Doing a Canonical tranformation to the new Hamiltonian

(1) $T = \frac{1}{2} m \left[\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \cos^2\theta \, \dot{\phi}^2 \right]$

(2) $V = -\frac{\mu m}{r}$ $\quad \mu = G(m + m_\odot)$

(3) $P_r = m\dot{r}$; $P_\phi = m r^2 \cos^2\theta \dot{\phi}$ $\quad P_\theta = m r^2 \dot{\theta}$

(4) $\mathcal{H} = \frac{1}{2m} \left[P_r^2 + \frac{P_\theta^2}{r^2} + \frac{P_\phi^2}{r^2 \cos^2\theta} \right] - \frac{\mu m}{r}$

sinse the generating function derivative are:

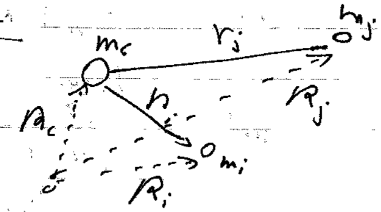
(5) $\frac{\partial S}{\partial q_i} = P_i$ $\quad q_1 = r \; q_2 = \theta \; q_3 = \phi \quad P_1 = P_r \; P_2 = P_\theta \; P_3 = P_\phi$

We can solve for $\frac{\partial S}{\partial q_i}$; $S = S(r, \theta, \phi, t)$

the hamiltonian is the total energy at the system $\boxed{\mathcal{H} = -\frac{m\mu}{2a}}$

P_θ = total angular momentum $\quad P_\phi \rightarrow$ z component of the total angular momentum

The Disturbing function



(1) $m_c \ddot{\bar R}_c = G m_c m_i \frac{\bar r_i}{r_i^3} + G m_c m_j \frac{\bar r_j}{r_j^3}$

(2) $m_i \ddot{\bar R}_i = G m_i m_j \frac{\bar r_j - \bar r_i}{|\bar r_j - \bar r_i|^3} - G m_i m_c \frac{\bar r_i}{r_i^3}$

(3) $m_j \ddot{\bar R}_j = G m_j m_i \frac{\bar r_i - \bar r_j}{|\bar r_i - \bar r_j|^3} - G m_j m_c \frac{\bar r_j}{r_j^3}$

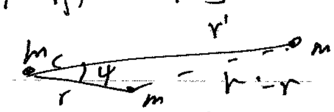
(4) where $\ddot{\bar r}_i = \ddot{\bar R}_i - \ddot{\bar R}_c$; $\ddot{\bar r}_j = \ddot{\bar R}_j - \ddot{\bar R}_c$

So we got:

(5) $$\ddot{\bar r}_i + G(m_c + m_i)\frac{\bar r_i}{r_i^3} = \underbrace{G m_j \left[\frac{\bar r_j - \bar r_i}{|\bar r_j - \bar r_i|^3} - \frac{\bar r_j}{r_j^3}\right]}_{\text{a disturbing function}} = R$$

(6) $$\ddot{\bar r}_j + G(m_c + m_j)\frac{\bar r_j}{r_j^3} = G m_i \left[\frac{\bar r_i - \bar r_j}{|\bar r_i - \bar r_j|^3} - \frac{\bar r_i}{r_i^3}\right]$$

So in general:



m, m'
$r < r'$

(7) $$R = \frac{\mu'}{|\bar r' - \bar r|} - \mu' \frac{\bar r \cdot \bar r'}{r'^3}$$

(8) $$R = \frac{\mu'}{r'} \sum_{l=2}^{\infty} \left(\frac{r}{r'}\right)^l P_l(\cos\psi)$$

(9) $$R = \mu' \sum S(a, a', e, e', I, I') \cos \ell$$

12 degrees of freedom

$\ell \Rightarrow$ linear combination of all the angles

New chapter

Hamiltonian formulation - 2 body.

(1) $\mathcal{H} = T + V$

In general given $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$ For

n Coordinates q_i

n momentum p_i

(2) $\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$

(3) $-\dot{p}_i = \frac{\partial \mathcal{H}}{\partial q_i}$

(4) $\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}$ $\mathcal{L}$ - the Langrangian

Let us define the following varilles:

(5) $L = \sqrt{\mu a}$ $\mu = G(m_1 + m_2)$

(6) $G = \sqrt{\mu a(1-e^2)}$ → Specific orbital angular momentum.

(7) $H = \sqrt{\mu a(1-e^2)} \cos i$ → ~~per~~ perpendicular G to the orbit not to be confuse with $\mathcal{H}$

(8) $\mathcal{L} = \sqrt{\frac{\mu^2}{2\varepsilon}}$

We define new angle: The mean longitude at epoch

define from:

(9) $M + \varpi = n(t - \tau) + \varpi = nt + \varepsilon$

(10) $\varepsilon = \varpi - n\tau$

~~[illegible]~~ ↓

Delaunay variables:

So the coordinates with their momentum are:

(11) $l = M \qquad g = \omega \qquad h = \Omega$

(12) $L = \sqrt{\mu a} \qquad G = \sqrt{\mu a(1-e^2)} \qquad H = \sqrt{\mu a(1-e^2)}\cos i$

The Hamiltonian for two body problem can be expressed as:

(13) $\mathcal{H} = -\dfrac{\mu^2}{2L^2}$ Delaunay Hamiltonian.

In the presence of a perturbing force with a perturbing function R we write:

(14) $\mathcal{H} = -\dfrac{\mu^2}{2L^2} - R$

(note that from eq. (13) we find that since g, h are not in there G, and H are conserved.)

From eq (13)

(note: $q_i = \tau$ but $M = n(t-\tau) \equiv l$)

$n = \dfrac{2\pi}{P_1} = \sqrt{\dfrac{\mu}{a^3}}$

(15) $\dot{L} = -\dfrac{\partial \mathcal{H}}{\partial l} = \dfrac{\partial R}{\partial l}$

(16) $\dot{G} = -\dfrac{\partial \mathcal{H}}{\partial g} = \dfrac{\partial R}{\partial g}$

(17) $\dot{H} = -\dfrac{\partial \mathcal{H}}{\partial h} = \dfrac{\partial R}{\partial h}$

and from eq (12)

(18) $\dot{l} = \dfrac{\partial \mathcal{H}}{\partial L} = \dfrac{\mu^2}{L^3} - \dfrac{\partial R}{\partial L}$

(19) $\dot{g} = \dfrac{\partial \mathcal{H}}{\partial G} = -\dfrac{\partial R}{\partial G}$

(20) $\dot{h} = \dfrac{\partial \mathcal{H}}{\partial H} = -\dfrac{\partial R}{\partial H}$

from Delaunay variables we can also find:

(27) $$a = \frac{L^2}{\mu}$$

(27) $$e = \sqrt{1 - \left(\frac{G}{L}\right)^2}$$

(28) $$\cos i = \frac{H}{G}$$

So for example since $H = \sqrt{\mu a(1-e^2)}\,\cos i$

(24) $$\frac{\partial H}{\partial i} = -\sin i \sqrt{\mu a(1-e^2)} = -\sin i \; a^2 n \sqrt{1-e^2}$$

$$n = \sqrt{\frac{\mu}{a^3}}$$

Thus,

(25) $$\dot{\Omega} = \dot{h} = -\frac{\partial R}{\partial H} = -\frac{\partial R}{\partial i}\frac{\partial i}{\partial H} = \frac{1}{n a^2 \sqrt{1-e^2}\,\sin i}\frac{\partial R}{\partial i}$$