Complex Analysis Homework 5

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Question 1

Let Log(z) be a principal value of the logarithm, i.e.,

$$\text{Log}(z) = \ln(r) + i\theta \qquad\qquad \text{for } z = re^{i\theta}, \quad r > 0, \ -\pi < \theta \leq \pi.$$

Show that for any two nonzero complex numbers z_1 and z_2, that

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2) + 2N\pi i \tag{1}$$

where N has the value 0, 1, or -1.

Answer.
First, let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ for $r_1, r_2 > 0$ and $-\pi < \theta_1, \theta_2 \leq \pi$. Then we have that

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

Clearly $r_1 r_2 > 0$ since $r_1, r_2 > 0$. However, with $\theta_1 + \theta_2$ we have three different cases:

<u>Case 1: $-2\pi < \theta_1 + \theta_2 \leq -\pi$</u>
Since the argument of $z_1 z_2$ is not in the desired interval for Log($\cdot$), we have to adjust $z_1 z_2$ to be $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2+2\pi)}$ which now puts the argument of $z_1 z_2$ in the interval $(0, \pi]$ which is acceptable. Thus, $\text{Log}(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2 + 2\pi) = (\ln(r_1) + i\theta_1) + (\ln(r_2) + i\theta_2) + 2\pi i = \text{Log}(z_1) + \text{Log}(z_2) + 2\pi i$ which satisfies (1) with $N = 1$.

<u>Case 2: $-\pi < \theta_1 + \theta_2 \leq \pi$</u>
Now the argument of $z_1 z_2$ is clearly in the desired interval so we need not make any changes and can simply note that $\text{Log}(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2) = (\ln(r_1) + i\theta_1) + (\ln(r_2) + i\theta_2) = \text{Log}(z_1) + \text{Log}(z_2)$ which satisfies (1) with $N = 0$.

<u>Case 3: $\pi < \theta_1 + \theta_2 \leq 2\pi$</u>
Once again, the argument of $z_1 z_2$ is not in the desired interval, so we must make the adjustment $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2-2\pi)}$ which now puts the argument of $z_1 z_2$ in the interval $(-\pi, 0]$ which is acceptable. Thus, $\text{Log}(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2 - 2\pi) = (\ln(r_1) + i\theta_1) + (\ln(r_2) + i\theta_2) - 2\pi i = \text{Log}(z_1) + \text{Log}(z_2) - 2\pi i$ which satisfies (1) with $N = -1$.

Thus, since these 3 cases determine all possibilities, and we have shown that all three cases relate to (1) with only $N = 1, 0, 1$, we have satisfied the question.