4. Consider a hypothetical star of radius R, with density ρ that is constant, i.e., independent of radius. The star is composed of a classical, non-relativistic, ideal gas of fully ionized hydrogen.
a. Solve the equations of stellar structure for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.
Answer: $P(r) = (2\pi/3)G\rho^2(R^2 - r^2)$
b. Find the temperature profile, $T(r)$.
c. Assume the nuclear energy production rate depends on temperature as $\epsilon \sim T^4$. (This is the approximate dependence of the rate for the pp chain at the temperature in the core of the Sun.) At what radius does ϵ decrease to 0.1 of its central value, and what fraction of the star's volume is included within this radius?

5. Suppose a star of total mass M and radius R has a density profile $\rho = \rho_c(1 - r/R)$, where ρ_c is the central density.
a. Find $M(r)$.
b. Express the total mass M in terms of R and ρ_c.
c. Solve for the pressure profile, $P(r)$, with the boundary condition $P(R) = 0$.
Answer:

$$P(r) = -4\pi G\rho_c^2 R^2 \left[\frac{5}{36} - \frac{2}{3}\left(\frac{r}{R}\right)^2 + \frac{7}{9}\left(\frac{r}{R}\right)^3 - \frac{1}{4}\left(\frac{r}{R}\right)^4\right].$$

6. Consider a star of mass $M = 10M_\odot$, composed entirely of fully ionized ^{12}C. Its core temperature is $T_c = 6 \times 10^8$ K (compared to $T_{c,\odot} = 1.5 \times 10^7$ K for the Sun).
a. What is the mean particle mass $\bar{m}$, in units of m_H?
Answer: 12/7.
b. Use the classical ideal gas law, the dimensional relation between mass, density, and radius, and the virial theorem, to find the scaling of the stellar radius r_* with total mass M, mean particle mass $\bar{m}$, and core temperature T_c. Using the values of these parameters for the Sun, derive the radius of the star.
Answer: $0.86r_\odot$.
c. If the luminosity of the star is $L = 10^7 L_\odot$, what is the effective surface temperature?
d. Suppose the star produces energy via the reaction

$$^{12}\mathrm{C} + {}^{12}\mathrm{C} \rightarrow {}^{24}\mathrm{Mg}.$$

The atomic weight of ^{12}C is 12, and that of ^{24}Mg is 23.985. (The atomic weight of a nucleus is defined as the ratio of its mass to 1/12 the mass of a ^{12}C nucleus). What fraction of the star's mass can be converted into thermal energy?
Answer: 6.3×10^{-4}.
e. How much time does it take for the star to use up 10% of its carbon?
Answer: 950 yr.