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a) If there is no spike in $[t, t+\Delta t]$, then the membrane potential changes from $u(t)=u'$ to $u(t+\Delta t)=u' e^{-\frac{\Delta t}{\tau_m}}$

On the other hand, If we have a spike in $[t, t+\Delta t]$ then we would have $u(t)=u'$ and $u(t+\Delta t)=u' e^{-\frac{\Delta t}{\tau_m}} + w_k$ where the spike arrives at synapse k. We have also

$$\text{Prob}(\text{no spike } [t, t+\Delta t]) = 1-\sum_k \nu_k(t)\Delta t$$

&

$$\text{Prob}(\text{a spike at } k) = \sum_k \nu_k(t)\Delta t$$

Therefore, equation 8.37 stands for the probability of being at state ~~(u,t+Δt)~~ $(u, t+\Delta t)$ given we ~~were there~~ were in the (u', t)

$$P^{trans}(u, t+\Delta t) = \underbrace{[1-\Delta t \sum_k \nu_k(t)]\, \delta(u-u' e^{-\frac{\Delta t}{\tau_m}})}_{\text{Either there is no spike}} + \underbrace{\Delta t \sum_k \nu_k(t)\, \delta(u-u' e^{-\frac{\Delta t}{\tau_m}} - w_k)}_{\text{or there was spike}}$$

Equation 8.38 stands for law of probability :

$$P(u,t+\Delta t) = P\left(u,t+\Delta t \,\middle|\, \begin{matrix} u',t \\ \text{no-spike} \end{matrix}\right) + P\left(u,t+\Delta t \,\middle|\, \begin{matrix} u',t \\ \text{spike} \end{matrix}\right)$$

$$= \int P^{trans}(u,t+\Delta t \mid u',t)\, P(u',t)\, du'$$

$$= \int \Big(1-\Delta t \sum_k \nu_k(t)\Big)\, \delta\big(u-u'e^{-\frac{\Delta t}{\tau_m}}\big)\, P(u',t)\, du' +$$

$$\int \Big(\Delta t \sum_k \nu_k(t)\Big)\, \delta\big(u-u'e^{-\frac{\Delta t}{\tau_m}}-w_k\big)\, P(u',t)\, du'$$

$\underline{\underline{x = u'e^{-\frac{\Delta t}{\tau_m}}}}$ $\underline{\underline{du' = e^{\frac{\Delta t}{\tau_m}}dx}}$

$dx = e^{-\frac{\Delta t}{\tau_m}}du'$

$$\Big(1-\Delta t \sum_k \nu_k(t)\Big)\, e^{\frac{\Delta t}{\tau_m}} \int \delta(u-x)\, P\big(xe^{\frac{\Delta t}{\tau_m}},t\big)\, dx$$

$$+ \Big(\Delta t \sum_k \nu_k(t)\Big)\, e^{\frac{\Delta t}{\tau_m}} \int \delta\big((u-w_k)-x\big)\, P\big(xe^{\frac{\Delta t}{\tau_m}},t\big)\, dx$$

we know that $\int \delta(a-b) f(a)\, da = f(b)$ and $\delta(au) = a^{-1}\delta(u)$

$$\Rightarrow P(u,t+\Delta t) = \Big(1-\Delta t \sum_k \nu_k(t)\Big)\, e^{\frac{\Delta t}{\tau_m}}\, P\big(ue^{\frac{\Delta t}{\tau_m}},t\big)$$

$$+ \Big(\Delta t \sum_k \nu_k(t)\Big)\, e^{\frac{\Delta t}{\tau_m}}\, P\big(ue^{\frac{\Delta t}{\tau_m}}+w_k,t\big)$$

which is equation 8.39.

Using Tylor expansion of $e^{\frac{\Delta t}{\tau_m}} = 1 + \frac{\Delta t}{\tau_m} + \underbrace{\frac{1}{2}\left(\frac{\Delta t}{\tau_m}\right)^2 + \cdots}_{\Delta t \to 0 \Rightarrow \text{ we ignore this terms.}}$

$$\Rightarrow P(ue^{\frac{\Delta t}{\tau_m}}, t) = P(u + \frac{\Delta t}{\tau_m}u, t) = \left\{P(u,t) + \frac{\partial}{\partial u}P(u,t)\, u\right\} \times \frac{\Delta t}{\tau_m}$$

and

$$P(ue^{\frac{\Delta t}{\tau_m}} - w_k, t) = P(u + \frac{\Delta t}{\tau_m}u - w_k, t) \xrightarrow{\Delta t \to 0} P(u - w_k, t)$$

rewriting equation 8.39:

$$P(u, t+\Delta t) = \left(1 - \Delta t \sum_k \nu_k(t)\right)\left(1 + \frac{\Delta t}{\tau_m}\right)\left(P(u,t) + \frac{\partial}{\partial u}P(u,t)\, u \frac{\Delta t}{\tau_m}\right)$$

$$+ \left(\Delta t \sum_k \nu_k(t)\right)\left(1 + \frac{\Delta t}{\tau_m}\right)\left(P(u - w_k, t)\right)$$

$$\xrightarrow{(\Delta t)^2 = 0} P(u,t) + \frac{\partial}{\partial u}P(u,t)\, u \frac{\Delta t}{\tau_m} - \left(\Delta t \sum_k \nu_k(t)\right)P(u,t) + \frac{\Delta t}{\tau_m}P(u,t)$$

$$+ \Delta t \sum_k \nu_k(t)\, P(u - w_k, t)$$

$$\Rightarrow P(u, t+\Delta t) - P(u,t) = \Delta t \Bigg[\left(\frac{1}{\tau_m}P(u,t) + \frac{\partial}{\partial u}P(u,t)\frac{u}{\tau_m}\right) \cancel{- \left(\sum_k \nu_k(t)\right)}$$

$$+ \left(\sum_k \nu_k(t)\right)\left(P(u - w_k, t) - P(u,t)\right)\Bigg]$$

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$$\Rightarrow \frac{P(u,t+\Delta t)-P(u,t)}{\Delta t} = \frac{1}{\tau_M} P(u,t) + \frac{u}{\tau_m}\frac{\partial}{\partial u} P(u,t)$$

$$+ \sum_n v_n(t)\big(P(u-\omega_n,t) - P(u,t)\big)$$

which is ~~exact~~ equation 8-40. The left side is partial derivation with respect to "t" and for small ω_n, we could expand $P(u-\omega_n,t)$ as $P(u,t) - \frac{\partial}{\partial u} P(u,t)\omega_n + \frac{\partial^2}{\partial^2 u} \frac{P(u,t)\omega_n^2}{2} + \dots$

we can rewrite equation 8.40 as

$$\tau_m \frac{\partial P(u,t)}{\partial t} = P(u,t) + u\frac{\partial}{\partial u} P(u,t) + \Big[\sum_n v_n(t)\Big(\cancel{P(u,t)} - \frac{\partial}{\partial u} P(u,t)\omega_n - \cancel{P(u,t)} + \frac{\partial^2}{\partial u^2}\frac{P(u,t)\omega_n^2}{2}\Big)\Big]\tau_m$$

$$\Rightarrow \tau_m \frac{\partial P(u,t)}{\partial t} = P(u,t) + \frac{\partial}{\partial u}\Big(u - \tau_m \sum_n v_n(t)\,\omega_n\Big) P(u,t)$$

$$+ \frac{\partial^2}{\partial u^2} \sum_n v_n(t)\,\omega_n^2\, \frac{P(u,t)}{2}\,\tau_m$$

which gives us equation 8.41.

b) we want to show that if

$$P(u,t) = \frac{1}{\sqrt{2\pi \langle \Delta u^2(t) \rangle}} e^{-\frac{[u(t|\hat{t}) - u_0(t)]^2}{2\langle \Delta u^2(t)\rangle}}$$

is a solution for equation 8.41, where the initial condition is $P(u,\hat{t}) = \delta(u - u_r)$ is a Gaussian with means $u_0(t)$ and variance $\langle \Delta u^2(t) \rangle$

$$\Rightarrow \frac{\partial P(u,t)}{\partial t} = \frac{-\sqrt{2\pi}\frac{\partial \langle \Delta u(t)\rangle}{\partial t}}{\sqrt{2\pi \langle \Delta u^2(t)\rangle}} e^{*} + (*)' e^{*} \frac{1}{\sqrt{2\pi \langle \Delta u^2(t)\rangle}} \quad (I)$$

where $(*)' = \dfrac{\frac{\partial u_0}{\partial t} \times 2 \times (u(t|\hat{t}) - u_0(t)) - \cancel{4}\, 2\frac{\partial \langle \Delta u^2(t)\rangle}{\partial t} \text{ (numerator)}}{2\langle \Delta u^3(t)\rangle}$

and $\dfrac{\partial P(u,t)}{\partial u} = P(u,t) \times \dfrac{2(u(t|\hat{t}) - u_0(t))}{2\langle \Delta u^2(t)\rangle} \quad II$

$$\frac{\partial^2 P(u,t)}{\partial u^2} = \frac{\partial P(u,t)}{\partial u} \times \frac{(u(t|\hat{t}) - u_0(t))}{\langle \Delta u^2(t)\rangle} + \frac{1}{\langle \Delta u^2(t)\rangle} P(u,t) \quad III$$

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Setting $\sigma^2(t) = \tau_M \sum_k v_k^{(t)} w_k^2$ and plug in ~~Eq~~ equation II and III in 8.41, we will get: (and $RI(t) = \tau_M \sum_k v_k(t) w_k$)

the right side of equation

$$-RI(t)\left(P(u,t)\right)\left(\frac{u(t)-u_0(t)}{\langle\Delta u^2(t)\rangle}\right) \cancel{\frac{1}{2}\sigma^2}$$

$$+\frac{1}{2}\sigma^2\left(P(u,t)\,\frac{(u(t)-u_0(t))^2}{\langle\Delta u^2(t)\rangle^2} + \frac{1}{\langle\Delta u^2(t)\rangle}\,P(u,t)\right)$$

$$= P(u,t)\left[-RI(t)\,\frac{u(t)-u_0(t)}{\langle\Delta u^2(t)\rangle} + \frac{1}{2}\sigma^2\left(\frac{(u(t)-u_0(t))^2}{\langle\Delta u^2(t)\rangle^2} + \frac{1}{\langle\Delta u^2(t)\rangle}\right)\right]$$

and rewriting the left side using equatio I, we will get

$$\tau_M e^{\frac{-[u(t)-u_0(t)]^2}{2\langle\Delta u^2(t)\rangle}}\left(-\frac{\frac{\partial\langle\Delta u(t)\rangle}{\partial t}\sqrt{2\pi}}{\sqrt{2\pi}\langle\Delta u^2(t)\rangle} + \left(\frac{1}{\sqrt{2\pi\langle\Delta u^2(t)\rangle}}\right)\times\left[\left(\frac{\frac{\partial u_0}{\partial t}(u(t)-u_0(t))}{2\langle\Delta u^3(t)\rangle}\right) - \left(\frac{\frac{\partial\langle\Delta u(t)\rangle}{\partial t}(u(t)-u_0(t))^2}{\Delta u^4(t)}\right)\right]\right)$$

$$= \tau_M P(u,t)\left(-\frac{\partial\langle\Delta u(t)\rangle}{\partial t} + \frac{\partial u_0}{\partial t}\times\frac{u(t)-u_0(t)}{2\langle\Delta u^3(t)\rangle} - \frac{\partial\Delta u(t)}{\partial t}\times\frac{(u(t)-u_0(t))^2}{\langle\Delta u^4(t)\rangle}\right.$$

(Q3)

For ~~γ~~ gamma distribution $\Gamma(\alpha,\beta)$ we have

$\mu = \frac{\alpha}{\beta}$ and $\sigma = \frac{\sqrt{\alpha}}{\beta} \Rightarrow CV = \frac{\sigma}{\mu} = \frac{1}{\sqrt{\alpha}}$

For Erlang distribution we have similar values:

$\mu = \frac{k}{\lambda}$ variance $\frac{k}{\lambda^2} \rightarrow CV = \frac{\sigma}{\mu} = \frac{1}{\sqrt{k}}$

By increasing k, the size ω_n of each jump will decrease and by $k \to \infty$ the jump process turns into a diffussion process.

Q4)

$$(I) \quad \frac{dP(n,t)}{dt} = \sum_{n' \neq n} T(n|n')P(n',t) - \sum_{n' \neq n} T(n'|n)P(n,t)$$

we could rewrite two equations as

$$T(n+1|n) = (1-u-v)\left(\frac{n}{N}\right)\left(1-\frac{n}{N}\right) + v\left(1-\frac{n}{N}\right)$$

$$T(n-1|n) = (1-u-v)\left(1-\frac{n}{N}\right)\left(\frac{n}{N}\right) + u\left(\frac{n}{N}\right)$$

Now, define $U = \frac{uN}{2}$ and $V = \frac{vN}{2}$

$$\Rightarrow u = \frac{2U}{N} \text{ and } v = \frac{2V}{N}$$

$$\Rightarrow T(n+1|n) = \left(1 - \frac{2U}{N} - \frac{2V}{N}\right)\left(1-\frac{n}{N}\right)\left(\frac{n}{N}\right) + \frac{2V}{N}\left(1-\frac{n}{N}\right)$$

$$T(n-1|n) = \left(1 - \frac{2U}{N} - \frac{2V}{N}\right)\left(1-\frac{n}{N}\right)\left(\frac{n}{N}\right) + \frac{2U}{N}\left(\frac{n}{N}\right)$$

Setting $x = \frac{n}{N}$ and expanding all terms in equation I in a $\frac{1}{N}$ expansion:

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$$\frac{\partial P}{\partial t} = P(n+1,t)\,T(n|n+1) + P(n-1,t)\,T(n|n-1) - P(n,t)\,T(n+1|n) - P(n,t)\,T(n-1|n)$$

$$= -\underbrace{P(n,t)\Big(T(n+1|n) + T(n-1|n)\Big)}_{\mathrm{II}} + \underbrace{P(n+1,t)\,T(n|n+1)}_{\mathrm{III}} + \underbrace{P(n-1,t)\,T(n|n-1)}_{\mathrm{IV}}$$

$\Downarrow$ II

$$\mathrm{II} \overset{\frac{n}{N}=x}{=} -P(x,t)\left(\frac{2V}{N}(1-x) + \frac{2U}{N}x\right) + O\left(\frac{1}{N^3}\right) \quad (1)$$

$$\mathrm{III} = \left(P(x,t) + \frac{1}{N}\frac{\partial P}{\partial x} + \frac{1}{2N^2}\frac{\partial^2 P}{\partial x^2}\right)\left(\frac{2U}{N}\left(x+\frac{1}{N}\right)\right) + O\left(\frac{1}{N^3}\right) \quad (2)$$

$$\mathrm{IV} = \left(P(x,t) - \frac{1}{N}\frac{\partial P}{\partial x} + \frac{1}{2N^2}\frac{\partial^2 P}{\partial x^2}\right)\left(\frac{2V}{N}\left(1-x+\frac{1}{N}\right)\right) + O\left(\frac{1}{N^3}\right) \quad (3)$$

from (1), (2) and (3) for $P(x,t)$ we have

$$P(x,t)\left(-\frac{2V}{N}(1-x) - \frac{2U}{N}x + \frac{2U}{N}\left(x+\frac{1}{N}\right) + \frac{2V}{N}\left(1-x+\frac{1}{N}\right)\right)$$

$$= P(x,t)\left(\frac{2U}{N^2} + \frac{2V}{N^2}\right) = \underbrace{\frac{2}{N^2}\times(U+V)}_{\text{coefficient of } P(x,t)} \times P(x,t)$$

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and from (2) and (3) we compute the coefficient of $\frac{\partial P}{\partial x}$ (omitting the term with $\frac{1}{N^3}$)

$$\frac{\partial P}{\partial x}\left(\frac{1}{N}\right)\left(\frac{2U}{N}x - \frac{2V}{N} + \frac{2V}{N}x\right)$$

$$= \frac{2}{N^2}\frac{\partial P}{\partial x}(U+V)x - \frac{2V}{N^2}\frac{\partial P}{\partial x}$$

$$= \underbrace{\frac{2}{N^2}\left(Ux - V(1-x)\right)}_{\text{coefficient of } \frac{\partial P}{\partial x}}\frac{\partial P}{\partial x}$$

and coefficient of $\frac{\partial^2 P}{\partial x^2}$ will have $\frac{1}{N^3}$ which will be neglected. The only $\frac{\partial^2 P}{\partial x^2}$ ~~which remains is~~ coefficient with degree $\frac{1}{N^2}$ is $\frac{1}{2}\frac{\partial^2}{\partial x^2}[x(1-x)P]$ which come from the Moran model without mutation. Then, the Fokker-Planck equation is:

$$\frac{\partial P}{\partial \tau} = \frac{\partial}{\partial x}\left[(Ux - V(1-x))P\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[x(1-x)P\right]$$

where we substitude $t = \frac{\tau N^2}{2}$. (4-3)