Portfolio Optimization - Markowitz Portfolio Theory
CO 370 Final Project Report - Winter 2020

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Part 1: Introduction

In 1952, Harry Markowitz published a paper called ‘Portfolio Selection’ which changed the way financiers look at their portfolios. It revolved around the Modern Portfolio Theory (MFT) which is a theory on how risk-averse investors can construct portfolios to optimize or maximize expected return based on a given level of market risk which is an integral part of achieving a higher reward, or minimize their risk based on a level of fixed expected return (MFT, Investopedia). Markowitz emphasized diversification which states that investors should not put all their eggs in one basket and proved the possibility to construct an efficient set of portfolios that offer the minimum possible risk for a given level of expected return. The higher the risk, the higher will be the return which means that investors are faced with a trade-off between risk and expected return.

I have used this portfolio theory as the thesis for my project as I plan to pursue this as a career in the future. I will show the relationship between risk and expected return. We will specifically use the method of *minimizing risk* to decide which stocks are ideal amongst our choice of 10 stocks through optimizing our model and using the efficient frontier which is explained in detail later in the report. The combination of securities with little correlation allows investors to optimize their return without assuming additional risk. This gives an overall portfolio risk which is lower than the risks of the individual stocks. Markowitz said that diversification reduces the variance of the portfolio (Project Proposal, 2020). That is exactly what I have done here, as our 10 stocks are extremely well diversified and are from different industries. Optimization can be roughly defined as a quantitative approach for decision making where we seek to determine a best decision from a set of possible decisions. The report will utilize the 3 key components of an optimization problem i.e. decision variables, constraints and an optimization problem to help decide an efficient portfolio with stocks that pose the least risk for the given return.

A portfolio consists of at least one kind of security. It can consist of bonds solely which provide low returns with low risks, or consist of equities solely, which provide high returns for high risks. It can also consist of a combination of them, or of any other kind of stock as well. Our portfolio consists of 9 equities and an index fund which I have mentioned in Part 2. Another aspect to keep in mind when making decisions of portfolios or investments is volatility. In finance, volatility equals risk and volatility can be found by taking the standard deviation of the returns. Risk or volatility is basically the measure of the likelihood of missing expected returns.The risk is approximated using a sample dataset of recent historical data points (time-series data points consisting of prices).

For our portfolio optimization problem, I want to minimize the risk for a target return. Some of the inputs which usually go into an optimization model are:

1. Expected return for each stock in the portfolio
2. Volatility (risk) for each stock
3. Target return
4. Covariance matrix

The output is the portfolio weights that minimize risk for a target return which I am trying to achieve.

Part 2: Data:

The data for our model consists of 10 companies: PG (P&G), ^GSPC (S&P500), MSFT (Microsoft), AAPL (Apple), AC.TO (Air Canada), SU (Suncore Energy), BA (Boeing), WMT (Walmart), TD (TD Bank) and ABT (Abbott Laboratories).

The data has been collected from Yahoo Finance and consists of daily stock prices for these 10 companies for a 2 year period, from 2018-04-01 till 2020-03-27. It is stored in the file "sample_data_daily.csv".

Part 3: Decision variables, constraints and objective function.

Assumptions:

- No short selling is allowed
- There are no transaction costs for the trades

Variables: Our output will be the optimal weight x_i allocated to each stock, while r_i is a parameter which goes into the model.

Decision Variables: Vector $X \in R^n$ for the weight of each stock.
Let $x_i \ \forall i \in \{1, 2, \ldots, 10\}$ is the amount allocated for each stock i.
Let $r_i \ \forall i \in \{1, 2, \ldots, 10\}$ be random return of stock i

Constraints:
I utilize constraints to satisfy our optimization problem.

1- The first constraint requires the portfolio to generate a fixed 'expected' target return of, say, 'z'. I take the summation of the product of the amount invested in a certain stock and the expected return of that particular stock for all stocks in our portfolio. I set this equal to the required return value of 'z' to ensure our portfolio can minimize its risk for this required return value.

2- In addition, another constraint is that the sum of the weights of the stock investments must sum up to 1.

3- Lastly, I ensure that a certain amount of capital must be invested in our portfolio. I set a lower bound of all stock weights in our portfolio equal to 0, to ensure there are no negative investments, and no short-selling is allowed. This ensures I have adequate capital invested in the equities to generate investment returns.

Constraints:

1) Return on the portfolio $\sum_{i=1}^{10} r_i x_i$.

We want an expected target return of z.Thus

$E[\sum_{i=1}^{10} r_i x_i] = \sum_{i=1}^{10} r_i \bar{x_i} = z$

2) Sum of weights = 1

$\sum_{i=1}^{10} x_i = 1$

Objective Function:

The goal of our objective function is to minimize the risk of our portfolio by minimizing the variance of the investment portfolio. As mentioned above, volatility equals risk and volatility can be found by taking the standard deviation of the returns. Risk or volatility is basically the measure of the likelihood of missing expected returns. The formula is as follows (Optimization Process, 2002):

$$\mathrm{Var}[\sum_{i=1}^{10} r_i x_i] = E[(\sum_{i=1}^{10} r_i x_i - \sum_{i=1}^{10} \overline{r}_i x_i)^2]$$
$$= E[(\sum_{i=1}^{10}(r_i - \overline{r}_i)x_i)(\sum_{j=1}^{10}(r_j - \overline{r}_j)x_j)] = \sum_{i=1}^{10}\sum_{j=1}^{10} x_i x_j E[(r_i - \overline{r}_i)(r_j - \overline{r}_j)]$$
$$= \sum_{i=1}^{10}\sum_{j=1}^{10} x_i x_j \sigma_{ij}$$

Sigma_ij is the covariance of the return between stock i and stock j for i,j in {1,2,...,10}.

Final program:

$$\min \sum_{i=1}^{10} \sum_{i=1}^{10} x_i x_j \sigma_{ij}$$

$s.t.$

$\sum_{i=1}^{10} x_i = 1$

$\sum_{i=1}^{10} r_i \bar{x_i} = z$

$\mathrm{x}_i >= 0$ for all $i = 1, 2, ..., 10$

Part 4: Findings:

Our output from the Python model is supposed to be the optimal weight for each stock in order to minimize the risk, subject to the constraints mentioned above. I am analyzing the daily prices of 10 well known companies. I have found the individual weights twice: once when no expected return target is fixed, and second when it is fixed.

1- When NO expected return target is fixed: As per the Python output, we see optimal weights for each of the 10 stocks:

```
Stock, Optimal weight
AAPL     0.000000
ABT      0.022230
AC.TO    0.060743
BA       0.000000
MSFT     0.000000
PG       0.180328
SU       0.002162
TD       0.161116
WMT      0.469448
^GSPC    0.103972
```

The model tells us to invest almost half of our budget (47%) on the Walmart stock, 18% on P&G, 16% on TD and 10% on the S&P500 (^GSPC) stock. The expected return for this investment when I set no fixed return is 4.39% with minimum risk (variance) of 1.66. These numbers are further analyzed down below.

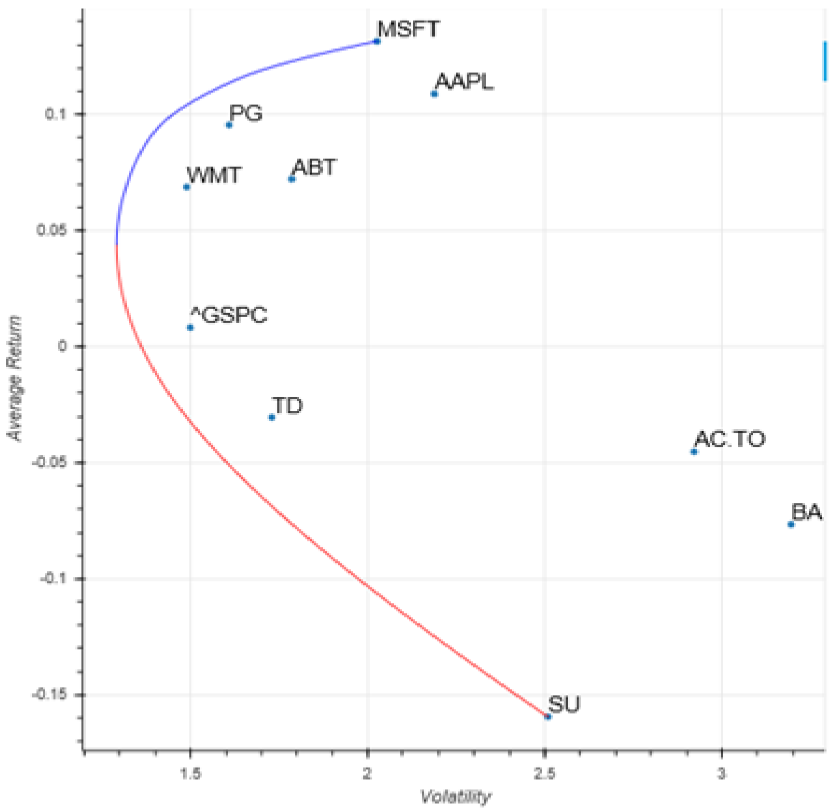
2- When an expected return target is fixed: *What I have done over here is that I have set the return target as 50% of the maximum average return of the 10 stocks.* The maximum avg return is 13.14% for Microsoft so the expected target return is 6.6% (=0.5 * 13.14%). With this constraint, the new optimal weights to minimize the risk of the portfolio are:

```
Stock, Optimal weight
AAPL     0.005869
ABT      0.123095
AC.TO    0.060279
BA       0.000000
MSFT     0.000000
PG       0.289116
SU       0.000000
TD       0.045344
WMT      0.476296
^GSPC    0.000000
```

These new numbers show that we should invest 48% of our budget on Walmart stock, 29% on P&G, 12% on Abbott Laboratories and 5% on TD. Since these numbers consist of data from the past 2 years, they reflect the volatile market as a result of COVID-19 as well. Walmart is a safe option as consumers continue to make everyday purchases from here during the recessionary period. P&G is also a Consumer Product Goods company and the rate of return is high with low risk as consumers continue to make purchases from here at all times. Abbott Laboratories is a healthcare company and their returns have gone up as consumers are in need of healthcare products these days. They released a test which can tell patients the result of their COVID-19 results in 5 minutes. And finally TD is a safe option as it is one of Canada's largest banks, the company has done well and if needed, there will be a government bailout.

As compared to the previous point when no expected return is fixed, our risk (variance) goes slightly higher to 1.72 as seen from the Python output, which makes sense as I am setting a target return in this case.

Efficient frontier (Pareto-Optimal Curve):



Based on the mathematical model I have implemented, we can find the efficient frontier. Most investment choices involve a tradeoff between risk and reward. This is where the efficient frontier comes in. It is a modern portfolio theory tool that shows investors the best possible return they can expect from their portfolio, given the level of volatility they are willing to accept. The vertical axis represents the expected rate of return and the x-axis shows the risk tolerance. The frontier/line curve shows the yield of portfolios given their degree of risk. Optimal portfolios lie on the curve while any portfolio lying below the curve represents a less-than-ideal investment.

Based on this, we see Walmart, P&G and Abbott stocks as good investments. Walmart has the same level of risk as the S&P500 but a much higher return, and so is the comparison between Abbott and TD Bank approximately. Apple is not a very good investment as it has high risk even though it has high return. Comparing it with Microsoft, Microsoft has lower risk and higher returns. Even though the curve shows Microsoft as an optimal stock with high returns and high risk, the python model told us not to allocate any amount of our budget towards it for the given expected target return of 6.6%. This is most likely because the risk level for Microsoft is high, and other stocks with lower risk but decent returns meet the criteria. If the target return was higher, say around 8-9%, then most likely, the algorithm would tell us to allocate some funds towards Microsoft. From this curve, we can see that Air Canada (AC.TO), Boeing (BA), Suncore Energy (SU), S&P500 and TD stocks are less-than-ideal investments and most likely, we should avoid them compared to the rest.

Limitations: The data includes stock price movements reflected from COVID-19 and have, therefore, been very volatile over the past one month. A better analysis could be conducted if data from the past one month would be excluded to see how the companies have performed. Another way to conduct an effective analysis would be to include equal time periods of data collected with and without COVID-19. For instance, if we analyze 6 months of data with COVID-19 and 6 months without, we will be able to effectively compare what kind of an effect it has had on the markets as compared to when there was no coronavirus.

References

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