Complex Analysis Homework 2

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Question 5

Question.
For each of the following subsets of $\mathbb{C}$, determine if they are closed, open, or neither:

a.) $|z - i| < |z - 1|$

b.) $|z + 2i| \geq 2$

c.) $1 < Re(z) \leq 2$

d.) $Re(z) \neq 0$.

For the non-closed sets, find the closure.

Answer. a.)
I claim that this set is open.

Proof.
Let's call the above set of points $S = \{z \in \mathbb{C} : |z - i| < |z - 1|\}$. As shown in Question 4*a*.), if we let $z = x + iy$, this is the set of all points above the line $y = x$.

If we choose an arbitrary point $z_0 \in S$, then the distance from z_0 to the line $y = x$ is given by the length of the perpendicular line segment from the line $y = x$ to the point z_0. This line would have slope -1; thus, is of the form $y = -x + c$ for some constant c. If we plug in the (x, y) coordinates of $z_0 = x_0 + iy_0$, then we get $y_0 = -x_0 + c \implies c = y_0 + x_0$.

Therefore, we can find the intersection of the line $y = x$ and $y = -x + (y_0 + x_0)$. The x-coordinate can be found by equating the two lines:

$$\begin{aligned} x &= -x + (y_0 + x_0) \\ 2x &= y_0 + x_0 \\ x &= \frac{y_0 + x_0}{2} \end{aligned}$$

It immediately follows that the y-coordinate is given by $y = \frac{y_0 + x_0}{2}$. Therefore, the distance between z_0 and the line $y = x$ is given by:

$$\begin{aligned} d(z_0, y = x) &= \sqrt{\left(x_0 - \frac{y_0 + x_0}{2}\right)^2 + \left(y_0 - \frac{y_0 + x_0}{2}\right)^2} \\ &= \sqrt{\left(\frac{x_0 - y_0}{2}\right)^2 + \left(\frac{y_0 - x_0}{2}\right)^2} \\ &= \sqrt{\left(\frac{x_0 - y_0}{2}\right)^2 + \left(\frac{x_0 - y_0}{2}\right)^2} \\ &= \sqrt{2\left(\frac{x_0 - y_0}{2}\right)^2} \\ &= \frac{|x_0 - y_0|}{\sqrt{2}} \\ &= \frac{y_0 - x_0}{\sqrt{2}} \qquad \text{since} \quad y_0 > x_0 \end{aligned}$$

Therefore, we can find an open disk $D(z_0, r) \subset S$ with $r > 0$ given by

$$r < \frac{y_0 - x_0}{\sqrt{2}}$$

Thus, S is, by definition, an open set. $\square$

Since S is non-closed, we want to find its closure as well. First, we need to find all of the limit points of S. I claim that all of the limit points of S are given by the set $T = \{z = x + iy \in \mathbb{C} : y \geq x\}$.

Proof.
First, note that any point $w = u + iv$ with $v < u$ (i.e. $w \notin T$) is not a limit point since $\exists\, r > 0$ such that $D'(w, r) \cap S = \emptyset$. Namely, any $r < \frac{u-v}{\sqrt{2}}$ by the same procedure used before to show S is open.

Next, note that for any point $z_0 = x_0 + iy_0 \in T$, we have that $D'(z_0, r) \cap S \neq \emptyset$ since, for example, $z_1 = x_0 + i(y_0 + \frac{r}{2}) \in D'(z_0, r) \cap S$. Thus, T is, indeed, the set of all limit points. $\square$

Thus, the closure of S, $\overline{S} = S \cup T = \{z = x + iy \in \mathbb{C} : y \geq x\}$.

Answer. b.)
I claim that this set is closed.

Proof.
Let's call the above set of points $S = \{z \in \mathbb{C} : |z + 2i| \geq 2\}$. I will show that S is closed by showing that S contains all of its limit points. Note that for any point $w \notin S$, we can find an $r > 0$ such that $D'(w, r) \cap S = \emptyset$. For convenience, let's express w as a polar point centered at $0 - 2i$, say $w = \rho e^{i\phi} - 2i$ for $0 \leq \rho < 2$ the distance from $0 - 2i$ and $-\pi < \phi \leq \pi$ the argument of the point when measured from the right side of the line $Im(z) = -2$. Therefore, if we choose $r < 2 - \rho$, then $D'(w, r) \cap S = \emptyset$. This means all points not in S are not limit points.

To show that all points in S are limit points, we take some arbitrary point $z_0 \in S$, and we show that it must be a limit point. Just as before, we can express it as the distance from $0 - 2i$, say $z_0 = r_0 e^{i\theta_0} - 2i$, but this time $r_0 \geq 2$ and $-\pi < \theta_0 \leq \pi$. For any $r > 0$, we know that $D'(z_0, r) \cap S \neq \emptyset$ since, for example, $z_1 = (r_0 + \frac{r}{2})e^{i\theta_0} \in D'(z_0, r) \cap S$. Thus, S contains all of its limit points, so S must be closed. $\square$

Answer. c.)
I claim that this set is neither closed nor open.

Proof.
Let's call the above set of points $S = \{z \in \mathbb{C} : 1 < Re(z) \leq 2\}$.

First, I will show that S is not open. Let z_0 be a point along the line $Re(z) = 2$, i.e. $z_0 = 2 + iy_0$. However, if we look at $D(z_0, r)$, we cannot find an $r > 0$ such that $D(z_0, r) \subset S$ because for any r, we have $z_1 = (2 + \frac{r}{2}) + iy_0 \in D(z_0, r)$ but $z_1 \notin S$, so $D(z_0, r) \not\subset S$. Therefore, S is not open.

Next, I will show that S is not closed. To do this I will find a point that is not in S, but is a limit point of S. Take w_0 to be on the line $Re(z) = 1$, i.e. $w_0 = 1 + iv_0$. For any $r > 0$, we know that $D'(w_0, r) \cap S \neq \emptyset$ since, for example if we let $r_1 = \min\{r/2, 1\}$, then $w_1 = (1 + r_1) + iv_0 \in D'(w_0, r) \cap S$. Thus, $D'(w_0, r) \cap S \neq \emptyset$ for all choices of r, so w_0 is a limit point of S. However, $w_0 \notin S$, so S does not contain all of its limit points, so S is not closed.

Thus, S is not closed nor open, proving my claim. $\square$

Since S is non-closed, we want to find its closure as well. We have already shown that points along the line $Re(z) = 1$ are limit points. I claim that these, along with S itself are all of the limit points of S.

Proof.
To prove my claim above, I need to show that all points in S are actually limit points, and that all points not in S or along the line $Re(z) = 1$ are not limit points.

First, let $z_0 = x_0 + iy_0 \in S$, i.e. $1 < x_0 \leq 2$. For any $r > 0$, we have that $D'(z_0, r) \cap S \neq \emptyset$ since, for example, $z_1 = x_0 + i(y_0 + \frac{r}{2}) \in D'(z_0, r) \cap S$. Thus, $D'(z_0, r) \cap S \neq \emptyset$ for any choice of r, so all points in S are limit points.

Next, take $w_1, w_2 \notin S \cup \{z \in \mathbb{C} : Re(z) = 1\}$. Let $w_1 = u_1 + iv_1$ and $w_2 = u_2 + iv_2$ for $u_1 < 1$, $u_2 > 2$, and $v_1, v_2 \in \mathbb{R}$. Now, note that $D'(w_1, r_1) \cap S = \emptyset$ for $r_1 < 1 - u_1$ and that $D'(w_2, r_2) \cap S = \emptyset$ for $r_2 < u_2 - 2$. Since w_1 and w_2 describe all possibles numbers not in S or on the line $Re(z) = 1$ and we have shown that neither w_1 nor w_2 could be limit points, we have shown that all points not in S or along the line $Re(z) = 1$ are not limit points; thus, proving the claim. $\square$

Thus, the closure of S, $\overline{S} = S \cup \{z \in \mathbb{C} : Re(z) = 1\} = \{z \in \mathbb{C} : 1 \leq Re(z) \leq 2\}$

Answer. d.)
I claim that this set is open

Proof.
Let's call the above set of points $S = \{z \in \mathbb{C} : Re(z) \neq 0\}$.

Let z_0 be a point in S, i.e. $z_0 = x_0 + iy_0$ for $x_0 \neq 0$. No matter what z_0 is, we can find the distance from z_0 to the line $Re(z) = 0$ easily as $d(z_0, Re(z) = 0) = |x_0|$. In fact, we can find an open disk $D(z_0, r) \subset S$ if we let $0 < r < |x_0|$. This means, by definition, S is open. $\square$

Since S is non-closed, we want to find its closure as well. I claim that S, along with the line $Re(z) = 0$ represent all of S's limit points.

Proof. First, for any point $z_0 = x_0 + iy_0 \in S$, notice that $D'(z_0, r) \cap S \neq \emptyset$ for any value of $r > 0$. This is because, for example, $z_1 = x_0 + i(y_0 + \frac{r}{2}) \in D'(z_0, r) \cap S$. Thus, all points in S are limits points of S.

Next, take any point $w_0 = 0 + iy_0 \notin S$. Again, we can see that $D'(w_0, r) \cap S \neq \emptyset$ for any $r > 0$ since we can find $w_1 = \frac{r}{2} + iy_0 \in D'(w_0, r) \cap S$. Therefore, all points on the line $Re(z) = 0$ are also limit points. $\square$

Thus, the closure of S, $\overline{S} = S \cup \{z \in \mathbb{C} : Re(z) = 0\} = \mathbb{C}$