Discrete Mathematics and Algorithms (CSE611)

Lecture No: 6

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on

Topic: Functions

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$Problem$: If X and Y be 2 non-empty sets and f() be a mapping of X into Y then for any subsets $A, B \subseteq X$

Prove that :

$$f(A \cup B) = f(A) \cup f(B)$$

R.T.P :

$$i) f(A \cup B) \subseteq f(A) \cup f(B)$$

$$ii) f(A) \cup f(B) \subseteq f(A \cup B)$$

i) Let $y \in f(A \cup B)$ be an arbitrary element. Therefore, $y \in f(A \cup B)$

$\Rightarrow y = f(x)$ for some $x \in A \cup B$

$\Rightarrow y = f(x)$ for some $x \in A$ or $x \in B$

$\Rightarrow y = f(x)$ for some $x \in A$ or $y = f(x)$ for some $x \in B$

$\Rightarrow y \in f(A)$ or $y \in f(B)$

$\Rightarrow y = f(A) \cup f(B)$ for some $x \in A \cup B$

ii) Let $y \in f(A) \cup f(B)$

$\Rightarrow y \in f(A)$ or $y \in f(B)$

$\Rightarrow y = f(x)$ for some $x \in A$ or $y = f(x)$ for some $x \in B$

$\Rightarrow y = f(x)$ for some $x \in A$ or $x \in B$

$\Rightarrow y = f(x)$ for some $x \in A \cup B$

$\Rightarrow y \in f(A \cup B)$

$Definition$:

A Partial Function $f : X \to Y$ is a relation which assigns every element $x \in D \subset X$ to a unique element in Y

$Definition$:

For all $f : S \to S$,$f^2 = f \circ f = f \diamond f$ if $f^2 = f$,then f is said to be an idempotent or projection function.

$Example$:

Let $f : S \to S$ be a function defined by $f(x) = x, \forall x \in S$.

Now, $f^2(x) = (f \circ f)(x)$
$= f[f(x)]$
$= f(x) \forall x \in S.$

$Definition$:

A set S is "infinite" if there is a one-one correspondence (bijection) between S and a proper subset of S

$Problem$:

Prove that the set of real numbers is infinite. S=set of real numbers $= (-\infty, \infty)$
$A = (-1, 1) \subset S$

Define:

$$f(x) = \begin{cases} x/(1+x) & \text{if } x \geq 0 \\ x/(1-x) & \text{if } x < 0 \end{cases}$$

range of f= $[0, 1)$,when $x \geq 0$
$(-1, 0)$ when $x < 0$

R.T.P : f is one-one and onto

Claim 1 : f is one-one

Case1: $x \geq 0[0, \infty)$
let $x_1, x_2 \in S$ such that $f(x_1) = f(x_2)$
$f(x_1) = f(x_2)$
$\Rightarrow x_1/(a + x_1) = x_2/(1 + x_2)$
$\Rightarrow x_1 + x_1x_2 = x_2 + x_1x_2$
$\Rightarrow x_1 = x_2$

Case2: $x < 0(-\infty, 0)$
let $f(x_1) = f(x_2)$
$\Rightarrow x_1/(1 - x_1) = x_2/(1 - x_2)$
$\Rightarrow x_1 - x_1x_2 = x_2 - x_1x_2$
$\Rightarrow x_1 = x_2$

Claim 2 : f is onto

RTP: $\forall y \in A, \exists x \in S$ such that f(x)=y.

Case1: let f(x)=y,when y $\in [0,1)$

$\Rightarrow x/(1+x)=y$

$\Rightarrow (1+x)/x=1/y$

$\Rightarrow 1+(1/x)=1/y$

$\Rightarrow 1/x=(1/y)-1$

Case2: let f(x)=y when $y \in (-1,0)$

$\Rightarrow x/(1-x)=y$

or $\Rightarrow x=y-xy$

$\Rightarrow x=y/(1+y) \in S$

$Problem:$

Let $f: x \rightarrow y$ be a function.

Let A= $\{f|f: X \rightarrow Y\}$

B= $\{f|f: X \rightarrow Y isone-one\}$

C=$\{f|f: X \rightarrow Y isonto\}$

D=$\{f|f: X \rightarrow Y isbijective\}$

Compute $|A|, |B|, |C|, |D|$?

1. $|A|$

Let Set A contains m elements and Set B contains n elements

Number of possibilites of mapping 1 element from $A \rightarrow B$ is n ways

Number of possibilites of mapping 2 elements from $A \rightarrow B$ is n^2 ways

Number of possibilites of mapping m elements from $A \rightarrow B$ is n^m ways

Number of functions from $f: X \rightarrow Y$ is n^m ways

2. $|B|$

Let Set A contains m elements and Set B contains n elements

Number of possibilites of mapping 1st element is n ways

Number of possibilites of mapping 2nd element is n-1 ways

Number of possibilites of mapping mth element is n-m ways

if $m \geq n$ number of one-one functions is 0

otherwise number of one-one functions,$|B| =^n P_m$

3. $|C|$

Let Set A contains m elements and Set B contains n elements

If m >n, there is no simple closed formula that describes the number of onto functions.

We need to count the number of partitions of A into n blocks by S(m,n). Each of these partitions then describes a function from A to B. Once we've counted the partitions we multiply by n!, just as in the bijection case, because for each partition the first block can map to any of the n elements of B, and so on.

So, Number of onto Funtions = n! S(m,n), where S(m,n) is a Stirling number of the second kind.

$S(m,n) = \frac{1}{m!}\sum_{r=0}^{n}(-1)^(n-r)\binom{n}{r}r^m$

Number of onto Funtions ,$|C| = m!.S(m,n) = \sum_{r=0}^{n}(-1)^(n-r)\binom{n}{r}r^m$

4. $|D|$

For a function $f : A \rightarrow B$to be bijective, f should be both one-one and onto

Let Set A contains m elements and Set B contains n elements

n should be equal to m

Number of possibilites of mapping 1st element is n ways

Number of possibilites of mapping 2nd element is n-1 ways

Number of possibilites of mapping mth element is n-m ways

but n=m

So,number of Bijection Functions,$|D| = n!$

$Definition$:(Countable or Denumerable Sets)

A set S is countable if it is either finite or denumerable.

If a set S be cardinally equivalent to the set of natural numbers, N = {1,2,3,4......} then S is countable.

⇒ S is countable if there exists a one-one correspondence between N and S.

⇒ S is countable if there exists a bijection $f : N \rightarrow Z$.

$Example 1.$Prove or Disprove that the set Z of all integers is countable.

Claim : Z is countable

R.T.P: There exists a bijection $f : N \to Z$

Proof. Define:

$$\text{f(n)} = \begin{cases} \frac{n}{2}, \text{ if n is even} \\ \frac{1-n}{2}, \text{ if n is odd} \end{cases}$$

$$\text{range of f} = \begin{cases} \{1, 2, 3, 4...\}, \text{ if n is even} \\ \{-\infty, ..., -2, -1, 0\}, \text{ if n is odd} \end{cases}$$

f is one-one and onto. Hence f is bijection which makes Z countable. □

$Example 2.$Prove or Disprove that $A = \{a^2 | a \in N\}$ is countable.

Claim : A is countable

R.T.P: There exists a bijection f : N → A

Proof. the function f is one one. Let a_1 and a_2 be two natural numbers. now f(a_1) = a_1^2 and f(a_2) = a_2^2

assume that f(a_1) = f(a_2)

⇒ a_1^2 = a_2^2

⇒ a_1^2 - a_2^2 = 0

⇒ $(a_1 - a_2)(a_1 + a_2)$ = 0

⇒ $a_1 = a_2$ or $a_1 + a_2 = 0$

But we know that a_1 and $a_2 \in N$

⇒ $a_1 = a_2$

⇒ f is one one function

Now

let y = a^2

$\Rightarrow$ a = $\sqrt{y}$

Now y $\in$ { 1 , 4 , 9 , 16 , 25 ... }

$\Rightarrow$ a $\in$ { 1 , 2 , 3 , 4 , 5 ... }

$\Rightarrow$ a $\in$ N

$\Rightarrow$ f is onto function

Since f is both one one and onto therefore we can say f is bijective function

Hence the set A is countable $\square$

Theorem 1. *Union of a countable collection of countable sets is again countable.*

Proof. CASE I : Let $S_1, S_2, ..., S_k$ be a finite collection of countable sets which are finite sets.

Required to prove : $S_1 \cup S_2 \cup S_3 ... \cup S_k$ is countably finite set.

From Inclusion Exclusion Principle

$|S_1 \cup S_2 \cup S_3 ... \cup S_k| = \sum_i |S_i| - \sum_{i<j} |S_i \cap S_j| + \sum_{i<j<k} |S_i \cap S_j \cap S_k| - ...$

$<= |S_1| + |S_2| + ...|S_k|$ (According to principle of mathematical induction)

where k is also finite

Therefore $|S_1 \cup S_2 \cup S_3 ... \cup S_k|$ is finite size.

CASE II : Let $S_1, S_2, ..., S_k$ be finite collection of countable sets which are themselves infinite sets.

Let $S_i = \{S_{i1}, S_{i2}, ...\}$

Consider $S_1 \cup S_2 \cup S_3 ... \cup S_k$ Now construct a table

$$\begin{array}{c||l} S_1 & S_{11}, S_{12}, S_{13}, S_{14}, ... \\ S_2 & S_{21}, S_{22}, S_{23}, S_{24}, ... \\ S_3 & S_{31}, S_{32}, S_{33}, S_{34}, ... \\ . & \\ . & \\ S_k & S_{k1}, S_{k2}, S_{k3}, S_{k4}, ... \end{array}$$

We traverse the elements of the sets in columnwise which are finite in number .Now to check if any element is repeated or not we can go back to check all previously visited elements in finite amount of time. Therefore $S_1 \cup S_2 \cup S_3 ... \cup S_k$ is countably infinite

$\square$

Theorem 2. *The set of real numbers is uncountable.*

Proof. Let R = set of real numbers $\in (-\infty, \infty)$ = $\cup_i (a_i, b_i)$ for all i $\in (-\infty, \infty)$. To show R is uncountable it suffices to show a proper subset $(a_i, b_i) \subset R$ is also uncountable.

Let (a , b) $\subset$ R where b $>$ a. Then we can have f : $(a, b) \to (0, 1)$ as a bijective function defined by w = f(x) = $\frac{x-a}{b-a}$ where a $<$ x $<$ b

R.T.P : (0, 1) is uncountable

If real numbers between (0 , 1) were countable then they could be written in a succession $x_1, x_2, x_3 ..., x_n$. - (I)

Let us express each x_n as a decimal . If we agree not to use recurring 9's then this can be done in only 1 way. Let us agree to this and write decimals in (I) as follows -

x_1 = $0.a_1 a_2 a_3 a_4 ...$

x_2 = $0.b_1 b_2 b_3 b_4 ...$

x_3 = $0.c_1 c_2 c_3 c_4 ...$

Let us take the diognal $a_1 b_2 c_3 ...$ and form a decimal y = 0.$\alpha\beta\gamma\delta$.. where

$$\alpha = \begin{cases} 5, a_1 \neq 1 \\ 6, a_1 = 1 \end{cases}$$

$$\beta = \begin{cases} 5, b_2 \neq 1 \\ 6, b_2 = 1 \end{cases}$$

$$\gamma = \begin{cases} 5, c_3 \neq 1 \\ 6, c_3 = 1 \end{cases}$$

$$\delta = \begin{cases} 5, d_4 \neq 1 \\ 6, d_4 = 1 \end{cases}$$

Now y $\neq x_1$ since it differs at 1st place. y $\neq x_2$ since it differs at second place of decimal and so on for all x's. Therefore a new decimal number can be generated in the given range other than above said x's. Therefore we can say that the set (0 ,1) is uncountable and therefore the set of real numbers R is uncountable. □