Dependent Types, Twelf and Its Application in Proof

Seyed Mohammad Mehdi Ahmadpanah

smahmadpanah@aut.ac.ir

Supervisor

Dr. Mehran S. Fallah

Formal Security Lab.

CEIT@AUT

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Outline

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- The Curry-Howard Correspondence
- Logical Frameworks
- Pure First-Order Dependent Types
- Dependent Sum Types
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- Twelf in Practice
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- Conclusion

Motivation

- Dependent types are type-valued functions
  - Those functions which send $terms$ to types
- Type family of vectors (one-dimensional arrays)

$$\texttt{Vector :: Nat} \rightarrow *$$

- **Kinding assertion:** Vector maps a $\mathrm{k{:}Nat}$ to a type
  - type $\mathrm{Vector\ k}$ contains vectors of length k of elements of some fixed type, say data
- Initialization function:

$$\mathrm{init} : \Pi \mathrm{n{:}Nat.\ data} \rightarrow \mathrm{Vector\ n}$$

Motivation (cont.)

$$\text{init} : \Pi \text{n:Nat. data} \rightarrow \text{Vector n}$$

- Dependent Product Type (Pi type)

$$\Pi x : S \, . \, T$$

  - Generalizes the arrow type of the simply typed λ-calculus

$$S \rightarrow [x \mapsto s] \; T$$

  - The result type can vary according to the argument supplied
  - Π-type is almost as old as the lambda calculus

Motivation (cont.)

- **Another way of building up vectors (constructor)**

$$\text{empty} : \text{Vector } 0$$

$$\text{cons} : \Pi\text{n:Nat. data} \rightarrow \text{Vector n} \rightarrow \text{Vector (n+1)}$$

- **Example**

$$\text{v : Vector 5 , x : data } \textbf{then} \text{ cons 5 x v : Vector 6}$$

Πx : S . T (Dependent Product Type)

$$S \rightarrow [x \mapsto s]\ T$$

vs.

Universal Type ∀X. T of System F

(If t: $\forall X.T$ **and** A **is a** **type**, **then** $t\ A : X \rightarrow [X \mapsto A]\ T$**)**

Motivation (cont.)

- Why dependent typing?
  - It reveals more information about the behavior of the term
  - More precious typing
  - Exclude more of the badly behaved terms in a type system
- We can type a function that returns the first element of a non-empty vector:

$$\text{first} : \Pi\text{n:Nat. Vector(n+1)} \to \text{data}$$

  - Non-emptiness is expressed within the type system itself!

Motivation (cont.)

- Another example: sprintf

$$\text{sprintf} : \Pi \text{f:Format. Data(f)} \rightarrow \text{String}$$

  - Format: type of valid print formats
  - Data(f): type of data corresponding to format f

```
Data([])          =  Unit
Data("%d"::cs)    =  Nat * Data(cs)
Data("%s"::cs)    =  String * Data(cs)
Data(c::cs)       =  Data(cs)
```

  - Vectors are uniform, here is non-uniform
    - More challenging!

Curry-Howard Correspondence

- Proposition-as-Type (and Proof-as-Term)
  - A formula has a proof iff the corresponding type is inhabited
- Example:

$$((A \to B) \to A) \to (A \to B) \to B \quad \text{(formula, type)}$$

$$\lambda f.\ \lambda u.\ u\ (f\ u) \quad \text{(proof, term)}$$

- Constructive proof of $A \Rightarrow B$ should be understood as a procedure that transforms any given proof of A into a proof of B
  - A proof of $A \Rightarrow B$ is simply any term of type $A \to B$

Curry-Howard Correspondence (cont.)

- Generalizing the correspondence to first-order predicate logic leads to dependent types
- A proof of the universal quantification $\forall \mathrm{x}{:}\mathrm{A}.\ \mathrm{B}(\mathrm{x})$ is constructively a procedure that given an arbitrary element x of type A produces a proof of $\mathrm{B}(\mathrm{x})$
- Identification of universal quantification with dependent product
  - A proof of $\forall \mathrm{x}{:}\mathrm{A}.\ \mathrm{B}(\mathrm{x})$ is a member of $\Pi \mathrm{x}{:}\mathrm{A}.\ \mathrm{B}(\mathrm{x})$
- Existential quantification $\equiv$ Σ-types
- Equality $\equiv$ Identity types

Curry-Howard Correspondence (cont.)

- Application: freely mixing propositions and types

Example - indexing function

$$\mathrm{ith(n)} : \Pi \mathrm{n{:}Nat}.\ \Pi \mathrm{l{:}Nat}.\ \mathrm{Lt(l,n)} \to \mathrm{Vector(n)} \to \mathrm{T}$$

Example – type of binary, associative operations on some type T

$$\Sigma \mathrm{m}{:}\ \mathrm{T} \to \mathrm{T} \to \mathrm{T}.\ \Pi \mathrm{x} : \mathrm{T}.\ \Pi \mathrm{y} : \mathrm{T}.\ \Pi \mathrm{z} : \mathrm{T}.$$
$$\mathrm{Id}\ (\mathrm{m(x,m(y,z))})\ (\mathrm{m(m(x,y),z)})$$

- A proof of $\exists \mathrm{x{:}A}.\ \mathrm{B(x)}$ would consist of a member a of type A and a proof (a member) of $\mathrm{B(a)}$
  - an element of $\Sigma \mathrm{a{:}A}.\ \mathrm{B(a)}$

Logical Frameworks

- Another application: representation of other type theories and formal systems
- Example – typechecker for simply typed λ-calculus

```
Ty    :: *
Tm    :: Ty → *
base  : Ty
arrow : Ty → Ty → Ty
app   : ΠA:Ty.ΠB:Ty.Tm(arrow A B) → Tm A → Tm B
lam   : ΠA:Ty.ΠB:Ty.(Tm A → Tm B) → Tm(arrow A B)
```

- Higher-order abstract syntax
- e.g. representation of $\mathrm{identity}$ term: $(\mathrm{A}\colon \mathrm{Ty})$

$$\mathrm{idA} = \mathrm{lam}\ \mathrm{A}\ \mathrm{A}\ (\lambda x\colon \mathrm{Tm}\ \mathrm{A}.\ x)$$

Logical Frameworks (cont.)

- **Definition: systems which provide mechanisms for representing** syntax **and** proof systems
  - **which makes up a logic**
- **It provides a means to define a logic as a "signature" in a higher-order type theory**
  - Provability of a formula **in the original logic reduces to a** type inhabitation problem **in the framework type theory**
- **To describe a logical framework, one must provide:**
  - **A characterization of the class of object-logics to be represented**
  - **An appropriate meta-language**
  - **A characterization of the mechanism by which object-logics are represented**

Logical Frameworks (cont.)

- One approach: Edinburgh Logical Framework (LF)
  - Judgments-as-Types
    - Judgments: types o
    - Derivations of judgm
  - Meta-language: dependently typed λ-calculus (λΠ-calculus)
    - Three-level entities: terms, types, kinds
- Twelf is an implementation of the logical framework LF
  - Twelf code describes logical systems
  - written in Standard ML
    - write out a statement
    - use Twelf to write out a proof (justification of why that statement is true)
    - Twelf will check your proof, making sure that what you said actually is true!

Logical Frameworks (cont.)

- Twelf includes:
  - an implementation of the LF logical framework, which can be used to type check LF representations
  - a logic programming language based on LF
  - a metatheorem checker, which can be used to verify proofs of theorems about LF representations
- Other systems that will let you define logical systems and prove things with them:
  - ACL2, AUTOMATH, Coq, HOL, HOL Light, LEGO, Isabelle, MetaPRL, NuPRL PVS, and TPS
  - In Twelf: *programming languages* are also logical systems

Pure First-Order Dependent Types

- λLF
  - Type system based on a simplified variant of the type system underlying LF
  - Generalizes simply typed λ-calculus by replacing the arrow type $S \to T$ with the dependent product type $\Pi x : S \,.\, T$ and by introducing type families
  - Pure
    - Only has Π-types
  - First-Order
    - Does not include higher-order type operators
  - Corresponds to $\forall$, $\to$-fragment of first-order predicate calculus

Pure First-Order Dependent Types (cont.)

λLF

Syntax

$\mathtt{t}$ ::=		*terms:*
	$\mathtt{x}$	*variable*
	$\lambda\mathtt{x}{:}\mathtt{T}.\mathtt{t}$	*abstraction*
	$\mathtt{t}\ \mathtt{t}$	*application*
$\mathtt{T}$ ::=		*types:*
	$\mathtt{X}$	*type/family variable*
	$\Pi\mathtt{x}{:}\mathtt{T}.\mathtt{T}$	*dependent product type*
	$\mathtt{T}\ \mathtt{t}$	*type family application*
$\mathtt{K}$ ::=		*kinds:*
	$*$	*kind of proper types*
	$\Pi\mathtt{x}{:}\mathtt{T}.\mathtt{K}$	*kind of type families*
Γ ::=		*contexts:*
	$\varnothing$	*empty context*
	$\Gamma, \mathtt{x}{:}\mathtt{T}$	*term variable binding*
	$\Gamma, \mathtt{X}{::}\mathtt{K}$	*type variable binding*

Well-formed kinds $\boxed{\Gamma \vdash \mathtt{K}}$

$$\Gamma \vdash * \qquad \text{(WF-STAR)}$$

$$\frac{\Gamma \vdash \mathtt{T} :: * \qquad \Gamma, \mathtt{x}{:}\mathtt{T} \vdash \mathtt{K}}{\Gamma \vdash \Pi\mathtt{x}{:}\mathtt{T}.\mathtt{K}} \qquad \text{(WF-PI)}$$

Kinding $\boxed{\Gamma \vdash \mathtt{T} :: \mathtt{K}}$

$$\frac{\mathtt{X} :: \mathtt{K} \in \Gamma \qquad \Gamma \vdash \mathtt{K}}{\Gamma \vdash \mathtt{X} :: \mathtt{K}} \qquad \text{(K-VAR)}$$

$$\frac{\Gamma \vdash \mathtt{T}_1 :: * \qquad \Gamma, \mathtt{x}{:}\mathtt{T}_1 \vdash \mathtt{T}_2 :: *}{\Gamma \vdash \Pi\mathtt{x}{:}\mathtt{T}_1.\mathtt{T}_2 :: *} \qquad \text{(K-PI)}$$

$$\frac{\Gamma \vdash \mathtt{S} :: \Pi\mathtt{x}{:}\mathtt{T}.\mathtt{K} \qquad \Gamma \vdash \mathtt{t} : \mathtt{T}}{\Gamma \vdash \mathtt{S}\ \mathtt{t} : [\mathtt{x} \mapsto \mathtt{t}]\mathtt{K}} \qquad \text{(K-APP)}$$

$$\frac{\Gamma \vdash \mathtt{T} :: \mathtt{K} \qquad \Gamma \vdash \mathtt{K} \equiv \mathtt{K}'}{\Gamma \vdash \mathtt{T} :: \mathtt{K}'} \qquad \text{(K-CONV)}$$

Typing $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\mathtt{x}{:}\mathtt{T} \in \Gamma \qquad \Gamma \vdash \mathtt{T} :: *}{\Gamma \vdash \mathtt{x} : \mathtt{T}} \qquad \text{(T-VAR)}$$

$$\frac{\Gamma \vdash \mathtt{S} :: * \qquad \Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{t} : \mathtt{T}}{\Gamma \vdash \lambda\mathtt{x}{:}\mathtt{S}.\mathtt{t} : \Pi\mathtt{x}{:}\mathtt{S}.\mathtt{T}} \qquad \text{(T-ABS)}$$

$$\frac{\Gamma \vdash \mathtt{t}_1 : \Pi\mathtt{x}{:}\mathtt{S}.\mathtt{T} \qquad \Gamma \vdash \mathtt{t}_2 : \mathtt{S}}{\Gamma \vdash \mathtt{t}_1\ \mathtt{t}_2 : [\mathtt{x} \mapsto \mathtt{t}_2]\mathtt{T}} \qquad \text{(T-APP)}$$

$$\frac{\Gamma \vdash \mathtt{t} : \mathtt{T} \qquad \Gamma \vdash \mathtt{T} \equiv \mathtt{T}' :: *}{\Gamma \vdash \mathtt{t} : \mathtt{T}'} \qquad \text{(T-CONV)}$$

Pure First-Order Dependent Types (cont.)

λLF

Kind Equivalence $\boxed{\Gamma \vdash K \equiv K'}$

$$\frac{\Gamma \vdash T_1 \equiv T_2 :: * \qquad \Gamma, x{:}T_1 \vdash K_1 \equiv K_2}{\Gamma \vdash \Pi x{:}T_1.K_1 \equiv \Pi x{:}T_2.K_2} \quad \text{(QK-PI)}$$

$$\frac{\Gamma \vdash K}{\Gamma \vdash K \equiv K} \quad \text{(QK-REFL)}$$

$$\frac{\Gamma \vdash K_1 \equiv K_2}{\Gamma \vdash K_2 \equiv K_1} \quad \text{(QK-SYM)}$$

$$\frac{\Gamma \vdash K_1 \equiv K_2 \qquad \Gamma \vdash K_2 \equiv K_3}{\Gamma \vdash K_1 \equiv K_3} \quad \text{(QK-TRANS)}$$

Type Equivalence $\boxed{\Gamma \vdash S \equiv T :: K}$

$$\frac{\Gamma \vdash S_1 \equiv T_1 :: * \qquad \Gamma, x{:}T_1 \vdash S_2 \equiv T_2 :: *}{\Gamma \vdash \Pi x{:}S_1.S_2 \equiv \Pi x{:}T_1.T_2 :: *} \quad \text{(QT-PI)}$$

$$\frac{\Gamma \vdash S_1 \equiv S_2 :: \Pi x{:}T.K \qquad \Gamma \vdash t_1 \equiv t_2 : T}{\Gamma \vdash S_1\, t_1 \equiv S_2\, t_2 : [x \mapsto t_1]K} \quad \text{(QT-APP)}$$

$$\frac{\Gamma \vdash T : K}{\Gamma \vdash T \equiv T :: K} \quad \text{(QT-REFL)}$$

$$\frac{\Gamma \vdash T \equiv S :: K}{\Gamma \vdash S \equiv T :: K} \quad \text{(QT-SYM)}$$

$$\frac{\Gamma \vdash S \equiv U :: K \qquad \Gamma \vdash U \equiv T :: K}{\Gamma \vdash S \equiv T :: K} \quad \text{(QT-TRANS)}$$

Term Equivalence $\boxed{\Gamma \vdash t_1 \equiv t_2 : T}$

$$\frac{\Gamma \vdash S_1 \equiv S_2 :: * \qquad \Gamma, x{:}S_1 \vdash t_1 \equiv t_2 : T}{\Gamma \vdash \lambda x{:}S_1.t_1 \equiv \lambda x{:}S_2.t_2 : \Pi x{:}S_1.T} \quad \text{(Q-ABS)}$$

$$\frac{\Gamma \vdash t_1 \equiv s_1 : \Pi x{:}S.T \qquad \Gamma \vdash t_2 \equiv s_2 : S}{\Gamma \vdash t_1\, t_2 \equiv s_1\, s_2 : [x \mapsto t_2]T} \quad \text{(Q-APP)}$$

$$\frac{\Gamma, x{:}S \vdash t : T \qquad \Gamma \vdash s : S}{\Gamma \vdash (\lambda x{:}S.t)\, s \equiv [x \mapsto s]t : [x \mapsto s]T} \quad \text{(Q-BETA)}$$

$$\frac{\Gamma \vdash t : \Pi x{:}S.T \qquad x \notin FV(t)}{\Gamma \vdash \lambda x{:}T.t\, x \equiv t : \Pi x{:}S.T} \quad \text{(Q-ETA)}$$

$$\frac{\Gamma \vdash t : T}{\Gamma \vdash t \equiv t : T} \quad \text{(Q-REFL)}$$

$$\frac{\Gamma \vdash t \equiv s : T}{\Gamma \vdash s \equiv t : T} \quad \text{(Q-SYM)}$$

$$\frac{\Gamma \vdash s \equiv u : T \qquad \Gamma \vdash u \equiv t : T}{\Gamma \vdash s \equiv t : T} \quad \text{(Q-TRANS)}$$

Strong Normalization

$$\frac{\mathtt{t}_1 \longrightarrow_\beta \mathtt{t}_1'}{\lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_1 \longrightarrow_\beta \lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_1'} \qquad \text{(Beta-Abs)}$$

$$\frac{\mathtt{t}_1 \longrightarrow_\beta \mathtt{t}_1'}{\mathtt{t}_1\ \mathtt{t}_2 \longrightarrow_\beta \mathtt{t}_1'\ \mathtt{t}_2} \qquad \text{(Beta-App1)}$$

$$\frac{\mathtt{t}_2 \longrightarrow_\beta \mathtt{t}_2'}{\mathtt{t}_1\ \mathtt{t}_2 \longrightarrow_\beta \mathtt{t}_1\ \mathtt{t}_2'} \qquad \text{(Beta-App2)}$$

$$(\lambda \mathtt{x}{:}\mathtt{T}_1.\mathtt{t}_1)\ \mathtt{t}_2 \longrightarrow_\beta [\mathtt{x} \mapsto \mathtt{t}_2]\mathtt{t}_1 \qquad \text{(Beta-AppAbs)}$$

- **Reduction does not go inside the type labels of λ abstractions**
- **Theorem - The relation $\rightarrow_\beta$ is** strongly normalizing **on well-typed terms. More precisely, if $\Gamma \vdash \mathrm{t} : \mathrm{T}$ then there is no infinite sequence of terms $(\mathrm{t_i})_{\mathrm{i}\geq 1}$ such that $t = t_1$ and $\mathrm{t_i} \rightarrow_\beta t_{i+1}$ for $i \geq 1$.**

Algorithmic Typing and Equality

- Needed to be formulated closer to an algorithm
  - Syntax-directed rules (going from premises to conclusions)
- It is shown that the typechecking algorithm is sound, complete, and terminates on all inputs
  - This also demonstrates the decidability of the original judgments

- Theorem (Preservation) – If $\Gamma \vdash \mathrm{t} : \mathrm{T}$ and $\mathrm{t} \rightarrow_\beta \mathrm{t}'$, then $\Gamma \vdash \mathrm{t}' : \mathrm{T}$.

Dependent Sum Types

- $\Sigma x : T_1 . T_2$ (Σ-types)
- Generalize ordinary product types $(T_1 \times T_2)$
- If x does not appear in T$_2$
  - $\Sigma x : T_1 . T_2 \equiv T_1 \times T_2$
  - $\Pi x : T_1 . T_2 \equiv T_1 \rightarrow T_2$
- $(t , t \text{:} \Sigma x : T . T)$
  - Typed pair (annotated explicitly)
  - If $S : T \rightarrow *$ and $x : T$ and $y : S\ x$, then the pair (x , y) could have both $\Sigma z : T.\ S\ z$ and $\Sigma z : T.\ S\ x$ as a type.

Dependent Sum Types (cont.)

Extends λLF (2-1 and 2-2)

New syntax

$\mathtt{t}$	::=	...	*terms:*
		$(\mathtt{t},\ \mathtt{t}{:}\Sigma\mathtt{x}{:}\mathtt{T}.\mathtt{T})$	*typed pair*
		$\mathtt{t.1}$	*first projection*
		$\mathtt{t.2}$	*second projection*
$\mathtt{T}$	::=	...	*types:*
		$\Sigma\mathtt{x}{:}\mathtt{T}.\mathtt{T}$	*dependent sum type*

Kinding $\boxed{\Gamma \vdash \mathtt{T} :: \mathtt{K}}$

$$\frac{\Gamma \vdash \mathtt{S} :: * \qquad \Gamma, \mathtt{x}{:}\mathtt{S} \vdash \mathtt{T} :: *}{\Gamma \vdash \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T} :: *} \quad \text{(K-SIGMA)}$$

Typing $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\Gamma \vdash \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T} :: * \qquad \Gamma \vdash \mathtt{t}_1 : \mathtt{S} \qquad \Gamma \vdash \mathtt{t}_2 : [\mathtt{x} \mapsto \mathtt{t}_1]\mathtt{T}}{\Gamma \vdash (\mathtt{t}_1,\mathtt{t}_2{:}\Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}) : \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}} \quad \text{(T-PAIR)}$$

$$\frac{\Gamma \vdash \mathtt{t} : \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}}{\Gamma \vdash \mathtt{t.1} : \mathtt{S}} \quad \text{(T-PROJ1)}$$

$$\frac{\Gamma \vdash \mathtt{t} : \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}}{\Gamma \vdash \mathtt{t.2} : [\mathtt{x} \mapsto \mathtt{t.1}]\mathtt{T}} \quad \text{(T-PROJ2)}$$

Term Equivalence $\boxed{\Gamma \vdash \mathtt{t}_1 \equiv \mathtt{t}_2 : \mathtt{T}}$

$$\frac{\Gamma \vdash \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T} :: * \qquad \Gamma \vdash \mathtt{t}_1 : \mathtt{S} \qquad \Gamma \vdash \mathtt{t}_2 : [\mathtt{x} \mapsto \mathtt{t}_1]\mathtt{T}}{\Gamma \vdash (\mathtt{t}_1,\mathtt{t}_2{:}\Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}).1 \equiv \mathtt{t}_1 : S} \quad \text{(Q-PROJ1)}$$

$$\frac{\Gamma \vdash \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T} :: * \qquad \Gamma \vdash \mathtt{t}_1 : \mathtt{S} \qquad \Gamma \vdash \mathtt{t}_2 : [\mathtt{x} \mapsto \mathtt{t}_1]\mathtt{T}}{\Gamma \vdash (\mathtt{t}_1,\mathtt{t}_2{:}\Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}).2 \equiv \mathtt{t}_2 : [\mathtt{x} \mapsto \mathtt{t}_1]\mathtt{T}} \quad \text{(Q-PROJ2)}$$

$$\frac{\Gamma \vdash \mathtt{t} : \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}}{\Gamma \vdash (\mathtt{t.1},\ \mathtt{t.2}{:}\Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}) \equiv \mathtt{t} : \Sigma\mathtt{x}{:}\mathtt{S}.\mathtt{T}} \quad \text{(Q-SURJPAIR)}$$

The Calculus of Construction

- One of the most famous systems of dependent types
- A setting for all of constructive mathematics
- Simple and very expressive

Extends λLF (2-1 and 2-2)

New syntax

`t`	::=	...	*terms:*
		`all x:T.t`	*universal quantification*
`T`	::=	...	*types:*
		`Prop`	*propositions*
		`Prf`	*family of proofs*

Kinding $\boxed{\Gamma \vdash \mathtt{T} :: \mathtt{K}}$

$$\Gamma \vdash \mathtt{Prop} :: * \qquad \text{(K-PROP)}$$

$$\Gamma \vdash \mathtt{Prf} :: \Pi \mathtt{x}{:}\mathtt{Prop}.\ * \qquad \text{(K-PRF)}$$

Typing $\boxed{\Gamma \vdash \mathtt{t} : \mathtt{T}}$

$$\frac{\Gamma \vdash \mathtt{T} :: * \qquad \Gamma, \mathtt{x} : \mathtt{T} \vdash \mathtt{t} : \mathtt{Prop}}{\Gamma \vdash \mathtt{all}\ \mathtt{x}{:}\mathtt{T}.\mathtt{t} : \mathtt{Prop}} \qquad \text{(T-ALL)}$$

Type Equivalence $\boxed{\Gamma \vdash \mathtt{S} \equiv \mathtt{T} :: K}$

$$\frac{\Gamma \vdash \mathtt{T} :: * \qquad \Gamma, \mathtt{x}{:}\mathtt{T} \vdash \mathtt{t} : \mathtt{Prop}}{\Gamma \vdash \mathtt{Prf}\ (\mathtt{all}\ \mathtt{x}{:}\mathtt{T}.\mathtt{t}) \equiv \Pi \mathtt{x}{:}\mathtt{T}.\mathtt{Prf}\ \mathtt{t} :: *} \qquad \text{(QT-ALL)}$$

The Calculus of Construction

Example –

nat = all a:Prop. all z:Prf a. all s: Prf a $\rightarrow$ Prf a. a

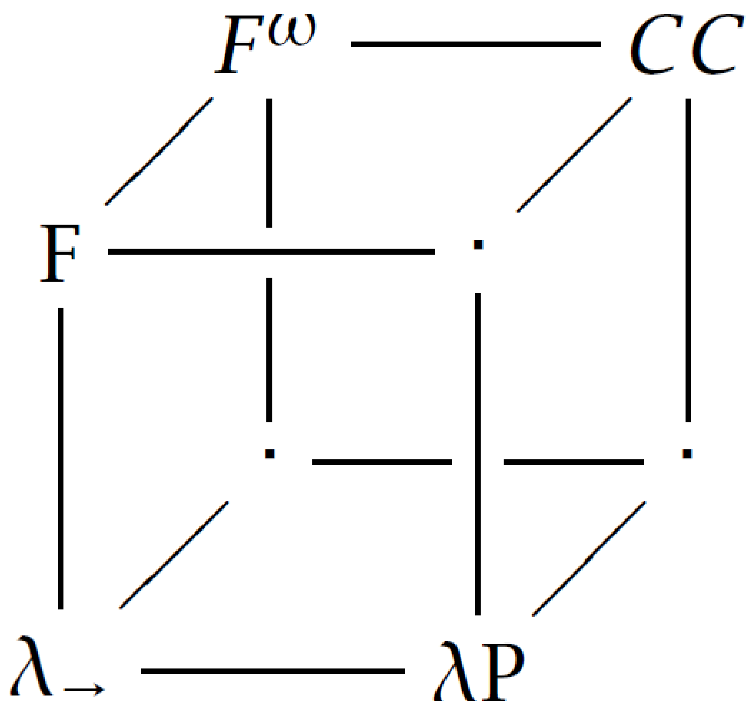
(nat is a member of type Prop)

zero = λa:Prop. λz:Prf a. λs:Prf a $\rightarrow$ Prf a. z : Prf nat

succ = λn:Prf nat. λa:Prop. λz:Prf a. λs:Prf a $\rightarrow$ Prf a.
s (n a z s) : Prf nat $\rightarrow$ Prf nat

add = λm:Nat. λn:Nat. m nat n succ : Prf nat $\rightarrow$ Prf nat $\rightarrow$ Prf nat

Lambda Cube



Twelf in Practice

```
nat: type.
z: nat.
s: nat -> nat.

plus: nat -> nat -> nat -> type.
p-z: plus z N N.
p-s: plus (s N1) N2 (s N3)
      <- plus N1 N2 N3.

> 2+1=3 : plus (s (s z)) (s z) (s (s (s z))) = p-s (p-s p-z).
```

http://twelf.plparty.org/live/

Programming with Dependent Types

Languages: Pebble, Cardelli's Quest, Cayenne, Dependent ML, Haskell, Agda

Example - a zip **function which can only be applied to a pair of lists of the same length**

```
datatype 'a list with nat =
    nil(0)
  | {n:nat} cons(n+1) 'a * 'a list(n)
```

The withtype **clause is a type annotation supplied by the programmer.**

```
fun zip ([], []) = []
  | zip (x :: xs, y :: ys) = (x, y) :: zip (xs, ys)
withtype {n:nat} => 'a list(n) * 'b list(n) -> ('a * 'b) list(n)
```

This leads to a safer and faster implementation of zip than a corresponding one in Standard ML.

Conclusion

- Dependent Types
  - Product Type and Sum Type
- Logical Framework LF and λLF
- Algorithms for Typechecking
- Programming in Twelf and DML

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Any Questions?