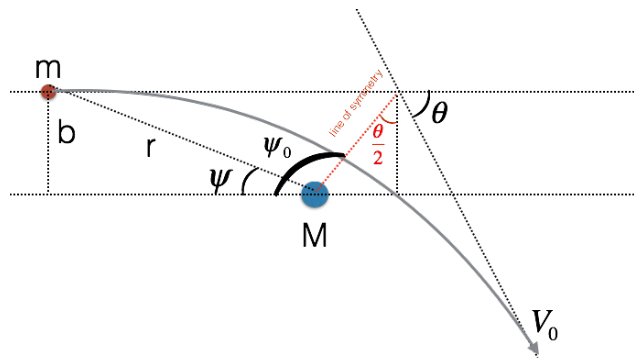
Astro-286 - Week 5

1. **Dynamical Friction** (30pt, 15 each)

   (a) Let us start with one encounter between mass M and mass m. Consider the following system: Have $\vec{v}_m$ and $\vec{v}_M$ be the velocity vector of m and M respectively,

   and have $\Delta\vec{V}$ be the change of the velocity vector of the reduced particle due to the encounter (where the mass of the reduced particle is $mM/(m+M)$). In other words $\Delta\vec{V} = \Delta\vec{v}_m - \Delta\vec{v}_M = (\vec{v}^i_m - \vec{v}^f_m) - (\vec{v}^i_M - \vec{v}^f_M)$, where v^i and v^f is the initial and final velocity. This change of velocity has two components, one which is perpendicular to the initial velocity vector of m and the other one parallel. Given the geometry of the system, the solution of the equation of motion for a Keplerian orbit as we found in class is:

   $$\frac{1}{r} = C\cos(\psi - \psi_0) + \frac{G(M+m)}{b^2V_0^2} \ . \tag{1}$$

   Find what are the two components of the change in the reduced particle velocity, i.e., $\Delta\vec{V}_{||} = V_0(1-\cos\theta)$ and $\Delta\vec{V}_{\perp} = V_0\sin\theta$.
   Hint: use the and linear angular momentum conservation laws and find the relation between θ and ψ_0

   (b) Assume a mass M traveling in a homogeneous sea of small bodies with mass m and velocity $\vec{v}_m$. Due to interaction with the small bodies The mass M suffers a steady deceleration, creating an over-density of small bodies behind it, which slows it down. Given the distribution function $f(\vec{v})$ which gives the number of small bodies encountered with $\vec{v}_m \pm d\vec{v}_m$ and $b \pm db$, where b is the impact parameter. Thus the rate that M encounters these small bodies is:

   $$2\pi b db v_0 f(\vec{v}_m) d\vec{v}_m \ . \tag{2}$$

   And the change of $\vec{v}_M$ is

   $$\frac{d\vec{v}_M}{dt} = \vec{v}_0 f(\vec{v}_m) d^3 v_m \int_0^{b_{max}} \Delta v_M(b) 2\pi b db \ . \tag{3}$$

The perpendicular part of Δv_M cancels out (**explain why!**). Show that the change rate of $\vec{v}_M$ is

$$\frac{d\vec{v}_M}{dt} = 2\pi \ln\left(1+\Lambda^2\right) G^2 m(m+M) f(\vec{v}_m) \frac{\vec{v}_m - \vec{v}_M}{|\vec{v}_m - \vec{v}_M|^3} d\vec{v}_m \ . \tag{4}$$

Hint: Note that $\vec{v}_m - \vec{v}_M = \vec{v}_0$.

2. **Small and Big bodies case** (40pt)

   (a) **Regime:** $u > v_{esc}$ (10pt)
   Assume that $u > v_{esc}$, where u is the small bodies velocity dispersion and v_{esc} is the escape velocity of the big bodies. The small bodies will be affected by dynamical fraction from the big bodies (both heating and cooling) and by viscous stirring. Find which of these effects dominates.

   (b) **Regime:** $v_{esc} > u > v_H$ (15pt)
   In our approximate way and using our notation, an integral over equation (4) (question 1b), for all possible small body velocities can be written as (now you know how to derive this equation!)

$$M\frac{d\vec{v}}{dt} \sim (GM)^2 m \ln\Lambda \int f(\vec{u}) \frac{\vec{u} - \vec{v}}{|\vec{u} - \vec{v}|^3} d^3u \ . \tag{5}$$

   In a protoplanetary disk, $f(u)$ may be approximated as a triaxial Gaussian. The rms velocities in the radial, azimuthal, and vertical dimensions are each comparable to u, although they differ by order-unity factors.

   i. (3pt) Assuming that $u << v$ expending the Gaussian to the lows order we get that $f(u) \to f(0)$. Find the dependence of $\frac{dv}{dt}$ in v

   ii. (9pt) But not all particles have small velocities, the small velocities represent a small fraction of v^3/u^3 from all particles. But these low-velocity bodies contribute $\sim -vf(0)$ to the integral, i.e., their fractional contribution to the integral is of order unity, even though they represent only a small fraction, v^3/u^3, of all the small bodies. Take only the fraction of particles that contribute to the integral (i.e., $f(0)v^3/u^3$, think how does $f(0)$ relates to the number density?) and find an expression for

$$\frac{1}{v}\frac{dv}{dt} \ . \tag{6}$$

   Is this similar to the expression we found in class for the dynamical friction cooling?

   iii. (3pt) In the oligarchy regime (i.e., $\Sigma < \sigma$) the big bodies dispersion velocity is larger than their hill velocity. Consider the dynamical friction heating and cooling rates, and estimate which one dominates. (For simplicity, ignore the coulomb logarithm)

   (c) **Regime:** $u < v_H$ (15pt, 5 each) Let us consider the dynamical friction heating:

i. For particles with velocity u what is the rate of crossing the Hill sphere? (*Hint: Be careful and don't replace* u *with* v_H)

ii. Find

$$\left.\frac{1}{v}\frac{dv}{dt}\right|_{DF\ heating} \tag{7}$$

iii. Consider the heating by dynamical friction and viscus stirring, which one of them is dominant ?

3. **Random walk in 1D** (15pt) A random walk is the process by which randomly-moving objects wander away from where they started. Assume 1D situation and a particle can have a scattering event (or take a step) either forward or backward, with equal probability. It keeps taking steps either forward or backward each time. We'll call the first step l_1 and the second step l_2 and so on. The probability to take the step is $\pm l$ ($+l$ forward and $-l$ back. Note that l is our mean free path). Starting from zero let the particle take N steps. Calculate the root-mean-squared distance ($\sqrt{< D >}$) the particle has taken after N steps.

4. **In a conference** (15pt)
During a big important conference a famous astronomer announced the detection of kuiper belt like binary at a distance of $\sim 30AU$ near Neptune's orbit, in fact he announced that this binary is currently less than 1 million km away from Neptune. The primary of this binary has a mass of 2×10^{24} g and a secondary with a mass of 2×10^{23} g and separated from each other by 2×10^{10} cm. He continued to explain how they know that this is a bound binary and why it wasn't observed so far. In the question session a student raised her hand and said that she doesn't think that this binary is bound. She then stepped to the blackboard and showed that this is impossible. Is she correct? If yes than what was her derivation? If not what do you think was her derivation and what was the error in it?