Complex Analysis Homework 4

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Question 3

Evaluate $\int_\gamma f(z)dz$ when:

(a) $f(z) = \frac{1}{z}, \gamma(t) = e^{-it}, t \in [0, 8\pi]$

(b) $f(z) = \text{Re}(z), \gamma(t) = t + it^2, t \in [0,1]$

(c) $f(z) = |z|^4, \gamma$ is the line segment $[-1+i, 1+i]$

First, note the definition of the path integral for a path $\gamma : [a,b] \to \mathbb{C}$ of a function $f : \gamma^* \to \mathbb{C}$:

$$\int_\gamma f(z)dz := \int_a^b f(\gamma(t))\gamma'(t)dt \tag{1}$$

This makes the calculations pretty straightforward:

Answer. (a)
Before we begin, note that $f(\gamma(t)) = e^{it}$ and $\gamma'(t) = -ie^{-it}$; thus, we have

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt && \text{by (1)} \\
&= \int_0^{8\pi} [e^{it}] \cdot [-ie^{-it}]dt && \text{by above comments} \\
&= \int_0^{8\pi} (-i)dt \\
&= -it\Big|_{t=0}^{t=8\pi} \\
&= -i(8\pi) - (-i(0)) \\
&= \boxed{-8\pi i}
\end{aligned}$$

Answer. (b)
Before we begin, note that $f(\gamma(t)) = t$ and $\gamma'(t) = 1 + 2it$. Therefore,

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt && \text{by (1)} \\
&= \int_0^1 [t] \cdot [1 + 2it]dt && \text{by the above comments} \\
&= \int_0^1 (t + 2it^2)dt \\
&= \left(\frac{t^2}{2} + \frac{2it^3}{3}\right)\Bigg|_{t=0}^{t=1} \\
&= \left(\frac{1}{2} + \frac{2i}{3}\right) - (0+0) \\
&= \boxed{\frac{1}{2} + \frac{2i}{3}}
\end{aligned}$$

Answer. (c)
First, let's parameterize γ in the following manner: $\gamma(t) = t + i, t \in [-1, 1]$. Now, we can see that $f(\gamma(t)) = |\gamma(t)|^4 = \left(\sqrt{t^2+1}\right)^4 = (t^2+1)^2 = t^4 + 2t^2 + 1$. Furthermore, $\gamma'(t) = 1$, so we can compute the integral as follows:

$$
\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt && \text{by (1)} \\
&= \int_{-1}^1 [t^4 + 2t^2 + 1] \cdot [1]dt && \text{by the above comments} \\
&= \int_{-1}^1 (t^4 + 2t^2 + 1)dt \\
&= \left(\frac{t^5}{5} + \frac{2t^3}{3} + t\right)\Bigg|_{t=-1}^{t=1} \\
&= \left(\frac{1}{5} + \frac{2}{3} + 1\right) - \left(\frac{-1}{5} + \frac{-2}{3} - 1\right) \\
&= \frac{2}{5} + \frac{4}{3} + 2 \\
&= \boxed{\frac{56}{15}}
\end{aligned}
$$