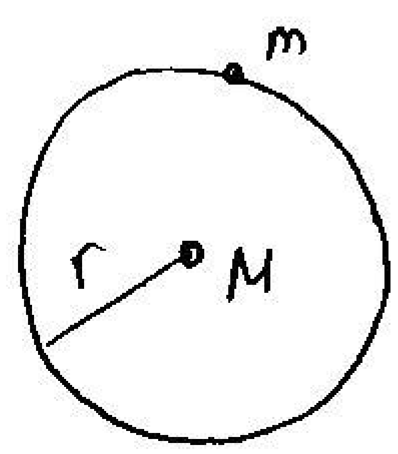
פיתוח משפט ויריאלי:



m in circular orbit around M ($m \ll M$):

$$E_{gr} = -\frac{GMm}{r} \; ; \; E_{KE} = \frac{1}{2} m v^2$$

Equation of motion:

$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$

$$\Rightarrow E_{KE} = \frac{1}{2}\frac{GMm}{r} = -\frac{1}{2} E_{gr}$$

This gives "virial theorem": $\boxed{E_{KE} = -\frac{1}{2} E_{gr}}$ (relation between KE and grav. potential energy)

$$E_{total} = E_{KE} + E_{gr} = -E_{KE} = \frac{1}{2} E_{gr}$$

$$\Rightarrow \boxed{E_{total} = \frac{1}{2} E_{gr} = -E_{KE} < 0}$$

particle is bound to M.

← עד כה סיימנו את פרקים 1 ו-2 בספר של דן מעוז.

19/03/07 6.

- Hydrostatic Balance
- Virial Theorem

1). stars dominated by ideal gas pressure (nonrelativistic) (E.g. the sun)

2). radiation pressure (relativistic).

Equation of hydrostatic equilibrium is:

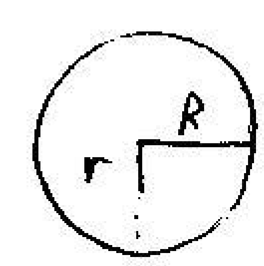
משוואה שכבר פיתחנו:

$$\boxed{\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2} < 0}$$

לחץ - $P(r)$

צפיפות - $\rho(r)$

mass within r - $M(r)$



הלחץ קטן ככל שיוצאים החוצה.

The pressure gradient is negative, so pressure naturally descreases outward ($P(R)=0$)

$$\int_{P(r)}^{P(R)} dP = \underbrace{P(R)}_{=0} - P(r) = -G\int_r^R M(r)\rho(r)/r^2 \cdot dr \Rightarrow \boxed{P(r) = G\int_r^R \frac{M(r)\rho(r)}{r^2} dr}$$

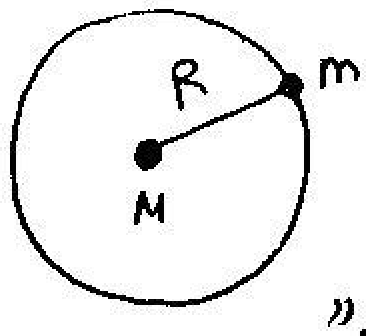
"משקל של עמודת חומר".

$$[P(r)] = N gm^{-1} \cancel{gm}\, cm^{-2} = N cm^{-2}$$

In HSE the pressure at each level is equal to the weight of a column of material of unit cross sectional area above the level.

נשכח לרגע מכדור הגז ונתבונן בחלקיק הבא:

for a point mass m in circular orbit around M ($m \ll M$):



$$E_{KE} = \frac{1}{2} m v^2$$

$$E_{gr} = -\frac{GMm}{R}$$

"virial theorem" for point particle: $\boxed{E_{KE} = -\frac{1}{2} E_{gr}}$

$$E_{tot} = E_{KE} + E_{gr} = -E_{KE} < 0 \Rightarrow \text{bound!}$$

החלקיק m קשור לחלקיק M.

כעת נפתח את המשפט הויריאלי הכללי:

נתחיל שוב עם המשוואה ההידרוסטטית ונעשה עליה כמה מניפולציות:

Multiply both sides of equation of HSE by $4\pi r^3$ and integrate:

$$\underbrace{\int_0^R \frac{dP}{dr} 4\pi r^3 dr = -\int_0^R G \frac{M(r)}{r} \underbrace{\rho(r) \cdot 4\pi r^2 dr}_{dm = \text{ the mass in a shell from } r \text{ to } r+dr}}_{\text{so this is the gravitational potential energy } E_{gr}.}$$

$$\int_0^R 4\pi r^3 dP = \underbrace{4\pi r^3 P \Big|_0^R}_{=0} - 3\int_0^R P \, 4\pi r^2 dr$$

אינטגרציה בחלקים

(since $P(R) = 0$)

$$-3\int_0^R P \, 4\pi r^2 dr = -3 \langle P \rangle V \qquad \text{נפח} - V = \int_0^R 4\pi r^2 dr$$

$$\text{הלחץ הממוצע} - \langle P \rangle = \frac{\int_0^R P \, 4\pi r^2 dr}{\int_0^R 4\pi r^2 dr}$$

so we have shown that in general for a system in hydrostatic equilibrium:

$$E_{gr} = -3 \langle P \rangle V$$

"virial theorem": $\boxed{\langle P \rangle = -\frac{1}{3} \cdot \frac{E_{gr}}{V}}$

נשים לב כי לא הנחנו כאן שום דבר על מהו מקור הלחץ. אזי, לא משנה מהו מקור הלחץ שיש לנו, במצב ש"מ הידרוסטטי המערכת צריכה לקיים את המשוואה. כעת נניח משהו לגבי הלחץ:

נתייחס למקרה בו מדובר <u>בגז אידיאלי</u>:

let's apply virial theorem for an ideal classical non-relativistic monoatomic gas:

$$PV = Nk_BT$$
$$P = nk_BT \qquad n = \frac{N}{V} \text{ (particle number density)}$$

$$E_{KE} = N\left\langle \frac{3}{2}k_BT \right\rangle \qquad \Rightarrow \langle P \rangle = \frac{2}{3}\frac{E_{KE}}{V}$$
$$\Rightarrow \langle P \rangle V = \frac{2}{3}E_{KE}$$

So we find that: $\boxed{E_{KE} = -\frac{1}{2}E_{gr}}$

$\Rightarrow$ exactly as for point mass m in circular orbit!

$$E_{tot} = E_{gr} + E_{KE} = -E_{KE} < 0 \quad \Rightarrow \text{bound!}$$

$\Leftarrow$ המערכת בש"מ גרביטציוני.

נשתמש בקשרים הללו כדי להעריך את הלחץ והטמפרטורה בתוך השמש:

Let's estimate mean pressure and temperature inside the sun:

$$E_{gr} = -\_\,\frac{GM^2}{R} \approx -\frac{GM^2}{R}$$

$$\langle P \rangle \approx \frac{1}{3}\frac{GM^2}{R}\frac{3}{4\pi R^3} = \frac{GM^2}{4\pi R^4}$$

$$M = 2.0\cdot 10^{33}\,gm$$

$$R = 7.0\cdot 10^{11}\,cm$$

$$\Rightarrow \langle P \rangle \cong 10^{15}\,dyne\,cm^{-2} = \boxed{10^9\,atm}$$

נניח כי הגז כולו עשוי ממימן מיונן (פרוטונים+אלקטרונים חופשיים) ונחשב את הטמפ' במרכז השמש:

מסה כוללת של מערכת: $M = \bar{m}N$

$\bar{m}$ = mean mass per particle

assume pure hydrogen gas fully ionized:

$$\bar{m} = \frac{1}{2} m_H$$

so: $k_B \langle T \rangle = \frac{1}{3} G \frac{M \bar{m}}{R} \Rightarrow \boxed{\langle T \rangle \cong 4 \cdot 10^6 K}$

Assume I'm always in hydrostatic equilibrium, but energy E_{tot} decreases (e.g. is radiated away).

From virial theorem know that:

$$dE_{KE} = -dE_{tot}$$

$$dE_{gr} = 2 dE_{tot}$$

so, if $dE_{tot} < 0$ (have lost energy):

$$E_{tot} + dE_{tot} < E_{tot}$$

$$dE_{KE} > 0$$

so system gains thermal energy!

$\langle T \rangle$ rises!

also $dE_{gr} < 0$, so the system loses gravitational energy, so R decreases, system shrinks!

So E.g. a 1% decrease in total energy during a contraction comes about from a 2% decrease in gravitational energy and a 1% increase in kinetic energy, or, of the gravitational energy released half heats up the gas and the other half is "lost" (radiated away).

We can now address the following important question: Could gravitational potential energy be the source of the Sun's luminosity?

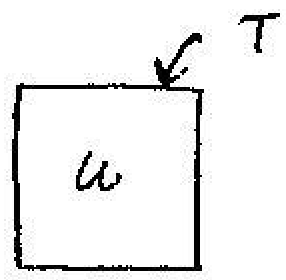
$$L_\odot = 3.8 \cdot 10^{33} \; erg \cdot s^{-1}$$

$$L t_{KH} = \frac{GM^2}{R} \cdot \underset{\substack{\downarrow \\ \text{virial theorem}}}{\frac{1}{2}} \Rightarrow t_{KH} = \frac{1}{L} \frac{GM^2}{R} \frac{1}{2} = 5 \cdot 10^{14} \sec = 1.6 \cdot 10^7 y$$

t_{KH} - "kelvin-Helmholtz time" → short relative to Biological time-scale.

ל הזמן הזה קטן מדי, משום שידוע שהחיים על כדוה"א מתקיימים כמה מיליארדי שנים.

<u>Radiation Pressure</u>: נניח קופסא בטמפ' T ובה צפיפות קרינת גוף שחור u:



$$P_{rad} = \frac{1}{3} u$$

$$u(T) = aT^4$$

In sun we estimated $\langle T \rangle = 4 \cdot 10^6 K$

$$\langle P_{thermal} \rangle = 10^{15} dyne \cdot cm^{-2}$$

In sun, $P_{rad} = \frac{1}{3} aT^4 = 6.5 \cdot 10^{11} dyne \cdot cm^{-2}$

so, in sun: $P_{rad} \ll P_{thermal}$

Let's assume hydrostatic equilibrium maintained by radiation pressure (i.e. by relativistic particles such as photons)

$\langle P \rangle = \frac{1}{3} \langle u \rangle = \frac{1}{3} \cdot \frac{U}{V}$ (U is total radiation energy $= uV$)

מהמשפט הויריאלי: $\langle P \rangle = -\frac{1}{3} \frac{E_{gr}}{V}$

so when radiation pressure dominates:

$$U_{rad} = -E_{gr}$$

$$E_{tot} = E_{gr} + U = 0 \Rightarrow \underline{\text{unstable!}}$$

A star supported by radiation pressure is unstable!

So we expect stars to become unstable when

$$P_{rad} / P_{thermal\ gas} \text{ becomes large.}$$

$$\frac{P_r}{P_t} = \frac{\frac{1}{3} aT^4}{n K_B T} = \frac{1}{3} \frac{1}{K_B} \frac{\bar{m}}{M} aT^3 V$$

Recall for classical themal gas found: $K_B T = \frac{1}{3} G \frac{M \bar{m}}{R}$

$$T^3 = \frac{4\pi}{3^4} G^3 \frac{1}{k^3} M^3 \bar{m} \frac{1}{V} \quad \left(a = \frac{8\pi^5 k^4}{15 c^3 h^3}\right), \quad \bar{m} = \frac{1}{2} m_H$$

$$\Longrightarrow \frac{P_r}{P_t} = \frac{2}{5}\left(\frac{\pi}{3}\right)^6 \frac{G^3}{c^3 h^3} m_H^4 M^2$$

so we can estimate the mass M for which $\frac{P_r}{P_t} = 1$ at which we expect stars to be gravitationally unstable:

$$M_{crit} = \underbrace{\left(\frac{5}{2}\right)^{\frac{1}{2}} \left(\frac{3}{\pi}\right)^3}_{1.4} \underbrace{\left(\frac{ch}{G}\right)^{\frac{3}{2}} \frac{1}{m_H^2}}_{29 M_\odot} = 40 M_\odot$$

$\Longrightarrow$ so don't expect the existence of stars with masses much above this.

20/03/07 (7)

Radiative Diffusion of photons out of the sun.

opacity: Thomson scattering

Kramers Law

Virial theorem + radiative diffusion $\Big\} \Rightarrow$ mass-luminosity relations.

<u>review</u>: Kelvin - Helmholtz time

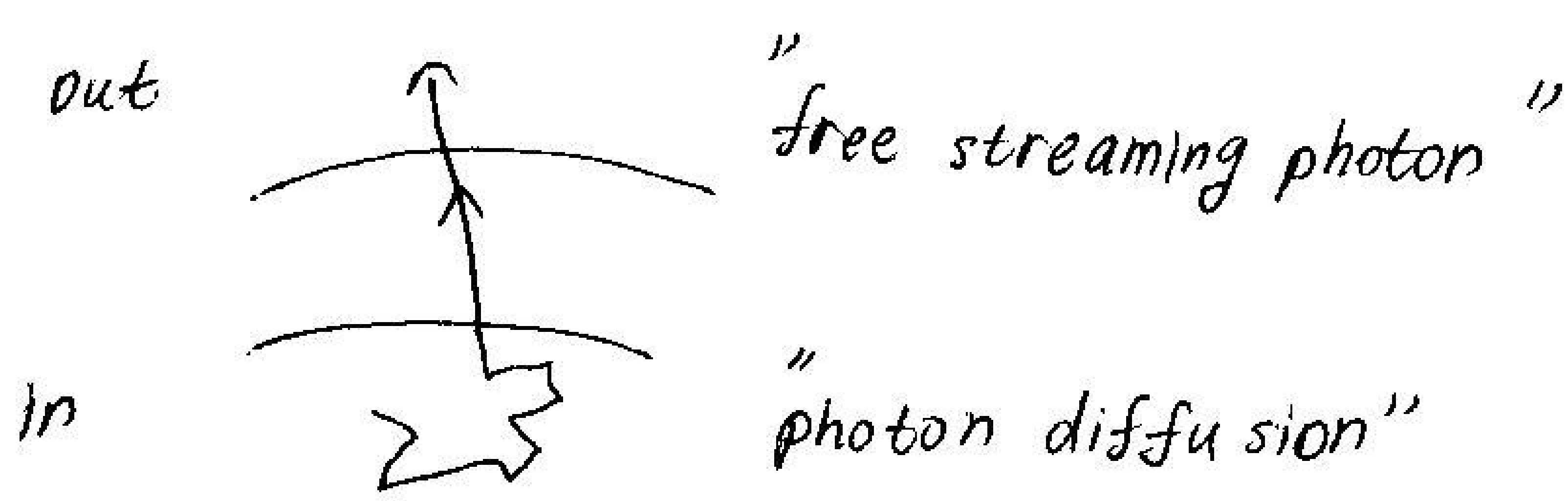
virial theorem: $\boxed{E_{tot} = -E_{KE} = \frac{1}{2} E_{gr}}$ current T

$$\boxed{\Delta E_{tot} = \frac{1}{2}\Delta E_{gr} = -G\frac{M^2}{R}\cdot\frac{1}{2} = -L t_{KH} = -\Delta E_{KE} = -\frac{3}{2} N k_B T}$$

$$\Longrightarrow \boxed{t_{KH} = \frac{1}{2} G \frac{M^2}{R} \cdot \frac{1}{L} = 1.6 \cdot 10^7 \text{yr for sun.}}$$

$$\boxed{t_{KH} = \frac{3}{2} N k_B T \frac{1}{L}}$$

כעת נתייחס להתנהגות הפוטונים הנפלטים מהשמש.

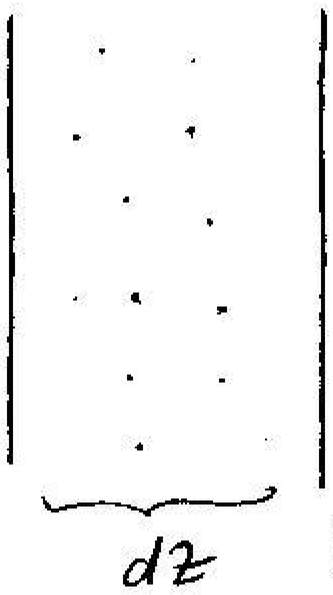


הפוטון מבצע תנועה בראונית.

כעת נחשב את הזמן הממוצע שלוקח לפוטון שנוצר בשמש להיפלט ממנו.

נניח תווך של חלקיקים מפוזרים עם צפיפות n:

$F\,[s^{-1}cm^{-2}] \longrightarrow$ incoming flux of particles (e.g. photons)



$F+dF$

n-density $[cm^{-3}]$ (# of particles per unit volume)

each particle has a "cross section" σ (cm^2) for interaction with an incoming flux of particles.

$$dF = -F n \underbrace{\sigma}_{} dz$$

חתך פעולה

דיפרנציאלי לבליעה.

define $d\tau = n\sigma dz$

$$\frac{dF}{d\tau} = -F \Rightarrow \boxed{F(\tau) = F_0\, \underbrace{e^{-\tau}}_{}}$$

Incident flux at edge (where $\tau = 0$)

מדד לסיכוי שהפוטון יחדור לעומק τ.

(probability of photon reaching a depth τ)

$$n\sigma \sim \frac{1}{\text{length}}$$

$$\langle\tau\rangle = \frac{\int_0^\infty \tau e^{-\tau} d\tau}{\int_0^\infty e^{-\tau} d\tau} = 1$$

$$\Longrightarrow \langle\tau\rangle = n\sigma\langle z\rangle = 1 \Rightarrow \boxed{\langle z\rangle = \frac{1}{n\sigma}}$$

"mean free path"

In one dimensional random walk:

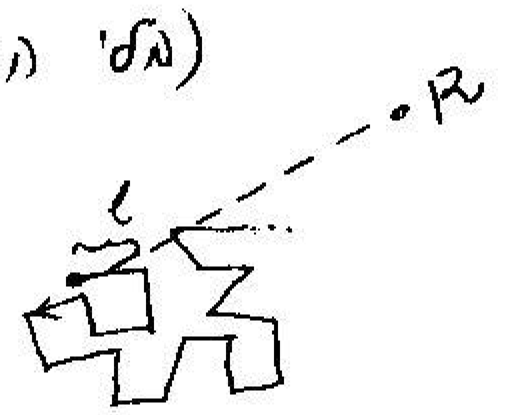
l = size of single step

L = total "root-mean-square" distance from origin.

N = number of steps

$$\Longrightarrow \boxed{N = \left(\frac{L}{l}\right)^2} \quad \text{(בלי הוכחה)}$$

In 3 dimensions: $\boxed{N = 3\left(\frac{R}{l}\right)^2}$



photons undergo many scatterings with gas particles in the sun and gradually diffuse out in a "random walk".

$$\boxed{L = \frac{U_{rad}}{t_d} = \frac{\frac{4\pi}{3} R^3 a \langle T \rangle^4}{t_d}}$$

Lets define ℓ as the mean-free path of the photon per scattering event.

$\frac{\ell}{c}$ is time it takes photon to move one mean free path

$\#$ of scatterings to reach surface $= 3\left(\frac{R}{\ell}\right)^2$

$$t_d = 3\left(\frac{R}{\ell}\right)^2 \frac{\ell}{c} = 3\left(\frac{R}{\ell}\right)\frac{R}{c}$$

$$\Longrightarrow \boxed{L = \frac{4\pi}{9} R a \langle T \rangle^4 \ell c}$$

נניח: $\langle T \rangle = 4 \cdot 10^6 K$, $R = 1 R_\odot$, $L = 1 L_\odot$

$$\Longrightarrow \boxed{\ell = 0{,}7 \, cm}$$

$$\Longrightarrow \boxed{t_d = 2.2 \cdot 10^4 \text{ year}}$$

הזמן שלוקח לפוטון להגיע בקו ישר : $R/c = 2.3$ seconds

מהשמש (בלי מהלכים חופשיים)

$$\ell = \frac{1}{n\sigma} \Rightarrow L = \frac{4\pi}{9} R a \langle T \rangle^4 c \cdot \frac{1}{n\sigma}$$