Complex Analysis Homework 3

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Question 5

Question.
Prove that f defined by

$$f(z) = \sqrt{|Re(z)Im(z)|}$$

satisfies the Cauchy-Riemann Equations at $z = 0$, but is not differentiable there.

First, recall that the Cauchy Riemann Equations for $f(z) = u(x,y) + iv(x,y)$ are the following:

$$u_x(x,y) = v_y(x,y) \qquad\qquad v_x(x,y) = -u_y(x,y)$$

Next, recall that for $f(z)$ to be differentiable at $z = 0$, the following limit must exist:

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0}$$

Proof.
If $z = x + iy$, then $f(z) = \sqrt{|\mathrm{Re}(z)\mathrm{Im}(z)|}$ reduces to $f(z) = \sqrt{|xy|}$. Thus, $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$. First, I will calculate $u_x(0,0)$:

$$\begin{aligned} u_x(0,0) &= \lim_{\Delta x\to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} && \text{by the definition of a partial derivative} \\ &= \lim_{\Delta x\to 0} \frac{\sqrt{|\Delta x \cdot 0|} - \sqrt{|0\cdot 0|}}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{0}{\Delta x} = \lim_{\Delta x\to 0} 0 = 0. \end{aligned}$$

Similarly, I will calculate $u_y(0,0)$:

$$\begin{aligned} u_y(0,0) &= \lim_{\Delta y\to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y} && \text{by the definition of a partial derivative} \\ &= \lim_{\Delta y\to 0} \frac{\sqrt{|0 \cdot \Delta y|} - \sqrt{|0\cdot 0|}}{\Delta y} \\ &= \lim_{\Delta y\to 0} \frac{0}{\Delta y} = \lim_{\Delta y\to 0} 0 = 0. \end{aligned}$$

Thus, it is clear that $u_x(0,0) = 0 = v_y(0,0)$ and $v_x(0,0) = 0 = -u_y(0,0)$, so the Cauchy-Riemann Equations are satisfied at the point $z = 0$. Next, I will examine the differentiability of f at this point by seeing if the required limit exists:

$$\begin{aligned} \lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} &= \lim_{z\to 0} \frac{\sqrt{|xy|} - \sqrt{|0\cdot 0|}}{x + iy} \\ &= \lim_{z\to 0} \frac{\sqrt{|xy|}}{x + iy} \end{aligned}$$

Let's first look at this limit as z approaches 0 along the line $\mathrm{Im}(z) = 0$, i.e. z is of the form $z = x + i(0) = x$, a purely Real Number.

$$\begin{aligned} \lim_{z\to 0} \frac{\sqrt{|xy|}}{x + iy} &= \lim_{x\to 0} \frac{\sqrt{|x\cdot 0|}}{x + i(0)} \\ &= \lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0. \end{aligned}$$

On the other hand, let's look at this limit along the line $\mathrm{Re}(z) = \mathrm{Im}(z)$ with $\mathrm{Re}(z) \geq 0$. In other words, $z = x + ix$ for $x \geq 0$.

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\begin{aligned}
\lim_{z \to 0} \frac{\sqrt{|xy|}}{x + iy} &= \lim_{x \to 0} \frac{\sqrt{|x \cdot x|}}{x + ix} \\
&= \lim_{x \to 0} \frac{\sqrt{|x^2|}}{x + ix} \\
&= \lim_{x \to 0} \frac{\sqrt{x^2}}{x + ix} && \text{since for all } x \in \mathbb{R}, x^2 \geq 0. \\
&= \lim_{x \to 0} \frac{|x|}{x + ix} \\
&= \lim_{x \to 0} \frac{x}{x(1 + i)} && \text{since } x > 0 \implies |x| = x \\
&= \lim_{x \to 0} \frac{1}{1 + i} = \frac{1}{1 + i}
\end{aligned}
$$

Thus, we see that in one case, this limit is equal to 0, and in another case, this limit is equal to $\frac{1}{1+i}$ and $0 \neq \frac{1}{1+i}$, so the limit does not exist. This means that $f(z)$ must not be differentiable at $z = 0$. $\square$