Conic section

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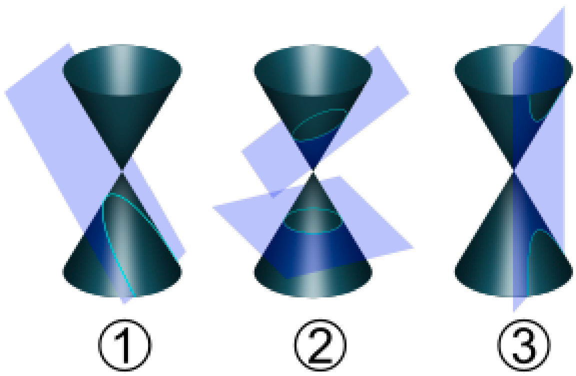
In mathematics, a **conic section** (or simply **conic**) is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse, and is of sufficient interest in its own right that it was sometimes called a fourth type of conic section. The conic sections have been studied by the ancient Greek mathematicians with this work culminating around 200 BC, when Apollonius of Perga undertook a systematic study of their properties.



Types of conic sections:
1. Parabola
2. Circle and ellipse
3. Hyperbola

The conic sections of the Euclidean plane have various distinguishing properties. Many of these have been used as the basis for a definition of the conic sections. One such property defines a non-circular conic[1] to be the set of those points whose distances to some particular point, called a *focus*, and some particular line, called a *directrix*, are in a fixed ratio, called the *eccentricity*. The type of conic is determined by the value of the eccentricity. In analytic geometry, a conic may be defined as a plane algebraic curve of degree 2; that is, as the set of points whose coordinates satisfy a quadratic equation in two variables. This equation may be written in matrix form, and some geometric properties can be studied as algebraic conditions.

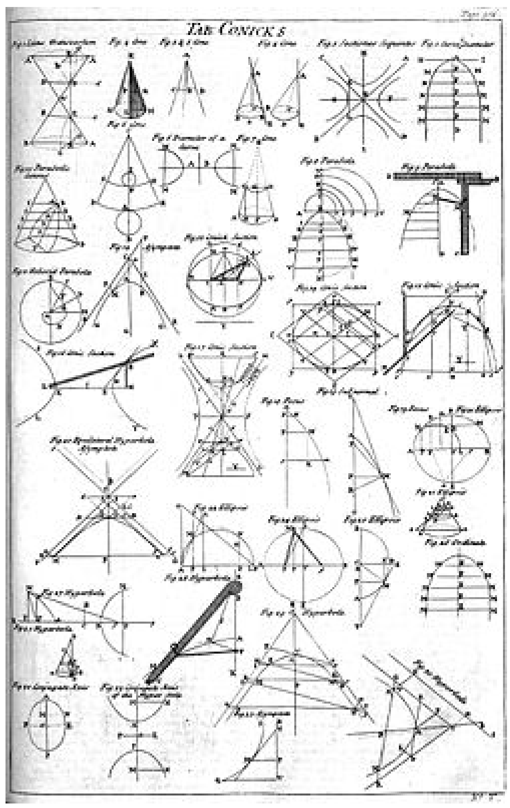


Table of conics, *Cyclopaedia*, 1728

In the Euclidean plane, the conic sections appear to be quite different from one another, but share many properties. By extending the geometry to a projective plane (adding a line at infinity) this apparent difference vanishes, and the commonality becomes evident. Further extension, by expanding the real coordinates to admit complex coordinates, provides the means to see this unification algebraically.

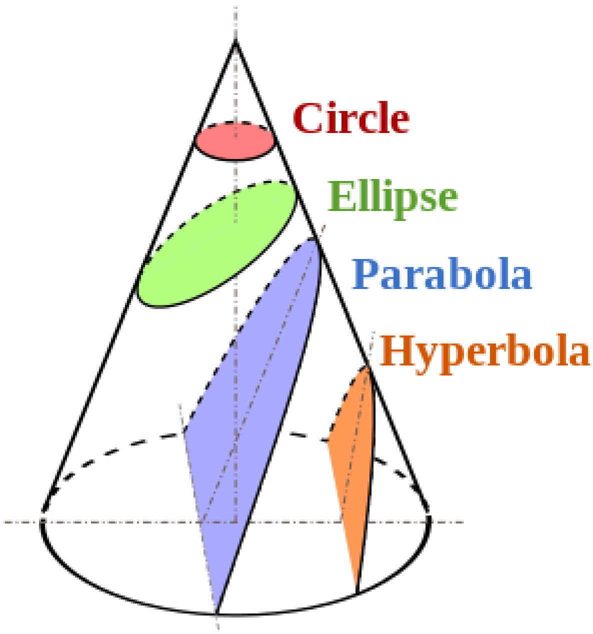
Contents

Euclidean geometry

The conic sections have been studied for thousands of years and have provided a rich source of interesting and beautiful results in Euclidean geometry.

Definition

A **conic** is the curve obtained as the intersection of a plane, called the *cutting plane*, with the surface of a double cone (a cone with two *nappes*). We shall assume that the cone is a right circular cone for the purpose of easy description, but this is not required; any double cone with some circular cross-section will suffice. Planes that pass through the vertex of the cone will intersect the cone in a point, a line or a pair of intersecting lines. These are called **degenerate conics** and some authors do not consider them to be conics at all. Unless otherwise stated, we shall assume that "conic" refers to a non-degenerate conic.

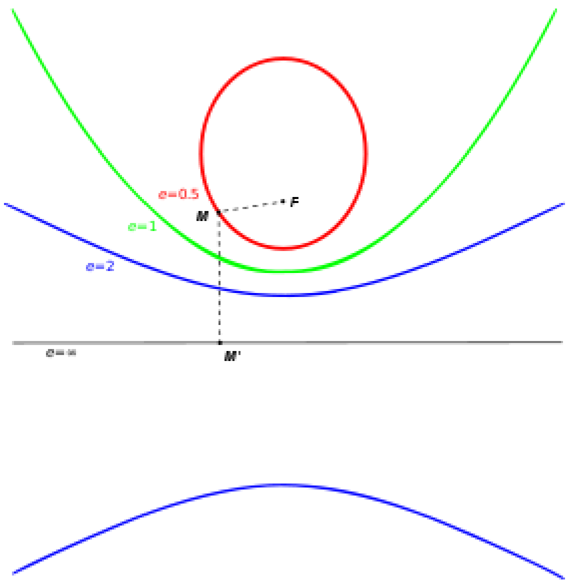


The black boundaries of the colored regions are conic sections.

There are three types of conics, the ellipse, parabola, and hyperbola. The circle is a special kind of ellipse, although historically it had been considered as a fourth type (as it was by Apollonius). The circle and the ellipse arise when the intersection of the cone and plane is a closed curve. The circle is obtained when the cutting plane is parallel to the plane of the generating circle of the cone – for a right cone, see diagram, this means that the cutting plane is perpendicular to the symmetry axis of the cone. If the cutting plane is parallel to exactly one generating line of the cone, then the conic is unbounded and is called a *parabola*. In the remaining case, the figure is a *hyperbola*. In this case, the plane will intersect *both* halves of the cone, producing two separate unbounded curves.

Eccentricity, focus and directrix

A property that the conic sections share is often presented as the following definition. A *conic section* is the locus of all points P whose distance to a fixed point F (called the ***focus*** of the conic) is a constant multiple (called the ***eccentricity***, e) of the distance from P to a fixed line L (called the ***directrix*** of the conic). For $0 < e < 1$ we obtain an ellipse, for $e = 1$ a parabola, and for $e > 1$ a hyperbola.



Ellipse (e=1/2), parabola (e=1) and hyperbola (e=2) with fixed focus F and directrix (e=∞).

A circle is a limiting case and is not defined by a focus and directrix, *in the plane* (however, see the section on the extension to projective planes). The eccentricity of a circle is defined to be zero and its focus is the center of the circle, but there is no line in the Euclidean plane that is its directrix.[2]

An ellipse and a hyperbola each have two foci and distinct directrices for each of them. The line joining the foci is called the **principal axis** and the points of intersection of the conic with the principal axis are called the **vertices** of the conic. The line segment joining the vertices of a conic is called the **major axis**, also called **transverse axis** in the hyperbola. The midpoint of this line segment is called the **center** of the conic.[3] Let a denote the distance from the center to a vertex of an ellipse or hyperbola. The distance from the center to a directrix is $\frac{a}{e}$ while the distance from the center to a focus is ae.[4]

A parabola does not have a center.

The eccentricity of an ellipse can be seen as a measure of how far the ellipse deviates from being circular.

A proof that the conic sections given by the focus-directrix property are the same as those given by planes intersecting a cone is facilitated by the use of Dandelin spheres.[5]

Conic parameters

Various parameters are associated with a conic section. Recall that the *principal axis is* the line joining the foci of an ellipse or hyperbola, and the *center* in these cases is the midpoint of the line segment joining the foci. Some of the other common features and/or parameters of conics are given below.

The **linear eccentricity** (c) is the distance between the center and the focus (or one of the two foci).

The **latus rectum** is the chord parallel to the directrix and passing through the focus (or one of the two foci). Its length is denoted by 2ℓ.

The **semi-latus rectum** (ℓ) is half of the length of the latus rectum.

The **focal parameter** (p) is the distance from the focus (or one of the two foci) to the directrix.

When an ellipse or hyperbola are in standard position (the principle axis is the x-axis and the center is the origin) the vertices of the conic have coordinates $(-a, 0)$ and $(a,0)$, with a non-negative.

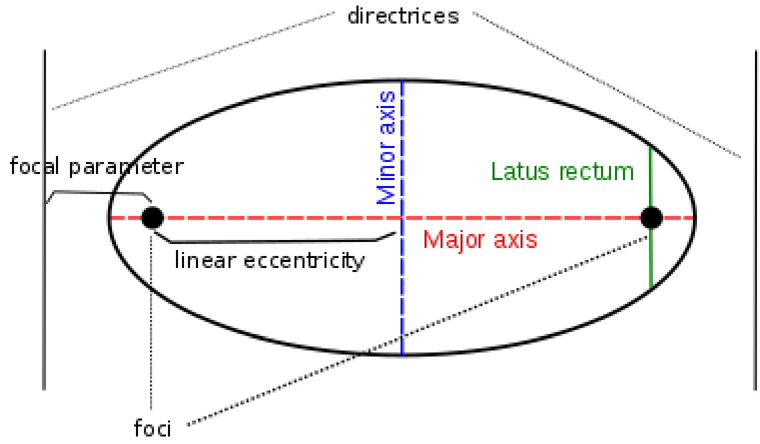
The **semi-major axis** is the value a.

The **semi-minor axis** is the value b in the standard Cartesian equation of the ellipse or hyperbola.

The following relations hold:

- $= \ell$
- $ae = c.$

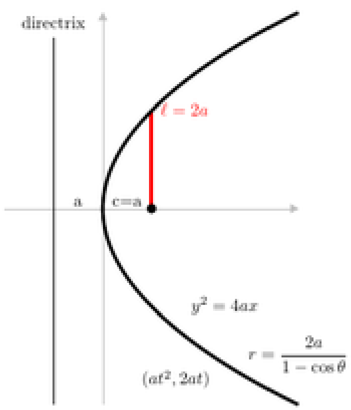
These parameters are related as shown in the following table, where the standard position is assumed. In all cases, a and b are positive.



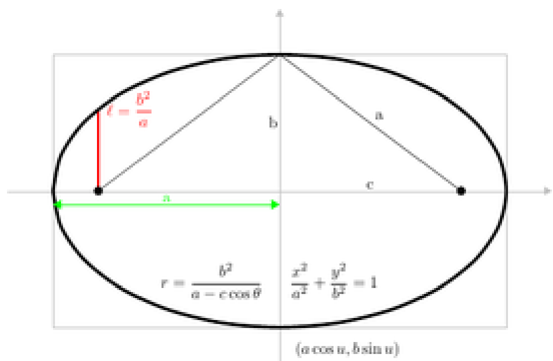
Conic parameters in the case of an ellipse

conic section	equation	eccentricity (e)	linear eccentricity (c)	semi-latus rectum (ℓ)	focal parameter (p)
circle	$x^2+y^2=a^2$	0	0	a	∞
ellipse	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$\sqrt{1-\frac{b^2}{a^2}}$	$\sqrt{a^2-b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2-b^2}}$
parabola	$y^2=4ax$	1	N/A	$2a$	$2a$
hyperbola	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$\sqrt{1+\frac{b^2}{a^2}}$	$\sqrt{a^2+b^2}$	$\frac{b^2}{a}$	$\frac{b^2}{\sqrt{a^2+b^2}}$

Standard forms in Cartesian coordinates



Standard forms of a parabola

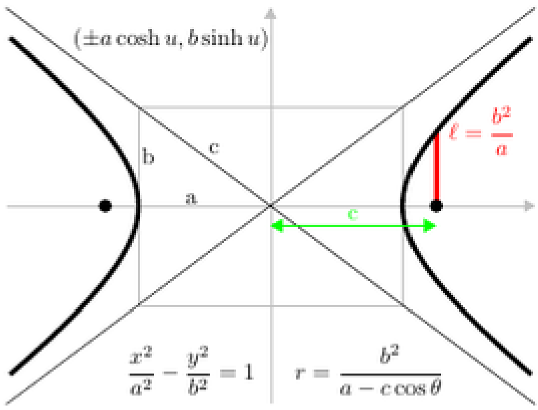


Standard forms of an ellipse

After introducing Cartesian coordinates the focus-directrix property can be used to produce equations that the coordinates of the points of the conic section must satisfy.[6] By means of a change of coordinates (a rotation of axes and a translation of axes) these equations can be put into *standard forms*.[7] For ellipses and (general) hyperbolas a standard form would have the x-axis as the principal axis and the origin (the point (0,0)) as the center. The vertices would have coordinates $(\pm a,\ 0)$ and foci coordinates $(\pm c, 0)$. Define b by the equations $c^2 = a^2 - b^2$ for an ellipse and $c^2 = a^2 + b^2$ for a hyperbola. For a circle, $c = 0$ so $a^2 = b^2$. For the parabola, the standard form has the focus on the x-axis at the point $(a,\ 0)$ and the directrix the line with equation $x = -a$. In standard form the parabola will always pass through the origin. A special case of the hyperbola occurs when its asymptotes are perpendicular. This special case is called a **rectangular** or **equilateral** hyperbola. In this case, the standard form is obtained by taking the asymptotes as the coordinate axes and the line $x = y$ as the principal axis. The foci would have coordinates $(c,\ c)$ and $(-c,\ -c)$.[8]

- Circle: $x^2 + y^2 = a^2$

- Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
- Parabola: $y^2 = 4ax$ with $a > 0$
- Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
- Rectangular hyperbola:[9] $xy = \frac{c^2}{2}$



Standard forms of a hyperbola

The first four of these forms are symmetric about both the x-axis and y-axis (for the circle, ellipse and hyperbola), or about the x-axis only (for the parabola). The rectangular hyperbola, however, is instead symmetric about the lines $y = x$ and $y = -x$.

These standard forms can be written parametrically as,

- Circle: $(a\cos\theta, a\sin\theta)$,
- Ellipse: $(a\cos\theta, b\sin\theta)$,
- Parabola: $(at^2, 2at)$,
- Hyperbola: $(a\sec\theta, b\tan\theta)$ or $(\pm a\cosh u, b\sinh u)$,
- Rectangular hyperbola: $\left(dt, \frac{d}{t}\right)$ where $d = \frac{c}{\sqrt{2}}$.

General Cartesian form

In the Cartesian coordinate system, the graph of a quadratic equation in two variables is always a conic section (though it may be degenerate[10]), and all conic sections arise in this way. The most general equation is of the form[11]

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

with all coefficients real numbers and A, B, C not all zero.

Matrix notation

The above equation can be written in matrix notation as[12]

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} D & E \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + F = 0.$$

The general equation can also be written as

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0.$$

This form is a specialization of the homogeneous form used in the more general setting of projective geometry (see below).

Discriminant

The conic sections described by this equation can be classified in terms of the value $B^2 - 4AC$, called the discriminant of the equation.[13] Thus, the discriminant is -4Δ where Δ is the matrix determinant $\begin{vmatrix} A & B/2 \\ B/2 & C \end{vmatrix}$.

If the conic is non-degenerate, then:[14]

- if $B^2 - 4AC < 0$, the equation represents an ellipse;
  - if $A = C$ and $B = 0$, the equation represents a circle, which is a special case of an ellipse;
- if $B^2 - 4AC = 0$, the equation represents a parabola;
- if $B^2 - 4AC > 0$, the equation represents a hyperbola;
  - if we also have $A + C = 0$, the equation represents a rectangular hyperbola.

In the notation used here, A and B are polynomial coefficients, in contrast to some sources that denote the semimajor and semiminor axes as A and B.

Invariants

The discriminant $B^2 – 4AC$ of the conic section's quadratic equation (or equivalently the determinant $AC – B^2/4$ of the 2×2 matrix) and the quantity $A + C$ (the trace of the 2×2 matrix) are invariant under arbitrary rotations and translations of the coordinate axes,[14][15][16] as is the determinant of the 3×3 matrix above.[17]:pp. 60–62 The constant term F and the sum D^2+E^2 are invariant under rotation only.[17]:pp. 60–62

Eccentricity in terms of coefficients

When the conic section is written algebraically as

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

the eccentricity can be written as a function of the coefficients of the quadratic equation.[18] If $4AC = B^2$ the conic is a parabola and its eccentricity equals 1 (provided it is non-degenerate). Otherwise, assuming the equation represents either a non-degenerate hyperbola or ellipse, the eccentricity is given by

$$e = \sqrt{\frac{2\sqrt{(A-C)^2 + B^2}}{\eta(A+C) + \sqrt{(A-C)^2 + B^2}}},$$

where $\eta = 1$ if the determinant of the 3×3 matrix above is negative and $\eta = -1$ if that determinant is positive.

It can also be shown[17]:p. 89 that the eccentricity is a positive solution of the equation

$$\Delta e^4 + [(A+C)^2 - 4\Delta]e^2 - [(A+C)^2 - 4\Delta] = 0,$$

where again $\Delta = AC - \frac{B^2}{4}$. This has precisely one positive solution—the eccentricity— in the case of a parabola or ellipse, while in the case of a hyperbola it has two positive solutions, one of which is the eccentricity.

Conversion to canonical form

In the case of an ellipse or hyperbola, the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

can be converted to canonical form in transformed variables x', y' as[19]

$$\frac{x'^2}{-S/\lambda_1^2\lambda_2} + \frac{y'^2}{-S/\lambda_1\lambda_2^2} = 1,$$

or equivalently

$$\frac{x'^2}{-S/\lambda_1\Delta} + \frac{y'^2}{-S/\lambda_2\Delta} = 1,$$

where λ_1 and λ_2 are the eigenvalues of the matrix $\begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix}$ — that is, the solutions of the equation

$$\lambda^2 - (A+C)\lambda + (AC - (B/2)^2) = 0$$

— and S is the determinant of the 3×3 matrix above, and $\Delta = \lambda_1\lambda_2$ is again the determinant of the 2×2 matrix. In the case of an ellipse the squares of the two semi-axes are given by the denominators in the canonical form.

Polar coordinates

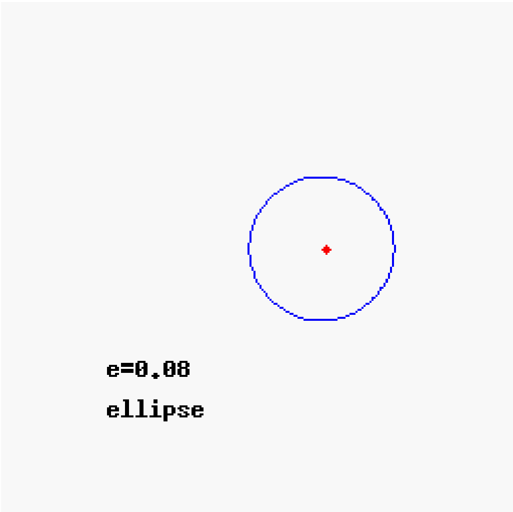
In polar coordinates, a conic section with one focus at the origin and, if any, the other at a negative value (for an ellipse) or a positive value (for a hyperbola) on the x-axis, is given by the equation

$$r = \frac{l}{1 + e\cos\theta},$$

where e is the eccentricity and l is the semi-latus rectum.

As above, for $e = 0$, we have a circle, for $0 < e < 1$ we obtain an ellipse, for $e = 1$ a parabola, and for $e > 1$ a hyperbola.

The polar form of the equation of a conic is often used in dynamics; for instance, determining the orbits of objects revolving about the Sun. [20]



Development of the conic section as the eccentricity e increases

Properties

Just as two (distinct) points determine a line, five points determine a conic. Formally, given any five points in the plane in general linear position, meaning no three collinear, there is a unique conic passing through them, which will be non-degenerate; this is true in both the Euclidean plane and its extension, the real projective plane. Indeed, given any five points there is a conic passing through them, but if three of the points are collinear the conic will be degenerate (reducible, because it contains a line), and may not be unique; see further discussion.

Four points in the plane in general linear position determine a unique conic passing through the first three points and having the fourth point as its center. Thus knowing the center is equivalent to knowing two points on the conic for the purpose of determining the curve.[21]

Furthermore, a conic is determined by any combination of *k* points in general position that it passes through and $5 - k$ lines that are tangent to it, for $0 \le k \le 5$.[22]

Any point in the plane is on either zero, one or two tangent lines of a conic. A point on just one tangent line is on the conic. A point on no tangent line is said to be an **interior point** (or **inner point**) of the conic, while a point on two tangent lines is an **exterior point** (or **outer point**).

All the conic sections share a *reflection property* that can be stated as: All mirrors in the shape of a non-degenerate conic section reflect light coming from or going toward one focus toward or away from the other focus. In the case of the parabola, the second focus needs to be thought of as infinitely far away, so that the light rays going toward or coming from the second focus are parallel.[23][24]

Pascal's theorem concerns the collinearity of three points that are constructed from a set of six points on any non-degenerate conic. The theorem also holds for degenerate conics consisting of two lines, but in that case it is known as Pappus's theorem.

Non-degenerate conic sections are always "smooth". This is important for many applications, such as aerodynamics, where a smooth surface is required to ensure laminar flow and to prevent turbulence.

History

Menaechmus and early works

It is believed that the first definition of a conic section is due to Menaechmus (died 320 BCE) as part of his solution[25] of the Delian problem (Duplicating the cube).[26] His work did not survive, not even the names he used for these curves, and is only known through secondary accounts.[27] The definition used at that time differs from the one commonly used today. Cones were constructed by rotating a right triangle about one of its legs so the hypotenuse generates the surface of the cone (such a line is called a generatrix). Three types of cones were determined by their vertex angles (measured by twice the angle formed by the hypotenuse and the leg being rotated about in the right triangle). The conic section was then determined by intersecting one of these cones with a plane drawn perpendicular to a generatrix. The type of the conic is determined by the type of cone, that is, by the angle formed at the vertex of the cone: If the angle is acute then the conic is an ellipse; if the angle is right then the conic is a parabola; and if the angle is obtuse then the conic is a hyperbola (but only one branch of the curve).[28]

Euclid (fl. 300 BCE) is said to have written four books on conics but these were lost as well.[29] Archimedes (died c. 212 BCE) is known to have studied conics, having determined the area bounded by a parabola and a chord in *Quadrature of the Parabola*. His main interest was in terms of measuring areas and volumes of figures related to the conics and part of this work survives in his book on the solids of revolution of conics, *On Conoids and Spheroids*.[30]

Apollonius of Perga

The greatest progress in the study of conics by the ancient Greeks is due to Apollonius of Perga (died c. 190 BCE), whose eight-volume *Conic Sections* or *Conics* summarized and greatly extended existing knowledge. Apollonius's study of the properties of these curves made it possible to show that any plane cutting a fixed double cone (two napped), regardless of its angle, will produce a conic according to the earlier definition, leading to the definition commonly used today. Circles, not constructible by the earlier method, are also obtainable in this way. This may account for why Apollonius considered circles a

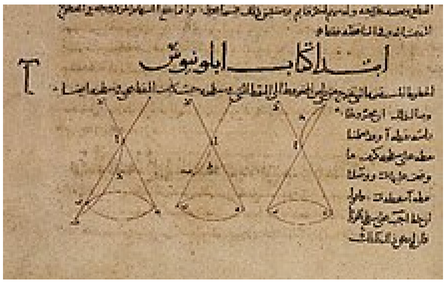


Diagram from Apollonius'*Conics*, in a 9th-century Arabic translation fourth type of conic section, a distinction that is no longer made. Apollonius used the names *ellipse*, *parabola* and *hyperbola* for these curves, borrowing the terminology from earlier Pythagorean work on areas.[31]

Pappus of Alexandria (died c. 350 CE) is credited with expounding on the importance of the concept of a conic's focus, and detailing the related concept of a directrix, including the case of the parabola (which is lacking in Apollonius's known works).[32]

Al-Kuhi

An instrument for drawing conic sections was first described in 1000 CE by the Islamic mathematician Al-Kuhi.[33][34]

Omar Khayyám

Apollonius's work was translated into Arabic and much of his work only survives through the Arabic version. Persians found applications to the theory; the most notable of these was the Persian[35] mathematician and poet Omar Khayyám who used conic sections to solve algebraic equations.

Europe

Johannes Kepler extended the theory of conics through the "principle of continuity", a precursor to the concept of limits. Kepler first used the term *foci* in 1604.[36]

Girard Desargues and Blaise Pascal developed a theory of conics using an early form of projective geometry and this helped to provide impetus for the study of this new field. In particular, Pascal discovered a theorem known as the hexagrammum mysticum from which many other properties of conics can be deduced.

René Descartes and Pierre Fermat both applied their newly discovered analytic geometry to the study of conics. This had the effect of reducing the geometrical problems of conics to problems in algebra. However, it was John Wallis in his 1655 treatise *Tractatus de sectionibus conicis* who first defined the conic sections as instances of equations of second degree.[37] Written earlier, but published later, Jan de Witt's *Elementa curvarum* starts with Kepler's kinematic construction of the conics and then develops the algebraic equations. This work, which uses Fermat's methodology and Descartes' notation has been described as the first textbook on the subject.[38] De Witt invented the term *directrix*.[38]

Applications

Conic sections are important in astronomy: the orbits of two massive objects that interact according to Newton's law of universal gravitation are conic sections if their common center of mass is considered to be at rest. If they are bound together, they will both trace out ellipses; if they are moving apart, they will both follow parabolas or hyperbolas. See two-body problem.

The paraboloid shape of Archeocyathids produces conic sections on rock faces

The reflective properties of the conic sections are used in the design of searchlights, radio-telescopes and some optical telescopes.[39] A parabolic mirror is used as the reflector, with a bulb at the focus, in a searchlight. The 4.2 meter Herschel optical telescope on La Palma, in the Canary islands, uses a primary parabolic mirror to reflect light towards a secondary hyperbolic mirror, which reflects it again to a focus behind the first mirror.

In the real projective plane

The conic sections have some very similar properties in the Euclidean plane and the reasons for this become clearer when the conics are viewed from the perspective of a larger geometry. The Euclidean plane may be embedded in the real projective plane and the conics may be considered as objects in this projective geometry. One way to do this is to introduce homogeneous coordinates and define a conic to be the set of points whose coordinates satisfy an irreducible quadratic equation in three variables (or equivalently, the zeros of an irreducible quadratic form). More technically, the set of points that are zeros of a quadratic form (in any number of variables) is called a quadric, and the irreducible quadrics in a two dimensional projective space (that is, having three variables) are traditionally called conics.

The Euclidean plane R^2 is embedded in the real projective plane by adjoining a line at infinity (and its corresponding points at infinity) so that all the lines of a parallel class meet on this line. On the other hand, starting with the real projective plane, a Euclidean plane is obtained by distinguishing some line as the line at infinity and removing it and all its points.

Intersection at infinity

We can classify the conic sections, as they appear in the Euclidean plane, by how they intersect the line at infinity.

- ellipses intersect the line at infinity in 0 points;
- parabolas intersect the line at infinity in 1 double point, corresponding to the axis— that is, they are tangent to the line at infinity, and close at a point at infinity forming an ellipse;
- hyperbolas intersect the line at infinity in 2 points, corresponding to the asymptotes—hyperbolas pass through infinity, with a twist. Going to infinity along one branch passes through the point at infinity corresponding to the asymptote, then re-emerges on the other branch at the other side but with the inside of the hyperbola (the direction of curvature) on the other side – left vs. right (corresponding to the non-orientability of the real projective plane)—and then passing through the other point at infinity returns to the first branch. Hyperbolas can thus be seen as ellipses that have been pulled through infinity and re-emerged on the other side, flipped.

In projective space, over any division ring, but in particular over either the real or complex numbers, all non-degenerate conics are equivalent, and thus in projective geometry one simply speaks of "a conic" without specifying a type, as type is not meaningful in this context. Geometrically, the line at infinity is not special, so while some conics intersect the line at infinity differently, this can be changed by a projective transformation – pulling an ellipse out to infinity or pushing a parabola off infinity to an ellipse or a hyperbola.

Homogeneous coordinates

In homogeneous coordinates a conic section can be represented as:

$$Ax^2 + Bxy + Cy^2 + Dxz + Eyz + Fz^2 = 0.$$

Or in matrix notation

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} A & B/2 & D/2 \\ B/2 & C & E/2 \\ D/2 & E/2 & F \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0.$$

The 3 × 3 matrix above is called *the matrix of the conic section*.

Some authors prefer to write the general homogeneous equation as

$$Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0,$$

(or some variation of this) so that the matrix of the conic section has the simpler form,

$$M = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix},$$

but we shall not use this notation.[40]

If the determinant of the matrix of the conic section is zero, the conic section is degenerate.

As multiplying all six coefficients by the same non-zero scalar yields an equation with the same set of zeros, one can consider conics, represented by (A, B, C, D, E, F) as points in the five-dimensional projective space $\mathbf{P}^5$.

Projective definition of a circle

Metrical concepts of Euclidean geometry (concepts concerned with measuring lengths and angles) can not be immediately extended to the real projective plane.[41] They must be redefined (and generalized) in this new geometry. This can be done for arbitrary projective planes, but to obtain the real projective plane as the extended Euclidean plane, some specific choices have to be made.[42]

Fix an arbitrary line in a projective plane that shall be referred to as the **absolute line**. Select two distinct points on the absolute line and refer to them as **absolute points**. Several metrical concepts can be defined with reference to these choices. For instance, given a line containing the points A and B, the **midpoint** of line segment AB is defined as the point C which is the projective harmonic conjugate of the point of intersection of AB and the absolute line, with respect to A and B.

A conic in a projective plane that contains the two absolute points is called a **circle**. Since five points determine a conic, a circle (which may be degenerate) is determined by three points. To obtain the extended Euclidean plane, the absolute line is chosen to be the line at infinity of the Euclidean plane and the absolute points are two special points on that line called the circular points at infinity. Lines containing two points with real coordinates do not pass through the circular points at infinity, so in the Euclidean plane a circle, under this definition, is determined by three points that are not collinear.[43]

It has been mentioned that circles in the Euclidean plane can not be defined by the focus-directrix property. However, if one were to consider the line at infinity as the directrix, then by taking the eccentricity to be $e = 0$ a circle will have the focus-directrix property, but it is still not defined by that property.[44] One must be careful in this situation to correctly use the definition of eccentricity as the ratio of the distance of a point on the circle to the focus (length of a radius) to the distance of that point to the directrix (this distance is infinite) which gives the limiting value of zero.

Steiner's projective conic definition

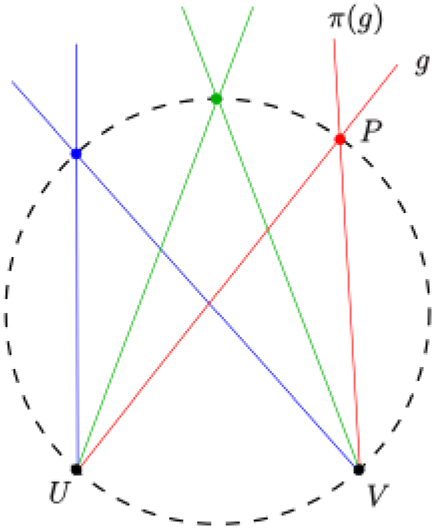
A synthetic (without the use of coordinates) approach to defining the conic sections in a projective plane was given by Jakob Steiner in 1867.

- Given two pencils $B(U), B(V)$ of lines at two points U, V (all lines containing U and V resp.) and a projective but not perspective mapping π of $B(U)$ onto $B(V)$. Then the intersection points of corresponding lines form a non-degenerate projective conic section.[45][46][47][48]

A *perspective* mapping π of a pencil $B(U)$ onto a pencil $B(V)$ is a bijection (1-1 correspondence) such that corresponding lines intersect on a fixed line a, which is called the *axis* of the perspectivity π.

A *projective* mapping is a finite sequence of perspective mappings.

As a projective mapping in a projective plane over a field (pappian plane) is uniquely determined by prescribing the images of three lines,[49] for the Steiner generation of a conic section, besides two points U, V only the images of 3 lines have to be given. These 5 items (2 points, 3 lines) uniquely determine the conic section.



Definition of the Steiner generation of a conic section

Line conics

By the Principle of Duality in a projective plane, the dual of each point is a line, and the dual of a locus of points (a set of points satisfying some condition) is called an *envelope* of lines. Using Steiner's definition of a conic (this locus of points will now be referred to as a *point conic*) as the meet of corresponding rays of two related pencils, it is easy to dualize and obtain the corresponding envelope consisting of the joins of corresponding points of two related ranges (points on a line) on different bases (the lines the points are on). Such an envelope is called a **line conic** (or **dual conic**).

In the real projective plane, a point conic has the property that every line meets it in two points (which may coincide, or may be complex) and any set of points with this property is a point conic. It follows dually that a line conic has two of its lines through every point and any envelope of lines with this property is a line conic. At every point of a point conic there is a unique tangent line, and dually, on every line of a line conic there is a unique point called a *point of contact*. An important theorem states that the tangent lines of a point conic form a line conic, and dually, the points of contact of a line conic form a point conic.[50]

Von Staudt's definition

Karl Georg Christian von Staudt defined a conic as the point set given by all the absolute points of a polarity that has absolute points. Von Staudt introduced this definition in *Geometrie der Lage* (1847) as part of his attempt to remove all metrical concepts from projective geometry.

A **polarity**, π, of a projective plane, P, is an involutory (i.e., of order two) bijection between the points and the lines of P that preserves the incidence relation. Thus, a polarity relates a point Q with a line q and, following Gergonne, q is called the **polar** of Q and Q the **pole** of q.[51] An **absolute point** (**line**) of a polarity is one which is incident with its polar (pole).[52]

A von Staudt conic in the real projective plane is equivalent to a Steiner conic.[53]

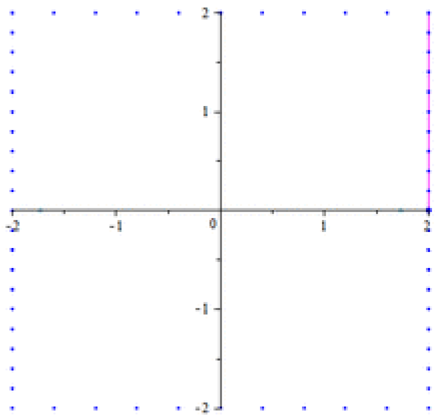
Constructions

A conic can not be constructed as a continuous curve (or two) with straightedge and compass. However, there are several methods that are used to construct as many individual points on a conic, with straightedge and compass, as desired.

One of them is based on the converse of Pascal's theorem, namely, *if the points of intersection of opposite sides of a hexagon are collinear, then the six vertices lie on a conic.* Specifically, given five points, A, B, C, D, E and a line passing through E, say $\overline{EG}$, we can construct a point F that lies on this line and is on the conic determined by the five points. Let $\overline{AB}$ meet $\overline{DE}$ in L, $\overline{BC}$ meet $\overline{EG}$ in M and let $\overline{CD}$ meet $\overline{LM}$ at N. Then $\overline{AN}$ meets $\overline{EG}$ at the required point F.[54] By varying the line through E, we can construct as many additional points on the conic as desired.

Another method, based on Steiner's construction and which is useful in engineering applications, is the **parallelogram method**, where a conic is constructed point by point by means of connecting certain equally spaced points on a horizontal line and a vertical line.[55] Specifically, to construct the ellipse with equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, first construct the rectangle $ABCD$ with vertices $A(a, 0)$, $B(a, 2b)$, $C(-a, 2b)$ and $D(-a, 0)$. Divide the side $\overline{BC}$ into n equal segments and use parallel projection, with respect to the diagonal $\overline{AC}$, to form equal segments on side $\overline{AB}$ (the lengths of these segments will be $\frac{b}{a}$ times the length of the segments on $\overline{BC}$). On the side $\overline{BC}$ label the left-hand endpoints of the segments with A_1 to A_n starting at B and going towards C. On the side $\overline{AB}$ label the upper endpoints D_1 to D_n starting at A and going towards B. The points of intersection, $\overline{AA_i} \cap \overline{DD_i}$ for $1 \leq i \leq n$ will be points of the ellipse between A and $P(0, b)$. The labeling associates the lines of the pencil through A with the lines of the pencil through D projectively but not perspectively. The sought for conic is obtained by this construction since three points A, D and P and two tangents (the vertical lines at A and D) uniquely determine the conic. If another diameter (and its conjugate diameter) are used instead of the major and minor axes of the ellipse, a parallelogram that is not a rectangle is used in the construction, giving the name of the method. The association of lines of the pencils can be extended to obtain other points on the ellipse. The constructions for hyperbolas[56] and parabolas[57] are similar.



Parallelogram method for constructing an ellipse

Yet another general method uses the polarity property to construct the tangent envelope of a conic (a line conic).[58]

In the complex projective plane

Further unification is possible if one allows complex numbers as coefficients. In the complex projective plane the non-degenerate conics can not be distinguished from one another.

Over the complex numbers ellipses and hyperbolas are not distinct, since −1 is a square; precisely, the ellipse $x^2 + y^2 = 1$ becomes a hyperbola under the substitution $y = iw$, geometrically a complex rotation, yielding $x^2 - w^2 = 1$ – a hyperbola is simply an ellipse with an imaginary axis length. Thus there is a 2-way classification: ellipse/hyperbola and parabola. Geometrically, this corresponds to intersecting the line at infinity in either 2 distinct points (corresponding to two asymptotes) or in 1 double point (corresponding to the axis of a parabola), and thus the real hyperbola is a more suggestive image for the complex ellipse/hyperbola, as it also has 2 (real) intersections with the line at infinity.

It can be proven that in the complex projective plane $\mathbf{CP}^2$ two conic sections have four points in common (if one accounts for multiplicity), so there are never more than 4 intersection points and there is always one *intersection point* (possibilities: four distinct intersection points, two singular intersection points and one double intersection points, two double intersection points, one singular intersection point and 1 with multiplicity 3, 1 intersection point with multiplicity 4). If there exists at least one intersection point with multiplicity > 1, then the two conic sections are said to be tangent. If there is only one intersection point, which has multiplicity 4, the two conic sections are said to be osculating.[59]

Furthermore, each straight line intersects each conic section twice. If the intersection point is double, the line is said to be tangent and it is called the tangent line. Because every straight line intersects a conic section twice, each conic section has two points at infinity (the intersection points with the line at infinity). If these points are real, the conic section must be a hyperbola, if they are imaginary conjugated, the conic section must be an ellipse, if the conic section has one double point at infinity it is a parabola. If the points at infinity are (1,i,0) and (1,-i,0), the conic section is a circle (see circular points at infinity). If a conic section has one real and one imaginary point at infinity or it has two imaginary points that are not conjugated then it not a real conic section (its coefficients are complex).

Degenerate cases

What should be considered as a **degenerate case** of a conic depends on the definition being used and the geometric setting for the conic section. There are some authors who define a conic as a two-dimensional nondegenerate quadric. With this terminology there are no degenerate conics (only degenerate quadrics), but we shall use the more traditional terminology and avoid that definition.

In the Euclidean plane, using the geometric definition, a degenerate case arises when the cutting plane passes through the apex of the cone. The degenerate conic is either: a point, when the plane intersects the cone only at the apex; a straight line, when the plane is tangent to the cone (it contains exactly one generator of the cone); or a pair of intersecting lines (two generators of the cone).[60] These correspond respectively to the limiting forms of an ellipse, parabola, and a hyperbola.

If a conic in the Euclidean plane is being defined by the zeros of a quadratic equation (that is, as a quadric), then the degenerate conics are: the empty set, a point, or a pair of lines which may be parallel, intersect at a point, or coincide. The empty set case may correspond either to a pair of complex conjugate parallel lines such as with the equation $x^2 + 1 = 0,$ or to an *imaginary ellipse*, such as with the equation $x^2 + y^2 + 1 = 0.$ An imaginary ellipse does not satisfy the general definition of a degeneracy, and is thus not normally considered as degenerated. The two lines case occurs when the quadratic expression factors into two linear factors, the zeros of each giving a line. In the case that the factors are the same, the corresponding lines coincide and we refer to the line as a *double* line (a line with multiplicity 2) and this is the previous case of a tangent cutting plane.

In the real projective plane, since parallel lines meet at a point on the line at infinity, the parallel line case of the Euclidean plane can be viewed as intersecting lines. However, as the point of intersection is the apex of the cone, the cone itself degenerates to a cylinder, i.e. with the apex at infinity. Other sections in this case are called *cylindric sections*.[61] The non-degenerate cylindrical sections are ellipses (or circles).

When viewed from the perspective of the complex projective plane, the degenerate cases of a real quadric (i.e., the quadratic equation has real coefficients) can all be considered as a pair of lines, possibly coinciding. The empty set may be the line at infinity considered as a double line, a (real) point is the intersection of two complex conjugate lines and the other cases as previously mentioned.

To distinguish the degenerate cases from the non-degenerate cases (including the empty set with the latter) using matrix notation, let Δ be the determinant of the 3×3 matrix of the conic section: that is, $\Delta = (AC - \frac{B^2}{4})F + \frac{BED - CD^2 - AE^2}{4}$ and let $\alpha = B^2 - 4AC$ be the discriminant. Then the conic section is non-degenerate if and only if $\Delta \neq 0$. If $\Delta = 0$ we have a point when $\alpha < 0$, two parallel lines (possibly coinciding) when $\alpha = 0$, or two intersecting lines when $\alpha > 0$.[62]

Pencil of conics

A (non-degenerate) conic is completely determined by five points in general position (no three collinear) in a plane and the system of conics which pass through a fixed set of four points (again in a plane and no three collinear) is called a **pencil of conics**.[63] The four common points are called the *base points* of the pencil. Through any point other than a base point, there passes a single conic of the pencil. This concept generalizes a pencil of circles.

In a projective plane defined over an algebraically closed field any two conics meet in four points (counted with multiplicity) and so, determine the pencil of conics based on these four points. Furthermore, the four base points determine three line pairs (degenerate conics through the base points, each line of the pair containing exactly two base points) and so each pencil of conics will contain at most three degenerate conics.[64]

A pencil of conics can represented algebraically in the following way. Let C_1 and C_2 be two distinct conics in a projective plane defined over an algebraically closed field K. For every pair λ, μ of elements of K, not both zero, the expression:

$$\lambda C_1 + \mu C_2$$

represents a conic in the pencil determined by C_1 and C_2. This symbolic representation can be made concrete with a slight abuse of notation (using the same notation to denote the object as well as the equation defining the object.) Thinking of C_1, say, as a ternary quadratic form, then $C_1 = 0$ is the equation of the "conic C_1". Another concrete realization would be obtained by thinking of C_1 as the 3×3 symmetric matrix which represents it. If C_1 and C_2 have such concrete realizations then every member of the above pencil will as well. Since the setting uses homogeneous coordinates in a projective plane, two concrete representations (either equations or matrices) give the same conic if they differ by a non-zero multiplicative constant.

Intersecting two conics

The solutions to a system of two second degree equations in two variables may be viewed as the coordinates of the points of intersection of two generic conic sections. In particular two conics may possess none, two or four possibly coincident intersection points. An efficient method of locating these solutions exploits the homogeneous matrix representation of conic sections, i.e. a 3x3 symmetric matrix which depends on six parameters.

The procedure to locate the intersection points follows these steps, where the conics are represented by matrices:

- given the two conics C_1 and C_2, consider the pencil of conics given by their linear combination $\lambda C_1 + \mu C_2$.
- identify the homogeneous parameters (λ, μ) which correspond to the degenerate conic of the pencil. This can be done by imposing the condition that $\det(\lambda C_1 + \mu C_2) = 0$ and solving for λ and μ. These turn out to be the solutions of a third degree equation.
- given the degenerate conic C_0, identify the two, possibly coincident, lines constituting it.
- intersect each identified line with either one of the two original conics; this step can be done efficiently using the dual conic representation of C_0
- the points of intersection will represent the solutions to the initial equation system.

Generalizations

Conics may be defined over other fields (that is, in other pappian geometries). However, some care must be used when the field has characteristic 2, as some formulas can not be used. For example, the matrix representations used above require division by 2.

A generalization of a non-degenerate conic in a projective plane is an oval. An oval is a point set that has the following properties, which are held by conics: 1) any line intersects an oval in none, one or two points, 2) at any point of the oval there exists a unique tangent line.

Generalizing the focus properties of conics to the case where there are more than two foci produces sets called generalized conics.

In other areas of mathematics

The classification into elliptic, parabolic, and hyperbolic is pervasive in mathematics, and often divides a field into sharply distinct subfields. The classification mostly arises due to the presence of a quadratic form (in two variables this corresponds to the associated discriminant), but can also correspond to eccentricity.

Quadratic form classifications:

Quadratic forms

Quadratic forms over the reals are classified by Sylvester's law of inertia, namely by their positive index, zero index, and negative index: a quadratic form in *n* variables can be converted to a diagonal form, as $x_1^2 + x_2^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_{k+\ell}^2,$ where the number of +1 coefficients, *k*, is the positive index, the number of −1 coefficients, ℓ, is the negative index, and the remaining variables are the zero index *m*, so $k + \ell + m = n.$ In two variables the non-zero quadratic forms are classified as:

- $x^2 + y^2$ – positive-definite (the negative is also included), corresponding to ellipses,
- x^2 – degenerate, corresponding to parabolas, and
- $x^2 - y^2$ – indefinite, corresponding to hyperbolas.

In two variables quadratic forms are classified by discriminant, analogously to conics, but in higher dimensions the more useful classification is as *definite,* (all positive or all negative), *degenerate,* (some zeros), or *indefinite* (mix of positive and negative but no zeros). This classification underlies many that follow.

Curvature

The Gaussian curvature of a surface describes the infinitesimal geometry, and may at each point be either positive – elliptic geometry, zero – Euclidean geometry (flat, parabola), or negative – hyperbolic geometry; infinitesimally, to second order the surface looks like the graph of $x^2 + y^2$, x^2 (or 0), or $x^2 - y^2$. Indeed, by the uniformization theorem every surface can be taken to be globally (at every point) positively curved, flat, or negatively curved. In higher dimensions the Riemann curvature tensor is a more complicated object, but manifolds with constant sectional curvature are interesting objects of study, and have strikingly different properties, as discussed at sectional curvature.

Second order PDEs

Partial differential equations (PDEs) of second order are classified at each point as elliptic, parabolic, or hyperbolic, accordingly as their second order terms correspond to an elliptic, parabolic, or hyperbolic quadratic form. The behavior and theory of these different types of PDEs are strikingly different – representative examples is that the Poisson equation is elliptic, the heat equation is parabolic, and the wave equation is hyperbolic.

Eccentricity classifications include:

Möbius transformations

Real Möbius transformations (elements of $PSL_2(\mathbf{R})$ or its 2-fold cover, $SL_2(\mathbf{R})$) are classified as elliptic, parabolic, or hyperbolic accordingly as their half-trace is $0 \leq |\operatorname{tr}|/2 < 1,$ $|\operatorname{tr}|/2 = 1,$ or $|\operatorname{tr}|/2 > 1,$ mirroring the classification by eccentricity.

Variance-to-mean ratio

The variance-to-mean ratio classifies several important families of discrete probability distributions: the constant distribution as circular (eccentricity 0), binomial distributions as elliptical, Poisson distributions as parabolic, and negative binomial distributions as hyperbolic. This is elaborated at cumulants of some discrete probability distributions.

See also

- Circumconic and inconic
- Conic Sections Rebellion, protests by Yale university students
- Director circle
- Elliptic coordinate system
- Equidistant set
- Nine-point conic
- Parabolic coordinates
- Quadratic function

Notes

1. Eves 1963, p. 319
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4. Brannan, Esplen & Gray 1999, pp. 13–16
5. Brannan, Esplen & Gray 1999, p. 19; Kendig 2005, pp. 86, 141
6. Brannan, Esplen & Gray 1999, pp. 11–16
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8. Protter & Morrey 1970, pp. 290–314
9. Wilson & Tracey 1925, p. 130
10. the empty set is included as a degenerate conic since it may arise as a solution of this equation
11. Protter & Morrey 1970, p. 316
12. Brannan, Esplen & Gray 1999, p. 30
13. Fanchi, John R. (2006), *Math refresher for scientists and engineers* (https://books.google.com/books?id=75mAJPcAWT8C), John Wiley and Sons, pp. 44–45, ISBN 0-471-75715-2, Section 3.2, page 45 (https://books.google.com/books?id=75mAJPcAWT8C&pg=PA45)
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25. This solution was rejected by Plato on the grounds that it could not be achieved using only straightedge and compass.
26. Boyer 2004, pp. 17–18
27. Boyer 2004, p. 18
28. Katz 1998, p. 117
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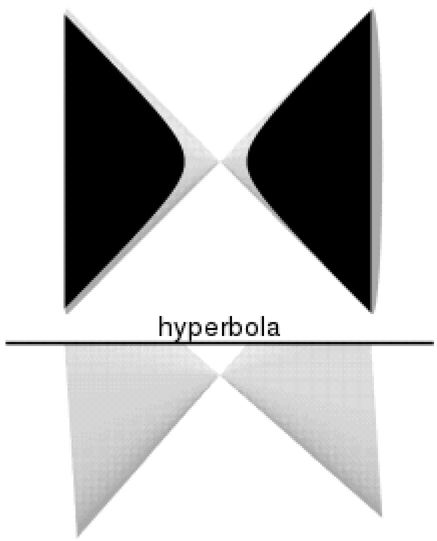

In this interactive SVG,(http://upload.wikimedia.org/wikipedia/commons/9/9a/Conic_section_interactive_visualisation.svg) move left and right over the SVG image to rotate the double cone

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37. Boyer 2004, p. 110
38. Boyer 2004, p. 114
39. Brannan, Esplen & Gray 1999, p. 27
40. This form of the equation does not generalize to fields of characteristic two (see below)
41. Consider finding the midpoint of a line segment with one endpoint on the line at infinity.
42. Faulkner 1952, p. 71
43. Faulkner 1952, p. 72
44. Eves 1963, p. 320
45. Coxeter 1993, p. 80
46. Hartmann, p. 38
47. Merserve 1983, p. 65
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53. Coxeter 1964, p. 80
54. Faulkner 1952, pp. 52–53
55. Downs 2003, p. 5
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External links

- Conic section (Geometry) (https://www.britannica.com/EBchecked/topic/132684) at *Encyclopædia Britannica*
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- Second-order implicit equation locus (http://archive.geogebra.org/en/upload/files/nikenuke/conics04b.html) An interactive Java conics grapher; uses a general second-order implicit equation.

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