Guidance Theory and Applications (Lecture 4)

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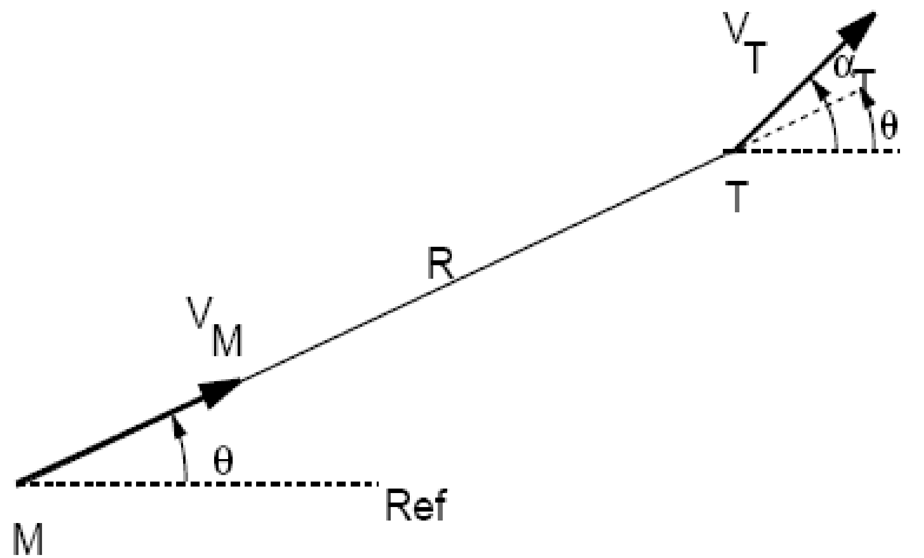
Pursuit Guidance

Types of Pursuit Guidance

- Pure pursuit
- Deviated pusuit

Pure Pursuit Guidance Law

The engagement geometry



The engagement equations

- Non-maneuvering target
- Missile and target speeds constant
- Missile uses perfect pursuit guidance, which means it should point towards the target all the time

$$V_R = \dot{R} \quad = \quad V_T \cos(\alpha_T - \theta) - V_M$$

$$V_\theta = R\dot{\theta} \quad = \quad V_T \sin(\alpha_T - \theta)$$

Some maths

Divide the first equation by the second

$$\frac{1}{R}\,dR = \{\cot(\alpha_T - \theta) - \nu\csc(\alpha_T - \theta)\}\,d\theta$$

$$\nu = \frac{V_M}{V_T} \qquad\qquad R = f(\theta)$$

- But this is no help!
- It is possible to write this expression and use it to get some trajectory parameters. But we will not follow this route.
- We will use the relative velocity components.

Trajectory in the (V_θ, V_R)-space

$$
\begin{aligned}
V_R + V_M &= V_T \cos(\alpha_T - \theta) \\
V_\theta &= V_T \sin(\alpha_T - \theta)
\end{aligned}
$$

$$(V_R + V_M)^2 + V_\theta^2 = V_T^2$$

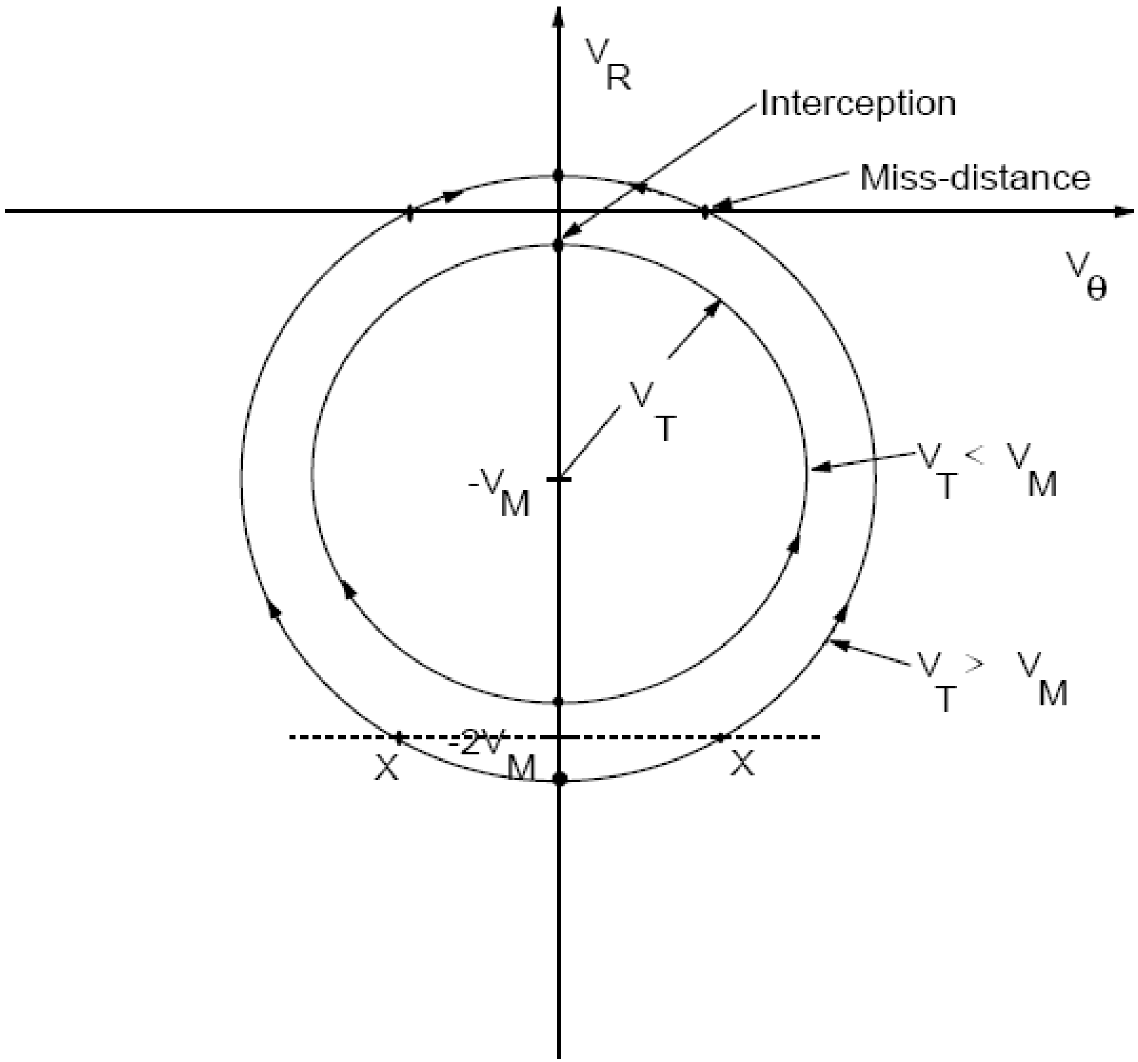


Figure 7.2: *The* (V_θ, V_R) *trajectory*

Direction of movement

$$\dot{V}_R = -V_T \sin(\alpha_T - \theta)(-\dot{\theta}) = \dot{\theta} V_\theta$$

$$\dot{V}_\theta = V_T \cos(\alpha_T - \theta)(-\dot{\theta}) = -\dot{\theta}(V_R + V_M)$$

$$R\dot{V}_R = V_\theta^2$$

$$R\dot{V}_\theta = -V_\theta(V_R + V_M)$$

$$\dot{V}_R > 0$$

Some simple analysis

In the case of pure pursuit, the missile is required to always point towards the current position of the target, and so collision can take place either in the

***tail-chase* mode** (missile pursuing the target with both the velocity vectors aligned along the LOS)

or in the

***head-on* mode** (missile and target approaching each other with both the velocity vectors aligned along the LOS).

Analysis contd…

Further, in the **tail-chase mode** collision occurs only if $VM > VT$, whereas in the **head-on mode** collision is possible for all values of VT and VM.

The collision triangle in the pure pursuit case is actually a **straight line** since the missile and target velocity vectors are both aligned along the LOS.

Similarly, points on the positive VR axis may correspond to missile and target travelling in opposite directions away from each other, or in a tail-chase mode when $VM < VT$.

Analysis contd…

- The main idea is that a point on the V_R axis essentially corresponds to the situation when both the missile and target velocities are aligned with the LOS. This we can also see by using the condition that on the *VR* axis we have $V\theta = 0$, which implies that,

$$\begin{aligned} V_\theta &= V_T \sin(\alpha_T - \theta) = 0 \\ \Rightarrow &\quad \alpha_T = \theta \text{ or } \theta + \pi \end{aligned}$$

Analysis contd…

If VT < VM capture is guaranteed since the engagement ends up in the tail chase mode, unless the initial geometry itself is head on.

If VT > VM capture occurs only if the initial geometry itself is head on.

The Capture Region

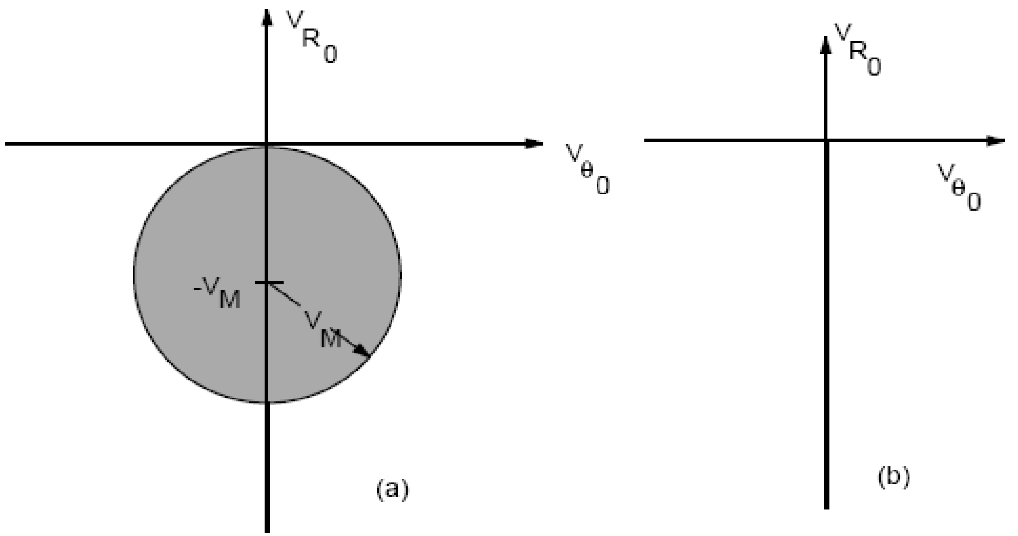


Figure 7.3: *Capture region for* (a) *Pure pursuit* (b) *Unguided missile*

However, note that the capture region for pure pursuit is obtained here with *VT* as the free parameter and the initial geometry restricted to the cases where *VM* points towards the target.

Time of interception

$$\left(V_R + V_M\right)^2 + V_\theta^2 = V_T^2$$

$$\Rightarrow \quad V_R^2 + V_M^2 + 2V_M V_R + V_\theta^2 = V_T^2$$

$$\Rightarrow \quad \dot{R}^2 + R\ddot{R} + 2V_M\dot{R} = V_T^2 - V_M^2$$

$$R\left(V_R + 2V_M\right) = \left(V_T^2 - V_M^2\right)t + b$$

$$b = R_0\left(V_{R0} + 2V_M\right) \qquad t_f = \frac{R_0\left(V_{R0} + 2V_M\right)}{V_M^2 - V_T^2}$$

Lateral acceleration history

$$\alpha_M = \theta \qquad \Rightarrow \quad \dot{\alpha}_M = \dot{\theta}$$

$$\Rightarrow \quad V_M \dot{\alpha}_M = V_M \dot{\theta} \;\; \Rightarrow \quad a_M = V_M \dot{\theta}$$

$$\Rightarrow a_M = \frac{V_M V_T}{R} \sin(\alpha_T - \theta)$$

Latax as a function of R and θ

$$\frac{1}{R}\,dR \;=\; \left\{\cot(\alpha_T-\theta)-\nu\csc(\alpha_T-\theta)\right\}d\theta \qquad \Rightarrow \quad R = K\frac{\left\{\tan\left(\frac{\alpha_T-\theta}{2}\right)\right\}^{\nu}}{\sin(\alpha_T-\theta)} = K\frac{\{\sin(\alpha_T-\theta)\}^{\nu-1}}{\{1+\cos(\alpha_T-\theta)\}^{\nu}}$$

$$K = R_0\frac{\sin(\alpha_T-\theta_0)}{\left\{\tan\left(\frac{\alpha_T-\theta_0}{2}\right)\right\}^{\nu}} = R_0\frac{\{1+\cos(\alpha_T-\theta_0)\}^{\nu}}{\{\sin(\alpha_T-\theta_0)\}^{\nu-1}}$$

$$a_M = \frac{V_M V_T}{K}\frac{\sin^2(\alpha_T-\theta)}{\left\{\tan\left(\frac{\alpha_T-\theta}{2}\right)\right\}^{\nu}}$$

t as a function of R and θ

$$t = \frac{b - R(V_R + 2V_M)}{V_M^2 - V_T^2}$$

$$t = \frac{R_0\{V_T\cos(\alpha_T - \theta_0) + V_M\} - R\{V_T\cos(\alpha_T - \theta) + V_M\}}{V_M^2 - V_T^2}$$

What happens at terminal time?

$$\lim_{t \to t_f} a_M = \lim_{\theta \to \alpha_T} a_M$$

As $t \to t_f$, if

$$1 < \nu < 2 \quad a_M \to 0$$

$$\nu = 2 \quad a_M \to \frac{4V_M V_T}{K}$$

$$\nu > 2 \quad a_M \to \infty$$

Miss-distance

$$\text{when } V_T > V_M, \qquad V_R = 0 \qquad \Rightarrow V_T \cos\left(\alpha_{T_{\text{miss}}} - \theta_{\text{miss}}\right) = V_M$$

$$\Rightarrow \alpha_{T_{\text{miss}}} - \theta_{\text{miss}} = \cos^{-1}\nu$$

$$R_{\text{miss}} = K \frac{\left\{\tan\left(\frac{\cos^{-1}\nu}{2}\right)\right\}^{\nu}}{\sin(\cos^{-1}\nu)} = K \frac{\{\sin(\cos^{-1}\nu)\}^{\nu-1}}{(1+\nu)^{\nu}}$$

$$2V_M R_{\text{miss}} = \left(V_T^2 - V_M^2\right) t_{\text{miss}} + b \qquad t_{\text{miss}} = \frac{2V_M R_{\text{miss}} - b}{V_T^2 - V_M^2}$$

Deviated Pursuit

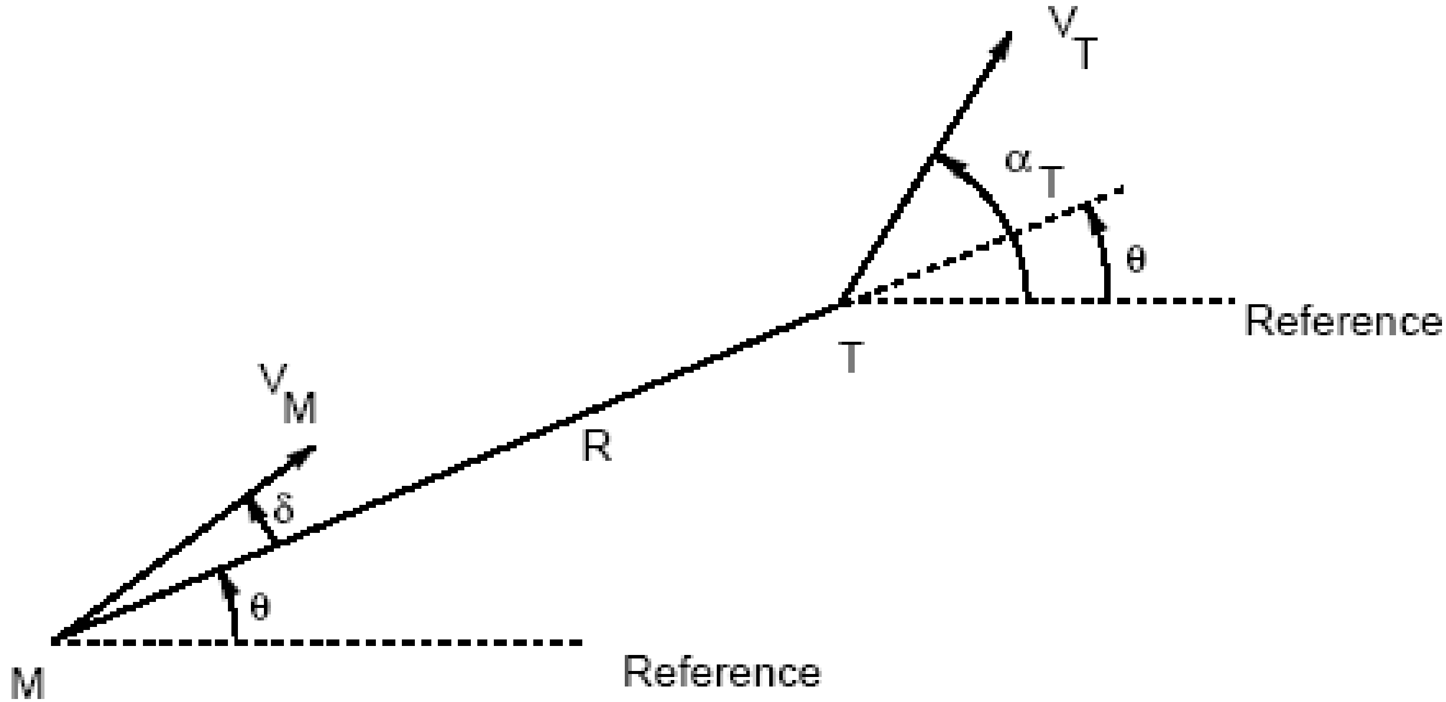


Figure 7.4: *Engagement geometry for deviated pursuit*

Engagement Equations

$$V_R = \dot{R} \quad = \quad V_T \cos(\alpha_T - \theta) - V_M \cos\delta$$

$$V_\theta = R\dot{\theta} \quad = \quad V_T \sin(\alpha_T - \theta) - V_M \sin\delta$$

Some algebra ...

$$V_R + V_M \cos\delta = V_T \cos(\alpha_T - \theta)$$

$$V_\theta + V_M \sin\delta = V_T \sin(\alpha_T - \theta)$$

$$(V_R + V_M \cos\delta)^2 + (V_\theta + V_M \sin\delta)^2 = V_T^2$$

Some algebra ...

$$\begin{aligned}\dot{V}_R &= -V_T \sin(\alpha_T - \theta)(-\dot{\theta}) = \dot{\theta}(V_\theta + V_M \sin\delta) \\ \dot{V}_\theta &= V_T \cos(\alpha_T - \theta)(-\dot{\theta}) = -\dot{\theta}(V_R + V_M \cos\delta)\end{aligned}$$

$$\begin{aligned}R\dot{V}_R &= V_\theta(V_\theta + V_M \sin\delta) \\ R\dot{V}_\theta &= -V_\theta(V_R + V_M \cos\delta)\end{aligned}$$

Some algebra ...

$$\dot{V}_R > 0 \;\text{ if }\; \{V_\theta > 0 \text{ and } V_\theta > -V_M \sin\delta\}$$

$$\text{OR }\; \{V_\theta < 0 \text{ and } (V_\theta < -V_M \sin\delta\}$$

$$\dot{V}_R < 0 \;\text{ if }\; \{V_\theta > 0 \text{ and } V_\theta < -V_M \sin\delta\}$$

$$\text{OR }\; \{V_\theta < 0 \text{ and } (V_\theta > -V_M \sin\delta\}$$

Trajectory in relative velocity space

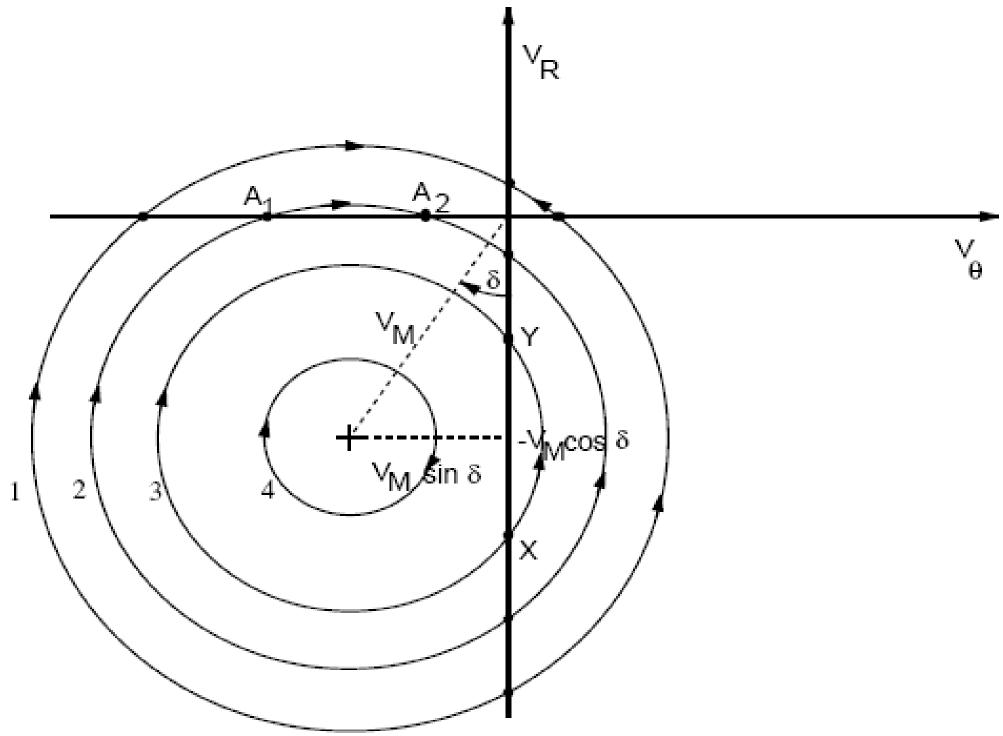


Figure 7.5: *The (V_θ, V_R) trajectories for deviated pursuit with $\delta < \pi/4$*

Some algebra ...

The points where the circle cuts the V_R-axis

are stationary points since at these

points $V_\theta = 0$ and so

$$\dot{V}_\theta = 0 \text{ and } \dot{V}_R = 0.$$

Collision Triangle

This collision triangle for deviated pursuit is defined by the requirement that the missile has to always point at an angle deviated by ± *delta* from the current LOS, and so is given by that value of α that satisfies,

$$V_T \sin(\alpha_T - \theta) = V_M \sin\delta$$

Collision triangles …

- There are two possibilities for the collision triangle at the point of interception

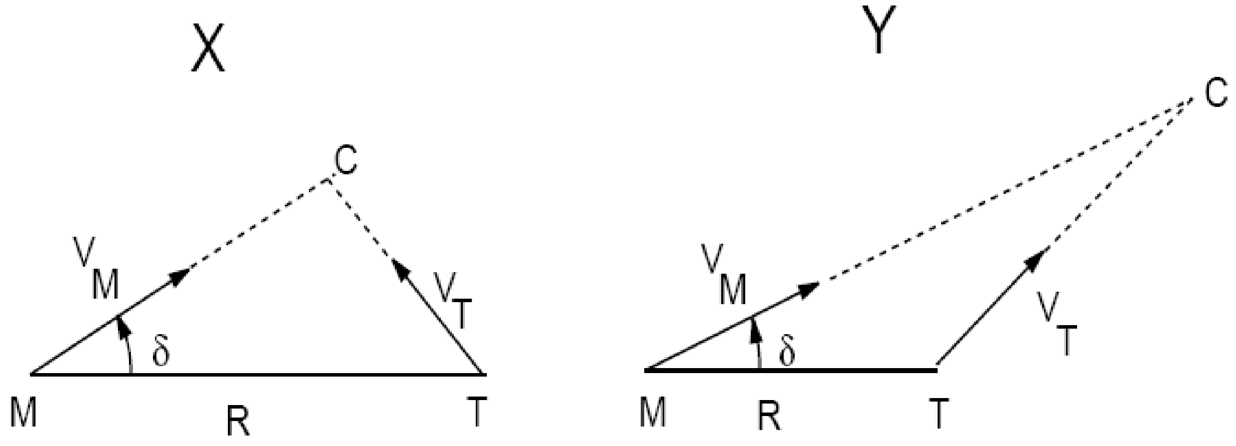


Figure 7.6: *The two possible collision triangles at interception*

- In the circle Figure we have shown four circles marked as 1, 2, 3, and 4. They correspond to the following conditions:

$$1 \quad : \quad V_T > V_M$$

$$2 \quad : \quad V_M \cos\delta < V_T < V_M$$

$$3 \quad : \quad V_M \sin\delta < V_T < V_M \cos\delta$$

$$4 \quad : \quad V_T < V_M \sin\delta$$

Another case

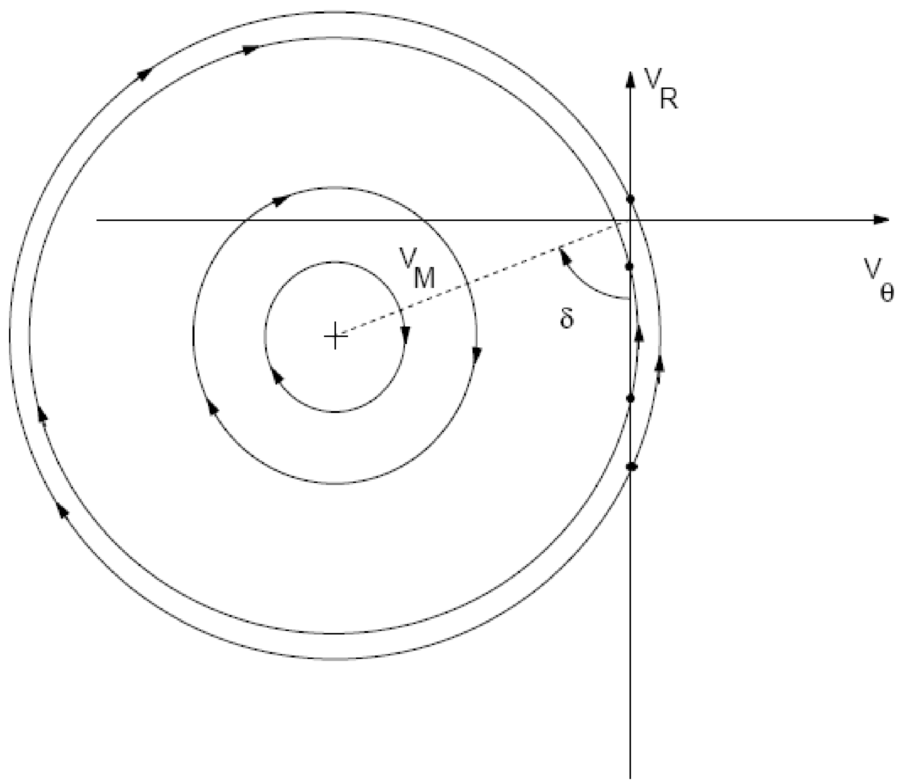


Figure 7.7: *The* (V_θ, V_R) *trajectories for deviated pursuit with* $\delta > \pi/4$

Circle 2

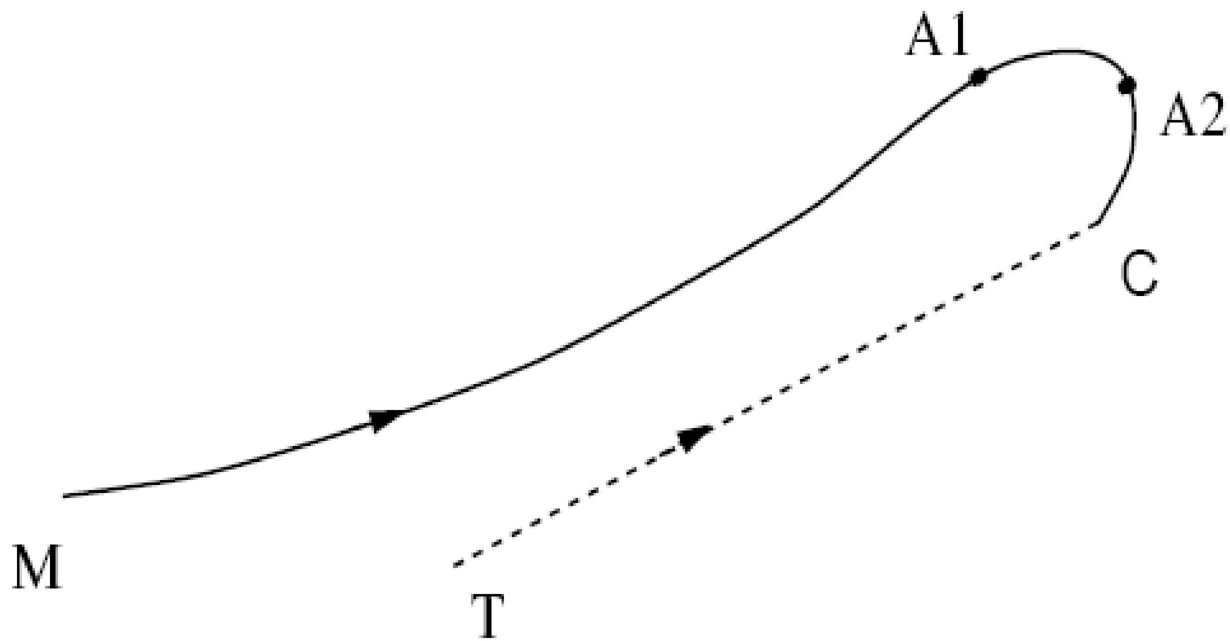


Figure 7.8: *A trajectory corresponding to Circle 2*

Some analysis

- Points corresponding to Circle 4 also lead to interception, but in this case the interception is somewhat different from the previous cases. If we monitor the rate of rotation of the ($V\mu$; VR)-point about the center of the circle with respect to time we will see that as time increases the angular velocity of this point also increases and tends toward infinity as the time tends to the interception time value.

Circle 4

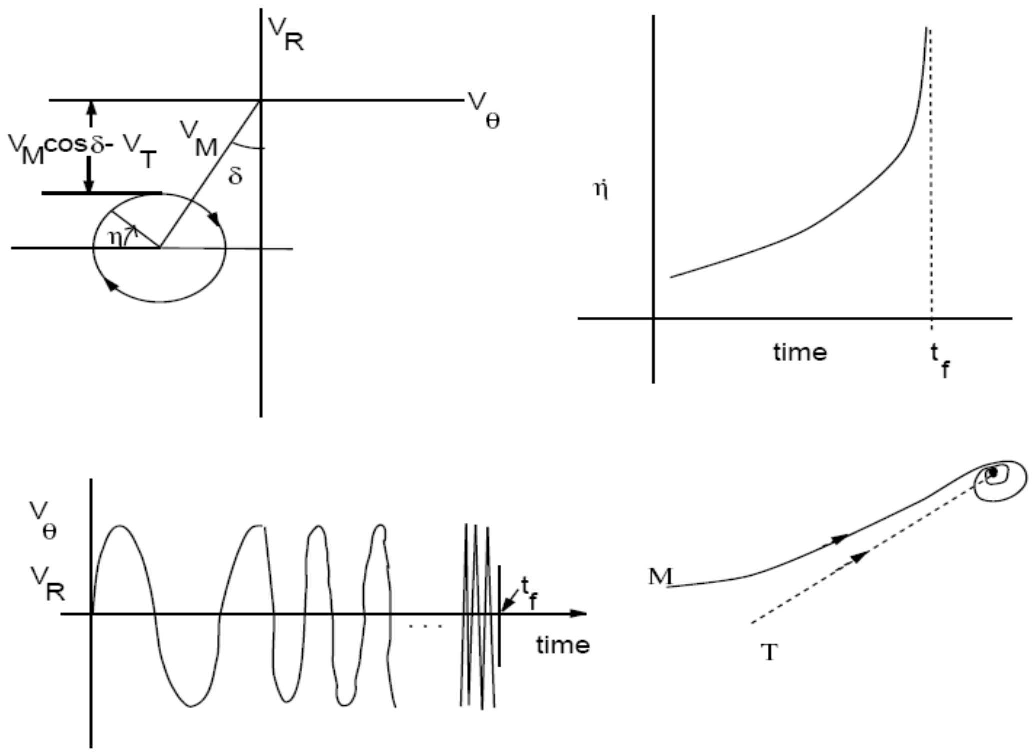


Figure 7.9: *Trajectory and angular velocity for Circle 4*

Upper bound on final time

$$t_f \leq \frac{R_0}{V_M \cos\delta - V_T}$$

Capture Region

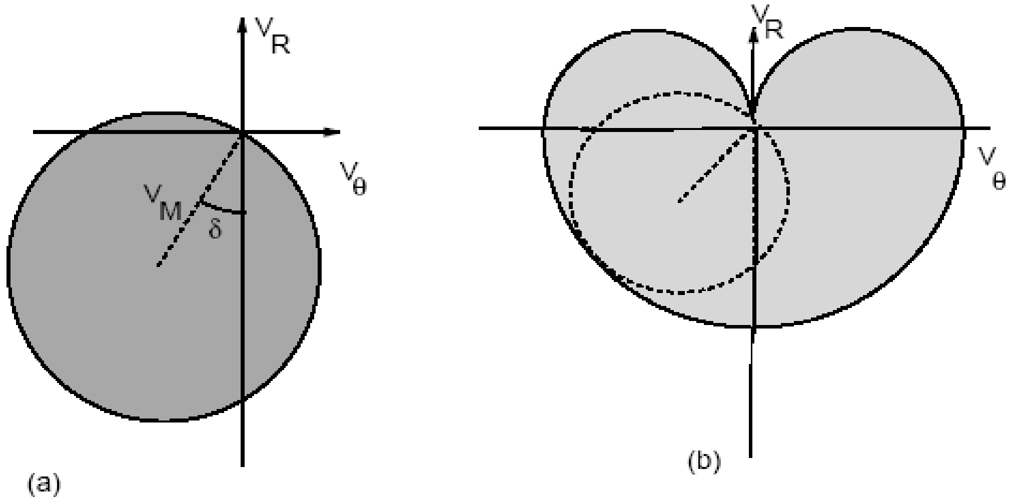


Figure 7.10: *Capture region for the deviated pursuit guidance law* (a) *Fixed* δ (b) $-\pi/2 < \delta < \pi/2$

A question

- Why were the cases of

  delta > 90 deg

Or

  delta < - 90 deg

not considered?

Time of interception

$$(V_R + V_M \cos\delta)^2 + (V_\theta + V_M \sin\delta)^2 = V_T^2$$

$$\Rightarrow \quad V_R^2 + V_M^2 + 2V_M V_R \cos\delta + V_\theta^2 + 2V_M V_\theta \sin\delta = V_T^2$$

$$\Rightarrow \quad V_R^2 + 2V_M V_R \cos\delta + V_\theta(V_\theta + V_M \sin\delta) + V_M V_\theta \sin\delta = V_T^2 - V_M^2$$

$$\Rightarrow \quad \dot{R}^2 + R\ddot{R} + 2V_M \cos\delta \dot{R} + V_M V_\theta \sin\delta = V_T^2 - V_M^2$$

Contd …

$$
\begin{aligned}
R\dot{V}_\theta &= -V_\theta V_R - V_\theta V_M \cos\delta \\
\Rightarrow &\quad V_\theta V_M = \frac{R\dot{V}_\theta + V_\theta V_R}{-\cos\delta} \\
\Rightarrow &\quad V_\theta V_M \sin\delta = \left(R\dot{V}_\theta + V_\theta V_R\right)(-\tan\delta) = \frac{d}{dt}(RV_\theta)(-\tan\delta)
\end{aligned}
$$

Contd.

Substituting the above in (7.37), we obtain,

$$\frac{d}{dt}(RV_R) + 2V_M \cos\delta \frac{d}{dt}(R) - \tan\delta \frac{d}{dt}(RV_\theta) = V_T^2 - V_M^2$$

which, on integration, yields

$$R(V_R + 2V_M \cos\delta - V_\theta \tan\delta) = \left(V_T^2 - V_M^2\right) t + c$$

Contd …

where,

$$c = R_0 \left(V_{R0} + 2V_M \cos\delta - V_{\theta 0} \tan\delta \right)$$

If interception occurs, then at $t = t_f$ we have $R = 0$, which yields

$$t_f = \frac{-c}{V_T^2 - V_M^2} = \frac{R_0 \left(V_{R0} + 2V_M \cos\delta - V_{\theta 0} \tan\delta \right)}{V_M^2 - V_T^2}$$

Lateral Acceleration

$$1 < \nu < 2 \quad a_M \to \text{A finite value}$$

$$\nu \geq 2 \quad a_M \to \infty$$

Implementation

Missile applies the maximum lateral acceleration till it is on a pursuit course and then applies the pursuit lateral acceleration.

If there is no bound on the missile lateral acceleration, then the missile can turn instantaneously and then apply the pursuit acceleration.

Contd…

- However, note that both these alternatives are open-loop in nature and requires lot of computations to make them feasible.

- Thus, errors in measurements, and mismatch between the missile flight angle and the LOS angle, will lead to large miss-distances.

Contd. ….

- Moreover, even if a closed-loop implementation is devised, based upon continuous measurements of the states, latax oscillations will be caused by the high demand on latax. This is bound to occur due to the dynamics of the system.

Contd…

$$a_M = -K(\alpha_M - \theta)$$

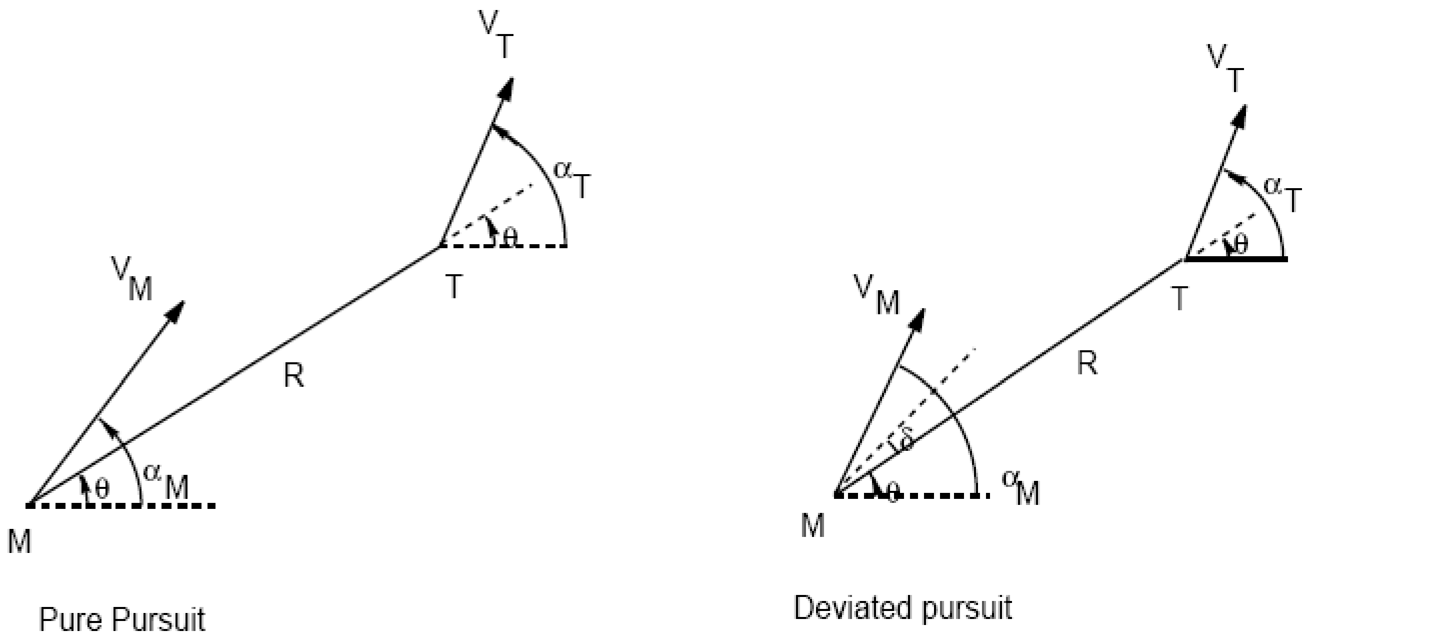


Figure 7.11: Pursuit guidance law: Implementation geometry

Latax

$$\dot{\alpha}_M = \dot{\theta}$$

$$\dot{\alpha}_M = \frac{a_M}{V_M}$$

$$a_M = V_M \dot{\theta}$$

Latax for pure and deviated pursuit

$$a_M = V_M \dot{\theta} - K(\alpha_M - \theta)$$

$$a_M = V_M \dot{\theta} - K(\alpha_M - \theta - \delta)$$

Comparison with PN

- Pure pursuit is PN with navigation constant equal to 1.

- We will understand this comparison better later.

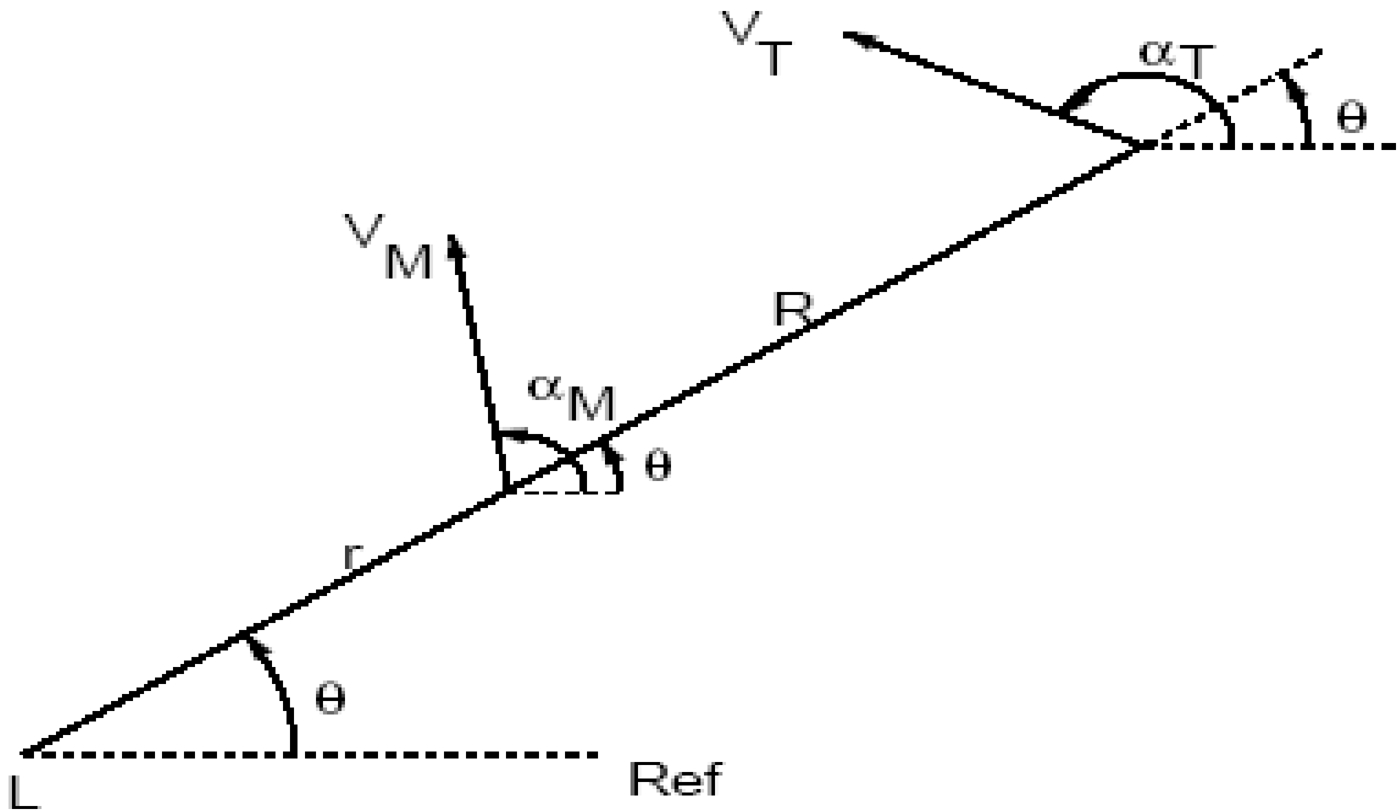
Line-of-Sight Guidance Law

Two Types of Implementation

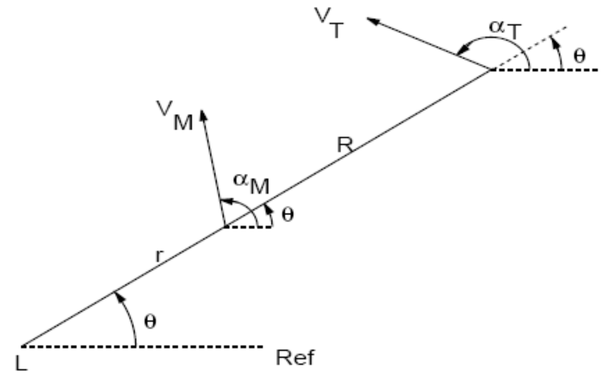
- Beam Rider (BR)
- Command-to-Line-of-Sight (CLOS)

BR and CLOS are the two different mechanizations of the basic LOS guidance philosophy

Engagement Geometry



Equations of Motion



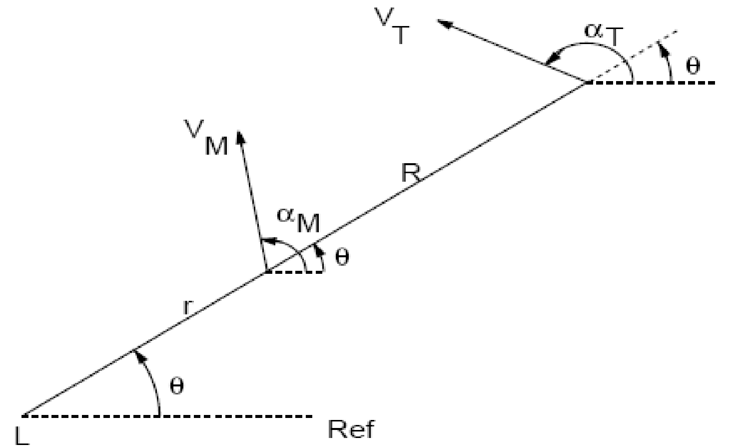
$$V_r = \dot{r} \quad = \quad V_M \cos(\alpha_M - \theta)$$

$$V_R = \dot{R} \quad = \quad V_T \cos(\alpha_T - \theta) - V_M \cos(\alpha_M - \theta)$$

$$V_{\theta r} = r\dot{\theta} \quad = \quad V_M \sin(\alpha_M - \theta)$$

$$V_{\theta R} = R\dot{\theta} \quad = \quad V_T \sin(\alpha_T - \theta) - V_M \sin(\alpha_M - \theta)$$

Algebraic manipulations …



$$\dot{\theta} = \frac{V_M \sin(\alpha_M - \theta)}{r}$$

$$= \frac{V_T \sin(\alpha_T - \theta)}{R + r}$$

$$= \frac{V_T \sin(\alpha_T - \theta) - V_M \sin(\alpha_M - \theta)}{R}$$

$$\Rightarrow \quad (R + r)V_M \sin(\alpha_M - \theta) = rV_T \sin(\alpha_T - \theta)$$

Algebraic manipulations …

Differentiating and using the equations of motion and

$$\dot{\alpha_M} = a_M / V_M$$

$$(\dot{R} + \dot{r}) V_M \sin(\alpha_M - \theta) + (R + r) V_M \cos(\alpha_M - \theta)(\dot{\alpha_M} - \dot{\theta})$$

$$= \dot{r} V_T \sin(\alpha_T - \theta) + r V_T \cos(\alpha_T - \theta)(-\dot{\theta})$$

Algebraic manipulations …

$$\Rightarrow \quad V_T \cos(\alpha_T - \theta) V_M \sin(\alpha_M - \theta) + (R + r) a_M \cos(\alpha_M - \theta)$$

$$- V_T \sin(\alpha_T - \theta) V_M \cos(\alpha_M - \theta)$$

$$= V_T \sin(\alpha_T - \theta) V_M \cos(\alpha_M - \theta) - V_T \cos(\alpha_T - \theta) V_M \sin(\alpha_M - \theta)$$

$$\Rightarrow \quad (R + r) a_M \cos(\alpha_M - \theta) = 2 V_T V_M \sin(\alpha_T - \alpha_M)$$

$$\Rightarrow \quad a_M = \frac{2 V_T V_M \sin(\alpha_T - \alpha_M)}{(R + r) \cos(\alpha_M - \theta)}$$

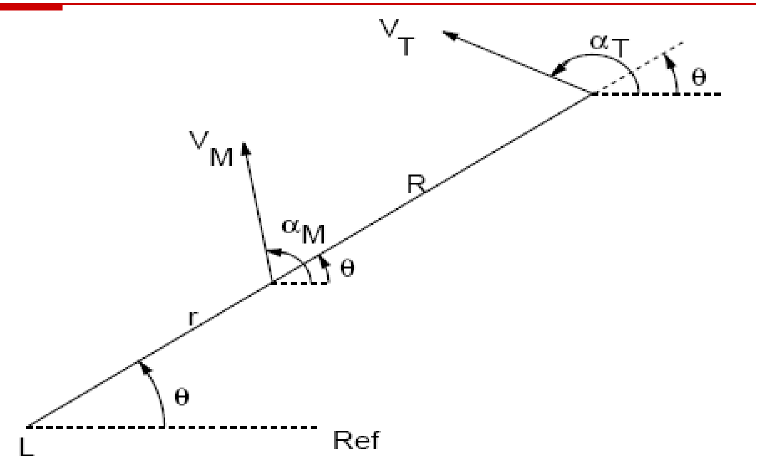
Lateral Acceleration

However, the expression for a_M is a function of several time-varying quantities: R, r, α_M, and θ.

express θ and $(R + r)$ as functions of time

Algebraic manipulations …

$$\tan\theta = \frac{(R_0+r_0)\sin\theta_0 + V_T t \sin\alpha_T}{(R_0+r_0)\cos\theta_0 + V_T t \cos\alpha_T}$$



$$\theta = \tan^{-1}\frac{(R_0+r_0)\sin\theta_0 + V_T t \sin\alpha_T}{(R_0+r_0)\cos\theta_0 + V_T t \cos\alpha_T}$$

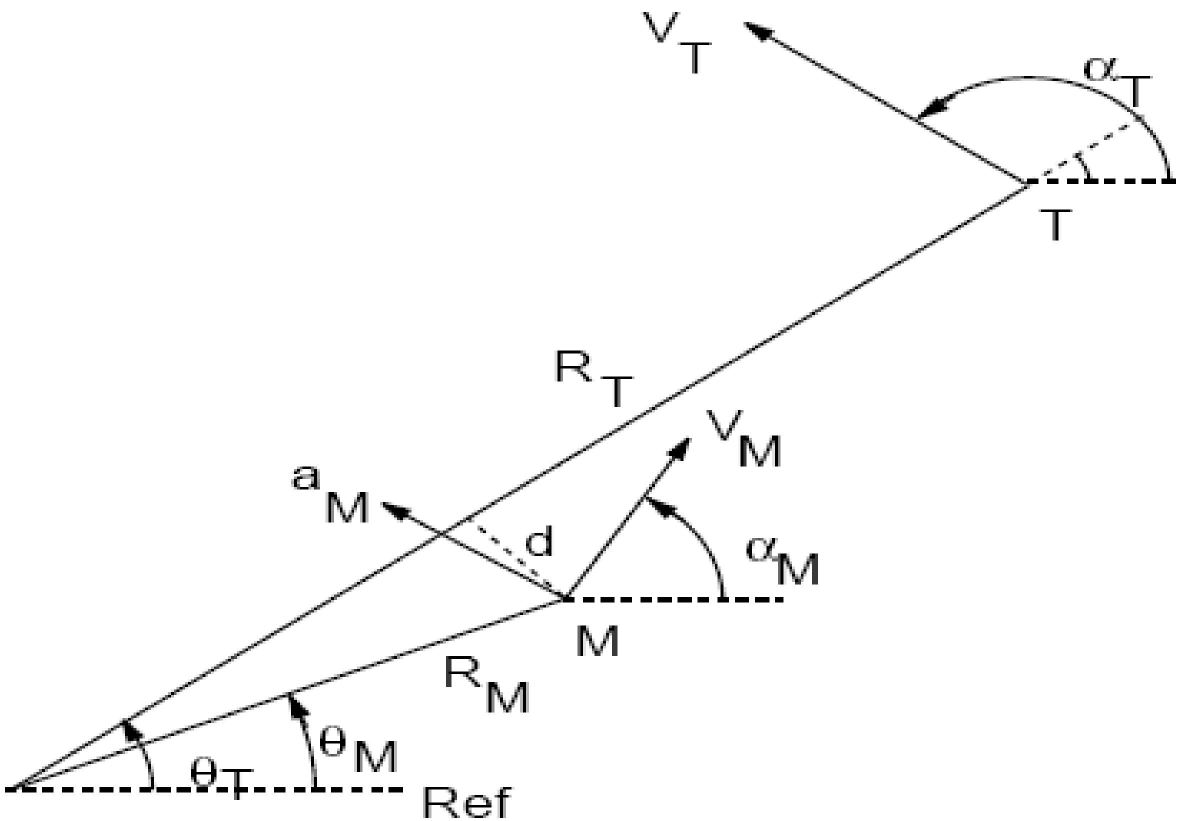
$$R + r = \sqrt{(R_0+r_0)^2 + (V_T t)^2 + 2(R_0+r_0)V_T t\cos(\theta_0-\alpha_T)}$$

Implementation of LOS Guidance

The implementation of LOS guidance law in an actual missile system differs from the analytical derivation of the lateral acceleration because of:

- *Non-availability of guidance parameters*
- *Autopilot dynamics*
- *Target maneuver*
- *Launch at off-nominal conditions*

Implementation of Beam Rider Guidance



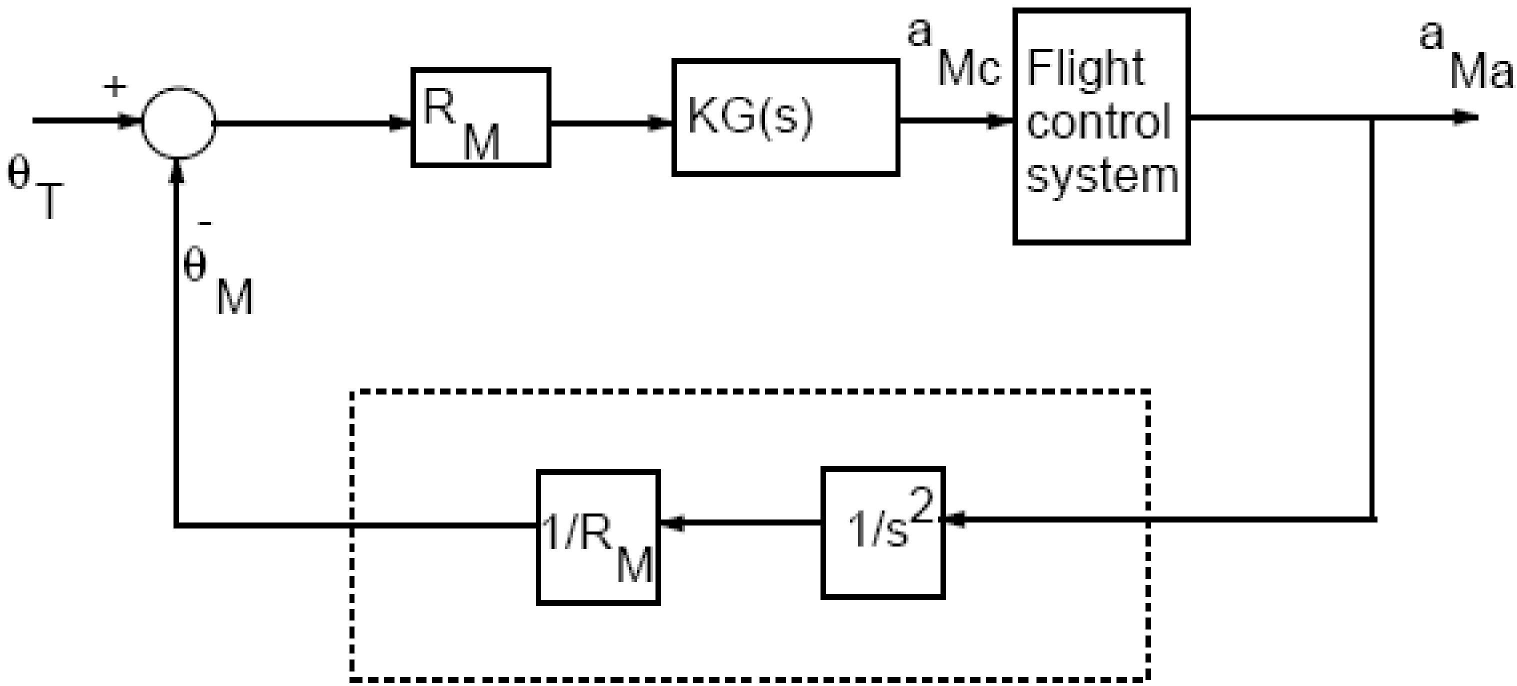
BR Latax Command

$$d \cong R_M(\theta_T - \theta_M)$$

$$a_M \quad = \quad Kd = KR_M(\theta_T - \theta_M)$$

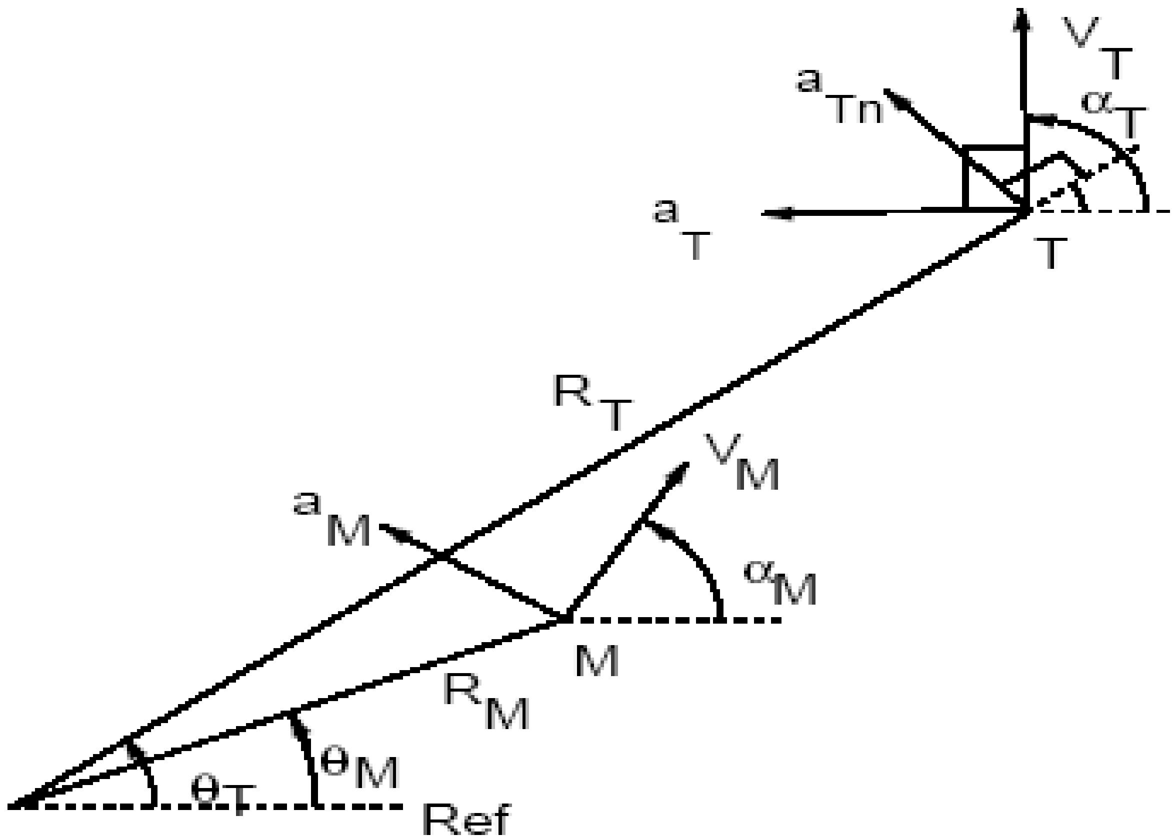
$$a_M = K\hat{R}_M(\theta_T - \theta_M)$$

Block Diagram Form



This may be replaced by a nonlinear block representing the engagement kinematics

CLOS Implementation



Latax Command

$$
\begin{aligned}
\dot{\theta}_M &= \dot{\theta}_T \\
\ddot{\theta}_M &= \ddot{\theta}_T
\end{aligned}
$$

$$R_T\dot{\theta}_T = V_T \sin(\alpha_T - \theta_T)$$

$$\dot{R}_T\dot{\theta}_T + R_T\ddot{\theta}_T = -V_T\dot{\theta}_T \cos(\alpha_T - \theta_T) + V_T\dot{\alpha}_T \cos(\alpha_T - \theta_T)$$

$$
\begin{aligned}
\ddot{\theta}_T &= \frac{-2\dot{R}_T\dot{\theta}_T - a_T \cos(\alpha_T - \theta_T)}{R_T} \\
&= \frac{a_{Tn} - 2\dot{R}_T\dot{\theta}_T}{R_T}
\end{aligned}
$$

$$a_{Tn} = a_T \cos(\alpha_T - \theta_T)$$

Latax Command …

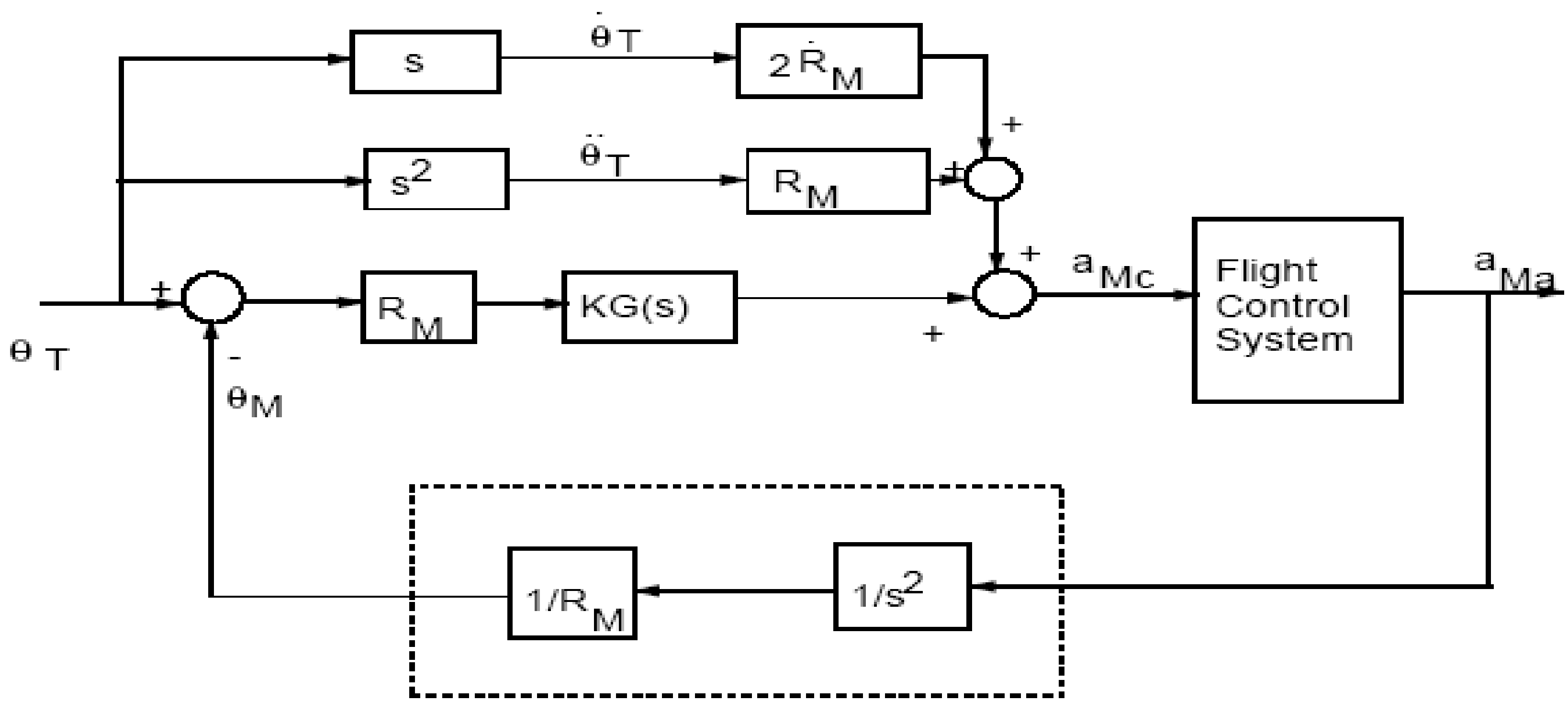
$$a_{Tn} = R_T\ddot{\theta}_T + 2\dot{R}_T\dot{\theta}_T$$

$$a_{Mn} = R_M\ddot{\theta}_T + 2\dot{R}_M\dot{\theta}_T$$

$$a_M = KR_M(\theta_T - \theta_M) + R_M\ddot{\theta}_T + 2\dot{R}_M\dot{\theta}_T$$

$$a_M = KG(s)R_M(\theta_T - \theta_M) + R_M\ddot{\theta}_T + 2\dot{R}_M\dot{\theta}_T$$

CLOS Block Diagram



This block may be replaced by the non-linear engagement kinematics

Capturability of LOS Guidance Laws

$$a_M = 2V_M\dot{\theta} + \frac{r}{\cos(\alpha_M - \theta)}\ddot{\theta}$$

At $t = 0$, we have $r = 0$, and so,

$$a_{M0} = 2V_M\dot{\theta}_0$$

$$V_M \sin(\alpha_{M0} - \theta_0) = 0 \Rightarrow \alpha_{M0} = \theta_0$$

For interception

$$V_M \sin(\alpha_{Mf} - \theta_f) = V_T \sin(\alpha_{Tf} - \theta_f)$$

$$V_{\theta Rf} = 0$$

Capturability Theorem

Theorem 8.1. If $V_M > V_T$ then the missile captures the target.

Proof. We will prove this by contradiction. Assume that interception does not occur. In that case the target continues to fly for infinite time. Now, as $t \to \infty$, we have $\theta \to \alpha_T$ which, from (8.5), implies that that $\alpha_{M\infty} = \theta_\infty = \alpha_T$. This implies that the missile converges to a tail-chase situation with the target as $t \to \infty$. From (8.2) and (8.4), we have $V_{\theta\infty} = 0$ and $V_{R\infty} = V_T - V_M$. Now, if $V_M > V_T$, then the missile will eventually intercept the target. This is a contradiction. And so, the missile will intercept the target whenever $V_M > V_T$. $\square$

Sufficient Condition only

This result can be interpreted as follows: The theorem gives a sufficient condition only. Based on the sufficient condition one can say that the capture region for the LOS guidance is at least as large as that of the pure pursuit guidance law, provided that the initial conditions are the ideal initial conditions for LOS guidance, that is the initial missile velocity vector points directly at the target (note that these initial conditions are the same as that of pure pursuit). However, the theorem does not give a necessary condition and so we cannot say if $V_M \leq V_T$ implies no interception. There could be conditions under which M may intercept T even when $V_M \leq V_T$.

PN Class of Guidance Laws

- True Proportional Navigation (TPN)
- Pure Proportional Navigation (PPN)
- Generalized TPN (GTPN)
- Ideal PN (IPN)

PPN

$$\dot{\alpha}_M = \frac{a_M}{V_M}$$

$$a_M = N V_M \dot{\theta}$$

TPN

$$a_M = c\dot{\theta}$$

$$a_M = N'V_c\dot{\theta} = -N'V_R\dot{\theta}$$

N' is called the *effective navigation ratio*

PN Class

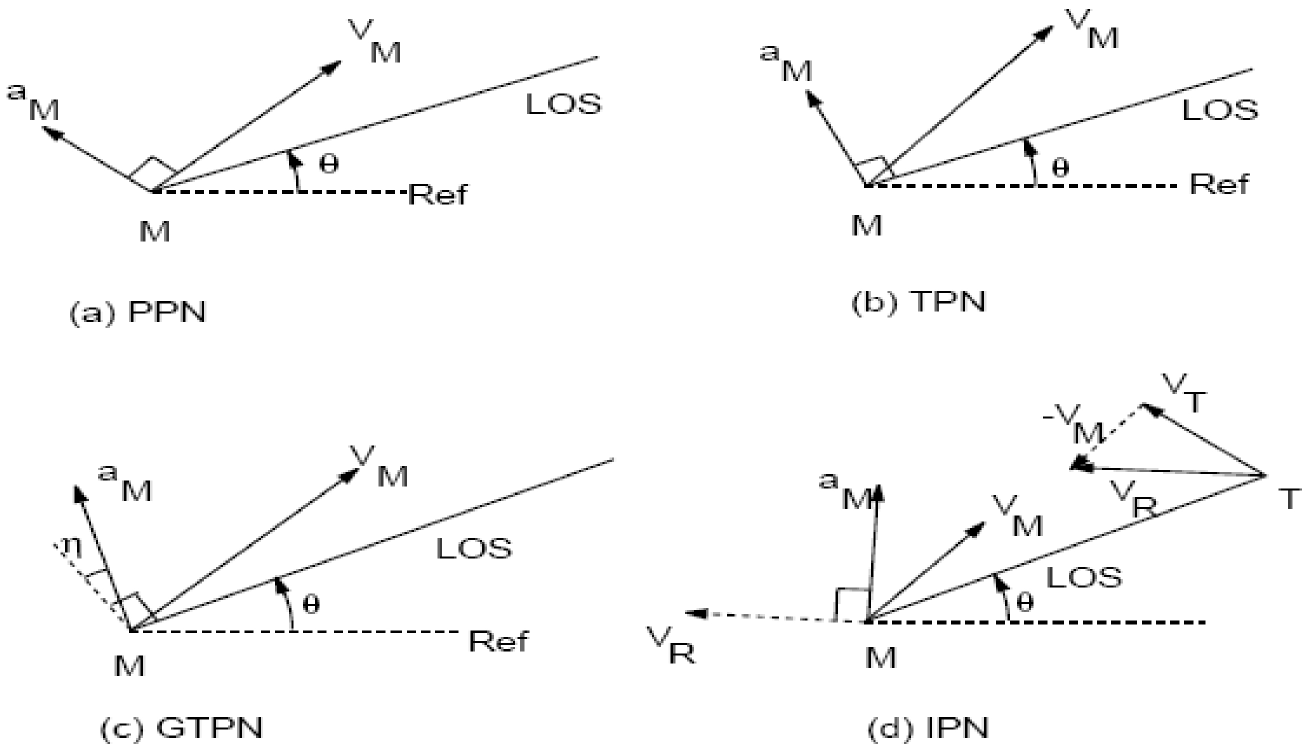


Figure 9.2: *(a) PPN Latax (b) TPN Latax (c) GTPN latax (d) IPN Latax*

Pure Proportional Navigation with Non-Maneuvering Target

Engagement Geometry

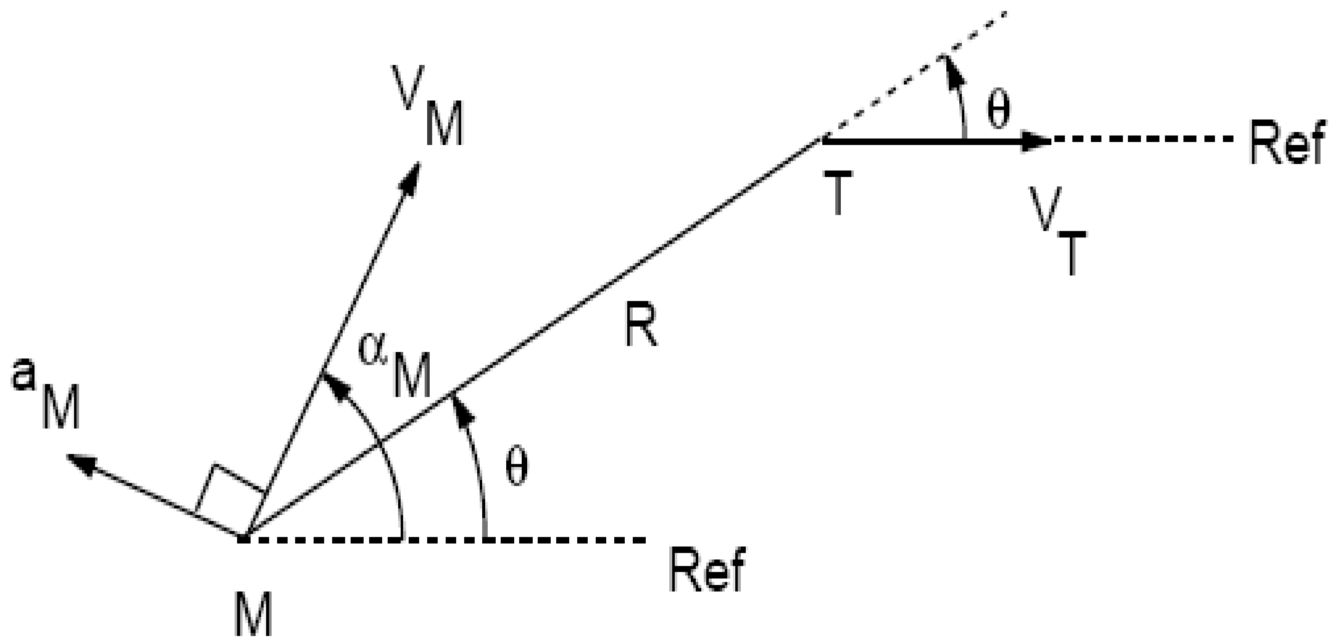


Figure 11.1: *Missile-target engagement geometry: Non-maneuvering target*

Equations of Motion

$$\begin{aligned} V_R = \dot{R} &= V_T \cos(-\theta) - V_M \cos(\alpha_M - \theta) \\ V_\theta = R\dot{\theta} &= V_T \sin(-\theta) - V_M \sin(\alpha_M - \theta) \end{aligned}$$

$$\begin{aligned} V_R = \dot{R} &= V_T \cos(\theta) - V_M \cos(\alpha_M - \theta) \\ V_\theta = R\dot{\theta} &= -V_T \sin(\theta) - V_M \sin(\alpha_M - \theta) \end{aligned}$$

PPN Guidance

$$\dot{\alpha}_M = N\dot{\theta}$$

$$\alpha_M - \alpha_{M0} = N\theta - N\theta_0$$

$$\alpha_M - \theta = k\theta + \phi_0$$

$$k = N - 1 \text{ and } \phi_0 = -N\theta_0 + \alpha_{M0}.$$

Substituting …

$$V_R(\theta) = \dot{R} \quad = \quad V_T \cos(\theta) - V_M \cos(k\theta + \phi_0)$$

$$V_\theta(\theta) = R\dot{\theta} \quad = \quad -V_T \sin(\theta) - V_M \sin(k\theta + \phi_0)$$

These equations are functions of THETA only

Let us try to solve …

$$\frac{\dot{R}}{R\dot{\theta}} = \frac{V_R(\theta)}{V_\theta(\theta)}$$

$$R = R_0 \exp\left\{\int_{\theta_0}^{\theta} \frac{V_R(\theta)}{V_\theta(\theta)} d\theta\right\}$$

$$\frac{1}{R}\,\frac{dR}{d\theta} = \frac{V_R(\theta)}{V_\theta(\theta)}$$

Although the variables have been separated, these equations can be solved for N=1 and N=2 (partially) but difficult to solve for higher N.

$$\int \frac{dR}{R} = \int \frac{V_R(\theta)}{V_\theta(\theta)} d\theta$$

They do have a solution, but it is very complicated and we will not go into the details of that in this class.

Qualitative Analysis

$$v_R(\theta) = \frac{V_R(\theta)}{V_T} = \frac{\dot{R}}{V_T} \quad = \quad \cos(\theta) - \nu\cos(k\theta + \phi_0)$$

$$v_\theta(\theta) = \frac{V_\theta(\theta)}{V_T} = \frac{R\dot{\theta}}{V_T} \quad = \quad -\sin(\theta) - \nu\sin(k\theta + \phi_0)$$

$$\nu = \frac{V_M}{V_T}$$

A Lemma

Lemma 11.1. If $\nu > 1$ and $k\nu > 1$, then the roots of the equations,

$$v_R(\theta) = \frac{V_R(\theta)}{V_T} = \frac{\dot{R}}{V_T} \quad = \quad \cos(\theta) - \nu\cos(k\theta + \phi_0) = 0$$

$$v_\theta(\theta) = \frac{V_\theta(\theta)}{V_T} = \frac{R\dot{\theta}}{v_T} \quad = \quad -\sin(\theta) - \nu\sin(k\theta + \phi_0) = 0$$

alternate along the θ axis.

$$\theta_{\theta 1} < \theta_{R1} < \theta_{\theta 2} < \theta_{R2} < \cdots < \theta_{\theta i} < \theta_{Ri} < .....$$

$$\theta_{R1} < \theta_{\theta 1} < \theta_{R2} < \theta_{\theta 2} < \cdots < \theta_{Ri} < \theta_{\theta i} < .....$$

Another Lemma

Lemma 11.2. If $\nu > 1$ and $k\nu > 1$; and θ_θ is a root of the equation $v_\theta(\theta) = 0$ then,

$$v_R(\theta_\theta)\ \frac{dv_\theta(\theta_\theta)}{d\theta} > 0$$

According to Lemma 11.2, if the slope of the *vtheta* curve is negative at one of its roots then the value of *vR* at that point is also negative.

Similarly, if the slope of *vtheta* is positive then the value of *vR* is also positive.

These Lemmas mean

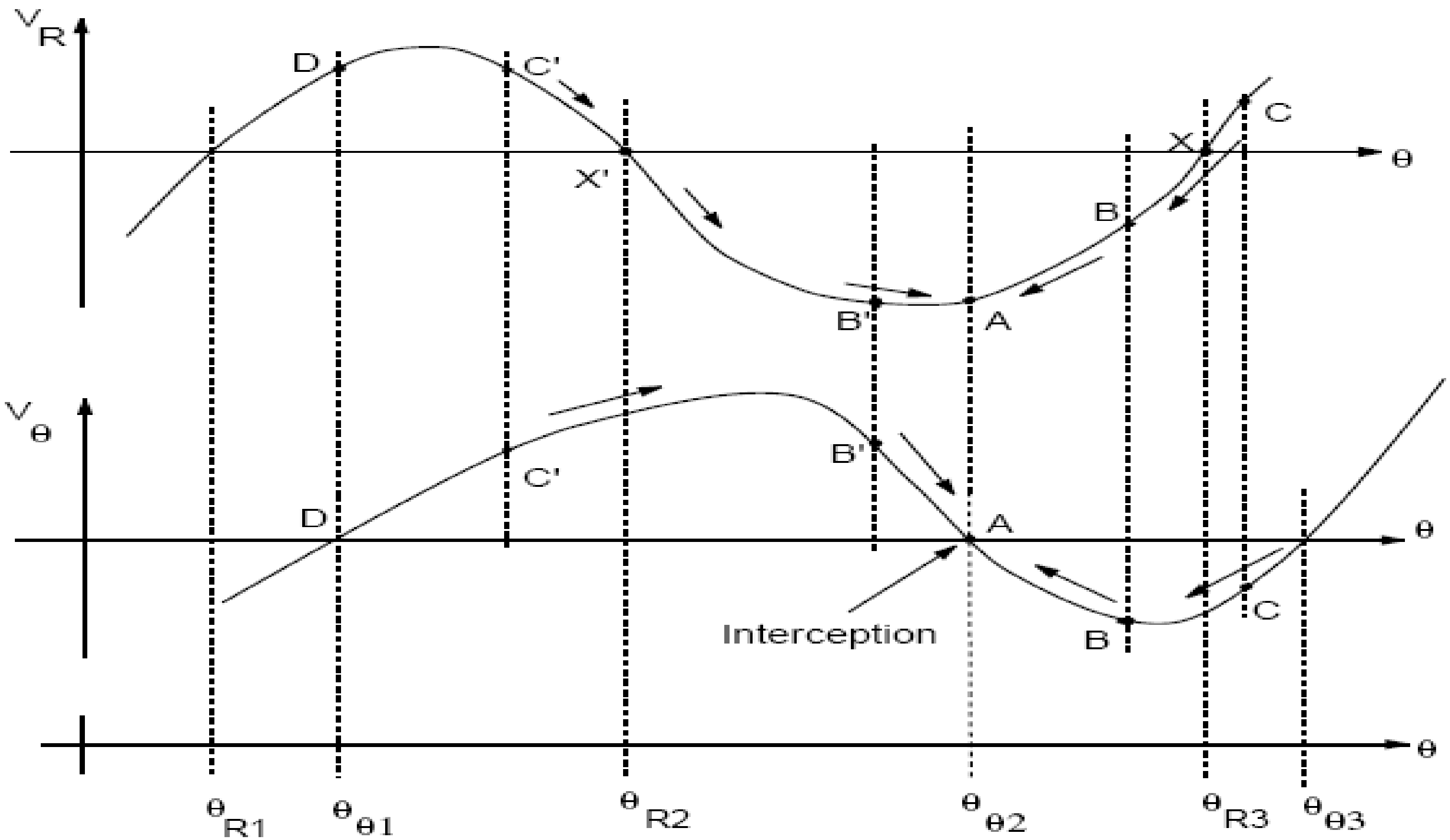


Figure 11.2: *A qualitative description of* $v_R(\theta)$ *and* $v_\theta(\theta)$

Polar Coordinate

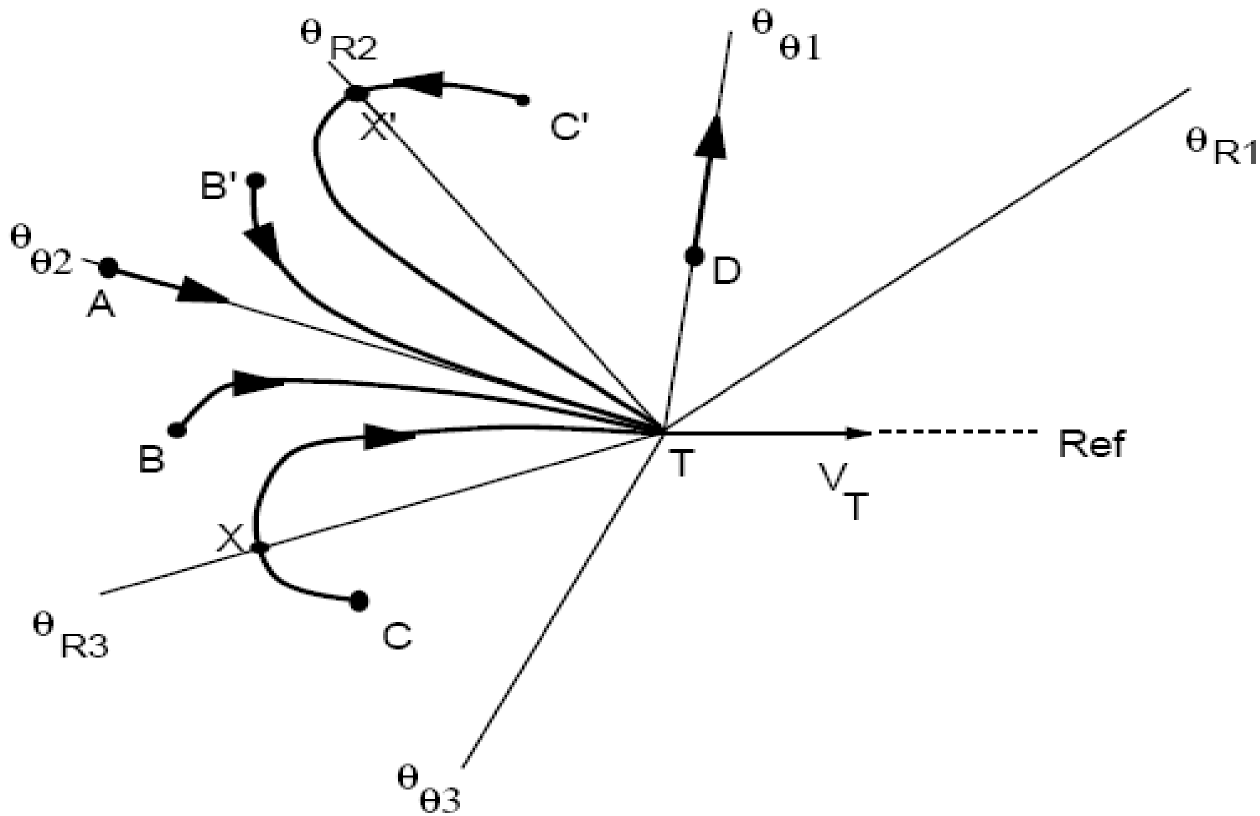


Figure 11.3: *A representation of the roots of* $v_\theta(\theta)$ *and* $v_R(\theta)$ *in the polar plane*

Capturability Theorem

A missile pursuing a non-maneuvering target, and following a PPN law with $\nu > 1$ and $k\nu > 1$ will be able to capture the target from all initial conditions except those for which *v_theta* = 0 and *v_R* > 0.

Capture Region

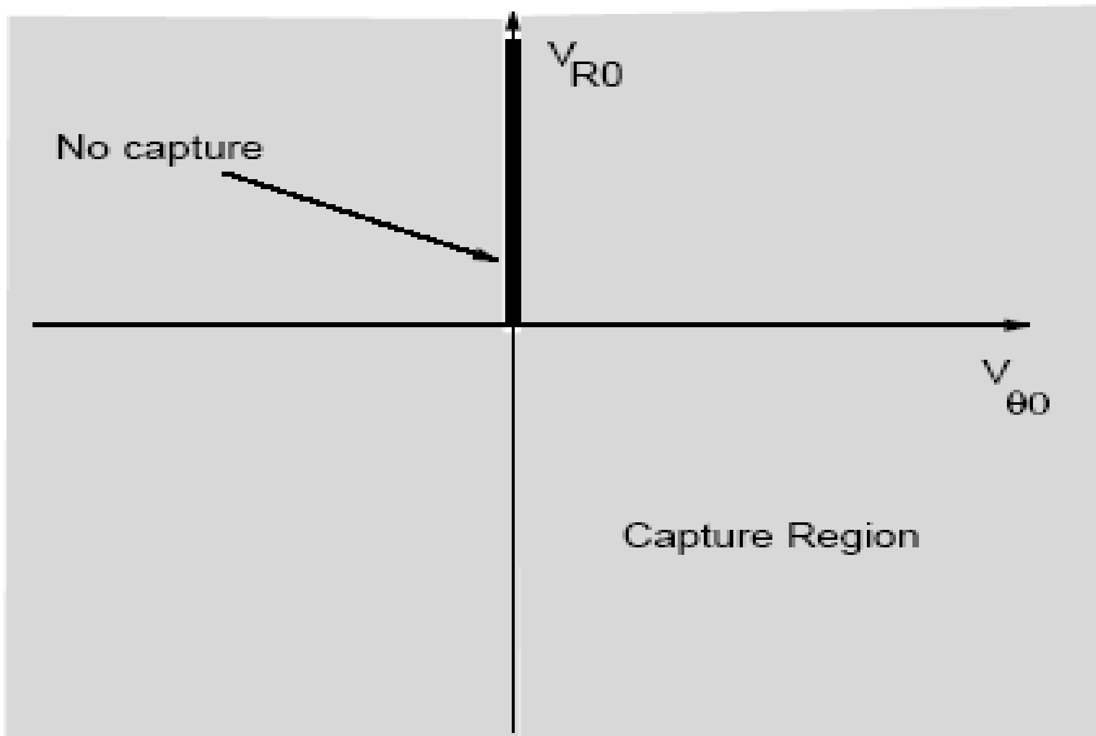


Figure 11.4: *Capture region of PPN against non-maneuvering target*

Pure Proportional Navigation against Maneuvering Targets

Engagement Geometry

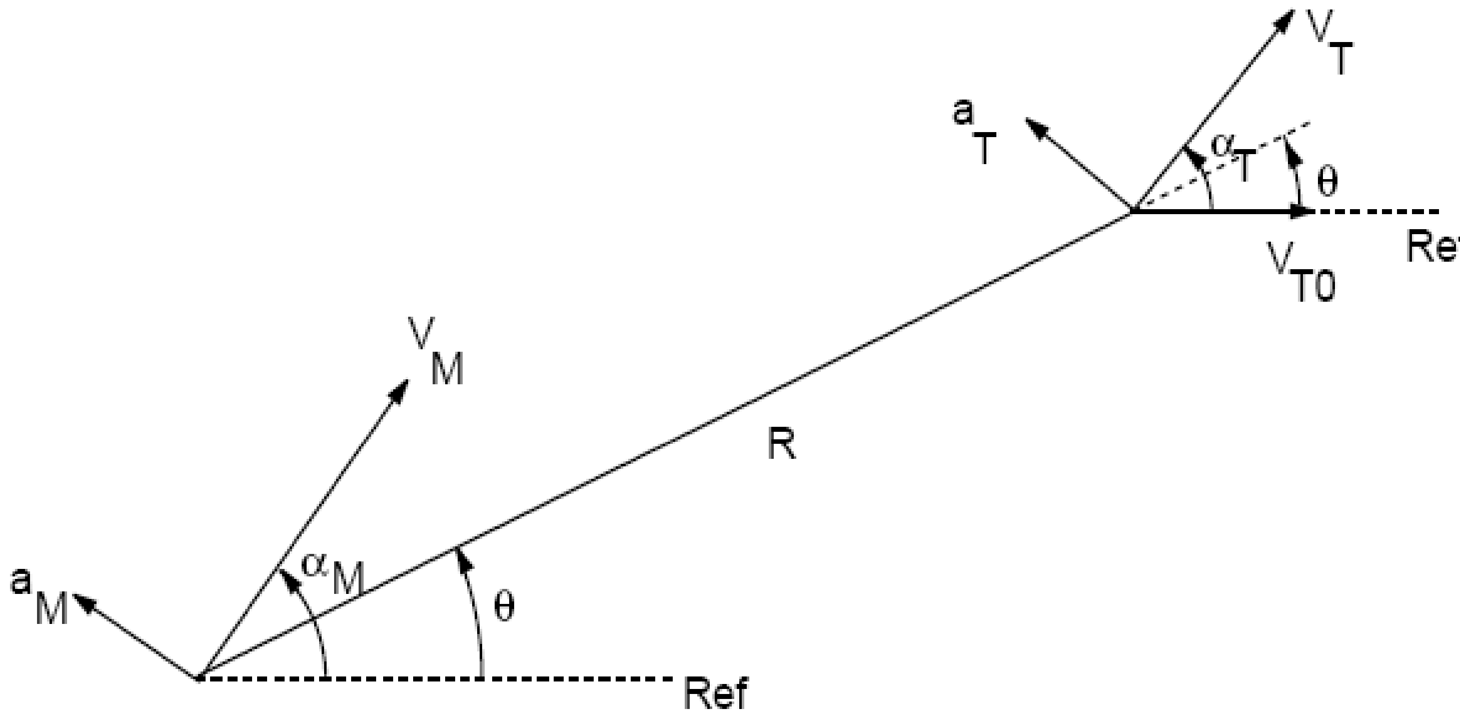


Figure 11.9: Missile-target engagement geometry: Maneuvering target

Constant target maneuver

$$\alpha_T = \left(\frac{a_T}{V_T}\right) t = a_{\nu T} t$$

Normalized latax

$$a_{\nu T} = (a_T / V_T)$$

Equations of Motion

$$
\begin{aligned}
V_R = \dot{R} &= V_T \cos(\alpha_T - \theta) - V_M \cos(\alpha_M - \theta) \\
V_\theta = R\dot{\theta} &= V_T \sin(\alpha_T - \lambda) - V_M \sin(\alpha_M - \lambda)
\end{aligned}
$$

$$
\begin{aligned}
V_R = \dot{R} &= V_T \cos(\theta - a_{\nu T} t) - V_M \cos(\alpha_M - \theta) \\
V_\theta = R\dot{\theta} &= -V_T \sin(\theta - a_{\nu T} t) - V_M \sin(\alpha_M - \theta)
\end{aligned}
$$

As before ….

$$\alpha_M - \theta = k\theta + \phi_0$$

$$k = N - 1 \text{ and } \phi_0 = -N\theta_0 + \alpha_{M0}.$$

$$\begin{aligned} V_R(\theta,t) = \dot{R} &= V_T\cos(\theta - a_{\nu T}t) - V_M\cos(k\theta + \phi_0) \\ V_\theta(\theta,t) = R\dot{\theta} &= -V_T\sin(\theta - a_{\nu T}t) - V_M\sin(k\theta + \phi_0) \end{aligned}$$

Normalizing

$$
\begin{aligned}
v_R(\theta, t) = \frac{V_R(\theta, t)}{V_T} = \frac{\dot{R}}{V_T} \quad &= \quad \cos(\theta - a_{\nu T} t) - \nu \cos(k\theta + \phi_0) \\
v_\theta(\theta, t) = \frac{V_\theta(\theta, t)}{V_T} = \frac{R\dot{\theta}}{V_T} \quad &= \quad -\sin(\theta - a_{\nu T} t) - \nu \sin(k\theta + \phi_0)
\end{aligned}
$$

A Lemma

Lemma 11.3. For a given t (say, $t = t_1$), if $\nu > 1$ and $k\nu > 1$, then the roots of the equations,

$$\begin{aligned} v_R(\theta, t_1) &= \cos(\theta - a_{\nu T} t_1) - \nu \cos(k\theta + \phi_0) = 0 \\ v_\theta(\theta, t_1) &= -\sin(\theta - a_{\nu T} t_1) - \nu \sin(k\theta + \phi_0) = 0 \end{aligned}$$

alternate along the θ axis. $\square$

Another Lemma

Lemma 11.4. For a given t (say, $t = t_1$), if $\nu > 1$ and $k\nu > 1$; and θ_θ is a root of the equation $v_\theta(\theta, t_1) = 0$ then,

$$v_R(\theta_\theta, t_1)\,\frac{dv_\theta(\theta_\theta, t_1)}{d\theta} > 0 \tag{11.29}$$

$\square$

The relative velocities against theta for a fixed time

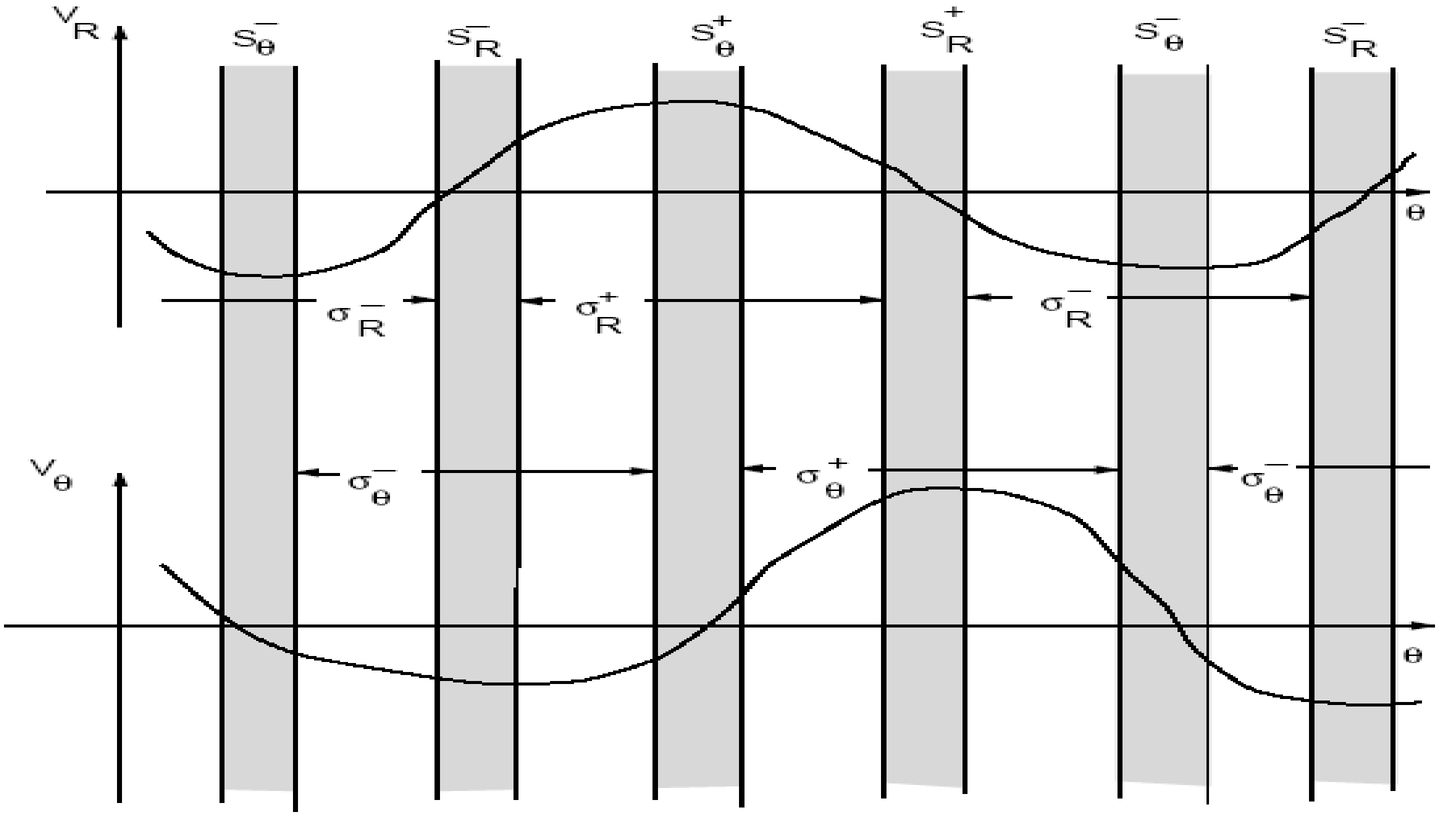


Figure 11.10: A qualitative description of $v_R(\theta, t)$ and $v_\theta(\theta, t)$

For a root of V theta to exist for a given t

$$-\sin(\theta_\theta - a_{\nu T}t) - \nu\sin(k\theta_\theta + \phi_0) = 0$$

$$t = \left(\frac{1}{a_{\nu T}}\right)\left[\theta_\theta - \sin^{-1}\{-\nu\sin(k\theta_\theta + \phi_0)\}\right]$$

$$|\sin(k\theta_\theta + \phi_0)| \leq \frac{1}{\nu}$$

$$-\frac{1}{\nu} \leq \sin(k\theta_\theta + \phi_0) \leq \frac{1}{\nu}$$

Some relationships

$$\sin^{-1}(-x) \quad = \quad -\sin^{-1}(x)$$

$$-a \leq \sin x \leq a \quad \Rightarrow \quad -a \leq -\sin x \leq a \;\Rightarrow$$

$$-a \leq \sin(n\pi + x) \leq a, \text{for } n = 0, \pm 1, \pm 2, \ldots$$

Using them …

$$-\sin^{-1}\left(\frac{1}{\nu}\right) \le k\theta_\theta + \phi_0 + n\pi \le \sin^{-1}\left(\frac{1}{\nu}\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$-\frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \le \theta_\theta - \theta_{n0} \le \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right)$$

$$\theta_{n0} = -\frac{\phi_0 + n\pi}{k}$$

Rearranging …

$$\theta_{n0} - \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \leq \theta_\theta \leq \theta_{n0} + \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right)$$

Similar analysis for roots of VR

$$\cos(\theta_R - a_{\nu T} t) - \nu \cos(k\theta_R + \phi_0) = 0$$

$$t = \left(\frac{1}{a_{\nu T}}\right)\left[\theta_R - \cos^{-1}\{\nu \cos(k\theta_R + \phi_0)\}\right]$$

$$|\cos(k\theta_R + \phi_0)| \leq \frac{1}{\nu}$$

$$-\frac{1}{\nu} \leq \cos(k\theta_R + \phi_0) \leq \frac{1}{\nu}$$

1 as,

$$-\frac{1}{\nu} \leq \sin\left(\frac{\pi}{2} + k\theta_R + \phi_0\right) \leq \frac{1}{\nu}$$

$$-\sin^{-1}\left(\frac{1}{\nu}\right) \le \frac{\pi}{2} + k\theta_R + \phi_0 + m\pi \le \sin^{-1}\left(\frac{1}{\nu}\right), \quad m = 0, \pm 1, \pm 2, \ldots$$

$$\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{\nu}\right) \le k\theta_R + \phi_0 + n\pi \le \frac{\pi}{2} + \sin^{-1}\left(\frac{1}{\nu}\right), \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\frac{\pi}{2k} - \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \leq \theta_R - \theta_{n0} \leq \frac{\pi}{2k} + \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right)$$

lity, on rearrangement, yields

$$\theta_{n0} + \frac{\pi}{2k} - \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right) \leq \theta_R \leq \theta_{n0} + \frac{\pi}{2k} + \frac{1}{k}\sin^{-1}\left(\frac{1}{\nu}\right)$$

End of Lecture 4 THANK YOU