Advanced Calc. Homework 9

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15.1

Determine which of the following series converge. Justify your answers.

(a) $\sum \frac{(-1)^n}{n}$

- This series $\boxed{\text{converges}}$ by the Alternating Series Test since $\frac{1}{n} \to 0$ as $n \to \infty$.

(b) $\sum \frac{(-1)^n n!}{2^n}$

- This series $\boxed{\text{diverges}}$ by the Ratio test, since $\lim |\frac{a_{n+1}}{a_n}| = \lim \left(\frac{(n+1)!}{2^{n+1}} \cdot \frac{2^n}{n!}\right) = \lim \left(\frac{n+1}{2}\right) = +\infty > 1$.

15.2

Determine which of the following series converge. Justify your answers.

(a) $\sum \left[\sin\left(\frac{n\pi}{6}\right)\right]^n$

- Note, that in order for a series $\sum a_n$ to converge, that $\lim(a_n)$ must be 0. That also means that for any subsequence, $\lim(a_{n_k}) = 0$. However, in this case for $a_n = \left[\sin\left(\frac{n\pi}{6}\right)\right]^n$ and choosing $n_k = 12k + 3$, we get:

$$\begin{aligned}
a_{n_k} &= \left[\sin\left(\frac{(12k+3)\pi}{6}\right)\right]^{12k+3} \\
&= \left[\sin\left(2k\pi + \frac{\pi}{2}\right)\right]^{12k+3} \\
&= \left[\sin\left(\frac{\pi}{2}\right)\right]^{12k+3} \\
&= (1)^{12K+3} \\
&= 1
\end{aligned}$$

- Hence, $\lim(a_{n_k}) = 1 \neq 0$. Therefore, $\lim(a_n) \neq 0$. From this, we can conclude that $\boxed{\text{the series diverges}}$.

(b) $\sum \left[\sin\left(\frac{n\pi}{7}\right)\right]^n$

- For this series, I will use the root test. I will also note that $|\sin(x)| \leq 1$ for all x. However, $|\sin(x)| = 1$ only for x of the form $x = k\pi + \frac{\pi}{2}$. Thus, for $x = \frac{n\pi}{7}$, $|\sin(x)| < 1$ since x is never of the aforementioned form. Thus,

$$\begin{aligned}
\sqrt[n]{|a_n|} &= \sqrt[n]{\left|\left[\sin\left(\frac{n\pi}{7}\right)\right]^n\right|} \\
&= \sqrt[n]{\left|\sin\left(\frac{n\pi}{7}\right)\right|^n} \\
&= \left|\sin\left(\frac{n\pi}{7}\right)\right| \\
&< 1 && \text{by the above comments}
\end{aligned}$$

Thus, $\boxed{\text{this series converges by the root test}}$.

15.3

Show $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p}$ converges if and only if $p > 1$.

Answer.
First, let's assume that $p > 1$ and note that this sum is strictly less than $\dfrac{1}{2(\log(2))^p}$ + the area under the curve $\dfrac{1}{x(\log(x))^p}$ from 2 to ∞. This translates to:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p} &< \frac{1}{2(\log(2))^p} + \int_2^{\infty} \frac{1}{x(\log(x))^p} dx \\
&= \frac{1}{2(\log(2))^p} + \lim_{n\to\infty} \int_2^{n} \frac{1}{x(\log(x))^p} dx \\
&= \frac{1}{2(\log(2))^p} + \lim_{n\to\infty} \int_{\log(2)}^{\log(n)} \frac{1}{u^p} du \qquad \text{with } u = \log(x) \text{ and } du = \frac{1}{x}dx \\
&= \frac{1}{2(\log(2))^p} + \lim_{n\to\infty} \left.\frac{u^{-p+1}}{-p+1}\right|_{u=\log(2)}^{u=\log(n)} \\
&= \frac{1}{2(\log(2))^p} + \lim_{n\to\infty} \frac{1}{1-p}\left(\frac{1}{(\log(n))^{p-1}} - \frac{1}{(\log(2))^{p-1}}\right)
\end{aligned}$$

Here the limit only applies to the first term in the parentheses in which the denominator goes to ∞ as $n \to \infty$. Therefore, that term goes to 0 as $n \to \infty$, so the whole expression is finite, meaning the series converges. Since these calculations were only valid under the assumption that $p > 1$, we have shown one direction of the statement. Now assume that $p \leq 1$. We can now use a similar statement about the integrals and say the following:

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{1}{n(\log(n))^p} &> \int_2^{\infty} \frac{1}{x(\log(x))^p} dx \\
&= \lim_{n\to\infty} \int_2^{n} \frac{1}{x(\log(x))^p} dx \\
&= \lim_{n\to\infty} \int_{\log(2)}^{\log(n)} \frac{1}{u^p} du \qquad \text{with } u = \log(x) \text{ and } du = \frac{1}{x}dx \\
&= \lim_{n\to\infty} \begin{cases} \left.\log(u)\right|_{u=\log(2)}^{u=\log(n)} & \text{for } p = 1 \\ \left.\frac{u^{-p+1}}{-p+1}\right|_{u=\log(2)}^{u=\log(n)} & \text{for } p < 1 \end{cases} \\
&= \lim_{n\to\infty} \begin{cases} \log(\log(n)) - \log(\log(2)) & \text{for } p = 1 \\ \frac{1}{1-p}\left((\log(n))^{1-p} - (\log(2))^{1-p}\right) & \text{for } p < 1 \end{cases}
\end{aligned}$$

However, in both cases, these expressions diverge since $\log(\log(n)) \to \infty$ as $n \to \infty$ and $(\log(n))^{1-p} \to \infty$ as $n \to \infty$ since $1 - p > 0$. Thus, in this case, the series is greater than a divergent integral, so the series must also be divergent. Therefore, the series is convergent precisely when $p > 1$, just as desired.

15.4

Determine which of the following series converge. Justify your answers.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\log(n)}$

- As we have shown in 15.3, $\sum \frac{1}{n\log(n)}$ diverges. Furthermore, we know that $\sqrt{n}\log(n) < n\log(n)$ for all $n \geq 2$ since $\sqrt{n} < n$ for all $n \geq 2$. Thus, $\frac{1}{\sqrt{n}\log(n)} > \frac{1}{n\log(n)}$ for all $n \geq 2$ which implies that

$$\boxed{\text{this series diverges by the Comparison Test with } \frac{1}{n\log(n)}}$$

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{\log(n)}{n}$

- Here, we know that $\sum \frac{1}{n}$ diverges, and we also know that $\frac{\log(n)}{n} > \frac{1}{n}$ for all $n \geq 3$ (assuming this is the natural logarithm) since $\log(n) > 1$ for all $n \geq 3$. Thus, $\boxed{\text{this series diverges by the Comparison Test with } \frac{1}{n}}$

(c) $\displaystyle\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log\log n)}$

- Here we can note that this series is strictly greater than the integral of that expression from 4 to ∞. Thus, calculating the integral, we get:

$$\begin{aligned}
\int_4^{\infty} \frac{1}{x(\log x)(\log\log x)}dx &= \lim_{N\to\infty} \int_4^{N} \frac{1}{x(\log x)(\log\log x)}dx \\
&= \lim_{N\to\infty} \int_{\log\log 4}^{\log\log N} \frac{1}{u}du \qquad \text{with } u = \log\log x \text{ and } du = \frac{1}{x\log x}dx \\
&= \lim_{N\to\infty} \log(u)\Big|_{u=\log\log 4}^{u=\log\log N} \\
&= \lim_{N\to\infty} \log\log\log(N) - \log\log\log(4)
\end{aligned}$$

- However, with the limit, this term goes to ∞ as $N \to \infty$; thus, the integral is divergent. Therefore, since the series is strictly greater than this integral, we know that $\boxed{\text{the series also diverges}}$

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{\log(n)}{n^2}$

- I claim that this is convergent. To show this, I will first show that $\log(n) < \sqrt{n}$ for all $n \in \mathbb{N}$. Consider $f(x) = \sqrt{x} - \log(x)$, then $f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{x} = \frac{1}{\sqrt{x}}\left(\frac{1}{2} - \frac{1}{\sqrt{x}}\right) \geq 0 \iff \frac{1}{2} - \frac{1}{\sqrt{x}} \geq 0 \iff x \geq 4$. Also note that $f(1) = 1, f(2) \approx 0.721, f(3) \approx 0.622, f(4) \approx 0.614$. Thus, with these values and the fact that $f'(x) \geq 0$ for all $x \geq 4$, we can conclude that $f(x) > 0$ for all $x \geq 1$. In particular, this means that $\log(n) < \sqrt{n}$ for all $n \in \mathbb{N}$. Thus, $\frac{\log(n)}{n^2} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{1.5}}$. We know that the series using this last term converges by the p-series test since $1.5 > 1$. Thus, our series of interest $\boxed{\text{converges by Comparison Test with } \frac{1}{n^{1.5}}}$

15.6

(a) Give an example of a divergent series $\sum a_n$ for which $\sum a_n^2$ converges.

- We know that for $a_n = \frac{1}{n}$, $\sum a_n$ diverges. However, for $a_n^2 = \frac{1}{n^2}$, $\sum a_n^2$ converges, both by the p-series test.

(b) Observe that if $\sum a_n$ is a convergent series of nonnegative terms, then $\sum a_n^2$ also converges (see Exercise 14.7).

- Note that if $\sum a_n$ is convergent, then $\lim(a_n) = 0$ which implies that there exists some N such that $a_n < 1$ for all $n \geq N$. In particular, this means that $a_n^2 < a_n$ for all $n \geq N$, thus, $\sum_{n=N}^{\infty} a_n^2$ converges by Comparison Test, which also means that $\sum a_n^2$ converges since a finite number (N) of terms does not affect convergence.

(c) Give an example of a convergent series $\sum a_n$ for which $\sum a_n^2$ diverges.

- Let $a_n = \frac{(-1)^n}{\sqrt{n}}$, then $\sum a_n$ converges by the Alternating Series Test since $\lim(a_n) = 0$ and a_n is alternating. However, $a_n^2 = \frac{1}{n}$ and we know that $\sum a_n^2$ diverges by the p-series test.

17.5

(a) Prove that if $m \in \mathbb{N}$, then the function $f(x) = x^m$ is continuous on $\mathbb{R}$.

(b) Prove every polynomial function $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ is continuous on $\mathbb{R}$.

Proof. **(a)**
I will prove this by using the sequential definition of continuity. Thus, let (x_n) be some sequence that converges to $x_0 \in \mathbb{R}$

and let $f(x) = x^m$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) &= \lim_{n\to\infty} x_n^m \\
&= \lim_{n\to\infty} \underbrace{x_n \cdot x_n \cdots x_n}_{m \text{ times}} \\
&= \underbrace{\left(\lim_{n\to\infty} x_n\right) \cdot \left(\lim_{n\to\infty} x_n\right) \cdots \left(\lim_{n\to\infty} x_n\right)}_{m \text{ times}} && \text{by Limit Theorems} \\
&= \underbrace{x_0 \cdot x_0 \cdots x_0}_{m \text{ times}} && \text{since } x_n \to x_0 \text{ as } n \to \infty \\
&= x_0^m \\
&= f(x_0)
\end{aligned}$$

Thus, f satisfies the sequential definition of continuity at every point $x_0 \in \mathbb{R}$, so f is continuous on $\mathbb{R}$. □

Proof. **(b)**
First, note that the constant function $f(x) = a_0$ for $a_0 \in \mathbb{R}$, is clearly continuous since $|f(x) - f(x_0)| = 0 < \varepsilon$ for every $\varepsilon > 0$ and any $x_0 \in \mathbb{R}$. Furthermore, by using part (a) of this question and Theorem 17.3, we know that $a_i x^i$ is continuous for all $i \in \mathbb{N}$. Therefore, since $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is the sum of $n+1$ different continuous functions, then Theorem 17.4(i) assures us that $p(x)$ is also a continuous function. □

17.7

(a) Observe that if $k \in \mathbb{R}$, then the function $g(x) = kx$ is continuous by Exercise 17.5.

(b) Prove that $f(x) = |x|$ is continuous on $\mathbb{R}$.

(c) Use parts (a) and (b) and Theorem 17.5 to give another proof of Theorem 17.3.

Answer. (a)
Simply note that $g(x)$ is a first degree polynomial of the form $g(x) = a_0 + a_1 x$ for $a_0 = 0$ and $a_1 = k$. Thus, Exercise 17.5(b) assures us that $g(x)$ is a continuous function.

Proof. **(b)**
I will prove this using the ε-δ definition of limits. Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ be fixed. We wish to show that $\big||x| - |x_0|\big| < \varepsilon$ whenever $|x - x_0| < \delta$ for sufficiently small δ. First, note that

$$\begin{aligned}
\big||x| - |x_0|\big| &= \big||x - x_0 + x_0| - |x_0|\big| \\
&= \big||(x - x_0) + x_0| - |x_0|\big| \\
&\leq \big||x - x_0| + |x_0| - |x_0|\big| && \text{by the Triangle Inequality} \\
&= \big||x - x_0|\big| \\
&= |x - x_0|
\end{aligned}$$

Thus, if $\delta = \varepsilon$, then whenever $|x - x_0| < \delta$, $\big||x| - |x_0|\big| \leq |x - x_0| < \delta = \varepsilon$. Thus, $|x|$ is continuous on $\mathbb{R}$. □

Proof. **(c)**
Let $g(x) = kx$ and $f(x) = |x|$. Then, by parts (a) and (b) of this question, g and f are both continuous functions on $\mathbb{R}$. Furthermore, let h be any real-valued function with $\text{dom}(h) \subseteq \mathbb{R}$. Let's also assume that h is continuous at $x_0 \in \text{dom}(h)$.

First, since g is continuous on all of $\mathbb{R}$, then in particular g is continuous at $h(x_0)$. Thus, Theorem 17.5 guarantees that $g \circ h = kh$ is a continuous function at x_0.

Next, since f is continuous on all of $\mathbb{R}$, then in particular, f is continuous at $h(x_0)$. Thus, Theorem 17.5 guarantees that $f \circ h = |h|$ is a continuous function at x_0. This completes the proof of Theorem 17.3. □

17.9

Prove each of the following functions is continuous at x_0 by verifying the ε–δ property of Theorem 17.2.

(a) $f(x) = x^2, x_0 = 2$;

- Let $\varepsilon > 0$, then we wish to show that $|x^2 - 2^2| < \varepsilon$ whenever $|x - 2| < \delta$ for δ sufficiently small.

- Since δ will likely be very small, let us assume that $\delta \le 1$. Thus,

$$|x^2 - 2^2| = |x^2 - 4| = |(x-2)(x+2)| = |x-2||x+2| < |x-2| \cdot 5$$

- The last inequality here follows from the assumption that $\delta \le 1$. In other words, $|x-2| < 1 \implies 1 < x < 3 \implies |x+2| < 5$.
- Therefore, choosing $\delta = \min\{1, \frac{\varepsilon}{5}\}$ yields:

$$\begin{aligned} |x^2 - 2^2| &< |x-2| \cdot 5 \\ &< \delta \cdot 5 \\ &\le \frac{\varepsilon}{5} \cdot 5 \\ &= \varepsilon \end{aligned}$$

- Which proves that f is continuous at x_0.

(b) $f(x) = \sqrt{x}, x_0 = 0$;

- Let $\varepsilon > 0$, then we wish to show that $|\sqrt{x} - \sqrt{0}| < \varepsilon$ whenever $|x - 0| < \delta$ for δ sufficiently small.
- Note that $|\sqrt{x} - \sqrt{0}| = \sqrt{x} < \varepsilon$ if and only if $x < \varepsilon^2$.
- Thus, choose $\delta = \varepsilon^2$. This yields:

$$|\sqrt{x} - \sqrt{0}| = \sqrt{x} = \sqrt{|x-0|} < \sqrt{\varepsilon^2} = \varepsilon$$

- Which proves that f is continuous at x_0.

(c) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}, x_0 = 0$;

- Let $\varepsilon > 0$, then we wish to show that $|x \sin\left(\frac{1}{x}\right) - 0| < \varepsilon$ whenever $|x - 0| < \delta$ for δ sufficiently small.
- Let us choose $\delta = \varepsilon$ to obtain the following:

$$\begin{aligned} \left| x \sin\left(\frac{1}{x}\right) - 0 \right| &= \left| x \sin\left(\frac{1}{x}\right) \right| \\ &= |x| \left| \sin\left(\frac{1}{x}\right) \right| \\ &\le |x| && \text{since } |\sin(\theta)| \le 1 \text{ for all } \theta \\ &= |x - 0| \\ &< \delta \\ &= \varepsilon \end{aligned}$$

- Which proves that f is continuous at x_0.

(d) $g(x) = x^3, x_0$ arbitrary.

- Let $\varepsilon > 0$, then we wish to show that $|x^3 - x_0^3| < \varepsilon$ whenever $|x - x_0| < \delta$ for δ sufficiently small.
- Since δ will likely be very small, let us assume that $\delta \le 1$. Thus,

$$\begin{aligned} |x^3 - x_0^3| = |(x - x_0)(x^2 + xx_0 + x_0^2)| &= |x - x_0||x^2 + xx_0 + x_0^2| \\ &\le |x - x_0|(|x|^2 + |x||x_0| + |x_0|^2) && \text{by Triangle Inequality} \\ &< |x - x_0|(1 + 3|x_0| + 3|x_0|^2) \end{aligned}$$

- The last inequality follows from the assumption that $\delta \le 1$. In other words,

$$\begin{aligned} & |x - x_0| < 1 \\ \implies & -1 - |x_0| \le -1 + x_0 < x < 1 + x_0 \le 1 + |x_0| \\ \implies & |x| < 1 + |x_0| \\ \implies & |x|^2 + |x||x_0| + |x_0|^2 < (1 + |x_0|)^2 + (1 + |x_0|)|x_0| + |x_0|^2 \\ & \qquad = 1 + 2|x_0| + |x_0|^2 + |x_0| + |x_0|^2 + |x_0|^2 \\ & \qquad = 1 + 3|x_0| + 3|x_0|^2 \end{aligned}$$

- Therefore, choosing $\delta = \min\{1, \frac{\varepsilon}{1+3|x_0|+3|x_0|^2}\}$ yields:

$$\begin{aligned}|x^3 - x_0^3| &< |x - x_0|(1 + 3|x_0| + 3|x_0|^2)\\ &< \delta \cdot (1 + 3|x_0| + 3|x_0|^2)\\ &\leq \frac{\varepsilon}{1 + 3|x_0| + 3|x_0|^2} \cdot (1 + 3|x_0| + 3|x_0|^2)\\ &= \varepsilon\end{aligned}$$

- Which proves that g is continuous at x_0.

17.10

Prove the following functions are discontinuous at the indicated points. You may use either Definition 17.1 or the ε–δ property in Theorem 17.2.

(a) $f(x) = 1$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$, $x_0 = 0$;

- Let $x_n = \frac{1}{n}$. Then (x_n) is a sequence of positive real numbers that converges to $x_0 = 0$. Thus, $f(x_n) = 1$ for all n since $x_n > 0$ for all n which implies that $\lim_{n\to\infty} f(x_n) = 1 \neq 0 = f(0)$. Thus, f is discontinuous at x_0.

(b) $g(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $g(0) = 0, x_0 = 0$;

- Let $x_n = \frac{1}{(2n\pi + \frac{\pi}{2})}$. Then (x_n) is a sequence that converges to $x_0 = 0$. Thus, $g(x_n) = \sin\left(\frac{1}{\frac{1}{2n\pi+\frac{\pi}{2}}}\right) = \sin\left(2n\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$ which implies that $\lim_{n\to\infty} g(x_n) = 1 \neq 0 = g(0)$. Thus, g is discontinuous at x_0.

(c) $\operatorname{sgn}(x) = -1$ for $x < 0$, $\operatorname{sgn}(x) = 1$ for $x > 0$, and $\operatorname{sgn}(0) = 0, x_0 = 0$.

- Again, let $x_n = \frac{1}{n}$ making (x_n) a sequence of positive real numbers that converges to $x_0 = 0$. Thus, $\operatorname{sgn}(x_n) = 1$ for all n since $x_n > 0$ for all n. Therefore, $\lim_{n\to\infty} \operatorname{sgn}(x_n) = 1 \neq 0 = \operatorname{sgn}(0)$. Thus, $\operatorname{sgn}(\cdot)$ is discontinuous at x_0.

17.12

(a) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.

(b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove that $f(x) = g(x)$ for all $x \in (a, b)$.

Proof. **(a)**
I will prove this by contradiction: assume that f is continuous in (a, b) and $f(r) = 0$ for all $r \in (a, b) \cap \mathbb{Q}$, but there exists some $x_0 \in (a, b)$ such that $|f(x_0)| > 0$.
By Theorem 17.3 (or Question 17.7(c)), $|f|$ is also a continuous function on (a, b). In particular, $|f|$ is continuous at x_0. That means that for every $\varepsilon > 0$ there exists some $\delta > 0$ such that for all $x \in (a, b)$ satisfying $|x - x_0| < \delta$, we have $\big||f(x)| - |f(x_0)|\big| < \varepsilon$. In particular, we can choose $\varepsilon = \frac{|f(x_0)|}{2} > 0$ which gives us

$$-\frac{|f(x_0)|}{2} < |f(x)| - |f(x_0)| < \frac{|f(x_0)|}{2}$$

$$\implies \frac{|f(x_0)|}{2} < |f(x)| < \frac{3|f(x_0)|}{2}$$

$$\implies |f(x)| > 0$$

This means for all $x \in (x_0 - \delta, x_0 + \delta)$ we have $|f(x)| > 0$. However, by the Denseness of the Rationals inside of the Real Numbers, there exists some $r_0 \in \mathbb{Q}$ where also $r_0 \in (x_0 - \delta, x_0 + \delta)$ which implies that $|f(r_0)| > 0$, which is a contradiction to the definition of f being 0 for all rational numbers in (a, b), so our assumption must have been false, so there does not exist an x_0 such that $|f(x_0)| > 0 \implies f(x) = 0$ for all $x \in (a, b)$. □

Proof. **(b)**
Define $h(x) = f(x) - g(x)$. Then clearly h is a continuous function on (a, b) by Theorem 17.3/17.4(i). Also for $r \in (a, b) \cap \mathbb{Q}$, $h(r) = f(r) - g(r) = 0$ since $f(r) = g(r)$ for each rational r in (a, b). Then, by part (a) of this question, $h(x) = 0$ for all $x \in (a, b)$ which implies $f(x) - g(x) = 0 \implies f(x) = g(x)$ for all $x \in (a, b)$, completing the proof. □

17.13

(a) Let $f(x) = 1$ for rational numbers x and $f(x) = 0$ for irrational numbers. Show f is discontinuous at every x in $\mathbb{R}$.

- Let $x_0 \in \mathbb{R}$ be arbitrary. I will show that f is discontinuous at x_0. First, let d_n represent the integer part of x_0 plus the first n decimal places of x_0 (For example, if $x_0 = \pi$, then $d_1 = 3.1, d_2 = 3.14, d_3 = 3.141, d_4 = 3.1415, \ldots$) From this, define

$$x_n = \begin{cases} d_n & \text{if } n \text{ is even} \\ d_n + \frac{\sqrt{2}}{n} & \text{if } n \text{ is odd} \end{cases}$$

- It is clear that $\lim(d_n) = x_0$ and $\lim(\frac{\sqrt{2}}{n}) = 0$. Thus, $\lim(x_n) = x_0$. However, for even n we have $f(x_n) = 1$ since d_n is rational and for odd n we have $f(x_n) = 0$ since $d + \frac{\sqrt{2}}{n}$ is irrational. Therefore $\lim f(x_n)$ does not exist. In particular, $\lim f(x_n) \neq f(x_0)$, so f is not continuous for any $x_0 \in \mathbb{R}$.

(b) Let $h(x) = x$ for rational numbers x and $h(x) = 0$ for irrational numbers. Show h is continuous at $x = 0$ and at no other point.

- First, I will show continuity at $x_0 = 0$. Let $\varepsilon > 0$ be given, then let's examine $|h(x) - h(x_0)|$,

$$\begin{aligned} |h(x) - h(x_0)| = |h(x) - h(0)| = |h(x) - 0| &= |h(x)| \\ &= \begin{cases} |x| & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} \end{aligned}$$

- Thus, if we set $\delta = \varepsilon$ in the ε–δ definition of continuity, we get $|x - x_0| < \varepsilon \implies |x| < \varepsilon$. Furthermore, since $|x| \geq 0$, we know that $|h(x)| \leq |x|$ for all x which implies $|h(x)| \leq |x| < \varepsilon$, proving continuity at $x_0 = 0$.
- Now, I will show discontinuity at any $x_0 \neq 0$. I will prove this by contradiction: assume h is continuous at some $x_0 \neq 0$. In the ε–δ definition of continuity, choose $\varepsilon = \frac{|x_0|}{2}$ (> 0 since $x_0 \neq 0$), then there must exist some δ such that for any $x \in (x_0 - \delta, x_0 + \delta)$, we have $|h(x) - h(x_0)| < \frac{|x_0|}{2}$. However, this cannot possibly be true for all $x \in (x_0 - \delta, x_0 + \delta)$ because if x_0 is rational, there there is an irrational x in $(x_0 - \delta, x_0 + \delta)$ which forces $|h(x) - h(x_0)| = |0 - x_0| = |x_0| > \frac{|x_0|}{2} = \varepsilon$. Alternatively, if x_0 is irrational, then by the Denseness of the Rationals in the Reals, there exists a rational x in $(x_0 - \delta, x_0 + \delta)$ such that $|x| > |x_0|$ which forces $|h(x) - h(x_0)| = |x - 0| = |x| > |x_0| > \frac{|x_0|}{2} = \varepsilon$. Therefore, there is never an instance in which $|x - x_0| < \delta \implies |h(x) - h(x_0)| < \frac{|x_0|}{2}$ which means h can never be continuous (if $x_0 \neq 0$).

17.15

Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Show f is continuous at x_0 in $\text{dom}(f)$ if and only if, for every sequence (x_n) in $\text{dom}(f)\backslash\{x_0\}$ converging to x_0, we have $\lim f(x_n) = f(x_0)$.

Let's label these premises as (i) and (ii):

(i) f is continuous at $x_0 \in \text{dom}(f)$.

(ii) $\lim f(x_n) = f(x_0)$ for every sequence (x_n) in $\text{dom}(f)\backslash\{x_0\}$ that converges to x_0.

Proof. **(i) $\implies$ (ii)**
By the sequential definition of continuity, since (i) says f is continuous at x_0, we can conclude that $\lim f(x_n) = f(x_0)$ for all sequences $(x_n) \subseteq \text{dom}(f)$ that converge to x_0. In particular, since $\text{dom}(f)\backslash\{x_0\} \subset \text{dom}(f)$, we know that $\lim f(x_n) = f(x_0)$ for every sequence $(x_n) \subseteq \text{dom}(f)\backslash\{x_0\}$ that converges to x_0, proving this direction. □

Proof. **(ii) $\implies$ (i)**
Now assume that $\lim f(x_n) = f(x_0)$ for every sequence (x_n) in $\text{dom}(f)\backslash\{x_0\}$ that converges to x_0. However, assume that (i) fails, i.e. that f is not continuous at x_0. This means that there exists some sequence (x_n) in $\text{dom}(f)$ that converges to x_0 but that $\lim f(x_n) \neq f(x_0)$. In other words, there exists some $\varepsilon > 0$ such that $|f(x_n) - f(x_0)| \geq \varepsilon$ for all n. This statement means that $x_n \neq x_0$ for all n (since if $x_n = x_0$ for some n, then $|f(x_n) - f(x_0)| = 0 < \varepsilon$). Thus, we can see that $(x_n) \subseteq \text{dom}(f)\backslash\{x_0\}$ which contradicts (ii) being true. Thus, (i) must be true. □