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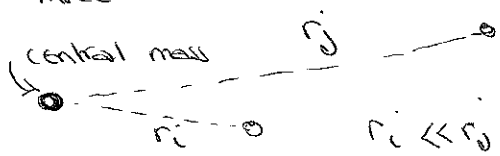
In the following I would derive the disturbing function for a three body problem:

I would treat my three bodies as an inner and outer binary



$r_i \ll r_j$

I would write the equations of motion directly:

$$\longrightarrow \quad \ddot{r} + \frac{G(m+m_c)}{r^3}\vec{r} = R$$

$$\ddot{r}' + \frac{G(m'+m_c)}{r'^3}\vec{r}' = R'$$

$$\begin{cases} R = \dfrac{\mu'}{|\vec{r}' - \vec{r}|} - \mu' \dfrac{\vec{r}\cdot\vec{r}'}{r'^3} \\[2ex] R' = \dfrac{\mu}{|r - r'|} - \mu \dfrac{r \cdot r'}{r^3} \end{cases}$$

two-body problem:

$$\ddot{\vec{r}} = -\frac{\mu \vec{r}}{r^3}$$

$$\ddot{\vec{r}}' = -\frac{\mu' \vec{r}'}{r'^3}$$

I start with the newton's laws of motion:

$$m_c \ddot{R}_c = G m_c m_i \frac{\vec{r}_i}{r_i^3} + G m_c m_j \frac{r_j}{r_j^3}$$

$$m_i \ddot{R}_i = G m_i m_j \frac{(\vec{r}_j - \vec{r}_i)}{|\vec{r}_i - \vec{r}_j|^3} - G m_i m_c \frac{r_i}{r_i^3}$$

$$m_j \ddot{R}_j = G m_j m_i \frac{(\vec{r}_i - \vec{r}_j)}{|r_j - r_i|^3} - G m_j m_c \frac{r_j}{r_j^3}$$

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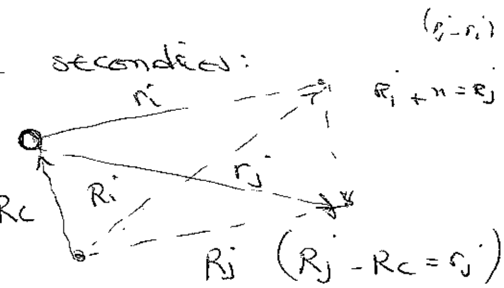
substituting for the secondaries:

$$\begin{cases} \ddot{r}_i = \ddot{R}_i - \ddot{R}_c \\ \ddot{r}_j = \ddot{R}_j - \ddot{R}_c \end{cases}$$



$$\to \ddot{\vec{r}}_i + G(m_c+m_i)\frac{\vec{r}_i}{r_i^3} = Gm_j\left[\frac{\vec{r}_j-\vec{r}_i}{|r_j-r_i|^3} - \frac{\vec{r}_j}{r_j^3}\right]$$

$$\to \ddot{\vec{r}}_j + G(m_c+m_j)\frac{\vec{r}_j}{r_j^3} = Gm_i\left[\frac{\vec{r}_i-\vec{r}_j}{|r_i-r_j|^3} - \frac{\vec{r}_i}{r_i^3}\right]$$

we are getting close to our disturbing functions

I can write the last two equations as

$$R = Gm_j\left[\frac{\vec{r}'}{r'^3}\right.$$

I have to go one step further and

I can derive the equations as gradients of some scalar function:

$$\ddot{\vec{r}}_i = \nabla_i(U_i+R_i) \qquad , U_i = \frac{G(m_c+m_i)}{r_i}$$

$$\ddot{\vec{r}}_j = \nabla_j(U_j+R_j) \qquad U_j = \frac{G(m_j+m_c)}{r_j}$$

$$R_i = Gm_j\left[\underbrace{\frac{1}{|r_j-r_i|}}_{\text{direct term}} - \underbrace{\frac{r_j\cdot r_i}{r_j^3}}_{\text{indirect term}}\right]$$

I would go back once again & derive the equations with respect to the central star:

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$\tilde{r}$ + (6.15)

(6.16) ←

(6.17)

(6.18) ←

Expansion using Legendre Polynomials:

$$|r'-r|^2 = r^2 + r'^2 - 2rr'\cos\psi$$

$$\rightarrow \frac{1}{|r-r'|} = \frac{1}{r'}\left\{1 - 2\frac{r}{r'}\cos\psi + \left(\frac{r}{r'}\right)^2\right\}^{-1/2}$$

$$= \frac{1}{r'}\sum_{l=0}^{\infty}\left(r/r'\right)^l P_l\cos\psi$$

$P_l\cos\psi$ ----

$$R = \frac{\mu'}{r'}\sum_{l=2}^{\infty}\left(\frac{r}{r'}\right)^l P_l\cos\psi \qquad 6.25$$

$$R = \mu' \sum S(a, a', t, t', I, I')\cos\psi$$

$$\left(\frac{a}{a'}\right) = \alpha \qquad \begin{cases} R = \frac{\mu'}{a'}R_D + \frac{\mu'}{a'}\alpha R_E \\ R' = \frac{\mu}{a'}R_D + \frac{\mu}{a'}\frac{1}{\alpha^2}R_I \end{cases}$$

$$\left.\begin{array}{l} R_D = \frac{a'}{|\vec{r}'-\vec{r}|} \\ R_E = -\left(\frac{r}{a}\right)\left(\frac{a'}{r'}\right)^2\cos\psi \\ R_I = -\left(\frac{r'}{a'}\right)\left(\frac{a}{r}\right)^2\cos\psi \end{array}\right\} \begin{array}{l} 6.44 \\ up\ 18 \end{array}$$

$$\sum_i \dot{j}_i = 0 \quad j_1\dot{\lambda}' + j_2\dot{\lambda} + j_3\dot{\bar{\omega}}' + j_4\dot{\bar{\omega}} \qquad 48$$

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~~we would like to solve the orbital equation~~

we would like to study the orbital evolution of a three-body problem now that we have introduced the disturbing function.

I would like to remind a few important formulas:

Hamiltonian : $\begin{cases} \partial H / \partial p_i = \dot{q}_i \\ \partial H / \partial q_i = -\dot{p}_i \end{cases}$

$H \longrightarrow q_i, p_i, t$

$K = H + dF/dt$

$\begin{cases} Q_i = Q_i(q_i, p_i, t) \\ P_i = P_i(q_i, p_i, t) \end{cases}$

I would describe the new hamiltonian in terms of new canonical variables

They satisfy the Hamiltonian equations of motion

$$\frac{\partial K}{\partial Q_i} = -\dot{P}_i \quad ; \quad \frac{\partial K}{\partial P_i} = \dot{Q}_i$$

$F_1 = F_1(q, Q, t)$

$F_2 = F_2(q, P, t)$

$F_3 = F_3(p, Q, t) \qquad F_4 = F_4(p, P, t)$

the generating function would give us the following transformation equation:

$F_2 = F_2(q, P, t) \qquad \partial F_2 / \partial q_i = p_i \quad \dfrac{\partial F_2}{\partial P_i} = Q_i$

I would choose the canonical variables in away so that

$H + \partial S / \partial t = 0$

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$S = S(q_1, q_2, \dots, q_m, P_1, \dots, P_m, t)$

$H + \partial S/\partial t = 0$ $\qquad$ $p_i = \partial S/\partial q_i$; $Q_i = \frac{\partial S}{\partial P_i}$

$H(q_1, \dots, q_m, p_1, \dots, p_m, t) + \partial S/\partial t = 0$

$H(q_1, \dots, q_m, \partial S/\partial q_1, \dots, \partial S/\partial q_m, t) + \partial S/\partial t = 0$

Hamilton – Jacobi equations ; depends on m coordinates & time. ↳ the solution depends on S itself.

m+1 → coordinates ⇒ Solution would have m+1 constants of integration

α_{m+1} constants of integration.

$S = S(q_1, \dots, q_m, \alpha_1, \dots, \alpha_m, t)$

↳ S is not a function of the momenta

~~its as if the momenta~~

α_i → take to be the momenta.

the new momenta are constants

The functions S can be determined up to a constant if i take S to be the constant α_i values.

I can define S as: the coordinates are:

$$Q_i = \frac{\partial S(q_1, \dots, q_m, \alpha_1, \dots \alpha_m, t)}{\cancel{\partial P_i}\ \partial \alpha_i} = \beta_i$$

if I know the q_i & p_i at a t_0

then: $p_i = \frac{\partial S(q_1, \dots, q_m, \alpha_1, \dots, \alpha_m, t)}{\partial q_i}$

Two body problem in Hamiltonian mechanics:

three dimensions (Spherical coordinates)

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Kinetic energy

$$T = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2\cos^2\theta\,\dot{\phi}^2)$$

Potential : $V = -\mu m / r$ $\quad \mu = G(m + M_\odot)$

Cojugate momenta : $p_r = m\dot{r}$

$$p_\theta = m r^2 \dot{\theta}$$

$$p_\phi = m r^2 \cos^2\theta\,\dot{\phi}$$

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\cos^2\theta}\right) - \mu m / r$$

Hamilton - Jacobi equation is:

$$\frac{1}{2m}\left\{ (\partial S/\partial r)^2 + \frac{1}{r^2}(\partial S/\partial \theta)^2 + \frac{1}{r^2\cos^2\theta}(\partial S/\partial \phi)^2 \right\}$$

$$-\mu m/r + \partial S/\partial t = 0$$

seperation of variables

$$S = S_t(t) + S_r(r) + S_\theta(\theta) + S_\phi(\phi)$$

partial to exact differentiation

$$\frac{dS_t}{dt} = -\alpha_1 \qquad \left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{\alpha_2^2}{\cos^2\theta} = \alpha_3^2$$

$$\frac{dS_\phi}{d\phi} = \alpha_2 \qquad \left(\frac{dS_r}{dr}\right)^2 + \frac{\alpha_3^2}{r^2} = 2m\left(\alpha_1 + \frac{\mu m}{r}\right)$$

I have to show you what ~~are~~ the α_i are

$H = mh \longrightarrow \alpha_1 = mh$

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$$\frac{\partial S}{\partial \phi} = \alpha_2 = P_\phi = m r^2 \cos^2\theta\, \dot{\phi}$$

$r\cos\theta$ = the projection of the radius in the xy plane

and the projection of the velocity is $r\cos\theta\,\dot{\phi}$

$\longrightarrow$ α_2 : we suspect it to be the projection of the product of $r\,\&\,\dot{r}$;

$L = m\vec{r}\times\dot{\vec{r}}$ $\longrightarrow$ the z component of the angular momentum

$$\alpha_2 = m\sqrt{a\mu(1-\epsilon^2)}\,\cos i$$

- - - - - - - - - - - - -

α_3 remains to be found:

$$\alpha_3 = \sqrt{\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{\alpha_2^2}{\cos^2\theta}} = \sqrt{P_\theta^2 + \frac{P_\phi^2}{\cos^2\theta}}$$

$$= \left[m^2 r^4 \dot{\theta}^2 + \frac{m^2 r^4 \cos^4\theta\,\dot{\phi}^2}{\cos^2\theta}\right]^{1/2} =$$

$$m r^2 \underbrace{\sqrt{\dot{\theta}^2 + \cos^2\theta\,\dot{\phi}^2}}$$

total angle measured in the direction of travel $= \dot{f}$ (true anomaly)

$\longrightarrow$ $\alpha_3 = m r^2 \dot{f}$ total angular momentum

$$= m\sqrt{a\mu(1-\epsilon^2)}$$

$P_1 = \alpha_1 = mh$; $P_2 = \alpha_2 = m\sqrt{a\mu(1-\epsilon^2)}\,\cos i$

$P_3 = \alpha_3 = m\sqrt{\mu a(1-e^2)}$

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$$S = -t\alpha_1 + \phi\alpha_2 + \int d\theta \sqrt{\alpha_3^2 - \frac{\alpha_2^2}{\cos^2\theta}} +$$
$$\int dr \sqrt{2m\left(\alpha_1 + \frac{\mu m}{r}\right) - \frac{\alpha_3^2}{r^2}}$$

$$Q_1 = \partial S/\partial\alpha_1 \quad ; \quad Q_2 = \partial S/\partial\alpha_2 \quad ; \quad Q_3 = \partial S/\partial\alpha_3$$

$$Q_1 = -\tau \qquad P_1 = mh = -m\mu/2a$$
$$Q_2 = \Omega \qquad P_2 = m\sqrt{a\mu(1-e^2)}\cos i$$
$$Q_3 = \omega \qquad P_3 = m\sqrt{a\mu(1-\epsilon^2)}$$

We have found a way to describe the two body system in a manner which the Hamiltonian vanishes and all the variables are constants of integration.

$$\begin{cases} q_1 = -\tau & \longrightarrow P_1 = -\mu/2a \\ q_2 = \Omega & P_2 = \sqrt{a\mu(1-\epsilon^2)}\cos i \\ q_3 = \omega & P_3 = \sqrt{a\mu(1-\epsilon^2)} \end{cases}$$

Except for q_1 the rest of the canonical coordinates are angles.

We replace q_1 with M the mean anomaly

$M = n(t-\tau)$ and keep all other coordinates intact

~~$q_1 = n(t+\tau)$~~ $\quad l = n(t+q_1)$

$$H = \sqrt{a\mu(1-\epsilon^2)}\cos i \qquad G = \sqrt{a\mu(1-\epsilon^2)} \qquad L = ?$$

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$L = ?$

the new hamiltonian of angular momentum should be described in a way such that the variables would remain canonical. I would guess generating function

$$F = \left(nL - \frac{3\mu}{2a}\right)(t + q_1) + q_2 H + q_3 G$$

$$P_1 = \partial F / \partial q_1 = -3\mu/2a + nL$$

from which we get

$$L = \frac{1}{n}\left(-\mu/2a + 3\mu/2a\right) = \mu/an = \mu / a\sqrt{\mu}\, a^{-3/2}$$

$$= \sqrt{a\mu}$$

the new Hamiltonian

$$K = \partial F / \partial t = -3\mu/2a + \mu/a = -\mu/2a = -\mu^2/2L^2$$

now the orbit of the planet can be written in the following way:

$l = n(t - \tau) = M$

$g = \omega$

$h = \Omega$

$L = \sqrt{a\mu}$

$G = \sqrt{a\mu(1-\epsilon^2)}$

$H = \sqrt{a\mu(1-\epsilon^2)}\cos i$

$K = -\mu^2/2L^2$

Delauny Elements