Complex Analysis Homework 7

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Question 4

Each of the following integrals is zero where $\gamma(a, r)$ denotes the circle with center at a and radius r. Give a reason or evaluate the integral in each case.

(a) $\displaystyle\int_{\gamma(1,1)} \frac{1}{z-3} dz,$

- If we let $\Omega = D(1, 1.5)$ be a convex set, then $\frac{1}{z-3}$ is holomorphic in Ω (since the only discontinuity, $z = 3 \notin \Omega$). Furthermore, $\gamma^* \subset \Omega$, so we can apply Cauchy's Integral Theorem for Convex Sets to conclude this integral is equal to 0.

(b) $\displaystyle\int_{\gamma(i,4)} \frac{1}{(z-3)^2} dz,$

- Recall Cauchy's Integral Formula (for derivatives) that states

$$\int_{\gamma(a,r)} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0) \qquad \text{for all } z_0 \in D(a, r) \subset U, \text{ for } U \text{ an open set and } f \text{ analytic in } U$$

In our case, $n = 1$, $a = i$, $r = 4$, $z_0 = 3 \in D(i, 4)$, $f(z) \equiv 1$. Thus, the integral we are interested in is

$$\int_{\gamma(a,r)} \frac{1}{(z-3)^2} dz = \frac{2\pi i}{n!} f'(3) = 0$$

since $f'(z) \equiv 0$ for all z.

(c) $\displaystyle\int_{\gamma(0,1)} z|z|^4 dz,$

- I will compute this integral by definition of the path integral:

$$\begin{aligned}
\int_{\gamma(0,1)} z|z|^4 dz &= \int_0^{2\pi} e^{it}|e^{it}|^4 (e^{it})' dt \\
&= i \int_0^{2\pi} e^{2it} dt \\
&= \frac{i}{2i} e^{2it} \Big|_{t=0}^{t=2\pi} \\
&= \frac{1}{2} \left(e^{4\pi i} - e^0 \right) \\
&= \frac{1}{2}(1-1) = 0
\end{aligned}$$

(d) $\displaystyle\int_{\gamma(1,1)} (1 + e^{-z})^{-1} dz,$

- This function is discontinuous at $z = ik\pi$ for k an odd integer. However, within the disk $\Omega = D(1, 1.5)$, there does not exist any of these such points. Thus, $(1 + e^{-z})^{-1}$ is holomorphic in Ω and also $\gamma^* \subset \Omega$. Furthermore, since Ω is a convex set, we can apply Cauchy's Integral Theorem for Convex Sets to conclude that this integral is equal to zero.