Test 2 practice problems, Math 1554, Fall 2017

In what follows "explain" means "give instructions on how to"; it doesn't mean "recite a definition". For example, a good answer to *Explain how to decide if a collection of vectors* $v_1, \dots, v_m$ *spans* $\mathbb{R}^n$ would be: Check that every row of the matrix $[v_1, \dots, v_m]$ is pivotal. A bad answer would be to recite the definition of the span, i.e., to say that we have to check that every vector in $\mathbb{R}^n$ is a linear combination of $v_1, \dots, v_m$ without explaining how to do that.

Convention: Below A denotes a matrix, which is sometimes thought of as the linear transformation $T(x) = Ax$.

1. Explain how to find a basis in the kernel of A, also known as $\operatorname{Nul} A$.

2. Explain how to find a basis in the image of A, also known as $\operatorname{Col} A$.

3. Explain how to decide if a collection of vectors $v_1, \dots, v_m$ spans $\mathbb{R}^n$.

4. Explain how to find a basis in $\operatorname{span}\{v_1, \dots, v_m\}$.

5. Explain how to complete a linearly independent collection of vectors $v_1, \dots, v_k$ to a basis of $\mathbb{R}^n$. (Hint: form the matrix $[v_1, \dots, v_k, e_1, \dots, e_n]$ and find pivotal columns. Why are the first k columns pivotal?)

6. If $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, and S is a subspace of $\mathbb{R}^n$, why is $T(S)$ a subspace of $\mathbb{R}^m$?

7. If a subspace S of $\mathbb{R}^n$ has dimension k, explain how to decide whether vectors $v_1, \dots, v_k$ in S form a basis.

8. Explain how to compute coordinates of a vector x in a given basis $B = \{v_1, \dots, v_k\}$.

9. Do the subpaces $\operatorname{Col}(A)$ and $\operatorname{Nul}(A)$ change when we do row operations on A? If you answer no, explain why. If you answer yes, give an example of A and a row operation which changes the subspace.

10. If S is a subspace of $\mathbb{R}^n$ what is the relationship between n and the dimension of S? Explain.

11. What is the relationship between the rank of an $m \times n$ matrix A, and the dimensions of $\operatorname{Nul}(A)$, $\operatorname{Col} A$, and the numbers m, n?

Practice how to estimate one in terms of the others (along the lines of #19-14 of sections 2.9).

12. Explain how to compute $\det(A)$ by cofactor expansion and by row operations. Which method is computationally better if A is $n \times n$ and n is large? (You have to know an estimate on how many arithmetic operations it takes in the worst case but you won't have to prove it).

13. Explain how to compute the determinant of a triangular matrix?

14. How does determinant change under row operations?

15. Express $\det(A^T B^{-1} A)$ in terms of $\det(A)$, $\det(B)$.

16. How does one see from $\det(A^\top)$ whether A is invertible?

17. If A is an invertible $n \times n$ matrix, what is an explicit formula for A^{-1} in terms of enties of A? (Considering the case $n = 2$ is not enough). Why is the formula not an efficient way to compute A^{-1}?

18. If columns of A are linearly dependent, compute $\det(A)$.

19. How is the area of a region R in $\mathbb{R}^2$ related to the area of its image under the linear transformation A? Under A^2? Under A^{-3}?

20. Discuss computational efficiency of LU factorization (p.129) versus solving $Ax = b$ directly (p.20). In what situation is the LU factorization useful?

21. Recall that eigenvalues of A are defined as the real numbers λ such that $(A - \lambda I_n)X = 0$ has a nonzero solution. Explain why eigenvalues are precisely the real roots of $\det(A - \lambda I_n)$, the characteristic polynomial of A.

22. If $Av = \lambda v$ with $v \neq 0$ and A is invertible, can you find an eigenvalue/eigenvector of A^{-1}? Can A has a zero eigenvalue?

23. If $Av = \lambda v$ and k is a positive integer, explain why v is an eigenvector for A^k, and what is the corresponding eigenvalue?

24. If v_1, v_2 are eigenvectors of A with the same eigenvalues λ. Is $7v_1 - 3v_2$ an eigenvector of A?

25. If λ_1, λ_2 are distinct eigenvalues of A explain why the corresponding eigenvectors v_1, v_2 are linearly independent.

26. Show that if a nonzero vector lies in two eigenspaces for A, then the eigenspaces coinside.

27. Explain why BAB^{-1} and $A^\top$ have the same eigenvalues.

28. Explain why eigenvalues of an upper or lower triangular matrix (a_{ij}) are its entries on the main diagonal, that is, $a_{11}, \dots, a_{nn}$.

29. Let P be a stochastic matrix with positive entries. Give two different methods of finding the steady state solution, i.e., a probability vector q with $Pq = q$.

30. Suppose A is a 4×4 matrix with eigenvalues $3,\ 1,\ -2,\ 2$. Is A is diagonalizable?

31. If A has a basic of eigenvectors $v_1, \dots, v_n$, explain how to find a matrix Q such that the matrix $Q^{-1}AQ$ is diagonal. Is such Q uniquely determined by A?

32. If A and Q are $n \times n$ matrices such that $Q^{-1}AQ$ is a diagonal matrix with diagonal entries $d_1, \dots, d_n$. Explain how to find the eigenvalues and a basis of eigenvectors for A em without any computation.

33. Explain how to compute A^{-1} via row operations.

34. Give an example of a 2×2 matrix that is not diagonalizable even though all of its eigenvalues are real.

36. Suppose A is a diagonalizable $n \times n$ matrix such that each of its eigenvalues satisfies $|\lambda_i| < 1$. Fix an arbitrary vector v in $\mathbb{R}^n$. Can you approximately compute $A^k v$ for large k?

37. Explain how to write the result of a row operation as the product EA where E is an elementary matrix. See pp.108–109 of Lay. What is the conceptual reason for the fact that is E invertible?

38. Suppose A, B are not square matrices, and AB is invertible (and thus AB is square). Why is the map T given by $T(x) = Bx$ one-to-one? (Hint: multiply both sides of $Bx = 0$ by A and use invertibility of AB to conclude that $x = 0$). Why is the map S given by $S(x) = Ax$ onto? (Hint: $Ax = b$ is consistent because $x = B(AB)^{-1}b$ is a solution).

39. Give an example of non-square matrices A, B with $AB = I_n$?