ even if $P[X(0) = \pm 1] \neq 1/2$.

We need to show that the following two joint pmf's are equal:

$$P[X(t_1) = a_1, \dots, X(t_k) = a_k] = P[X(t_1 + \tau) = a_1, \dots, X(t_k + \tau) = a_k],$$

for any k, any $t_1 < \cdots < t_k$, and any $a_j = \pm 1$. The independent increments property of the Poisson process implies that

$$\begin{aligned} P[X(t_1) = a_1, \dots, X(t_k) = a_k] &= P[X(t_1) = a_1] \\ &\quad \times P[X(t_2) = a_2 \mid X(t_1) = a_1] \dots P[X(t_k) = a_k \mid X(t_{k-1}) = a_{k-1}], \end{aligned}$$

since the values of the random telegraph at the times $t_1, \dots, t_k$ are determined by the number of occurrences of events of the Poisson process in the time intervals (t_j, t_{j+1}). Similarly,

$$\begin{aligned} &P[X(t_1 + \tau) = a_1, \dots, X(t_k + \tau) = a_k] \\ &\quad = P[X(t_1 + \tau) = a_1]P[X(t_2 + \tau) = a_2 \mid X(t_1 + \tau) = a_1] \dots \\ &\qquad\qquad \times P[X(t_k + \tau) = a_k \mid X(t_{k-1} + \tau) = a_{k-1}]. \end{aligned}$$

The corresponding transition probabilities in the previous two equations are equal since

$$\begin{aligned} P[X(t_{j+1}) = a_{j+1} \mid X(t_j) = a_j] &= \begin{cases} \frac{1}{2}\{1 + e^{-2\alpha(t_{j+1}-t_j)}\} & \text{if } a_j = a_{j+1} \\ \frac{1}{2}\{1 - e^{-2\alpha(t_{j+1}-t_j)}\} & \text{if } a_j \neq a_{j+1} \end{cases} \\ &= P[X(t_{j+1} + \tau) = a_{j+1} \mid X(t_j + \tau) = a_j]. \end{aligned}$$

Thus the two joint probabilities differ only in the first term, namely, $P[X(t_1) = a_1]$ and $P[X(t_1 + \tau) = a_1]$.

From Example 9.24 we know that if $P[X(0) = \pm 1] = 1/2$ then $P[X(t) = \pm 1] = 1/2$, for all t. Thus $P[X(t_1) = a_1] = 1/2$, $P[X(t_1 + \tau) = a_1] = 1/2$, and

$$P[X(t_1) = a_1, \ldots, X(t_k) = a_k] = P[X(t_1 + \tau) = a_1, \ldots, X(t_k + \tau) = a_k].$$

Thus we conclude that the process is stationary when $P[X(0) = \pm 1] = 1/2$.

If $P[X(0) = \pm 1] \neq 1/2$, then the two joint pmf's are not equal because $P[X(t_1) = a_1] \neq P[X(t_1 + \tau) = a_1]$. Let's see what happens if we know that the process started at a specific value, say $X(0) = 1$, that is, $P[X(0) = 1] = 1$. The pmf for $X(t)$ is obtained from Eqs. (9.39) through (9.41):

$$P[X(t) = a] = P[X(t) = a \mid X(0) = 1]1$$
$$= \begin{cases} \dfrac{1}{2}\{1 + e^{-2\alpha t}\} & \text{if } a = 1 \\ \dfrac{1}{2}\{1 - e^{-2\alpha t}\} & \text{if } a = -1. \end{cases}$$

For very small t, the probability that $X(t) = 1$ is close to 1; but as t increases, the probability that $X(t) = 1$ becomes 1/2. Therefore as t_1 becomes large, $P[X(t_1) = a_1] \to 1/2$ and $P[X(t_1 + \tau) = a_1] \to 1/2$ and the two joint pmf's become equal. In other words, the process "forgets" the initial condition and settles down into "steady state," that is, stationary behavior.

9.6.1 Wide-Sense Stationary Random Processes

In many situations we cannot determine whether a random process is stationary, but we can determine whether the mean is a constant:

$$m_X(t) = m \qquad \text{for all } t, \tag{9.62}$$

and whether the autocovariance (or equivalently the autocorrelation) is a function of $t_1 - t_2$ only:

$$C_X(t_1, t_2) = C_X(t_1 - t_2) \qquad \text{for all } t_1, t_2. \tag{9.63}$$

*A discrete-time or continuous-time random process X(t) is **wide-sense stationary** (WSS) if it satisfies Eqs. (9.62) and (9.63).* Similarly, we say that the processes $X(t)$ and $Y(t)$ are **jointly wide-sense stationary** if they are both wide-sense stationary and if their cross-covariance depends only on $t_1 - t_2$. When $X(t)$ is wide-sense stationary, we write

$$C_X(t_1, t_2) = C_X(\tau) \qquad \text{and} \qquad R_X(t_1, t_2) = R_X(\tau),$$

where $\tau = t_1 - t_2$.

All stationary random processes are wide-sense stationary since they satisfy Eqs. (9.62) and (9.63). The following example shows that some wide-sense stationary processes are not stationary.

Example 9.34

Let X_n consist of two interleaved sequences of independent random variables. For n even, X_n assumes the values ± 1 with probability 1/2; for n odd, X_n assumes the values 1/3 and -3 with probabilities 9/10 and 1/10, respectively. X_n is not stationary since its pmf varies with n. It is easy to show that X_n has mean

$$m_X(n) = 0 \qquad \text{for all } n$$

and covariance function

$$C_X(i, j) = \begin{cases} E[X_i]E[X_j] = 0 & \text{for } i \neq j \\ E[X_i^2] = 1 & \text{for } i = j. \end{cases}$$

X_n is therefore wide-sense stationary.

We will see in Chapter 10 that the autocorrelation function of wide-sense stationary processes plays a crucial role in the design of linear signal processing algorithms. We now develop several results that enable us to deduce properties of a WSS process from properties of its autocorrelation function.

#1) First, *the autocorrelation function at* $\tau = 0$ *gives the* **average power** (second moment) of the process:

$$R_X(0) = E[X(t)^2] \qquad \text{for all } t. \tag{9.64}$$

#2) Second, *the autocorrelation function is an even function of* τ since

$$R_X(\tau) = E[X(t + \tau)X(t)] = E[X(t)X(t + \tau)] = R_X(-\tau). \tag{9.65}$$

$t' = t + \tau$

#3) Third, *the autocorrelation function is a measure of the rate of change of a random process* in the following sense. Consider the change in the process from time t to $t + \tau$:

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$$\begin{aligned} P[|X(t + \tau) - X(t)| > \varepsilon] &= P[(X(t + \tau) - X(t))^2 > \varepsilon^2] \\ &\leq \frac{E[(X(t + \tau) - X(t))^2]}{\varepsilon^2} \\ &= \frac{2\{R_X(0) - R_X(\tau)\}}{\varepsilon^2}, \end{aligned} \tag{9.66}$$

where we used the Markov inequality, Eq. (4.75), to obtain the upper bound. Equation (9.66) states that if $R_X(0) - R_X(\tau)$ is small, that is, $R_X(\tau)$ drops off slowly, then the probability of a large change in $X(t)$ in τ seconds is small.

#4) Fourth, *the autocorrelation function is maximum at* $\tau = 0$. We use the Cauchy-Schwarz inequality:[2]

$$E[XY]^2 \leq E[X^2]E[Y^2], \tag{9.67}$$

for any two random variables X and Y. If we apply this equation to $X(t + \tau)$ and $X(t)$, we obtain

$$R_X(\tau)^2 = E[X(t + \tau)X(t)]^2 \leq E[X^2(t + \tau)]E[X^2(t)] = R_X(0)^2.$$

Thus

$$|R_X(\tau)| \leq R_X(0). \tag{9.68}$$

[2]See Problem 5.74 and Appendix C.

#5) Fifth, *if* $R_X(0) = R_X(d)$, *then* $R_X(\tau)$ *is periodic with period d and* $X(t)$ *is* **mean square periodic**, that is, $E[(X(t + d) - X(t))^2] = 0$. If we apply Eq. (9.67) to $X(t + \tau + d) - X(t + \tau)$ and $X(t)$, we obtain

$$\begin{aligned} &E[(X(t + \tau + d) - X(t + \tau))X(t)]^2 \\ &\quad \le E[(X(t + \tau + d) - X(t + \tau))^2]E[X^2(t)], \end{aligned}$$

which implies that

$$\{R_X(\tau + d) - R_X(\tau)\}^2 \le 2\{R_X(0) - R_X(d)\}R_X(0).$$

Thus $R_X(d) = R_X(0)$ implies that the right-hand side of the equation is zero, and thus that $R_X(\tau + d) = R_X(\tau)$ for all τ. Repeated applications of this result imply that $R_X(\tau)$ is periodic with period d. The fact that $X(t)$ is mean square periodic follows from

$$E[(X(t + d) - X(t))^2] = 2\{R_X(0) - R_X(d)\} = 0.$$

#6) Sixth, let $X(t) = m + N(t)$, where $N(t)$ is a zero-mean process for which $R_N(\tau) \to 0$ as $\tau \to \infty$, then

$$\begin{aligned} R_X(\tau) &= E[(m + N(t + \tau))(m + N(t))] = m^2 + 2m\overbrace{E[N(t)]}^{=0} + R_N(\tau) \\ &= m^2 + R_N(\tau) \to m^2 \qquad \text{as } \tau \to \infty. \end{aligned}$$

In other words, $R_X(\tau)$ *approaches the square of the mean of* $X(t)$ *as* $\tau \to \infty$.

In summary, the autocorrelation function can have three types of components: (1) a component that approaches zero as $\tau \to \infty$; (2) a periodic component; and (3) a component due to a nonzero mean.

Example 9.35

Figure 9.13 shows several typical autocorrelation functions. Figure 9.13(a) shows the autocorrelation function for the random telegraph signal $X(t)$ (see Eq. (9.44)):

$$R_X(\tau) = e^{-2\alpha|\tau|} \qquad \text{for all } \tau.$$

$X(t)$ is zero mean and $R_X(\tau) \to 0$ as $|\tau| \to \infty$.

Figure 9.13(b) shows the autocorrelation function for a sinusoid $Y(t)$ with amplitude a and random phase (see Example 9.10):

$$R_Y(\tau) = \frac{a^2}{2}\cos(2\pi f_0 \tau) \qquad \text{for all } \tau.$$

$Y(t)$ is zero mean and $R_Y(\tau)$ is periodic with period $1/f_0$.

Figure 9.13(c) shows the autocorrelation function for the process $Z(t) = X(t) + Y(t) + m$, where $X(t)$ is the random telegraph process, $Y(t)$ is a sinusoid with random phase, and m is a constant. If we assume that $X(t)$ and $Y(t)$ are independent processes, then

$$\begin{aligned} R_Z(\tau) &= E[\{X(t + \tau) + Y(t + \tau) + m\}\{X(t) + Y(t) + m\}] \\ &= R_X(\tau) + R_Y(\tau) + m^2. \end{aligned}$$

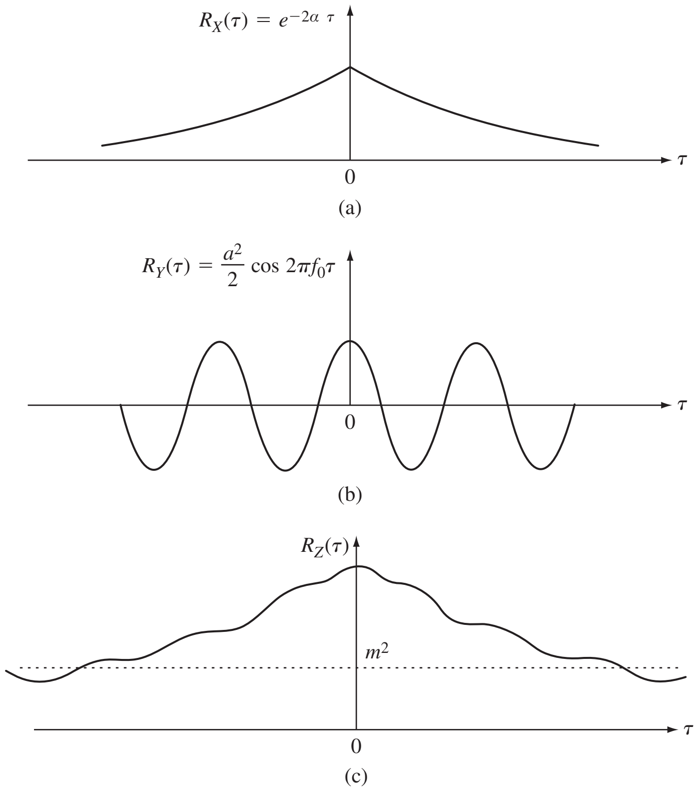

FIGURE 9.13
(a) Autocorrelation function of a random telegraph signal. (b) Autocorrelation function of a sinusoid with random phase. (c) Autocorrelation function of a random process that has nonzero mean, a periodic component, and a "random" component.

9.6.2 Wide-Sense Stationary Gaussian Random Processes

No lo vimos como tal
Lo saltio

If a Gaussian random process is wide-sense stationary, then it is also stationary. Recall from Section 9.5, Eq. (9.47), that the joint pdf of a Gaussian random process is completely determined by the mean $m_X(t)$ and the autocovariance $C_X(t_1, t_2)$. If $X(t)$ is wide-sense stationary, then its mean is a constant m and its autocovariance depends only on the difference of the sampling times, $t_i - t_j$. It then follows that the joint pdf of $X(t)$ depends only on this set of differences, and hence it is invariant with respect to time shifts. Thus the process is also stationary.

The above result makes WSS Gaussian random processes particularly easy to work with since all the information required to specify the joint pdf is contained in m and $C_X(\tau)$.

Example 9.36 A Gaussian Moving Average Process

Let X_n be an iid sequence of Gaussian random variables with zero mean and variance σ^2, and let Y_n be the average of two consecutive values of X_n:

$$Y_n = \frac{X_n + X_{n-1}}{2}.$$

The mean of Y_n is zero since $E[X_i] = 0$ for all i. The covariance is

$$\begin{aligned} C_Y(i, j) &= E[Y_i Y_j] = \frac{1}{4} E[(X_i + X_{i-1})(X_j + X_{j-1})] \\ &= \frac{1}{4}\{E[X_i X_j] + E[X_i X_{j-1}] + E[X_{i-1} X_j] + E[X_{i-1} X_{j-1}]\} \\ &= \begin{cases} \frac{1}{2}\sigma^2 & \text{if } i = j \\ \frac{1}{4}\sigma^2 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We see that Y_n has a constant mean and a covariance function that depends only on $|i - j|$, thus Y_n is a wide-sense stationary process. Y_n is a Gaussian random variable since it is defined by a linear function of Gaussian random variables (see Section 6.4, Eq. 6.45). Thus the joint pdf of Y_n is given by Eq. (9.47) with zero-mean vector and with entries of the covariance matrix specified by $C_Y(i, j)$ above.

9.6.3 Cyclostationary Random Processes

→ Lo vimos, pero como una curiosidad. Lo salteo

Many random processes arise from the repetition of a given procedure every T seconds. For example, a data modulator ("modem") produces a waveform every T seconds according to some input data sequence. In another example, a "time multiplexer" interleaves n separate sequences of information symbols into a single sequence of symbols. It should not be surprising that the periodic nature of such processes is evident in their probabilistic descriptions. A discrete-time or continuous-time random process $X(t)$ is said to be **cyclostationary** if the joint cumulative distribution function of any set of samples is invariant with respect to shifts of the origin by *integer multiples of some period T*. In other words, $X(t_1), X(t_2), \ldots, X(t_k)$ and $X(t_1 + mT), X(t_2 + mT), \ldots, X(t_k + mT)$ have the same joint cdf for all k, m, and all choices of sampling times $t_1, \ldots, t_k$:

$$\begin{aligned} &F_{X(t_1), X(t_2), \ldots, X(t_k)}(x_1, x_2, \ldots, x_k) \\ &\quad = F_{X(t_1+mT), X(t_2+mT), \ldots, X(t_k+mT)}(x_1, x_2, \ldots, x_k). \end{aligned} \tag{9.69}$$

We say that $X(t)$ is **wide-sense cyclostationary** if the mean and autocovariance functions are invariant with respect to shifts in the time origin by integer multiples of T, that is, for every integer m,

$$m_X(t + mT) = m_X(t) \tag{9.70a}$$

$$C_X(t_1 + mT, t_2 + mT) = C_X(t_1, t_2). \tag{9.70b}$$

Note that if $X(t)$ is cyclostationary, then it follows that $X(t)$ is also wide-sense cyclostationary.

Example 9.37

Consider a random amplitude sinusoid with period T:

$$X(t) = A\cos(2\pi t/T).$$

Is $X(t)$ cyclostationary? wide-sense cyclostationary?

Consider the joint cdf for the time samples $t_1, \ldots, t_k$:

$$\begin{aligned}
P[X(t_1) &\le x_1, X(t_2) \le x_2, \ldots, X(t_k) \le x_k)] \\
&= P[A\cos(2\pi t_1/T) \le x_1, \ldots, A\cos(2\pi t_k/T) \le x_k] \\
&= P[A\cos(2\pi(t_1 + mT)/T) \le x_1, \ldots, A\cos(2\pi(t_k + mT)/T) \le x_k] \\
&= P[X(t_1 + mT) \le x_1, X(t_2 + mT) \le x_2, \ldots, X(t_k + mT) \le x_k].
\end{aligned}$$

Thus $X(t)$ is a cyclostationary random process and hence also a wide-sense cyclostationary process.

In the above example, the sample functions of the random process are always periodic. The following example shows that, in general, the sample functions of a cyclostationary random process need not be periodic.

Example 9.38 Pulse Amplitude Modulation

A modem transmits a binary iid equiprobable data sequence as follows: To transmit a binary 1, the modem transmits a rectangular pulse of duration T seconds and amplitude 1; to transmit a binary 0, it transmits a rectangular pulse of duration T seconds and amplitude -1. Let $X(t)$ be the random process that results. Is $X(t)$ wide-sense cyclostationary?

Figure 9.14(a) shows a rectangular pulse of duration T seconds, and Fig. 9.14(b) shows the waveform that results for a particular data sequence. Let A_i be the sequence of amplitudes (± 1)

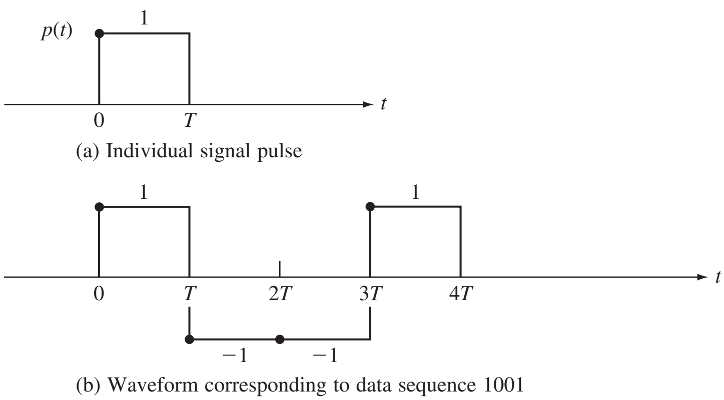

FIGURE 9.14
Pulse amplitude modulation.

corresponding to the binary sequence, then $X(t)$ can be represented as the sum of amplitude-modulated time-shifted rectangular pulses:

$$X(t) = \sum_{n=-\infty}^{\infty} A_n p(t - nT). \tag{9.71}$$

The mean of $X(t)$ is

$$m_X(t) = E\left[\sum_{n=-\infty}^{\infty} A_n p(t - nT)\right] = \sum_{n=-\infty}^{\infty} E[A_n]p(t - nT) = 0$$

since $E[A_n] = 0$. The autocovariance function is

$$\begin{aligned} C_X(t_1, t_2) &= E[X(t_1)X(t_2)] - 0 \\ &= \begin{cases} E[X(t_1)^2] = 1 & \text{if } nT \le t_1, t_2 < (n+1)T \\ E[X(t_1)]E[X(t_2)] = 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Figure 9.15 shows the autocovariance function in terms of t_1 and t_2. It is clear that $C_X(t_1 + mT, t_2 + mT) = C_X(t_1, t_2)$ for all integers m. Therefore the process is wide-sense cyclostationary.

We will now show how a stationary random process can be obtained from a cyclostationary process. Let $X(t)$ be a cyclostationary process with period T. We "stationarize" $X(t)$ by observing a randomly phase-shifted version of $X(t)$:

$$X_s(t) = X(t + \Theta) \qquad \Theta \text{ uniform in } [0, T], \tag{9.72}$$

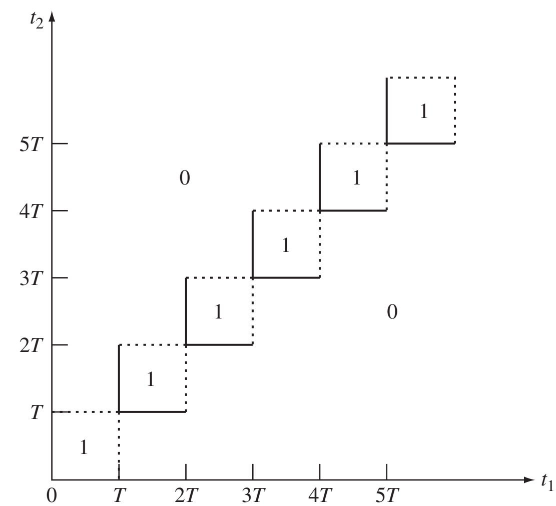

FIGURE 9.15
Autocovariance function of pulse amplitude-modulated random process.

where Θ is independent of $X(t)$. $X_s(t)$ can arise when the phase of $X(t)$ is either unknown or not of interest. *If $X(t)$ is a cyclostationary random process, then $X_s(t)$ is a stationary random process.* To show this, we first use conditional expectation to find the joint cdf of $X_s(t)$:

$$
\begin{aligned}
&P[X_s(t_1) \le x_1, X_s(t_2) \le x_2, \ldots, X_s(t_k) \le x_k] \\
&\quad = P[X(t_1 + \Theta) \le x_1, X(t_2 + \Theta) \le x_2, \ldots, X(t_k + \Theta) \le x_k] \\
&\quad = \int_0^T P[X(t_1 + \Theta) \le x_1, \ldots, X(t_k + \Theta) \le x_k \,|\, \Theta = \theta] f_\Theta(\theta)\, d\theta \\
&\quad = \frac{1}{T}\int_0^T P[X(t_1 + \theta) \le x_1, \ldots, X(t_k + \theta) \le x_k]\, d\theta.
\end{aligned} \tag{9.73}
$$

Equation (9.73) shows that the joint cdf of $X_s(t)$ is obtained by integrating the joint cdf of $X(t)$ over one time period. It is easy to then show that a time-shifted version of $X_s(t)$, say $X_s(t_1 + \tau), X_s(t_2 + \tau), \ldots, X_s(t_k + \tau)$, will have the same joint cdf as $X_s(t_1), X_s(t_2), \ldots, X_s(t_k)$ (see Problem 9.80). Therefore $X_s(t)$ is a stationary random process.

By using conditional expectation (see Problem 9.81), it is easy to show that *if $X(t)$ is a wide-sense cyclostationary random process, then $X_s(t)$ is a wide-sense stationary random process*, with mean and autocorrelation given by

$$E[X_s(t)] = \frac{1}{T}\int_0^T m_x(t)\, dt \tag{9.74a}$$

$$R_{X_s}(\tau) = \frac{1}{T}\int_0^T R_X(t + \tau, t)\, dt. \tag{9.74b}$$

Example 9.39 Pulse Amplitude Modulation with Random Phase Shift

Let $X_s(t)$ be the phase-shifted version of the pulse amplitude–modulated waveform $X(t)$ introduced in Example 9.38. Find the mean and autocorrelation function of $X_s(t)$.

$X_s(t)$ has zero mean since $X(t)$ is zero-mean. The autocorrelation of $X_s(t)$ is obtained from Eq. (9.74b). From Fig. 9.15, we can see that for $0 < t + \tau < T$, $R_X(t + \tau, t) = 1$ and $R_X(t + \tau, t) = 0$ otherwise. Therefore:

$$\text{for } 0 < \tau < T: \qquad R_{X_s}(\tau) = \frac{1}{T}\int_0^{T-\tau} dt = \frac{T - \tau}{T};$$

$$\text{for } -T < \tau < 0: \qquad R_{X_s}(\tau) = \frac{1}{T}\int_{-\tau}^{T} dt = \frac{T + \tau}{T}.$$

Thus $X_s(t)$ has a triangular autocorrelation function:

$$R_{X_s}(\tau) = \begin{cases} 1 - \dfrac{|\tau|}{T} & |\tau| \le T \\ 0 & |\tau| > T. \end{cases}$$

24/4 Hasta acá llegué. Voy al cap #10 a DEP (ó SPD)

9.7 CONTINUITY, DERIVATIVES, AND INTEGRALS OF RANDOM PROCESSES

Many of the systems encountered in electrical engineering have dynamics that are described by linear differential equations. When the input signals to these systems are deterministic, the solutions of the differential equations give the output signals of the systems. In developing these solutions we make use of the results of calculus for deterministic functions. Since each sample function of a random process can be viewed as a deterministic signal, it is only natural to apply continuous-time random processes as input signals to the above systems. The output of the systems then consists of a sample function of another random process. On the other hand, if we view a system as acting on an input random process to produce an output random process, we find that we need to develop a new "calculus" for continuous-time random processes. In particular we need to develop probabilistic methods for addressing the continuity, differentiability, and integrability of random processes, that is, of the ensemble of sample functions as a whole. In this section we develop these concepts.

9.7.1 Mean Square Continuity

A natural way of viewing a random process is to imagine that each point ζ in S produces a particular deterministic sample function $X(t, \zeta)$. The standard methods from calculus can be used to determine the continuity of the sample function at a point t_0 for each point ζ. Intuitively, we say that $X(t, \zeta)$ is continuous at t_0 if the difference $|X(t, \zeta) - X(t_0, \zeta)|$ approaches zero as t approaches t_0. More formally, we say that:

> $X(t, \zeta)$ is continuous at t_0 if given any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|t - t_0| < \delta$ implies that $|X(t, \zeta) - X(t_0, \zeta)| < \varepsilon$, and we write:
>
> $$\lim_{t \to t_0} X(t, \zeta) = X(t_0, \zeta).$$

In some simple cases, such as the random sinusoid discussed in Example 9.2, we can establish that all sample functions of the random process are continuous at a point t_0, and so we can conclude that the random process is continuous at t_0. In general, however, we can only address the continuity of a random process in a probabilistic sense. In this section, we concentrate on convergence in the mean square sense, introduced in Section 7.4, because of its tractability and its usefulness in the study of linear systems subject to random inputs.

> **Mean Square Continuity:** The random process $X(t)$ is continuous at the point t_0 *in the mean square sense* if
>
> $$E[(X(t) - X(t_0))^2] \to 0 \qquad \text{as } t \to t_0. \tag{9.75}$$

We denote mean square continuity by (limit in the mean)

$$\underset{t \to t_0}{\text{l.i.m.}}\, X(t) = X(t_0).$$

We say that $X(t)$ is mean square continuous if it is mean square continuous for all t_0. Note that if all sample functions of a random process are continuous at a point t_0, then

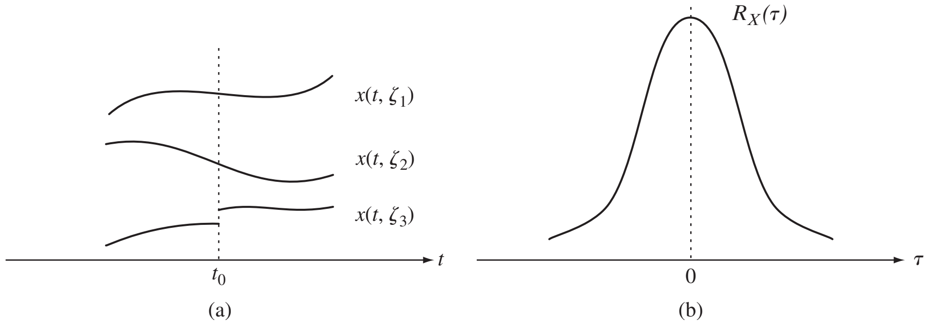

FIGURE 9.16
(a) Mean square continuity at t_0 does not imply all sample functions are continuous at t_0. (b) If $X(t)$ is WSS and $R_X(\tau)$ is continuous at $\tau = 0$, then $X(t)$ is mean square continuous for all t.

the process will also be mean square continuous at the point t_0. In the examples we will see that mean square continuity does not imply that all the sample functions are continuous. Thus, in general, we may have the situation in Fig. 9.16.

In order to determine what conditions are required for mean square continuity, consider the mean square difference between $X(t)$ and $X(t_0)$:

$$E[(X(t) - X(t_0))^2] = R_X(t, t) - R_X(t_0, t) - R_X(t, t_0) + R_X(t_0, t_0). \tag{9.76}$$

Hence, if the autocorrelation function $R_X(t_1, t_2)$ is continuous at the point (t_0, t_0), then letting $t \rightarrow t_0$, the right-hand side of Eq. (9.76) will vanish. Thus we conclude that *if $R_X(t_1, t_2)$ is continuous in both t_1 and t_2 at the point (t_0, t_0), then $X(t)$ is mean square continuous at the point t_0.*

At this point it is worth recalling that a function of two variables $f(x, y)$ is continuous at a point (a, b) if the limit $f(x, y)$ reaches the *same* value for *any* mode of approach from (x, y) to (a, b). In particular, in order for $R_X(t_1, t_2)$ to be continuous at (t_0, t_0), $R_X(t_1, t_2)$ must approach the same value as t_1 and t_2 approach (t_0, t_0) from *any* direction.

A discontinuity in the mean function $m_X(t)$ at some point t_0 indicates that the sample functions must be discontinuous at t_0 with nonzero probability. Therefore, we must have that *if $X(t)$ is mean square continuous at t_0, then the mean function $m_X(t)$ must be continuous at t_0:*

$$\lim_{t \rightarrow t_0} m_X(t) = m_X(t_0). \tag{9.77a}$$

To show this, we note that the variance of the difference $X(t) - X(t_0)$ is nonnegative, thus

$$0 \leq \text{VAR}[X(t) - X(t_0)] = E[(X(t) - X(t_0))^2] - E[X(t) - X(t_0)]^2.$$

Therefore

$$E[(X(t) - X(t_0))^2] \geq E[X(t) - X(t_0)]^2 = (m_X(t) - m_X(t_0))^2.$$

If $X(t)$ is mean square continuous at t_0, then as $t \to t_0$ the left-hand side of the above equation approaches zero. This implies that the right-hand side approaches zero, and hence $m_X(t) \to m_X(t_0)$. Equation (9.77a) can be rewritten as follows:

$$\lim_{t \to t_0} E[X(t)] = E\Big[\underset{t \to t_0}{\text{l.i.m.}}\, X(t)\Big]. \tag{9.77b}$$

Therefore if $X(t)$ is mean square continuous at t_0, then we can interchange the order of the limit and the expected value.

If the random process $X(t)$ is wide-sense stationary, then Eq. (9.76) becomes

$$E[(X(t_0 + \tau) - X(t_0))^2] = 2(R_X(0) - R_X(\tau)). \tag{9.78}$$

Therefore *if $R_X(\tau)$ is continuous at $\tau = 0$, then the wide-sense stationary random process $X(t)$ is mean square continuous at every point t_0.*

Example 9.40 Wiener and Poisson Processes

Are the Wiener and Poisson processes mean square continuous?

The autocorrelation of the Wiener process $X(t)$ is given by

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2).$$

Consider the limit as t_1 and t_2 approach (t_0, t_0):

$$|R_X(t_0 + \varepsilon_1, t_0 + \varepsilon_2) - R_X(t_0, t_0)|$$
$$= \alpha\,|\min(t_0 + \varepsilon_1, t_0 + \varepsilon_2) - t_0| \leq \alpha \max(\varepsilon_1, \varepsilon_2).$$

As ε_1 and ε_2 approach zero, the above difference vanishes. Therefore the autocorrelation function is continuous at the point (t_0, t_0), and the Wiener process is mean square continuous.

The autocorrelation of the Poisson process $N(t)$ is given by

$$R_N(t_1, t_2) = \lambda \min(t_1, t_2).$$

This is exactly the same as that of the Wiener process. Therefore the Poisson process is also mean square continuous.

The above example shows clearly how mean square continuity does not imply continuity of the sample functions. The Poisson and Wiener processes have the same autocorrelation function and are both mean square continuous. However, the Poisson process has a countably infinite number of discontinuities, while it can be shown that almost all sample functions of the Wiener process are continuous.

Example 9.41 Pulse Amplitude Modulation

Let $X(t)$ be the pulse amplitude modulation random process introduced in Example 9.38. Is $X(t)$ mean square continuous?

The process has discontinuity at $t = mT$ with nonzero probability, so we expect the process not to be mean square continuous. The autocorrelation function of $X(t)$ is shown in Fig. 9.15 and is given by

$$R_X(t_1, t_2) = \begin{cases} 1 & nT \le t_1 < (n+1)T \text{ and } nT \le t_2 < (n+1)T \\ 0 & \text{otherwise.} \end{cases}$$

The autocorrelation function is continuous at all points $t_1 = t_0 \neq nT$, and hence $X(t)$ is mean square continuous at all points within the signaling intervals, $nT < t < (n+1)T$. However, the autocorrelation function is not continuous at the points $t_1 = t_0 = nT$, which correspond to the points where the transitions between pulses occur. For example, if we approach the point (nT, nT) along the line $t_1 = t_2$, we obtain the limit 1; if we approach (nT, nT) along a line perpendicular to the above, the limit is zero. Thus $X(t)$ is not mean square continuous at the point $t = nT$.

9.7.2 Mean Square Derivatives

Suppose we take a sample function of a random process $X(t, \zeta)$ and carry out the limiting procedure that defines the derivative of a deterministic function:

$$\lim_{\varepsilon \to 0} \frac{X(t+\varepsilon, \zeta) - X(t, \zeta)}{\varepsilon}.$$

This limit may exist for some sample functions and it may fail to exist for other sample functions of the same random process. We define the derivative of a random process in terms of mean square convergence:

> **Mean Square Derivative:** The random process $X(t)$ has *mean square derivative $X'(t)$* at t defined by
>
> $$\underset{\varepsilon \to 0}{\text{l.i.m.}} \frac{X(t+\varepsilon) - X(t)}{\varepsilon}, \tag{9.79}$$
>
> provided that the mean square limit exists, that is,
>
> $$\lim_{\varepsilon \to 0} E\left[\left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - X'(t)\right)^2\right] = 0. \tag{9.80}$$

We also denote the mean square derivative by $dX(t)/dt$. Note that if all sample functions of $X(t)$ are differentiable at the point t, then the mean square derivative exists because Eq. (9.80) is satisfied. However, the existence of the mean square derivative does not imply the existence of the derivative for all sample functions.

It can be shown that *the mean square derivative of $X(t)$ at the point t exists if*

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$$

exists at the point $(t_1, t_2) = (t, t)$. We examine the special case where $X(t)$ is WSS. Consider the mean square value of the first difference in $X(t)$:

$$E\left[\left(\frac{X(t+h) - X(t)}{h}\right)^2\right] = \frac{1}{h^2}(R_X(0) - R_X(h) - R_X(-h) + R_X(0))$$
$$= -\frac{1}{h}\left\{\frac{R_X(h) - R_X(0)}{h} - \frac{R_X(0) - R_X(-h)}{h}\right\}$$
$$\rightarrow)-\frac{d^2}{d\tau^2}R_X(\tau)\bigg|_{\tau=0}. \tag{9.81}$$

Therefore *the mean square derivative of a WSS random process $X(t)$ exists for all t if $R_X(\tau)$ has derivatives up to order two at $\tau = 0$.*

If $X(t)$ is a Gaussian random process for which the mean square derivative $X'(t)$ exists, then $X'(t)$ must also be a Gaussian random process. To show this, consider $Y_\varepsilon(t) = (X(t+\varepsilon) - X(t))/\varepsilon$. The k time samples $Y_\varepsilon(t_1), Y_\varepsilon(t_2), \ldots, Y_\varepsilon(t_k)$ are given by a linear transformation of the jointly Gaussian random variables $X(t_1 + \varepsilon)$, $X(t_1), X(t_2 + \varepsilon), X(t_2), \ldots, X(t_k + \varepsilon), X(t_k)$. It then follows that $Y_\varepsilon(t_1), Y_\varepsilon(t_2), \ldots,$ $Y_\varepsilon(t_k)$ are jointly Gaussian random variables and hence that $Y_\varepsilon(t)$ is a Gaussian random process. $X'(t)$, the limit of $Y_\varepsilon(t)$ as ε approaches zero, is then also a Gaussian random process since (from Section 7.4) mean square convergence implies convergence in distribution.

Once we have determined the existence of the mean square derivative $X'(t)$, we can proceed to find its mean and autocorrelation functions. Using the same reasoning that led to Eq. (9.77b), we can show that we can interchange the order of expectation and mean square differentiation. Therefore

$$E[X'(t)] = E\left[\underset{\varepsilon\to 0}{\text{l.i.m.}}\frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right]$$
$$= \lim_{\varepsilon\to 0} E\left[\frac{X(t+\varepsilon) - X(t)}{\varepsilon}\right]$$
$$= \lim_{\varepsilon\to 0}\frac{m_X(t+\varepsilon) - m_X(t)}{\varepsilon} = \frac{d}{dt}m_X(t). \tag{9.82}$$

Note that if $X(t)$ is a wide-sense stationary process, then $m_X(t) = m$, a constant, and therefore $E[X'(t)] = 0$.

Next we find the cross-correlation between $X(t)$ and $X'(t)$:

$$R_{X,X'}(t_1, t_2) = E\left[X(t_1)\underset{\varepsilon\to 0}{\text{l.i.m.}}\frac{X(t_2+\varepsilon) - X(t_2)}{\varepsilon}\right]$$
$$= \lim_{\varepsilon\to 0}\frac{R_X(t_1, t_2+\varepsilon) - R_X(t_1, t_2)}{\varepsilon}$$
$$= \frac{\partial}{\partial t_2}R_X(t_1, t_2).$$

Finally, we obtain the autocorrelation of $X'(t)$:

$$R_{X'}(t_1, t_2) = E\left[\underset{\varepsilon \to 0}{\text{l.i.m.}}\left\{\frac{X(t_1 + \varepsilon) - X(t_1)}{\varepsilon}\right\} X'(t_2)\right]$$

$$= \lim_{\varepsilon \to 0} \frac{R_{X,X'}(t_1 + \varepsilon, t_2) - R_{X,X'}(t_1, t_2)}{\varepsilon}$$

$$= \frac{\partial}{\partial t_1} R_{X,X'}(t_1, t_2) \qquad = \frac{\partial^2}{\partial t_1\, \partial t_2} R_X(t_1, t_2). \tag{9.83}$$

If $X(t)$ is a wide-sense stationary process, we have

$$R_{X,X'}(\tau) = \frac{\partial}{\partial t_2} R_X(t_1 - t_2) = -\frac{d}{d\tau} R_X(\tau), \tag{9.84}$$

where $\tau = t_1 - t_2$, and then

$$R_{X'}(\tau) = \frac{\partial}{\partial t_1}\left\{\frac{\partial}{\partial t_2} R_X(t_1 - t_2)\right\} = \frac{\partial}{\partial t_1} - \frac{d}{d\tau} R_X(t_1 - t_2)$$

$$= -\frac{d^2}{d\tau^2} R_X(\tau). \tag{9.85}$$

Example 9.42

Let $X(t)$ be the random amplitude sinusoid introduced in Example 9.9. Does $X(t)$ have a mean square derivative?

The autocorrelation function for $X(t)$ is

$$R_X(t_1, t_2) = E[A^2] \cos 2\pi t_1 \cos 2\pi t_2.$$

The second mixed partial derivative with respect to t_1 and t_2 exists at every point (t, t), and is given by

$$\frac{\partial^2}{\partial t_1\, \partial t_2} R_X(t_1, t_2)\big|_{t_1 = t_2 = t} = 4\pi^2 E[A^2] \sin^2 2\pi t.$$

Therefore $X(t)$ has a mean square derivative at every point t.

Example 9.43 Wiener Process and White Gaussian Noise

Does the Wiener process have a mean square derivative?

Recall that the Wiener process is Gaussian, so we expect that its derivative is also Gaussian. We first show that this process does not have a mean square derivative. The Wiener process has autocorrelation function given by

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2) = \begin{cases} \alpha t_2 & t_2 < t_1 \\ \alpha t_1 & t_2 \geq t_1. \end{cases}$$

The first derivative with respect to t_2 is

$$\frac{\partial}{\partial t_2} R_X(t_1, t_2) = \begin{cases} \alpha & t_2 < t_1 \\ 0 & t_2 > t_1 \end{cases} = \alpha u(t_1 - t_2).$$

The derivative of a step function does not exist at its point of discontinuity. We therefore conclude that the second mixed partial derivative does not exist at any point t, and hence the Wiener process does not have a mean square derivative at any point.

We can generalize the notion of derivative of a random process if we use delta functions. Recall that the delta function is defined so that its integral is a unit step function (see Eq. 4.18). We can therefore interpret the derivative of a unit step function as yielding a delta function. This suggests that the process $X'(t)$ has autocorrelation function given by

$$R_{X'}(t_1, t_2) = \frac{\partial}{\partial t_1}\alpha u(t_1 - t_2) = \alpha\delta(t_1 - t_2). \tag{9.86}$$

The properties of the delta function give the random process $X'(t)$ some unusual properties. First, since the delta function is infinite at $t_1 = t_2$, it follows that the mean square value of $X'(t)$ is infinite, that is, $X'(t)$ has infinite power. Also, since the delta function is zero whenever $t_1 \neq t_2$, it follows that any two distinct time samples, $X'(t_1)$ and $X'(t_2)$, are uncorrelated regardless of how close t_1 is to t_2. This suggests that $X'(t)$ varies extremely rapidly in time. Recall that the Wiener process was obtained in Section 9.5 as the limit of the random walk process. Thus it is not surprising that the derivative of the process has these properties.

The random process that results from taking the derivative of the Wiener process is called **white Gaussian noise**. It is very useful in modeling broadband noise in communication and radar systems. We discuss it further in the next chapter.

9.7.3 Mean Square Integrals

The integral of a continuous-time random process arises naturally when computing time averages. It also arises as the solution to systems described by linear differential equations. In this section, we develop the notion of the integral of a random process in the sense of mean square convergence.

Suppose we are interested in the integral of the random process $X(t)$ over the interval (t_0, t). We partition the interval into n subintervals and form the sum

$$I_n = \sum_{k=1}^{n} X(t_k)\,\Delta_k.$$

We define the integral of $X(t)$ as the mean square limit of the sequence I_n as the width of the subintervals approaches zero. When the limit exists, we denote the limiting random process by

$$Y(t) = \int_{t_0}^{t} X(t')\,dt' = \underset{\Delta_k\to 0}{\text{l.i.m.}} \sum_k X(t_k)\Delta_k. \tag{9.87}$$

The Cauchy criterion provides us with conditions that ensure the existence of the mean square integral in Eq. (9.87):

$$E\left[\left\{\sum_j X(t_j)\,\Delta_j - \sum_k X(t_k)\,\Delta_k\right\}^2\right] \to 0 \qquad \text{as } \Delta_j, \Delta_k \to 0. \tag{9.88}$$

As in the case of the mean square derivative, we obtain three terms when we expand the square inside the expected value. Each of these terms leads to an expression of the form

$$E\left[\sum_j\sum_k X(t_j)X(t_k)\,\Delta_j\,\Delta_k\right] = \sum_j\sum_k R_X(t_j, t_k)\,\Delta_j\,\Delta_k. \tag{9.89}$$

If the limit of the expression on the right-hand side exists, then it can be shown that the three terms resulting from Eq. (9.88) add to zero. On the other hand, the limit of the right-hand side of Eq. (9.89) approaches a double integral of the autocorrelation function. We have thus shown that *the mean square integral of $X(t)$ exists if the following double integral exists:*

$$\int_{t_0}^{t}\int_{t_0}^{t} R_X(u, v)\, du\, dv. \tag{9.90}$$

It can be shown that if $X(t)$ is a mean square continuous random process, then its integral exists.

If $X(t)$ is a Gaussian random process, then its integral $Y(t)$ is also a Gaussian random process. This follows from the fact that the I_n's are linear combinations of jointly Gaussian random variables.

The mean and autocorrelation function of $Y(t)$ are given by

$$\begin{aligned} m_Y(t) &= E\left[\int_{t_0}^{t} X(t')\, dt'\right] = \int_{t_0}^{t} E[X(t')]\, dt' \\ &= \int_{t_0}^{t} m_X(t')\, dt' \end{aligned} \tag{9.91}$$

and

$$\begin{aligned} R_Y(t_1, t_2) &= E\left[\int_{t_0}^{t_1} X(u)\, du \int_{t_0}^{t_2} X(v)\, dv\right] \\ &= \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u, v)\, du\, dv. \end{aligned} \tag{9.92}$$

Finally, we note that if $X(t)$ is wide-sense stationary, then the integrands in Eqs. (9.90) and (9.92) are replaced by $R_X(u - v)$.

Example 9.44 Moving Average of $X(t)$

Find the mean and variance of $M(t)$, the moving average over half a period of a random amplitude sinusoid $X(t)$ with period T:

$$M(t) = \frac{2}{T}\int_{t-T/2}^{t} X(t')\, dt'.$$

The mean of $M(t)$ is given by

$$E[M(t)] = \frac{2}{T}\int_{t-T/2}^{t} E[A]\cos\frac{2\pi t'}{T}\, dt' = E[A]\frac{2}{\pi}\sin\frac{2\pi t}{T}.$$

Its second moment at time t is given by

$$E[M^2(t)] = R_M(t, t) = \frac{4}{T^2}\int_{t-T/2}^{t}\int_{t-T/2}^{t} E[A^2] \times \cos\frac{2\pi u}{T}\cos\frac{2\pi v}{T}\, du\, dv = E[A^2]\frac{4}{\pi^2}\sin^2\frac{2\pi t}{T}.$$

The variance is then

$$\begin{aligned} \text{VAR}[M(t)] &= E[A^2]\frac{4}{\pi^2}\sin^2\frac{2\pi t}{T} - E[A]^2\frac{4}{\pi^2}\sin^2\frac{2\pi t}{T} \\ &= \text{VAR}[A]\frac{4}{\pi^2}\sin^2\frac{2\pi t}{T}. \end{aligned}$$

Example 9.45 Integral of White Gaussian Noise

Let $Z(t)$ be the white Gaussian noise process introduced in Example 9.43. Find the autocorrelation function of $X(t)$, the integral of $Z(t)$ over the interval $(0, t)$.

From Example 9.43, the white Gaussian noise process has autocorrelation function

$$R_Z(t_1, t_2) = \alpha\delta(t_1 - t_2).$$

The autocorrelation function of $X(t)$ is then given by

$$\begin{aligned} R_X(t_1, t_2) &= \int_0^{t_1}\int_0^{t_2} \alpha\delta(w - v)\, dw\, dv = \alpha\int_0^{t_2} u(t_1 - v)\, dv \\ &= \alpha\int_0^{\min(t_1,t_2)} dv = \alpha\min(t_1, t_2). \end{aligned}$$

We thus find that $X(t)$ has the same autocorrelation as the Wiener process. In addition we have that $X(t)$ must be a Gaussian random process since $Z(t)$ is Gaussian. It then follows that $X(t)$ must be the Wiener process because it has the joint pdf given by Eq. (9.52).

9.7.4 Response of a Linear System to Random Input

We now apply the results developed in this section to develop the solution of a linear system described by a first-order differential equation. The method can be generalized to higher-order equations. In the next chapter we develop transform methods to solve the general problem.

Consider a linear system described by the first-order differential equation:

$$X'(t) + \alpha X(t) = Z(t) \qquad t \geq 0, X(0) = 0. \tag{9.93}$$

For example, $X(t)$ may represent the voltage across the capacitor of an RC circuit with current input $Z(t)$. We now show how to obtain $m_X(t)$ and $R_X(t_1, t_2)$. If the input process $Z(t)$ is Gaussian, then the output process will also be Gaussian. Therefore, in the case of Gaussian input processes, we can then characterize the joint pdf of the output process.

We obtain a differential equation for $m_X(t)$ by taking the expected value of Eq. (9.93):

$$E[X'(t)] + E[X(t)] = m'_X(t) + m_X(t) = m_Z(t) \qquad t \geq 0 \tag{9.94}$$

with initial condition $m_X(0) = E[X(0)] = 0$.

As an intermediate step we next find a differential equation for $R_{Z,X}(t_1, t_2)$. If we multiply Eq. (9.93) by $Z(t_1)$ and take the expected value, we obtain

$$E[Z(t_1)X'(t_2)] + \alpha E[Z(t_1)X(t_2)] = E[Z(t_1)Z(t_2)] \qquad t_2 \geq 0$$

with initial condition $E[Z(t_1)X(0)] = 0$ since $X(0) = 0$. The same derivation that led to the cross-correlation between $X(t)$ and $X'(t)$ (see Eq. 9.83) can be used to show that

$$E[Z(t_1)X'(t_2)] = \frac{\partial}{\partial t_2} R_{Z,X}(t_1, t_2).$$

Thus we obtain the following differential equation:

$$\frac{\partial}{\partial t_2} R_{Z,X}(t_1, t_2) + \alpha R_{Z,X}(t_1, t_2) = R_Z(t_1, t_2) \qquad t_2 \geq 0 \tag{9.95}$$

with initial condition $R_{Z,X}(t_1, 0) = 0$.

Finally we obtain a differential equation for $R_Z(t_1, t_2)$. Multiply Eq. (9.93) by $X(t_2)$ and take the expected value:

$$E[X'(t_1)X(t_2)] + \alpha E[X(t_1)X(t_2)] = E[Z(t_1)X(t_2)] \qquad t_1 \geq 0$$

with initial condition $E[X(0)X(t_2)] = 0$. This leads to the differential equation

$$\frac{\partial}{\partial t_1} R_X(t_1, t_2) + \alpha R_X(t_1, t_2) = R_{Z,X}(t_1, t_2) \qquad t_1 \geq 0 \tag{9.96}$$

with initial condition $R_{Z,X}(0, t_2) = 0$. Note that the solution to Eq. (9.95) appears as the forcing function in Eq. (9.96). Thus we conclude that by solving the differential equations in Eqs. (9.94), (9.95), and (9.96) we obtain the mean and autocorrelation function for $X(t)$.

Example 9.46 Ornstein-Uhlenbeck Process

Equation (9.93) with the input given by a zero-mean, white Gaussian noise process is called the *Langevin equation*, after the scientist who formulated it in 1908 to describe the Brownian motion of a free particle. In this formulation $X(t)$ represents the velocity of the particle, so that Eq. (9.93) results from equating the acceleration of the particle $X'(t)$ to the force on the particle due to friction $-\alpha X(t)$ and the force due to random collisions $Z(t)$. We present the solution developed by Uhlenbeck and Ornstein in 1930.

First, we note that since the input process $Z(t)$ is Gaussian, the output process $X(t)$ will also be a Gaussian random process. Next we recall that the first-order differential equation

$$x'(t) + ax(t) = g(t) \qquad t \geq 0, x(0) = 0$$

has solution

$$x(t) = \int_0^t e^{-a(t-\tau)} g(\tau)\, d\tau \qquad t \geq 0.$$

Therefore the solution to Eq. (9.94) is

$$m_X(t) = \int_0^t e^{-\alpha(t-\tau)} m_Z(\tau)\, d\tau = 0.$$

The autocorrelation of the white Gaussian noise process is

$$R_Z(t_1, t_2) = \sigma^2 \delta(t_1 - t_2).$$

Equation (9.95) is also a first-order differential equation, and it has solution

$$\begin{aligned} R_{Z,X}(t_1, t_2) &= \int_0^{t_2} e^{-\alpha(t_2-\tau)} R_Z(t_1, \tau)\, d\tau \\ &= \int_0^{t_2} e^{-\alpha(t_2-\tau)} \sigma^2 \delta(t_1 - \tau)\, d\tau \\ &= \begin{cases} 0 & 0 \leq t_2 < t_1 \\ \sigma^2 e^{-\alpha(t_2-t_1)} & t_2 \geq t_1 \end{cases} \\ &= \sigma^2 e^{-\alpha(t_2-t_1)} u(t_2 - t_1), \end{aligned}$$

where $u(x)$ is the unit step function.

The autocorrelation function of the output process $X(t)$ is the solution to the first-order differential equation Eq. (9.96). The solution is given by

$$\begin{aligned} R_X(t_1, t_2) &= \int_0^{t_1} e^{-\alpha(t_1-\tau)} R_{Z,X}(\tau, t_2)\, d\tau \\ &= \sigma^2 \int_0^{t_1} e^{-\alpha(t_1-\tau)} e^{-\alpha(t_2-\tau)} u(t_2 - \tau)\, d\tau \\ &= \sigma^2 \int_0^{\min(t_1,t_2)} e^{-\alpha(t_1-\tau)} e^{-\alpha(t_2-\tau)}\, d\tau \\ &= \frac{\sigma^2}{2\alpha}\left(e^{-\alpha|t_1-t_2|} - e^{-\alpha(t_1+t_2)}\right) \qquad t_1 \geq 0, t_2 \geq 0. \end{aligned} \tag{9.97a}$$

A Gaussian random process with this autocorrelation function is called an **Ornstein-Uhlenbeck process**. Thus we conclude that the output process $X(t)$ is an Ornstein-Uhlenbeck process.

If we let $t_1 = t$ and $t_2 = t + \tau$, then as t approaches infinity,

$$R_X(t + \tau, t) \rightarrow \frac{\sigma^2}{2\alpha} e^{-\alpha|\tau|}. \tag{9.97b}$$

This shows that the effect of the zero initial condition dies out as time progresses, and the process becomes wide-sense stationary. Since the process is Gaussian, this also implies that the process becomes strict-sense stationary.

9.8 TIME AVERAGES OF RANDOM PROCESSES AND ERGODIC THEOREMS

At some point, the parameters of a random process must be obtained through measurement. The results from Chapter 7 and the statistical methods of Chapter 8 suggest that we repeat the random experiment that gives rise to the random process a large number of times and take the arithmetic average of the quantities of interest. For example, to estimate the mean $m_X(t)$ of a random process $X(t, \zeta)$, we repeat the random experiment and take the following average:

$$\hat{m}_X(t) = \frac{1}{N}\sum_{i=1}^{N} X(t, \zeta_i), \tag{9.98}$$

where N is the number of repetitions of the experiment, and $X(t, \zeta_i)$ is the realization observed in the ith repetition.

In some situations, we are interested in estimating the mean or autocorrelation functions from the **time average** of a single realization, that is,

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} X(t, \zeta)\, dt. \tag{9.99}$$

An **ergodic theorem** states conditions under which a time average converges as the observation interval becomes large. In this section, we are interested in ergodic theorems that state when time averages converge to the ensemble average (expected value).

The strong law of large numbers, presented in Chapter 7, is one of the most important ergodic theorems. It states that if X_n is an iid discrete-time random process with finite mean $E[X_n] = m$, then the time average of the samples converges to the ensemble average with probability one:

$$P\left[\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = m\right] = 1. \tag{9.100}$$

This result allows us to estimate m by taking the time average of a single realization of the process. We are interested in obtaining results of this type for a larger class of random processes, that is, for non-iid, discrete-time random processes, and for continuous-time random processes.

The following example shows that, in general, time averages do not converge to ensemble averages.

Example 9.47

Let $X(t) = A$ for all t, where A is a zero-mean, unit-variance random variable. Find the limiting value of the time average.

The mean of the process is $m_X(t) = E[X(t)] = E[A] = 0$. However, Eq. (9.99) gives

$$\langle X(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} A\, dt = A.$$

Thus the time-average mean does not always converge to $m_X(t) = 0$. Note that this process is stationary. Thus this example shows that stationary processes need not be ergodic.

Consider the estimate given by Eq. (9.99) for $E[X(t)] = m_X(t)$. The estimate yields a single number, so obviously it only makes sense to consider processes for which $m_X(t) = m$, a constant. We now develop an ergodic theorem for the time average of wide-sense stationary processes.

Let $X(t)$ be a WSS process. The expected value of $\langle X(t) \rangle_T$ is

$$E[\langle X(t)\rangle_T] = E\left[\frac{1}{2T}\int_{-T}^{T} X(t)\,dt\right] = \frac{1}{2T}\int_{-T}^{T} E[X(t)]\,dt = m. \tag{9.101}$$

Equation (9.101) states that $\langle X(t)\rangle_T$ is an unbiased estimator for m.

Consider the variance of $\langle X(t)\rangle_T$:

$$\begin{aligned}\text{VAR}[\langle X(t)\rangle_T] &= E[(\langle X(t)\rangle_T - m)^2]\\ &= E\left[\left\{\frac{1}{2T}\int_{-T}^{T}(X(t) - m)\,dt\right\}\left\{\frac{1}{2T}\int_{-T}^{T}(X(t') - m)\,dt'\right\}\right]\\ &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} E[(X(t) - m)(X(t') - m)]\,dt\,dt'\\ &= \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t, t')\,dt\,dt'. \end{aligned} \tag{9.102}$$

Since the process $X(t)$ is WSS, Eq. (9.102) becomes

$$\text{VAR}[\langle X(t)\rangle_T] = \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_X(t - t')\,dt\,dt'. \tag{9.103}$$

Figure 9.17 shows the region of integration for this integral. The integrand is constant along the line $u = t - t'$ for $-2T < u < 2T$, so we can evaluate the integral as the

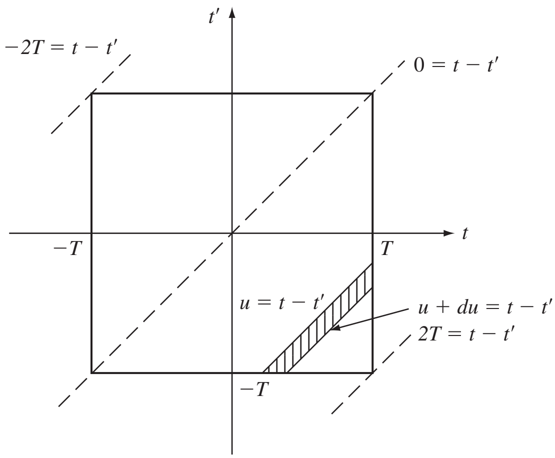

FIGURE 9.17
Region of integration for integral in Eq. (9.102).

sums of infinitesimal strips as shown in the figure. It can be shown that each strip has area $(2T - |u|)\,du$, so the contribution of each strip to the integral is $(2T - |u|)C_X(u)\,du$. Thus

$$\begin{aligned}\mathrm{VAR}[\langle X(t)\rangle_T] &= \frac{1}{4T^2}\int_{-2T}^{2T}(2T - |u|)C_X(u)\,du \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|u|}{2T}\right)C_X(u)\,du. \end{aligned} \tag{9.104}$$

Therefore, $\langle X(t)\rangle_T$ will approach m in the mean square sense, that is, $E[(\langle X(t)\rangle_T - m)^2] \to 0$, if the expression in Eq. (9.104) approaches zero with increasing T. We have just proved the following ergodic theorem.

Theorem

Let $X(t)$ be a WSS process with $m_X(t) = m$, then

$$\lim_{T\to\infty}\langle X(t)\rangle_T = m$$

in the mean square sense, if and only if

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1 - \frac{|u|}{2T}\right)C_X(u)\,du = 0.$$

In keeping with engineering usage, we say that a WSS process is **mean ergodic** if it satisfies the conditions of the above theorem.

The above theorem can be used to obtain ergodic theorems for the time average of other quantities. For example, if we replace $X(t)$ with $Y(t + \tau)Y(t)$ in Eq. (9.99), we obtain a time-average estimate for the autocorrelation function of the process $Y(t)$:

$$\langle Y(t + \tau)Y(t)\rangle_T = \frac{1}{2T}\int_{-T}^{T} Y(t + \tau)Y(t)\,dt. \tag{9.105}$$

It is easily shown that $E[\langle Y(t + \tau)Y(t)\rangle_T] = R_Y(\tau)$ if $Y(t)$ is WSS. The above ergodic theorem then implies that the time-average autocorrelation converges to $R_Y(\tau)$ in the mean square sense if the term in Eq. (9.104) with $X(t)$ replaced by $Y(t)Y(t + \tau)$ converges to zero.

Example 9.48

Is the random telegraph process mean ergodic?

The covariance function for the random telegraph process is $C_X(\tau) = e^{-2\alpha|\tau|}$, so the variance of $\langle X(t)\rangle_T$ is

$$\begin{aligned}\mathrm{VAR}[\langle X(t)\rangle_T] &= \frac{2}{2T}\int_0^{2T}\left(1 - \frac{u}{2T}\right)e^{-2\alpha u}\,du \\ &< \frac{1}{T}\int_0^{2T} e^{-2\alpha u}\,du = \frac{1 - e^{-4\alpha T}}{2\alpha T}.\end{aligned}$$

The bound approaches zero as $T \to \infty$, so $\mathrm{VAR}[\langle X(t)\rangle_T] \to 0$. Therefore the process is mean ergodic.

If the random process under consideration is discrete-time, then the time-average estimate for the mean and the autocorrelation functions of X_n are given by

$$\langle X_n \rangle_T = \frac{1}{2T + 1} \sum_{n=-T}^{T} X_n \tag{9.106}$$

$$\langle X_{n+k} X_n \rangle_T = \frac{1}{2T + 1} \sum_{n=-T}^{T} X_{n+k} X_n. \tag{9.107}$$

If X_n is a WSS random process, then $E[\langle X_n \rangle_T] = m$, and so $\langle X_n \rangle_T$ is an unbiased estimate for m. It is easy to show that the variance of $\langle X_n \rangle_T$ is

$$\text{VAR}[\langle X_n \rangle_T] = \frac{1}{2T + 1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T + 1}\right) C_X(k). \tag{9.108}$$

Therefore, $\langle X_n \rangle_T$ approaches m in the mean square sense and is mean ergodic if the expression in Eq. (9.108) approaches zero with increasing T.

Example 9.49 Ergodicity and Exponential Correlation

Let X_n be a wide-sense stationary discrete-time process with mean m and covariance function $C_X(k) = \sigma^2 \rho^{-|k|}$, for $|\rho| < 1$ and $k = 0, \pm 1, +2, \ldots$. Show that X_n is mean ergodic.

The variance of the sample mean (Eq. 9.106) is:

$$\begin{aligned} \text{VAR}[\langle X_n \rangle_T &= \frac{1}{2T + 1} \sum_{k=-2T}^{2T} \left(1 - \frac{|k|}{2T + 1}\right) \sigma^2 \rho^{|k|} \\ &< \frac{2}{2T + 1} \sum_{k=0}^{\infty} \sigma^2 \rho^k = \frac{2\sigma^2}{2T + 1} \frac{1}{1 - \rho}. \end{aligned}$$

The bound on the right-hand side approaches zero as T increases and so X_n is mean ergodic.

Example 9.50 Ergodicity of Self-Similar Process and Long-Range Dependence

Let X_n be a wide-sense stationary discrete-time process with mean m and covariance function

$$C_X(k) = \frac{\sigma^2}{2} \{|k + 1|^{2H} - 2|k|^{2H} + |k - 1|^{2H}\} \tag{9.109}$$

for $1/2 < H < 1$ and $k = 0, \pm 1, +2, \ldots$ X_n is said to be **second-order self-similar**. We will investigate the ergodicity of X_n.

We rewrite the variance of the sample mean in (Eq. 9.106) as follows:

$$\begin{aligned} \text{VAR}[\langle X_n \rangle_T] &= \frac{1}{(2T + 1)^2} \sum_{k=-2T}^{2T} (2T + 1 - |k|) C_X(k) \\ &= \frac{1}{(2T + 1)^2} \{(2T + 1) C_X(0) + 2(2T C_X(1)) + \ldots + 2 C_X(2T)\}. \end{aligned}$$

It is easy to show (See Problem 9.132) that the sum inside the braces is $\sigma^2(2T + 1)^{2H}$. Therefore the variance becomes:

$$\text{VAR}[\langle X_n \rangle_T] = \frac{1}{(2T + 1)^2}\sigma^2(2T + 1)^{2H} = \sigma^2(2T + 1)^{2H-2}. \tag{9.110}$$

The value of H, which is called the **Hurst parameter**, affects the convergence behavior of the sample mean. Note that if $H = 1/2$, the covariance function becomes $C_X(k) = 1/2\sigma^2\delta_k$ which corresponds to an iid sequence. In this case, the variance becomes $\sigma^2/(2T + 1)$ which is the convergence rate of the sample mean for iid samples. However, for $H > 1/2$, the variance becomes:

$$\text{VAR}[\langle X_n \rangle_T] = \frac{\sigma^2}{2T + 1}(2T + 1)^{2H-1}, \tag{9.111}$$

so the convergence of the sample mean is slower by a factor of $(2T + 1)^{2H-1}$ than for iid samples.

The slower convergence of the sample mean when $H > 1/2$ results from the long-range dependence of X_n. It can be shown that for large k, the covariance function is approximately given by:

$$C_X(k) = \sigma^2 H(2H - 1)k^{2H-2}. \tag{9.112}$$

For $1/2 < H < 1$, $C(k)$ decays as $1/k^\alpha$ where $0 < \alpha < 1$, which is a very slow decay rate. Thus the dependence between values of X_n decreases slowly and the process is said to have a long memory or long-range dependence.

*9.9 FOURIER SERIES AND KARHUNEN-LOEVE EXPANSION

Let $X(t)$ be a wide-sense stationary, mean square periodic random process with period T, that is, $E[(X(t + T) - X(t))^2] = 0$. In order to simplify the development, we assume that $X(t)$ is zero mean. We show that $X(t)$ can be represented in a mean square sense by a **Fourier series**:

$$X(t) = \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}, \tag{9.113}$$

where the coefficients are random variables defined by

$$X_k = \frac{1}{T}\int_0^T X(t')e^{-j2\pi kt'/T}\,dt'. \tag{9.114}$$

Equation (9.114) implies that, in general, the coefficients are complex-valued random variables. *For complex-valued random variables, the correlation between two random variables X and Y is defined by* $E[XY^*]$. We also show that *the coefficients are orthogonal random variables*, that is, $E[X_k X_m^*] = 0$ for $k \neq m$.

Recall that if $X(t)$ is mean square periodic, then $R_X(\tau)$ is a periodic function in τ with period T. Therefore, it can be expanded in a Fourier series:

$$R_X(\tau) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\tau/T}, \tag{9.115}$$

where the coefficients a_k are given by

$$a_k = \frac{1}{T}\int_0^T R_X(t')e^{-j2\pi kt'/T}\,dt'. \tag{9.116}$$

The coefficients a_k appear in the following derivation.

First, we show that the coefficients in Eq. (9.113) are orthogonal random variables, that is, $E[X_k X_m^*] = 0$:

$$E[X_k X_m^*] = E\left[X_k \frac{1}{T}\int_0^T X^*(t')e^{j2\pi mt'/T}\,dt'\right]$$
$$= \frac{1}{T}\int_0^T E[X_k X^*(t')]e^{j2\pi mt'/T}\,dt'.$$

The integrand of the above equation has

$$E[X_k X^*(t)] = E\left[\frac{1}{T}\int_0^T X(u)e^{-j2\pi ku/T}\,du\, X^*(t)\right]$$
$$= \frac{1}{T}\int_0^T R_X(u-t)e^{-j2\pi ku/T}\,du$$
$$= \left\{\frac{1}{T}\int_{-t}^{T-t} R_X(v)e^{-j2\pi kv/T}\,dv\right\}e^{-j2\pi kt/T}$$
$$= a_k e^{-j2\pi kt/T},$$

where we have used the fact that the Fourier coefficients can be calculated over any full period. Therefore

$$E[X_k X_m^*] = \frac{1}{T}\int_0^T a_k e^{-j2\pi kt'/T}e^{j2\pi mt'/T}\,dt' = a_k\delta_{k,m}, \tag{9.117}$$

where $\delta_{k,m}$ is the Kronecker delta function. Thus X_k and X_m are orthogonal random variables. Note that the above equation implies that $a_k = E\left[|X_k|^2\right]$, that is, the a_k are real-valued.

To show that the Fourier series equals $X(t)$ in the mean square sense, we take

$$E\left[\left|X(t) - \sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}\right|^2\right]$$
$$= E[|X(t)|^2] - E\left[X(t)\sum_{k=-\infty}^{\infty} X_k^* e^{-j2\pi kt/T}\right]$$
$$- E\left[X^*(t)\sum_{k=-\infty}^{\infty} X_k e^{j2\pi kt/T}\right] + E\left[\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} X_k X_m^* e^{j2\pi(k-m)t/T}\right]$$
$$= R_X(0) - \sum_{k=-\infty}^{\infty} a_k - \sum_{k=-\infty}^{\infty} a_k^* + \sum_{k=-\infty}^{\infty} a_k.$$

The above equation equals zero, since the a_k are real and since $R_X(0) = \Sigma a_k$ from Eq. (9.115).

If $X(t)$ is a wide-sense stationary random process that is *not* mean square periodic, we can still expand $X(t)$ in the Fourier series in an arbitrary interval $[0, T]$. Mean square equality will hold only inside the interval. Outside the interval, the expansion repeats itself with period T. The Fourier coefficients will no longer be orthogonal; instead they are given by

$$E[X_k X_m^*] = \frac{1}{T^2}\int_0^T \int_0^T R_X(t - u)e^{-j2\pi kt/T}e^{j2\pi mu/T}\,dt\,du. \qquad (9.118)$$

It is easy to show that if $X(t)$ is mean square periodic, then this equation reduces to Eq. (9.117).

9.9.1 Karhunen-Loeve Expansion

In this section we present the **Karhunen-Loeve expansion**, which allows us to expand a (possibly nonstationary) random process $X(t)$ in a series:

$$X(t) = \sum_{k=1}^{\infty} X_k \phi_k(t) \qquad 0 \le t \le T, \qquad (9.119a)$$

where

$$X_k = \int_0^T X(t)\phi_k^*(t)\,dt, \qquad (9.119b)$$

where the equality in Eq. (9.119a) is in the mean square sense, where the coefficients $\{X_k\}$ are orthogonal random variables, and where the functions $\{\phi_k(t)\}$ are orthonormal:

$$\int_0^T \phi_i(t)\phi_j(t)\,dt = \delta_{i,j} \qquad \text{for all } i, j.$$

In other words, *the Karhunen-Loeve expansion provides us with many of the nice properties of the Fourier series for the case where $X(t)$ is not mean square periodic.* For simplicity, we again assume that $X(t)$ is zero mean.

In order to motivate the Karhunen-Loeve expansion, we review the Karhunen-Loeve transform for vector random variables as introduced in Section 6.3. Let $\mathbf{X}$ be a zero-mean, vector random variable with covariance matrix K_X. The eigenvalues and eigenvectors of K_X are obtained from

$$K_X \mathbf{e}_i = \lambda_i \mathbf{e}_i, \qquad (9.120)$$

where the $\mathbf{e}_i$ are column vectors. The set of normalized eigenvectors are orthonormal, that is, $\mathbf{e}_i^{\mathrm{T}}\mathbf{e}_j = \delta_{i,j}$. Define the matrix P of eigenvectors and Λ of eigenvalues as

$$P = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] \qquad \Lambda = \mathrm{diag}[\lambda_i],$$

then

$$K_X = P\Lambda P^{\mathrm{T}} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}\begin{bmatrix} \mathbf{e}_1^{\mathrm{T}} \\ \mathbf{e}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{e}_n^{\mathrm{T}} \end{bmatrix}$$

$$= [\lambda_1\mathbf{e}_1, \lambda_2\mathbf{e}_2, \ldots, \lambda_n\mathbf{e}_n]\begin{bmatrix}\mathbf{e}_1^{\mathrm{T}}\\ \mathbf{e}_2^{\mathrm{T}}\\ \vdots\\ \mathbf{e}_n^{\mathrm{T}}\end{bmatrix}$$

$$= \sum_{k=1}^{n}\lambda_i\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}. \tag{9.121a}$$

Therefore we find that the covariance matrix can be expanded as a weighted sum of matrices, $\mathbf{e}_i\mathbf{e}_i^{\mathrm{T}}$. In addition, if we let $\mathbf{Y} = P^{\mathrm{T}}\mathbf{X}$, then the random variables in $\mathbf{Y}$ are orthogonal. Furthermore, since $PP^{\mathrm{T}} = I$, then

$$\mathbf{X} = P\mathbf{Y} = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]\begin{bmatrix}Y_1\\ Y_2\\ \vdots\\ Y_n\end{bmatrix} = \sum_{k=1}^{n}Y_k\mathbf{e}_k. \tag{9.121b}$$

Thus we see that the arbitrary vector random variable $\mathbf{X}$ can be expanded as a weighted sum of the eigenvectors of K_X, where the coefficients are orthogonal random variables. Furthermore the eigenvectors form an orthonormal set. These are exactly the properties we seek in the Karhunen-Loeve expansion for $X(t)$. If the vector random variable $\mathbf{X}$ is jointly Gaussian, then the components of $\mathbf{Y}$ are independent random variables. This results in tremendous simplification in a wide variety of problems.

In analogy to Eq. (9.120), we begin by considering the following eigenvalue equation:

$$\int_0^T K_X(t_1, t_2)\phi_k(t_2)\,dt_2 = \lambda_k\phi_k(t_1) \qquad 0 \le t_1 \le T. \tag{9.122}$$

The values λ_k and the corresponding functions $\phi_k(t)$ for which the above equation holds are called the eigenvalues and eigenfunctions of the covariance function $K_X(t_1, t_2)$. Note that it is possible for the eigenfunctions to be complex-valued, e.g., complex exponentials. It can be shown that if $K_X(t_1, t_2)$ is continuous, then the normalized eigenfunctions form an orthonormal set and satisfy Mercer's theorem:

$$K_X(t_1, t_2) = \sum_{k=1}^{\infty}\lambda_k\phi_k(t_1)\phi_k^*(t_2). \tag{9.123}$$

Note the correspondence between Eq. (9.121) and Eq. (9.123). Equation (9.123) in turn implies that

$$K_X(t, t) = \sum_{k=1}^{\infty}\lambda_k|\phi_k(t)|^2. \tag{9.124}$$

We are now ready to show that the equality in Eq. (9.119a) holds in the mean square sense and that the coefficients X_k are orthogonal random variables. First consider $E[X_kX_m^*]$:

$$E[X_kX_m^*] = E\left[X_m^*\int_0^T X(t')\phi_k^*(t)\,dt'\right] = \int_0^T E[X(t')X_m^*]\phi_k^*(t')\,dt'.$$

The integrand of the above equation has

$$E[X(t)X_m^*] = E\left[X(t)\int_0^T X^*(u)\phi_m(u)\,du\right] = \int_0^T K_X(t,u)\phi_m(u)\,du$$
$$= \lambda_m\phi_m(t).$$

Therefore

$$E[X_kX_m^*] = \int_0^T \lambda_m\phi_k^*(t')\phi_m(t')\,dt' = \lambda_k\delta_{k,m},$$

where $\delta_{k,m}$ is the Kronecker delta function. Thus X_k and X_m are orthogonal random variables. Note that the above equation implies that $\lambda_k = E\left[|X_k|^2\right]$, that is, the eigenvalues are real-valued.

To show that the Karhunen-Loeve expansion equals $X(t)$ in the mean square sense, we take

$$E\left[\left|X(t) - \sum_{k=-\infty}^{\infty} X_k\phi_k(t)\right|^2\right]$$
$$= E[|X(t)|^2] - E\left[X(t)\sum_{k=-\infty}^{\infty} X_k^*\phi_k^*(t)\right]$$
$$- E\left[X^*(t)\sum_{k=-\infty}^{\infty} X_k\phi_k(t)\right]$$
$$+ E\left[\sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} X_kX_m^*\phi_k(t)\phi_m^*(t)\right]$$
$$= R_X(t,t) - \sum_{k=-\infty}^{\infty}\lambda_k|\phi_k(t)|^2$$
$$- \sum_{k=-\infty}^{\infty}\lambda_k^*|\phi_k(t)|^2 + \sum_{k=-\infty}^{\infty}\lambda_k|\phi_k(t)|^2.$$

The above equation equals zero from Eq. (9.124) and from the fact that the λ_k are real. Thus we have shown that Eq. (9.119a) holds in the mean square sense.

Finally, we note that in the important case where $X(t)$ is a Gaussian random process, then the components X_k will be independent Gaussian random variables. This result is extremely useful in solving certain signal detection and estimation problems. [Van Trees.]

Example 9.51 Wiener Process

Find the Karhunen-Loeve expansion for the Wiener process.

Equation (9.122) for the Wiener process gives, for $0 \le t_1 \le T$,

$$\lambda\phi(t_1) = \int_0^T \sigma^2 \min(t_1, t_2)\phi(t_2)\,dt_2$$
$$= \sigma^2\int_0^{t_1} t_2\phi(t_2)\,dt_2 + \sigma^2\int_{t_1}^T t_1\phi(t_2)\,dt_2.$$

We differentiate the above integral equation once with respect to t_1 to obtain an integral equation and again to obtain a differential equation:

$$\sigma^2 \int_{t_1}^{T} \phi(t_2)\, dt_2 = \lambda \frac{d}{dt_1}\phi(t_1)$$

$$-\phi(t_1) = \frac{\lambda}{\sigma^2}\frac{d^2}{dt_1^2}\phi(t_1).$$

This second-order differential equation has a sinusoidal solution:

$$\phi(t_1) = a \sin\frac{\sigma t_1}{\sqrt{\lambda}} + b\cos\frac{\sigma t_1}{\sqrt{\lambda}}.$$

In order to solve the above equation for a, b, and λ, we need boundary conditions for the differential equation. We obtain these by substituting the general solution for $\phi(t)$ into the integral equation:

$$\frac{\lambda}{\sigma^2}\left(a\sin\frac{\sigma t_1}{\sqrt{\lambda}} + b\cos\frac{\sigma t_1}{\sqrt{\lambda}}\right) = \int_0^{t_1} t_2\phi(t_2)\, dt_2 + \int_{t_1}^{T} t_1\phi(t_2)\, dt_2.$$

As t_1 approaches zero, the right-hand side approaches zero. This implies that $b = 0$ in the left-hand side of the equation. A second boundary condition is obtained by letting t_1 approach T in the equation obtained after the first differentiation of the integral equation:

$$0 = \lambda\frac{d}{dt_1}\phi(T) = \frac{\sigma a}{\sqrt{\lambda}}\cos\frac{\sigma T}{\sqrt{\lambda}}.$$

This implies that

$$\frac{\sigma T}{\sqrt{\lambda}} = \left(n - \frac{1}{2}\right)\pi \qquad n = 1, 2, \ldots.$$

Therefore the eigenvalues are given by

$$\lambda_n = \frac{\sigma^2 T^2}{\left(n - \frac{1}{2}\right)^2 \pi^2} \qquad n = 1, 2, \ldots.$$

The normalization requirement implies that

$$1 = \int_0^T \left(a\sin\frac{\sigma t}{\sqrt{\lambda}}\right)^2 dt = a^2\frac{T}{2},$$

which implies that $a = (2/T)^{1/2}$. Thus the eigenfunctions are given by

$$\phi_n(t) = \sqrt{\frac{2}{T}}\sin\left(n - \frac{1}{2}\right)\frac{\pi}{T}t \qquad 0 \le t \le T,$$

and the Karhunen-Loeve expansion for the Wiener process is

$$X(t) = \sum_{n-1}^{\infty} X_n\sqrt{\frac{2}{T}}\sin\left(n - \frac{1}{2}\right)\frac{\pi}{T}t \qquad 0 \le t < T,$$

where the X_n are zero-mean, independent Gaussian random variables with variance given by λ_n.

Example 9.52 White Gaussian Noise Process

Find the Karhunen-Loeve expansion of the white Gaussian noise process.

The white Gaussian noise process is the derivative of the Wiener process. If we take the derivative of the Karhunen-Loeve expansion of the Wiener process, we obtain

$$X'(t) = \sum_{n=1}^{\infty} \frac{\sigma}{\sqrt{\lambda}} X_n \sqrt{\frac{2}{T}} \cos\left(n - \frac{1}{2}\right)\frac{\pi}{T}t$$

$$= \sum_{n=1}^{\infty} W_n \sqrt{\frac{2}{T}} \cos\left(n - \frac{1}{2}\right)\frac{\pi}{T}t \qquad 0 \leq t < T,$$

where the W_n are independent Gaussian random variables with the same variance σ^2. This implies that the process has infinite power, a fact we had already found about the white Gaussian noise process. In the Problems we will see that any orthonormal set of eigenfunctions can be used in the Karhunen-Loeve expansion for white Gaussian noise.

9.10 GENERATING RANDOM PROCESSES

Many engineering systems involve random processes that interact in complex ways. It is not always possible to model these systems precisely using analytical methods. In such situations computer simulation methods are used to investigate the system dynamics and to measure the performance parameters of interest. In this section we consider two basic methods to generating random processes. The first approach involves generating the sum process of iid sequences of random variables. We saw that this approach can be used to generate the binomial and random walk processes, and, through limiting procedures, the Wiener and Poisson processes. The second approach involves taking the linear combination of deterministic functions of time where the coefficients are given by random variables. The Fourier series and Karhunen-Loeve expansion use this approach. Real systems, e.g., digital modulation systems, also generate random processes in this manner.

9.10.1 Generating Sum Random Processes

The generation of sample functions of the sum random process involves two steps:

1. Generate a sequence of iid random variables that drive the sum process.
2. Generate the cumulative sum of the iid sequence.

Let D be an array of samples of the desired iid random variables. The function `cumsum(D)` in Octave and MATLAB then provides the cumulative sum, that is, the sum process, that results from the sequence in D.

The code below generates m realizations of an n-step random walk process.

```
>p=1/2
>n=1000
>m=4
```

```
> V=-1:2:1;
> P=[1-p,p];
> D=discrete_rnd(V, P, m, n);
> X=cumsum (D);
> plot (X)
```

Figures 9.7(a) and 9.7(b) in Section 9.3 show four sample functions of the symmetric random walk process for $p = 1/2$. The sample functions vary over a wide range of positive and negative values. Figure 9.7(c) shows four sample functions for $p = 3/4$. The sample functions now have a strong linear trend consistent with the mean $n(2p - 1)$. The variability about this trend is somewhat less than in the symmetric case since the variance function is now $n4p(1 - p) = 3n/4$.

We can generate an approximation to a Poisson process by summing iid Bernoulli random variables. Figure 9.18(a) shows ten realizations of Poisson processes with $\lambda = 0.4$ arrivals per second. The sample functions for $T = 50$ seconds were generated using a 1000-step binomial process with $p = \lambda T/n = 0.02$. The linear increasing trend of the Poisson process is evident in the figure. Figure 9.18(b) shows the estimate of the mean and variance functions obtained by averaging across the 10 realizations. The linear trend in the sample mean function is very clear; the sample variance function is also linear but is much more variable. The mean and variance functions of the realizations are obtained using the commands `mean(transpose(X))` and `var(transpose(X))`.

We can generate sample functions of the random telegraph signal by taking the Poisson process $N(t)$ and calculating $X(t) = 2(N(t) \text{ modulo } 2) - 1$. Figure 9.19(a) shows a realization of the random telegraph signal. Figure 9.19(b) shows an estimate of the covariance function of the random telegraph signal. The exponential decay in the covariance function can be seen in the figure. See Eq. (9.44).

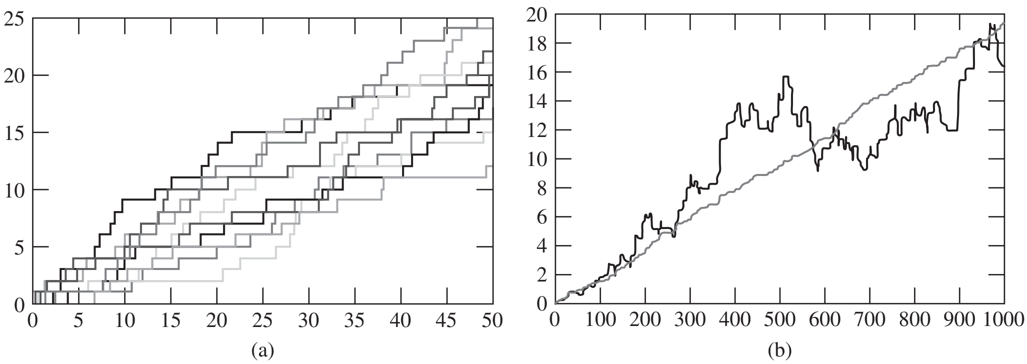

FIGURE 9.18
(a) Ten sample functions of a Poisson random process with $\lambda = 0.4$. (b) Sample mean and variance of ten sample functions of a Poisson random process with $\lambda = 0.4$.

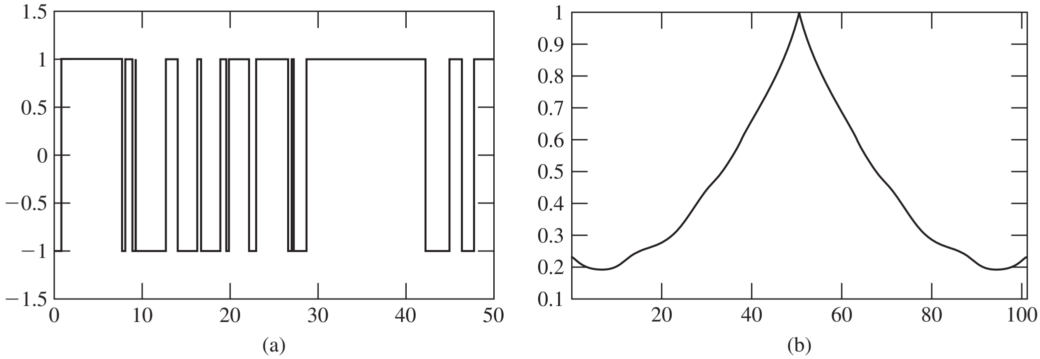

FIGURE 9.19
(a) Sample function of a random telegraph process with $\lambda = 0.4$. (b) Estimate of covariance function of a random telegraph process.

The covariance function is computed using the function `CX_est` below.

```
function [CXall]=CX_est (X, L, M_est)
N=length(X);                                    % N is number of samples
CX=zeros (1,L+1);                               % L is maximum lag
M_est=mean(X)                                   % Sample mean
for m=1:L+1,                                    % Add product terms
  for n=1:N-m+1,
    CX(m)=CX(m) + (X(n) - M_est) * (X(n+m-1)- M_est);
  end;
   CX (m)=CX(m) / (N-m+1);                      % Normalize by number of terms
end;
for i=1:L,
   CXall(i)=CX(L+2-i);                          % Lags 1 to L
end
CXall(L+1:2*L+1)=CX(1:L+1);                     % Lags L + 1 to 2L + 1
```

The Wiener random process can also be generated as a sum process. One approach is to generate a properly scaled random walk process, as in Eq. (9.50). A better approach is to note that the Wiener process has independent Gaussian increments, as in Eq. (9.52), and therefore, to generate the sequence D of increments for the time subintervals, and to then find the corresponding sum process. The code below generates a sample of the Wiener process:

```
> a=2
> delta=0.001
> n=1000
> D=normal_rnd(0,a*delta,1,n)
> X=cumsum(D);
> plot(X)
```

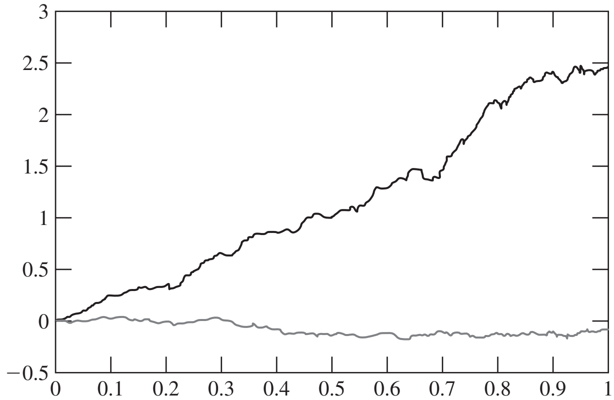

FIGURE 9.20
Sample mean and variance functions from 50 realizations of Wiener process.

Figure 9.12 in Section 9.5 shows four sample functions of a Brownian motion process with $\alpha = 2$. Figure 9.20 shows the sample mean and sample variance of 50 sample functions of the Wiener process with $\alpha = 2$. It can be seen that the mean across the 50 realizations is close to zero which is the actual mean function for the process. The sample variance across the 50 realizations increases steadily and is close to the actual variance function which is $\alpha t = 2t$.

9.10.2 Generating Linear Combinations of Deterministic Functions

In some situations a random process can be represented as a linear combination of deterministic functions where the coefficients are random variables. The Fourier series and the Karhunen-Loeve expansions are examples of this type of representation.

In Example 9.51 let the parameters in the Karhunen-Loeve expansion for a Wiener process in the interval $0 \le t \le T$ be $T = 1, \sigma^2 = 1$:

$$X(t) = \sum_{n=1}^{\infty} X_n \sqrt{\frac{2}{T}} \sin\left(n - \frac{1}{2}\right)\frac{\pi t}{T} = \sum_{n=1}^{\infty} X_n \sqrt{2} \sin\left(n - \frac{1}{2}\right)\pi t$$

where the X_n are zero-mean, independent Gaussian random variables with variance

$$\lambda_n = \frac{\sigma^2 T^2}{(n - 1/2)^2 \pi^2} = \frac{1}{(n - 1/2)^2 \pi^2}.$$

The following code generates the 100 Gaussian coefficients for the Karhunen-Loeve expansion for the Wiener process.

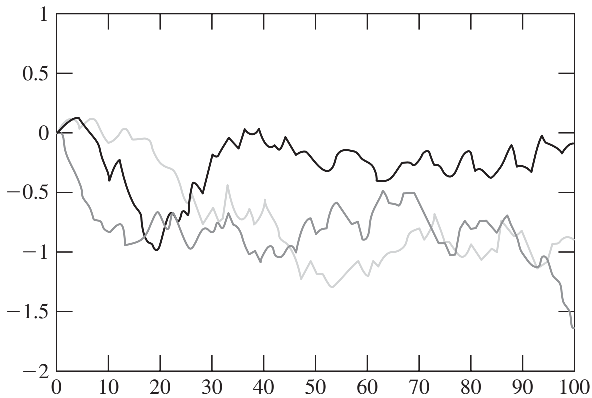

FIGURE 9.21
Sample functions for Wiener process using 100 terms in Karhunen-Loeve expansion.

```
> M=zeros(100,1);
> n=1:1:100;                          % Number of coefficients
> N=transpose(n);
> v=1./((N-0.5).^2 *pi ^2);           % Variances of coefficients
> t=0.01:0.01:1;
> p=(N-0.5)*t;                        % Argument of sinusoid
> x=normal_rnd(M,v,100,1);            % Gaussian coefficients
> y=sqrt(2)*sin(pi *p);               % sin terms
> z=transpose(x)*y
> plot(z)
```

Figure 9.21 shows the Karhunen-Loeve expansion for the Wiener process using 100 terms. The sample functions generally exhibit the same type behavior as in the previous figures. The sample functions, however, do not exhibit the jaggedness of the other examples, which are based on the generation of many more random variables.

SUMMARY

- A random process or stochastic process is an indexed family of random variables that is specified by the set of joint distributions of any number and choice of random variables in the family. The mean, autocovariance, and autocorrelation functions summarize some of the information contained in the joint distributions of pairs of time samples.
- The sum process of an iid sequence has the property of stationary and independent increments, which facilitates the evaluation of the joint pdf/pmf of the

process at any set of time instants. The binomial and random processes are sum processes. The Poisson and Wiener processes are obtained as limiting forms of these sum processes.

- The Poisson process has independent, stationary increments that are Poisson distributed. The interarrival times in a Poisson process are iid exponential random variables.
- The mean and covariance functions completely specify all joint distributions of a Gaussian random process.
- The Wiener process has independent, stationary increments that are Gaussian distributed. The Wiener process is a Gaussian random process.
- A random process is stationary if its joint distributions are independent of the choice of time origin. If a random process is stationary, then $m_X(t)$ is constant, and $R_X(t_1, t_2)$ depends only on $t_1 - t_2$.
- A random process is wide-sense stationary (WSS) if its mean is constant and if its autocorrelation and autocovariance depend only on $t_1 - t_2$. A WSS process need not be stationary.
- A wide-sense stationary Gaussian random process is also stationary.
- A random process is cyclostationary if its joint distributions are invariant with respect to shifts of the time origin by integer multiples of some period T.
- The white Gaussian noise process results from taking the derivative of the Wiener process.
- The derivative and integral of a random process are defined as limits of random variables. We investigated the existence of these limits in the mean square sense.
- The mean and autocorrelation functions of the output of systems described by a linear differential equation and subject to random process inputs can be obtained by solving a set of differential equations. If the input process is a Gaussian random process, then the output process is also Gaussian.
- Ergodic theorems state when time-average estimates of a parameter of a random process converge to the expected value of the parameter. The decay rate of the covariance function determines the convergence rate of the sample mean.

CHECKLIST OF IMPORTANT TERMS

Autocorrelation function
Autocovariance function
Average power
Bernoulli random process
Binomial counting process
Continuous-time process
Cross-correlation function
Cross-covariance function
Cyclostationary random process
Discrete-time process
Ergodic theorem
Fourier series
Gaussian random process
Hurst parameter
iid random process
Independent increments
Independent random processes
Karhunen-Loeve expansion
Markov random process
Mean ergodic random process

Mean function
Mean square continuity
Mean square derivative
Mean square integral
Mean square periodic process
Ornstein-Uhlenbeck process
Orthogonal random processes
Poisson process
Random process
Random telegraph signal
Random walk process
Realization, sample path, or sample function
Shot noise
Stationary increments
Stationary random process
Stochastic process
Sum random process
Time average
Uncorrelated random processes
Variance of $X(t)$
White Gaussian noise
Wide-sense cyclostationary process
Wiener process
WSS random process

ANNOTATED REFERENCES

References [1] through [6] can be consulted for further reading on random processes. Larson and Shubert [ref 7] and Yaglom [ref 8] contain excellent discussions on white Gaussian noise and Brownian motion. Van Trees [ref 9] gives detailed examples on the application of the Karhunen-Loeve expansion. Beran [ref 10] discusses long memory processes.

1. A. Papoulis and S. Pillai, *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, New York, 2002.
2. W. B. Davenport, *Probability and Random Processes: An Introduction for Applied Scientists and Engineers*, McGraw-Hill, New York, 1970.
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8. A. M. Yaglom, *Correlation Theory of Stationary and Related Random Functions*, vol. 1: *Basic Results*, Springer-Verlag, New York, 1987.
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10. J. Beran, *Statistics for Long-Memory Processes*, Chapman & Hall/CRC, New York, 1994.

PROBLEMS

Sections 9.1 and 9.2: Definition and Specification of a Stochastic Process

9.1. In Example 9.1, find the joint pmf for X_1 and X_2. Why are X_1 and X_2 independent?

9.2. A discrete-time random process X_n is defined as follows. A fair die is tossed and the outcome k is observed. The process is then given by $X_n = k$ for all n.

(a) Sketch some sample paths of the process.
(b) Find the pmf for X_n.
(c) Find the joint pmf for X_n and X_{n+k}.
(d) Find the mean and autocovariance functions of X_n.

9.3. A discrete-time random process X_n is defined as follows. A fair coin is tossed. If the outcome is heads, $X_n = (-1)^n$ for all n; if the outcome is tails, $X_n = (-1)^{n+1}$ for all n.

(a) Sketch some sample paths of the process.
(b) Find the pmf for X_n.
(c) Find the joint pmf for X_n and X_{n+k}.
(d) Find the mean and autocovariance functions of X_n.

9.4. A discrete-time random process is defined by $X_n = s^n$, for $n \geq 0$, where s is selected at random from the interval $(0, 1)$.

(a) Sketch some sample paths of the process.
(b) Find the cdf of X_n.
(c) Find the joint cdf for X_n and X_{n+1}.
(d) Find the mean and autocovariance functions of X_n.
(e) Repeat parts a, b, c, and d if s is uniform in $(1, 2)$.

9.5. Let $g(t)$ be the rectangular pulse shown in Fig. P9.1. The random process $X(t)$ is defined as

$$X(t) = Ag(t),$$

where A assumes the values ± 1 with equal probability.

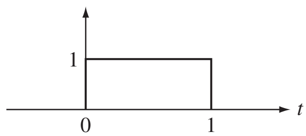

FIGURE P9.1

(a) Find the pmf of $X(t)$.
(b) Find $m_X(t)$.
(c) Find the joint pmf of $X(t)$ and $X(t + d)$.
(d) Find $C_X(t, t + d), d > 0$.

9.6. A random process is defined by

$$Y(t) = g(t - T),$$

where $g(t)$ is the rectangular pulse of Fig. P9.1, and T is a uniformly distributed random variable in the interval $(0, 1)$.

(a) Find the pmf of $Y(t)$.

(b) Find $m_Y(t)$ and $C_Y(t_1, t_2)$.

9.7. A random process is defined by

$$X(t) = g(t - T),$$

where T is a uniform random variable in the interval $(0, 1)$ and $g(t)$ is the periodic triangular waveform shown in Fig. P9.2.

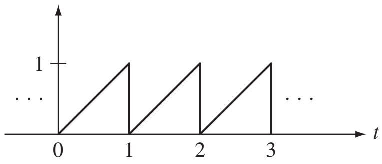

FIGURE P9.2

(a) Find the cdf of $X(t)$ for $0 < t < 1$.

(b) Find $m_X(t)$ and $C_X(t_1, t_2)$.

9.8. Let $Y(t) = g(t - T)$ as in Problem 9.6, but let T be an exponentially distributed random variable with parameter α.

(a) Find the pmf of $Y(t)$.

(b) Find the joint pmf of $Y(t)$ and $Y(t + d)$. Consider two cases: $d > 1$, and $0 < d < 1$.

(c) Find $m_Y(t)$ and $C_Y(t, t + d)$ for $d > 1$ and $0 < d < 1$.

9.9. Let $Z(t) = At^3 + B$, where A and B are independent random variables.

(a) Find the pdf of $Z(t)$.

(b) Find $m_Z(t)$ and $C_Z(t_1, t_2)$.

9.10. Find an expression for $E[|X_{t_2} - X_{t_1}|^2]$ in terms of autocorrelation function.

9.11. The random process $H(t)$ is defined as the "hard-limited" version of $X(t)$:

$$H(t) = \begin{cases} +1 & \text{if} \quad X(t) \geq 0 \\ -1 & \text{if} \quad X(t) < 0. \end{cases}$$

(a) Find the pdf, mean, and autocovariance of $H(t)$ if $X(t)$ is the sinusoid with a random amplitude presented in Example 9.2.

(b) Find the pdf, mean, and autocovariance of $H(t)$ if $X(t)$ is the sinusoid with random phase presented in Example 9.9.

(c) Find a general expression for the mean of $H(t)$ in terms of the cdf of $X(t)$.

9.12. **(a)** Are independent random processes orthogonal? Explain.

(b) Are orthogonal random processes uncorrelated? Explain.

(c) Are uncorrelated processes independent?

(d) Are uncorrelated processes orthogonal?

9.13. The random process $Z(t)$ is defined by

$$Z(t) = 2Xt - Y,$$

where X and Y are a pair of random variables with means m_X, m_Y, variances σ_X^2, σ_Y^2, and correlation coefficient $\rho_{X,Y}$. Find the mean and autocovariance of $Z(t)$.

9.14. Let $H(t)$ be the output of the hard limiter in Problem 9.11.

(a) Find the cross-correlation and cross-covariance between $H(t)$ and $X(t)$ when the input is a sinusoid with random amplitude as in Problem 9.11a.

(b) Repeat if the input is a sinusoid with random phase as in Problem 9.11b.

(c) Are the input and output processes uncorrelated? Orthogonal?

9.15. Let $Y_n = X_n + g(n)$ where X_n is a zero-mean discrete-time random process and $g(n)$ is a deterministic function of n.

(a) Find the mean and variance of Y_n.

(b) Find the joint cdf of Y_n and Y_{n+1}.

(c) Find the autocovariance function of Y_n.

(d) Plot typical sample functions forX_n and Y_n if: $g(n) = n$; $g(n) = 1/n^2$; $g(n) = 1/n$.

9.16. Let $Y_n = c(n)X_n$ where X_n is a zero-mean, unit-variance, discrete-time random process and $c(n)$ is a deterministic function of n.

(a) Find the mean and variance of Y_n.

(b) Find the joint cdf of Y_n and Y_{n+1}.

(c) Find the autocovariance function of Y_n.

(d) Plot typical sample functions forX_n and Y_n if: $c(n) = n$; $c(n) = 1/n^2$; $c(n) = 1/n$.

9.17. **(a)** Find the cross-correlation and cross-covariance for X_n and Y_n in Problem 9.15.

(b) Find the joint pdf of X_n and Y_{n+1}.

(c) Determine whether X_n and Y_n are uncorrelated, independent, or orthogonal random processes.

9.18. **(a)** Find the cross-correlation and cross-covariance for X_n and Y_n in Problem 9.16.

(b) Find the joint pdf of X_n and Y_{n+1}.

(c) Determine whether X_n and Y_n are uncorrelated, independent, or orthogonal random processes.

9.19. Suppose that $X(t)$ and $Y(t)$ are independent random processes and let

$$U(t) = X(t) - Y(t)$$
$$V(t) = X(t) + Y(t).$$

(a) Find $C_{UX}(t_1, t_2)$, $C_{UY}(t_1, t_2)$, and $C_{UV}(t_1, t_2)$.

(b) Find the $f_{U(t_1)X(t_2)}(u, x)$, and $f_{U(t_1)V(t_2)}(u, v)$. *Hint:* Use auxiliary variables.

9.20. Repeat Problem 9.19 if $X(t)$ and $Y(t)$ are independent discrete-time processes and $X(t)$ and $Y(t)$ have different iid random processes.

Section 9.3: Sum Process, Binomial Counting Process, and Random Walk

9.21. **(a)** Let Y_n be the process that results when individual 1's in a Bernoulli process are erased with probability α. Find the pmf of S'_n, the counting process for Y_n. Does Y_n have independent and stationary increments?

(b) Repeat part a if in addition to the erasures, individual 0's in the Bernoulli process are changed to 1's with probability β.

9.22. Let S_n denote a binomial counting process.

(a) Show that $P[S_n = j, S_{n'} = i] \neq P[S_n = j]P[S_{n'} = i]$.

(b) Find $P[S_{n_2} = j \mid S_{n_1} = i]$, where $n_2 > n_1$.

(c) Show that $P[S_{n_2} = j \mid S_{n_1} = i, S_{n_0} = k] = P[S_{n_2} = j \mid S_{n_1} = i]$, where $n_2 > n_1 > n_0$.

9.23. **(a)** Find $P[S_n = 0]$ for the random walk process.

(b) What is the answer in part a if $p = 1/2$?

9.24. Consider the following *moving average* processes:

$$Y_n = 1/2(X_n + X_{n-1}) \qquad X_0 = 0$$

$$Z_n = 2/3\,X_n + 1/3\,X_{n-1} \qquad X_0 = 0$$

(a) Find the mean, variance, and covariance of Y_n and Z_n if X_n is a Bernoulli random process.

(b) Repeat part a if X_n is the random step process.

(c) Generate 100 outcomes of a Bernoulli random process X_n, and find the resulting Y_n and Z_n. Are the sample means of Y_n and Z_n in part a close to their respective means?

(d) Repeat part c with X_n given by the random step process.

9.25. Consider the following autoregressive processes:

$$W_n = 2W_{n-1} + X_n \qquad W_0 = 0$$

$$Z_n = 3/4\,Z_{n-1} + X_n \quad Z_0 = 0.$$

(a) Suppose that X_n is a Bernoulli process. What trends do the processes exhibit?

(b) Express W_n and Z_n in terms of $X_n, X_{n-1}, \ldots, X_1$ and then find $E[W_n]$ and $E[Z_n]$. Do these results agree with the trends you expect?

(c) Do W_n or Z_n have independent increments? stationary increments?

(d) Generate 100 outcomes of a Bernoulli process. Find the resulting realizations of W_n and Z_n. Is the sample mean meaningful for either of these processes?

(e) Repeat part d if X_n is the random step process.

9.26. Let M_n be the discrete-time process defined as the sequence of sample means of an iid sequence:

$$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

(a) Find the mean, variance, and covariance of M_n.

(b) Does M_n have independent increments? stationary increments?

9.27. Find the pdf of the processes defined in Problem 9.24 if the X_n are an iid sequence of zero-mean, unit-variance Gaussian random variables.

9.28. Let X_n consist of an iid sequence of Cauchy random variables.

(a) Find the pdf of the sum process S_n. *Hint:* Use the characteristic function method.

(b) Find the joint pdf of S_n and S_{n+k}.

9.29. Let X_n consist of an iid sequence of Poisson random variables with mean α.

(a) Find the pmf of the sum process S_n.

(b) Find the joint pmf of S_n and S_{n+k}.

9.30. Let X_n be an iid sequence of zero-mean, unit-variance Gaussian random variables.

(a) Find the pdf of M_n defined in Problem 9.26.

(b) Find the joint pdf of M_n and M_{n+k}. *Hint:* Use the independent increments property of S_n.

9.31. Repeat Problem 9.26 with $X_n = 1/2(Y_n + Y_{n-1})$, where Y_n is an iid random process. What happens to the variance of M_n as n increases?

9.32. Repeat Problem 9.26 with $X_n = 3/4X_{n-1} + Y_n$ where Y_n is an iid random process. What happens to the variance of M_n as n increases?

9.33. Suppose that an experiment has three possible outcomes, say 0, 1, and 2, and suppose that these occur with probabilities p_0, p_1, and p_2, respectively. Consider a sequence of independent repetitions of the experiment, and let $X_j(n)$ be the indicator function for outcome j. The vector

$$\mathbf{X}(n) = (X_0(n), X_1(n), X_2(n))$$

then constitutes a vector-valued Bernoulli random process. Consider the counting process for $\mathbf{X}(n)$:

$$\mathbf{S}(n) = \mathbf{X}(n) + \mathbf{X}(n-1) + \cdots + \mathbf{X}(1) \qquad \mathbf{S}(0) = 0.$$

(a) Show that $\mathbf{S}(n)$ has a multinomial distribution.

(b) Show that $\mathbf{S}(n)$ has independent increments, then find the joint pmf of $\mathbf{S}(n)$ and $\mathbf{S}(n+k)$.

(c) Show that components $S_j(n)$ of the vector process are binomial counting processes.

Section 9.4: Poisson and Associated Random Processes

9.34. A server handles queries that arrive according to a Poisson process with a rate of 10 queries per minute. What is the probability that no queries go unanswered if the server is unavailable for 20 seconds?

9.35. Customers deposit \$1 in a vending machine according to a Poisson process with rate λ. The machine issues an item with probability p. Find the pmf for the number of items dispensed in time t.

9.36. Noise impulses occur in a radio transmission according to a Poisson process of rate λ.

(a) Find the probability that no impulses occur during the transmission of a message that is t seconds long.

(b) Suppose that the message is encoded so that the errors caused by up to 2 impulses can be corrected. What is the probability that a t-second message cannot be corrected?

9.37. Packets arrive at a multiplexer at two ports according to independent Poisson processes of rates $\lambda_1 = 1$ and $\lambda_2 = 2$ packets/second, respectively.

(a) Find the probability that a message arrives first on line 2.

(b) Find the pdf for the time until a message arrives on either line.

(c) Find the pmf for $N(t)$, the total number of messages that arrive in an interval of length t.

(d) Generalize the result of part c for the "merging" of k independent Poisson processes of rates $\lambda_1, \ldots, \lambda_k$, respectively:

$$N(t) = N_1(t) + \cdots + N_k(t).$$

9.38. **(a)** Find $P[N(t-d) = j \mid N(t) = k]$ with $d > 0$, where $N(t)$ is a Poisson process with rate λ.

(b) Compare your answer to $P[N(t+d) = j \mid N(t) = k]$. Explain the difference, if any.

9.39. Let $N_1(t)$ be a Poisson process with arrival rate λ_1 that is started at $t = 0$. Let $N_2(t)$ be another Poisson process that is independent of $N_1(t)$, that has arrival rate λ_2, and that is started at $t = 1$.

(a) Show that the pmf of the process $N(t) = N_1(t) + N_2(t)$ is given by:

$$P[N(t+\tau) - N(t) = k] = \frac{(m(t+\tau) - m(t))^k}{k!} e^{-(m(t+\tau)-m(t))} \qquad \text{for } k = 0, 1, \ldots$$

where $m(t) = E[N(t)]$.

(b) Now consider a Poisson process in which the arrival rate $\lambda(t)$ is a piecewise constant function of time. Explain why the pmf of the process is given by the above pmf where

$$m(t) = \int_0^t \lambda(t')\,dt'.$$

(c) For what other arrival functions $\lambda(t)$ does the pmf in part a hold?

9.40. **(a)** Suppose that the time required to service a customer in a queueing system is a random variable T. If customers arrive at the system according to a Poisson process with parameter λ, find the pmf for the number of customers that arrive during one customer's service time. *Hint:* Condition on the service time.

(b) Evaluate the pmf in part a if T is an exponential random variable with parameter β.

9.41. **(a)** Is the difference of two independent Poisson random processes also a Poisson process?

(b) Let $N_p(t)$ be the number of complete pairs generated by a Poisson process up to time t. Explain why $N_p(t)$ is or is not a Poisson process.

9.42. Let $N(t)$ be a Poisson random process with parameter λ. Suppose that each time an event occurs, a coin is flipped and the outcome (heads or tails) is recorded. Let $N_1(t)$ and $N_2(t)$ denote the number of heads and tails recorded up to time t, respectively. Assume that p is the probability of heads.

(a) Find $P[N_1(t) = j, N_2(t) = k \mid N(t) = k + j]$.

(b) Use part a to show that $N_1(t)$ and $N_2(t)$ are independent Poisson random variables of rates $p\lambda t$ and $(1-p)\lambda t$, respectively:

$$P[N_1(t) = j, N_2(t) = k] = \frac{(p\lambda t)^j}{j!} e^{-p\lambda t} \frac{((1-p)\lambda t)^k}{k!} e^{-(1-p)\lambda t}.$$

9.43. Customers play a \$1 game machine according to a Poisson process with rate λ. Suppose the machine dispenses a random reward X each time it is played. Let $X(t)$ be the total reward issued up to time t.

(a) Find expressions for$P[X(t) = j]$ if X_n is Bernoulli.

(b) Repeat part a if X assumes the values $\{0, 5\}$ with probabilities (5/6, 1/6).

(c) Repeat part a if X is Poisson with mean 1.

(d) Repeat part a if with probability p the machine returns all the coins.

9.44. Let $X(t)$ denote the random telegraph signal, and let $Y(t)$ be a process derived from $X(t)$ as follows: Each time $X(t)$ changes polarity, $Y(t)$ changes polarity with probability p.

(a) Find the $P[Y(t) = \pm 1]$.

(b) Find the autocovariance function of $Y(t)$. Compare it to that of $X(t)$.

9.45. Let $Y(t)$ be the random signal obtained by switching between the values 0 and 1 according to the events in a Poisson process of rate λ. Compare the pmf and autocovariance of $Y(t)$ with that of the random telegraph signal.

9.46. Let $Z(t)$ be the random signal obtained by switching between the values 0 and 1 according to the events in a counting process $N(t)$. Let

$$P[N(t) = k] = \frac{1}{1 + \lambda t}\left(\frac{\lambda t}{1 + \lambda t}\right)^k \qquad k = 0, 1, 2, \ldots.$$

(a) Find the pmf of $Z(t)$.

(b) Find $m_Z(t)$.

9.47. In the filtered Poisson process (Eq. (9.45)), let $h(t)$ be a pulse of unit amplitude and duration T seconds.

(a) Show that $X(t)$ is then the increment in the Poisson process in the interval $(t - T, t)$.

(b) Find the mean and autocorrelation functions of $X(t)$.

9.48. (a) Find the second moment and variance of the shot noise process discussed in Example 9.25.

(b) Find the variance of the shot noise process if $h(t) = e^{-\beta t}$ for $t \geq 0$.

9.49. Messages arrive at a message center according to a Poisson process of rate λ. Every hour the messages that have arrived during the previous hour are forwarded to their destination. Find the mean of the total time waited by all the messages that arrive during the hour. *Hint:* Condition on the number of arrivals and consider the arrival instants.

Section 9.5: Gaussian Random Process, Wiener Process and Brownian Motion

9.50. Let $X(t)$ and $Y(t)$ be jointly Gaussian random processes. Explain the relation between the conditions of independence, uncorrelatedness, and orthogonality of $X(t)$ and $Y(t)$.

9.51. Let $X(t)$ be a zero-mean Gaussian random process with autocovariance function given by

$$C_X(t_1, t_2) = 4e^{-2|t_1 - t_2|}.$$

Find the joint pdf of $X(t)$ and $X(t + s)$.

9.52. Find the pdf of $Z(t)$ in Problem 9.13 if X and Y are jointly Gaussian random variables.

9.53. Let $Y(t) = X(t + d) - X(t)$, where $X(t)$ is a Gaussian random process.

(a) Find the mean and autocovariance of $Y(t)$.

(b) Find the pdf of $Y(t)$.

(c) Find the joint pdf of $Y(t)$ and $Y(t + s)$.

(d) Show that $Y(t)$ is a Gaussian random process.

9.54. Let $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are iid Gaussian random variables with zero mean and variance σ^2.

(a) Find the mean and autocovariance of $X(t)$.

(b) Find the joint pdf of $X(t)$ and $X(t + s)$.

9.55. Let $X(t)$ and $Y(t)$ be independent Gaussian random processes with zero means and the same covariance function $C(t_1, t_2)$. Define the "amplitude-modulated signal" by

$$Z(t) = X(t)\cos\omega t + Y(t)\sin\omega t.$$

(a) Find the mean and autocovariance of $Z(t)$.

(b) Find the pdf of $Z(t)$.

9.56. Let $X(t)$ be a zero-mean Gaussian random process with autocovariance function given by $C_X(t_1, t_2)$. If $X(t)$ is the input to a "square law detector," then the output is

$$Y(t) = X(t)^2.$$

Find the mean and autocovariance of the output $Y(t)$.

9.57. Let $Y(t) = X(t) + \mu t$, where $X(t)$ is the Wiener process.

(a) Find the pdf of $Y(t)$.

(b) Find the joint pdf of $Y(t)$ and $Y(t + s)$.

9.58. Let $Y(t) = X^2(t)$, where $X(t)$ is the Wiener process.

(a) Find the pdf of $Y(t)$.

(b) Find the conditional pdf of $Y(t_2)$ given $Y(t_1)$.

9.59. Let $Z(t) = X(t) - aX(t - s)$, where $X(t)$ is the Wiener process.

(a) Find the pdf of $Z(t)$.

(b) Find $m_Z(t)$ and $C_Z(t_1, t_2)$.

9.60. **(a)** For $X(t)$ the Wiener process with $\alpha = 1$ and $0 < t < 1$, show that the joint pdf of $X(t)$ and $X(1)$ is given by:

$$f_{X(t),X(1)}(x_1, x_2) = \frac{\exp\left\{-\frac{1}{2}\left[\frac{x_1^2}{t} + \frac{(x_2 - x_1)^2}{(1 - t)}\right]\right\}}{2\pi\sqrt{t(1 - t)}}.$$

(b) Use part a to show that for $0 < t < 1$, the conditional pdf of $X(t)$ given $X(0) = X(1) = 0$ is:

$$f_{X(t)}(x \mid X(0) = X(1) = 0) = \frac{\exp\left\{-\frac{1}{2}\left[\frac{x^2}{t(1 - t)}\right]\right\}}{2\pi\sqrt{t(1 - t)}}.$$

(c) Use part b to find the conditional pdf of $X(t)$ given $X(t_1) = a$ and $X(t_2) = b$ for $t_1 < t < t_2$. *Hint:* Find the equivalent process in the interval $(0, t_2 - t_1)$.

Section 9.6: Stationary Random Processes

9.61. **(a)** Is the random amplitude sinusoid in Example 9.9 a stationary random process? Is it wide-sense stationary?

(b) Repeat part a for the random phase sinusoid in Example 9.10.

9.62. A discrete-time random process X_n is defined as follows. A fair coin is tossed; if the outcome is heads then $X_n = 1$ for all n, and $X_n = -1$ for all n, otherwise.

(a) Is X_n a WSS random process?

(b) Is X_n a stationary random process?

(c) Do the answers in parts a and b change if p is a biased coin?

9.63. Let X_n be the random process in Problem 9.3.

(a) Is X_n a WSS random process?

(b) Is X_n a stationary random process?

(c) Is X_n a cyclostationary random process?

9.64. Let $X(t) = g(t - T)$, where $g(t)$ is the periodic waveform introduced in Problem 9.7, and T is a uniformly distributed random variable in the interval $(0, 1)$. Is $X(t)$ a stationary random process? Is $X(t)$ wide-sense stationary?

9.65. Let $X(t)$ be defined by

$$X(t) = A \cos \omega t + B \sin \omega t,$$

where A and B are iid random variables.

(a) Under what conditions is $X(t)$ wide-sense stationary?

(b) Show that $X(t)$ is not stationary. *Hint:* Consider $E[X^3(t)]$.

9.66. Consider the following moving average process:

$$Y_n = 1/2(X_n + X_{n-1}) \qquad X_0 = 0.$$

(a) Is Y_n a stationary random process if X_n is an iid integer-valued process?

(b) Is Y_n a stationary random process if X_n is a stationary process?

(c) Are Y_n and X_n jointly stationary random processes if X_n is an iid process? a stationary process?

9.67. Let X_n be a zero-mean iid process, and let Z_n be an autoregressive random process

$$Z_n = 3/4Z_{n-1} + X_n \qquad Z_0 = 0.$$

(a) Find the autocovariance of Z_n and determine whether Z_n is wide-sense stationary. *Hint:* Express Z_n in terms of $X_n, X_{n-1}, \ldots, X_1$.

(b) Does Z_n eventually settle down into stationary behavior?

(c) Find the pdf of Z_n if X_n is an iid sequence of zero-mean, unit-variance Gaussian random variables. What is the pdf of Z_n as $n \to \infty$?

9.68. Let $Y(t) = X(t + s) - \beta X(t)$, where $X(t)$ is a wide-sense stationary random process.

(a) Determine whether $Y(t)$ is also a wide-sense stationary random process.

(b) Find the cross-covariance function of $Y(t)$ and $X(t)$. Are the processes jointly wide-sense stationary?

(c) Find the pdf of $Y(t)$ if $X(t)$ is a Gaussian random process.
(d) Find the joint pdf of $Y(t_1)$ and $Y(t_2)$ in part c.
(e) Find the joint pdf of $Y(t_1)$ and $X(t_2)$ in part c.

9.69. Let $X(t)$ and $Y(t)$ be independent, wide-sense stationary random processes with zero means and the same covariance function $C_X(\tau)$. Let $Z(t)$ be defined by

$$Z(t) = 3X(t) - 5Y(t).$$

(a) Determine whether $Z(t)$ is also wide-sense stationary.
(b) Determine the pdf of $Z(t)$ if $X(t)$ and $Y(t)$ are also jointly Gaussian zero-mean random processes with $C_X(\tau) = 4e^{-|\tau|}$.
(c) Find the joint pdf of $Z(t_1)$ and $Z(t_2)$ in part b.
(d) Find the cross-covariance between $Z(t)$ and $X(t)$. Are $Z(t)$ and $X(t)$ jointly stationary random processes?
(e) Find the joint pdf of $Z(t_1)$ and $X(t_2)$ in part b. *Hint:* Use auxilliary variables.

9.70. Let $X(t)$ and $Y(t)$ be independent, wide-sense stationary random processes with zero means and the same covariance function $C_X(\tau)$. Let $Z(t)$ be defined by

$$Z(t) = X(t)\cos\omega t + Y(t)\sin\omega t.$$

(a) Determine whether $Z(t)$ is a wide-sense stationary random process.
(b) Determine the pdf of $Z(t)$ if $X(t)$ and $Y(t)$ are also jointly Gaussian zero-mean random processes with $C_X(\tau) = 4e^{-|\tau|}$.
(c) Find the joint pdf of $Z(t_1)$ and $Z(t_2)$ in part b.
(d) Find the cross-covariance between $Z(t)$ and $X(t)$. Are $Z(t)$ and $X(t)$ jointly stationary random processes?
(e) Find the joint pdf of $Z(t_1)$ and $X(t_2)$ in part b.

9.71. Let $X(t)$ be a zero-mean, wide-sense stationary Gaussian random process with autocorrelation function $R_X(\tau)$. The output of a "square law detector" is

$$Y(t) = X(t)^2.$$

Show that $R_Y(\tau) = R_X(0)^2 + 2R_X^2(\tau)$. *Hint:* For zero-mean, jointly Gaussian random variables $E[X^2Z^2] = E[X^2]E[Z^2] + 2E[XZ]^2$.

9.72. A WSS process $X(t)$ has mean 1 and autocorrelation function given in Fig. P9.3.

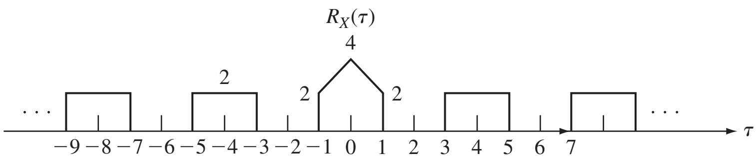

FIGURE P9.3

(a) Find the mean component of $R_X(\tau)$.
(b) Find the periodic component of $R_X(\tau)$.
(c) Find the remaining component of $R_X(\tau)$.

9.73. Let X_n and Y_n be independent random processes. A multiplexer combines these two sequences into a combined sequence U_k, that is,

$$U_{2n} = X_n, \qquad U_{2n+1} = Y_n.$$

(a) Suppose that X_n and Y_n are independent Bernoulli random processes. Under what conditions is U_k a stationary random process? a cyclostationary random process?

(b) Repeat part a if X_n and Y_n are independent stationary random processes.

(c) Suppose that X_n and Y_n are wide-sense stationary random processes. Is U_k a wide-sense stationary random process? a wide-sense cyclostationary random process? Find the mean and autocovariance functions of U_k.

(d) If U_k is wide-sense cyclostationary, find the mean and correlation function of the randomly phase-shifted version of U_k as defined by Eq. (9.72).

9.74. A ternary information source produces an iid, equiprobable sequence of symbols from the alphabet $\{a, b, c\}$. Suppose that these three symbols are encoded into the respective binary codewords 00, 01, 10. Let B_n be the sequence of binary symbols that result from encoding the ternary symbols.

(a) Find the joint pmf of B_n and B_{n+1} for n even; n odd. Is B_n stationary? cyclostationary?

(b) Find the mean and covariance functions of B_n. Is B_n wide-sense stationary? wide-sense cyclostationary?

(c) If B_n is cyclostationary, find the joint pmf, mean, and autocorrelation functions of the randomly phase-shifted version of B_n as defined by Eq. (9.72).

9.75. Let $s(t)$ be a periodic square wave with period $T = 1$ which is equal to 1 for the first half of a period and -1 for the remainder of the period. Let $X(t) = As(t)$, where A is a random variable.

(a) Find the mean and autocovariance functions of $X(t)$.

(b) Is $X(t)$ a mean-square periodic process?

(c) Find the mean and autocovariance of $X_s(t)$ the randomly phase-shifted version of $X(t)$ given by Eq. (9.72).

9.76. Let $X(t) = As(t)$ and $Y(t) = Bs(t)$, where A and B are independent random variables that assume values $+1$ or -1 with equal probabilities, where $s(t)$ is the periodic square wave in Problem 9.75.

(a) Find the joint pmf of $X(t_1)$ and $Y(t_2)$.

(b) Find the cross-covariance of $X(t_1)$ and $Y(t_2)$.

(c) Are $X(t)$ and $Y(t)$ jointly wide-sense cyclostationary? Jointly cyclostationary?

9.77. Let $X(t)$ be a mean square periodic random process. Is $X(t)$ a wide-sense cyclostationary process?

9.78. Is the pulse amplitude modulation random process in Example 9.38 cyclostationary?

9.79. Let $X(t)$ be the random amplitude sinusoid in Example 9.37. Find the mean and autocorrelation functions of the randomly phase-shifted version of $X(t)$ given by Eq. (9.72).

9.80. Complete the proof that if $X(t)$ is a cyclostationary random process, then $X_s(t)$, defined by Eq. (9.72), is a stationary random process.

9.81. Show that if $X(t)$ is a wide-sense cyclostationary random process, then $X_s(t)$, defined by Eq. (9.72), is a wide-sense stationary random process with mean and autocorrelation functions given by Eqs. (9.74a) and (9.74b).

Section 9.7: Continuity, Derivatives, and Integrals of Random Processes

9.82. Let the random process $X(t) = u(t - S)$ be a unit step function delayed by an exponential random variable S, that is, $X(t) = 1$ for $t \geq S$, and $X(t) = 0$ for $t < S$.
- **(a)** Find the autocorrelation function of $X(t)$.
- **(b)** Is $X(t)$ mean square continuous?
- **(c)** Does $X(t)$ have a mean square derivative? If so, find its mean and autocorrelation functions.
- **(d)** Does $X(t)$ have a mean square integral? If so, find its mean and autocovariance functions.

9.83. Let $X(t)$ be the random telegraph signal introduced in Example 9.24.
- **(a)** Is $X(t)$ mean square continuous?
- **(b)** Show that $X(t)$ does not have a mean square derivative, and show that the second mixed partial derivative of its autocorrelation function has a delta function. What gives rise to this delta function?
- **(c)** Does $X(t)$ have a mean square integral? If so, find its mean and autocovariance functions.

9.84. Let $X(t)$ have autocorrelation function

$$R_X(\tau) = \sigma^2 e^{-\alpha\tau^2}.$$

- **(a)** Is $X(t)$ mean square continuous?
- **(b)** Does $X(t)$ have a mean square derivative? If so, find its mean and autocorrelation functions.
- **(c)** Does $X(t)$ have a mean square integral? If so, find its mean and autocorrelation functions.
- **(d)** Is $X(t)$ a Gaussian random process?

9.85. Let $N(t)$ be the Poisson process. Find $E[(N(t) - N(t_0))^2]$ and use the result to show that $N(t)$ is mean square continuous.

9.86. Does the pulse amplitude modulation random process discussed in Example 9.38 have a mean square integral? If so, find its mean and autocovariance functions.

9.87. Show that if $X(t)$ is a mean square continuous random process, then $X(t)$ has a mean square integral. *Hint:* Show that

$$R_X(t_1, t_2) - R_X(t_0, t_0) = E[(X(t_1) - X(t_0))X(t_2)] + E[X(t_0)(X(t_2) - X(t_0))],$$

and then apply the Schwarz inequality to the two terms on the right-hand side.

9.88. Let $Y(t)$ be the mean square integral of $X(t)$ in the interval $(0, t)$. Show that $Y'(t)$ is equal to $X(t)$ in the mean square sense.

9.89. Let $X(t)$ be a wide-sense stationary random process. Show that $E[X(t)X'(t)] = 0$.

9.90. A linear system with input $Z(t)$ is described by

$$X'(t) + \alpha X(t) = Z(t) \qquad t \geq 0, X(0) = 0.$$

Find the output $X(t)$ if the input is a zero-mean Gaussian random process with autocorrelation function given by

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}.$$

Section 9.8: Time Averages of Random Processes and Ergodic Theorems

9.91. Find the variance of the time average given in Example 9.47.

9.92. Are the following processes WSS and mean ergodic?

(a) Discrete-time dice process in Problem 9.2.

(b) Alternating sign process in Problem 9.3.

(c) $X_n = s^n$, for $n \geq 0$ in Problem 9.4.

9.93. Is the following WSS random process $X(t)$ mean ergodic?

$$R_X(\tau) = \begin{cases} 0 & |\tau| > 1 \\ 5(1 - |\tau|) & |\tau| \leq 1. \end{cases}$$

9.94. Let $X(t) = A\cos(2\pi f t)$, where A is a random variable with mean m and variance σ^2.

(a) Evaluate $<X(t)>_T$, find its limit as $T \to \infty$, and compare to $m_X(t)$.

(b) Evaluate $<X(t + \tau)X(t)>$, find its limit as $T \to \infty$, and compare to $R_X(t + \tau, t)$.

9.95. Repeat Problem 9.94 with $X(t) = A\cos(2\pi f t + \Theta)$, where A is as in Problem 9.94, Θ is a random variable uniformly distributed in $(0, 2\pi)$, and A and Θ are independent random variables.

9.96. Find an exact expression for $\text{VAR}[<X(t)>_T]$ in Example 9.48. Find the limit as $T \to \infty$.

9.97. The WSS random process X_n has mean m and autocovariance $C_X(k) = (1/2)^{|k|}$. Is X_n mean ergodic?

9.98. **(a)** Are the moving average processes Y_n in Problem 9.24 mean ergodic?

(b) Are the autoregressive processes Z_n in Problem 9.25a mean ergodic?

9.99. **(a)** Show that a WSS random process is mean ergodic if

$$\int_{-\infty}^{\infty} |C(u)| < \infty.$$

(b) Show that a discrete-time WSS random process is mean ergodic if

$$\sum_{k=-\infty}^{\infty} |C(k)| < \infty.$$

9.100. Let $<X^2(t)>_T$ denote a time-average estimate for the mean power of a WSS random process.

(a) Under what conditions is this time average a valid estimate for $E[X^2(t)]$?

(b) Apply your result in part a for the random phase sinusoid in Example 9.2.

9.101. **(a)** Under what conditions is the time average $<X(t + \tau)X(t)>_T$ a valid estimate for the autocorrelation $R_X(\tau)$ of a WSS random process $X(t)$?

(b) Apply your result in part a for the random phase sinusoid in Example 9.2.

9.102. Let $Y(t)$ be the indicator function for the event $\{a < X(t) \leq b\}$, that is,

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \in (a, b] \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $<Y(t)>_T$ is the proportion of time in the time interval $(-T, T)$ that $X(t) \in (a, b]$.

(b) Find $E[<Y(t)>_T]$.

(c) Under what conditions does $<Y(t)>_T \to P[a < X(t) \le b]$?

(d) How can $<Y(t)>_T$ be used to estimate $P[X(t) \le x]$?

(e) Apply the result in part d to the random telegraph signal.

9.103. **(a)** Repeat Problem 9.102 for the time average of the discrete-time Y_n, which is defined as the indicator for the event $\{a < X_n \le b]\}$.

(b) Apply your result in part a to an iid discrete-valued random process.

(c) Apply your result in part a to an iid continuous-valued random process.

9.104. For $n \ge 1$, define $Z_n = u(a - X_n)$, where $u(x)$ is the unit step function, that is, $X_n = 1$ if and only if $X_n \le a$.

(a) Show that the time average $<Z_n>_N$ is the proportion of X_n's that are less than a in the first N samples.

(b) Show that if the process is ergodic (in some sense), then this time average is equal to $F_X(a) = P[X \le a]$.

9.105. In Example 9.50 show that $\text{VAR}[\langle X_n \rangle_T] = (\sigma^2)(2T + 1)^{2H-2}$.

9.106. Plot the covariance function vs. k for the self-similar process in Example 9.50 with $\sigma^2 = 1$ for: $H = 0.5$, $H = 0.6$, $H = 0.75$, $H = 0.99$. Does the long-range dependence of the process increase or decrease with H?

9.107. **(a)** Plot the variance of the sample mean given by Eq. (9.110) vs. T with $\sigma^2 = 1$ for: $H = 0.5$, $H = 0.6$, $H = 0.75$, $H = 0.99$.

(b) For the parameters in part a, plot $(2T + 1)^{2H-1}$ vs. T, which is the ratio of the variance of the sample mean of a long-range dependent process relative to the variance of the sample mean of an iid process. How does the long-range dependence manifest itself, especially for H approaching 1?

(c) Comment on the width of confidence intervals for estimates of the mean of long-range dependent processes relative to those of iid processes.

9.108. Plot the variance of the sample mean for a long-range dependent process (Eq. 9.110) vs. the sample size T in a log-log plot.

(a) What role does H play in the plot?

(b) One of the remarkable indicators of long-range dependence in nature comes from a set of observations of the minimal water levels in the Nile river for the years 622–1281 [Beran, p. 22] where the log-log plot for part a gives a slope of -0.27. What value of H corresponds to this slope?

9.109. Problem 9.99b gives a sufficient condition for mean ergodicity for discrete-time random processes. Use the expression in Eq. (9.112) for a long-range dependent process to determine whether the sufficient condition is satisfied. Comment on your findings.

*Section 9.9: Fourier Series and Karhunen-Loeve Expansion

9.110. Let $X(t) = Xe^{j\omega t}$ where X is a random variable.

(a) Find the correlation function for $X(t)$, which for complex-valued random processes is defined by $R_X(t_1, t_2) = E[X(t_1)X^*(t_2)]$, where * denotes the complex conjugate.

(b) Under what conditions is $X(t)$ a wide-sense stationary random process?

9.111. Consider the sum of two complex exponentials with random coefficients:

$$X(t) = X_1 e^{j\omega_1 t} + X_2 e^{j\omega_2 t} \quad \text{where } \omega_1 \neq \omega_2.$$

(a) Find the covariance function of $X(t)$.

(b) Find conditions on the complex-valued random variables X_1, and X_2 for $X(t)$ to be a wide-sense stationary random process.

(c) Show that if we let $\omega_1 = -\omega_2$, $X_1 = (U - jV)/2$ and $X_2 = (U + jV)/2$, where U and V are real-valued random variables, then $X(t)$ is a real-valued random process. Find an expression for $X(t)$ and for the autocorrelation function.

(d) Restate the conditions on X_1 and X_2 from part b in terms of U and V.

(e) Suppose that in part c, U and V are jointly Gaussian random variables. Show that $X(t)$ is a Gaussian random process.

9.112. (a) Derive Eq. (9.118) for the correlation of the Fourier coefficients for a non-mean square periodic process $X(t)$.

(b) Show that Eq. (9.118) reduces to Eq. (9.117) when $X(t)$ is WSS and mean square periodic.

9.113. Let $X(t)$ be a WSS Gaussian random process with $R_X(\tau) = e^{-|\tau|}$.

(a) Find the Fourier series expansion for $X(t)$ in the interval $[0, T]$.

(b) What is the distribution of the coefficients in the Fourier series?

9.114. Show that the Karhunen-Loeve expansion of a WSS mean-square periodic process $X(t)$ yields its Fourier series. Specify the orthonormal set of eigenfunctions and the corresponding eigenvalues.

9.115. Let $X(t)$ be the white Gaussian noise process introduced in Example 9.43. Show that any set of orthonormal functions can be used as the eigenfunctions for $X(t)$ in its Karhunen-Loeve expansion. What are the eigenvalues?

9.116. Let $Y(t) = X(t) + W(t)$, where $X(t)$ and $W(t)$ are orthogonal random processes and $W(t)$ is a white Gaussian noise process. Let $\phi_n(t)$ be the eigenfunctions corresponding to $K_X(t_1, t_2)$. Show that $\phi_n(t)$ are also the eigenfunctions for $K_Y(t_1, t_2)$. What is the relation between the eigenvalues of $K_X(t_1, t_2)$ and those of $K_Y(t_1, t_2)$?

9.117. Let $X(t)$ be a zero-mean random process with autocovariance

$$R_X(\tau) = \sigma^2 e^{-\alpha|\tau|}.$$

(a) Write the eigenvalue integral equation for the Karhunen-Loeve expansion of $X(t)$ on the interval $[-T, T]$.

(b) Differentiate the above integral equation to obtain the differential equation

$$\frac{d^2}{dt^2}\phi(t) = \frac{\alpha^2\left(\lambda - 2\dfrac{\sigma^2}{\alpha}\right)}{\lambda}\phi(t).$$

(c) Show that the solutions to the above differential equation are of the form $\phi(t) = A\cos bt$ and $\phi(t) = B\sin bt$. Find an expression for b.

(d) Substitute the $\phi(t)$ from part c into the integral equation of part a to show that if $\phi(t) = A\cos bt$, then b is the root of $\tan bT = \alpha/b$, and if $\phi(t) = B\sin bt$, then b is the root of $\tan bT = -b/\alpha$.

(e) Find the values of A and B that normalize the eigenfunctions.

***(f)** In order to show that the frequencies of the eigenfunctions are not harmonically related, plot the following three functions versus bT: $\tan bT$, $bT/\alpha T$, $-\alpha T/bT$. The intersections of these functions yield the eigenvalues. Note that there are two roots per interval of length π.

*Section 9.10: Generating Random Processes

9.118. **(a)** Generate 10 realizations of the binomial counting process with $p = 1/4$, $p = 1/2$, and $p = 3/4$. For each value of p, plot the sample functions for $n = 200$ trials.

(b) Generate 50 realizations of the binomial counting process with $p = 1/2$. Find the sample mean and sample variance of the realizations for the first 200 trials.

(c) In part b, find the histogram of increments in the process for the interval $[1, 50]$, $[51, 100]$, $[101, 150]$, and $[151, 200]$. Compare these histograms to the theoretical pmf. How would you check to see if the increments in the four intervals are stationary?

(d) Plot a scattergram of the pairs consisting of the increments in the interval $[1, 50]$ and $[51, 100]$ in a given realization. Devise a test to check whether the increments in the two intervals are independent random variables.

9.119. Repeat Problem 9.118 for the random walk process with the same parameters.

9.120. Repeat Problem 9.118 for the sum process in Eq. (9.24) where the X_n are iid unit-variance Gaussian random variables with mean: $m = 0$; $m = 0.5$.

9.121. Repeat Problem 9.118 for the sum process in Eq. (9.24) where the X_n are iid Poisson random variables with $\alpha = 1$.

9.122. Repeat Problem 9.118 for the sum process in Eq. (9.24) where the X_n are iid Cauchy random variables with $\alpha = 1$.

9.123. Let $Y_n = \alpha Y_{n-1} + X_n$ where $Y_0 = 0$.

(a) Generate five realizations of the process for $\alpha = 1/4, 1/2, 9/10$ and with X_n given by the $p = 1/2$ and $p = 1/4$ random step process. Plot the sample functions for the first 200 steps. Find the sample mean and sample variance for the outcomes in *each* realization. Plot the histogram for outcomes in each realization.

(b) Generate 50 realizations of the process Y_n with $\alpha = 1/2$, $p = 1/4$, and $p = 1/2$. Find the sample mean and sample variance of the realizations for the first 200 trials. Find the histogram of Y_n across the realizations at times $n = 5$, $n = 50$, and $n = 200$.

(c) In part b, find the histogram of increments in the process for the interval $[1, 50]$, $[51, 100]$, $[101, 150]$, and $[151, 200]$. To what theoretical pmf should these histograms be compared? Should the increments in the process be stationary? Should the increments be independent?

9.124. Repeat Problem 9.123 for the sum process in Eq. (9.24) where the X_n are iid unit-variance Gaussian random variables with mean: $m = 0$; $m = 0.5$.

9.125. **(a)** Propose a method for estimating the covariance function of the sum process in Problem 9.118. Do not assume that the process is wide-sense stationary.

(b) How would you check to see if the process is wide-sense stationary?

(c) Apply the methods in parts a and b to the experiment in Problem 9.118b.

(d) Repeat part c for Problem 9.123b.

9.126. Use the binomial process to approximate a Poisson random process with arrival rate $\lambda = 1$ customer per second in the time interval (0, 100]. Try different values of n and come up with a recommendation on how n should be selected.

9.127. Generate 100 repetitions of the experiment in Example 9.21.

(a) Find the relative frequency of the event $P[N(10) = 3 \text{ and } N(60) - N(45) = 2]$ and compare it to the theoretical probability.

(b) Find the histogram of the time that elapses until the second arrival and compare it to the theoretical pdf. Plot the empirical cdf and compare it to the theoretical cdf.

9.128. Generate 100 realizations of the Poisson random process $N(t)$ with arrival rate $\lambda = 1$ customer per second in the time interval (0, 10]. Generate the pair $(N_1(t), N_2(t))$ by assigning arrivals in $N(t)$ to $N_1(t)$ with probability $p = 0.25$ and to $N_2(t)$ with probability 0.75.

(a) Find the histograms for $N_1(10)$ and $N_2(10)$ and compare them to the theoretical pmf by performing a chi-square goodness-of-fit test at a 5% significance level.

(b) Perform a chi-square goodness-of-fit test to test whether $N_1(10)$ and $N_2(10)$ are independent random variables. How would you check whether $N_1(t)$ and $N_2(t)$ are independent random processes?

9.129. Subscribers log on to a system according to a Poisson process with arrival rate $\lambda = 1$ customer per second. The ith customer remains logged on for a random duration of T_i seconds, where the T_i are iid random variables and are also independent of the arrival times.

(a) Generate the sequence S_n of customer arrival times and the corresponding departure times given by $D_n = S_n + T_n$, where the connections times are all equal to 1.

(b) Plot: $A(t)$, the number of arrivals up to time t; $D(t)$, the number of departures up to time t; and $N(t) = A(t) - D(t)$, the number in the system at time t.

(c) Perform 100 simulations of the system operation for a duration of 200 seconds. Assume that customer connection times are an exponential random variables with mean 5 seconds. Find the customer departure time instants and the associated departure counting process $D(t)$. How would you check whether $D(t)$ is a Poisson process? Find the histograms for $D(t)$ and the number in the system $N(t)$ at $t = 50, 100, 150, 200$. Try to fit a pmf to each histogram.

(d) Repeat part c if customer connection times are exactly 5 seconds long.

9.130. Generate 100 realizations of the Wiener process with $\alpha = 1$ for the interval (0, 3.5) using the random walk limiting procedure.

(a) Find the histograms for increments in the intervals (0, 0.5], (0.5, 1.5], and (1.5, 3.5] and compare these to the theoretical pdf.

(b) Perform a test at a 5% significance level to determine whether the increments in the first two intervals are independent random variables.

9.131. Repeat Problem 9.130 using Gaussian-distributed increments to generate the Wiener process. Discuss how the increment interval in the simulation should be selected.

Problems Requiring Cumulative Knowledge

9.132. Let $X(t)$ be a random process with independent increments. Assume that the increments $X(t_2) - X(t_1)$ are gamma random variables with parameters $\lambda > 0$ and $\alpha = t_2 - t_1$.

(a) Find the joint density function of $X(t_1)$ and $X(t_2)$.

(b) Find the autocorrelation function of $X(t)$.

(c) Is $X(t)$ mean square continuous?

(d) Does $X(t)$ have a mean square derivative?

9.133. Let $X(t)$ be the pulse amplitude modulation process introduced in Example 9.38 with $T = 1$. A phase-modulated process is defined by

$$Y(t) = a \cos\left(2\pi t + \frac{\pi}{2} X(t)\right).$$

(a) Plot the sample function of $Y(t)$ corresponding to the binary sequence 0010110.

(b) Find the joint pdf of $Y(t_1)$ and $Y(t_2)$.

(c) Find the mean and autocorrelation functions of $Y(t)$.

(d) Is $Y(t)$ a stationary, wide-sense stationary, or cyclostationary random process?

(e) Is $Y(t)$ mean square continuous?

(f) Does $Y(t)$ have a mean square derivative? If so, find its mean and autocorrelation functions.

9.134. Let $N(t)$ be the Poisson process, and suppose we form the phase-modulated process

$$Y(t) = a \cos(2\pi f t + \pi N(t)).$$

(a) Plot a sample function of $Y(t)$ corresponding to a typical sample function of $N(t)$.

(b) Find the joint density function of $Y(t_1)$ and $Y(t_2)$. *Hint:* Use the independent increments property of $N(t)$.

(c) Find the mean and autocorrelation functions of $Y(t)$.

(d) Is $Y(t)$ a stationary, wide-sense stationary, or cyclostationary random process?

(e) Is $Y(t)$ mean square continuous?

(f) Does $Y(t)$ have a mean square derivative? If so, find its mean and autocorrelation functions.

9.135. Let $X(t)$ be a train of amplitude-modulated pulses with occurrences according to a Poisson process:

$$X(t) = \sum_{k=1}^{\infty} A_k h(t - S_k),$$

where the A_k are iid random variables, the S_k are the event occurrence times in a Poisson process, and $h(t)$ is a function of time. Assume the amplitudes and occurrence times are independent.

(a) Find the mean and autocorrelation functions of $X(t)$.

(b) Evaluate part a when $h(t) = u(t)$, a unit step function.

(c) Evaluate part a when $h(t) = p(t)$, a rectangular pulse of duration T seconds.

9.136. Consider a linear combination of two sinusoids:

$$X(t) = A_1 \cos(\omega_0 t + \Theta_1) + A_2 \cos(\sqrt{2}\omega_0 t + \Theta_2),$$

where Θ_1 and Θ_2 are independent uniform random variables in the interval $(0, 2\pi)$, and A_1 and A_2 are jointly Gaussian random variables. Assume that the amplitudes are independent of the phase random variables.

(a) Find the mean and autocorrelation functions of $X(t)$.

(b) Is $X(t)$ mean square periodic? If so, what is the period?

(c) Find the joint pdf of $X(t_1)$ and $X(t_2)$.

9.137. **(a)** A *Gauss-Markov random process* is a Gaussian random process that is also a Markov process. Show that the autocovariance function of such a process must satisfy

$$C_X(t_3, t_1) = \frac{C_X(t_3, t_2)C_X(t_2, t_1)}{C_X(t_2, t_2)},$$

where $t_1 \leq t_2 \leq t_3$.

(b) It can be shown that if the autocovariance of a Gaussian random process satisfies the above equation, then the process is Gauss-Markov. Is the Wiener process Gauss-Markov? Is the Ornstein-Uhlenbeck process Gauss-Markov?

9.138. Let A_n and B_n be two independent stationary random processes. Suppose that A_n and B_n are zero-mean, Gaussian random processes with autocorrelation functions

$$R_A(k) = \sigma_1^2 \rho_1^{|k|} \qquad R_B(k) = \sigma_2^2 \rho_2^{|k|}.$$

A block multiplexer takes blocks of two from the above processes and interleaves them to form the random process Y_m:

$$A_1 A_2 B_1 B_2 A_3 A_4 B_3 B_4 A_5 A_6 B_5 B_6 \ldots.$$

(a) Find the autocorrelation function of Y_m.

(b) Is Y_m cyclostationary? wide-sense stationary?

(c) Find the joint pdf of Y_m and Y_{m+1}.

(d) Let $Z_m = Y_{m+T}$, where T is selected uniformly from the set $\{0, 1, 2, 3\}$. Repeat parts a, b, and c for Z_m.

9.139. Let A_n be the Gaussian random process in Problem 9.138. A decimator takes every other sample to form the random process V_m:

$$A_1 A_3 A_5 A_7 A_9 A_{11}$$

(a) Find the autocorrelation function of V_m.

(b) Find the joint pdf of V_m and V_{m+k}.

(c) An interpolator takes the sequence V_m and inserts zeros between samples to form the sequence W_k:

$$A_1 0 A_3 0 A_5 0 A_7 0 A_9 0 A_{11} \ldots.$$

Find the autocorrelation function of W_k. Is W_k a Gaussian random process?

9.140. Let A_n be a sequence of zero-mean, unit-variance independent Gaussian random variables. A block coder takes pairs of A's and linearly transforms them to form the sequence Y_n:

$$\begin{bmatrix} Y_{2n} \\ Y_{2n+1} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} A_{2n} \\ A_{2n+1} \end{bmatrix}.$$

(a) Find the autocorrelation function of Y_n.

(b) Is Y_n stationary in any sense?

(c) Find the joint pdf of Y_n, Y_{n+1}, and Y_{n+2}.

9.141. Suppose customer orders arrive according to a Bernoulli random process with parameter p. When an order arrives, its size is an exponential random variable with parameter λ. Let S_n be the total size of all orders up to time n.

(a) Find the mean and autocorrelation functions of S_n.

(b) Is S_n a stationary random process?

(c) Is S_n a Markov process?

(d) Find the joint pdf of S_n and S_{n+k}.