Advanced Calc. Homework 3

Colin Williams

September 15, 2020

Prove Theorem 3.5

Theorem. ***3.5***
The following holds $\forall\, a, b \in \mathbb{R}$:

(i) $|a| \geq 0$

(ii) $|ab| = |a| \cdot |b|$

(iii) $|a + b| \leq |a| + |b|$

Proof. **(i)**
Case 1: If $a \geq 0$, then $|a| = a$, so clearly $|a| \geq 0$.
Case 2: If $a \leq 0$, then $|a| = -a$, so by Theorem 3.2 (i), we can say $-a \geq -0$. Since we've already shown that $0 = -0$ and we know that $|a| = -a$, it is clear that this implies $|a| \geq 0$.
Since $a \leq 0$ or $a \geq 0$ constitute all possible cases for a, we have shown that $|a| \geq 0$ for all a. □

Proof. **(ii)**
Case 1: If $a \geq 0$ and $b \geq 0$, then $ab \geq 0$ by Theorem 3.2 (iii). Thus, $|a| = a$, $|b| = b$, and $|ab| = ab$. So,

$$\begin{aligned} ab &= a \cdot b \\ \implies |ab| &= |a| \cdot |b| \end{aligned}$$

Case 2: If $a \geq 0$ and $b \leq 0$, then $ab \leq 0 \cdot b$ by Theorem 3.2 (ii) $\implies$ $ab \leq 0$ by Theorem 3.1 (ii) and the commutative law. Thus, $|a| = a$, $|b| = -b$, and $|ab| = -ab$. So,

$$\begin{aligned} ab &= a \cdot b \\ \implies (ba) &= b \cdot a && \text{by the commutative law} \\ \implies -(ba) &= (-b) \cdot a && \text{by Theorem 3.1 (iii)} \\ \implies -(ab) &= a \cdot (-b) && \text{by the commutative law} \\ \implies |ab| &= |a| \cdot |b| && \text{by above comments} \end{aligned}$$

Case 3: If $a \leq 0$ and $b \geq 0$, then $ba \leq 0 \cdot a$ by Theorem 3.2 (ii) $\implies$ $ab \leq 0$ by Theorem 3.1 (ii) and the commutative law. Thus, $|a| = -a$, $|b| = b$, and $|ab| = -ab$. So,

$$\begin{aligned} ab &= a \cdot b \\ \implies -ab &= (-a) \cdot b && \text{by Theorem 3.1 (iii)} \\ \implies |ab| &= |a| \cdot |b| && \text{by above comments} \end{aligned}$$

Case 4: If $a \leq 0$ and $b \leq 0$, then $-a \geq 0$ and $-b \geq 0$ by Theorem 3.2 (i) and the fact that $0 = -0$. Thus, by Theorem 3.2 (iii) $(-a)(-b) \geq 0 \implies ab \geq 0$ by Theorem 3.1 (iv). Thus, $|a| = -a$, $|b| = -b$, and $|ab| = ab$. So,

$$\begin{aligned} ab &= a \cdot b \\ \implies ab &= (-a) \cdot (-b) && \text{by Theorem 3.1 (iv)} \\ \implies |ab| &= |a| \cdot |b| && \text{by above comments} \end{aligned}$$

□

Proof. **(iii)**
Since part (i) of this proof tells us that $|a| \geq 0$ for all a, we can also say that $-|a| \leq 0$ by Theorem 3.2 (i) and the fact

that $0 = -0$. Also, since $a = |a|$ or $a = -|a|$ for every a, we can conclude that $-|a| \leq a \leq |a|$. Similarly, $-|b| \leq b \leq |b|$ for any b as well. Starting with $-|a| \leq a$, we can get:

$$\begin{aligned}
& -|a| + (-|b|) \leq a + (-|b|) && \text{by O4} \\
\implies & -(|a| + |b|) \leq a + b && \text{by DL on the left, and the fact that } -|b| \leq b \\
\implies & -(a + b) \leq |a| + |b| && \text{by Theorem 3.2 (i) and the fact that } -(-c) = c
\end{aligned}$$

Similarly, if we start with $a \leq |a|$, we can get:

$$\begin{aligned}
a + b &\leq |a| + b && \text{by O4} \\
&\leq |a| + |b| && \text{since } b \leq |b|
\end{aligned}$$

Thus, since $|a + b|$ either equals $a + b$ or $-(a + b)$ and both of those we have shown are less than or equal to $|a| + |b|$, we can conclude that $|a + b| \leq |a| + |b|$. $\square$

4.1, 4.2, 4.3, 4.4

Question. *For each of the following sets:*

- ***Question 4.1*** *If it is bounded above, list 3 upper bounds; otherwise, write Not Bounded Above.*
- ***Question 4.2*** *If it is bounded below, list 3 lower bounds; otherwise, write Not Bounded Below.*
- ***Question 4.3*** *Give its supremum if it has one; otherwise, write No Supremum.*
- ***Question 4.4*** *Give its infimum if it has one; otherwise, write No Infimum.*

(a) $S = [0, 1]$
  - $1, 1.5, \sqrt{2}$
  - $0, -0.5, -2$
  - $\sup(S) = 1$
  - $\inf(S) = 0$

(b) $S = (0, 1)$
  - $1, 17, \pi$
  - $0, -5, -22$
  - $\sup(S) = 1$
  - $\inf(S) = 0$

(c) $S = \{2, 7\}$
  - $7, 11, 8.8$
  - $2, 1, 0$
  - $\sup(S) = 7$
  - $\inf(S) = 2$

(d) $S = \{\pi, e\}$
  - $4, 5, 6$
  - $0, 1, 2$
  - $\sup(S) = \pi$
  - $\inf(S) = e$

(e) $S = \{\frac{1}{n} : n \in \mathbb{N}\}$
  - $2, 3, 4$
  - $0, -1, -2$
  - $\sup(S) = 1$ since the largest element occurs when $n = 1$ and corresponds to $\frac{1}{1} = 1$
  - $\inf(S) = 0$ since any $r > 0$ also satisfies $r \geq \frac{1}{N}$ for $N \geq \lceil \frac{1}{r} \rceil$, so r cannot be a lower bound.

(f) $S = \{0\}$

- $1, 2, 3$
- $-1, -2, -3$
- $\sup(S) = 0$
- $\inf(S) = 0$

(g) $S = [0, 1] \cup [2, 3]$

- $3, 4, 5$
- $0, -1, -2$
- $\sup(S) = 3$
- $\inf(S) = 0$

(h) $S = \cup_{n=1}^{\infty}[2n, 2n + 1]$

- Not Bounded Above
- $2, 1, 0$
- No Supremum
- $\inf(S) = 2$ since the interval with the smallest elements occurs when $n = 1$ and corresponds to $[2, 3]$ with minimum $= 2$.

(i) $S = \cap_{n=1}^{\infty}[-\frac{1}{n}, 1 + \frac{1}{n}]$
First, note that S simply equals $[0, 1]$ since all $r \in [0, 1]$ are in S and any $r < 0$ or > 1 of the form $r = -\varepsilon$ or $r = 1 + \varepsilon$ is not contained in the set $[-\frac{1}{N}, 1 + \frac{1}{N}]$ for $N \geq \lceil \frac{1}{\varepsilon} \rceil$ which is a requirement to belong to S.

- $2, 3, 4$
- $-1, -2, -3$
- $\sup(S) = 1$
- $\inf(S) = 0$

(j) $S = \{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$

- $1, 2, 3$
- $0, -1, -2$
- $\sup(S) = 1$ since any $r < 1$ cannot be an upper bound since $r < 1 - \frac{1}{3^N}$ for $N \geq \lceil \log_3(\frac{1}{1-r}) \rceil$. This value of N was found by solving $r < 1 - \frac{1}{3^N}$ for N.
- $\inf(S) = \frac{2}{3}$ since the smallest element in the list occurs at $n = 1$ and corresponds to $1 - \frac{1}{3} = \frac{2}{3}$.

(k) $S = \{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

- Not Bounded Above
- $0, -1, -2$
- No Supremum
- $\inf(S) = 0$ since the smallest element occurs at $n = 1$ and corresponds to $1 + \frac{(-1)^1}{1} = 1 - 1 = 0$.

(l) $S = \{r \in \mathbb{Q} : r < 2\}$

- $3, 4, 5$
- Not Bounded Below
- $\sup(S) = 2$ since 2 is greater than all elements of S and any $r < 2$ also has some $r < \frac{p}{q} < 2$
- No Infimum

(m) $S = \{r \in \mathbb{Q} : r^2 < 4\}$

- $3, 4, 5$
- $-3, -4, -5$
- $\sup(S) = 2$ by the same argument as before
- $\inf(S) = -2$ by a similar argument as before

(n) $S = \{r \in \mathbb{Q} : r^2 < 2\}$

- $2, 3, 4$
- $-2, -3, -4$
- $\sup(S) = \sqrt{2}$ by the same argument as (m) but with $\sqrt{2}$ instead of 2
- $\inf(S) = -\sqrt{2}$ by the same argument as (m) but with $\sqrt{2}$ instead of 2

(o) $S = \{x \in \mathbb{R} : x < 0\}$

- $0, 1, 2$
- Not Bounded Below
- $\sup(S) = 0$ since 0 is greater than all elements in S and any $r < 0$ also satisfies $r < s < 0$ for $s \in S$. In particular, choose $s = \frac{r}{2}$.
- No Infimum

(p) $S = \{1, \frac{\pi}{3}, \pi^2, 10\}$

- $11, 12, 13$
- $0, -1, -2$
- $\sup(S) = 10$
- $\inf(S) = 1$

(q) $S = \{0, 1, 2, 4, 8, 16\}$

- $32, 64, 128$
- $-1, -2, -4$
- $\sup(S) = 16$
- $\inf(S) = 0$

(r) $S = \cap_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})$
First, note that $S = 1$. This is because 1 is in every set we are taking the intersection of and any number r of the form $r = 1 \pm \varepsilon \notin S$ because that r does not belong in the set $(1 - \frac{1}{N}, 1 + \frac{1}{N})$ for $N \geq \lceil \frac{1}{\varepsilon} \rceil$.

- $2, 3, 4$
- $0, -1, -2$
- $\sup(S) = 1$
- $\inf(S) = 1$

(s) $S = \{\frac{1}{n} : n \in \mathbb{N}$ and n is prime$\}$

- $2, 3, 4$
- $-1, -2, -3$
- $\sup(S) = \frac{1}{2}$ since the largest element of the set occurs when $n = 2$ and corresponds to $\frac{1}{2}$
- $\inf(S) = 0$ since 0 is less than all elements of S and since there are infinitely many primes, for any $r > 0$ we can also find a prime N such that $r > \frac{1}{N}$.

(t) $S = \{x \in \mathbb{R} : x^3 < 8\}$

- $3, 4, 5$
- Not Bounded Below
- $\sup(S) = 2$ by similar reasoning as part (l).
- No Infimum

(u) $S = \{x^2 : x \in \mathbb{R}\}$

- Not Bounded Above
- $-1, -2, -3$
- No Supremum
- $\inf(S) = 0$ since the smallest element of this set happens when $x = 0$ and corresponds to $0^2 = 0$.

(v) $S = \{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

- $2, 3, 4$
- $-2, -3, -4$

- $\sup(S) = 1$ since the largest element of this set happens when n is of the form $n = 6k$ for $k \in \mathbb{N}$ and corresponds to $\cos(\frac{6k\pi}{3}) = \cos(2k\pi) = 1$.
- $\inf(S) = -1$ since the smallest element of this set happens when n is of the form $n = 6k + 3$ for $k \in \mathbb{N}$ and corresponds to $\cos(\frac{(6k+3)\pi}{3}) = \cos(2k\pi + \pi) = \cos(\pi) = -1$.

(w) $S = \{\sin(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

- $1, 2, 3$
- $-1, -2, -3$
- $\sup(S) = \frac{\sqrt{3}}{2}$ since the largest element of this set happens when n is of the form $n = 6k+1$ or $6k+2$ for $k \in \mathbb{N}$ and this corresponds to $\sin(\frac{(6k+1)\pi}{3}) = \sin(2k\pi + \frac{\pi}{3}) = \sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$ or $\sin(\frac{(6k+2)\pi}{3}) = \sin(2k\pi + \frac{2\pi}{3}) = \sin(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$
- $\inf(S) = -\frac{\sqrt{3}}{2}$ since the smallest element of this set happens when n is of the form $n = 6k+4$ or $6k+5$ for $k \in \mathbb{N}$ and this corresponds to $\sin(\frac{(6k+4)\pi}{3}) = \sin(2k\pi + \frac{4\pi}{3}) = \sin(\frac{4\pi}{3}) = -\frac{\sqrt{3}}{2}$ or $\sin(\frac{(6k+5)\pi}{3}) = \sin(2k\pi + \frac{5\pi}{3}) = \sin(\frac{5\pi}{3}) = -\frac{\sqrt{3}}{2}$

4.5

Question. *Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Prove that if* $\sup(S)$ *belongs to S, then* $\sup(S) = \max(S)$.

Proof.
Since $\sup(S)$ must be an upper bound of S, then by definition of an upper bound, $s \leq \sup(S)$ for all $s \in S$. Also since, by assumption, $\sup(S) \in S$, then $\sup(S)$ satisfies the definition of $\max(S)$, so $\sup(S) = \max(S)$. $\square$

4.7

Question. *Let S and T be nonempty bounded subsets of* $\mathbb{R}$.

(a) Prove that if $S \subseteq T$, then $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$.

(b) Prove that $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$.

Proof. **(a)**
By definition of $\inf(T)$, we know that $\inf(T) \leq t$ for all $t \in T$. In particular, since $S \subseteq T$, we know that $s \in T$ for all $s \in S$. This means $\inf(T) \leq s$ for all $s \in S$, telling us that $\inf(T)$ is a lower bound of S. Furthermore since $\inf(S)$ is the Greatest Lower Bound of S, we know that $\inf(T) \leq \inf(S)$, the first of these inequalities.

Next, we know that $\inf(S) \leq s$ for all $s \in S$ and that $s \leq \sup(S)$ for all $s \in S$. Thus, fix some $s_0 \in S$ to give us that $\inf(S) \leq s_0$ and $s_0 \leq \sup(S) \implies \inf(S) \leq \sup(S)$ by O3. Therefore, we have our second of the desired inequalities.

Lastly, we know that $t \leq \sup(T)$ for all $t \in T$. Therefore, since $S \subseteq T$ tells us that $s \in T$ for all $s \in S$, we can conclude that $s \leq \sup(T)$ for all $s \in S$. This tells us that $\sup(T)$ is an upper bound of S. However, $\sup(S)$ is the Least Upper Bound, so we must have that $\sup(S) \leq \sup(T)$, the last of the desired inequalities.

We now know that $\inf(T) \leq \inf(S)$, $\inf(S) \leq \sup(S)$, and $\sup(S) \leq \sup(T)$. More compactly, this is equivalent to $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$ exactly what we wished to prove. $\square$

Proof. **(b)**
It is clear that $\sup(S) \leq \max\{\sup(S), \sup(T)\}$ and $\sup(T) \leq \max\{\sup(S), \sup(T)\}$. By definition, we know that $s \leq \sup(S)$ for all $s \in S$ and $t \leq \sup(T)$ for all $t \in T$. We can use O3 to deduce that $s \leq \max\{\sup(S), \sup(T)\}$ and $t \leq \max\{\sup(S), \sup(T)\}$ for all $s \in S$ and $t \in T$. This means that for any $r \in S \cup T$, we have that $r \leq \max\{\sup(S), \sup(T)\}$, i.e. $\max\{\sup(S), \sup(T)\}$ is an upper bound for $S \cup T$. Furthermore, we know that $\sup(S \cup T) \leq \max\{\sup(S), \sup(T)\}$ since $\sup(S \cup T)$ is the Least Upper Bound of $S \cup T$.

On the other hand, since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, then by part (a) of this questions, we know that $\sup(S) \leq \sup(S \cup T)$ and $\sup(T) \leq \sup(S \cup T)$. Also, since $\max\{\sup(S), \sup(T)\}$ equals either $\sup(S)$ or $\sup(T)$, we know that $\max\{\sup(S), \sup(T)\} \leq \sup(S \cup T)$.

Since we have shown this previous inequality holds in both directions, then by O2, we know that $\sup(S \cup T) = \max\{\sup(S), \sup(T)\}$. $\square$

4.9

Question.
Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Define $-S$ to be the set $\{-s : s \in S\}$ and let $s_0 = \sup(-S)$. Prove the following:

(1) $-s_0 \leq s$ for all $s \in S$.

(2) If $t \leq s$ for all $s \in S$, then $t \leq -s_0$.

Proof. **(1)**
Since $s_0 = \sup(-S)$, then by the definition of Supremum, we know that $\tilde{s} \leq s_0$ for every $\tilde{s} \in -S$. By the definition of $-S$, we know this is equivalent to saying $-s \leq s_0$ for every $s \in S$. Lastly, by Theorem 3.2 (i), we can say that $-s_0 \leq -(-s)$ for every $s \in S$ which is equivalent to $-s_0 \leq s$ for every $s \in S$ since $-(-c) = c$ for all $c \in \mathbb{R}$. Thus, we have proven the desired inequality. □

Proof. **(2)**
Let t be such that $t \leq s$ for all $s \in S$. Then, by Theorem 3.2 (i), we can say that $-s \leq -t$ for all $s \in S$. However, by the definition above of $-S$, this is the same as saying $\tilde{s} \leq -t$ for all $\tilde{s} \in -S$. Therefore, this shows that $-t$ is an upper bound for $-S$. However, since s_0 is the Least Upper Bound of $-S$, we know that $s_0 \leq -t$. Then, again by Theorem 3.2 (i), we can say that $-(-t) \leq -s_0$ which is the same as saying $t \leq -s_0$, exactly what we wanted to show. □

4.14

Question.
Let A and B be nonempty bounded subsets of $\mathbb{R}$ and define $A + B = \{a + b : a \in A, b \in B\}$.

(a) Prove that $\sup(A + B) = \sup(A) + \sup(B)$.

(b) Prove that $\inf(A + B) = \inf(A) + \inf(B)$.

Proof. **(a)**
For any $x \in A + B$, x is of the form $x = a + b$ for $a \in A$ and $b \in B$. Clearly, $a + b \leq \sup(A) + \sup(B)$ since $a \leq \sup(A)$ and $b \leq \sup(B)$. Thus, for any $x \in A + B$, $x \leq \sup(A) + \sup(B)$. In other words, $\sup(A) + \sup(B)$ is an upper bound for $A + B$. However, since $\sup(A + B)$ is the Least Upper Bound of $A + B$, we have that $\sup(A + B) \leq \sup(A) + \sup(B)$.

To show the inequality in the other direction, fix b_0 as some arbitrary element of B. Next, note that $x \leq \sup(A + B)$ for all $x \in A + B$, i.e. $a + b \leq \sup(A + B)$ for all $a \in A$ and $b \in B$. In particular, $a + b_0 \leq \sup(A + B)$ for all $a \in A$. Thus, $a + b_0 - b_0 \leq \sup(A + B) - b_0$ for all $a \in A$ by O4. Using A4 and A3, we can finally conclude that $a \leq \sup(A + B) - b_0$ for all $a \in A$. In other words, $\sup(A + B) - b_0$ is an upper bound for A. However, since $\sup(A)$ is the Least Upper Bound for A, we have that $\sup(A) \leq \sup(A + B) - b_0$. From this we can get:

$$
\begin{aligned}
& \sup(A) \leq \sup(A + B) - b_0 \\
\implies & -(\sup(A + B) - b_0) \leq -\sup(A) && \text{by Theorem 3.2 (i)} \\
\implies & -\sup(A + B) + b_0 \leq -\sup(A) && \text{by DL} \\
\implies & -\sup(A + B) + b_0 + \sup(A + B) \leq -\sup(A) + \sup(A + B) && \text{by O4} \\
\implies & b_0 - \sup(A + B) + \sup(A + B) \leq \sup(A + B) - \sup(A) && \text{by A2} \\
\implies & b_0 \leq \sup(A + B) - \sup(A) && \text{by A2, A4, and A3 in that order}
\end{aligned}
$$

Therefore, since this b_0 can be any arbitrary element of B, we have shown that $\sup(A+B) - \sup(A)$ is an upper bound for B. However, since $\sup(B)$ is the Least Upper Bound for B, we have that $\sup(B) \leq \sup(A+B) - \sup(A)$. Thus, by applying O4 with $\sup(A)$, then applying A4 and A3 on the right and A2 on the left, we get precisely $\sup(A) + \sup(B) \leq \sup(A+B)$

Therefore, we have shown that $\sup(A + B) \leq \sup(A) + \sup(B)$ and $\sup(A) + \sup(B) \leq \sup(A + B)$. Therefore, by O2, we have that $\sup(A + B) = \sup(A) + \sup(B)$, exactly what we wanted to prove. □

Proof. **(b)**
Similarly to in (a), we know that for any $x = a + b \in A + B$, we have that $\inf(A) + \inf(B) \leq a + b$ since $\inf(A) \leq a$ and $\inf(B) \leq b$. Thus, $\inf(A) + \inf(B) \leq x$ for all $x \in A + B$ meaning that $\inf(A) + \inf(B)$ is a lower bound for $A + B$. However, since $\inf(A + B)$ is the Greatest Lower Bound, we know that $\inf(A) + \inf(B) \leq \inf(A + B)$.

Motivated from above, we will fix b_0 as some arbitrary element of B and notice that $\inf(A + B) \leq x$ for all $x \in A + B$. In particular, $\inf(A + B) \leq a + b_0$ for all $a \in A$. By adding $-b_0$ to both sides according to O4 and then using A4 and

A3, we can see that $\inf(A+B) - b_0 \leq a$ for all $a \in A$. In other words, $\inf(A+B) - b_0$ is a lower bound for A. However, since $\inf(A)$ is the Greatest Lower Bound, we see that $\inf(A+B) - b_0 \leq \inf(A)$. Following this we can see that:

$$\begin{aligned} & \inf(A+B) - b_0 \leq \inf(A) \\ \Longrightarrow\ & \inf(A+B) - \inf(A) \leq b_0 && \text{by using a nearly identical set of steps used in part (a)} \end{aligned}$$

Therefore, since b_0 can be any arbitrary element of B, this shows us that $\inf(A+B) - \inf(A)$ is a lower bound for B. However, since $\inf(B)$ is the Greatest lower bound for B, we get that $\inf(A+B) - \inf(A) \leq \inf(B)$. By applying O4, A4, A3, and A2, we immediately get that $\inf(A+B) \leq \inf(A) + \inf(B)$.

Therefore, we have shown that $\inf(A) + \inf(B) \leq \inf(A+B)$ and $\inf(A+B) \leq \inf(A) + \inf(B)$. Thus, by O2, we get $\inf(A+B) = \inf(A) = \inf(B)$, exactly what we wanted to prove. $\square$