Discrete Mathematics and Algorithms (CSE611)
Lecture No: *4*
Prepared by

Madala V V Satyanarayna (201305508)

Arpit Kumar Jaiswal (201305509)

Priyanshu Agrawal (201305511)

Mohd Salman Khan (201305513)

on

Topic: Relations

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1 Problem (Equivalence Classes)

Let Z be the set of integers. Let R be a relation in Z defined by the open sentence ($x - y$) is divisible by m, where $x, y \in Z$. Verify whether R is an equivalence relation. If so, find the equivalence classes.

Part 1 Claim : R is an equivalence relation.

Part 2 Equivalence classes.

Solution (Part 1)

I) **Reflexive**

: Let $(x, x\ ) \in R$.

Therefore, $x - x$ is divisible by $m.$ and 0 is divisible by all the integers except 0 itself.

Hence Relation R is Reflexive.

II) **Symmetric**

Let $(x, y) \in R$.

Therefore, $x - y$ = (k)m — equation (1) where K is any integer.

Equation 1 can be also be written as :

$y - x = (-k)m - k$ is also a constant.

Hence the relation R is symmetric.

III)**Trasitive Relation** Let $(x, y), (y, z) \in R$.

Therefore,

$x - y = m * k_1$

and $y - z = m * k_2$

Hence, $x - z = x - y + y - z = mk_1 + mk_2 = m(k_1 + k_2)$

here $k_1 + k_2$ is a constant.

Hence $x - z$ is also divisible by m

$(x, z) \in R$.

Therefore, R is a transitive relation.

solution (part 2)

For($m = 6$) we define the equivalent classes as : -

$\implies [p] = S_p = \{6k + p|p = 0, 1, 2, 3.......and\ k = \pm 1, \pm 2, \pm 3......$

$x - y = 6k$ (x-y is divisible by m)

$\implies x = 6k + y$

So, $[0] = S_0 = \{.... - 12, -6, 0, 6, 12....\}$

$\implies [1] = S_1 = \{... - 11, -5, 1, 7, 13....\}$

$\implies [2] = S_2 = \{... - 10, -4, 2, 8, 14....\}$

$\implies [3] = S_3 = \{... - 9, -3, 3, 9....\}$

$\implies [4] = S_4 = \{... - 8, -2, 4, 10....\}$

$\implies [5] = S_5 = \{... - 7, -1, 5, 11....\}$

$S_0, S_1, S_2, S_3, S_4, S_5$ forms a partition of Z under modulo 6 operation,since

1) $S_i \bigcap S_j = \emptyset\ \forall i \neq j, i, j = 0, 1, 2, 3, 4, 5$

2) $\bigcup_{i=0to5} \cup_i = Z$

2 Theorem on Equivalence relations and partition

An Equivalence relation R in a non-empty set A partitions A and conversely a partition of A defines an Equivalence relation.

2.1 Theorem : Every partition of a set induces an equivalence relation on it

Proof: Let $P = \{A1, A2, A3....An\}$ be a partition on a set A. Define a relation R on set A as ${}_aR_b$ if a belongs to the same block as b.

RTP: R is an equivalence relation.

(i) Since every element in a belongs to the same block as itself, ${}_aR_b$ holds, $\forall a \in A$.

$\Rightarrow R$ is Reflexive.

(ii) Let ${}_aR_b$ hold.

$\Rightarrow$ a belongs to the same block as b.

$\Rightarrow$ b belongs to the same block as a.

$\Rightarrow$ ${}_bR_a$ holds

$\Rightarrow$ R is Symmetric.

(iii) Let ${}_aR_b$ and ${}_bR_c$ hold

${}_aR_b \Rightarrow a$ belongs to the same block as b.

${}_bR_c \Rightarrow a$ belongs to the same block as c.

${}_aR_b \wedge {}_bR_c$

$\Rightarrow$ a also belongs to the same block as c.

$\Rightarrow$ ${}_aR_b$ holds.

$\therefore$ R is transitive.

2.2 Number of partitions of finite set

Number of partitions(and hence the equivalence relation) of a set with size n

$= \sum\limits_{r=1}^{n} S(n,r)$

where $S(n,r)$ is the string number of the second kind.

$$S(n,r) = \begin{cases} 1 \text{ if r=1 or n=r} \\ \text{S(n-1,r-1) + r.S(n-1,r) if 1<r<n} \end{cases}$$

Let R be a binary relation on a set A, where $|A| = n$.

$|A \times A| = n \times n = n^2$

$|P(A \times A)| = 2^{n^2}$

$\Rightarrow$ Total no. of relation on $A = 2^{n^2}$

$\Rightarrow$ Total no. of reflexive relation on $A = 2^{n(n-1)}$

$\Rightarrow$ Total no. of symmetric relation on $= 2^{n(n+1)/2}$

$\Rightarrow$ Total no. of anti-symmetric relation on $= 2^n.3^{n(n-1)/2}$

$\Rightarrow$ Total no. of both reflexive and symmetric (compatible) relations on $A = 2^p$,where $p = ^n C_2 = n(n-1)/2$

$\Rightarrow$ Total no. of equivalence relation on $A = \sum_{r=1}^{n} S(n,r)$
example: $S(5,r) = S(5,1) + S(5,2) + S(5,3) + S(5,4) + S(5,5) = 52$

3 Closure of Relations

3.1 Compatibility Relation

Def: Let R be a relation in a non-empty set A (i.e., $R \subseteq A \times A$). Then, R is said to be a compatibility relation if it is both reflexive and symmetric.

Problem: Let A be a set of people, and R a binary relation on A such that $(a,b) \in R$ if a is a friend of b. Verify whether R is a compatibilty relation.

Solution:

(i) R is reflexive, since a is always a friend of $a \in A$ (i.e., himself/herself), that is, ${}_aR_b$ holds, $a \in A$.

(ii) R is symmetric, since, if a is a friend of b, then obviously b is also a friend of a, that is,
if ${}_aR_b$ holds, then ${}_bR_a$ also holds, $\forall a, b \in A$.
Hence, R is a compatibility relation.

Important Observations:

All equivalence relations are compatibility relations.
Let R and S be two compatibility relations on a set A. Then $R \cap S$ is a compatibility relation, but $R \cup S$ may or may not be a compatibility relation (True/False).

4 Closure of Relations

4.1 Reflexive Closure

Def: A relation R is the reflexive closure of a relation R if and only if

(a) R is reflexive,

(b) $R \subseteq R'$,

(c) For any relation R'' , if $R \subseteq R''$ and R'' is reflexive, then $R' \subseteq R''$, i.e., R' is the smallest relation that satisfies the conditions (a) and (b).

The reflexive closure of a relation R is denoted by $r(R)$.

Problem (Closure of Relations): Given the relation R = $\{(a, b), (b, a), (b, b), (c, b)\}$ on the set A = $\{a, b, c\}$. Compute the reflexive closure $r(R)$ of R.

It is clear that R is not reflexive, since $(a, a) \notin R$ and $(c, c) \subseteq R$.

Consider a relation R' which contains R as well as the tuples (a, a) and (c, c) that is,

R' = $R \cup \{(a, a), (c, c)\} = \{(a, a), (a, b), (b, a), (b, b), (c, b), (c, c)\}$

Then, clearly R' is reflexive and $R \subseteq R'$.

Furthermore, any other relation, say R'' , containing R must also contain (a, a) and (c, c); otherwise it will not be reflexive. So, $R' \subseteq R''$. As R' contains R, andR' is reflexive, and is contained in every reflexive relation that contains R, so R' is the smallest relation satisfies conditions (a) and (b). Hence, $r(R) = R'$.

4.2 Symmteric Closure

Def:

A relation R' is the symmetric closure of a relation R if and only if

(a)R' is symmetric,

(b) $R \subseteq$R', $(c) For any relation$R'', ifR' $\subseteq R''$ and R is symmetric, then $R' \subseteq R''$, i.e., R' is the smallest relation that satisfies the conditions (a) and (b).
The symmetric closure of a relation $R$is denoted by$s(R)$.

Problem (Closure of Relations): Given the relation $R = \{(a, a), (a, b), (c, c), (b, c), (b, a), (a, c)\}$ on the set $A = \{a, b, c\}$. Compute the symmetric closure $s(R)$ of R.
It is clear that R is not symmetric.

To be symmetric, R needs the pairs(c, b) and (c, a). Consider a relation R' which contains R as well as the tuples (c, b) and (c, a), that is,
R' = $R \cup \{(c, b), (c, a)\}$ =$\{(a, a), (a, b), (c, c), (b, c), (b, a), (a, c), (c, b), (c, a)\}$

Then, clearly R' is symmetric and $R \subseteq R'$.

Furthermore, any other relation, say R'' , containing R must also contain (c, b) and (c, a); otherwise it will not be symmetric. So, $R' \subseteq R''$. So, R' is the smallest relation satisfies conditions (a) and (b). Hence, $s(R) = R'$.

4.3 Transitive Closure

Def: A relation R' is the transitive closure of a relation R if and only if

(a) R' is transitive,

(b) $R \subseteq R'$,

(c) For any relation R'' , if $R \subseteq R''$ and R is transitive, then $R' \subseteq R''$, i.e., R' is the smallest relation that satisfies the conditions (a) and (b).

The transitive closure of a relation R is denoted by $t(R)$ orR^t.

Problem (Closure of Relations): Let R be the less than ($<$) relation on the set Z of integers. Compute the transitive closure $t(R)$ of R.

The transitive closure of the less than (<) relation on Z is the less than (<) relation itself.

How to find Transitive Closure of a given Relation R ?

We need to add the minimum number of tuples to R giving us R^t such that if$(a, b) \subseteq R^t$ and $(b, c) \subseteq R^t$, then $(a, c) \subseteq R^t$.

Thus,$R^t = R \cup \{(a, b) \subseteq R^t \wedge (b, c) \subseteq R^t \Longrightarrow (a, c) \subseteq R^t\}$.

Problem (Closure of Relations): Let A = 1, 2, 3 and $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1)\}$ be a relation on A.

Compute the transitive closure R^t of R.

Solution :

Clearly, R is not transitive. For example,

$(2, 3) \in R \wedge (3, 1) \in R$

$(2, 1) \subseteq$R.

Add the following mininum number of tuples in R to construct R such that $R \subseteq R$ and R is transitive:

$(2, 3) \subseteq R \wedge (3, 1) \subseteq R \longrightarrow (2, 1) \subseteq R^t$

$(3, 1) \subseteq R \wedge (1, 2) \subseteq R \longrightarrow (3, 2) \subseteq R^t$

$(3, 1) \subseteq R \wedge (1, 3) \subseteq R \longrightarrow (3, 3) \subseteq R6t$

$(2, 1) \subseteq R^t \wedge (1, 2) \subseteq R \longrightarrow (2, 2) \subseteq R^t$

Thus, $R^t = t(R) = R' = R(2, 1), (2, 2), (3, 2), (3, 3)$.

5 Functions

A Function is defined by two sets X and Y and a rule (relation) f which assigns every element of X to an element of Y.

Mathematically, $f : X$is a function from X to Y defined by

$y = f(x), \forall x \in X.$

X: Domain of f

Y: Co-Domain (range) of f

The image $y \in Y$ of an element $x \in X$ is denoted by $y = f(x)$.

The pre-image $x \in X$ of an element, $y \in Y$ is an element such that $f(x) = y$.

The set of all elements in Y which have atleast one pre-image is called the image of f, denoted by

Im(f)= $\{f(x)|x \in X\} \subset Y$.

Example : Let $X = \{a, b, c\}$ $Y = \{1, 2, 3, 4\}$ $f : X \in Y$

Im(f) = $\{1, 3, 4\} \subset Y$.