Complex Analysis Homework 3

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September 17, 2020

Question 3

Question. *Which of the following functions are differentiable at z = 0?*

(a) $f(z) = (Re(z))(Im(z))$

(b) $f(z) = Re(z) + Im(z)$

First, I will note that we can express any $z \in \mathbb{C}$ as $z = x + iy$ implying that $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Next, note that if we let $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy$ in some open set G, then if $u(x, y)$ and $v(x, y)$ have continuous first order partial derivatives in G and satisfy both $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $v_x(x_0, y_0) = -u_y(x_0, y_0)$ at the point $z_0 = x_0 + iy_0$, then f is differentiable at z_0.

Answer. (a)
If we use $z = x + iy$, then $f(z) =$(Re(z))(Im(z)) can be reduced to $f(z) = xy$. This means the functions u and v that we are interested in are $u(x, y) = xy$ and $v(x, y) = 0$. Thus, the partial derivatives that we are interested in are:

$$u_x(x, y) = y \qquad v_y(x, y) = 0 \qquad v_x(x, y) = 0 \qquad u_y(x, y) = x$$

Clearly all of these functions are continuous. Additionally, $u_x(0, 0) = 0 = v_y(0, 0)$ and $v_x(0, 0) = 0 = -u_y(0, 0)$; thus, $f(z)$ at $z = 0$ satisfies all of the above conditions for differentiability, so $\boxed{f(z) \text{ must be differentiable at } z = 0.}$

Answer. (b)
If we use $z = x + iy$, then $f(z) =$Re(z) + Im(z) can be reduced to $f(z) = x + y$. This means the functions u and v that we are interested in are $u(x, y) = x + y$ and $v(x, y) = 0$. Thus, the partial derivatives that we are interested in are:

$$u_x(x, y) = 1 \qquad v_y(x, y) = 0 \qquad v_x(x, y) = 0 \qquad u_y(x, y) = 1$$

Clearly all of these functions are continuous. However, $u_x(0, 0) = 1 \neq 0 = v_y(0, 0)$ and $v_x(0, 0) = 0 \neq -1 = -u_y(0, 0)$; thus, $f(z)$ at $z = 0$ does not satisfy the above conditions for differentiability, so $\boxed{f(z) \text{ must not be differentiable at } z = 0.}$