Introduction to Astrophysics 0321.3108
Exercise 5

1. Nuclear reactions occur when nuclei are within range of the "strong nuclear force", i.e., "strong interactions". In the first step of the p-p chain one proton is transformed to neutron. The reaction is mediated by the exchange of **virtual** pairs of pions, $m_\pi c^2 = 135$ MeV. The Heisenberg uncertainty principle for **virtual** is $\Delta t \cdot \Delta E \leq \hbar$. *Note that for real particles it is* $\geq \hbar$. Find the maximum distance that the pions pair can travel. This is an estimate for the range of the strong interaction.

2. We saw in class that the nuclear reaction rate in a star between particles A and B is: $R_{AB} = n_A n_B < \sigma v >$ where

$$< \sigma v >= \frac{1}{(k_B T)^{3/2}} \left(\frac{8}{\pi \mu} \right)^{1/2} \int_0^\infty S(E) f(E) dE \ , \qquad (1)$$

where σ is the cross section, v is the relative velocity of the particles, $S(E)$ is a function that contains the "physics details" in units of $energy \times area$, μ is the reduced mass and

$$f(E) = e^{-E/(k_B T)} e^{-\sqrt{E_G/E}} \ , \qquad (2)$$

and E_G is the Gamow energy.

a. By taking the derivative of $f(E)$ and equating to zero, show that $f(E)$ has a maximum at

$$E_0 = \left(\frac{kT}{2} \right)^{2/3} E_G^{1/3} \ . \qquad (3)$$

b. Perform a Taylor expansion, to second order, of $f(E)$ around E_0, to approximate $f(E)$ with a Gaussian, i.e., $f(E) \sim \exp[-(E - E_0)^2/(\Delta/2)^2]$. Show that the width parameter (i.e., the Δ) of the Gaussian is

$$\Delta = \frac{4}{2^{1/3}\sqrt{3}} E_G^{1/6} (k_B T)^{5/6} \qquad (4)$$

Hint: Take the logarithm of f(E), before Taylor expanding, and then exponentiate again the Taylor expansion.

c. Show that

$$\int_0^\infty f(E) dE = \sqrt{\pi} f(E_0) \frac{\Delta}{2}. \qquad (5)$$

Hint: Pay attention that in doing the integral the lower limit must be below 0, and since it a Gaussian we incur hardly any error by extending to minus infinity.

d. Use equation 1 and $E_G = (\pi\alpha Z_A Z_B)^2 2\mu c^2$, where α is the fine structure constant, to show that the rate can be written as:

$$R = n_A n_B S(E_0) \frac{4\sqrt{2}}{2^{1/3}\sqrt{3}} \frac{1}{(k_B T)^{2/3}} \frac{\sqrt{E_G}}{\pi\alpha Z_A Z_B c\sqrt{2}} \frac{E_G^{1/6}}{\mu} \exp\left(-3\left(\frac{E_G}{4k_B T}\right)^{1/3}\right). \quad (6)$$

Hint: Set $S(E) = S(E_0)$ and pull in out of the integral, since $S(E)$ is slowly varying.

e. Finally writing $\mu = A_r m_p$, where A_r is the reduced mass in units of the proton mass, show that:

$$R = 6.5 \times 10^{-18} \frac{n_A n_B}{A_r Z_A Z_B} S(E_0) \left(\frac{E_G}{4k_B T}\right)^{2/3} \exp\left(-3\left(\frac{E_G}{4k_B T}\right)^{1/3}\right) \mathrm{cm}^{-3}\mathrm{s}^{-1}. \quad (7)$$

For $S(E_0)$ in units of KeV-barns.

3. **CNO cycle**

a. Calculate the Gamow energy, $E_G = (\pi\alpha Z_A Z_B)^2 2\mu c^2$, for the reaction:

$$p +^{14}_{7} N \rightarrow^{15}_{8} O + \gamma, \quad (8)$$

which is the rate limiting step in the CNO cycle due to the large E_G.

b. Recall that the luminosity of a star can be written as: $L = \epsilon M_{core}$, where ϵ is the rate of energy production per unit mass. In class we saw that

$$\epsilon \propto \rho T^{-2/3} \exp\left(-3\left(\frac{E_G}{4k_B T}\right)^{1/3}\right). \quad (9)$$

Show that for $T = 2 \times 10^7$ K, the CNO energy production rate varies as approximately T^{18}, i.e. is extremely sensitive to the temperature.

4. **White Dwarfs**

a. Calculate the thermal and degeneracy pressure of our Sun, i.e., for our Sun density and temperature parameters. Show that the degeneracy pressure in the Sun is negligible.

b. Show that the thermal pressure in a White Dwarf is negligible. Use the following parameters: $\rho = 10^6$ g cm^{-3} and $T = 10^7$ K.

5. **Neutron Star**

a. Derive the relation between the radius and mass for a neutron star, assuming the neutrons are non-relativistic.

b. What is the radius for a mass of a 1.4 M$_\odot$?