Complex Analysis Homework 6

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Question 4

Write down an expression of the form $\sum_{n=0}^{\infty} a_n z^n$ for

(a) $\dfrac{1}{1+z^4}$

(b) $\dfrac{1}{2z+5}$

In each case, specify where the expansion is valid.

To find these expansions, I will use the geometric series given by $\sum_{n=0}^{\infty} w^n = \dfrac{1}{1-w}$ for all $|w| < 1$.

Answer. (a)

$$\begin{aligned}
\frac{1}{1+z^4} &= \frac{1}{1-(-z^4)} \\
&= \sum_{n=0}^{\infty} (-z^4)^n \\
&= \boxed{\sum_{n=0}^{\infty} (-1)^n z^{4n}}
\end{aligned}$$

Since our w in this case was equal to $-z^4$, this is valid whenever $|-z^4| < 1 \implies \boxed{|z| < 1}$

Note, this is not quite in the form we desire since the exponent of z is $4n$ instead of just n. However, if we re-define a_n as the following:

$$\begin{cases} a_n = 0 & \text{if } n = 4k+m,\ m \in \{1,2,3\} \\ a_n = (-1)^k & \text{if } n = 4k \end{cases} \quad k \in \mathbb{Z}$$

Then we get $\dfrac{1}{1+z^4} = \sum_{n=0}^{\infty} a_n z^n$ just as desired. However, I think the first answer is easier to work with.

Answer. (b)

$$\begin{aligned}
\frac{1}{2z+5} &= \frac{1}{5} \cdot \frac{1}{1+\frac{2z}{5}} \\
&= \frac{1}{5} \cdot \frac{1}{1-\left(-\frac{2z}{5}\right)} \\
&= \frac{1}{5} \sum_{n=0}^{\infty} \left(-\frac{2z}{5}\right)^n \\
&= \boxed{\sum_{n=0}^{\infty} \frac{(-2)^n}{5^{n+1}} z^n}
\end{aligned}$$

Since our w in this case was equal to $-\frac{2z}{5}$, this is valid whenever $\left|-\frac{2z}{5}\right| < 1 \implies \boxed{|z| < \frac{5}{2}}$

Note this is the form we desired with $a_n = \dfrac{(-2)^n}{5^{n+1}}$.