B sections Test 1, Math 1554, Fall 2017

Name:

TA's name or section number:

Rules: This is an closed book, closed notes test. A one-page (two-sided if needed) "help sheet" is allowed. No electronic equipment (such as calculators, computers, cell phones, headphones) is allowed. This test includes 7 problems. Give as much detail as is given in the lectures, and justify all answers. In justifying your answers, please be concise and write in complete grammatically correct sentences. And take it easy!

problem	points	out of
1		3
2		2
3		3
4		3
5		3
6		3
7		3
total		20

1. [3 points] Let A be an 7×8 matrix whose columns span $\mathbb{R}^7$. For a vector b in $\mathbb{R}^7$, how many solutions can the system $Ax = b$ have?

Solution: Since columns of A span $\mathbb{R}^7$, the system $Ax = b$ is consistent for every b. Hence A has a pivot in each row, i.e., 7 pivots. Thus there is a free variable, and hence $Ax = b$ has infinitely many solutions.

2. [1 points] Give an example of a nonzero 2×2 matrix A such that A^2 is the zero matrix. (Here A^2 is the product of A and A).

Solution: $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ works.

3. [3 points] Can the rows of a 6×5 matrix be linearly independent? Explain.

Solution: Six vectors in $\mathbb{R}^5$ cannot be independent. Indeed, if $v_1, \dots, v_k$ are independent in $\mathbb{R}^5$, and $A = [v_1, \dots, v_k]$, then $Ax = 0$ has a pivot in every column, i.e., there are k pivots. Since pivots lie in different rows we get $k \leq 5$.

4. Let A be a matrix and $v_1, \dots, v_n$ be linearly dependent vectors. Are the vectors $Av_1, \dots, Av_k$ also linearly dependent?

Solution: Since $v_1, \dots, v_n$ are linearly dependent, there is a nonzero vector x with coordinates $x_1, \dots, x_k$ such that $x_1 v_1 + \dots x_k v_k = 0$. Multiplying both sides by A we get

$$x_1 A v_1 + \cdots + x_k A v_k = A(x_1 v_1 + \cdots + x_k v_k) = A0 = 0.$$

Thus $Av_1, \dots, Av_k$ also linearly dependent.

5. [3 points] Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Give step by step instructions on how to decide whether T is onto.

Solution: Write $A = [T(e_1), \dots, , T(e_n)]$ where e_i is the ith column of the identity $n \times n$ matrix. Compute $RREF(A)$ and see if every row is pivotal. As we know every row is pivotal if and only if $Ax = b$ is consistent for every b if and only if T is onto.

6. [3 points] Let A, B, C be invertible $n \times n$ matrices. Derive the formula for the inverse of ABC. (Be sure to check that the formula works).

Solution: Let's try $C^{-1}B^{-1}A^{-1}$:

$$C^{-1}B^{-1}A^{-1}ABC = C^{-1}B^{-1}IBC = C^{-1}B^{-1}BC = C^{-1}IC = I$$

and similarly, $ABCC^{-1}B^{-1}A^{-1} = I$.

7. Let AB be the product of two matrixes A, B. If the system $Bx = 0$ has a free variable, show that the system $ABx = 0$ has a free variable.

Solution: Multiplying both sides of the equation $Bx = 0$ by A gives $ABx = A0 = 0$. Thus any solution of $Bx = 0$ is also a solution $ABx = 0$. By assumption $Bx = 0$ has a nonzero solution, and hence so does $ABx = 0$. Thus $ABx = 0$ has a free variable.

SCRATCH PAPER