Appendix A

Review of Vectors

This appendix is a summary of the mathematical aspects of vectors used in electricity and magnetism. For a more detailed introduction to vectors, see Chapter 1.

A.1 DESCRIBING THE 3D WORLD: VECTORS

Physical phenomena take place in the 3D world around us. In order to be able to make quantitative predictions and give detailed, quantitative explanations, we need tools for describing precisely the positions and velocities of objects in 3D, and the changes in position and velocity due to interactions. These tools are mathematical entities called 3D "vectors."

3D Coordinates

We will use a 3D coordinate system to specify positions in space and other vector quantities. Usually we will orient the axes of the coordinate system as shown in Figure A.1: $+x$ axis to the right, $+y$ axis upward, and $+z$ axis coming out of the page, toward you. This is a "right-handed" coordinate system: if you hold the thumb, first, and second fingers of your right hand perpendicular to each other, and align your thumb with the x axis and your first finger with the y axis, your second finger points along the z axis. (In some math and physics textbook discussions of 3D coordinate systems, the x axis points out, the y axis points to the right, and the z axis points up, but we will also use a 2D coordinate system with y up, so it makes sense always to have the y axis point up.)

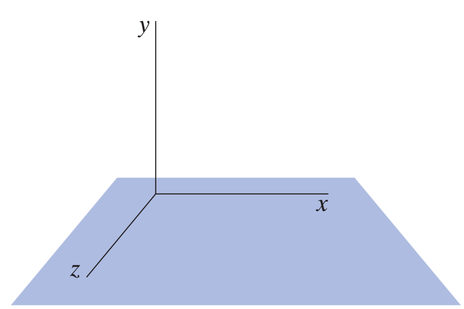

Figure A.1 Right-handed 3D coordinate system.

Basic Properties of Vectors: Magnitude and Direction

A vector is a quantity that has a magnitude and a direction. For example, the velocity of a baseball is a vector quantity. The magnitude of the baseball's velocity is the speed of the baseball, for example 20 meters/second. The direction of the baseball's velocity is the direction of its motion at a particular instant, for example "up" or "to the right" or "west" or "in the $+y$ direction." A symbol denoting a vector is written with an arrow over it:

$$\vec{v} \text{ is a vector.}$$

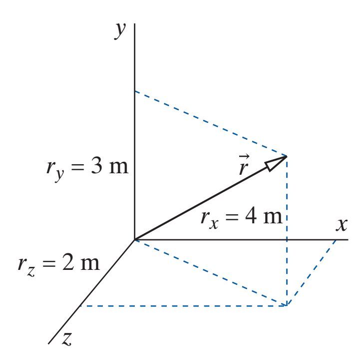

Figure A.2 A position vector $\vec{r} = \langle 4, 3, 2 \rangle$ m and its x, y, and z components.

Position

A position in space can also be considered to be a vector, called a *position vector*, pointing from an origin to that location. Figure A.2 shows a position vector that might represent your final position if you started at the origin and walked 4 meters along the x axis, then 2 meters parallel to the z axis, then climbed a ladder so you were 3 meters above the ground. Your new position relative to the origin is a vector that can be written like this:

$$\vec{r} = \langle 4, 3, 2\rangle\,\text{m}$$
$$x \text{ component } r_x = 4\,\text{m}$$
$$y \text{ component } r_y = 3\,\text{m}$$
$$z \text{ component } r_z = 2\,\text{m}$$

In three dimensions a vector is a triple of numbers $\langle x, y, z\rangle$. Quantities like the position of an object and the velocity of an object can be represented as vectors:

$$\vec{r} = \langle x, y, z\rangle \text{ (a position vector)}$$
$$\vec{r}_1 = \langle 3.2, -9.2, 66.3\rangle\,\text{m (a position vector)}$$
$$\vec{v} = \langle v_x, v_y, v_z\rangle \text{ (a velocity vector)}$$
$$\vec{v}_1 = \langle -22.3, 0.4, -19.5\rangle\,\text{m/s (a velocity vector)}$$

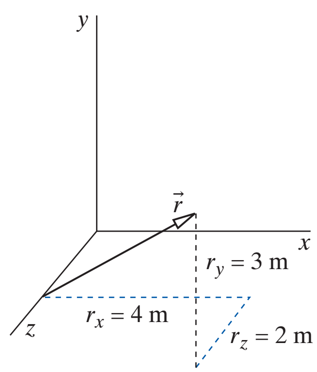

Figure A.3 The arrow represents the vector $\vec{r} = \langle 4, 3, 2\rangle$ m, drawn with its tail at location $\langle 0, 0, 2\rangle$.

Components of a Vector

Each of the numbers in the triple is referred to as a *component* of the vector. The x component of the vector $\vec{v}$ is the number v_x. The z component of the vector $\vec{v}_1 = \langle -22.3, 0.4, -19.5\rangle$ m/s is -19.5 m/s. A component such as v_x is not a vector, since it is only one number.

It is important to note that the x component of a vector specifies the difference between the x coordinate of the tail of the vector and the x coordinate of the tip of the vector. It does not give any information about the location of the tail of the vector (compare Figure A.2 and Figure A.3).

Drawing Vectors

In Figure A.2 we represented your position vector relative to the origin graphically by an arrow whose tail is at the origin and whose arrowhead is at your position. The length of the arrow represents the distance from the origin, and the direction of the arrow represents the direction of the vector, which is the direction of a direct path from the initial position to the final position (the “displacement”; by walking and climbing you “displaced” yourself from the origin to your final position).

Since it is difficult to draw a 3D diagram on paper, when working on paper you will usually be asked to draw vectors which all lie in a single plane. Figure A.4 shows an arrow in the xy plane representing the vector $\langle -3, -1, 0\rangle$.

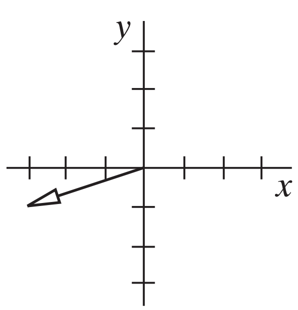

Figure A.4 The position vector $\langle -3, -1, 0\rangle$, drawn at the origin, in the xy plane. The components of the vector specify the displacement from the tail to the tip. The z axis, which is not shown, comes out of the page, toward you.

Vectors and Scalars

A quantity which is represented by a single number is called a *scalar*. A scalar quantity does not have a direction. Examples include the mass of an object, such as 5 kg, or the temperature, such as -20C. Vectors and scalars are very different entities; a vector can never be equal to a scalar, and a scalar cannot be added to a vector. Scalars can be positive or negative:

$$m = 50\,\text{kg}$$
$$T = -20\,\text{C}$$

Although a component of a vector such as v_x is not a vector, it’s not a scalar either, despite being only one number. An important property of a true scalar is that its value doesn’t change if we orient the xyz coordinate axes differently. Rotating the axes doesn’t change an object’s mass, or the temperature, but it does change what we mean by the x component of the velocity since the x axis now points in a different direction.

Magnitude of a Vector

In Figure A.5 we again show the vector from Figure A.2, showing your displacement from the origin. Using a 3D extension of the Pythagorean theorem for right triangles (Figure A.6), the net distance you have moved from the starting point is

$$\sqrt{(4\,\mathrm{m})^2 + (3\,\mathrm{m})^2 + (2\,\mathrm{m})^2} = \sqrt{29}\,\mathrm{m} = \sqrt{5.39}\,\mathrm{m}$$

We say that the *magnitude* $|\vec{r}|$ of the position vector $\vec{r}$ is

$$|\vec{r}| = 5.39\,\mathrm{m}$$

The magnitude of a vector is written either with absolute-value bars around the vector as $|\vec{r}|$, or simply by writing the symbol for the vector without the little arrow above it, r.

The magnitude of a vector can be calculated by taking the square root of the sum of the squares of its components (see Figure A.6).

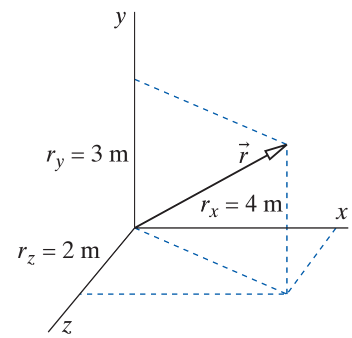

Figure A.5 A vector representing a displacement from the origin.

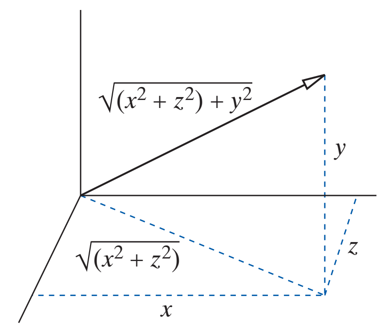

Figure A.6 The magnitude of a vector is the square root of the sum of the squares of its components (3D version of the Pythagorean theorem).

MAGNITUDE OF A VECTOR

If the vector $\vec{r} = \langle r_x, r_y, r_z \rangle$ then $|\vec{r}| = \sqrt{r_x^2 + r_y^2 + r_z^2}$ (a scalar).

The magnitude of a vector is always a positive number. The magnitude of a vector is a single number, not a triple of numbers, and it is a scalar, not a vector.

The magnitude of a vector is a true scalar, because its value doesn't change if you rotate the coordinate axes. Rotating the axes changes the individual components, but the length of the arrow representing the vector doesn't change.

Can a *Vector* be Positive or Negative?

QUESTION Consider the vector $\vec{v} = \langle 8 \times 10^6, 0, -2 \times 10^7 \rangle$ m/s. Is this vector positive? Negative? Zero?

None of these descriptions is appropriate. The x component of this vector is positive, the y component is zero, and the z component is negative. Vectors aren't positive, or negative, or zero. Their components can be positive or negative or zero, but these words just don't mean anything when used with the vector as a whole.

On the other hand, the *magnitude* of a vector such as $|\vec{v}|$ is always positive.

Mathematical Operations Involving Vectors

Although the algebra of vectors is similar to the scalar algebra with which you are very familiar, it is not identical. There are some algebraic operations that cannot be performed on vectors.

Algebraic operations that *are* legal for vectors include the following operations, which we will discuss in this chapter:

- adding one vector to another vector: $\vec{a} + \vec{w}$
- subtracting one vector from another vector: $\vec{b} - \vec{d}$
- finding the magnitude of a vector: $|\vec{r}|$
- finding a unit vector (a vector of magnitude 1): $\hat{r}$
- multiplying (or dividing) a vector by a scalar: $3\vec{v}$ or $\vec{w}/2$
- finding the rate of change of a vector: $\Delta\vec{r}/\Delta t$ or $d\vec{r}/dt$.

In later chapters we will also see that there are two more ways of combining two vectors:

the vector dot product, whose result is a scalar

the vector cross product, whose result is a vector

Operations that are *Not* Legal for Vectors

Although vector algebra is similar to the ordinary scalar algebra you have used up to now, there are certain operations that are not legal (and not meaningful) for vectors:

A vector cannot be set equal to a scalar.

A vector cannot be added to or subtracted from a scalar.

A vector cannot occur in the denominator of an expression. (Although you can't divide by a vector, note that you can legally divide by the *magnitude* of a vector, which is a scalar.)

Multiplying a Vector by a Scalar

A vector can be multiplied (or divided) by a scalar. If a vector is multiplied by a scalar, each of the components of the vector is multiplied by the scalar:

$$\text{If } \vec{r} = \langle x, y, z \rangle \text{ then } a\vec{r} = \langle ax, ay, az \rangle$$

$$\text{If } \vec{v} = \langle v_x, v_y, v_z \rangle \text{ then } \frac{\vec{v}}{b} = \langle \frac{v_x}{b}, \frac{v_y}{b}, \frac{v_z}{b} \rangle$$

$$(\frac{1}{2})\langle 6, -20, 9 \rangle = \langle 3, -10, 4.5 \rangle$$

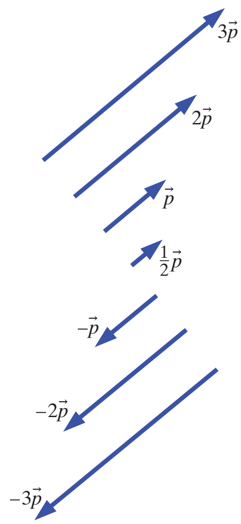

Figure A.7 Multiplying a vector by a scalar changes the magnitude of the vector. Multiplying by a negative scalar reverses the direction of the vector.

Multiplication by a scalar "scales" a vector, keeping its direction the same but making its magnitude larger or smaller (Figure A.7). Multiplying by a negative scalar reverses the direction of a vector.

Magnitude of a Scalar

You may wonder how to find the magnitude of a quantity like $-3\vec{r}$, which involves the product of a scalar and a vector. This expression can be factored:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}|$$

The magnitude of a scalar is its absolute value, so:

$$|-3\vec{r}| = |-3| \cdot |\vec{r}| = 3\sqrt{r_x^2 + r_y^2 + r_z^2}$$

Direction of a Vector: Unit Vectors

One way to describe the direction of a vector is by specifying a *unit vector*. A unit vector is a vector of magnitude 1, pointing in some direction. A unit vector is written with a "hat" (caret) over it instead of an arrow. The unit vector $\hat{a}$ is called "a-hat".

QUESTION Is the vector $\langle 1, 1, 1 \rangle$ a unit vector?

The magnitude of $\langle 1, 1, 1 \rangle$ is $\sqrt{1^2 + 1^2 + 1^2} = 1.73$, so this is not a unit vector.

The vector $\langle 1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3} \rangle$ is a unit vector, since its magnitude is 1:

$$\sqrt{(\frac{1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2 + (\frac{1}{\sqrt{3}})^2} = 1$$

Note that every component of a unit vector must be less than or equal to 1.

In our 3D Cartesian coordinate system, there are three special unit vectors, oriented along the three axes. They are called i-hat, j-hat, and k-hat, and they point along the x, y, and z axes, respectively (Figure A.8):

$$\hat{\imath} = \langle 1, 0, 0 \rangle$$
$$\hat{\jmath} = \langle 0, 1, 0 \rangle$$
$$\hat{k} = \langle 0, 0, 1 \rangle$$

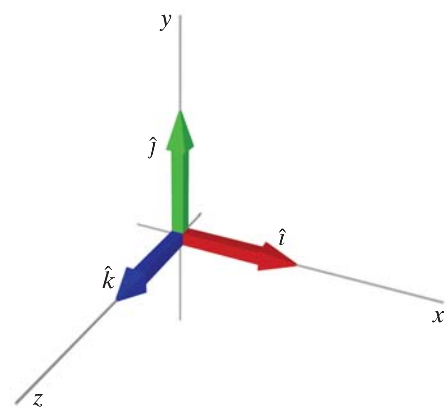

Figure A.8 The unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

One way to express a vector is in terms of these special unit vectors:

$$\langle 0.02, -1.7, 30.0 \rangle = 0.02\hat{\imath} + (-1.7)\hat{\jmath} + 30.0\hat{k}$$

We will usually use the $\langle x, y, z \rangle$ form rather than the $\hat{\imath}\hat{\jmath}\hat{k}$ form in this book, because the familiar $\langle x, y, z \rangle$ notation, used in many calculus textbooks, emphasizes that a vector is a single entity.

Not all unit vectors point along an axis, as shown in Figure A.9. For example, the vectors

$$\hat{g} = \langle 0.5774, 0.5774, 0.5774 \rangle \text{ and } \hat{F} = \langle 0.424, 0.566, 0.707 \rangle$$

are both unit vectors, since the magnitude of each is equal to 1. Note that every component of a unit vector is less than or equal to 1.

Calculating Unit Vectors

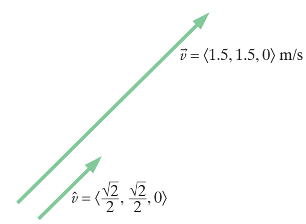

Figure A.9 The unit vector $\hat{v}$ has the same direction as the vector $\vec{v}$, but its magnitude is 1, and it has no physical units.

Any vector may be factored into the product of a unit vector in the direction of the vector, multiplied by a scalar equal to the magnitude of the vector.

$$\vec{w} = |\vec{w}| \cdot \hat{w}$$

For example, a vector of magnitude 5, aligned with the y axis, could be written as:

$$\langle 0, 5, 0 \rangle = 5\langle 0, 1, 0 \rangle$$

Therefore, to find a unit vector in the direction of a particular vector, we just divide the vector by its magnitude:

CALCULATING A UNIT VECTOR

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\langle x, y, z \rangle}{\sqrt{(x^2 + y^2 + z^2)}}$$
$$\hat{r} = \left\langle \frac{x}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{y}{\sqrt{(x^2 + y^2 + z^2)}}, \frac{z}{\sqrt{(x^2 + y^2 + z^2)}} \right\rangle$$

EXAMPLE Unit Vector

If $\vec{v} = \langle -22.3, 0.4, -19.5 \rangle$ m/s, then

$$\hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -22.3, 0.4, -19.5 \rangle \text{ m/s}}{\sqrt{(-22.3)^2 + (0.4)^2 + (-19.5)^2} \text{ m/s}} = \langle -0.753, 0.0135, -0.658 \rangle$$

Remember that to divide a vector by a scalar, you divide each component of the vector by the scalar. The result is a new vector. Note also that a unit vector has no physical units (such as meters per second), because the units in the numerator and denominator cancel.

Equality of Vectors

EQUALITY OF VECTORS

A vector is equal to another vector if and only if all the components of the vectors are equal.

$$\vec{w} = \vec{r} \text{ means that}$$
$$w_x = r_x \text{ and } w_y = r_y \text{ and } w_z = r_z$$

The magnitudes and directions of two equal vectors are the same:

$$|\vec{w}| = |\vec{r}| \text{ and } \hat{w} = \hat{r}$$

EXAMPLE Equal Vectors

$$\vec{r} = \langle 4, 3, 2 \rangle$$
$$|\vec{r}| = \sqrt{(4^2 + 3^2 + 2^2)} = 5.39$$
$$\hat{r} = \frac{\langle 4, 3, 2 \rangle}{5.39} = \langle 0.742, 0.557, 0.371 \rangle$$
$$\text{If } \vec{w} = \vec{r}$$
$$\vec{w} = \langle 4, 3, 2 \rangle$$
$$|\vec{w}| = 5.39$$
$$\hat{w} = \langle 0.742, 0.557, 0.371 \rangle$$

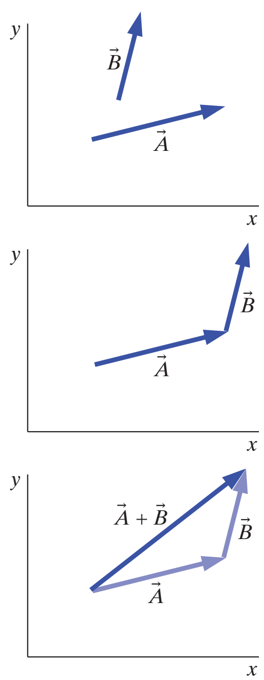

Figure A.10 The procedure for adding two vectors graphically: draw vectors tip to tail. To add $\vec{A} + \vec{B}$ graphically, move $\vec{B}$ so the tail of $\vec{B}$ is at the tip of $\vec{A}$ then draw a new arrow starting at the tail of $\vec{A}$ and ending at the tip of $\vec{B}$.

Vector Addition

ADDING VECTORS

The sum of two vectors is another vector, obtained by adding the components of the vectors.

$$\vec{A} = \langle A_x, A_y, A_z \rangle$$
$$\vec{B} = \langle B_x, B_y, B_z \rangle$$
$$\vec{A} + \vec{B} = \langle (A_x + B_x), (A_y + B_y), (A_z + B_z) \rangle$$

EXAMPLE Adding Vectors

$$\langle 1, 2, 3 \rangle + \langle -4, 5, 6 \rangle = \langle -3, 7, 9 \rangle$$

Warning: Don't Add Magnitudes!

The magnitude of a vector is *not* in general equal to the sum of the magnitudes of the two original vectors! For example, the magnitude of the vector $\langle 3, 0, 0 \rangle$ is 3, and the magnitude of the vector $\langle -2, 0, 0 \rangle$ is 2, but the magnitude of the vector $(\langle 3, 0, 0 \rangle + \langle -2, 0, 0 \rangle)$ is 1, not 5!

Adding Vectors Graphically: Tip to Tail

The sum of two vectors has a geometric interpretation. In Figure A.10 you first walk along displacement vector $\vec{A}$, followed by walking along displacement vector $\vec{B}$. What is your net displacement vector $\vec{C} = \vec{A} + \vec{B}$? The x component C_x

of your net displacement is the sum of A_x and B_x. Similarly, the y component C_y of your net displacement is the sum of A_y and B_y.

GRAPHICAL ADDITION OF VECTORS

To add two vectors $\vec{A}$ and $\vec{B}$ graphically (Figure A.10):

- Draw the first vector $\vec{A}$
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the *tip* of the first vector
- Draw a new vector from the tail of vector $\vec{A}$ to the tip of vector $\vec{B}$

Vector Subtraction

The difference of two vectors will be very important in this and subsequent chapters. To subtract one vector from another, we subtract the components of the second from the components of the first:

$$\vec{A} - \vec{B} = \langle (A_x - B_x), (A_y - B_y), (A_z - B_z) \rangle$$
$$\langle 1, 2, 3 \rangle - \langle -4, 5, 6 \rangle = \langle 5, -3, -3 \rangle$$

Subtracting Vectors graphically: Tail to Tail

To subtract one vector $\vec{B}$ from another vector $\vec{A}$ graphically:

- Draw the first vector $\vec{A}$
- Move the second vector $\vec{B}$ (without rotating it) so its tail is located at the *tail* of the first vector
- Draw a new vector from the tip of vector $\vec{B}$ to the tip of vector $\vec{A}$

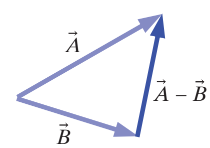

Figure A.11 The procedure for subtracting vectors graphically: draw vectors tail to tail; draw new vector from tip of second vector to tip of first vector.

Note that you can check this algebraically and graphically. As shown in Figure A.11, since the tail of $\vec{A} - \vec{B}$ is located at the tip of $\vec{B}$, then the vector $\vec{A}$ should be the sum of $\vec{B}$ and $\vec{A} - \vec{B}$, as indeed it is:

$$\vec{B} + (\vec{A} - \vec{B}) = \vec{A}$$

Commutativity and Associativity

Vector addition is commutative:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Vector subtraction is *not* commutative:

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$$

The associative property holds for vector addition and subtraction:

$$(\vec{A} + \vec{B}) - \vec{C} = \vec{A} + (\vec{B} - \vec{C})$$

The Zero Vector

It is convenient to have a compact notation for a vector whose components are all zero. We will use the symbol $\vec{0}$ to denote a zero vector, in order to distinguish it from a scalar quantity that has the value 0.

$$\vec{0} = \langle 0, 0, 0 \rangle$$

For example, the sum of two vectors $\vec{B} + (-\vec{B}) = \vec{0}$.

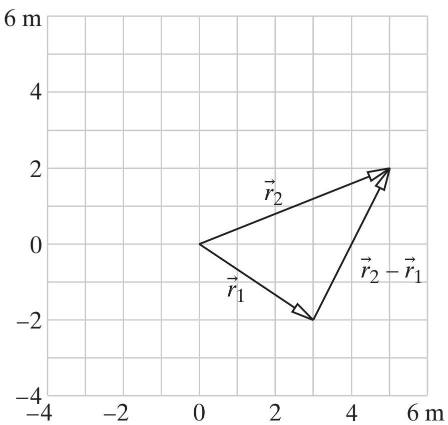

Figure A.12 Relative position vector.

Change in a Quantity: The Greek Letter Δ

Frequently we will want to calculate the change in a quantity. For example, we may want to know the change in a moving object's position or the change in its velocity during some time interval. The Greek letter Δ (capital delta suggesting "d for difference") is used to denote the change in a quantity (either a scalar or a vector).

We use the subscript i to denote an *initial* value of a quantity, and the subscript f to denote the *final* value of a quantity. If a vector $\vec{r}_i$ denotes the initial position of an object relative to the origin (its position at the beginning of a time interval), and $\vec{r}_f$ denotes the final position of the object, then

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_i$$

$\Delta\vec{r}$ means "change of $\vec{r}$" or $\vec{r}_f - \vec{r}_i$ (displacement)
Δt means "change of t" or $t_f - t_i$ (time interval)
The symbol Δ (delta) always means "final minus initial", not "initial minus final". For example, when a child's height changes from $1.1\,\mathrm{m}$ to $1.2\,\mathrm{m}$, the change is $\Delta y = +0.1\,\mathrm{m}$, a positive number. If your bank account dropped from \$150 to \$130, what was the change in your balance? Δ(bank account)$= -20$ dollars.

Relative Position Vectors

Vector subtraction is used to calculate relative position vectors, vectors which represent the position of an object relative to another object. In Figure A.12 object 1 is at location $\vec{r}_1$ and object 2 is at location $\vec{r}_2$. We want the components of a vector that points from object 1 to object 2. This is the vector obtained by subtraction: $\vec{r}_{2\text{ relative to }1} = \vec{r}_2 - \vec{r}_1$. Note that the form is always "final" minus "initial" in these calculations.

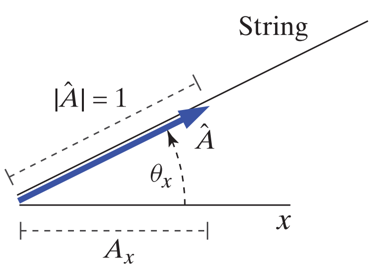

Figure A.13 A unit vector whose direction is at a known angle from the $+x$ axis.

Unit Vectors and Angles

Suppose a taut string is at an angle θ_x to the $+x$ axis, and we need a unit vector in the direction of the string. Figure A.13 shows a unit vector $\hat{A}$ pointing along the string. What is the x component of this unit vector? Consider the triangle whose base is A_x and whose hypotenuse is $|\hat{A}| = 1$. From the definition of the cosine of an angle we have this:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{A_x}{1}, \text{ so } A_x = \cos\theta_x$$

In Figure A.13 the angle θ_x is shown in the first quadrant (θ_x less than $90°$), but this works for larger angles as well. For example, in Figure **??** the angle from the $+x$ axis to a unit vector $\hat{B}$ is in the second quadrant (θ_x greater than $90°$) and $\cos\theta_x$ is negative, which corresponds to B_x being negative.

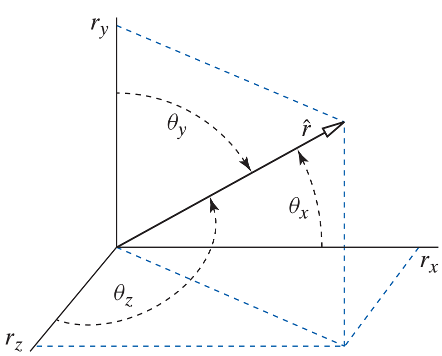

Figure A.14 A 3D unit vector and its angles to the x, y, and z axes.

What is true for x is also true for y and z. Figure A.14 shows a 3D unit vector $\hat{r}$ and indicates the angles between the unit vector and the x, y, and z axes. Evidently we can write

$$\hat{r} = \langle\cos\theta_x, \cos\theta_y, \cos\theta_z\rangle$$

These three cosines of the angles between a vector (or unit vector) and the coordinate axes are called the "direction cosines" of the vector. The cosine function is never greater than 1, just as no component of a unit vector can be greater than 1.

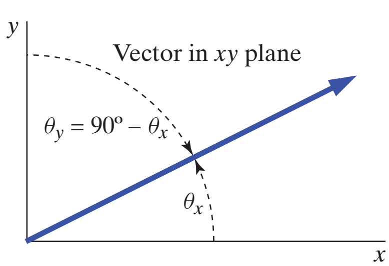

Figure A.15 If a vector lies in the xy plane, $\cos\theta_y = \sin\theta_x$.

A common special case is that of a unit vector lying in the xy plane, with zero z component (Figure A.15). In this case $\theta_x + \theta_y = 90°$, so that $\cos\theta_y = \cos(90° - \theta_x) = \sin\theta_x$, so that you can express the cosine of θ_y as the sine of θ_x, which is often convenient. However, in the general 3D case shown in Figure A.14 there is no such simple relationship among the direction angles, nor among their cosines.

FINDING A UNIT VECTOR FROM ANGLES

To find a unit vector if angles are given:

- Redraw the vector of interest with its tail at the origin, and determine the angles between this vector and the axes.
- Imagine the vector $\langle 1,0,0\rangle$, which lies on the $+x$ axis. θ_x is the angle through which you would rotate the vector $\langle 1,0,0\rangle$ until its direction matched that of your vector. θ_x is positive, and $\theta_x \leq 180°$.
- θ_y is the angle through which you would rotate the vector $\langle 0,1,0\rangle$ until its direction matched that of your vector. θ_y is positive, and $\theta_y \leq 180°$.
- θ_z is the angle through which you would rotate the vector $\langle 0,0,1\rangle$ until its direction matched that of your vector. θ_z is positive, and $\theta_z \leq 180°$.

EXAMPLE From Angle to Unit Vector

A rope lying in the xy plane, pointing up and to the right, supports a climber at an angle of $20°$ to the vertical (Figure A.16). What is the unit vector pointing up along the rope?

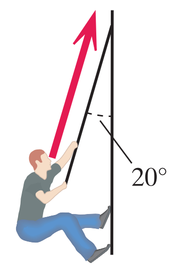

Figure A.16 A climber supported by a rope.

Solution Follow the procedure given above for finding a unit vector from angles. In Figure A.17 we redraw the vector with its tail at the origin, and we determine the angles between the vector and the axes. If we rotate the unit vector $\langle 1,0,0\rangle$ from along the $+x$ axis to the vector of interest, we see that we have to rotate through an angle $\theta_x = 70°$. To rotate the unit vector $\langle 0,1,0\rangle$ from along the $+y$ axis to the vector of interest, we have to rotate through an angle of $\theta_y = 20°$. The angle from the $+z$ axis to our vector is $\theta_z = 90°$. Therefore the unit vector that points along the rope is this:

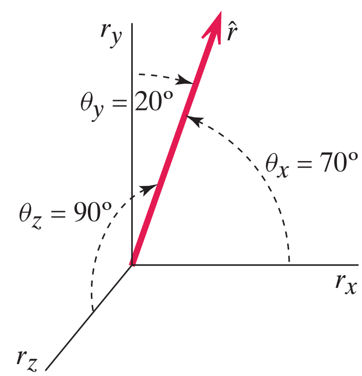

Figure A.17 Redraw the vector with its tail at the origin. Identify the angles between the positive axes and the vector. In this example the vector lies in the xy plane.

$$\langle \cos 70°, \cos 20°, \cos 90°\rangle = \langle 0.342, 0.940, 0\rangle$$

FURTHER DISCUSSION You may have noticed that the y component of the unit vector can also be calculated as $\sin 70° = 0.940$, and it is often useful to recognize that a vector component can be obtained using sine instead of cosine. There is however some advantage always to calculate in terms of direction cosines. This is a method that always works, including in 3D, and which avoids having to decide whether to use a sine or a cosine. Just use the cosine of the angle from the relevant positive axis to the vector.

EXAMPLE From Unit Vector to Angles

A vector $\vec{r}$ points from the origin to the location $\langle -600, 0, 300\rangle$ m. What is the angle that this vector makes to the x axis? To the y axis? To the z axis?

Solution

$$\hat{r} = \frac{\langle -600, 0, 300\rangle}{\sqrt{(-600)^2 + (0)^2 + (300)^2}\,\text{m}} = \langle -0.894, 0, 0.447\rangle$$

But we also know that $\hat{r} = \langle \cos\theta_x, \cos\theta_y, \cos\theta_z\rangle$, so
$\cos\theta_x = -0.894$, and the arccosine gives $\theta_x = 153.4^\circ$.
Similarly,
$\cos\theta_y = 0$, $\theta_y = 90^\circ$ (which checks; no y component)
$\cos\theta_z = 0.447, \theta_z = 63.4^\circ$

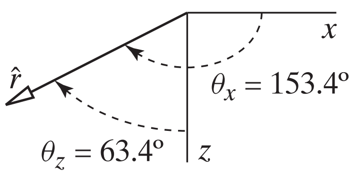

Figure A.18 Look down on the xz plane. The difference in the two angles is 90°, as it should be.

FURTHER DISCUSSION Looking down on the xz plane in Figure A.18, you can see that the difference between $\theta_x = 153.4^\circ$ and $\theta_z = 63.4^\circ$ is 90°, as it should be.

A.2 VECTOR MULTIPLICATION

Vectors can be added and subtracted, and they can be multiplied by a scalar. Two vectors can also be multiplied, but two different kinds of vector multiplication are defined: the dot product and the cross product. In the previous volume the dot product was introduced in the context of work, and the cross product was introduced in the context of angular momentum.

The Dot Product

The dot product is an operation involving two vectors. This is encountered in the expression for work in Chapter 6:

$$W = \vec{F} \bullet \Delta\vec{r} = (F_x\Delta x + F_y\Delta y + F_z\Delta z)$$
$$\text{If } \vec{F} = \langle 3, -2, 4\rangle \text{ N and } \Delta\vec{r} = \langle 2, 0, -5\rangle \text{ m, then}$$
$$\vec{F} \bullet \Delta\vec{r} = ((3\cdot 2) + (-2\cdot 0) + (4\cdot -5))\ \text{N}\ \cdot\ \text{m}\ = -14\,\text{N}\cdot\text{m}$$

The result of a dot product operation is a scalar (like the quantity work). Note that the dot product of a vector with itself gives the square of the magnitude of the vector:

$$\langle r_x, r_y, r_z\rangle \bullet \langle r_x, r_y, r_z\rangle = (r_x^2, r_y^2, r_z^2) = |\vec{r}|^2$$

The magnitude of the dot product can also be calculated as:

$$\vec{F} \bullet \Delta\vec{r} = F\Delta r\cos\theta = F_{||}\Delta r = F\Delta r_{||}$$

where θ is the angle between the two vectors, placed tail to tail. In the VPython programming language, dot(vector1,vector2) gives the dot product of two vectors.

The Cross Product

The cross product is discussed in detail in Chapter 18 in the context of the Biot-Savart law for finding the magnetic field of moving charges. In the VPython programming language, cross(vector1,vector2) gives the cross product of two vectors.

It is possible to evaluate the cross product in terms of unit vectors along the three axes (Figure A.19). First, note that $\hat{\imath}\times\hat{\imath} = 0$, $\hat{\jmath}\times\hat{\jmath} = 0$, and $\hat{k}\times\hat{k} = 0$, since when we cross a vector with itself the angle between the two vectors is zero, and $\sin 0^\circ = 0$.

Second, $\hat{\imath}\times\hat{\jmath} = \hat{k}$, since the angle is 90° and the right-hand rule gives a result in the $+z$ direction (out of the page; Figure A.19). On the other hand, $\hat{\jmath}\times\hat{\imath} = -\hat{k}$, because the right-hand rule gives a result in the $-z$ direction (into the

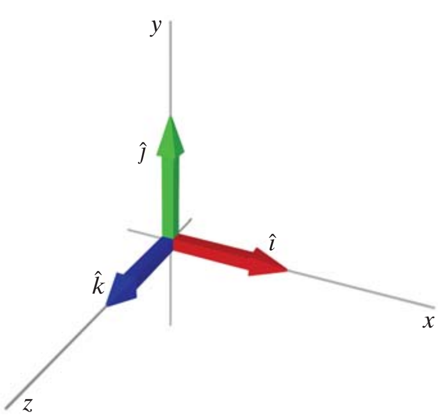

Figure A.19 Cross products of unit vectors.

page). Similarly, $\hat{\jmath} \times \hat{k} = \hat{\imath}$, $\hat{k} \times \hat{\jmath} = -\hat{\imath}$, $\hat{k} \times \hat{\imath} = \hat{\jmath}$, and $\hat{\imath} \times \hat{k} = -\hat{\jmath}$. Putting this all together, we obtain the following general result:

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k} \text{ or}$$
$$\vec{A} \times \vec{B} = \langle (A_y B_z - A_z B_z), (A_z B_x - A_x B_z), (A_x B_y - A_y B_x) \rangle$$

This approach to calculating a cross product is particularly useful in computer calculations. Note the cyclic nature of the subscripts: xyz, yzx, zxy.

Common Errors in Vector Multiplication

(1) A dot product of two vectors results in a scalar, not another vector.
(2) A cross product of two vectors results in another vector, not a scalar.

Technically, although a component of a vector is a single number, it is not a scalar. If you rotate your coordinate axes, the x, y, and z components of a vector change, but a true scalar such as $m = 5$ kg doesn't change.

A.3 SUMMARY

Vectors

A 3D *vector* is a quantity with magnitude and a direction, which can be expressed as a triple $\langle x, y, z \rangle$. A vector is indicated by an arrow: $\vec{r}$.

A *scalar* is a single number.

Legal mathematical operations involving vectors include:

- adding one vector to another vector
- subtracting one vector from another vector
- multiplying or dividing a vector by a scalar
- finding the magnitude of a vector
- taking the derivative of a vector

Operations that are *not* legal with vectors include:

- A vector cannot be added to a scalar
- A vector cannot be set equal to a scalar
- A vector cannot appear in the denominator
  (you can't divide by a vector)

The symbol Δ denotes subtraction

The symbol Δ (delta) means "change of": $\Delta t = t_f - t_i$, $\Delta\vec{r} = \vec{r}_f - \vec{r}_i$.

Δ always means "final minus initial".