Advanced Calc. Homework 8

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14.1

Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{n^4}{2^n}$

- Using the ratio test,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^4}{2^{n+1}} \cdot \frac{2^n}{n^4}\right| = \frac{1}{2}\left|\frac{(n+1)^4}{n^4}\right| \\
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+1}{n}\right)^4 \\
&= \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^4 \\
&= \frac{1}{2}\cdot 1^4 = \frac{1}{2} < 1
\end{aligned}$$

- Thus, $\boxed{\text{the series converges due to the ratio test}}$

(b) $\sum \frac{2^n}{n!}$

- Using the ratio test,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \left|\frac{2}{n+1}\right| \\
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} 2\cdot\frac{1}{n+1} \\
&= 2\cdot\lim_{n\to\infty}\frac{1}{n+1} \\
&= 2\cdot 0 = 0 < 1
\end{aligned}$$

- Thus, $\boxed{\text{the series converges due to the ratio test}}$

(c) $\sum \frac{n^2}{3^n}$

- Using the ratio test,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^2}{3^{n+1}} \cdot \frac{3^n}{n^2}\right| = \frac{1}{3}\left|\frac{(n+1)^2}{n^2}\right| \\
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{1}{3}\left(\frac{n+1}{n}\right)^2 \\
&= \frac{1}{3}\cdot\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \\
&= \frac{1}{3}\cdot 1^2 = \frac{1}{3} < 1
\end{aligned}$$

- Thus, $\boxed{\text{the series converges due to the ratio test}}$

(d) $\sum \frac{n!}{n^4+3}$

- Using the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^4+3} \cdot \frac{n^4+3}{n!}\right| = \left|\frac{(n+1)(n^4+3)}{(n+1)^4+3}\right| = \left|\frac{n^5+n^4+3n+3}{n^4+4n^3+6n^2+4n+4}\right|$$

$$\begin{aligned}
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{n^5+n^4+3n+3}{n^4+4n^3+6n^2+4n+4} \cdot \frac{\frac{1}{n^5}}{\frac{1}{n^5}} \\
&= \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{3}{n^4}+\frac{3}{n^5}}{\frac{1}{n}+\frac{4}{n^2}+\frac{6}{n^3}+\frac{4}{n^4}+\frac{4}{n^5}} \\
&= \infty \quad \text{since we are of the form } \frac{1}{0} \\
&\not< 1
\end{aligned}$$

- Thus, $\boxed{\text{the series diverges due to the ratio test}}$

(e) $\sum \dfrac{\cos^2(n)}{n^2}$

- Consider $|a_n|$, then we have:

$$|a_n| = \left|\frac{\cos^2(n)}{n^2}\right| = \frac{|\cos(n)|^2}{|n|^2} = \frac{|\cos(n)|^2}{n^2} \leq \frac{1^2}{n^2} = \frac{1}{n^2} \qquad \text{for all } n \text{ since } \cos(n) \leq 1 \text{ for all } n.$$

- Furthermore, $\sum \frac{1}{n^2}$ converges by the p-series test (example 2 pg 96-97), so $\boxed{\text{the series converges by the comparison test with } \frac{1}{n^2}}$

(f) $\sum\limits_{n=2}^{\infty} \dfrac{1}{\log(n)}$

- Consider $|a_n|$, then we have:

$$\begin{aligned}
|a_n| = \left|\frac{1}{\log(n)}\right| &= \frac{1}{\log(n)} && \text{since } \log(n) > 0 \text{ for all } n \geq 2 \\
&> \frac{1}{n} && \text{since } \log(n) < n \text{ for all } n \geq 2
\end{aligned}$$

- However, $\sum \frac{1}{n}$ diverges according to the p-series test (example 2 pg 96-97), so $\boxed{\text{the series diverges by the comparison test with } \frac{1}{n}}$

14.2

Determine which of the following series converge. Justify your answers.

(a) $\sum \dfrac{n-1}{n^2}$

- Note that $\sum \frac{n-1}{n^2} = \sum \left(\frac{n}{n^2} - \frac{1}{n^2}\right) = \sum \frac{1}{n} - \sum \frac{1}{n^2}$
- We know that the second sum converges, say to $L \in \mathbb{R}$. Then we are left with a divergent sum minus L, so we can conclude that the original sum $\boxed{\text{diverges due to the divergence of } \sum \frac{1}{n}}$

(b) $\sum (-1)^n$

- We require for $\sum a_n$ to converge, that $\lim(a_n) = 0$. However, in this case for $a_n = (-1)^n$, $\liminf(a_n) = -1 \neq 1 = \limsup(a_n)$. Therefore, $\lim(a_n)$ does not exist, so $\boxed{\text{the series does not converge since } \lim(-1)^n \neq 0}$

(c) $\sum \dfrac{3n}{n^3}$

- First, note that $\frac{3n}{n^3} = \frac{3}{n^2}$ for all n.

- Thus, $\sum \frac{3n}{n^3} = \sum \frac{3}{n^2} = \left(\frac{3}{1} + \frac{3}{4} + \frac{3}{9} + \cdots\right) = 3\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \cdots\right) = 3 \cdot \sum \frac{1}{n^2}$
- We know that $\sum \frac{1}{n^2}$ converges due to p-series test (example 2 pg 96-97), so in other words $\sum \frac{1}{n^2} = L \in \mathbb{R}$; thus, the series we're interested in converges to $3L \in \mathbb{R}$, so $\boxed{\text{the series converges due to the convergence of } \sum \frac{1}{n^2}}$

(d) $\sum \frac{n^3}{3^n}$

- Using the ratio test,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \frac{1}{3}\left|\frac{(n+1)^3}{n^3}\right| \\ \implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right)^3 \\ &= \lim_{n\to\infty} \frac{1}{3}\left(1 + \frac{1}{n}\right)^3 \\ &= \frac{1}{3} \cdot 1^3 = \frac{1}{3} < 1 \end{aligned}$$

- Thus, $\boxed{\text{the series converges by the ratio test}}$

(e) $\sum \frac{n^2}{n!}$

- Using the ratio test,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{(n+1)^2}{(n+1)!} \cdot \frac{n!}{n^2}\right| = \left|\frac{1}{n+1} \cdot \frac{(n+1)^2}{n^2}\right| \\ \implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{n+1}{n^2}\right) \\ &= \lim_{n\to\infty}\left(\frac{\frac{1}{n} + \frac{1}{n^2}}{1}\right) \\ &= 0 < 1 \end{aligned}$$

- Thus, $\boxed{\text{the series converges by the ratio test}}$

(f) $\sum \frac{1}{n^n}$

- Using the root test,

$$\begin{aligned} \sqrt[n]{|a_n|} &= \sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \\ \implies \lim_{n\to\infty} \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \frac{1}{n} \\ &= 0 < 1 \end{aligned}$$

- Thus, $\boxed{\text{the series converges by the root test}}$

(g) $\sum \frac{n}{2^n}$

- Using the ratio test,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}\right| = \frac{1}{2}\left|\frac{n+1}{n}\right| \\ \implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) \\ &= \frac{1}{2} \cdot 1 = \frac{1}{2} < 1 \end{aligned}$$

- Thus, $\boxed{\text{the series converges by the ratio test}}$

14.3

Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{1}{\sqrt{n!}}$

- Using the ratio test,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\sqrt{n!}}{\sqrt{(n+1)!}}\right| = \left|\frac{\sqrt{n!}}{\sqrt{n+1}\sqrt{n!}}\right| \\
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{1}{\sqrt{n+1}} \\
&= 0 < 1
\end{aligned}$$

- Thus, $\boxed{\text{the series converges by the ratio test}}$

(b) $\sum \frac{2+\cos(n)}{3^n}$

- Let's consider $|a_n|$,

$$\begin{aligned}
|a_n| = \left|\frac{2+\cos(n)}{3^n}\right| = \frac{|2+\cos(n)|}{|3^n|} &\leq \frac{|2|+|\cos(n)|}{3^n} && \text{by the Triangle Inequality} \\
&\leq \frac{2+1}{3^n} && \text{since } \cos(n) \leq 1 \text{ for all } n \\
&= \frac{1}{3^{n-1}}
\end{aligned}$$

- Furthermore, we know that $\sum \frac{1}{3^{n-1}}$ converges as a geometric series; thus,
$\boxed{\text{the series converges by the comparison test with } \frac{1}{3^{n-1}}}$

(c) $\sum \frac{1}{2^n+n}$

- Let's consider $|a_n|$,

$$|a_n| = \left|\frac{1}{2^n+n}\right| = \frac{1}{|2^n+n|} = \frac{1}{2^n+n} \leq \frac{1}{2^n} \qquad \text{since } 2^n+n \geq 2^n \text{ for all } n \in \mathbb{N}$$

- Furthermore, we know that $\sum \frac{1}{2^n}$ converges as a geometric series; thus,
$\boxed{\text{the series converges by the comparison test with } \frac{1}{2^n}}$

(d) $\sum \left(\frac{1}{2}\right)^n \left(50+\frac{2}{n}\right)$

- Using the ratio test,

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{\left(\frac{1}{2}\right)^{n+1}\left(50+\frac{2}{n+1}\right)}{\left(\frac{1}{2}\right)^n\left(50+\frac{2}{n}\right)}\right| = \frac{1}{2}\left(\frac{50+\frac{2}{n+1}}{50+\frac{2}{n}}\right) \\
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{1}{2}\lim_{n\to\infty}\left(\frac{50+\frac{2}{n+1}}{50+\frac{2}{n}}\right) \\
&= \frac{1}{2}\frac{\lim_{n\to\infty}(50+\frac{2}{n+1})}{\lim_{n\to\infty}(50+\frac{2}{n})} \\
&= \frac{1}{2}\cdot\frac{50}{50} = \frac{1}{2} < 1
\end{aligned}$$

- Thus, $\boxed{\text{the series converges by the ratio test}}$

(e) $\sum \sin\left(\frac{n\pi}{9}\right)$

- We require for $\sum a_n$ to converge that $\lim(a_n) = 0$. However, in this case for $a_n = \sin\left(\frac{n\pi}{9}\right)$, we have $\liminf(a_n) = \sin\left(\frac{-4\pi}{9}\right) \approx -0.9848 \neq 0.9848 \approx \sin\left(\frac{4\pi}{9}\right) = \limsup(a_n)$. Therefore, $\lim(a_n)$ does not exist, so $\boxed{\text{the series does not converge since } \lim\left(\sin\left(\frac{n\pi}{9}\right)\right) \neq 0}$

(f) $\sum \dfrac{100^n}{n!}$

- Using the ratio test,

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n}\right| &= 100 \cdot \left|\frac{1}{n+1}\right| \\ \implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= 100 \lim_{n\to\infty} \frac{1}{n+1} \\ &= 100 \cdot 0 = 0 < 1 \end{aligned}$$

- Thus, $\boxed{\text{this series converges due to the ratio test}}$

14.4

Determine which of the following series converge. Justify your answers.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$

- Let's consider $|a_n|$,

$$|a_n| = \left|\frac{1}{[n+(-1)^n]^2}\right| = \frac{1}{[n+(-1)^n]^2} \leq \frac{1}{(n-1)^2} \qquad \text{since } n+(-1)^n \geq n-1 \text{ for all } n \geq 2$$

- We also know that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \sum_{m=1}^{\infty} \frac{1}{m^2}$ and we know that this second series is a convergent p-series; thus, the first series must also converge as well. Using this, we can conclude that $\boxed{\text{the original series must converge by the comparison test with } \dfrac{1}{(n-1)^2}}$

(b) $\sum \left[\sqrt{n+1} - \sqrt{n}\right]$

- First, note that $\left[\sqrt{n+1}-\sqrt{n}\right] = \left[\sqrt{n+1}-\sqrt{n}\right] \cdot \dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{n+1+\sqrt{n}\sqrt{n+1}-\sqrt{n}\sqrt{n+1}-n}{\sqrt{n+1}+\sqrt{n}} = \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$.
- Thus, if we analyze $|a_n|$,

$$\begin{aligned} |a_n| = \left|\sqrt{n+1}-\sqrt{n}\right| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| &= \frac{1}{\sqrt{n+1}+\sqrt{n}} \\ &> \frac{1}{2\sqrt{n+1}} && \text{since } \sqrt{n} < \sqrt{n+1} \text{ for all } n \in \mathbb{N} \\ &\geq \frac{1}{2\sqrt{2n}} && \text{since } n+1 \leq 2n \text{ for all } n \in \mathbb{N} \\ &= \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{n}} \end{aligned}$$

- However, we know that $\sum \frac{1}{\sqrt{n}}$ diverges due to the p-series test, so $\sum \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{n}}$ diverges as well. Thus, $\boxed{\text{this series diverges due to comparison test with } \dfrac{1}{2\sqrt{2n}}}$

(c) $\sum \dfrac{n!}{n^n}$

- Using the ratio test,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}\right| = \left|(n+1) \cdot \frac{n^n}{(n+1)^{n+1}}\right| = \left|\frac{n^n}{(n+1)^n}\right| = \left(\frac{n}{n+1}\right)^n$$

$$\begin{aligned}
\implies \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n \\
&= \frac{1}{\lim_{n\to\infty}\frac{1}{(n/(n+1))^n}} && \text{by Lemma 9.5} \\
&= \frac{1}{\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n} \\
&= \frac{1}{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n} \\
&= \frac{1}{e}
\end{aligned}$$

- This last equality here follows from the discussion on Example 3(e) on page 37 of section 7. Therefore, since $\frac{1}{e} \approx 0.3679 < 1$, then $\boxed{\text{this series converges due to the ratio test}}$

14.5

Suppose $\sum a_n = A$ and $\sum b_n = B$ where A and B are real numbers. Use limit theorems to quickly prove the following:

(a) $\sum(a_n + b_n) = A + B$

(b) $\sum ka_n = kA$ for $k \in \mathbb{R}$

(c) Is $\sum a_n b_n = AB$ a reasonable conjecture? Discuss.

Proof. **(a)**
I will start by examining the series on the left and assuming their indices start at some arbitrary point m:

$$\begin{aligned}
\sum_{n=m}^{\infty}(a_n + b_n) &= \lim_{N\to\infty}\sum_{n=m}^{N}(a_n + b_n) \\
&= \lim_{N\to\infty}\left[(a_m + b_m) + (a_{m+1} + b_{m+1}) + \cdots + (a_N + b_N)\right] \\
&= \lim_{N\to\infty}\left[(a_m + a_{m+1} + \cdots + a_N) + (b_m + b_{m+1} + \cdots + b_N)\right] \\
&= \lim_{N\to\infty}\left[(a_m + a_{m+1} + \cdots + a_N)\right] + \lim_{N\to\infty}\left[(b_m + b_{m+1} + \cdots + b_N)\right] && \text{by Theorem 9.3} \\
&= \lim_{N\to\infty}\sum_{n=m}^{N} a_n + \lim_{N\to\infty}\sum_{n=m}^{N} b_n \\
&= \sum_{n=m}^{\infty} a_n + \sum_{n=m}^{\infty} b_n \\
&= A + B && \text{by the given information}
\end{aligned}$$

□

Proof. **(b)**

I will start by examining the series on the left and assuming the index starts at some arbitrary point m:

$$\begin{aligned}
\sum_{n=m}^{\infty} ka_n &= \lim_{N\to\infty} \sum_{n=m}^{N} ka_n \\
&= \lim_{N\to\infty} [ka_m + ka_{m+1} + \cdots + ka_N] \\
&= \lim_{N\to\infty} [k(a_m + a_{m+1} + \cdots + a_N)] \\
&= k \cdot \lim_{N\to\infty} [(a_m + a_{m+1} + \cdots + a_N)] && \text{by Theorem 9.2} \\
&= k \cdot \lim_{N\to\infty} \sum_{n=m}^{N} a_n \\
&= k \cdot \sum_{n=m}^{\infty} a_n \\
&= kA && \text{by the given information}
\end{aligned}$$

□

Answer. (c)
This is not a reasonable conjecture. For example, this does not even hold for finite sums (e.g. for 2 terms) since $a_1b_1 + a_2b_2 \neq (a_1 + a_2)(b_1 + b_2)$.

14.6

(a) Prove that if $\sum |a_n|$ converges and (b_n) is a bounded sequence, then $\sum a_nb_n$ converges.

(b) Observe that Corollary 14.7 is a special case of part (a).

Proof. **(a)**
First, since (b_n) is bounded, there exists some $M \in \mathbb{R}$ such that $b_n \leq M$ for all n. Next, I will prove the statement by showing that $\sum_{k=1}^{n} a_kb_k$ is a Cauchy sequence. Let $n \geq m$, then we have:

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_kb_k - \sum_{k=0}^{m} a_kb_k\right| = \left|\sum_{k=m+1}^{n} a_kb_k\right| &= |a_{m+1}b_{m+1} + a_{m+2}b_{m+2} + \cdots + a_nb_n| \\
&\leq |a_{m+1}b_{m+1}| + |a_{m+2}b_{m+2}| + \cdots + |a_nb_n| && \text{by the Triangle Inequality} \\
&= |a_{m+1}||b_{m+1}| + |a_{m+2}||b_{m+2}| + \cdots + |a_n||b_n| \\
&\leq |a_{m+1}| \cdot M + |a_{m+2}| \cdot M + \cdots + |a_n| \cdot M && \text{by } (b_n)\text{'s boundedness} \\
&= M \cdot \sum_{k=m+1}^{n} |a_k|
\end{aligned}$$

Furthermore, since $\sum |a_n|$ converges (i.e. is Cauchy), then for every $\varepsilon > 0$, there exists some N such that $n, m > N$ implies $\left|\sum_{k=0}^{n} |a_k| - \sum_{k=0}^{m} |a_k|\right| = \left|\sum_{k=m+1}^{n} |a_k|\right| < \frac{\varepsilon}{M}$. Using this, we can conclude that:

$$\begin{aligned}
\left|\sum_{k=0}^{n} a_kb_k - \sum_{k=0}^{m} a_kb_k\right| &\leq M \cdot \sum_{k=m+1}^{n} |a_k| \\
&< M \cdot \frac{\varepsilon}{M} = \varepsilon.
\end{aligned}$$

Thus, $\sum_{k=1}^{n} a_kb_k$ is a Cauchy sequence, so it converges. □

Answer. (b)
Indeed, Corollary 14.7 asserts that absolutely convergent sequences are convergent. This can follow immediately from part (a) if we let $(b_n) := 1$.

14.7

Prove that if $\sum a_n$ is a convergent series of nonnegative numbers and $p > 1$, then $\sum a_n^p$ converges.

Proof.
Corollary 14.5 tells us that if $\sum a_n$ converges, then $\lim(a_n) = 0$. Furthermore, this tells us that for any $\varepsilon > 0$, there exists some N such that $|a_n| < \varepsilon$ for all $n > N$. In particular, there exists some N such that $|a_n| < 1$ for all $n > N$. We can therefore make the following conclusions for all $n > N$: $|a_n^p| = |a_n||a_n^{p-1}| < |a_n^{p-1}| < |a_n^{p-2}| < \cdots < |a_n| \leq a_n$. In other words, $|a_n^p| < a_n$ for all $n > N$. Thus, by the comparison test, $\sum a_n^p$ converges. $\square$

14.8

Show that if $\sum a_n$ and $\sum b_n$ are convergent series of nonnegative numbers, then $\sum \sqrt{a_n b_n}$ converges.

Proof.
Claim: $\sqrt{ab} \leq a + b$ for $a, b \in [0, \infty)$
Proof of claim:
We know that $0 \leq a^2 + ab + b^2$ since $0 \leq a^2$, $0 \leq ab$, and $0 \leq b^2$. From this, we can add ab to both sides to get $ab \leq a^2 + 2ab + b^2 = (a + b)^2$. Thus, taking square roots give $\sqrt{ab} \leq a + b$ the desired inequality.

Thus, we can now conclude that $\sqrt{a_n b_n} \leq a_n + b_n$ for all n since $(a_n), (b_n) \in [0, \infty)$ for all n. Furthermore, since we showed that $\sum(a_n + b_n)$ converges in question 14.5(a), then we can use the comparison test with that series to conclude that $\sum \sqrt{a_n b_n}$ also converges. $\square$

14.9

The convergence of a series does not depend on any finite number of the terms, though of course the value of the limit does. More precisely, consider series $\sum a_n$ and $\sum b_n$ and suppose the set $\{n \in \mathbb{N} : a_n \neq b_n\}$ is finite. Then the series both converge or else they both diverge. Prove this.

Proof.
If we let $N_0 = \max\{n \in \mathbb{N} : a_n \neq b_n\}$, then the fact that that set is finite implies that $N_0 < \infty$. Thus, for $n \geq m > N_0$ we know that $a_k = b_k$ for all $k \geq m$. Thus, $\displaystyle\sum_{k=m}^{n} a_k = \sum_{k=m}^{n} b_k$. Furthermore, $\displaystyle\lim_{n\to\infty} \sum_{k=m}^{n} a_k = \lim_{n\to\infty} \sum_{k=m}^{n} b_k$. Since the convergence of a series does not depend on any finite number of terms, then $\sum a_n$ converges if and only if $\displaystyle\lim_{n\to\infty} \sum_{k=m}^{n} a_k$ converges and similarly $\sum b_n$ converges if and only if $\displaystyle\lim_{n\to\infty} \sum_{k=m}^{n} b_k$ converges. Thus, if $\displaystyle\lim_{n\to\infty} \sum_{k=m}^{n} a_k = \lim_{n\to\infty} \sum_{k=m}^{n} b_k$, then $\sum a_n$ and $\sum b_n$ both converge if those limits are finite and $\sum a_n$ and $\sum b_n$ both do not converge if those limits are not a real (finite) number – proving the statement. $\square$