Perturbation in hierarchical system.

first → Kozai:

Lets remember that we can write the Hamiltonian including a small perturbing term R (perturbing function) as:

$$(1) \quad \mathcal{H} = -\frac{\mu^2}{2L^2} - R$$

Where the Delaunay elements are:

$$(2) \begin{cases} h = \Omega & H = \sqrt{a\mu(1-e^2)}\cos i \\ g = \omega & G = \sqrt{a\mu(1-e^2)} \\ l = n(t-\tau) = M & L = \sqrt{a\mu} \end{cases}$$

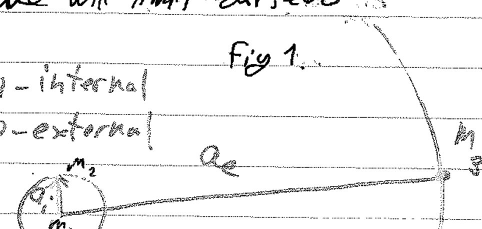
$\leftarrow$ z component of the angular momentum

For 3-body problem we will limit ourselves to a hierarchical system:

(i) - internal

(e) - external

Fig 1.



The full Hamiltonian is:

$$(3) \quad \mathcal{H} = -\frac{\mu^2}{2L_i^2} - \frac{m^2}{2L_e} + \\ + Gm_3\left(\frac{m_B}{R_3} - \frac{m_1}{r_{13}} - \frac{m_2}{r_{23}}\right)$$

and we have Delaunay elements for both the inner and outer orbits.

and the Hamiltonian eqs. of motion are →

$m = (m_1 + m_2 + m_3)G$

(4) $\dot{L}_{i,e} = \frac{\partial H}{\partial l_{i,e}}$ $\qquad$ $\dot{l}_{i,e} = \frac{\partial H}{\partial L_{i,e}}$

(5) $\dot{G}_{i,e} = -\frac{\partial H}{\partial g_{i,e}}$ $\qquad$ $\dot{g}_{i,e} = \frac{\partial H}{\partial G_{e,i}}$

(6) $\dot{H}_{i,e} = -\frac{\partial H}{\partial h_{i,e}}$ $\qquad$ $\dot{h}_{i,e} = \frac{\partial H}{\partial H_{i,e}}$

Now Note that $\bar{G}_{i,e}$ is the angular momentum of the inner/external orbits and the total angular momentum $\bar{G}_{tot}$ is simply

(7) $\bar{G}_{tot} = \bar{G}_i + \bar{G}_e$

So the projection on the fram where $\bar{G}_{tot} \parallel \hat{z}$ is:

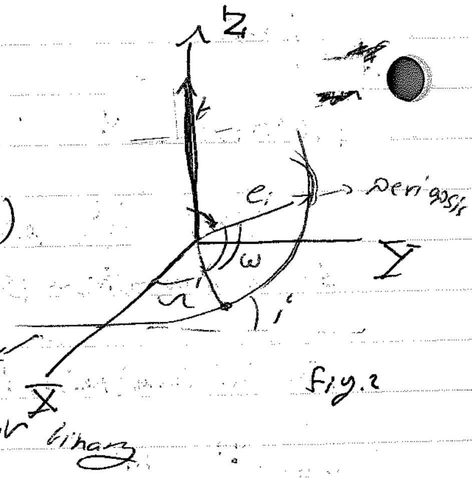


fig. 2

(8) $\bar{G}_{tot} = G(\sin i \sin\Omega, -\sin i \cos\Omega, \cos i)$

so

(9) $\bar{G}_{tot} = (G_i \sin i \sin h_i + G_e \sin i \sin h_e, -G_i \sin i \cos h_i - G_e \sin i \cos h_i, H_i + H_e)$

remember: $H = G\cos i$

Elimination of nodes: We can simplified the Hamiltonian by choosing an optimal coordinate system. We chose the invarible plane which is perpendicular to the total angular momentum.

So in this coordinate system the total angular momentum is:

(10) $\bar{G}_{tot} = (0, 0, G_{tot})$

new system

remember - longitude of acending nodes is define with respect to the referance direction

So in the invarible plane

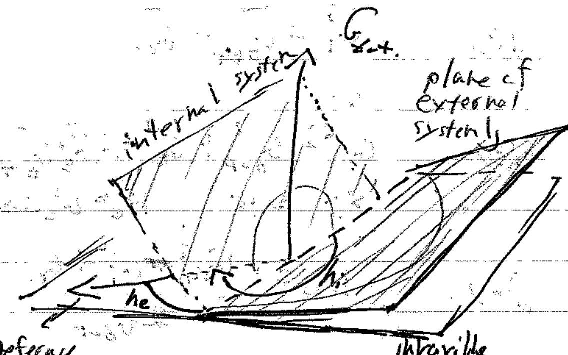


Fig 3

h_e and h_i are different by amount of π

(11) $h_i + \pi = h_e$

So From (10) and (9):

(12) $G_i \sin i \sin h_i = - G_e \sin i \sin h_e$

(13) $G_i^2 - H_i^2 = G_e^2 - H_e^2$ ⟩ the same as

(14) $G_i \sin i \cos h_i = G_e \sin i \cos h_i$

(15) $\bar{G}_i + \bar{G}_e = \bar{G}_{tot}$

(16) $G_{tot} = |\bar{G}_{tot}|^2 = |(\bar{G}_i + \bar{G}_e)|^2 = G_i^2 + G_e^2 + 2\bar{G}_i \cdot \bar{G}_e = G_i^2 + G_e^2 + 2G_i G_e \cos i$

Where i is the angle between $\bar{G}_1$ and $\bar{G}_2$ which is the inclination angle.

So eq. (16) is

(17) $$\boxed{G_{tot}^2 = G_i^2 + G_e^2 + 2G_iG_e\cos i}$$

from eq. (15)

(18) $$\bar{G}_i = \bar{G}_{tot} - \bar{G}_e$$

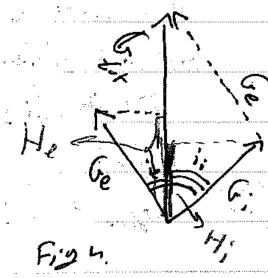


Fig 4.

so

(19) $$G_i^2 = G_{tot}^2 + G_e^2 - 2\bar{G}_e\cdot\bar{G}_{tot} = G_{tot}^2 + G_e^2 - 2G_eG_{tot}\cos i_e$$

so; since $H_e = G_e\cos i_e$

(20) $$\boxed{H_e = \frac{G_{tot}^2 - G_i^2 + G_e^2}{2G_{tot}}}$$

similarly:

(21) $$\boxed{H_i = \frac{G_{tot}^2 - G_e^2 + G_i^2}{2G_{tot}}}$$

$$\Rightarrow \boxed{H_e + H_i = G_{tot}}$$

So the Hamiltonian depends only on a combination of $h_i - h_e$. To see that lets remember the hamiltonian.

Lets take a look on the Hamiltonian

(22) $$H = T + V$$

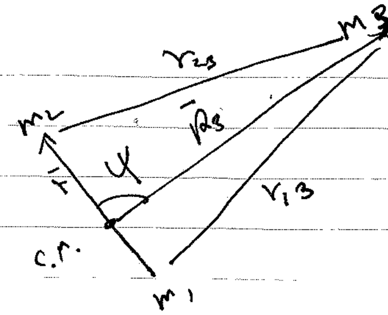
and

(23) $$V = -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{r_{13}} - \frac{Gm_2m_3}{r_{23}}$$ (remember Fig. 1)

The Hamiltonian is:

(24) $$H = -\frac{\mu^2}{2L_i^2} - \frac{m^2}{2L_e^2} + GM_3\left(\frac{m_1+m_2}{R_3} - \frac{m_1}{r_{13}} - \frac{m_2}{r_{23}}\right)$$

so also note:



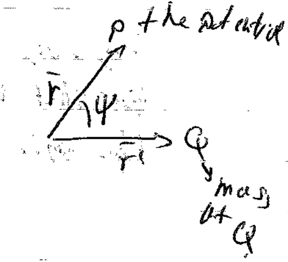
(25) $\bar{r}_{23} = \bar{R}_3 - \frac{m_1}{m_1+m_2}\bar{r}$

(26) $\bar{r}_{13} = \bar{R}_3 + \frac{m_2}{m_1+m_2}\bar{r}$

Let us remember than in "The Disturbing Function" chapter we had our disturbing function define as:

(27) $$R = \frac{\mu'}{r'}\sum_{l=2}^{\infty}\left(\frac{r}{r'}\right)^{l} P_l(\cos\psi)$$

Remeber:



we had:

(A.1) $$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cos\psi}} =$$

$$= \frac{1}{r}\left[1+\left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\psi\right]^{-\frac{1}{2}}$$

define:

(A.2) $t = \frac{r'}{r}$, so

(A.3) $$\frac{1}{|\bar{r}-\bar{r}'|} = \frac{1}{r}\left[1+t^2-2tx\right]^{-\frac{1}{2}} = \frac{1}{r}\sum_{n=0}^{\infty} t^n P_n(x)$$

We can write $P_n(x)$:

(A.4) $$P_n(x) = \sum_{k=0}^{n/2}\frac{(-1)^k(2n-2k)!}{2^{n}k!(n-k)!(n-2k)!}x^{n-2k}$$

$\rightarrow$

and we have: Legender pabnomials

4.5 $P_0(x) = 1$

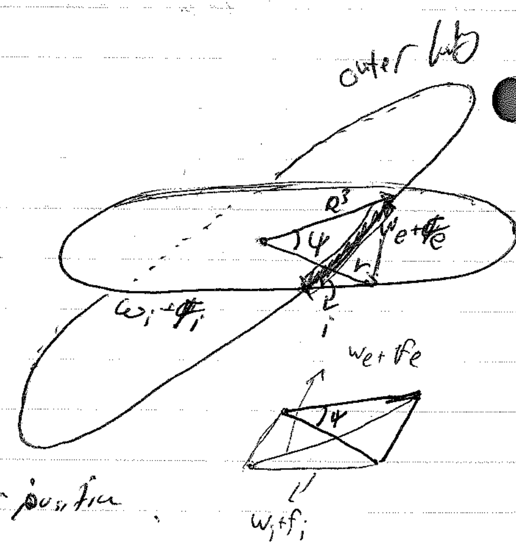
4.6 $P_1(x) = x$

4.7 $P_2(x) = \frac{1}{2}(3x^2 - 1)$

4.8 $P_3(x) = \frac{1}{2}(5x^3 - 3x)$

4.9 $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ and so on ...

So, Lets calculate R.
for that we need
the angle ψ



linary plane

$\phi_{e,i}$ → the phase (true anomaly)

$\omega_i + f_i$ = from the point of interaction (i.e., the line of nodes) to the position

So:

(28) $\cos\psi = \cos(\omega_e + \phi_e)\cos(\omega_i + \phi_i) + \sin(\omega_e + \phi_e)\sin(\omega_i + \phi_i)\cos i$

We can write V as:

$$V = -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{R_3}\sum_{n=0}^{\infty}\left(-\frac{m_2}{m_1+m_2}\right)^n\left(\frac{r}{R_3}\right)^n P_n(\cos\psi)$$

(29) $$-\frac{Gm_2m_3}{R_3}\sum_{n=0}^{\infty}\left(\frac{m_1}{m_1+m_2}\right)^n\left(\frac{r}{R_3}\right)^n P_n(\cos\psi) =$$

$$= -\frac{Gm_1m_2}{r} - \frac{Gm_1m_3}{R_3}\left[\left(-\frac{m_2}{m_1+m_2}\right)^0\left(\frac{r}{R_3}\right)^0 P_0(\cos\psi) + \right.$$

$$\left. + \left(\frac{-m_2}{m_1+m_2}\right)\frac{r}{R_3}P_1(\cos\psi)\right] - \frac{Gm_2m_3}{R_3}\left[\left(\frac{m_1}{m_1+m_2}\right)^0\left(\frac{r}{R_3}\right)^0 P_0 + \right.$$

$$\left. + \left(\frac{m_1}{m_1+m_2}\right)^1\left(\frac{r}{R_3}\right)^1 P_1\right] - \frac{Gm_1m_3}{R_3}\sum_{n=2}^{\infty}\left(-\frac{m_2}{m_1+m_2}\right)^n\left(\frac{r}{R_3}\right)^n P_n(\cos\psi)$$

$$-\frac{Gm_2m_3}{R_3}\sum_{n=2}^{\infty}\left(\frac{m_1}{m_1+m_2}\right)^n\left(\frac{r}{R_3}\right)^n P_n(\cos\psi)$$

$R \to$ the disturb. function.

Quadrupole $\to n=2$

$$V_{(to\ n=2)} = -\frac{Gm_1m_2}{r} - \frac{Gm_3(m_1+m_2)}{R_3} -$$

(30) $$-\frac{Gm_1m_2m_3}{2(m_1+m_2)R_3}\left(\frac{r}{R_3}\right)^2\left[3\cos^2\psi - 1\right]$$

The first two terms are the potential of the two body motion for the inner an outer binary. With the kinetic energy they give us the hamiltonian $-\frac{\mu^2}{2L_i} - \frac{m^2}{2L_e}$.

So for the $n=2$ we get that

(31) $$R = \frac{G m_1 m_2 m_3}{2(m_1+m_2) R_3}\left(\frac{r}{R_3}\right)^2 \left(3\cos^2\psi - 1\right)$$

We need to integrate twice. Once over the outer orbit and once over the inner orbit.

So we can write R as:

(32) $$R = \frac{G m_1 m_2 m_3}{2(m_1+m_2)} dV_i\, dV_e\, dM_i\, dM_e$$

So from Kepler eq. $M = E - e\sin E$

we have:

(33) $$dM_i = (1 - e_i\cos E)\, dE$$ inner orbit

and since

(34) $$R_3 = \frac{a_e(1-e_e^2)}{1+e_e\cos f_e}$$

We find (after some more manipulations)

(35) $$dM_e = \left(\frac{R_3}{a_e}\right)^2 \frac{df_e}{\sqrt{1-e_e^2}}$$

(36) $$dV_e = (3\cos^2\psi - 1)\frac{(1+e_e\cos f_e)}{a_e^3(1-e_e^2)^{3/2}} df_e$$

(37) $$\langle \dots \rangle = \frac{1}{2\pi}\int_0 \dots df_e$$ so the averaging

where:

(38)
$$\begin{cases} \langle \cos f_e \rangle = 0 & \langle \cos^2 f_e \rangle = \frac{1}{2}, \quad \langle \sin^2 f_e \rangle = \frac{1}{2} \\ \langle \sin f_e \cos f_e \rangle = 0 & e_e \langle \cos^3 f_e \rangle = 0 \\ e_e \langle \cos f_e \sin^2 f_e \rangle = 0 & e_e \langle \sin f_e \cos f_e \rangle = 0 \end{cases}$$

(39) $\langle V_e \rangle = \frac{1}{2a_e^3(1-e_e^2)^{3/2}} \left[3C_1^2 + 3C_2^2\cos^2 i - 2\right]$

where

(40) $C_1 = \cos(w_i + f_i)$

(41) $C_2 = \sin(w_i + f_i)$

Thus,

(42) $\langle R \rangle = \frac{Gm_1m_2m_3}{2(m_1+m_2)} \frac{r^2}{2a_e^3(1-e_e^2)^{3/2}} \left[3\left(\cos^2(w_i+f_i) + \sin^2(w_i+f_i)\cdot \cos^2 i\right) - 2\right]$

So we need now to avg :

(43) $N_i = \left(\frac{r}{a_i}\right)^2 \left\{3\left[\cos^2(w_i+f_i) + \sin^2(w_i+f_i)\cos^2 i\right] - 2\right\} dM_i$

So:

(44) $$\boxed{\langle\langle R \rangle\rangle = \frac{Gm_1m_2m_3 a_i^2}{8 m_p a_e^3 (1-e_e^2)^{3/2}} \left\{2 + 3e_i^2 - 3\sin^2 i \left[5e_i^2 \sin^2 w_i + 1 - e_i^2\right]\right\}}$$

Conserved parameters:

1. since we have integrated over f_e f_i, (l_e, l_i), (M_e, M_i) the corresponding canonical momenta L_i, L_e are constant (elimination of mean anomaly)

(45) $L_i = \text{cnst}$, $L_e = \text{cnst} \Rightarrow a_i = \text{cnst}$, $a_e = \text{cnst}$

2. note that fure this approx the Hamiltonian does not depend on ω_e (remark: no longer true when we do the expansions)

So

(46) $G_e = const$

(47) $e_e = const$) since $1-e_e^2 = \frac{G_e^2}{L_e^2}$

So the outer orbit does not change in its shape and size.

3. Elimination of the nodes.

The way we choose our fram of referance mean that

(48) $H_e + H_i = G_{tot}$ (eq. 20) and (21))

that mean that only $h_i - h_e$ enters the hamiltonian, i.e., $H_e + H_i$ enters throw

(49) $\cos i = \frac{G_{tot}^2 - G_i^2 - G_e^2}{2 G_i G_e}$ ← (eq. 17)

and $H_e = \frac{G_{tot}^2 - G_i^2 + G_e^2}{2 G_{tot}}$, $H_i = \frac{G_{tot}^2 - G_e^2 + G_i^2}{2 G_{tot}}$

Lets count degrees of fredume

Keplor has 6, so for each orbit $\times 2 = 12$ total

$a_{e,i}$, $l_{e,i}$, $w_{e,i}$, $\Omega_{e,i}$, $h_{e,i}$, $e_{e,i}$ [12] total

from ① $\dot{a}_{e,i} = 0$, $\dot{l}_{e,i} = 0$. $12 - 4 = 8$ total

from ② $\dot{w}_e = 0$, $\dot{e}_e = 0$ 6 total.

from ③ $\dot{h}_i = -\dot{h}_e$ ~ 5 total

also for fig 4 $i_i + i_e = i$ $\boxed{4 \text{ total}}$

now we will choose $i_e = 0$
This mean that we restric out outer orbit to be in the invirible plane. This is not allways realictic since if our outer orbit dosn't carry most of the angular momentum or even a lot it is not a very good assumption. non the less we will assume it. ~~thus~~

Thus

(50) $H_e = G_e \cos i_e = G_e \underset{(46)}{\overset{\epsilon}{=}} \text{const}$

since

(51) $H_e + H_i = G_{tot}$

(52) $H_i = \text{const}$

(53) $\boxed{G_e \cos i_i \underset{i_e=0}{\overset{\epsilon}{=}} G_i \cos i = L_i \sqrt{1-e_i^2} \cos i = \text{const}}$

so the inner binary is in a perfect Keppler orbit so its z component of the angular momentum is conserve

We will now describe the evolution in time of the orbital parameters.

From the hamiltonian formalism we know the evolution of $\dot{l}, \dot{g}, \dot{h}$ the the orbital parameter are

(54) $a = \frac{L^2}{\mu}$

(55) $e = \sqrt{1 - \frac{G^2}{L^2}}$

(56) $\cos i = \frac{H}{G}$

So after some math and remember that

(57) $\sqrt{\frac{\mu}{a}} = an \quad \sqrt{\mu a} = a^2 n$ we have:

remember the Lagrangian planetary equations

(58) $\dot{a} = \frac{2}{na} \frac{\partial R}{\partial M}$

(59) $\dot{e} = -\frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial \omega} + \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial M}$

(60) $\dot{(i)} = -\frac{1}{na^2\sqrt{1-e^2} \sin i} \frac{\partial R}{\partial \Omega} + \frac{\cos i}{na^2\sqrt{1-e^2} \sin i} \frac{\partial R}{\partial \omega}$

(61) $\dot{M} = n - \frac{2}{na} \frac{\partial R}{\partial a} - \frac{1-e^2}{na^2 e} \frac{\partial R}{\partial e}$

(62) $\dot{\omega} = \frac{\sqrt{1-e^2}}{na^2 e} \frac{\partial R}{\partial e} - \frac{\cos i}{na^2\sqrt{1-e^2} \sin i} \frac{\partial R}{\partial i}$

(63) $\dot{\Omega} = \frac{1}{na^2\sqrt{1-e^2} \sin i} \frac{\partial R}{\partial i}$

for all systems with all the observed taken into account:

(64) $$\dot{i} = -\frac{15}{8} \frac{e^2}{\sqrt{1-e^2}} \sin 2\omega \sin i \cos i \; \frac{A}{h}$$

(65) $$\dot{e} = \frac{15}{8} e\sqrt{1-e^2} \sin 2\omega \sin^2 i \; \frac{A}{h}$$

(66) $$\dot{\omega} = \frac{3}{4} \frac{1}{\sqrt{1-e^2}} \left[2(1-e^2) + 5\sin^2\omega\,(e^2-\sin^2 i)\right] \frac{A}{h}$$

(67) $$\dot{\Omega} = -\frac{\cos i}{4\sqrt{1-e^2}} \left[3 + 12 e^2 - 15 e^2 \cos^2\omega\right] \frac{A}{h}$$

where,

(68) $$A = \frac{G m_1 m_2 m_3}{m_3 a_e^3 (1-e_e^2)^{\frac{3}{2}}}$$

we can change the variable $t \to \tau$ so

(69) $$\tau = \frac{A}{h} t = \frac{G m_1 m_2 m_3}{m_3 n b_e^3} t$$ where b_e ~~se~~ semiminor axes

(70) $$e = \frac{\sqrt{a^2-b^2}}{a}$$

(71) $$\frac{di}{d\tau} = -\frac{15}{8} \frac{e^2}{\sqrt{1-e^2}} \sin 2\omega \sin i \cos i$$

(72) $$\frac{de}{d\tau} = \frac{15}{8} e\sqrt{1-e^2} \sin 2\omega \sin^2 i$$

(73) $$\frac{d\omega}{d\tau} = \frac{3}{4} \frac{1}{\sqrt{1-e^2}} \left[2(1-e^2) + 5\sin^2\omega(e^2-\sin^2 i)\right]$$

(74) $$\frac{d\Omega}{d\tau} = -\frac{\cos i}{4\sqrt{1-e^2}} \left(3+12e^2-15 e^2 \cos^2\omega\right)$$

close eq system solution is numerical

To get some intuition we will make an approximation that the eccentricity is very small, i.e., $e << 1$

Thus $O(e^2) \to 0$

Therefore we can write:

(75) $\frac{di}{d\tau} = 0 \Rightarrow i = \text{const}$

(76) $\frac{de}{d\tau} = \frac{15}{8} e \sin 2\omega \sin^2 i$

(77) $\frac{d\omega}{d\tau} = \frac{3}{4}(2 - 5\sin^2\omega \sin^2 i)$

(78) $\frac{d\Omega}{d\tau} = -\frac{3}{4}\cos i$

Since we want to solve for the evolution of e with time we need to know how ω changes with time.

(79) $\int \frac{4}{3} \frac{d\omega}{2 - 5\sin^2 i \sin^2\omega} = \int d\tau$

(80) $$\int \frac{dx}{a + b\sin^2 x} = \begin{cases} \frac{1}{\sqrt{a(a+b)}} \arctg \left[\frac{(a+b)\tan x}{\sqrt{a(a+b)}}\right] & a(a+b) > 0 \\ \frac{1}{2\sqrt{-a(a+b)}} \ln \left[\frac{(a+b)\tan x - \sqrt{-a(a+b)}}{(a+b)\tan x + \sqrt{-a(a+b)}}\right] & a(a+b) < 0 \end{cases}$$

here $a = 2$, $b = 5\sin^2 i$

So we have a critical angle which differs between the two behaviours:

(81) $2 - 5\sin^2 i = 0$

(82) $\sin^2 i = \frac{2}{5}$

(83) $i = 39.23°$ // ino.77

So:

(84) $$w = \begin{cases} \operatorname{arctg}\left[\sqrt{\frac{2}{A}}\, \frac{e^{\frac{3}{2}\sqrt{2A}\,\tau}+1}{e^{\frac{3}{2}\sqrt{2A}\,\tau}-1}\right] & \sin^2 i > 0.4,\ i > 39.23 \\ \operatorname{arctg}\left[\sqrt{\frac{2}{-A}}\, \tan\left(\frac{3}{4}\sqrt{-2A}\,\tau\right)\right] & \sin^2 i < 0.4 \quad i < 39.23 \end{cases}$$

$\Downarrow$

(85) $$W \rightarrow \begin{cases} w(\tau=0) = \frac{\pi}{2}, \quad w(\tau \to \infty) = \text{Const} & i > 39.23 \\ w(\tau=0) = 0 \quad w(\tau \to \infty) \to \text{fluctuate. periodic} & i < 39 \end{cases}$$

so for the regime where $i \gtrsim 39$ we can write

(86) $$\frac{dw}{d\tau} = 0$$

Thus eq. 77 for the evolution of w ~~\#~~ now gives:

(87) $$\boxed{5 \sin^2 w \sin^2 i = 2}$$

(88) $$\sin^2 w = \frac{2}{5}\,\frac{1}{\sin^2 i}$$

and

(89) $$\sin 2w = 2 \sin w \cos w = 2 \sin w \sqrt{1 - \sin^2 w}$$

So inserting that to eq. 76 we get

$$\frac{de}{d\tau} = \frac{15}{8} e\, 2 \sin w \sqrt{1 - \sin^2 w}\, \sin^2 i = \frac{15}{8} e \cdot 2 \sqrt{\frac{2}{5}}\, \frac{1}{\sin i} \cdot$$

$$\cdot \sqrt{1 - \frac{2}{5 \sin^2 i}}\, \sin^2 i = \frac{15}{4} e \sin i \sqrt{\frac{2}{5}\left(\sin^2 i - \frac{2}{5}\right)}\, \frac{1}{\sin i}$$

(90) $$\boxed{\frac{de}{d\tau} = \frac{15}{4} e \sqrt{\frac{2}{5}\left(\sin^2 i - \frac{2}{5}\right)}}$$

Where we take the positive value of $\sin\omega = \pm\sqrt{\frac{3}{5}}\,\frac{1}{\sin i}$ but $\frac{de}{d\tau}$ can be negative

The solution of eq. 90 is

(91) $$\tau = 0.42\ln\left(\frac{e}{e_0}\right)\sqrt{\sin^2 i - \frac{2}{5}}$$

where $e(\tau=0) = e_0$

Remember that this happens only in the assimptotic regime of ω.

But we can still have a general explanation using the angular momentum conservation

(92) $$\sqrt{1-e^2}\cos i = \text{const}$$

In the regime of $i > 39$, i.e., $\sin^2 i > \frac{2}{5} \Rightarrow \cos^2 i < \frac{3}{5}$ so this is the minimum value for i of which we get e_{max}

(i - grow, e decreases and visaversa)

so

(93) $$\sqrt{1-e_0^2}\cos i_0 = \sqrt{1-e_{max}^2}\sqrt{\frac{3}{5}}$$

(94) $$\boxed{e_{max} = \sqrt{1-\frac{5}{3}(1-e_0^2)\cos^2 i_0} \underset{e_0 \ll 1}{\approx} \sqrt{1-\frac{5}{3}\cos^2 i_0}}$$

if we will insert that into (91) to get an estimate for the time it takes to get to e_{max}

(95) $\tau = 0.42\, \ln \frac{e_{max}}{e_0} \sqrt{\sin^2 i_0 - \frac{2}{5}}$

we set $i = i_0$ since i is cnst in this approx. (eq. 75)
setting the value for τ (eq. 69) we have

(96) $t = \tau \frac{m_B}{G m_1 m_2 m_3} b_e^2 \sqrt{\frac{G m_B}{a^3}}\, \ln \frac{e_{max}}{e_0}$

Since this is an osilatory motion e_{max} is an e fold of e_0 thus $\ln \frac{e_{max}}{e} = 1$ and using $P = 2\pi \sqrt{\frac{a^3}{G m_B}}$ the inner orbit period we can write

(97) $t = \tau_0 \frac{m_B}{m_3} \frac{b_e^3}{a^3} \frac{1}{2\pi} P$

where we define

(98) $\tau_0 = 0.42 \sqrt{\sin^2 i_0 - 0.4}$

so the tspread time is

(99) $t = \tau_0 \frac{m_B}{m_3} \left(\frac{b_e}{a}\right)^3 \frac{1}{2\pi} P$ ← assuming $b_e \approx a_e$ outer orbit is close to circl

This acctualy gives a low estimation. in general

(100) $$\boxed{t_{kozai} \approx \left(\frac{a_e}{a}\right)^3 \frac{m_B}{m_3} P}$$

The functions that describe the evolution of Kozai are actually elliptical integral, thus the period is set by the extramum values

Lets find in general i_{max}, e_{min} i_{min}, e_{max} for a given i_0, e_0, ω_0

first note that after some mathematical manipulation

multiply eq. (72) by $2e[5\sin^2\omega - 2]\sin^2 i$

eq (71) by $[5e^2\sin^2\omega + 2(1-e^2)]\,2\sin i\cos i$

eq (73) by $5e^2\sin^2 i\,2\sin\omega\cos\omega$

adding all we get:

(101) $[5e^2\sin^2\omega + 2(1-e^2)]\,2\sin i\cos i\,\frac{di}{dt} + 5e^2\sin^2 i\cdot 2\sin\omega\cdot$
$\cos\omega\frac{d\omega}{dt} + 2e[5\sin^2\omega - 2]\sin^2 i\,\frac{de}{dt} = 0$

and than after arranging we get

(102) $$\boxed{[5e^2\sin^2\omega + 2(1-e^2)]\sin^2 i = \text{const}}$$

now we are looking for extramum

So:

(103) $\frac{de}{dt} = 0$, thus from eq. (65) it happens if

(104) $\omega = 0$, or, $\frac{\pi}{2}$ mode 2π

so if we define

(105) $X = [5 e_0^2 \sin^2 \omega_0 + 2(1-e_0^2)] \sin^2 i_0$

(106) $Y = (1-e_0^2) \cos^2 i_0$

we also know that X, Y are constant

so for

$\omega = 0$:

(107) $Y = (1-e_{min}^2) \cos^2 i_{max}$

(108) $X = (1-e_{min}^2) \cos^2 i_{max}$

$\Downarrow$

(109)
$$\boxed{\begin{cases} i_{max} = \arctan\sqrt{\dfrac{X}{2Y}} \\ e_{min} = \sqrt{1 - \dfrac{Y}{\cos^2 i_{max}}} \end{cases}}$$

and for $\omega = \frac{\pi}{2}$

(110) $X = [5 e_{max}^2 + 2(1-e_{max}^2)] \sin^2 i_{min} = (3e_{max}^2 + 2)(1-\cos^2 i_{min})$

(111) $Y = (1-e_{max}^2) \cos^2 i_{min}$

so e_{max} and i_{min} is the solution of this eqs.

(112) $\boxed{-3e_{max}^4 + (1-3Y+X)\, e_{max}^2 + 2 - 2Y - X = 0}$