Advanced Calculus Homework 1

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1.1

Question.
Prove that

$$P_n : 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

is true for all positive integers n.

Proof. (Using induction)
Base Case: $n = 1$,

$$1^2 = 1 \qquad \text{also,} \quad \frac{1(1+1)(2(1)+1)}{6} = \frac{(2)(3)}{6} = 1.$$

Inductive Hypothesis:
Assume that P_k is true for some integer $k \geq 1$. We want to show that P_{k+1} must also be true. i.e., we want to show that:

$$1^2 + 2^2 + \cdots + k^2 + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}.$$

Starting with the left hand side, we have:

$$\begin{aligned} 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by Inductive Hypothesis} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)(k(2k+1) + 6(k+1))}{6} \\ &= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} \\ &= \frac{(k+1)(2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

This is our desired equality. Therefore, by the principle of Mathematical Induction, P_n must be true for all $n \in \mathbb{N}$. $\square$

1.3

Question.
Prove that

$$P_n : 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

is true for all positive integers n.

Proof. (Using induction)
Base Case: $n = 1$,

$$1^3 = 1 \qquad \text{also,} \quad 1^2 = 1.$$

Inductive Hypothesis:
Assume that P_k is true for some integer $k \geq 1$. We want to show that P_{k+1} must also be true. i.e., we want to show that:

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = (1 + 2 + \cdots + k + (k+1))^2$$

Starting with the left hand side, we have:

$$\begin{aligned}
1^3 + 2^3 + \cdots + k^3 + (k+1)^3 &= (1 + 2 + \cdots + k)^2 + (k+1)^3 && \text{by Inductive Hypothesis} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{by Lemma 1 below} \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2(k^2 + 4(k+1))}{4} \\
&= \frac{(k+1)^2(k^2 + 4k + 4)}{4} \\
&= \frac{(k+1)^2(k+2)^2}{4} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\
&= (1 + 2 + \cdots + k + (k+1))^2 && \text{again by Lemma 1 below} \qquad (1)
\end{aligned}$$

Lemma 1.

$$P_n : 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad \textit{is true for all } n \in \mathbb{N}$$

Proof. Proof of Lemma 1, using induction
Base Case: $n = 1$,

$$1 = 1 \qquad \text{also,} \qquad \frac{1(1+1)}{2} = \frac{2}{2} = 1.$$

Inductive Hypothesis
Assume that P_k is true for some integer $k \geq 1$. We want to show that P_{k+1} must also be true. i.e., we want to show that:

$$1 + 2 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Starting with the left hand side, we have:

$$\begin{aligned}
1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + k + 1 \quad \text{by Inductive Hypothesis} \\
&= \frac{k(k+1) + 2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

This is the desired equality; thus, proving the Lemma by the Principle of Mathematical Induction. □

With the above Lemma proven, and the desired equality in line (1), we have proven the given statement due to the Principle of Mathematical Induction. □

1.5

Question.
Prove that

$$P_n : 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$$

is true for all positive integers n.

Proof. (Using indution)
Base Case: $n = 1$,

$$1 + \frac{1}{2^1} = \frac{3}{2} \qquad \text{also,} \qquad 2 - \frac{1}{2^1} = \frac{3}{2}$$

Inductive Hypothesis
Assume that P_k is true for some integer $k \geq 1$. We want to show that P_{k+1} must also be true. i.e., we want to show that:

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 2 - \frac{1}{2^{k+1}}$$

Starting with the left hand side, we have:

$$\begin{aligned}
1 + \frac{1}{2} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} &= 2 - \frac{1}{2^k} + \frac{1}{2^{k+1}} \quad \text{by the Inductive Hypothesis} \\
&= 2 - \frac{1}{2^k}\left(1 - \frac{1}{2}\right) \\
&= 2 - \frac{1}{2^k}\left(\frac{1}{2}\right) \\
&= 2 - \frac{1}{2^{k+1}}
\end{aligned}$$

This is our desired equality. Thus, by the Principle of Mathematical induction, P_n must be true for all positive integers n. □

2.1

Question.
Show that:

a) $\sqrt{3}$

b) $\sqrt{5}$

c) $\sqrt{7}$

d) $\sqrt{24}$

e) $\sqrt{31}$

are all not rational numbers.

For the following proofs, I will use the "Rational Zeros Theorem" as stated below:

Theorem 1. *Suppose $c_0, c_1, \ldots, c_n$ are integers and $r = \frac{c}{d}$ is a rational number (with $d \neq 0$ and c and d sharing no common factors) satisfying the following polynomial equation :*

$$c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

where $n \geq 1, c_n \neq 0$. Then, c divides c_0 and d divides c_n

Proof. a.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 2, c_2 = 1, c_1 = 0$, and $c_0 = -3$ because $\sqrt{3}$ is clearly a zero of $f(x) = x^2 - 3$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-3)$ and $d|1$. Since 3 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 3\}$. However, $f(\pm 1) = -2$ and $f(\pm 3) = 6$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt{3}$ is a root of f, it cannot be rational. □

Proof. b.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 2, c_2 = 1, c_1 = 0$, and $c_0 = -5$ because $\sqrt{5}$ is clearly a zero of $f(x) = x^2 - 5$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-5)$ and $d|1$. Since 5 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 5\}$. However, $f(\pm 1) = -4$ and $f(\pm 5) = 20$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt{5}$ is a root of f, it cannot be rational. □

Proof. c.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 2, c_2 = 1, c_1 = 0$, and $c_0 = -7$ because $\sqrt{7}$ is clearly a zero of $f(x) = x^2 - 7$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-7)$ and $d|1$. Since 7 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 7\}$. However, $f(\pm 1) = -6$ and $f(\pm 7) = 42$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt{7}$ is a root of f, it cannot be rational. □

Proof. d.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 2, c_2 = 1, c_1 = 0$, and $c_0 = -24$ because $\sqrt{24}$ is clearly a zero of $f(x) = x^2 - 24$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-24)$ and $d|1$. Since 24 has 1, 2, 3, 4, 6, 8, 12, and 24 as its positive factors, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 8, \pm 12, \pm 24\}$. However, $f(\pm 1) = -23$, $f(\pm 2) = -20$, $f(\pm 3) = -15$, $f(\pm 4) = -8$, $f(\pm 6) = 12$, $f(\pm 12) = 120$, and $f(\pm 24) = 552$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt{24}$ is a root of f, it cannot be rational. □

Proof. e.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 2, c_2 = 1, c_1 = 0$, and $c_0 = -31$ because $\sqrt{31}$ is clearly a zero of $f(x) = x^2 - 31$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-31)$ and $d|1$. Since 31 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 31\}$. However, $f(\pm 1) = -30$ and $f(\pm 31) = 930$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt{31}$ is a root of f, it cannot be rational. □

2.2

Question.
Show that:

a) $\sqrt[3]{2}$

b) $\sqrt[7]{5}$

c) $\sqrt[4]{13}$

are all not rational numbers.

For the following proofs, I will once again use the "Rational Zeros Theorem" from above.

Proof. a.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 3, c_3 = 1, c_2 = c_1 = 0$, and $c_0 = -2$ because $\sqrt[3]{2}$ is clearly a zero of $f(x) = x^3 - 2$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-2)$ and $d|1$. Since 2 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 2\}$. However, $f(\pm 1) = -1$ and $f(\pm 2) = 6$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt[3]{2}$ is a root of f, it cannot be rational. □

Proof. b.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 7, c_7 = 1, c_6 = c_5 = c_4 = c_3 = c_2 = c_1 = 0$, and $c_0 = -5$ because $\sqrt[7]{5}$ is clearly a zero of $f(x) = x^7 - 5$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-5)$ and $d|1$. Since 5 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 5\}$. However, $f(\pm 1) = -4$ and $f(\pm 5) = 78120$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt[7]{5}$ is a root of f, it cannot be rational. □

Proof. c.)
I will apply the Rational Zeros Theorem to the polynomial f with $n = 4, c_4 = 1, c_3 = c_2 = c_1 = 0$, and $c_0 = -13$ because $\sqrt[4]{13}$ is clearly a zero of $f(x) = x^4 - 13$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|(-13)$ and $d|1$. Since 13 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 13\}$. However, $f(\pm 1) = -12$ and $f(\pm 13) = 28548$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt[4]{13}$ is a root of f, it cannot be rational. □

2.3

Question.
Show that $\sqrt{2 + \sqrt{2}}$ *is not a rational number.*

Proof.
Here, I would again like to use the "Rational Zeros Theorem," but it is not quite as obvious as to what my polynomial should be. I will begin by setting $a = \sqrt{2 + \sqrt{2}}$ and manipulating the equation.

$$\begin{aligned}
a &= \sqrt{2 + \sqrt{2}} \\
a^2 &= 2 + \sqrt{2} \\
a^2 - 2 &= \sqrt{2} \\
(a^2 - 2)^2 &= 2 \\
a^4 - 4a^2 + 4 &= 2 \\
a^4 - 4a^2 + 2 &= 0
\end{aligned}$$

From this, it is clear that I can apply the Rational Zeros Theorem to the polynomial f with $n = 4, c_4 = 1, c_3 = 0, c_2 = -4, c_1 = 0$, and $c_0 = 2$ because we've just shown that $\sqrt{2 + \sqrt{2}}$ is a zero of $f(x) = x^4 - 4x^2 + 2$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|2$ and $d|1$. Since 2 is prime, and 1 only has itself and -1 as factors, we know that $r \in \{\pm 1, \pm 2\}$. However, $f(\pm 1) = -1$ and $f(\pm 2) = 2$, so none of those are roots of f. Therefore, f has no rational roots and since $\sqrt{2 + \sqrt{2}}$ is a root of f, it cannot be rational. □

2.8

Question.
Find all rational solutions of the equation $x^8 - 4x^5 + 13x^3 - 7x + 1 = 0$.

Proof. I will once again apply the Rational Zeros Theorem on this polynomial. In this case, $n = 8, c_8 = 1, c_7 = c_6 = 0, c_5 = -4, c_4 = 0, c_3 = 13, c_2 = 0, c_1 = -7$, and $c_0 = 1$. By the Rational Zeros Theorem, if this polynomial has any rational roots, they must be of the form $r = \frac{c}{d}$ where $c|1$ and $d|1$. Since 1 only has itself and -1 as factors, we know that $r \in \{+1, -1\}$. However, $(1)^8 - 4(1)^5 + 13(1)^3 - 7(1) + 1 = 1 - 4 + 13 - 7 + 1 = 4$, so $+1$ is not a solution of the equation. Thus, -1 is the only option for a rational root for the polynomial, and, indeed, $(-1)^8 - 4(-1)^5 + 13(-1)^3 - 7(-1) + 1 = 1 + 4 - 13 + 7 + 1 = 0$. Therefore, -1 is the only rational solution to the equation. $\square$