Analysis HW 7

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Question 1

Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by

$$f_n(x) = \begin{cases} \sin^2\left(\frac{\pi}{x}\right), & \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$$

Prove that the sequence $f_n(x)$ converges pointwise, but not uniformly, to a continuous function. Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely, but not uniformly.

Proof.
I claim that $f_n(x) \to 0$ as $n \to \infty$ for all $x \in \mathbb{R}$. If $x \le 0$, then $f_n(x)$ is the constant zero sequence. If $x > 1$, then $f_n(x)$ is also the constant zero sequence. If $x \in (0, 1]$, then by taking $N = \lceil 1/x \rceil + 1$, we have

$$f_N(x) = \begin{cases} \sin^2\left(\frac{\pi}{x}\right), & \frac{1}{\lceil 1/x \rceil + 2} \le x \le \frac{1}{\lceil 1/x \rceil + 1} \\ 0, & \text{otherwise} \end{cases}$$

Notice the first case never occurs so we have that $f_N(x) = 0$. This also holds true for all $n \ge N$. Therefore, if we fix some $r > 0$, we can see that $|f_n(x) - 0| = 0 < r$ for all $n \ge N$. Since this can be done for all $x \in (0, 1]$ (and trivially for all $x \le 0$ and all $x > 1$), then we have that f_n converges pointwise to the zero function which is trivially continuous. However, for uniform convergence, we need to examine the difference

$$||f_n - 0|| = ||f_n|| = \sup_{x \in \mathbb{R}}\{|f_n(x)|\}$$

Notice that this supremum is equal to 1, since $\sin^2\left(\frac{\pi}{x}\right)$ is equal to 1 for at least one $x \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, namely choosing x of the form $\frac{2}{2n+1}$ which is easily seen to be inside the desired interval and makes f_n evaluate to one. Therefore,

$$\lim_{n \to \infty} ||f_n - 0|| = \lim_{n \to \infty} 1 = 1$$

Since this limit does not converge to zero, we can say that f_n does NOT converge to zero uniformly.

Next, notice that f_n is a sequence of non-negative functions. Therefore, if we can simply show that the series of f_n's converges, then it automatically converges absolutely. Consider the partial sum

$$S_N(x) = \sum_{n=1}^{N} f_n(x)$$

Notice that each f_n is not identically zero in distinct open intervals since $\left(\frac{1}{n+1}, \frac{1}{n}\right) \cap \left(\frac{1}{m+1}, \frac{1}{m}\right) \ne \emptyset$ if and only if $n = m$. Furthermore, the endpoints of these intervals agree only if n and m are adjacent to one another. However, notice that at any point x which could be the endpoint of an interval, we have

$$f_n(x) = f_n\left(\frac{1}{n}\right) = \sin^2\left(\frac{\pi}{1/n}\right) = \sin^2(n\pi) = 0$$

and similarly for $x = 1/(n+1)$. Therefore, when taking the summation of the f_n's, there is at most one function who is non-zero for each distinct $x \in \mathbb{R}$. Therefore, we can explicitly write the partial sum as

$$S_N(x) = \begin{cases} \sin^2\left(\frac{\pi}{x}\right), & \frac{1}{N+1} \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that as $N \to \infty$ we have

$$\lim_{N \to \infty} S_N(x) = S(x) := \begin{cases} \sin^2\left(\frac{\pi}{x}\right), & 0 < x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

This is clear for any $x \notin (0,1]$. For $x \in (0,1]$, let us fix some $r > 0$, then for $M = \lfloor 1/x \rfloor$, we have for any $N \geq M$:

$$|S_N(x) - S(x)| = \left|\sin^2\left(\frac{\pi}{x}\right) - \sin^2\left(\frac{\pi}{x}\right)\right| = 0 < r$$

This shows that S_N converges to S pointwise which in turns means that $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely when considering the earlier comments. However, for uniform convergence, consider the difference

$$||S_N - S|| = \sup_{x \in \mathbb{R}} |S_N(x) - S(x)|$$

Notice this supremum is attained when $S_N(x) = 0$ and when $S(x) = 1$. A point where this occurs is at $x = \frac{2}{2N+3}$ since this x is less than $\frac{1}{N+1}$ and $\sin^2\left(\frac{\pi}{x}\right)$ is equal to one at this x. Therefore,

$$\lim_{N\to\infty} ||S_N - S|| = \lim_{N\to\infty} 1 = 1$$

Since this limit does not converge to zero, we can say that S_N does NOT converge to S uniformly. $\square$

Question 2

Let (X, d) be a compact metric space, and let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions such that the series $\sum_{n=1}^{\infty} f_n(x)$ is absolutely convergent. Prove that if $\sum_{n=1}^{\infty} |f_n(x)|$ is continuous, then so is $\sum_{n=1}^{\infty} f_n(x)$.

Proof.
Define $S_N : X \to \mathbb{R}$ as

$$S_N(x) = \sum_{n=1}^{N} |f_n(x)|.$$

Note that each f_n is continuous, so each $|f_n|$ is also continuous. Thus, since S_N is the finite sum of continuous functions, each S_N is continuous on X. Furthermore, each $|f_n|$ is a non-negative function, so $S_N(x)$ is clearly a monotonic sequence. By assumption $S := \lim_{N\to\infty} S_N$, is a continuous function on X. Therefore, by using that X is compact, then by a Theorem proved in class, this means that S_N converges to S uniformly. We have also shown that this is equivalent to the sequence S_N being a Cauchy Sequence, i.e. given some $r > 0$, we have some K such that

$$|S_M - S_N| < r$$

for all $M \geq N \geq K$. Using this, we have

$$\begin{aligned}
\left|\sum_{n=1}^{M} f_n(x) - \sum_{n=1}^{N} f_n(x)\right| &= \left|\sum_{n=N+1}^{M} f_n(x)\right| \\
&\leq \sum_{n=N+1}^{M} |f_n(x)| \\
&= \sum_{n=1}^{M} |f_n(x)| - \sum_{n=1}^{N} |f_n(x)| \\
&= \left|\sum_{n=1}^{M} |f_n(x)| - \sum_{n=1}^{N} |f_n(x)|\right| \\
&= |S_M - S_N| \\
&< r
\end{aligned}$$

Therefore, we have that $\sum_{n=1}^{\infty} f_n(x)$ is a Cauchy sequence; thus, uniformly convergent. Thus, by defining

$$T_N = \sum_{n=1}^{N} f_n(x)$$

we have that each T_N is a continuous function as it is the finite sum of continuous functions. Furthermore, we have shown that T_N is a Cauchy sequence. Therefore, by a Theorem proven in class, we know that

$$\lim_{N\to\infty} T_N = \sum_{n=1}^{\infty} f_n(x)$$

is a continuous function, finishing the proof. $\square$

Question 3

Let $f : \mathbb{R} \to \mathbb{R}$ be given. Assume that the sequence $f_n(x) = f(nx)$ is equicontinuous. What can you say about $f(x)$?

Answer.
By definition of equicontinuity, we have that for $x \in \mathbb{R}$ and every $r > 0$, there exists some $s > 0$ such that $|x - y| < s$ implies that $|f_n(x) - f_n(y)| < r$ for all $n \in \mathbb{N}$. I claim that this means f is a constant function.

Proof.
Assume that f is not a constant function. This means that there exist some $x \neq y \in \mathbb{R}$ such that $f(x) \neq f(y)$, i.e. that $|f(x) - f(y)| = r > 0$. However, by equicontinuity, we can say that there exists some $s > 0$ such that $|x_0 - y_0| < s$ implies $|f_n(x_0) - f_n(y_0)| < r$ for all n. In particular, choose

$$\begin{aligned} x_0 &= \frac{x}{n} \qquad\qquad y_0 = \frac{y}{n} \\ \implies |x_0 - y_0| &= \frac{1}{n}|x - y| \end{aligned}$$

Thus, choosing $n = \lceil |x - y|/s \rceil$, we have that this particular choice of x_0 and y_0 satisfies $|x_0 - y_0| < s$ which then gives

$$\begin{aligned} r > |f_n(x_0) - f_n(y_0)| &= \left| f_n\left(\frac{x}{n}\right) - f_n\left(\frac{y}{n}\right) \right| \\ &= |f(x) - f(y)|. \end{aligned}$$

We just obtained that $|f(x) - f(y)| < r$, but r was defined to be equal to $|f(x) - f(y)|$ and a quantity can't be strictly less than itself, so we have a contradiction. Therefore, our assumption that f was non-constant was incorrect. Therefore, f is a constant function. $\square$

Question 4

Let $P_n(x)$ be a sequence of polynomials: $P_n(x) = a_n + b_n x + c_n x^2 + d_n x^3$, where $\sup\{|a_n|, |b_n|, |c_n|, |d_n| : n \in \mathbb{N}\} \leq 1$. Prove that if the sequence $P_n(x)$ converges pointwise on $[0, 1]$, then it converges uniformly on $[0, 1]$.

Proof.
Assume that $P_n \to P$ pointwise as $n \to \infty$. This means that

$$\lim_{n\to\infty} |P_n(x) - P(x)| = 0$$

for every $x \in [0, 1]$. In particular, for $x = 0$, we have that

$$P_n(0) = a_n$$

is a convergent sequence. Therefore, a_n is a convergent sequence. Next, consider $x = 1$ which gives

$$\begin{aligned} P_n(1) &= a_n + b_n + c_n + d_n \\ \implies P_n(1) - P_n(0) &= b_n + c_n + d_n \end{aligned}$$

Therefore, $B_n := b_n + c_n + d_n$ is a convergent sequence since linear combinations of convergent sequences are convergent. Next, consider $x = \frac{1}{2}$ which gives

$$\begin{aligned} P_n\left(\frac{1}{2}\right) &= a_n + \frac{b_n}{2} + \frac{c_n}{4} + \frac{d_n}{8} \\ \implies P_n\left(\frac{1}{2}\right) - a_n - \frac{1}{8}B_n &= \frac{3}{8}b_n + \frac{1}{8}c_n \end{aligned}$$

Thus, $C_n := \frac{3}{8}b_n + \frac{1}{8}c_n$ is a convergent sequence. Lastly, consider $x = \frac{1}{3}$ which gives

$$\begin{aligned} P_n\left(\frac{1}{3}\right) &= a_n + \frac{b_n}{3} + \frac{c_n}{9} + \frac{d_n}{27} \\ \implies P_n\left(\frac{1}{3}\right) - a_n - \frac{1}{27}B_n &= \frac{8}{27}b_n + \frac{2}{27}c_n \end{aligned}$$

Thus, $D_n := \frac{8}{27}b_n + \frac{2}{27}c_n$ is a convergent sequence. Using this, we can get

$$12\left[\frac{8}{3}C_n - \frac{27}{8}D_n\right] = 12\left[b_n + \frac{1}{3}c_n - b_n - \frac{1}{4}c_n\right] = c_n$$

which means that c_n is a convergent sequence. Also,

$$\frac{8}{3}\left[C_n - \frac{1}{8}c_n\right] = \frac{8}{3}\cdot\frac{3}{8}b_n = b_n$$

so that b_n is also a convergent sequence. Lastly,

$$B_n - b_n - c_n = d_n$$

so we get that d_n is a convergent sequence. Let us say that $a_n \to A, b_n \to B, c_n \to C, d_n \to D$ as $n \to \infty$. Therefore, if we fix some $r > 0$, then there exists some N_1, N_2, N_3 and $N_4 \in \mathbb{N}$ such that

$$\begin{aligned} |a_n - A| &< \frac{r}{4} \text{ for all } n \geq N_1 & \qquad |b_n - B| &< \frac{r}{4} \text{ for all } n \geq N_2 \\ |c_n - C| &< \frac{r}{4} \text{ for all } n \geq N_3 & \qquad |d_n - D| &< \frac{r}{4} \text{ for all } n \geq N_4 \end{aligned}$$

which also gives

$$\lim_{n\to\infty} P_n(x) = \lim_{n\to\infty} a_n + b_n x + c_n x^2 + d_n x^3 = A + Bx + Cx^2 + Dx^3 = P(x)$$

Therefore if we note that $||x|| = ||x^2|| = ||x^3|| = 1$ over the domain $[0,1]$, we get that for all $n \geq \max\{N_1, N_2, N_3, N_4\}$:

$$\begin{aligned} ||P_n(x) - P(x)|| &= ||(a_n - A) + (b_n - B)x + (c_n - C)x^2 + (d_n - D)x^3|| \\ &\leq |a_n - A| + |b_n - B|\ ||x|| + |c_n - C|\ ||x^2|| + |d_n - D|\ ||x^3|| \\ &= |a_n - A| + |b_n - B| + |c_n - C| + |d_n - D| \\ &< \frac{r}{4} + \frac{r}{4} + \frac{r}{4} + \frac{r}{4} \\ &= r. \end{aligned}$$

Thus, we get that $P_n \to P$ uniformly on $[0,1]$ as $n \to \infty$. $\square$

Note that this proof was a bit clunky with tedious calculations and never in fact used that the coefficients a_n, b_n, c_n, d_n were bounded. A more elegant, albeit less constructive, way of approaching this problem would be as was done in class where we first show that P_n is equicontinuous for each n (which does use the boundedness of the coefficients), then apply the so-called "Adaptation" of the Arzela-Ascoli Theorem which says if we have a sequence of equicontinuous functions which converge pointwise on a compact space (which $[0,1]$ is compact), then we have that this sequence converges uniformly to a uniformly continuous function. The meat of this proof would simply be showing that a sequence of polynomials with bounded coefficients is equicontinuous which was done for degree 2 polynomials in class and would generalize to degree 3 polynomials with only slight modification.