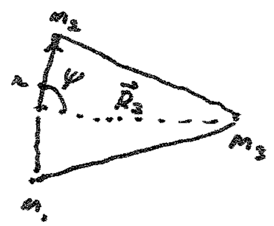
Sec. Dyn. Reading Group 13 July 2010 ①

Jean + Sara, Murray & Dermott Ch 7

Secular perturbations: those that never vanish on $\langle\ \rangle_{orbit}$ (as opposed to resonant pert., which are only nonzero $\langle\ \rangle$ when in resonance).

Disturbing f'cn: $R = \frac{G m_1 m_2 m_3}{2(m_1+m_2)R_3}\left(\frac{r}{R_3}\right)^2\left(3\cos^2\psi - 1\right)$



Want $\langle R\rangle = \frac{1}{T}\int_0^T R\,dT \overset{\text{define angle } \phi}{=} \frac{1}{2\pi}\int_0^{2\pi} R\,d\phi$

(Actually average twice: once over inner orbit, once over outer orbit.) See, e.g. 6.164 from M&D, which comes from 6.107.

Recall D'Lambert condition; $\sum_i j_i = 0$ (j keeps track of the orders of the various angles in disturbing function Fourier series.) If set $j_3 = 0$, have average over λ; recall remaining constraint.

Let's do it ⚡

Recall Poincaré variables: eq 2.179 $(\lambda, \gamma, z, \Lambda, \Gamma, Z)$.

$\gamma = -\omega - \overset{\varpi-\omega}{\Omega}$; $\Gamma = \alpha\left(1-\sqrt{1-e^2}\right) \approx e^2/2$ $\left(\Rightarrow e = \sqrt{2\Gamma}\right)$

$R = k_1 e^2 + k_2 e'^2 + k_3 e e' \cos(\varpi - \varpi')$

Equations of motion (Hamiltonian):

$\frac{d\gamma}{dt} = \partial R/\partial\Gamma$; $\frac{d\Gamma}{dt} = -\partial R/\partial\gamma$

~~Let~~ Transform to a complex canonical variable:

$$z = \sqrt{\rho}\, e^{i\delta} \implies \frac{dz}{dt} = +i\, \partial H / \partial z^* \qquad \text{(note } z^* = \sqrt{\rho}\, e^{-i\delta}.)$$

Apply to 1/1 system: $z = \sqrt{\Gamma}\, e^{i\gamma}$ or $z = \frac{e}{\sqrt{2}} e^{i\bar\omega}$

Then $\langle R \rangle = 2k_1 |z|^2 + k_3 e' \frac{\sqrt{2}}{2} \left\{ e^{-i(\gamma + \bar\omega')} + e^{i(\gamma + \bar\omega')} \right\} |z|$

↑ Note: 2 terms collapse into one because $k_1 = k_2$.

$$\langle R \rangle = \ldots + k_3 e \frac{\sqrt{2}}{2} \left\{ z e^{i\bar\omega'} + z^* e^{-i\bar\omega'} \right\}$$

$$\implies \frac{dz}{dt} = i\left[2k_1 z + \frac{\sqrt{2}}{2} k_3 e e' e^{-i\bar\omega'} \right] \quad \text{(combine 6.170, 6.171 in complex form)}$$

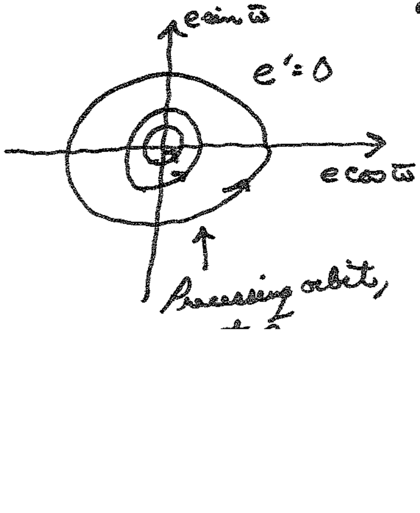
Re-write as $dz/dt = iaz + ibz'$, and from there to

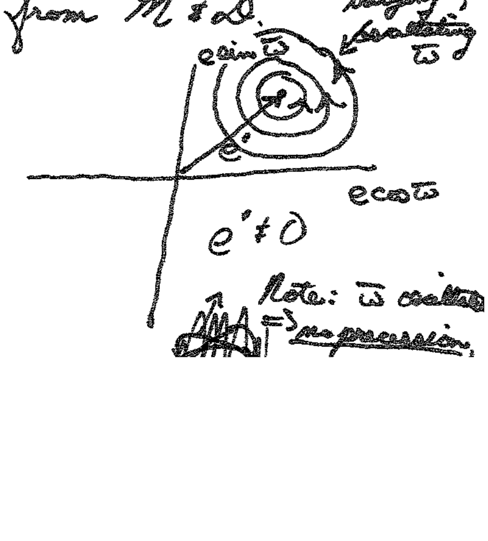
$$e^{-i\omega_0 t} \frac{d}{dt} \left\{ z e^{i\omega_0 t} \right\} = ibz' \quad \text{where } i\omega_0 = ia = 2ik_1.$$

Now seek particular solution for this equation:

$$z = \frac{b}{\omega_0} z' + C_0 e^{-i\omega_0 t}$$

This gives the motion of the orbit as a epicyclic oscillation (free) plus a forced term $\frac{b}{\omega_0} z'$ from the effects of perturber's e.
$\implies$ if $e' = 0$, then e ~~rotates in circ~~ has const magnitude;
if $e' \neq 0$ then e magnitude fluctuates about forced solution $z = \frac{b}{\omega_0} z'$. See Fig. 7.2 from M & D.





Now force outer planet to precess:

$$\frac{dz}{dt} = i\left(\omega_0 z + e' e^{igt}\right)$$

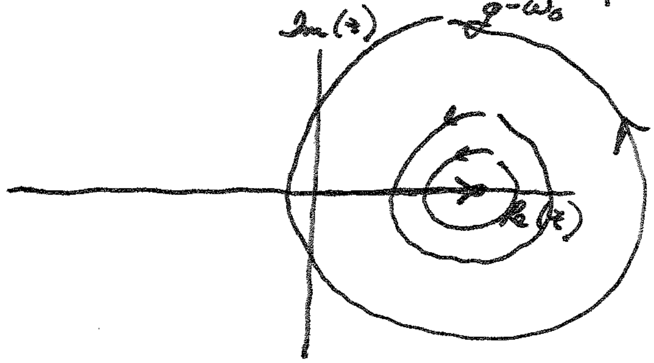
g is precession rate of perturbed planet.

Solution is $z(t) = k e^{i\omega_0 t} + \dfrac{\omega_0 e'}{g-\omega_0} e^{igt}$

The denominator $g-\omega_0$ allows for resonance if precession of perturber (g) occurs at the same rate as free precession of planet (ω_0).

If we go into "rotating frame" and plot $e^{-igt}z$:

$$e^{-igt}z = k e^{i(\omega_0-g)t} + \frac{\omega_0 e'}{g-\omega_0}$$



$\Rightarrow$ small Δe: librate about pert. $\}$ modified freq $\omega_0 - g$
Large Δe: Rotate

$\Delta e = |z| - |z'|$ controls the distance between fixed pt and circle.

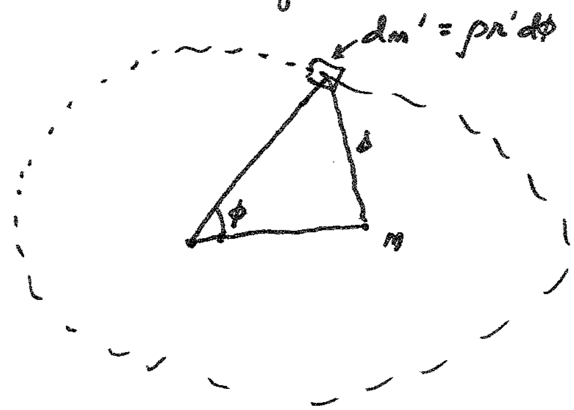
[Yoram told a story about nonlinear resonances that I don't have down here.] Yoram is preparing a paper about Mercury, Jupiter in this framework (LL plus non-linear terms) and

Gauss' Method

Treat bodies as rings distributed over their orbits proportional to the time they spend @ each location.

Then apply perturbative forces (see Ch 2) due to ring on orbit. (e.g. 2.165)

Gauss' method calculates the forces: $dF = \bar{R}\hat{r} + \bar{T}\hat{\theta} + \vec{N}\hat{z}$ on the orbit.



See 7.98. Gauss' method does $V = \int dm \frac{1}{\Delta}$, $F = -\nabla V$. When done in book, find 7.123. Can check prior expansions in e, i, etc. by expanding Gauss' approach (e.g. Gauss does not assume that $e \ll 1$ or $i \ll 1$).