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Mathematics

Are there polynomial equations that are equal to basic trig functions?

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submitted 2 years ago by SwftCurlz

Are there polynomial functions that are equal to basic trig functions (i.e: y=cos(x), y=sin(x))? If so what are they and how are they calculated? Also are there any limits on them (i.e only works when a<x<b)?

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[–] iorgfeflkd Biophysics 568 points 2 years ago (*last edited 2 years ago)

It's possible to express these functions as Taylor series, which are sums of polynomial terms of increasing power, getting more and more accurate.

(working in radians here)

For the sine function, it's sin(x)~=x-x^3 /6 + x^5 /120 - x^7 /5040... Each term is an odd power, divided by the factorial of the power, alternating positive and negative.

For cosine it's even powers instead of odd: cos(x)~=1-x^2 /2 +x^4 /24 ...

With a few terms, these are pretty accurate over the normal range that they are calculated for (0 to 360 degrees or x=0 to 2pi). However, with a finite number of terms they are never completely accurate. The smaller x is, the more accurate the series approximation is.

You can also fit a range of these functions to a polynomial of arbitrary order, which is what calculators use to calculate values efficiently (more efficient than Taylor series).

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> [–] [deleted] 60 points 2 years ago
>
> Would you mind elaborating a bit on that last paragraph?
>
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>
> > [–] iorgfeflkd Biophysics 103 points 2 years ago
> >
> > I could but I'd basically just be googling. This is the algorithm: http://en.wikipedia.org/wiki/CORDIC
> >
> > permalink embed parent
> >
> > > [–] Ganparse 87 points 2 years ago
> > >
> > > This is how calculators and computers used to calculate these functions. However, now that we want our calculators to have lots of fancy functionality a calculator practically requires hardware multiplication

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support. With hardware multiplication the Taylor series is often used instead.

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[–] **Firadin** 14 points 2 years ago

Cordic is still significantly faster than multiplication. Unless you are designing your trig functions on a software level, which you might depending on your speed requirements, cordic is still better.

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[–] **Ganparse** 60 points 2 years ago

From my understanding Cordic is only super fast when done using a specific Cordic hardware block. Since most calculators these days are simply cutting costs by using a standard micro processor which doesnt have a Cordic hardware block it is actually slower than doing the Taylor series when each method is done using typical RISC instructions.

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[–] **Firadin** 11 points 2 years ago

Yeah that's pretty much entirely true.

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[–] **djeinstine** 6 points 2 years ago

I did not know this. I probably should have checked that my micro processor had cordic hardware before switching all my trig functions to cordic in my simulink model thinking it would be faster.

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[–] **noggin-scratcher** 23 points 2 years ago

You should probably also have profiled it before attempting to optimise, so that you knew what you were starting from and could use that as a base to compare against to see the effects of changes.

Or maybe you did... but your post makes it sound like you didn't.

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[–] **georgejameson** 22 points 2 years ago

I used to write these sorts of subroutines.

As soon as you have a reasonably performant multiply, CORDIC is no longer the fastest option. Our processor had a single cycle 32-bit multiply, so CORDIC would have been maybe 30x slower than a polynomial fit.

We didn't actually use Taylor series, but it was pretty close. A Taylor series optimizes the error immediately around your reference

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point(s). We instead wanted to optimize maximal error across the entire band. So, we chopped up the range into subranges and then ran an optimizer to tweak the coefficients. This meant we could just use 3 or 4 terms in the polynomial for the same accuracy as a Taylor with many more terms.

For less well behaved functions (e.g. tangent, arcsine) we typically performed some sort of transform to avoid those awful pointy bits. For arcsine we logspaced our LUT in a way that would give more terms and resolution towards the ends.

Divide and square root were done with Newton Raphson

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[–] **srjones92** 1 point 2 years ago

Is square root ever still implemented using the "fast inverse" trick popularized by quake? Or, I guess a more general question - how common are tricks involving "magic numbers" (or similar) at this level of numerical analysis?

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[–] **b4b** 2 points 2 years ago (*last edited 2 years ago)

from what I know there are a ton of those tricks used in the so called "demoscene" where people try to create a video ("demo") that shows some cool graphics / graphic tricks + has some nice music

the so called demoscene was much more popular in the past around the time of commodore/atari and amiga computers, where all users had basically the same setup and the difference in graphics of the demo were caused by using more clever programming. Nowadays the companies just tell you to "get more ram / faster computer", so the demoscene somehow died - although there are for example 64 kilobyte games that can show "quake-like" graphics

demoscene guys had TONS of such tricks up their sleeves, nowadays such extreme programming techniques are mostly used in game development, sometimes database software (in order to deal with tons of data it needs to be optimized)... and as the guy above wrote programming of "old school" processors that are used in cheap appliances.

your typical "office program" (there as an expression for them, something like save / load / write / close) written in java is often not very optimized and written by someone who had maybe few years tops at some java school; the "real" programming does not often that much any more, everyone is using components made by smarter people and they usually just add buttons to your next program. Only guys that deal with critical systems really focus on such optimization

what is showed above, is not really programming, its more mathmatics. The best programmers usually know a lot of maths, but for your typical java program.. you dont really use much maths, just pass crap around

I dont want to even start the debate of some languages not having GOTO because it is harmful (͡° ͜ʖ ͡°)

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[–] **Tasgall** 1 point 2 years ago (*last edited 2 years ago)

There's no reason to on modern hardware. Here's a timing comparison between the standard `sqrt` function (using x87 fsqrt), the magic number, and a few different uses of SSE intrinsics.

Even if it *was* faster, people would still probably use the standard library functions, if only because the hardware itself is so much faster. Also, most situations where it was useful in the past are now done on the GPU anyway.

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[–] **muyuu** 2 points 2 years ago

Only if by "better" you mean strictly being faster at a given decimal precision (esp. with very limited hardware).

Taylor polynomials give you arbitrary precision without having to recompute any tables and you can basically choose to compute up to a given precision boundary or a given computation limit boundary.

You can also benefit from previous calculation if you have a big pool of memory like most computers and even calculators these days. For instance, all terms in sin(x) and in sinh(x) expansions are the exact same (in sinh(x) they are all added, in sin(x) they are added and subtracted in alternation - there are common computations with tan(x) as well, with exp(x), Pi, etc so all this is shared logic for fast arbitrary precision arithmetic).

Within numerical methods, CORDIC is rather niche while Taylor and similar expansions/series are all over the place.

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[+] ***TinTin0*** *comment score below threshold (3 children)*

[+] *[deleted] 2 years ago (3 children)*

[–] **_westcoastbestcoast** 9 points 2 years ago

Or additionally, you could also look at the Stone-Weirstrass theorem, which states that on a closed set, all continuous functions (here, sine and cosine are continuous) can be approximated very well by a polynomial.

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[–] **madhatta** 5 points 2 years ago

But note that the polynomial may have a very large number of terms and its coefficients may be difficult to calculate.

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[–] **SilverTabby** 20 points 2 years ago

If you have **n** points one-to-one points in 2-dimensional space, then there exists a polynomial of order **n** that passes thru all of those points.

There also exist methods to find that polynomial.

A polynomial of order n will look like:

a + b x + c x^2 + d x^3 + ... + constant * x ^{n}

So if you take enough samples of a sine curve, let's say 20 points, then you can fit a 20^{th} order polynomial that will pass thru all 20 of those points exactly. If those 20 points were chosen logically, then you can get a pretty damn good approximation of a sine wave.

It turns out that as the number of sample points you take approaches infinity, you end up with the Taylor Series mentioned above.

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[–] **sfurbo** 8 points 2 years ago

> It turns out that as the number of sample points you take approaches infinity, you end up with the Taylor Series mentioned above.

The Taylor series is derived from the derivatives at one point. What you describe is closer to Bernstein polynomials. This convergence is stronger than the convergence of Taylor series (it is uniform, not just point-wise).

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[–] goltrpoat 14 points 2 years ago

> So if you take enough samples of a sine curve, let's say 20 points, then you can fit a 20th order polynomial that will pass thru all 20 of those points exactly.

This is wrong. A 20th degree polynomial will swing wildly between the sample points. In general, the higher the degree, the less likely it is that it will do what you want when you fit it to a bunch of sample points.

What you want to do is take `int [sin(x) - p(x)]^2 dx` in some range, differentiate the result with respect to each of the coefficients, set the derivatives to 0 and solve the resulting system of equations.

For instance, the quadratic $ax^2 + bx + c$ that best approximates sin(x) on [0,pi] has the following coefficients:

```
a = (60*pi^2 - 720) / pi^5
b = -(60*pi^2 - 720) / pi^4
c = (12*pi^2 - 120) / pi^3
```

If you plot that quadratic in the [0,pi] range, it'll look like this.

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[–] [deleted] 15 points 2 years ago

What OP said is not wrong. What OP said is exactly accurate. Given twenty points, you can fit a polynomial that passes through them all exactly. OP gave no claim that the polynomial you found using this process would properly interpolate the sine curve (which, as you pointed out, it might well not).

The magic words in the u/SliverTabby's post are "If those 20 points were chosen logically" -- there are different methods of sampling points that will result in polynomials which interpolate the original curve better or worse.

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[–] goltrpoat 2 points 2 years ago

Yeah, I chose a bad quote to reply to. "Wrong" is of course the method of approximating a function on an interval, not the fact that you can fit an nth degree polynomial through n points.

> there are different methods of sampling points that will result in polynomials which interpolate the original curve better or worse.

Sure. With clever choices of sampling points, one could get arbitrarily close to the optimal method I've outlined. Not sure why one would do that, though.

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[–] [deleted] 2 points 2 years ago

> Not sure why one would do that, though.

That's an interesting question! The reason one would do that is that most of the time we're fitting a polynomial to data, we don't have the true function (in the above example, sin) available. Thus, your plan of minimizing $[\sin(x) - p(x)]^2$ doesn't work. Lots of times, though, we have a lot of discrete data to make our polynomial work, so what we do is choose a selection of points that we can guess will result in a well-behaved, non-wildly oscillating polynomial, and fit our function to those.

See: Chebyshev Nodes

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[–] **goltrpoat** 2 points 2 years ago

> The reason one would do that is that most of the time we're fitting a polynomial to data, we don't have the true function (in the above example, sin) available.

But we're specifically talking about the case when the true function is available. That's in the title of the post, and in the comment I replied to.

Fitting something reasonable to discrete data is generally treated as a whole different problem, and even there, you rarely fit an nth degree polynomial to n points. The usual approach is, roughly speaking, to fit it piecewise with low-degree polynomials whose derivatives match at the junctions (e.g. composite Bezier, B-splines, etc).

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[–] **[deleted]** 1 point 2 years ago

;) But if we're just talking about theory, why were you taking issue with u/SilverTabby, since his method works as the number of sampled points becomes the sine curve on the whole real line?

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[–] **goltrpoat** 1 point 2 years ago

What theory? I work in realtime graphics, coming up with optimal polynomial or rational approximations to ugly functions is something that pops up on a fairly regular basis for me.

As a nitpick, the number of sampled points can't become the sine curve, it can only become a countable subset of it. It's not immediately clear to me that fitting an nth degree polynomial to n points spits out the Taylor series as n goes to infinity, since I would expect the squared error to actually grow with n (assuming an arbitrary function and a choice of sample points that is independent of the function).

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[–] **trainbuff** 3 points 2 years ago

Don't n points determine a polynomial of degree n-1?

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[–] **SilverTabby** 1 point 2 years ago

a line thru two points would be

f(x) = a + b*x

...so yeah you're right it would be n-1.

I'm an engineering undergraduate, not a mathematician. Subtleties like this are considered "rounding error"

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[–] **plotifer** 1 point 2 years ago

Don't know if I'm stretching this, but is there any connection between this and the nyquist sampling rate? Assuming we are dealing with an oscillating function like sin or cos, if I were to somehow always pick 20 points such that I had more than one per cycle, would that be objectively better than 20 points picked once per cycle (or less)?

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[–] **Enthused_Llama** 2 points 2 years ago

You'll get a more accurate fit with more points, I'd imagine, but the Fourier Series/Transform stuff works on converting functions into distinct sine and cosine terms (with the transform taking it into the complex/frequency domain), so trying to use a polynomials sort of seems like a step backwards.

The Nyquist Frequency (at least as I learned it) is determined from the sampling rate, not the other way around. You look at the signal in the time domain, determine what the maximum frequency is, and then fsample >= 2fmax, with fnyquist=0.5fsample (ordinarily fsample is not just 2fmax but 3 or 5fmax). Any frequencies that you get out of the transform that exceed fnyquist are lies, basically.

So uh, no, I don't think they're really related.

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[–] brwbck 7 points 2 years ago

There are a number of things you can leverage to make the approximation cheaper without giving up accuracy. For instance, one cycle of a sine or cosine function can be broken up into four quarters which are just flips and inversions of each other. So you don't have to have high accuracy approximation over an entire cycle, just a quarter cycle.

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[–] Mazetron 3 points 2 years ago

In order to have your series of polynomials be exactly equal to the actual sin function everywhere, you need to have an infinite number of terms in your series (so you can never get there).

However, it is possible to get close. When you have more and more terms in your series, the values from the series get closer and closer to the actual sin values, and the range for which the values are fairly accurate gets bigger and bigger. With only a few terms, it is possible to get a fairly accurate approximation for small portions of the sin function (say -90 degrees to 90 degrees). Fortunately for us, the sin function repeats itself, so if we have a small piece of the function, we can calculate for all values of the function. This makes the Taylor series a very practical method for calculating the values of sin. In fact, computers and calculators often use Taylor series for trig calculations.

If you have a graphing calculator or program you can experiment with this yourself. If you have a Mac, type "grapher" into Spotlight. Otherwise, maybe try Wolfram Alpha or something. Graph the Sin function. Then, graph x on top of it. Then x-(x$^{3)/6.}$ Then x-(x$^{3)/6+(x^{5)/120.}}$ The pattern is that the nth term will be ((-1)$^{n)*(x^{2n+1})/((2n+1)!),}$ starting with n=0. You will see how the series approaches the sin function as you add more terms.

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[–] slicedclementines 6 points 2 years ago

If you were to sample a few hundred points over some interval a<=x<=b, and then find the interpolating polynomial that connects these points, would it be roughly equal to the taylor approximation or would it be something different altogether?

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[–] **[deleted]** 8 points 2 years ago

WhatWhatWhatYeahWhat is absolutely right about polynomial interpolation being inaccurate, they are useful but up to a point. To really use polynomial interpolation, you need to divide your domain into smaller sections that are reasonably small and at the most use a third power polynomial approximation. (This method can be used, also to simplify calculations, people also use what is called the "Spline Method.")

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[–] **fizban99** 2 points 2 years ago

That may be how *you* find polynomial interpolation useful, but there are certainly applications that don't do that. See error correcting codes, for example.

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[–] **grumbelbart2** 4 points 2 years ago

The difference is that error correcting codes operate on discrete spaces, such as Z_n, while sin, cos and the corresponding interpolating polynomials (and likely what /u/hpdicon1 had in mind) are defined over the continous set R.

If you fit a polynomial of order 20 into 20 points sampled from sin(x), you'll end up with a polynomial that is exactly sin(x) at those 20 locations, but oscillates pretty drastically in between those points. It's thus rather useless for most applications.

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[–] **iorgfeflkd** Biophysics 3 points 2 years ago

I don't know, try it out!

With a Taylor series each term gets smaller and smaller, that might not be the case with an arbitrary fit to some range.

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[+] ***AmyWarlock*** *comment score below threshold (4 children)*

[–] **sakurashinken** 8 points 2 years ago

I'm surprised nobody has mentioned Chebyschev Polynomials which are essentially higher order multiple angle formulas for cosine.

http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html

While these are not expansions, they are fascinating.

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[+] *[deleted] 2 years ago (2 children)*

[–] **joemalola** 4 points 2 years ago

This is exactly what I need for my C programming lab! Thank you.

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[–] **RIPphonebattery** 2 points 2 years ago

Out of curiosity, what's your programming lab?

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[–] **joemalola** 1 point 2 years ago

We weren't allowed to use #include<math.h> and their functions but had to calculate the sin, cos, and tangent of certain inputs in degrees. Ended up using for loops fah dayz.

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[–] plotifer 4 points 2 years ago

I finally truly understand why Sinx can be approximated as x for small angles. I was never told of or made the connection to the Taylor series.

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[–] Scenario_Editor 1 point 2 years ago

What's neat is that you can get it both from the Taylor series or by approximating it as arclength with r*theta=s by realizing that your triangle is close to a skinny isosceles triangle, which is almost like a circle. The skinny isosceles thing comes up again when dealing in infinitesimal changes in angle in curved coordinates.

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[–] plotifer 1 point 2 years ago

Oh, true! That's where I learned it first. I completely forgot about that. Now I realize I totally did know where it came from :(

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[–] B1ack0mega -3 points 2 years ago

It's not even Taylor series really, it's a lot simpler. The gradient of the sin curve at x = 0 is 1 (since d/dx(sin(x)) = cos(x)), so we can approximate it for small values of x (i.e., small angles), by the straight line of gradient 1 through the origin. Of course, that's just y = x.

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[–] physicsdood 3 points 2 years ago

That is the first term in the Taylor series. It's not any simpler; it's the Taylor series.

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[–] B1ack0mega 3 points 2 years ago

Well of course, but you can explain it the way I did without going into Taylor series. We don't do Taylor series in the UK until university (Maclaurin in Further Maths at college). I don't need any more knowledge than the ability to draw a tangent to sin(x) at x = 0 and calculate its gradient.

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[–] ximeraMath 2 points 2 years ago

Linear approximations essential to understanding the derivative. Taylor series are much higher on the abstraction scale compared to derivatives (you need to repeatedly differentiate, and understand series, integration to get the error terms, etc.). So I think that B1ackOmega is correct in saying the linear approximation is simpler.

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[–] plotifer 1 point 2 years ago

Ah, that makes sense too. Intuitively, for me at least, it's actually easier to understand using the Taylor Series, even if it may not necessarily be correct.

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[–] TheNiceGuy14 3 points 2 years ago

Could we represent a sin function by an infinite product of its root? When we factorize, we get the zeros. And we do know all of them in sin since they're periodic.

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[–] [deleted] 2 points 2 years ago

It's also pretty cool that the taylor series for the hyperbolic functions are related:

```
sinh(x) = x^1/1! + x^3/3! + x^5/5! + x^7/7! ...
cosh(x) = x^0/0! + x^2/2! + x^4/4! + x^6/6! ...
```

In fact, you can get from sin(x) to sinh(x) by introducing a complex factor:

```
sinh(x) = -i * sin(ix)
cosh(x) = cos(ix)
```

One of my favorite excersizes is to find the eigenvalues of a 2x2 rotation matrix and the related 2x2 "hyperbolic rotation" matrix:

```
[cos(x) -sin(x)]
[sin(x)  cos(x)]

[cosh(x) sinh(x)]
[sinh(x) cosh(x)]
```

The way these functions are related and what pops out is just too cool.

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[+] *[deleted] 2 years ago (6 children)*

[–] kennensie 1 point 2 years ago

this is true that you can get arbitrarily close to a trig function with a taylor series, but by definition you cannot express them as a polynomial because they are a transcendental function.

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[–] sakibomb222 1 point 2 years ago

Just a minor correction, I believe you meant "alternating positive and negative" instead of "alternating even and odd".

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[–] iorgfeflkd Biophysics 2 points 2 years ago

Yes, that's what I meant. Thanks.

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[–] ritz_are_the_shitz 1 point 2 years ago

Couldn't you use a limit to perfectly approximate this?

It's been a long time since I took calc 1/2, I don't do series much anymore...

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[–] iorgfeflkd Biophysics 1 point 2 years ago

You can take the sum to infinity.

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[–] Egren 1 point 2 years ago

I threw together a spreadsheet giving the correct function depending on "how high you want to go". Here it is.

The interesting column is G, where the resulting function can be found.

The function is copypasteable into fooplot and *should* work properly, although there doesn't seem to be much change after *(x^{45} / ($1.19622x10^{56}$))*. It should definitely be enough to show the concept, though.

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[–] iorgfeflkd Biophysics 1 point 2 years ago

And really one just cares about the 0 to pi/2 range; after that symmetry takes care of the rest.

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[–] B1ack0mega 0 points 2 years ago

To tag on, the answer is no, because sin(x) is a transcendental function. It "transcends algebra", because it can't be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and taking nth roots. In order to have such an expression, it must be infinite (i.e., a Taylor series).

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[–] DarylHannahMontana Mathematical Physics | Elastic Waves 104 points 2 years ago (*last edited 2 years ago)

No, the Taylor series is the closest thing, as others have pointed out.

To see that no polynomial (i.e. with a finite number of terms) can equal sine or cosine for all x, simply observe that both trig functions are always between -1 and 1, and that all (non-constant) polynomials are unbounded (any polynomial is dominated by its leading term x^n, and as x goes to infinity, the polynomial must go to either positive or negative infinity).

To show that no finite polynomial can be exactly equal to sine or cosine on a restricted interval a < x < b (with a < b) is a little more subtle, but here's the basic idea:

- Taylor series are unique*.
- Sine and cosine both have a Taylor series on any interval (a,b), and both series have infinitely many non-zero terms.
- If sine was equal to a polynomial (finitely many terms), then this would be a *different* Taylor series for sine (a polynomial can be viewed as an infinite series with only finitely many non-zero terms), contradicting the first fact. Same with cosine.

*: It's maybe worth noting that there can be different polynomial *approximations* to a function on an interval (i.e. distinct polynomials that are *close* to the original function), but no two distinct polynomials (infinite or otherwise) can be *equal* to the function.

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[–] swws 53 points 2 years ago (*last edited 2 years ago)

An easier proof of the second half (that no polynomial can equal sine or cosine even locally) is that if you repeatedly differentiate any polynomial, eventually all the derivatives will be identically zero. But the iterated derivatives of sine and cosine repeat cyclically (sin -> cos -> -sin -> -cos -> sin -> ...), so they will never become identically zero, even just on an interval.

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[–] DarylHannahMontana Mathematical Physics | Elastic Waves 5 points 2 years ago

Ahh, of course. Thanks for adding this.

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[–] NimbusBP1729 16 points 2 years ago

this is one of the few answers that has an ELI15 proof for why sin(x) can't be represented as a sum of finite polynomials. nicely done.

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[–] Oripy 10 points 2 years ago

An other attempt using an other approach:

A sum of finite polynomials have a finite number times it crosses the zero line whereas the sin(x) function crosses the zero line a infinite number of times.

In mathematical terms:

If P(x) is a polynomial of degree n then P(x) will have exactly n zeros (some of which may repeat).

sin(x) has an infinite number of zeros : sin(x) = 0 is true for x = 0 mod pi

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[–] OldWolf2 2 points 2 years ago

It takes the uniqueness of Taylor series as an axiom though; proving that is more complicated than the original question!

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[–] DarylHannahMontana Mathematical Physics | Elastic Waves 3 points 2 years ago

Another person chimed in with an even simpler proof:

Differentiating a polynomial repeatedly will eventually yield zero.

Differentiating sine or cosine repeatedly will not.

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[–] NimbusBP1729 2 points 2 years ago

it only takes that as a given for the proof of nonequality over a finite interval. his infinite interval proof is simpler and answers a portion of OP's question too.

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[–] GOD_Over_Djinn 30 points 2 years ago

The answer is no. No polynomial is equal to sin(x), for instance. However, the Taylor series of the sine function

$P(x) = x - x^3/6 + x^5/120 + ...$

can be thought of as kind of an "infinite polynomial", and it is exactly equal sin(x). If we take the first however many terms of this "infinite polynomial", we obtain a polynomial which approximates sin(x) for values "close enough" to 0. The more terms we take, the better the approximation is for terms close enough to 0, and the farther away from 0 the approximation works.

Lots of functions have Taylor series, and you learn how to construct them in a typical first year calculus class.

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[–] you-get-an-upvote 0 points 2 years ago

May be wrong but I'll make the stronger claim that "every function continuous on a given interval can be approximated by a Taylor series on that interval (centered on any value that belongs to the domain)".

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[–] browb3aten 19 points 2 years ago

Nope, it also has to be at least infinitely differentiable on that interval (well, also complex differentiable to guarantee analyticity).

For example, f(x) = |x| is continuous everywhere. But if you construct a Taylor series at x = 1, all you'll get is T(x) = x, obviously diverging for x < 0.

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[–] SnackRelatedMishap 12 points 2 years ago

Correct.

But, any continuous function on a closed interval *can* be uniformly approximated by polynomials, per the Stone-Weierstrass theorem.

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[–] swws 9 points 2 years ago

Infinite differentiability is also not sufficient to get a Taylor series approximation. For instance, let f(x)=exp(-1/x) for nonnegative x and f(x)=0 for negative x. This is infinitely differentiable everywhere, but its Taylor series around 0 does not converge to f(x) for any x>0 (the Taylor series is just identically 0).

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[–] browb3aten 6 points 2 years ago

I didn't say it was sufficient. It's still necessary though.

Complex differentiability is both.

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[–] GOD_Over_Djinn 2 points 2 years ago

This is not true. What is true as that and continuous function on a closed interval can be approximated by polynomials, but these polynomials might not be close to as easy to find as a Taylor polynomial. This result is called the Weierstrass Approximation Theorem. A more general result called the Stone-Weierstrass theorem looks at which kinds of sets of functions have members that can approximate arbitrary continuous functions; for instance, we know that polynomials can approximate functions via their Taylor series, but we also know that linear combinations of powers of trig functions can approximate functions via their Fourier series. What is it about polynomials and trig polynomials that allows this to happen? The Stone-Weierstrass theorem answers this question.

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[–] thatikey -1 points 2 years ago

Technically that's the Maclaurin Polynomial. I'd just like to add that's it's also possible to estimate how far the result is from the true answer, so you could construct the polynomial with a sufficient number of terms to be correct to within a certain number of decimal places

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[–] B1ack0mega 7 points 2 years ago

Maclaurin series is just the Taylor series at 0, though. I only ever heard people call them Maclaurin series at a very basic level (A-Level Further Maths). After that, it's just a Taylor series at 0.

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[–] lsdkljdsfsd 12 points 2 years ago (*last edited 2 years ago)

The other commenters have said anything I could say already, but I thought I'd add in this link for visualization purposes:

http://www.wolframalpha.com/input/?i=graph+sum+from+n+%3D+1+to+3+of+%28-1%29^%28n+%2B+1%29+*+x^%282n+-+1%29+%2F+%282n+-+1%29!+and+sin%28x%29+for+-10+%3C+x+%3C+10

That will make Wolfram|Alpha graph the Taylor series approximation of sin(x) to a certain degree, and also plot sin(x) for comparison. To make the Taylor approximation more accurate, just increase the "3" in the equation. It will calculate the first "3" (Or whatever you make it) terms of the Taylor series for sin(x). You'll see it gets extremely accurate for small x, and its range of accuracy increases as the number of terms do. By the time you add 14 terms, you can't even tell the difference anymore in the graph.

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[–] Frobeniu5 7 points 2 years ago (*last edited 2 years ago)

No, trigonometric functions are examples of transcendental functions, which not only can not be written as polynomials, but are also not solutions to polynomial equations.

The closest thing to what you ask for is a Taylor series, which is a kind of infinite polynomial. We have

$\sin(x) = x - x^3 / 3! + x^5 / 5! - x^7 / 7! + x^9 / 9! - \ldots$

$\cos(x) = 1 - x^2 /2! + x^4 / 4! - x^6 / 6! + x^8 / 8! - \ldots$

(here n! as usual is the product of the first n natural numbers)

Generally when you have a series representation, there are some limits on what x can be, but for these two x can be anything. You can derive these formulas yourself using

$e^x = 1 + x + x^2 / 2! + x^3 / 3! + x^4 / 4! + \ldots$

and the fact that $e^{ix} = \cos(x) + i \sin(x)$.

Just substitute ix for x in the formula for $e^{x,}$ and group the resulting real and imaginary terms on the right hand side together. The real part will be the series expansion of cos(x), the imaginary part will be the that of sin(x).

You can see from these series expansions that there can be no polynomial expression for cos(x) and sin(x). If there were, that polynomial would have to equal the series expansion, which is impossible.

Not just the basic trig functions, but all the rest, such as tan(x), cot(x), their inverses, and even their hyperbolic versions are all transcendental. This is one reason why we give them special names.

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[–] B1ack0mega 1 point 2 years ago

Can't believe I had to scroll down this much to find the word transcendental. I thought I had gone mad and forgotten what it really meant.

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[–] wall_words 1 point 2 years ago

This is the only post in the thread that actually answers the question.

question.

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[–] **Kymeri** 21 points 2 years ago

As many others have pointed out, an infinite Taylor Series is equal to the functions of sine and cosine.

However, it may be interesting to note that any polynomial (in fact any function at all) can also uniquely be represented by an infinite series of sine or cosine terms with varying periods, also called a Fourier Series.

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[–] **dogdiarrhea** Analysis | Hamiltonian PDE 16 points 2 years ago

> (in fact any function at all)

Function must be square integrable.

You do not need to use sine and cosine, just an infinite set of orthogonal functions under some weight. The Chebyshev polynomials would also work, for example.

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[–] **shaun252** 1 point 2 years ago

How is this idea compatible with the taylor series, is 1, x, x^2 etc a complete orthonormal basis for L^2 . If I take the inner product of a function with these basis functions will I get the formula for the taylor series coefficients?

Also why is square integrability necessary to expand a function in a basis?

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[–] **dogdiarrhea** Analysis | Hamiltonian PDE 1 point 2 years ago (*last edited 2 years ago)

It isn't, the person just mentioned it as another way of approximating functions. 1, x, $x^{2...}$ Cannot be made orthogonal under any weight I think, for example let 0= <x,x^3 >=int(x*x^3 *w(x) dx)=<x^2,x^2>

Making x and x^3 orthogonal would make the norm of x^2 0, unless I've made a mistake.

On second thought, I'm not sure what the requirements for a Fourier series were, you certainly need that int(f(x) sin(kx)) and iny(f(x) cos(kx)) to be bounded on whatever interval you're expanding on to get the Fourier coefficients, and I remember square integrability being needed but looking at it again absolute integrability should be what's needed. There's going to be other conditions needed for convergence as well, my main point was that it is not the case that *any* function can be expanded in a Fourier series.

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[–] **shaun252** 1 point 2 years ago

Given that 1,x,x^2 .... do form a linear independent basis of a vector space per http://en.wikipedia.org/wiki/Monomial_basis, what happens if I gram-schmidt it? Is there a problem with it being infinite dimensional?

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[–] **SnackRelatedMishap** 2 points 2 years ago

No, that's exactly what one would do.

Given a closed interval K on the real line, we start with the standard basis, and by

Gramm-Schmidt we can inductively build up a (Hilbertian) orthonormal basis for L^2 (K).

There's a free Functional Analysis course being offered on Coursera right now which you may wish to check out. The first few weeks of the course constructs the Hilbert space and its properties.

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[–] **shaun252** 1 point 2 years ago

Thanks, is there a special name for this specific basis?

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[–] **SnackRelatedMishap** 1 point 2 years ago

Not really. The orthonormal set produced by Gramm-Schmidt will depend entirely upon the closed interval K; different intervals will give different sets of polynomials. And, there's nothing particularly special about the basis one obtains through this process -- it's just one of many such orthonormal bases.

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[–] **shaun252** 1 point 2 years ago

Why do we have special orthogonal polynomials then. Is it just because when certain functions are projected onto to them they have nice coefficients?

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[–] **SnackRelatedMishap** 1 point 2 years ago (*last edited 2 years ago)

If you're referring to Hermite, Chebyshev, Legendre etc... polynomials, these are orthonormal sets that also happen to satisfy ordinary differential equations.

These are useful when you want to express a solution of an ODE in terms of orthonormal basis functions which also satisfy the ODE.

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[–] **dogdiarrhea** Analysis | Hamiltonian PDE 1 point 2 years ago

Gram-Scmidt away! There are certainly orthogonal polynomial bases out there. As I mentioned the Chebyshev polynomials are an example. Gram-Schmidt does certainly work in infinte dimensions, keep in mind here an important part is also choosing an appropriate weight function. There's probably better tools for finding these things and they'd typically be done in courses on functional analysis, Fourier analysis, or numerical analysis.

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[–] aczelthrow 1 point 2 years ago

> You do not need to use sine and cosine, just an infinite set of orthogonal functions under some weight. The Chebyshev polynomials would also work, for example.

Pedantic point: Orthogonality makes the analysis easier, connects solutions to areas of ODEs and PDEs, and imparts a useful interpretation of truncation, but a set of linearly independent basis functions need not be orthogonal to be able to represent other functions via infinite series.

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[–] timeforanargument 8 points 2 years ago

It's an infinite polynomial. But there is a transform that converts it to an imaginary exponential form.

cos(x) = (1/2)[exp(ix) + exp(-ix)]

sin(x) = (1/2i)[exp(ix) - exp(-ix)]

From a basic math point of view, cosine and sine have an infinite number of roots. Therefore, whatever polynomial represents these trig functions will also have an infinite number roots. And that's why we have the Taylor series.

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[–] vambot5 3 points 2 years ago

Applying calculus principles, you can use infinite series that equal the trigonometric functions. You can use a finite sum of these series to approximate values of the trig functions. I haven't used these in a few years, but a practicing mathematician or engineer would know the series formulae.

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[–] vambot5 3 points 2 years ago

My high school math mentor did not have us memorize the common series of this type, called Taylor Series. Instead, he just taught us how to derive them by taking repeated derivatives until we found a pattern. This was solid mathematics, but on the AP exam for BC Calc we were creamed by those who had simply memorized the common series and could apply them without any extra work.

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[–] microphylum 3 points 2 years ago

You can "derive" the basic ones quickly in your head using geometric intuition. For instance: the graph of cos x intersects the y axis at a maximum, y=1. So the series begins with 1, or $y=+1x^0$ / 0!

The next term can't be of x^1 order since the derivative of cos is sin, and sin 0=0. So it must go x^0, x^2, x^4...

is sin, and sin 0=0. So it must go x^0, x^2, x^4 ....

Thus you can use that fact to recall cos x = 1 - x^2 / 2! + x^4 / 4! - ... No memorization needed beyond remembering how the graph of cos x looks.

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[–] _TheRooseIsLoose_ 1 point 2 years ago

I'm teaching ap calc and this is the daily wreckage of my soul. I want to teach them, have them understand fully, and have them probe/derive everything they do. The ap curriculum structure strongly opposes that. It's not nearly as horrible of a test as people expect but it *is* very strongly oriented towards future engineers.

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[–] thbb 3 points 2 years ago

I'm surprised no one mentioned parametric methods to represent functions, and rational forms. While more powerful than polynomials, they let you represent (not just approximate) transcendental functions using just finite algebraic expressions. see http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/curves/rational.html for instance.

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[–] ReverseCombover 3 points 2 years ago

You know how you can factor the polynomials by their zeros like how you can write p(x)=x^2-3x+2=(x-1)(x-2)? well the sin function has infinite zeros so if you had a polynomial if you were to factor it you would end up with infinite factors, Euler just assumed he could, basically he factorized the zeros of the function sin(x)/x ending up with an infinite product, he used this to calculate the sum of 1/n^2=pi^2/6 it was 100 years after he calculated this value that it was shown he could actually do this by Weirestras you can read more about it here http://en.wikipedia.org/wiki/Basel_problem on the section Eulers approach.

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[–] Zosymandias 3 points 2 years ago

Everyone here is trying to show that there is some silly construction that is a polynomial approximation for the Sin or Cos functions but as many of us in the thread are aware there isn't one.

So lets do the important step and prove one doesn't exist! Now before anyone gets on me for being inexact this is a "hand wavey" proof just to get the idea out there.

So what do we know about the end behavior of polynomials? Eventually no matter how many terms they have to go off to Infinity of negative Infinity. But now what about the end behavior of the Sin and Cos functions? They continue to oscillate off into Infinity. Now I think from this we can all see a problem with construction of a polynomial we will never be able to get the same end behavior.

Side Note: The Taylor series expansion on the other hand isn't a polynomial because of the Infinite sum which allows it to get around my "proof" and it does equal the function if you where to evaluate it infinity. Which if you can... I have some stuff I need computed.

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[+] *[deleted] 2 years ago (5 children)*

[–] cheunger 5 points 2 years ago

No! Polynomials have the important property that they have at most as many roots as its degree. Sin(x) has infinity roots, so it cannot be a many roots as its degree. Sin(x) has infinity roots, so it cannot be a polynomial. Another thing is that if it were a polynomial, you could differentiate it up to degree n times and get the zero function! The second property is better for seeing that it cannot be agree with a polynomial even in any interval (a,b)

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[–] the_integral_of_man 13 points 2 years ago (*last edited 2 years ago)

Finally my Linear Algebra 2 class will pay off!

Many of you offer that the Taylor Series representation is the closest approximation to a trig function when in fact there is one that is EVEN closer! WARNING VERY ADVANCED MATH AHEAD!

Here's our goal: We are going to find a polynomial approximation to the sine function by using Inner Products. The Theorems used are long and require some background knowledge, if you are interested PM me.

Here we go: Let *v* in *C*[-π,π] be the function defined by *v(x)*= sin x. Let *U* denote the subspace of *C*[-π,π] consisting of the polynomials with real coefficients and degree at most 5. Our problem can now be reformulated as follows: find *u* in *U* such that ||*v-u*|| is as small as possible.

To compute the solution to our approximation problem, first apply the Gram-Schmidt procedure to the basis (1 ,x,x^2 ,x^3 ,x^4 ,x^5) of *U*, producing an orthonormal basis (*e1,e2,e3,e4,e5,e6*) of *U*.

Then, again using the inner product given: <*f,g*>= the integral from -π to π of *f(x)g(x)dx*, compute *Puv* by using: *Puv*= <*v,e1*>*e1*+...+<*v,en*>*en*.

Doing this computation shows that *Puv* is the function: 0.987862x-0.155271x^3+0.00564312x^5

Graph that and set your calculator to the interval [-π,π] and it should be almost EXACT!

This is only an approximation on a certain interval ([-π,π]). But the thing that makes this MORE accurate than a Taylor Series expansion is that this way uses an incredibly accurate computation called Inner Products.

PM me any questions on this I am an undergrad student and I have a very good understanding of Linear Algebra.

Edit: the Taylor Series expansion x-x3 /6 + x5 /120. Graph that on [-π,π] and you will notice the the Taylor Series isn't so accurate. For example look at x=3 our approximation estimates sin 3 with an error of 0.001 but the Taylor Series has an error of 0.4. So the Taylor Series expansion is hundreds of times larger than our error estimation!

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[–] marpocky 7 points 2 years ago

> Many of you offer that the Taylor Series representation is the closest approximation to a trig function when in fact there is one that is EVEN closer!

/u/tedbradly addressed why this is a nonsensical statement, but left out the point that the Taylor series representation is not an approximation at all. It's actually equal to the function, if you carry out the infinite summation of terms.

The Taylor *polynomial* of any given degree is an approximation, but nobody ever claimed it was the best one by all possible metrics. Of course no one function will be.

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[–] esmooth 1 point 2 years ago

> It's actually equal to the function, if you carry out the infinite summation of terms.

In the real case even that's not true for all infinitely differentiable functions.

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[–] the_integral_of_man -1 points 2 years ago

Please read. I gave you a closer approximation on an INTERVAL. Of course the Taylor Series is exact sine expanded to infinity.

Did you graph my function compared to the Taylor Series one? You can see the error on the given interval.

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[+] *[deleted] 2 years ago (8 children)*

[–] jedi-son 3 points 2 years ago

Nicely done. In terms of L2 closeness this is optimal

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[+] *[deleted] 2 years ago (2 children)*

[–] pokelover12 3 points 2 years ago

Nope, thats the definition of a transcendental function. A function that cant be expressed as a finite degree polynomial.

The best you can do is approximate using taylor aeries.

Look up taylor series if you have calculus under your belt. If not, learn calculus then come back to this question.

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[–] Nevermynde 3 points 2 years ago (*last edited 2 years ago)

Forget all the dribble about Taylor series. Taylor series are local properties: they make sense in an asymptotically small neighborhood of a point. I don't think that's what you are after.

Functions like cosine and sine have a much more powerful property: they are analytic, meaning that they are the limit of a power series. Intuitively speaking, they are a kind of "infinite-degree polynomials". Thanks to that property, you can do a bunch of algebra and calculus with them (almost) as easily as if they were polynomials.

So trig functions are almost as "regular" or "well-behaved" as polynomials, with the exception that they don't have a null finite-n-th derivative.

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[–] **[deleted]** 1 point 2 years ago

I think Khan Academy will be the best resource you can find to answer this question, I actually remembered this video, and this video was by far the best explanation of how to understand Taylor Series and the power they have to approximate things, (functions and other extremely small quantities). This video was literally made to answer and explain your question....What a Taylor Series is and How it works as an approximation method

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[–] PetaPetaa -1 points 2 years ago

Yes! A brilliant question my lad. This is the precise application of the Taylor series! Please, one quick google with a Kham Academy tag should enlighten you :) The application is not limited to trig functions, it can also just be used to write out small quantities!

It's a rather brilliant method that is used extensively in the derivation of common formulas. For example, when calculating the electric potential of a dipole(a system of a +charge and a -charge,) one's initial answer is

a rather ugly term, one with a trig on top and a demoninator written as a rather ugly term, one with a trig on top and a denominator written as the sum of some small quantities all under a square root sign. It turns out there is a taylor approximation for $(1+x)^{-1/2}$, where x is a small quantity, that allows us to rewrite the equation.

Now, this might seem trivial but at the end of the day we've taken a rather ugly definition that has little physical insight and we've rewritten it with a taylor expansion to get it into a form that lets us actually see important physical insight! In this case, relevant information that is derived from the taylor expansion that cannot be seen in the original equation would be that the potential of the dipole is proportional to ql, the product of charge and the distance between them, that it is proportional to 1^{r^3}, and that it is proportional to cos (theta).

In general, the Taylor series expansion shows up quite often in physical derivations to rewrite equations into a more useful, meaningful form.

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[–] GOD_Over_Djinn 6 points 2 years ago

The reason for the downvotes (I didn't downvote, by the way), is that the answer is actually not "yes". A Taylor series is not a polynomial. A polynomial is a **finite** sum of the form $ax^n + bx^{n-1} + ... + cx + d$. A Taylor series is an **infinite** sum of such terms. If you choose finitely many terms from a Taylor series, sure enough, you end up with a polynomial, and if you choose nice ones then you'll even end up with a polynomial that looks very much like the function like its Taylor series, but the two functions are not *equal* unless you take all infinitely many terms of the Taylor series, in which case you do not have a polynomial.

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[–] Mr_New_Booty 2 points 2 years ago

OP, another use of the Taylor series that is very well known is the proof of Euler's Identity. There are lots of things that have a Taylor Series thrown into the proof. I can't even begin to recall all the proofs I've seen with Taylor Series in them.

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[–] PetaPetaa 1 point 2 years ago

Yep. The deeper you get in a given field, using Taylor series in derivations really becomes less of an oddity and more of a consistent method of rewriting (really just approximating) ugly equations.

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[–] Tylerjb4 -2 points 2 years ago

Everyone seems to be going at this from a calc 101 point of view with taylor series. In differential equations we learn "using" (really its just manipulating) Eulers formula it is possible to solve for sin(x) where $\sin(x)= (e^{ix}-e^{-ix})/2i$

edit: The derivation or proof of eulers formula is about as beautiful as math can get. Everything you have learned in years of schooling pulls together into this Eureka moment.

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[–] AmyWarlock 4 points 2 years ago

They're probably doing that because the question was in regards to polynomials, not exponentials

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[–] Tylerjb4 -2 points 2 years ago

Technically yes, you are correct there. But I would assume this would still be an answer that op would be interested in. I kind of doubt he literally meant only polynomials and nothing but polynomials. I would infer that "polynomial" in his question meant some numerical expression

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[–] felixar90 -1 points 2 years ago

Euler's identity almost seems magical in some way. If there is such a thing as mathematical beauty, it's when three apparently completely unrelated constants come together to make $e^{i\pi} + 1 = 0$.

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[–] Gate_surf 0 points 2 years ago

By definition, the trig functions cannot be expressed exactly as a polynomial function. Check out this definition of a transcendental function from Wolfram:

> A function which is not an algebraic function. In other words, a function which "transcends," i.e., cannot be expressed in terms of, algebra. Examples of transcendental functions include the exponential function, the trigonometric functions, and the inverse functions of both.

Like most of the posts here are saying, you can get close enough with approximations, but you can't come up with an algebraic function that is equivalent. You can unwrap the definitions of algebraic functions, roots of polynomials, etc, to see exactly what this means. But, the gist of it is that there are no polynomials that will be exactly equal to a trig function at every point.

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[–] Frexxia 2 points 2 years ago (*last edited 2 years ago)

The fact that trigonometric functions aren't algebraic is a theorem, not a definition.

edit: However, the result that OP asks about is much simpler. For instance, you can immediately see that sin and cos aren't polynomials, because they are bounded (and not constant).

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