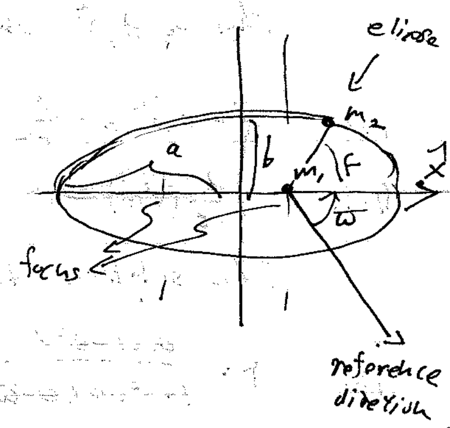
2-body Problem Notes

Some defenition:



(1) $\frac{b^2}{a^2} = (1-e^2)$

(2) $\theta = f + \varpi$

ϖ = longitude of pericenter

f = true anomaly

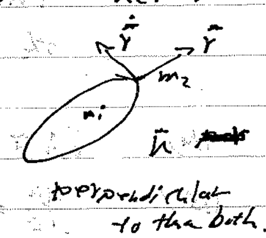
θ = true longitude

e = eccentricity

The position and velocity are in the same plane - the orbit plane.

In polar co:

$\hat{r}, \hat{\theta}$ unit vectors



(3) $\bar{r} = r\hat{r}$

(4) $\dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

(5) $\ddot{\bar{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\right]\hat{\theta}$

(6) $\bar{r} \times \dot{\bar{r}} = \bar{h}$

The motion eq:

(7) $\ddot{\bar{r}} = -\mu \frac{\bar{r}}{r^3}$

wher

(8) $\mu = G(m_1 + m_2)$

Thus, the $\hat{\theta}, \hat{r}$ components give:

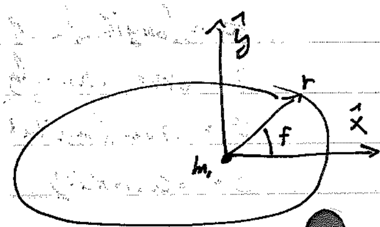
(9) $\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$: $\hat{r}$

(10) $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$: $\vec{\theta}$ (Note: $\dot{\theta} = \dot{f}$)

The solution to eq. (9) is: where, $h^2 = \mu a(1-e^2)$

(11) $$r = \frac{a(1-e^2)}{1+e\cos(\theta-\bar{\omega})} = \frac{a(1-e^2)}{1+e\cos f}$$

(12) $h = r^2\dot{\theta} = \text{const}$



(*) To Cartesian coordinate:

(13) $$\begin{cases} x = r\cos f \\ y = r\sin f \end{cases}$$

(6) The area of the elips and ~~the~~ Kepler's third law:

The area swept out by the radius in unit time:

(14) $\delta A \cong \frac{1}{2}r(r+dr)\sin\delta\theta \sim \frac{1}{2}r^2\delta\theta$

(15) $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} \underset{(12)}{=} \frac{1}{2}h$

Also remember that in order to derive eq. 11 we wrote:

(16) $\boxed{h^2 = \mu a(1-e^2)}$, so,

(17) $\frac{A^2}{T^2} = \frac{1}{4}\mu a(1-e^2)$

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also use:

(18) $A = \pi a b = \pi a^2\sqrt{1-e^2}$ (1)

So,

(19) $\frac{A^2}{T^2} = \frac{1}{4}h^2 = \frac{1}{4}\mu a(1-e^2) = \pi^2 a^4 (1-e^2)$ (18)

So:

(20) $$\boxed{T^2 = \frac{4\pi^2 a^3}{\mu}}$$ Kepler's 3rd law

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Some more relations:

Using eq. (4) we get:

(21) $V^2 = \dot{\vec{r}}\cdot\dot{\vec{r}} = \dot{r}^2 + r^2\dot{\theta}^2 = \dot{r}^2 + r^2\dot{f}^2$ $(\dot{\theta} = \dot{f})$

also, from eq. (11)

(22) $\dot{r} = \frac{r\dot{f}e\sin f}{1+e\cos f}$

From (1), (19)
$r^2\dot{f} = h = na^2\sqrt{1-e^2}$
, $n = \frac{2\pi}{T}$

$\frac{1}{a(1-e^2)} = \frac{1}{r(1+e\cos f)}$

(23) $\dot{r} = \frac{na}{\sqrt{1-e^2}}e\sin f$

so, from (22)

(24) $r\dot{f} = \frac{na}{\sqrt{1-e^2}}(1+e\cos f)$

So after some manipulations:

(25) $$\boxed{V^2 = \frac{n^2a^2}{1-e^2}(1+2e\cos f+e^2) = \mu\left(\frac{2}{r}-\frac{1}{a}\right)}$$

on the other hand
$\dot{\vec{r}}\cdot(7)$
$\dot{\vec{r}}\cdot\ddot{\vec{r}} + \mu\frac{\dot{r}}{r^2} = 0$
integrating
$\frac{1}{2}V^2 - \frac{\mu}{r} = C$

$$\boxed{C = -\frac{\mu}{2a}}$$ Constant Energy with (25)

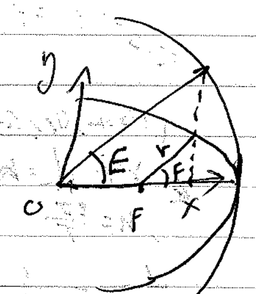
The mean Anomaly and
The eccentric anomaly

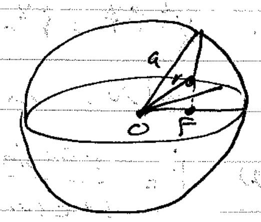
We found that for given true anomaly, f,
we know the orbital radius, r, and velocity, v;
if we also know the eccentricity and semi major axis. However, we want to know the location of a body in a given time.

First we define an angle which apart from being 2π-periodic will also be linear with time.
The mean anomaly: M

(26) $M = n(t - \tau)$ remember: $n = \frac{2\pi}{T}$

τ - the time of pericenter passage





The eccentric anomaly
E - the angle between the major axis, a, of the ellipse and the radius from the center to the intersection point on the circumscribed circle.

Transforming to Cartesian:

(27) $\begin{cases} x = a(\cos E - e) \\ y = a\sqrt{1-e^2}\,\sin E \end{cases}$

From (27):

(28) $r = a(1 - e\cos E)$

(29) $\cos f = \dfrac{\cos E - e}{1 - e\cos E}$

$r = \sqrt{x^2 + y^2}$

and with (11) $r = \dfrac{a(1-e^2)}{1+e\cos f}$

Than after some manipulations we get

(30) $\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\,\tan\frac{E}{2}$

However in order to find the location of a body we need to have a relation between E and M.

So, from eq (25) and (24) and some manipulations we can write:

(31) $\dfrac{dr}{dt} = \dfrac{na}{r}\sqrt{a^2e^2 - (r-a)^2}$

We'll use eq. (28) $r - a = -e\cos E$ and get:

(32) $\dfrac{dE}{dt} = \dfrac{n}{1 - e\cos E}$

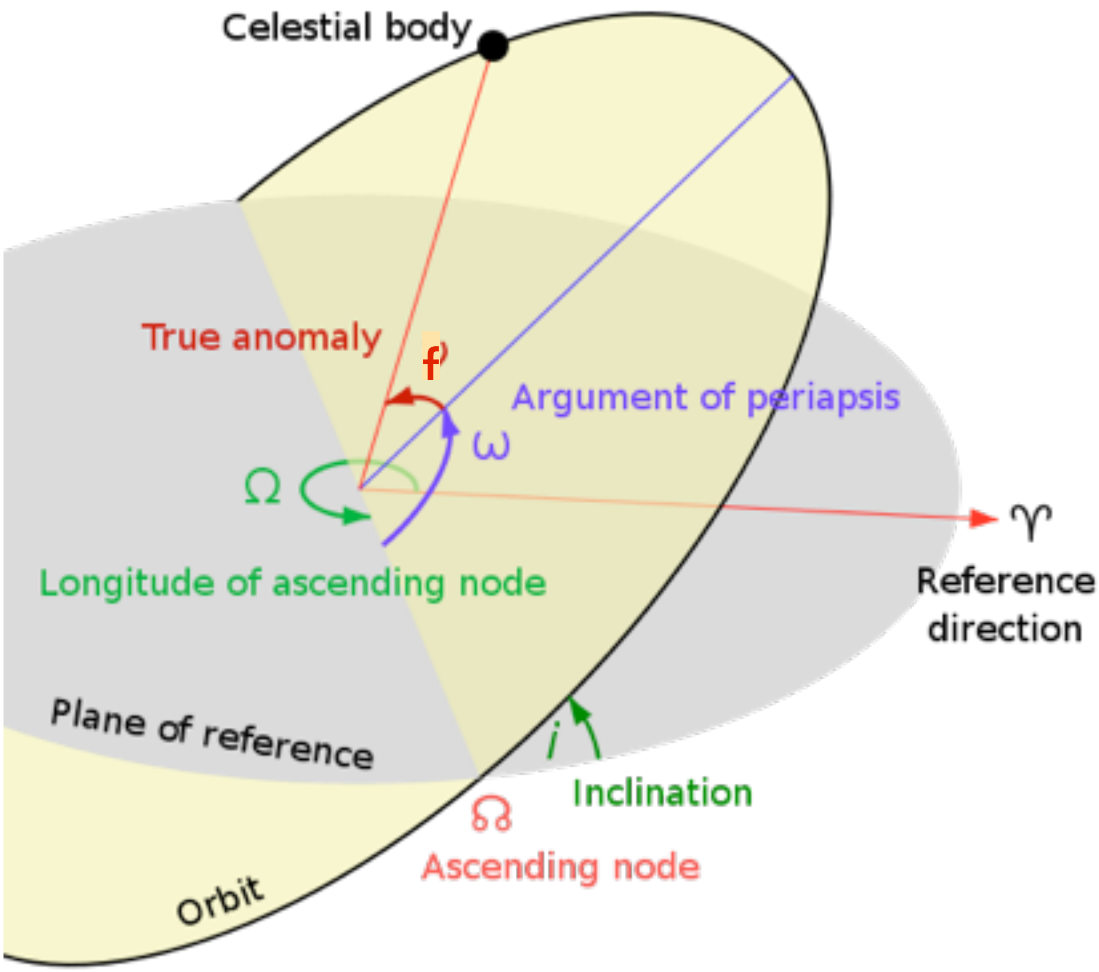
and the solution is:

(33) $n(t - \tau) = E - e\sin E$ where we set $E(t=\tau) = 0$

The left hand side is the mean anomaly (eq. 26)

(34) $\boxed{M = E - e\sin E}$ Kepler's equation

A useful figure with the important angles.



Summery - Yoram's important eqs.

$$r = \frac{a(1-e^2)}{1+e\cos(\theta+\varpi)}$$

$$h = \sqrt{GMa}\sqrt{1-e^2}$$

$$E_{neg} = -\frac{GM}{2a}$$

$$GM = n^2 a^3$$

$$n = \frac{2\pi}{T}$$