Discrete Mathematics and Algorithms (CSE611)
Lecture No: *2*
Prepared by

Ramesh Katoori (201250898)

Rashmi Mishra (201250855)

Varunkumar Pandey (201250918)

Raj Mehta (201305504)

on

Topic: Relations

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1 Relation

A relation between two sets A and B is a subset of the cartesian product $A \times B$ and is defined by R (or ρ or r).
$R \subseteq A \times B$.
We write ${}_xR_y$ or ${}_x\rho_y$ if and only if (iff) $(x, y) \in R$ (or ρ).
We also write $x(\sim R)y$ when x is NOT related to y in R.

2 Inverse Relation

If R be a relation from A to B, then the inverse relation of R is the relation from B to A and is denoted and defined by

$R^{-1} = \{(y, x) : y \in B, x \in A, (x, y) \in R\}$.

$\Longrightarrow (x, y) \in R \leftrightarrow (y, x) \in R^{-1}$

Theorem 1. *If R be a relation from A to B, then $(R^{-1})^{-1} = R$*

Proof. We need to prove

(i) $(R^{-1})^{-1} \subseteq R$

(ii) $R \subseteq (R^{-1})^{-1}$

(i) Let $(x, y) \in (R^{-1})^{-1}$

Required to prove (RTP) that $(x, y) \in R$

Let, $(x, y) \in (R^{-1})^{-1}$

$\Rightarrow (y, x) \in R^{-1}$ (By the definition of R^{-1})

$\Rightarrow (x, y) \in R$ (By the definition of R^{-1})

$Thus$,$(R^{-1})^{-1} \subseteq R$

(ii) Let $(x, y) \in R$

Required to prove (RTP) that $(x, y) \in (R^{-1})^{-1}$

Let, $(x, y) \in R$

$\Rightarrow (y, x) \in (R^{-1})$ (By the definition of R^{-1})

$\Rightarrow (x, y) \in (R^{-1})^{-1}$ (By the definition of R^{-1})

$Thus$,$R \subseteq (R^{-1})^{-1}$

$\square$

3 Reflexive Relation

Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$).

R is said to be reflexive, if $(a, a) \in R, \forall a \in A$

$\Rightarrow_a R_a$ holds for every $a \in A$.

Example. Consider the relation $R = \{(a, a), (a, c), (b, b), (c, c), (d, d)\}$ in the set $A = (a, b, c, d)$.

Then R is reflexive, since $(x, x) \in R, \forall x \in A$, that is, ${}_xR_x$ holds for every $x \in A$.

4 Symmetric Relation

Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$).

R is said to be symmetric, if $(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$

In other words, ${}_aR_b \Rightarrow_b R_a$ for every $a, b \in A$.

Example. Let N be the set of natural numbers and R the relation defined in it such that ${}_xR_y$ if x is a divisor of y (that is, $x|y$), $x, y \in N$.

Then R is NOT symmetric, since ${}_xR_y \not\Rightarrow_y R_x, \forall x, y \in N$.

For example, ${}_3R_9 \not\Rightarrow_9 R_3$.

Theorem 2. *For a symmetric relation* R,$R^{-1} = R$

Proof. We need to prove

(i) $R^{-1} \subseteq R$

(ii) $R \subseteq R^{-1}$

(i) Let $(x, y) \in R^{-1}$

Required to prove (RTP) that $(x, y) \in R$

Let, $(x, y) \in R^{-1}$

$\Rightarrow (y, x) \in (R^{-1})^{-1} = R$ (By the definition of R^{-1})

$\Rightarrow (x, y) \in R$ (Since R is symmetric)

Thus,$R^{-1} \subseteq R$

(ii) Let $(x, y) \in R$

Required to prove (RTP) that $(x, y) \in R^{-1}$

Let, $(x, y) \in R$

$\Rightarrow (y, x) \in R$ (Since R is symmetric)

$\Rightarrow (x, y) \in R^{-1}$ (By the definition of R^{-1})

$Thus$,R $\subseteq R^{-1}$

□

5 Anti-Symmetric Relation

Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$).

R is said to be anti-symmetric, if ${}_aR_b$ and ${}_bR_a \Rightarrow a = b$, for every $a, b \in A$.

Example. Let A be the set of real numbers and R the relation defined in it such that ${}_xR_y$ if $x \leq y$, that is,

$R = \{(x, y) \in A \times A \colon x \leq y\}$.

Then R is anti-symmetric, since

${}_xR_y$ and ${}_yR_x$

$\Rightarrow x \leq y$ and $y \leq x$

$\Rightarrow x = y$.

6 Transitive Relation

Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$).

R is said to be transitive, if ${}_aR_b$ and ${}_bR_c \Rightarrow_a R_c, \forall a, b, c \in A$.

Example. Let N be the set of natural numbers and R the relation defined in it such that ${}_xR_y$ if $x < y$, that is,

$R = \{(x, y) \in N \times N \colon x < y\}$.

Then R is transitive, since

${}_xR_y$ and ${}_yR_z$

$\Rightarrow x < y$ and $y < z$

$\Rightarrow x < z$

$\Rightarrow_x R_z$.

7 Equivalence Relation

Let A be a set and R the relation defined in it (i.e., $R \subseteq A \times A$).

R is said to be an equivalence relation, if and only if

(1) R is reflexive, that is, ${}_aR_a$ holds, for every $a \in A$.

(2) R is symmetric, that is, ${}_aR_b \Rightarrow_b R_a, \forall a, b \in A$.

(3) R is transitive, that is, ${}_aR_b$ and ${}_bR_c \Rightarrow_a R_c, \forall a, b, c \in A$.

Q1. A relation ρ is defined on the set Z (set of all integers) by ${}_a\rho_b$ if and only if $(2a + 3b)$ is divisible by 5.

Prove or disprove: ρ is an equivalence relation.

Sol:

Claim 1:

Let $a \in Z$.

Then, $2a + 3a = 5a$ is divisible by 5.

Hence, ${}_a\rho_a$ holds, $\forall a \in Z$

$\Rightarrow \rho$ is reflexive.

Claim 2:*If $a(\neq 0)$ divides b (i.e., $a|b$), $a, b \in Z$ being integers, then $\exists x \in Z$ such that $b = ax$.*

Lemma 4. *If ρ be prime and a, b are integers such that $\rho|ab$, then either $\rho|a$ or $\rho|b$.*

Let $a, b \in Z$. Assume that ${}_a\rho_b$ holds.

Then, $(2a + 3b)$ is divisible by 5.

By the Euclids division algorithm, we have,

$2a + 3b = 5k_1$, for some integer $k_1 \in Z$. $\Rightarrow 2(2a + 3b) = 10k_1$

$\Rightarrow 4a + 6b = 10k_1$

$\Rightarrow 3(2b + 3a) - 5a = 10k_1$

$\Rightarrow 3(2b + 3a) = 5(a + 2k_1) = 5k_2$, say, where $k_2 = (a + 2k_1)$ is an Integer

If ρ is prime and $\rho|ab$, then either $\rho|a$ or $\rho|b$.

Thus, $5|(2b+3a)$

$\Rightarrow_b \rho_a$ holds.

Hence, ρ is symmetric.

Claim 3:

Let ${}_a\rho_b$ and ${}_b\rho_c$ hold, for every $a,b,c \in Z$. Then

$\Rightarrow (2a+3b)$ is divisible by 5

$\Rightarrow 2a+3b = 5l_1$, for some $l_1 \in Z$, and $(2b+3c)$ is divisible by 5

$\Rightarrow 2b+3c = 5l_2$, for some $l_2 \in Z$.

Now $2(2a+3b) - 3(2b+3c) = 10l_1 - 15l_2$

$\Rightarrow 4a - 9c = 10l_1 - 15l_2$

$\Rightarrow 2(2a+3c) = 10l_1 - 15l_2 + 15c = 5(2l_1 - 3l_2 + 3c) = 5l_3$, say,Where $l_3 = 2l_1 - 3l_2 + 3c \in Z$

$\Rightarrow 5|(2a+3c)$

$\Rightarrow_a \rho_c$ holds and ρ is also transitive.

Since ρ is reflexive, symmetric and transitive, so ρ is an equivalence relation.

8 Partial-order Relation

Let S be a non-empty set and R the relation defined in it (i.e., $R \subseteq S \times S$). R is said to be an partial-order relation, if and only if it satisfies the following three conditions:

(1) R is reflexive, that is, ${}_aR_a$ holds, for every $a \in S$.

(2) R is anti-symmetric, that is, ${}_aR_b$ and ${}_bR_a \Rightarrow a = b, \forall a, b \in S$.

(3) R is transitive, that is, ${}_aR_b$ and ${}_bR_c \Rightarrow_a R_c, \forall a, b, c \in S$.

9 Equivalence Classes

Let A be a non-empty set and R be an equivalence relation defined in A. Let $a \in A$ be an arbitrary element. Then the elements $x \in A$ which satisfy ${}_xR_a$ form a subset of A which is called the equivalence class of $a \in A$ with respect to (w.r.to) R.

Thus, Aa or $[a]$ or $cl(a)$ or $a = \{x|_xR_a, x \in A\}$ is called the equivalence class of a in A w.r.to R.*Let R be an equivalence relation on set A with a and b any 2 elements in A. Then prove:*

(i) $a \in [a]$

(ii) $[a] = [b]$ *iff* ${}_aR_b$ *i.e* $(a, b) \in R$

(iii) If $[a] \neq [b]$, *then* $[a] \bigcap [b] = \emptyset$

Proof. $[a] = \{x|_xR_a \ \ i.e.(x, a) \in R, x \in A\}$

$[b] = \{x|_xR_b \ \ i.e.(x, b) \in R, x \in A\}$

(i)

Since R is reflexive

${}_aR_a$ holds for every $a \in A$

so, $a \in A$

(ii)

if part

Let $[a] = [b]$

Since $a \in [a]$ (by part i)

$\therefore a \in [b]$

i.e. $(a, b) \in R$ (by definition of equivalence classes)

$\Rightarrow_a R_b$ holds

only if part

${}_aR_b$ hold

RTP: $[a] = [b]$ i.e. $[a] \subseteq [b]$ and $[b] \subseteq [a]$

Let $x \in [a]$ Then, ${}_xR_a$ holds

$\therefore_x R_a$ and ${}_aR_b \Rightarrow_x R_b$ holds (since R is tranisitive)

$\Rightarrow x \in [b]$ (by definition of equivalence classes)

$\therefore [a] \subseteq [b]$

Let $x \in [b]$ Then, ${}_xR_b$ holds

$\therefore_x R_b$ and ${}_aR_b$ holds

$\Rightarrow_x R_b$ and ${}_bR_a$ holds (since R is symmetric)

$\Rightarrow_x R_a$ holds (since R is tranisitive)

$\Rightarrow x \in [a]$ (by definition of equivalence classes)

$\therefore [b] \subseteq [a]$

Thus $[a] = [b]$

(iii)

$[a] \neq [b] \Rightarrow [a] \bigcap [b] = \emptyset$

contrapositive statement

RTP : [a] $\bigcap [b] \neq \emptyset \Rightarrow [a] = [b]$

Let $[a] \bigcap [b] \neq \emptyset$

Then, $x \in [a] \bigcap [b]$

$\Rightarrow x \in [a]$ and $x \in [b]$

$\Rightarrow_x R_a$ and ${}_xR_b$ holds

$\Rightarrow_a R_x$ and $_xR_a$ holds (since R is symmetric)

$\Rightarrow_a R_b$ holds (since R is transitive)

$\Rightarrow [a] = [b]$ (by part ii)

□

10 Partition

Let S be a non-empty set. Then a partition of S is a collection of non-empty disjoint sub-sets of S whose union is S.

In other words, if $A_1, A_2, ..., A_n$ be the non-empty sub-sets of S, then the set $P = \{A_1, A_2, ..., A_n\}$ is said to be a partition of S, if

(1) $A_1 \cup A_2 \cup ... \cup A_n = S$,

(2) either $A_i = A_j$ or $A_i \cap A_j = \emptyset, \forall i, j = 1, 2, ..., n$.

Example Consider a set $S = \{1, 2, 3, ..., 22\}$. Now consider three subsets A, B and C of S as follows:

$A = \{1, 4, 7, ..., 22\}$,

$B = \{2, 5, 8, ..., 20\}$,

$C = \{3, 6, 9, ..., 21\}$.

See that

(1) $A \cup B \cup C = S$, and

(2) $A \cap B = B \cap C = C \cap A = \emptyset$.

Hence, the set $(P) = \{A, B, C\}$ forms a partition of S.