# Discrete Mathematics and Algorithms (CSE611)

# Lecture No: $5$

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on

**Topic: Functions**

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## 1 One to One or Injective Functions

A function $f : X- > Y$ is called 1-1, if each element in the co-domain Y is the image of atmost one element in the domain X.
In other words,
if $a, b \in X$ with a $\neq$ b
then $f(a) = f(b)$
or eqivalently if $f(a) = f(b)$ then a = b

# 2 Onto or Surjective Functions

A function $f : X- > Y$ is called onto, if each element in the co-domain Y is the image of atlest one element in the domain X.
In other words,
For all $b \in Y$, there exist $a \in X$, such that $f(a) = b$.

# 3 Bijective Functions

A function $f : X- > Y$ is called bijective, if it is one-one and onto.
Remark: Let $f : X- > Y$ be one-one then
Im($f$) subset $Y$
then, $f : X- > Im(f)$ is a bijective function

**Lemma 1.** *If f* $: X- > Y$, *is one-one and X,Y are finite sets having same number of elements then f is bijective.*

# 4 Identity Functions

The identity function $1_s : S- > S$ maps each element $x \in S$ (Domain) onto itself.

# 5 Equality of Functions

Let $f : A- > B$ and $g : A- > B$ be two functions,
then $f = g$, iff $f(x) = g(x)$, for all $x \in A$.

# 6 Composite Functions

Let $f : A- > B$ and $g : B- > C$ be two functions

then the composite function is defined as

$g \circ f : A- > C$ is defined as

$(g \circ f)(a)$ = $g[f(a)]$ for all $a \in A$

**Lemma 2.** *Let $f : A- > B$, $g : B- > C$ and $h : C- > D$ be three functions. Whenever the composite involved are defined, composition of function always obeys the following law:*
$(h \circ g) \circ f = h \circ (g \circ f)$

*Proof:*
$((h \circ g) \circ f)(x) = (h \circ g) \circ (f(x))$
$h[g(f(x))]$
$((h \circ g) \circ f)(x) = h[(g \circ f)(x)]$
$h[g(f(x))]$

**Theorem 3.** (Identity Law) *Let $f : S- > T$ be a function and $1_s : S- > S$,$1_s : T- > T$ are the identity functions respectively.*
*Then, $f.1_s = 1_t.f = f$*

*Proof.* RTP:
(I) $f.1_s = f$
(II) $1_t.f = f$

(I) Let $s \in S$
Then,
$f.1_s(s) = f[1_s(s)]$
$= f(s)$, since $1_s(s) = s$,for all $s \in S$
It implies $f.1_s = f$
(II) Let $s \in S$
Then,
$(1_t.f)(s) = 1_t(f(s))$

$= f(s)$, It implies $1_t.f = f$

$\square$

# 7 Characteristic Function

Any set $S$ which is a subset of a set U can be associated with a function called it's Characteristic function.

$e_s : U- > \{0,1\}$ defined by

In other words if $S \subset Uu_1, u_2, u_3.....u_n$ then,

$e_s : u- > 0,1$ is defined as

$$e_s(u_i) = \begin{cases} 1, u_i \in S \\ 0, u_i \notin S \end{cases}$$

Example:

Let U = 1,2,3,....,10 and

S = 4,7,9$\subset U$

Define

$e_s : U- > \{0,1\}$

$e_s(2) = 0, 2 \notin S$

$e_s(7) = 1, 7 \in S$

$e_s(12) = undefined$, as 12 not $\in U$

# 8 Inverse of function

Let $f : S- > T$ and $g : T- > S$ be two functions such that

$g \circ f$ =$1_S$ = $f \circ g$

where '$\circ$' is the left composition

'$\circ$' is the right composition

Then g is called "left invertible" f of w.r.t '$\circ$' and g is called "right invertible" of f w.r.t '$\circ$'

**Definition 1.** *A function which has a two-sided inverse is called "invertible".*

*A function is "invertible" if it is bijective.*

**Theorem 4.**

*a.) A function is left-invertible iff it is injective.*

*b.) A function is right-invertible iff it is surjective.*

*Proof.* a.) Let $f : A- > B$ be left-invertible.

RTP:

f is 1-1

By definition, there exist $g : B- > A$ such that

$g \circ f$=$1_A$

Let f($x_1$)= f($x_2$), for $x_1$ ,$x_2$ belongs to A.

Now,

g[f($x_1$)] = g[f($x_2$)]

$=>$ (g$\circ$f)($x_1$) = (g$\circ$f)($x_2$)

$=>$ $1_A(x_1)$ = $1_A(x_2)$

$=>$ $x_1$ = $x_2$

therefore, $f : A- > B$ is 1-1.

$\square$

*Proof.* a.) Let $f : A- > B$ be 1-1.

RTP:

f is left-invertible.

Since f is 1-1,

f($x_1$) = f($x_2$)

$=>$ $x_1$ = $x_2$ for all $x_1$ ,$x_2$ $\in A$

$\square$

*Let* $g : B- > A$ *be a function*

*Construction of g:*

*for all* $S \in A$*,*

*Let* $f(s) = t \in B$

$$g(t) \;\; = \;\; \begin{cases} s, iff, \text{\textit{f(s)=t}} \in B \\ s_1, iff, t \notin I_m(f) \text{\textit{for all }} t \in B \end{cases}$$

*Therefore (g∘f)(s) = g[f(s)]*

*= g(t)*

*= s, for all* $s \in A$

$=> (g \circ f)(s) = 1_A(s), s \in A => g \circ f = 1_A$

**Corollary 5.** *A function* $f : A- > B$ *is a bijection iff it has both a left-inverse and a right inverse.*

# 9 Problem

Let $f : A- > B$ and $g : B- > C$ be functions.

(I) If f and g are both injective, then prove that g∘f is injective too.

*Proof.* Given $f : A- > B$ is injective and $g : B- > C$ is injective.

RTP:

$g \circ f : A- > C$ is also injective.

Let $x_1, x_2 \in A$ such that $x_1 \neq x_2$

$x_1 \neq x_2 => f(x_1) \neq f(x_2)$, since f is injective.

$f(x_1) = y_1 \in B$ and $f(x_2) = y_2 \in B$

$=> y_1 \neq y_2$

$=> g(y_1 \neq g(y_2))$, since g is injective. $=> g[f(x_1)] \neq g[f(x_2)]$

$=> (g \circ f)(x_1) \neq (g \circ f)(x_2)$ by definition of g∘f.

$=> g \circ f$ is also injective.

$\square$