Complex Analysis Homework 3

Colin Williams

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Question 4

Question.
Show that the derivative $f'(z)$ does not exist at any point z when

(a) $f(z) = Re(z)$

(b) $f(z) = Im(z)$

Note that to be differentiable (i.e. for $f'(z)$ to exist), a function f must satisfy the Cauchy-Riemann Equations. In particular, for $f(z) = u(x, y) + iv(x, y)$ at $z = x + iy$ to be differentiable, u and v must satisfy:

$$u_x(x, y) = v_y(x, y) \qquad\qquad v_x(x, y) = -u_y(x, y) \tag{1}$$

Proof. **(a)**
Let $z = x + iy$, then $f(z) = \mathrm{Re}(z) = x$. Thus, $u(x, y) = x$ and $v(x, y) = 0$. However, this leads to $u_x(x, y) = 1$ and $v_y(x, y) = 0$, but $0 \neq 1$ no matter what z you choose so the Cauchy-Riemann Equations are never satisfied; thus, $f(z)$ cannot be differentiable for any $z \in \mathbb{C}$. $\square$

Proof. **(b)**
Let $z = x + iy$, then $f(z) = \mathrm{Im}(z) = y$. Thus, $u(x, y) = y$ and $v(x, y) = 0$. However, this leads to $-u_y(x, y) = -1$ and $v_x(x, y) = 0$, but $0 \neq -1$ no matter what z you choose so the Cauchy-Riemann Equations are never satisfied; thus, $f(z)$ cannot be differentiable for any $z \in \mathbb{C}$. $\square$