Discrete Mathematics and Algorithms (CSE611)

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Topic: **Set Theory**

SET THEORY

- A set is a well-defined collection of *distinct* objects, which are called members of the set or elements of the set.
- **Representation of a set**
  - **Tabular form:** If the set A consists of the elements 1, 2, 3, and 4, then we express the set in the "tabular form" as $A = \{1, 2, 3, 4\}$.
  - **Set-builder form:** A set is expressed in this form by displaying a typical element and by stating the properties which the elements of the set must satisfy.
    The symbol $A = \{x|P(x)\}$ or
    $A = \{x : P(x)\}$
    states that A is a set of elements x which satisfy the condition $P(x)$; the symbol ':' or '$|$' is read as 'such that'.

SET THEORY

- **Examples**

$$\begin{aligned} A &= \{1, 3, 5, \ldots, 39\} && (1)\\ &= \{x \mid x \text{ is a positive odd integer} < 40\}. \\ B &= \{2, 4, 6, \ldots\} && (2)\\ &= \{x \mid x = 2n, n \text{ being a natural number}\}. \\ X &= \{1, 8, 27, 64, \ldots\} && (3)\\ &= \{x \mid x = n^3, n \text{ being a positive integer}\}. \\ S &= \{\ldots, -15, -10, -5, 0, 5, 10, 15, \ldots\} && (4)\\ &= \{x \mid x = 5n, n \text{ is an integer}\}. \end{aligned}$$

SET THEORY

- **Null Set:** A set, having no elements, is defined as the null set or the empty set. An empty set is denoted by ϕ.
- **Finite Set:** A set is finite, if it be empty or contains a finite number of elements.
- **Infinite Set:** A set contains an infinite number of elements is called an infinite set.
  Example: The set $\{1, 2, 3, 4, 5\}$ is a finite set and the set $\{x_1, x_2, x_3, \ldots\}$ is an infinite set.
- **Order of a set:** The number of elements of a finite set A is called the order or cardinal number or cardinality of the set A and is symbolically denoted by $n(A)$ or $|A|$.
  Example 1: If $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then $|A| = 2$ and $|B| = 3$.
  Example 2: The null set is regarded as a finite set of order zero, that is $|\phi| = 0$.

SET THEORY

Notations for some well-known sets

- N the set of all natural numbers
- Z the set of all integers
- Q the set of all rational numbers, r such that $r = \frac{a}{b}$, $a, b \in Z$, with $b \neq 0$ and $\gcd(a, b) = 1$
- R the set of all real numbers
- C the set of all complex numbers $z = a + ib$, $a, b \in R$
- E the set of all even integers
- Z^+, Q^+, R^+ the corresponding sets of positive quantities only
- Z^-, Q^-, R^- the corresponding sets of negative quantities only

SET THEORY

- **Sub-set:** If every element of a set A be also an element of another set B, then A is called a subset of B and we write it as $A \subseteq B$. Mathematically, $A \subseteq B$ means if an arbitrary element $x \in A$, then $x \in B$ also.
- **Proper subset:** If, however, the set B contains some elements which are not the elements of a set A, then A is called a proper subset of B and we write it as $A \subset B$.
- **Comparable:** Two sets A and B are said to be comparable, if either $A \subseteq B$ or $B \subseteq A$.
- **Equality of sets:** Two sets A and B are said to be equal, that is $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.
- **Disjoint set:** Two sets A and B are said to be disjoint, if they have no element in common, that is $A \cap B = \emptyset$.

SET THEORY

- **Difference between sets:** The difference between two sets A and B in that order is the set of elements which belong to A, but do not belong to B.
  $A - B$ or $A \setminus B = \{x | x \in A, \text{but } x \notin B\}$
  $B - A$ or $B \setminus A = \{x | x \in B, \text{but } x \notin A\}$
  Example: Let $A = \{1, 2, 3, 4\}$ and $B = \{5, 2, 3, 6\}$. Then $A - B = \{1, 4\}$ and $B - A = \{5, 6\}$.
- **Theorem:** If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- **Theorem:** If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- **Theorem:** The null set $\emptyset$ is a proper subset of every set except $\emptyset$ itself.

SET THEORY

- **Power set:** A set formed of all the subsets of a set S as its element is called a power set of S and is symbolically denoted by $\mathcal{P}(S)$.
  Example: If $S = \{a, b, c\}$, then
  $\mathcal{P}(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$.
- *Notes:* (1) The null set $\emptyset$ is an element/member of $\mathcal{P}(S)$.
  (2) The set S being a subset of itself is also an element of the power set $\mathcal{P}(S)$.
- **Theorem:** If a finite set S has n elements, then its power set $\mathcal{P}(S)$ has 2^n elements. In other words, $|\mathcal{P}(S)| = 2^{|S|}$.
- **Quiz.** What will happen for the power set $\mathcal{P}(S)$, if S is itself a null set?

SET THEORY

- **Universal set:** A universal set, U is the set of elements from which elements may be chosen to form sets for a particular discussion.
  Example: The set of even numbers is a subset of the universal set of whole numbers.
- **Complement of a set:** Let S be a given subset of the universal set U. The complement of S relative to U is the set of all elements of U which are not elements of S, and it is denoted by $\sim S$ or S' or S^c or $\overline{S}$.
  Example: If $U = \{1, 2, 3, 4, 5, 6\}$ and $S = \{2, 3, 4\}$, then $S' = \{1, 5, 6\}$.
  Symbolically, $S' = \{x | x \in U \text{ and } x \notin S\}$.

Venn-Euler diagram

- It is a schematic representation of sets by certain areas containing the elements of the sets, being represented by the points of the respective areas.

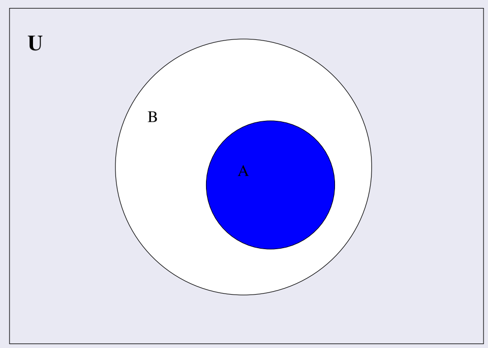

Figure: $A \subseteq B$

Venn-Euler diagram

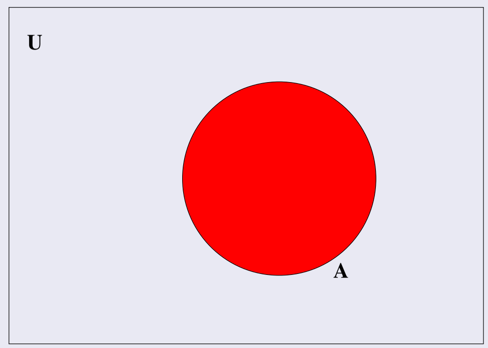

Figure: $A' = U - A$

Basic Set Operations

- **Union or Join** The union of two sets A and B is denoted and defined by

$$\begin{aligned} A \cup B &= \{x | x \in A \text{ or } x \in B\} \\ &= \{x | x \in A \vee x \in B\}. \end{aligned}$$

- If $A_1, A_2, \ldots, A_n$ be the subsets of X, where n is a positive integer, then

$$\begin{aligned} A_1 \cup A_2 \cup \ldots \cup A_n &= \cup_{i=1}^{n} A_i \\ &= \{x | x \in A_i \text{ for some value } i, 1 \leq i \leq n\}. \end{aligned}$$

- Example: If $A = \{1, 2, 3\}$ and $B = \{4, 3, 5, 6\}$, then $A \cup B = \{1, 2, 3, 4, 5, 6\}$.

Basic Set Operations (Continued...)

- From the Venn diagram, it is easy to observe the following theorems.
  1. $A \cup A = A$
  2. $A \cup U = U$
  3. If $A \subseteq B$, then $A \cup B = B$
  4. $A \cup B = B \cup A$
  5. $A \cup \emptyset = A$
  6. $A \cup A' = U$

Basic Set Operations (Continued...)

- **Intersection or Meet** The intersection of two sets A and B is denoted and defined by

$$\begin{aligned} A \cap B &= \{x | x \in A \text{ and } x \in B\} \\ &= \{x | x \in A \wedge x \in B\}. \end{aligned}$$

- If $A_1, A_2, \ldots, A_n$ be the subsets of X, where n is a positive integer, then

$$\begin{aligned} A_1 \cap A_2 \cap \ldots \cap A_n &= \cap_{i=1}^{n} A_i \\ &= \{x | x \in A_i, \forall i, 1 \leq i \leq n\}. \end{aligned}$$

- Example: If $A = \{a, b, c\}$ and $B = \{c, d, e\}$, then $A \cap B = \{c\}$.

Basic Set Operations (Continued...)

- From the Venn diagram, the following theorems are obvious:
  1. $A \cap A = A$
  2. $A \cap U = A$
  3. If $A \subseteq B$, then $A \cap B = A$
  4. $A \cap B = B \cap A$
  5. $A \cap \emptyset = \emptyset$
  6. $A \cap A' = \emptyset$

Laws of Algebra on Sets

Let A, B and C be three any sets.

- **Commutative laws**
  1. $A \cup B = B \cup A$
  2. $A \cap B = B \cap A$
- **Associative laws**
  1. $A \cup (B \cup C) = (A \cup B) \cup C$
  2. $A \cap (B \cap C) = (A \cap B) \cap C$
- **Idempotent laws**
  1. $A \cup A = A$
  2. $A \cap A = A$

Laws of Algebra on Sets (Continued...)

Let A, B and C be three any sets.

- **Distributive laws**
  1. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  2. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- **De Morgan's laws**
  1. $A - B = A \cap B'$
  2. $(A \cup B)' = A' \cap B'$
  3. $(A \cap B)' = A' \cup B'$
  4. $A - (B \cup C) = (A - B) \cap (A - C)$
  5. $A - (B \cap C) = (A - B) \cup (A - C)$

Symmetric Difference

Let A and B be two sets.

- The symmetric difference of A and B is denoted and defined by

$$\begin{aligned} A \triangle B &= (A - B) \cup (B - A) \\ &= \{x \mid [x \in A \text{ and } x \notin B] \text{ or } [x \in B \text{ and } x \notin A]\}. \end{aligned}$$

- Example: If $A = \{1, 2, 4, 7, 9\}$ and $B = \{2, 3, 7, 8, 9\}$, then $A - B = \{1, 4\}$, $B - A = \{3, 8\}$.
  Thus, $A \triangle B = \{1, 4\} \cup \{3, 8\} = \{1, 3, 4, 8\}$.
- It can be easily verified that
  (i) $A \triangle \emptyset = A$,
  (ii) $A \triangle A = \emptyset$,
  (iii) $A \triangle B = \emptyset \Rightarrow A = B$.

Cartesian product of sets

- The Cartesian product of two sets A and B is denoted and defined by

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

  More generally, the Cartesian product of n sets A_1, A_2, ..., A_n is

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i, 1 \leq i \leq n\}.$$

- Example: If $A = \{a, b, c\}$ and $B = \{m, n\}$, then $A \times B = \{(a, m), (a, n), (b, m), (b, n), (c, m), (c, n)\}$.
- It can be easily verified that if $|A| = m$ and $|B| = n$, then $|A \times B| = mn$.
- In general, $A \times B \neq B \times A$.

The Inclusion-Exclusion Principle

- Let $A_1, A_2, \ldots, A_n$ be n finite sets. Then

$$\begin{aligned} |\cup_{i=1}^{n} A_i| &= \sum_{i=1}^{n} |A_i| - \sum_{i,j=1; i\neq j}^{n} |A_i \cap A_j| \\ &\quad + \sum_{i,j,k=1; i\neq j\neq k}^{n} |A_i \cap A_j \cap A_k| - \ldots \\ &\quad + (-1)^{n+1} | \cap_{i=1}^{n} A_i| \end{aligned}$$

- Special cases
  - When $n = 2$, $|A \cup B| = |A| + |B| - |A \cap B|$
  - When $n = 3$, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

Problem: Prove that $(A - B)$, $(B - A)$ and $A \cap B$ are disjoint, where A and B are two sets.

Two sets X and Y are disjoint, if $X \cap Y = \emptyset$.
Now,

$$\begin{aligned}
(A - B) \cap (A \cap B) &= (A \cap B') \cap (A \cap B), \text{by De Morgan's laws} \\
&= (A \cap B') \cap (B \cap A), \text{by Commutative laws} \\
&= A \cap (B' \cap B) \cap A, \text{by Associative laws} \\
&= A \cap (\emptyset \cap A) \\
&= A \cap \emptyset \\
&= \emptyset
\end{aligned}$$

Similarly, it can be shown that
$(B - A) \cap (A \cap B) = \emptyset$
$(A - B) \cap (B - A) = \emptyset$

Problem

- The number of elements in a finite set S is denoted by $|S|$.
  (a) Starting from the fact that $|A \cup B| = |A| + |B|$ when A and B are two disjoint sets, show that in general, $|A \cup B| = |A| + |B| - |A \cap B|$.
  (b) For any three sets A, B, and C, show that
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$.