

Introduction to Programming (in C++)

Multi-dimensional vectors

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Matrices

- A matrix can be considered a two-dimensional vector, i.e. a vector of vectors.

my_matrix:

3	8	1	0
5	0	6	3
7	2	9	4

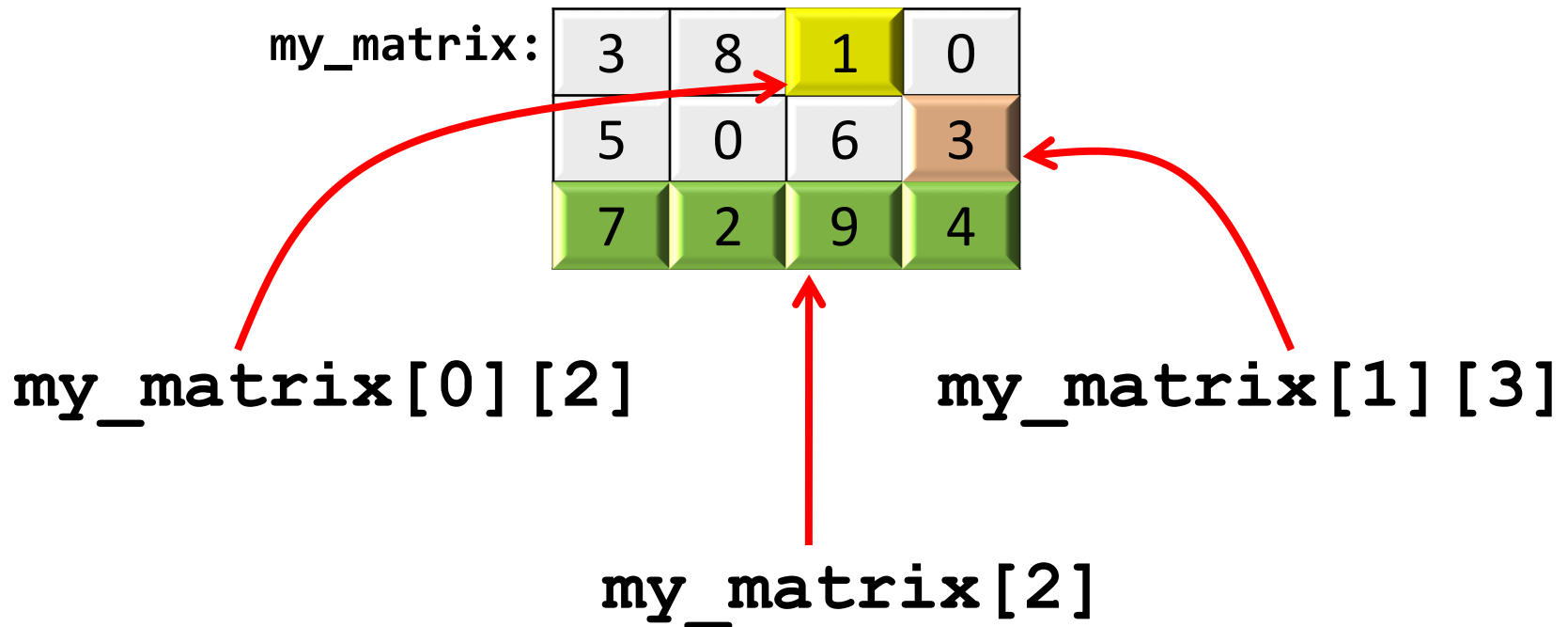
```
// Declaration of a matrix with 3 rows and 4 columns  
vector< vector<int> > my_matrix(3,vector<int>(4));
```

```
// A more elegant declaration  
typedef vector<int> Row;      // One row of the matrix  
typedef vector<Row> Matrix;  // Matrix: a vector of rows
```

```
Matrix my_matrix(3,Row(4));  // The same matrix as above
```

Matrices

- A matrix can be considered as a 2-dimensional vector, i.e., a vector of vectors.



n -dimensional vectors

- Vectors with any number of dimensions can be declared:

```
typedef vector<int> Dim1;  
typedef vector<Dim1> Dim2;  
typedef vector<Dim2> Dim3;  
typedef vector<Dim3> Matrix4D;  
  
Matrix4D my_matrix(5,Dim3(i+1,Dim2(n,Dim1(9))));
```

Sum of matrices

- Design a function that calculates the sum of two $n \times m$ matrices.

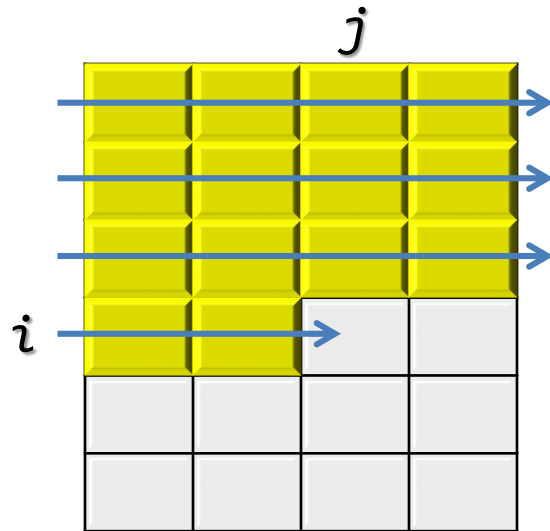
$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 2 & 0 \\ 1 & 1 \end{bmatrix}$$

```
typedef vector< vector<int> > Matrix;
```

```
Matrix matrix_sum(const Matrix& a,  
                  const Matrix& b);
```

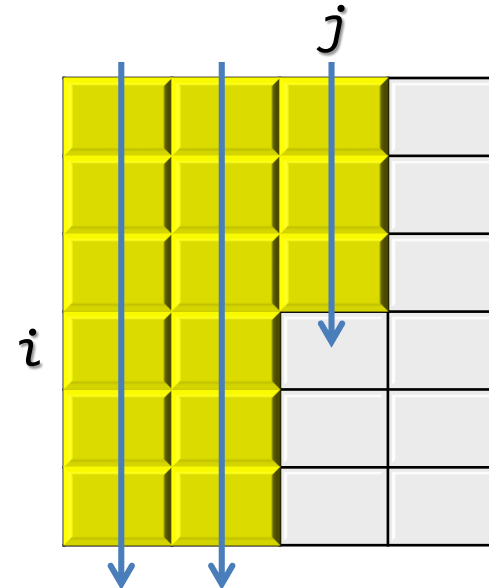
How are the elements of a matrix visited?

- By rows



For every row i
For every column j
Visit `Matrix[i][j]`

- By columns



For every column j
For every row i
Visit `Matrix[i][j]`

Sum of matrices (by rows)

```
typedef vector< vector<int> > Matrix;

// Pre: a and b are non-empty matrices and have the same size.
// Returns a+b (sum of matrices).

Matrix matrix_sum(const Matrix& a, const Matrix& b) {

    int nrows = a.size();
    int ncols = a[0].size();
    Matrix c(nrows, vector<int>(ncols));

    for (int i = 0; i < nrows; ++i) {
        for (int j = 0; j < ncols; ++j) {
            c[i][j] = a[i][j] + b[i][j];
        }
    }
    return c;
}
```

Sum of matrices (by columns)

```
typedef vector< vector<int> > Matrix;

// Pre: a and b are non-empty matrices and have the same size.
// Returns a+b (sum of matrices).

Matrix matrix_sum(const Matrix& a, const Matrix& b) {

    int nrows = a.size();
    int ncols = a[0].size();
    Matrix c(nrows, vector<int>(ncols));


    for (int j = 0; j < ncols; ++j) {
        for (int i = 0; i < nrows; ++i) {
            c[i][j] = a[i][j] + b[i][j];
        }
    }
    return c;
}
```


Transpose a matrix

- Design a procedure that transposes a square matrix in place:

```
void Transpose (Matrix& m);
```

3	8	1
0	6	2
4	5	9



3	0	4
8	6	5
1	2	9

- Observation: we need to swap the upper with the lower triangular matrix. The diagonal remains intact.

Transpose a matrix

// Interchanges two values

```
void swap(int& a, int& b) {  
    int c = a;  
    a = b;  
    b = c;  
}
```

// Pre: m is a square matrix

// Post: m contains the transpose of the input matrix

```
void Transpose(Matrix& m) {  
    int n = m.size();  
    for (int i = 0; i < n - 1; ++i) {  
        for (int j = i + 1; j < n; ++j) {  
            swap(m[i][j], m[j][i]);  
        }  
    }  
}
```

Is a matrix symmetric?

- Design a procedure that indicates whether a matrix is symmetric:

```
bool is_symmetric(const Matrix& m);
```

3	0	4
0	6	5
4	5	9

symmetric

3	0	4
0	6	5
4	2	9

not symmetric

- Observation: we only need to compare the upper with the lower triangular matrix.

Is a matrix symmetric?

```
// Pre: m is a square matrix  
// Returns true if m is symmetric, and false otherwise
```

```
bool is_symmetric(const Matrix& m) {  
    int n = m.size();  
    for (int i = 0; i < n - 1; ++i) {  
        for (int j = i + 1; j < n; ++j) {  
            if (m[i][j] != m[j][i]) return false;  
        }  
    }  
    return true;  
}
```

Search in a matrix

- Design a procedure that finds a value in a matrix. If the value belongs to the matrix, the procedure will return the location (i, j) at which the value has been found.

```
// Pre:  m is a non-empty matrix
// Post: i and j define the location of a cell
//       that contains the value x in m.
//       In case x is not in m, then i = j = -1.
```

```
void search(const Matrix& m, int x, int& i, int& j);
```

Search in a matrix

```
// Pre:  m is a non-empty matrix
// Post: i and j define the location of a cell
//       that contains the value x in M.
//       In case x is not in m, then i = j = -1
```

```
void search(const Matrix& m, int x, int& i, int& j) {
    int nrows = m.size();
    int ncols = m[0].size();
    for (i = 0; i < nrows; ++i) {
        for (j = 0; j < ncols; ++j) {
            if (m[i][j] == x) return;
        }
    }

    i = -1;
    j = -1;
}
```

Search in a sorted matrix

- A sorted matrix m is one in which

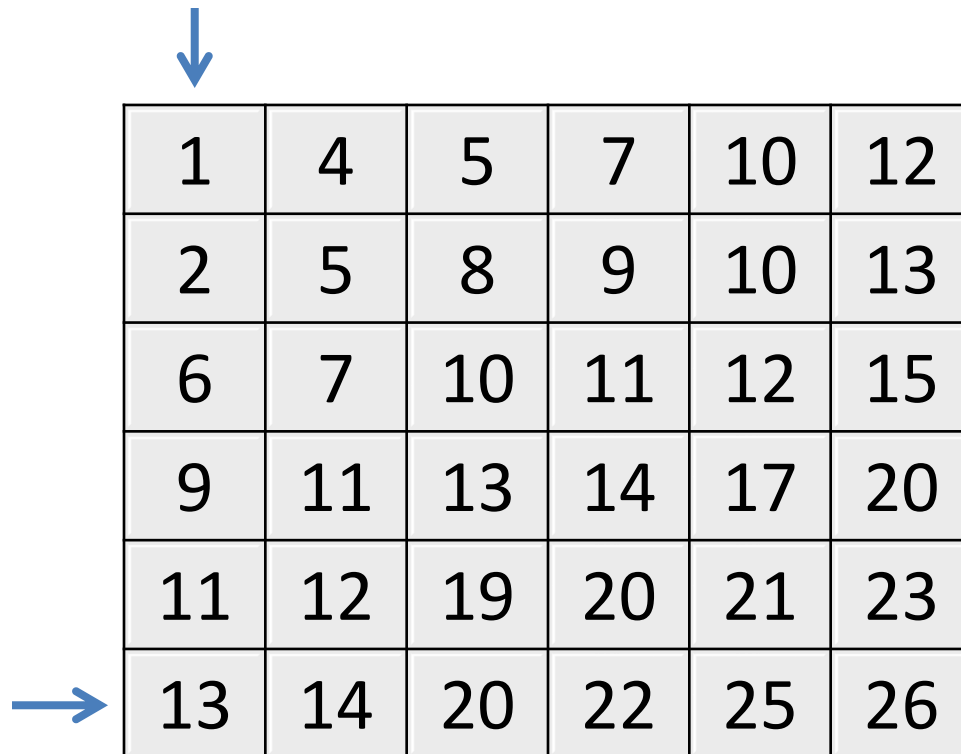
$$m[i][j] \leq m[i][j+1]$$

$$m[i][j] \leq m[i+1][j]$$

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

- Example: let us find 10 in the matrix. We look at the lower left corner of the matrix.
- Since $13 > 10$, the value cannot be found in the last row.

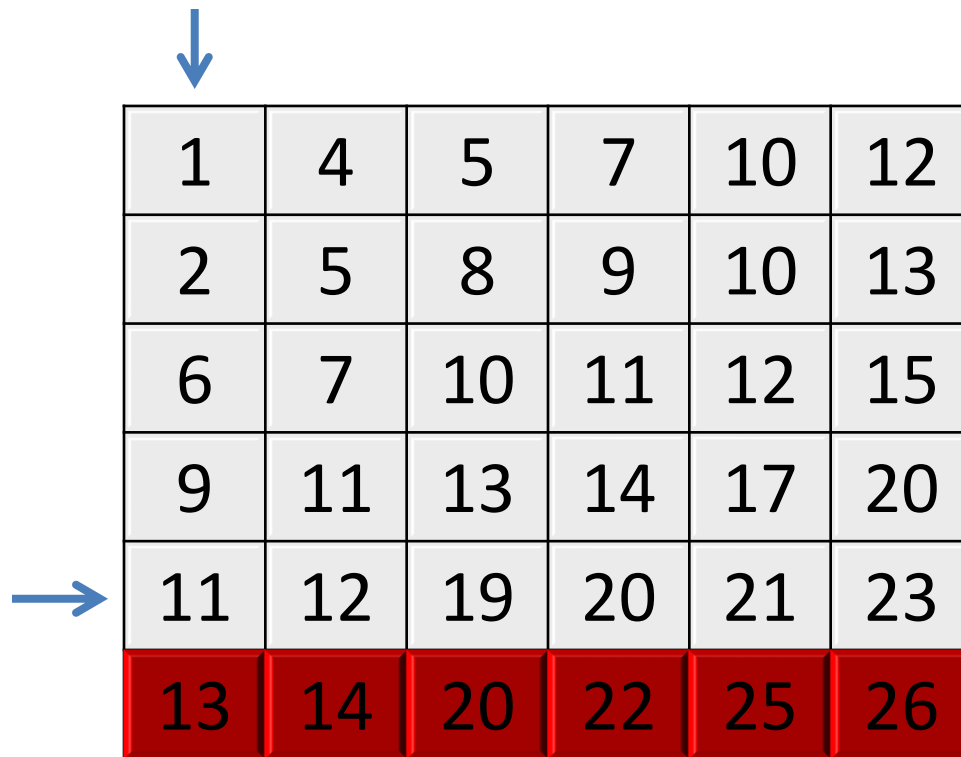


A 6x6 matrix of sorted numbers. The first column is [1, 2, 6, 9, 11, 13] and the last row is [13, 14, 20, 22, 25, 26]. A blue arrow points down to the first column, and another blue arrow points right to the last row.

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

- We look again at the lower left corner of the remaining matrix.
- Since $11 > 10$, the value cannot be found in the row.

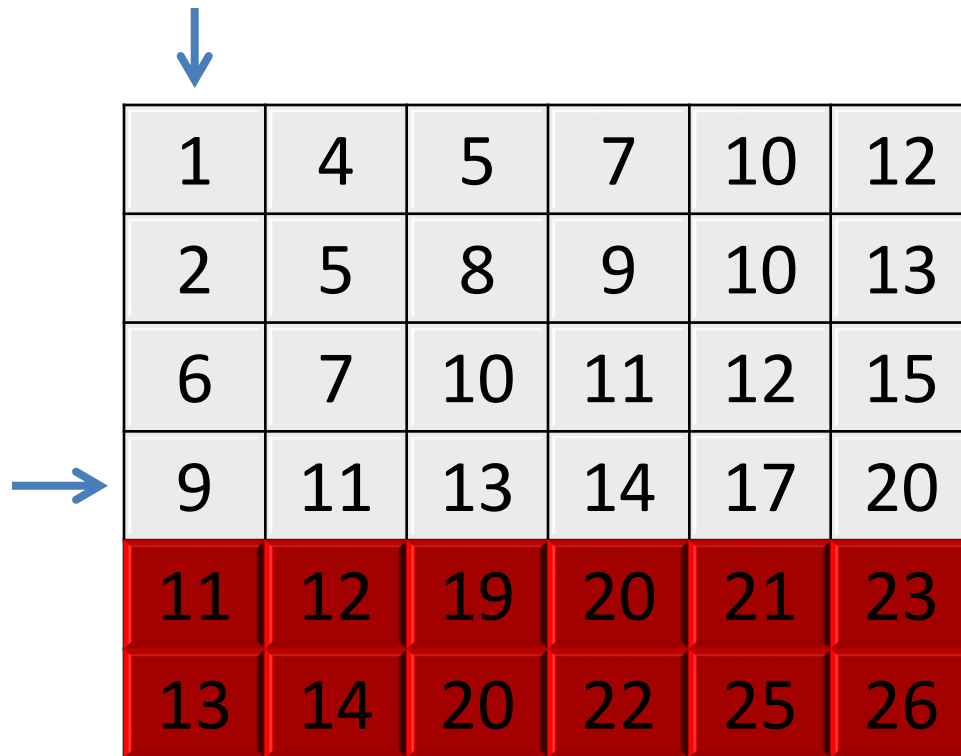


A 6x6 matrix of sorted numbers. The first column is pointed to by a blue arrow, and the fifth row is pointed to by another blue arrow. The last row is highlighted in red.

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

- Since $9 < 10$, the value cannot be found in the column.



A 6x6 matrix of sorted numbers. The first four rows are light gray, and the last two rows are dark red. A blue arrow points down to the first column, and another blue arrow points right to the fourth row.

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

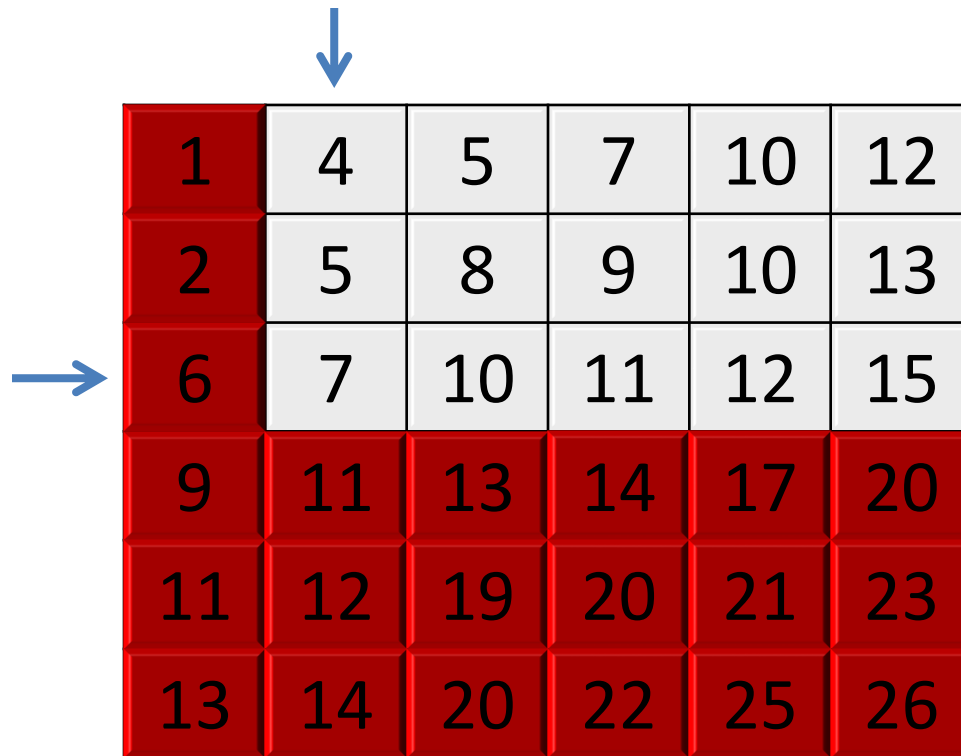
Search in a sorted matrix

- Since $11 > 10$, the value cannot be found in the row.

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

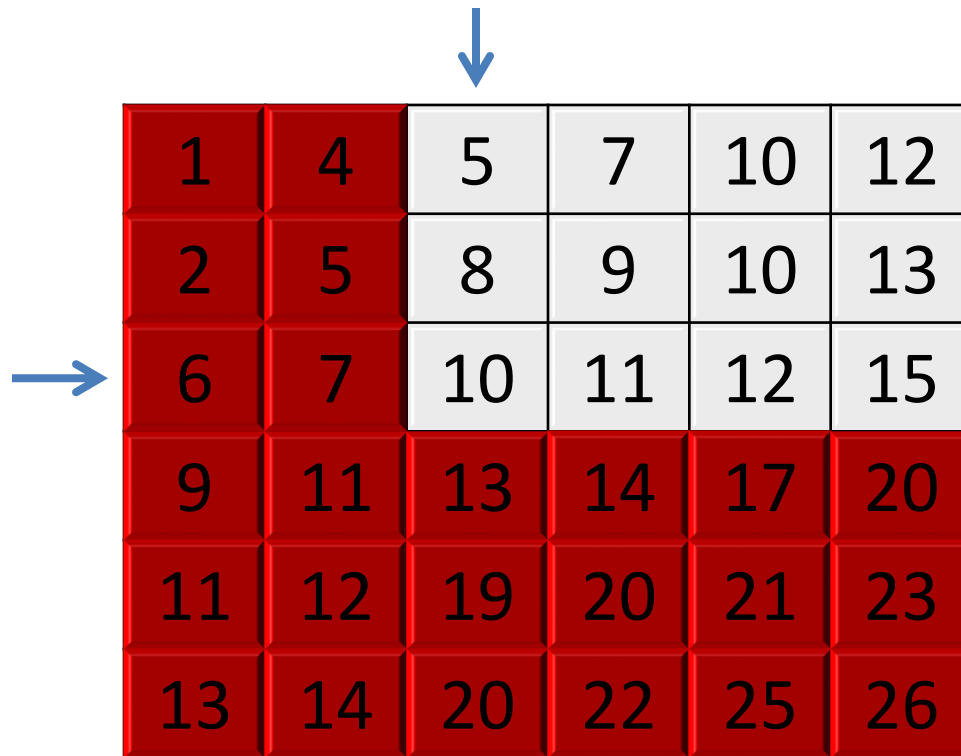
- Since $7 < 10$, the value cannot be found in the column.



1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

- The element has been found!

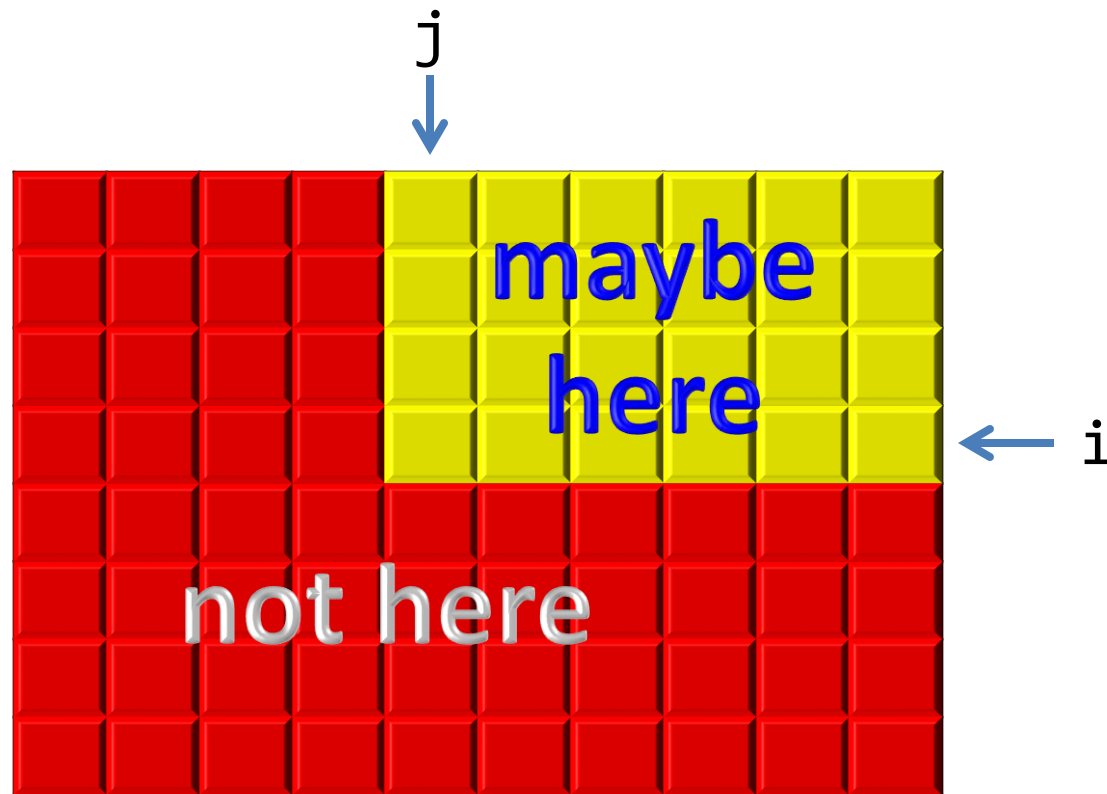


A 6x6 matrix of numbers, sorted row-wise and column-wise. The matrix is displayed with a 3D effect. A search path is highlighted in red, starting from the top-left (1) and moving right to the third column (5), then down to the third row (6), and finally right to the target element (10). The target element (10) is highlighted in light gray. A blue arrow points down to the first row, and another blue arrow points right to the third row.

1	4	5	7	10	12
2	5	8	9	10	13
6	7	10	11	12	15
9	11	13	14	17	20
11	12	19	20	21	23
13	14	20	22	25	26

Search in a sorted matrix

- *Invariant*: if the element is in the matrix, then it is located in the sub-matrix $[0...i, j...ncols-1]$



Search in a sorted matrix

```
// Pre:  m is non-empty and sorted by rows and columns
//       in ascending order.
// Post: i and j define the location of a cell that contains the value
//       x in m. In case x is not in m, then i=j=-1.
```

```
void search(const Matrix& m, int x, int& i, int& j) {
    int nrows = m.size();
    int ncols = m[0].size();

    i = nrows - 1;
    j = 0;

    // Invariant: x can only be found in M[0..i,j..ncols-1]
    while (i >= 0 and j < ncols) {
        if (m[i][j] < x) j = j + 1;
        else if (m[i][j] > x) i = i - 1;
        else return;
    }

    i = -1;
    j = -1;
}
```

Search in a sorted matrix

- What is the largest number of iterations of a search algorithm in a matrix?

Unsorted matrix	$nrows \times ncols$
Sorted matrix	$nrows + ncols$

- The search algorithm in a sorted matrix cannot start in all of the corners of the matrix. Which corners are suitable?

Matrix multiplication

- Design a function that returns the multiplication of two matrices.

2	-1	0	1
1	3	2	0

 ×

1	2	-1
3	0	2
-1	1	3
2	-1	4

 =

1	3	0
8	4	11

// Pre: a is a non-empty n×m matrix,

// b is a non-empty m×p matrix

// Returns a×b (an n×p matrix)

Matrix multiply(const Matrix& a, const Matrix& b);

Matrix multiplication

// Pre: a is a non-empty $n \times m$ matrix, b is a non-empty $m \times p$ matrix.
// Returns $a \times b$ (an $n \times p$ matrix).

```
Matrix multiply(const Matrix& a, const Matrix& b) {  
    int n = a.size();  
    int m = a[0].size();  
    int p = b[0].size();  
    Matrix c(n, vector<int>(p));  
  
    for (int i = 0; i < n; ++i) {  
        for (int j = 0; j < p; ++j) {  
            int sum = 0;  
            for (int k = 0; k < m; ++k) {  
                sum = sum + a[i][k]*b[k][j];  
            }  
            c[i][j] = sum;  
        }  
    }  
    return c;  
}
```

Matrix multiplication

// Pre: a is a non-empty $n \times m$ matrix, b is a non-empty $m \times p$ matrix.
// Returns $a \times b$ (an $n \times p$ matrix).

```
Matrix multiply(const Matrix& a, const Matrix& b) {  
    int n = a.size();  
    int m = a[0].size();  
    int p = b[0].size();  
    Matrix c(n, vector<int>(p, 0));
```

Initialized
to zero

```
    for (int i = 0; i < n; ++i) {  
        for (int j = 0; j < p; ++j) {  
            for (int k = 0; k < m; ++k) {  
                c[i][j] += a[i][k]*b[k][j];  
            }  
        }  
    }  
    return c;  
}
```

The loops can
be in any order

Accumulation

Matrix multiplication

// Pre: a is a non-empty $n \times m$ matrix, b is a non-empty $m \times p$ matrix.
// Returns $a \times b$ (an $n \times p$ matrix).

```
Matrix multiply(const Matrix& a, const Matrix& b) {  
    int n = a.size();  
    int m = a[0].size();  
    int p = b[0].size();  
    Matrix c(n, vector<int>(p, 0));  
  
    for (int j = 0; j < p; ++j) {  
        for (int k = 0; k < m; ++k) {  
            for (int i = 0; i < n; ++i) {  
                c[i][j] += a[i][k]*b[k][j];  
            }  
        }  
    }  
    return c;  
}
```