Power series

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In mathematics, a **power series** (in one variable) is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1 (x-c)^1 + a_2 (x-c)^2 + \dots$$

where a_n represents the coefficient of the *n*th term and c is a constant. This series usually arises as the Taylor series of some known function.

In many situations c (the *center* of the series) is equal to zero, for instance when considering a Maclaurin series. In such cases, the power series takes the simpler form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

These power series arise primarily in analysis, but also occur in combinatorics (as generating functions, a kind of formal power series) and in electrical engineering (under the name of the Z-transform). The familiar decimal notation for real numbers can also be viewed as an example of a power series, with integer coefficients, but with the argument x fixed at $1/_{10}$. In number theory, the concept of p-adic numbers is also closely related to that of a power series.

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Examples

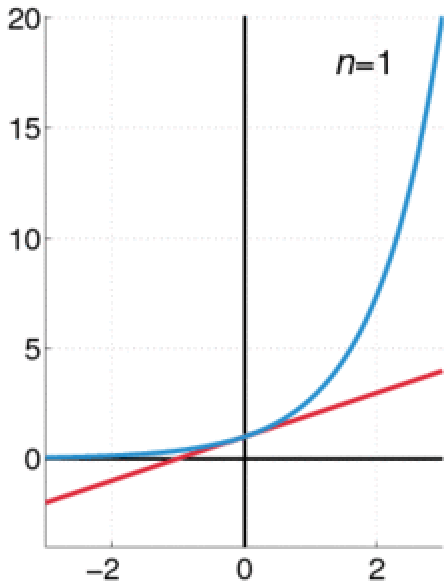
Any polynomial can be easily expressed as a power series around any center c, although most of the coefficients will be zero since a power series has infinitely many terms by definition. For instance, the polynomial $f(x) = x^2 + 2x + 3$ can be written as a power series around the center $c = 0$ as

$$f(x) = 3 + 2x + 1x^2 + 0x^3 + 0x^4 + \cdots$$

or around the center $c = 1$ as

$$f(x) = 6 + 4(x-1) + 1(x-1)^2 + 0(x-1)^3 + 0(x-1)^4 + \cdots$$

or indeed around any other center c.[1] One can view power series as being like "polynomials of infinite degree," although power series are not polynomials.



The exponential function(in blue), and the sum of the first n+1 terms of its Maclaurin power series(in red).

The geometric series formula

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$$

which is valid for $|x| < 1$, is one of the most important examples of a power series, as are the exponential function formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

and the sine formula

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

valid for all real x. These power series are also examples of Taylor series.

Negative powers are not permitted in a power series; for instance, $1 + x^{-1} + x^{-2} + \cdots$ is not considered a power series (although it is a Laurent series). Similarly, fractional powers such as $x^{1/2}$ are not permitted (but see Puiseux series). The coefficients a_n are not allowed to depend on x, thus for instance:

$\sin(x)x + \sin(2x)x^2 + \sin(3x)x^3 + \cdots$ is not a power series.

Radius of convergence

A power series will converge for some values of the variable x and may diverge for others. All power series $f(x)$ in powers of $(x-c)$ will converge at $x = c$. (The correct value $f(c) = a_0$ requires interpreting the expression 0^0 as equal to 1.) If c is not the only convergent point, then there is always a number r with $0 < r \le \infty$ such that the series converges whenever $|x - c| < r$ and diverges whenever $|x - c| > r$. The number r is called the **radius of convergence** of the power series; in general it is given as

$$r = \liminf_{n\to\infty} |a_n|^{-\frac{1}{n}}$$

or, equivalently,

$$r^{-1} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

(this is the Cauchy–Hadamard theorem; see limit superior and limit inferior for an explanation of the notation). A fast way to compute it is

$$r^{-1} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

if this limit exists.

The series converges absolutely for $|x - c| < r$ and converges uniformly on every compact subset of $\{x : |x - c| < r\}$. That is, the series is absolutely and compactly convergent on the interior of the disc of convergence.

For $|x - c| = r$, we cannot make any general statement on whether the series converges or diverges. However, for the case of real variables, Abel's theorem states that the sum of the series is continuous at x if the series converges at x. In the case of complex variables, we can only claim continuity along the line segment starting at c and ending at x.

Operations on power series

Addition and subtraction

When two functions f and g are decomposed into power series around the same center c, the power series of the sum or difference of the functions can be obtained by termwise addition and subtraction. That is, if:

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$
$$g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$$

then

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-c)^n.$$

Multiplication and division

With the same definitions above, for the power series of the product and quotient of the functions can be obtained as follows:

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n (x-c)^n\right)\left(\sum_{n=0}^{\infty} b_n (x-c)^n\right)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_i b_j (x-c)^{i+j}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)(x-c)^n.$$

The sequence $m_n = \sum_{i=0}^{n} a_i b_{n-i}$ is known as the convolution of the sequences a_n and b_n.

For division, if one defines the sequence d_n by

$$\frac{f(x)}{g(x)} = \frac{\sum_{n=0}^{\infty} a_n (x-c)^n}{\sum_{n=0}^{\infty} b_n (x-c)^n} = \sum_{n=0}^{\infty} d_n (x-c)^n$$

then

$$f(x)=\left(\sum_{n=0}^{\infty} b_n (x-c)^n\right)\left(\sum_{n=0}^{\infty} d_n (x-c)^n\right)$$

and one can solve recursively for the terms d_n by comparing coefficients.

Differentiation and integration

Once a function is given as a power series, it is differentiable on the interior of the domain of convergence. It can be differentiated and integrated quite easily, by treating every term separately:

$$f'(x)=\sum_{n=1}^{\infty} a_n n (x-c)^{n-1}=\sum_{n=0}^{\infty} a_{n+1}(n+1)(x-c)^n$$

$$\int f(x)\,dx=\sum_{n=0}^{\infty}\frac{a_n (x-c)^{n+1}}{n+1}+k=\sum_{n=1}^{\infty}\frac{a_{n-1}(x-c)^n}{n}+k.$$

Both of these series have the same radius of convergence as the original one.

Analytic functions

A function *f* defined on some open subset *U* of **R** or **C** is called analytic if it is locally given by a convergent power series. This means that every $a \in U$ has an open neighborhood $V \subseteq U$, such that there exists a power series with center *a* which converges to *f*(*x*) for every $x \in V$.

Every power series with a positive radius of convergence is analytic on the interior of its region of convergence. All holomorphic functions are complex-analytic. Sums and products of analytic functions are analytic, as are quotients as long as the denominator is non-zero.

If a function is analytic, then it is infinitely often differentiable, but in the real case the converse is not generally true. For an analytic function, the coefficients a_n can be computed as

$$a_n=\frac{f^{(n)}(c)}{n!}$$

where $f^{(n)}(c)$ denotes the *n*th derivative of *f* at *c*, and $f^{(0)}(c)=f(c)$. This means that every analytic function is locally represented by its Taylor series.

The global form of an analytic function is completely determined by its local behavior in the following sense: if *f* and *g* are two analytic functions defined on the same connected open set *U*, and if there exists an element $c \in U$ such that $f^{(n)}(c) = g^{(n)}(c)$ for all $n \geq 0$, then $f(x) = g(x)$ for all $x \in U$.

If a power series with radius of convergence *r* is given, one can consider analytic continuations of the series, i.e. analytic functions *f* which are defined on larger sets than $\{ x : |x - c| < r \}$ and agree with the given power series on this set. The number *r* is maximal in the following sense: there always exists a complex number *x* with $|x - c| = r$ such that no analytic continuation of the series can be defined at *x*.

The power series expansion of the inverse function of an analytic function can be determined using the Lagrange inversion theorem.

Formal power series

In abstract algebra, one attempts to capture the essence of power series without being restricted to the fields of real and complex numbers, and without the need to talk about convergence. This leads to the concept of formal power series, a concept of great utility in algebraic combinatorics.

Power series in several variables

An extension of the theory is necessary for the purposes of multivariable calculus. A **power series** is here defined to be an infinite series of the form

$$f(x_1,\dots,x_n)=\sum_{j_1,\dots,j_n=0}^{\infty}a_{j_1,\dots,j_n}\prod_{k=1}^{n}(x_k-c_k)^{j_k},$$

where $j = (j_1, ..., j_n)$ is a vector of natural numbers, the coefficients $a_{(j_1,...,j_n)}$ are usually real or complex numbers, and the center $c = (c_1, ..., c_n)$ and argument $x = (x_1, ..., x_n)$ are usually real or complex vectors. The symbol Π is the product symbol, denoting multiplication. In the more convenient multi-index notation this can be written

$$f(x)=\sum_{\alpha\in\mathbb{N}^n}a_\alpha(x-c)^\alpha.$$

where $\mathbb{N}$ is the set of natural numbers, and so $\mathbb{N}^n$ is the set of ordered n-tuples of natural numbers.

The theory of such series is trickier than for single-variable series, with more complicated regions of convergence. For instance, the power series $\sum_{n=0}^{\infty}x_1^n x_2^n$ is absolutely convergent in the set $\{(x_1,x_2):|x_1x_2|<1\}$ between two hyperbolas. (This is an example of a *log-convex set*, in the sense that the set of points $(\log|x_1|,\log|x_2|)$, where (x_1,x_2) lies in the above region, is a convex set. More generally, one can show that when c=0, the interior of the region of absolute convergence is always a log-convex set in this sense.) On the other hand, in the interior of this region of convergence one may differentiate and integrate under the series sign, just as one may with ordinary power series.

Order of a power series

Let α be a multi-index for a power series $f(x_1, x_2, \ldots, x_n)$. The **order** of the power series f is defined to be the least value $|\alpha|$ such that $a_\alpha \neq 0$, or 0 if $f \equiv 0$. In particular, for a power series $f(x)$ in a single variable x, the order of f is the smallest power of x with a nonzero coefficient. This definition readily extends to Laurent series.

See also

- Flat function
- Linear approximation
- Random variable
- Series multisection

References

1. Howard Levi (1967). *Polynomials, Power Series, and Calculus* (https://books.google.com/books?id=AcI-AAAAIAAJ). Van Nostrand. p. 24.

- Solomentsev, E.D. (2001) [1994], "Power series" (https://www.encyclopediaofmath.org/index.php?title=Power_series&oldid=15309), in Hazewinkel, Michiel, *Encyclopedia of Mathematics*, Springer

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- Weisstein, Eric W. "Formal Power Series" (http://mathworld.wolfram.com/FormalPowerSeries.html). *MathWorld*.
- Weisstein, Eric W. "Power Series" (http://mathworld.wolfram.com/PowerSeries.html). *MathWorld*.
- Powers of Complex Numbers (http://demonstrations.wolfram.com/PowersOfComplexNumbers/) by Michael Schreiber, Wolfram Demonstrations Project.