Hidden Figures

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Introduction

In this lab I'll try to show you an application of the Z transform to solving continuous time differential equations through discretization. We're going to be using forward difference (Euler's Method), and also the backwards difference method to perform the discritization and obtain the solution of a first order differential equation. For your entertainment, here's the scene from the Hidden Figures movie where Katherine and Sheldon Cooper wax poetic about the Euler's method. Euler's Method

The Stars of Our Show

The methods we're going to use today are the forward difference equation given below, let " $\Delta_x[n]$" be a discrete approximation of the derivative.

$$\frac{dy}{dt} \approx \Delta_f y[n] = \frac{y[n+1] - y[n]}{T} \tag{1}$$

We will also employ the backwards difference equation given below

$$\frac{dy}{dt} \approx \Delta_b y[n] = \frac{y[n] - y[n-1]}{T} \tag{2}$$

Given below is the equation we'd like to discretize and solve numerically. We'll also use the Z transform to determine a step size which will yield a stable result for each differentiation method.

$$\frac{dy}{dt} + ky = x(t) \tag{3}$$

This equation has the initial condition $y(t=0) = 0$.

Analysis

Step 1: Forwards Difference

For the terms $y(t)$, and $x(t)$, it is sufficient to substitute in $y[n]$ and $x[n]$ at this time to stand in our equation. We will also replace the derivative term $\frac{dy}{dt}$ with the forward difference equation. Please write the resulting discrete time equation.

Step 2: A pesky T in our denominator

For the reason that we don't really want to deal with fractions, and also that it will provide a later insight into the stability of our numerical solution, let's multiply both sides of the equation from Step 1 by T. Write that equation below.

Step 3: Y(z)

The time has come. Let's take the Z transform of this function, and solve it for $\frac{Y(z)}{X(z)}$.

Step 4: Pole Location

Express the pole location. Leave everything in a variable form. When is the discrete system stable? Assume K is a positive constant.

Step 1a, 2a, 3a, ...: Backwards Difference

Let's do the same steps to determine the pole location of the discrete time system using the backwards difference equation instead. We'll refer to these steps in our lab as Step 1a, Step 2a etc. It may take a little more love to coax the pole location out for this one but you will get it.

Simulation

The solution of

$$\frac{dy}{dt} + ky = 0; \tag{4}$$

With the initial condition that

$$y(0) = 1 \tag{5}$$

is

$$y(t) = e^{-kt} \tag{6}$$

Step 5: Z.T.F. Table

Use the Z transform table to verify that the solution to the forward difference equation is containing the term

$$y[n] = (1 - kT)^n \tag{7}$$

I'll just tell you that the closed form solution for the backwards difference equation is containing the term

$$y[n] = (1 + kT)^{-n} \tag{8}$$

Overall I just want you to have some idea where this process is coming from.

Step 6: Continuous Time Function

Plot e^{-kt} from $t = 0$ to $t = 11$ seconds, and use an interval of $1(10)^{-3}$ seconds. **For all upcoming simulations, we will set k = 1**.

Step 7: Comparison Plot

Generate the following plot. Use the subplot() command to achieve this goal. In your lab report, show calculations that verify the pole location that I've given. The blue line should be the plot of e^{-1t} from Step 6. Each red line is the approximation of the given forward (eqn 7) or backwards (eqn 8) difference equation with the given value of T.

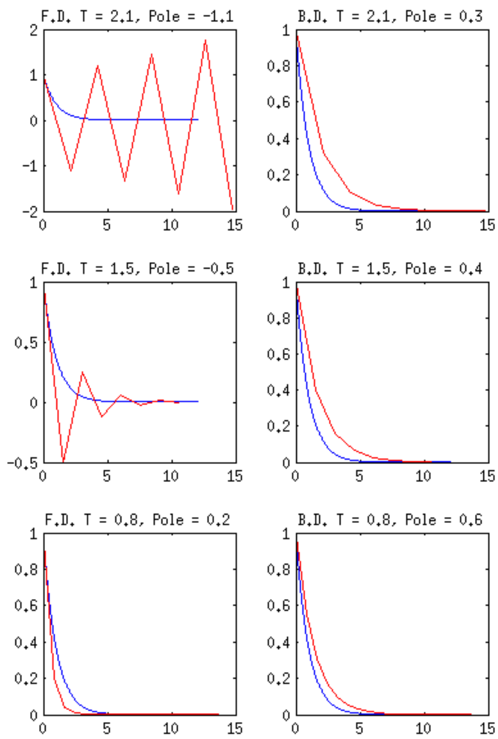

Figure 1: Comparison of Forward and Backward Difference for varying T

Step 8: Detailed Response

Please state detailed reasons explaining why you'd prefer to use one method or the other to approximate the solution to this differential equation.

Step 9: For Fun Not for Credit

Watch the video clip linked in the first paragraph of this document. Then read the first 1 sentence of Leonhard Euler's Wikipedia page. In your opinion, was Sheldon's description of Euler's method accurate?