Discrete Mathematics and Algorithms (CSE611)

Lecture No: 2

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on

Topic: Set Theory

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Problem [The Inclusion-Exclusion Principle]: Find the number of positive integers ≤ 2076 and divisible by neither 4 or 5.

Let $A = \{x \in N \mid x \leq 2076$ and divisible by 4$\}$, and
Let $B = \{x \in N \mid x \leq 2076$ and divisible by 5$\}$.

By the Inclusion-Exclusion Principle, we have,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= \lfloor \frac{2076}{4} \rfloor + \lfloor \frac{2076}{5} \rfloor - \lfloor \frac{2076}{4 \times 5} \rfloor$$

$$= 519 + 415 - 103$$

$$= 831$$

Thus, among the first 2076 positive numbers, there are 2076 − 831 = 1245 integers divisible by neither 4 or 5.

A Number-Theoretic Function

An integer $p(>1)$ is called a prime number or simply a prime, if its only positive divisors are 1 and itself. In other words, p does not have any non-trivial divisor d such that $1 < d < p$. Let x be a positive real number. Then $\pi(x)$ denotes the number of primes $\leq x$.

Prime Number Theorem: $\pi(x) \to \frac{x}{\ln(x)}$ as $x \to \infty$

Theorem: Let $p_1, p_2 \ldots . p_t$ be the primes $\leq \sqrt{n}$. Then

$$\pi(n) = n - 1 + \pi(\sqrt{n}) - \sum_i \lfloor \frac{n}{p_i} \rfloor + \sum_{i<j} \lfloor \frac{n}{p_i p_j} \rfloor - \sum_{i<j<k} \lfloor \frac{n}{p_i p_j p_k} \rfloor + \ldots .$$
$$+ (-1)^t \lfloor \frac{n}{p_1 p_2 \ldots p_t} \rfloor$$

Problem: Find the number of primes ≤ 100

Here n = 100. Then $\pi(\sqrt{n}) = \pi(\sqrt{100}) = \pi(10) = 4$. The four primes $\leq \sqrt{n} = 10$ are 2, 3, 5 and 7. Let $p_1 = 2, p_2 = 3, p_3 = 5$ and $p_4 = 7$, t = 4. From the previous theorem, we have ,

$$\pi(\sqrt{100}) = 100 - 1 + 4 - \left(\lfloor \frac{100}{2} \rfloor + \lfloor \frac{100}{3} \rfloor + \lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{7} \rfloor\right)$$
$$+ \left(\lfloor \frac{100}{2.3} \rfloor + \lfloor \frac{100}{2.5} \rfloor + \lfloor \frac{100}{2.7} \rfloor + \lfloor \frac{100}{3.5} \rfloor + \lfloor \frac{100}{3.7} \rfloor + \lfloor \frac{100}{5.7} \rfloor\right)$$
$$- \left(\lfloor \frac{100}{2.3.5} \rfloor + \lfloor \frac{100}{2.3.7} \rfloor + \lfloor \frac{100}{2.5.7} \rfloor + \lfloor \frac{100}{3.5.7} \rfloor\right) + \lfloor \frac{100}{2.3.5.7} \rfloor$$

$$= 103 - (50 + 33 + 20 + 14) + (16 + 10 + 7 + 6 + 4 + 2) - (3 + 2 + 1 + 0) + 0$$
$$= 25$$

This is consistent with the sieve of Eratosthenes.

Quiz: Find the number of primes in between 50 and 100

Step1: Find the number of primes ≤ 50. We have π(50) = 15.

Step 2:Find the number of primes ≤ 100. We have π(100) = 25.

Step3: Finally, Calculate the number of primes ≥ 50 and ≤ 100, which is
π(100) - π(50) = 25 -1 5 = 10

This is consistent with the sieve of Eratosthenes.

Note: Using the sieve if Eratosthenes, the primes ≤ 100 are:
2,3,5,7,11,13,17, 19, 23, 29, 31, 37, 41, 43, 47,53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

Problem: If A = {1, 2}, B = {2, 3} and C = {3, 4}, then find A $\times$ (B $\cup$ C). Further verify whether A $\times$ (B $\cup$ C) = (A $\times$ B) $\cup$ (A $\times$ C)

Part 1: We have, $B \cup C = \{2, 3, 4\}$.
Now,

$$A \times (B \cup C) = \{1, 2\} \times \{2, 3, 4\}$$
$$= \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$$

Part 2: We also have,

$$(A \times B) \cup (A \times C) = \{(1, 2), (1, 3), (2, 2), (2, 3)\}$$
$$\cup \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$
$$= \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$$
$$= A \times (B \cup C)$$

Problem: Let X, A and B be three sets such that X ∩ A = X ∩ B and X U A = X U B. Prove that A = B.

We have to prove that $A \subseteq B$ and $B \subseteq A$

Let $x \in A$,

We have two cases :-

Case1: Let $x \in X$,
then $x \in A \cap X = X \cap B$.
Thus $x \in B$

Case2: Let $x \notin X$,
then $x \in A$,
=> $x \in A \cup X = X \cup B$
Thus, $x \in B$, since $x \notin X$

Thus for all $x \in A$, $x \in B$. ------(1)

Let $x \in B$,

We have two cases :-

Case1: Let $x \in X$,
then $x \in B \cap X = X \cap A$.
Thus $x \in A$

Case2: Let $x \notin X$,
then $x \in B$,
=> $x \in B \cup X = X \cup A$
Thus, $x \in A$, since $x \notin X$

Thus for all $x \in B$, $x \in A$. -----(2)

Thus from (1) and (2), A = B.

Problem: For any two sets A and B, prove that $A \cap B = A$ if and only if $A \subseteq B$.

We first prove that if $A \cap B = A \Rightarrow A \subseteq B$
Since $A \cap B \subseteq B$, by definition $A \subseteq B$

Conversely, we prove that if $A \subseteq B \Rightarrow A \cap B = A$
By definition, $A \cap B \subseteq A$.
If $x \in A$, then $x \in B$, since $A \subseteq B$.
Hence, $x \in A \cap B$ and $A \subseteq A \cap B$
Thus $A \cap B = A$

Problem: Find a necessary and sufficient condition for $S + T = S \cup T$, where $S + T = (S \cap T^I) \cup (S^I \cap T)$, S^I is the complement of S.

$S \cap T^I$, $S \cap T$ and $S^I \cap T$ are pair wise disjoint, and

$S \cup T = (S^I \cap T) \cup (S \cap T) \cup (S^I \cap T)$

$= (S^I \cap T) \cup [(S^I \cap T) \cup (S \cap T)]$ -- Commutative law

$= [(S \cap T^I) \cup (S^I \cap T)] \cup (S \cap T)$ -- Associative law

$= (S + T) \cup (S \cap T)$

From given condition
$S \cup T = (S + T) \cup (S \cap T)$

$\Rightarrow A = A \cup B$, where $A = S \cup T = S + T$ and $B = S \cap T$

$\Rightarrow B \subseteq A$

i.e, $S \cap T \subseteq [(S \cap T^I) \cup (S^I \cap T)]$ is the necessary and sufficient condition.

Problem: Prove or disprove the following statements
a) $P(X) \cup P(Y) = P(X \cup Y)$ -- FALSE
b) $P(X) \cap P(Y) = P(X \cap Y)$ -- TRUE
C) $P(X) - P(Y) = P(X - Y)$ -- FALSE
where P(A) denotes power set of A

a) We can prove by counter example

Let $X = \{a, b\}$, $Y = \{a, c\}$
$X \cup Y = \{a, b, c\}$

$P(X \cup Y) = \{ \varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$

$P(X) = \{ \varnothing, \{a\}, \{b\}, \{a,b\} \}$

$P(Y) = \{ \varnothing, \{a\}, \{c\}, \{a,c\} \}$

As $P(X) \cup P(Y) \neq P(X \cup Y)$, the statement is FALSE

b) Let $A \in P(X \cap Y)$
$\iff A \subseteq (X \cap Y)$
$\iff A \subseteq X$ and $A \subseteq Y$
$\iff A \in P(X) \wedge A \in P(Y)$
$\iff A \in P(X) \cap P(Y)$
$\therefore P(X \cap Y) \subseteq P(X) \cap P(Y)$ --(1)

Let $A \in P(X) \cap P(Y)$
$\iff A \subseteq X$ and $A \subseteq Y$
$\iff A \subseteq X \cap Y$
$\iff A \in P(X \cap Y)$
$\therefore P(X) \cap P(Y) \subseteq P(X \cap Y)$ --(2)

From (1) and (2), $P(X) \cap P(Y) = P(X \cap Y)$. Thus the statement is TRUE

c) We can prove by counter example
Let $X = \{a, b\}$, $Y = \{a, c\}$
$X - Y = \{b\}$
$P(X - Y) = \{ \varnothing, \{b\} \}$ --(1)
$P(X) = \{ \varnothing, \{a\}, \{b\}, \{a,b\} \}$
$P(Y) = \{ \varnothing, \{a\}, \{c\}, \{a,c\} \}$
$P(X) - P(Y) = \{\varnothing, \{b\}, \{a,b\} \}$ --(2)

From (1) and (2) the statement $P(X) - P(Y) = P(X - Y)$ is FALSE

Problem: A survey of 1000 smokers reported that 850 smoked cigarettes, 200 smoked pipes and 300 smoked bidis, where as 130 smoked cigarettes and pipes, 220 smoked cigarettes and bidis, 30 smoked pipes and bidis and 20 smoked all three . Are these figures consistent?
Justify your answer.

Let A = the set of smokers who smoked cigarettes
Let B = the set of smokers who smoked pipes
Let C = the set of smokers who smoked bidis

$|A| = 850$, $|B| = 200$, $|C| = 300$
$|A \cup B \cup C| = 1000$
$|A \cap B| = 130$
$|A \cap C| = 220$
$|B \cap C| = 30$
$|A \cap B \cap C| = 20$

Apply inclusion-exclusion principle

$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|$$
$$850 + 200 + 300 - 130 - 220 - 30 + 20$$
$$990 \neq 1000$$

So the figures are not consistent.

Problem: Let X $\vartriangle$ Y denote the symmetric difference between two sets X and Y. Given three sets A,B and C, prove or disprove
a) $A \cap (B \vartriangle C) = (A \cap B) \vartriangle (A \cap C)$ -- TRUE
b) $A \cup (B \vartriangle C) = (A \cup B) \vartriangle (A \cup C)$ -- FALSE

a) RHS $= [(A \cap B) \cap (A \cap C)^{I}] \cup [(A \cap B)^{I} \cap (A \cap C)]$ by definition of $\vartriangle$

$= [(A \cap B) \cap (A^{I} \cup C^{I})] \cup [(A^{I} \cup B^{I}) \cap (A \cap C)]$ by Demorgan's Law

$= [(A \cap (A^{I} \cup C^{I})) \cap B] \cup [((A^{I} \cup B^{I}) \cap A) \cap C)]$ by associative Law

$= [[\underbrace{(A \cap A^{I})}_{\varnothing} \cup (A \cap C^{I})] \cap B] \cup [[\underbrace{(A^{I} \cap A)}_{\varnothing} \cup (B^{I} \cap A)] \cap C]$ by distributive Law

$= [(A \cap C^{I}) \cap B] \cup [(B^{I} \cap A) \cap C]$

$= [A \cap (B \cap C^{I})] \cup [A \cap (B^{I} \cap C)]$ by associative and commutative Law

$= A \cap [(B \cap C^{I}) \cup (B^{I} \cap C)]$ by distributive Law

$= A \cap (B \vartriangle C)$

= LHS

Therefore the statement is TRUE

b) We can prove by counter example
Let A = {a}, B = {a, b} and C = {a, b, c}

RHS = $(A \cup B) \vartriangle (A \cup C)$
= $(\{a,b\}) \vartriangle (\{a,b,c\})$
= $(\{a,b\} - \{a,b,c\}) \cup (\{a,b,c\} - \{a,b\})$ by definition of $\vartriangle$
= $(\varnothing \cup \{c\})$
= $\{c\}$

LHS = $A \cup (B \vartriangle C)$
= $\{a\} \cup [(\{a,b\} - \{a,b,c\}) \cup (\{a,b,c\} - \{a,b\})]$
= $\{a\} \cup [(\varnothing \cup \{c\})]$
= $\{a,c\}$

RHS ≠ LHS thus the statement is FALSE