Analysis Homework 1

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Question 1

Which of the following are metrics on $\mathbb{R}$?

(a) $d(x,y) = (x-y)^2$;

- Note that $d(x,y) \geq 0$ for all x, y since $(\cdot)^2$ is a non-negative function on the real numbers. Next, assume that $d(x,y) = (x-y)^2 = 0$ for some $x, y \in \mathbb{R}$. From this, we obtain $x - y = 0 \implies x = y$. Similarly, if $x = y \in \mathbb{R}$, then $d(x,x) = (x-x)^2 = 0^2 = 0$ which satisfies the first property of a metric.
- Next, if $x, y \in \mathbb{R}$, note that $d(x,y) = (x-y)^2 = (-1[y-x])^2 = (-1)^2(y-x)^2 = (y-x)^2 = d(y,x)$ meaning it satisfies the second property of a metric.
- Lastly, let $x = 1, y = -1$, and $z = 0$, then it is clear to see that $d(x,y) \not\leq d(x,z) + d(z,y)$ since $4 \not\leq 1 + 1$ so the triangle inequality does not apply to this function, so $\boxed{d(x,y) \text{ is not a metric}}$

(b) $d(x,y) = |x^2 - y^2|$;

- Note that the first property of a metric does not hold for this function since for $x = 1$ and $y = -1$, we have $d(x,y) = |1^2 - (-1)^2| = 0$ but $1 \neq -1$ meaning $\boxed{d(x,y) \text{ is not a metric}}$

(c) $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$

- Note that $d(x,y) \geq 0$ for all $x, y \in \mathbb{R}$ since the numerator and denominator are both non-negative functions on $\mathbb{R}$. Furthermore, it is clear that if $x = y$, then $d(x,y) = 0$ since the numerator would be zero. On the other hand, assume that $d(x,y) = 0$:

$$\begin{aligned}
&\frac{|x-y|}{1+|x-y|} = 0 \\
&\implies |x-y| = 0(1+|x-y|) \\
&\implies |x-y| = 0 \\
&\implies x = y && \text{since } |\cdot,\cdot| \text{ is a metric}
\end{aligned}$$

  Thus, the first property of a metric is satisfied.
- Next, let $x, y \in \mathbb{R}$, then note:

$$d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|(-1)(y-x)}{1+|(-1)(y-x)|} = \frac{|-1||y-x|}{1+|-1||y-x|} = \frac{|y-x|}{1+|y-x|} = d(y,x)$$

  which satisfies the second property of a metric.

(d) Lastly, we need to deal with the Triangle Inequality, I will do so in a reverse fashion: let $x, y, z \in \mathbb{R}$, then we have:

$$\begin{aligned}
d(x,z) + d(z,y) &= \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|} \\
&\geq \frac{|x-z|}{1+|x-z|+|z-y|} + \frac{|z-y|}{1+|x-z|+|z-y|} \\
&= \frac{|x-z|+|z-y|}{1+|x-z|+|z-y|} \\
&= \frac{1+|x-z|+|z-y|-1}{1+|x-z|+|z-y|} \\
&= 1 - \frac{1}{1+|x-z|+|z-y|} \\
&\geq 1 - \frac{1}{1+|x-y|} && \text{since } |x-y| \leq |x-z|+|z-y| \\
&= \frac{1+|x-y|-1}{1+|x-y|} \\
&= \frac{|x-y|}{1+|x-y|} \\
&= d(x,y)
\end{aligned}$$

Altogether, this states that $d(x,y) \leq d(x,z) + d(z,y)$ which is precisely the Triangle Inequality. Thus, $\boxed{d(x,y) \text{ is a metric.}}$

Question 2

Let (X, d) be a metric space, and let $Y \subset X$. Prove the following:

(a) Y° is open;

Proof.
Recall that a set $E \subset X$ is open if $E \subset E^\circ$. Thus, I need only check that $Y^\circ \subset (Y^\circ)^\circ$. Let $x \in Y^\circ$. This means that there exists some $r > 0$ such that $U_r(x) \subset Y$. Furthermore, let $z \in U_r(x)$, then if we take $\rho = r - d(x,z)$, it is clear that $U_\rho(z) \subset U_r(x) \subset Y$ since for any $y \in U_\rho(z)$, $d(x,y) \leq d(x,z) + d(z,y) < d(x,z) + r - d(x,z) = r$. Thus, z is an interior point of Y as well. In fact, since z was chosen arbitrarily from $U_r(x)$, we can conclude that $U_r(x) \subset Y^\circ$.

Thus, we have shown that for any $x \in Y^\circ$, there is some radius r where $U_r(x) \subset Y^\circ$. By definition, this means that $x \in (Y^\circ)^\circ$ so that $Y^\circ \subset (Y^\circ)^\circ$, i.e. Y° is an open set. □

(b) if $Z \subset Y$ and Z is open, then $Z \subset Y^\circ$;

Proof.
Let $x \in Z$. Since Z is open, we know that x is an interior point of Z, i.e. there exists some $r > 0$ such that $U_r(x) \subset Z$. However, since $Z \subset Y$, this means that $U_r(x) \subset Y$, so x is indeed an interior point of Y as well. Thus, we have some arbitrary point in Z must be an interior point of Y, so $Z \subset Y^\circ$. □

(c) $(Y^\circ)^c = \overline{Y^c}$.

Proof.
Let $x \in (Y^\circ)^c$. In other words, $x \in X$, but $x \notin Y^\circ$. Since $x \notin Y^\circ$, then for any $r > 0$, we know that $U_r(x) \not\subset Y$. Thus, there is some element in $U_r(x)$ which is not in Y, i.e. $U_r(x) \cap Y^c \neq \emptyset$. If the point $x \notin Y$, then $x \in Y^c$ and obviously $x \in \overline{Y^c} = Y^c \cup (Y^c)'$. If $x \in Y$, we can improve the previous result about the intersection to say that $\overset{\circ}{U}_r(x) \cap Y^c \neq \emptyset$ since x is not in Y^c, it is clearly not in the intersection. However, we have just shown that x is a limit point of Y^c which means that $x \in (Y^c)' \subset \overline{Y^c}$. Thus, whether $x \in Y$ or $x \notin Y$, we can say that $x \in \overline{Y^c}$ meaning that $(Y^\circ)^c \subset \overline{Y^c}$.

Next, let $x \in \overline{Y^c}$. If $x \in Y^c$, then clearly $x \in (Y^\circ)^c$ since $Y^\circ \subset Y$. Thus, let $x \in (Y^c)'$. By definition, this means that for all $r > 0$, $\overset{\circ}{U}_r(x) \cap Y^c \neq \emptyset$. In particular, since the punctured ball with the intersection is nonempty, so too must the standard ball with the intersection be nonempty. Thus, $U_r(x) \cap Y^c \neq \emptyset$. This means there is some element in $U_r(x)$ which is not in Y^c. In other words, $U_r(x) \not\subset Y$. Since this argument was made in terms of all $r > 0$, then we can say that there is no ball around x of any radius that is contained in Y, i.e. x is not an interior point of Y which means $x \in (Y^\circ)^c$ showing that $\overline{Y^c} \subset (Y^\circ)^c$.

Together, this shows that $(Y^\circ)^c = \overline{Y^c}$. □

Question 3

Let X be an infinite set. Define $d : X \times X \to \mathbb{R}$ as

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Prove that d is a metric. Which subsets of X are open? Closed? Compact?

Proof.
The first property of a metric is trivial based off of the definition of d. Clearly $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0 \iff x = y$. The second property is also trivial to show since $x = y \iff y = x$ and $x \neq y \iff y \neq x$ which clearly shows that $d(x,y) = d(y,x)$.

The Triangle Inequality takes a little bit more work. Let $x, y, z \in X$ and consider the following cases:

- $x = y = z$, then $d(x,y) \leq d(x,z) + d(z,y)$ since $0 \leq 0 + 0$.
- $x = y$ and $x \neq z$, then $d(x,y) \leq d(x,z) + d(z,y)$ since $0 \leq 1 + 1$.
- $x \neq y$ and $x = z$, then $d(x,y) \leq d(x,z) + d(z,y)$ since $1 \leq 0 + 1$.
- $x \neq y$ and $y = z$, then $d(x,y) \leq d(x,z) + d(z,y)$ since $1 \leq 1 + 0$.
- $x \neq y$, $y \neq z$, and $x \neq z$, then $d(x,y) \leq d(x,z) + d(z,y)$ since $1 \leq 1 + 1$.

Since this exhausts the list of possibilities, we have the Triangle Inequality holds as well Thus, d is a metric. $\square$

Answer.
To determine which subsets are open, closed, or compact, I will first establish those definitions using this metric.

A set $S \subset X$ will be open if for every $x \in S$, there exists some $r > 0$ such that $U_r(x) \subset S$. More precisely, there exists some $r > 0$ such that $\{y \in X \mid d(x,y) < r\} \subset S$. Note that if $r > 1$, then $U_r(x) = X$ and if $r < 1$, then $U_r(x) = \{y \mid y = x\} = \{x\}$. Thus, $U_r(x) \subset S$ for all $r < 1$ since $x \in S$. Thus, any subset S must be open since every point is an interior point.

A set $S \subset X$ will be closed if every point $x \in X$ satisfying $\overset{\circ}{U}_r(x) \cap S \neq \emptyset$ is also a point in S for any $r > 0$. In particular, this must be true for $r = 0.5$, but $\overset{\circ}{U}_{0.5}(x) = \emptyset$. This means there are no points x satisfying that criteria. Thus, there are no limits points in this metric space, so every subset S must be closed as they trivially contain all of their limit points.

Lastly, recall that for a subset S to be compact, that *any* open cover of S must admit a finite subcover. However, note that for each $x \in S$, $\{x\}$ is an open set. Therefore $\{\{x\}\}_{x \in S}$ is an open cover for S since

$$S \subset \bigcup_{x \in S} \{x\}$$

However, this cover has no strict subcover since removing any element from the set no longer makes it a cover. Therefore, the only "subcover" is the cover itself. Therefore, a finite subcover only exists whenever the set S itself is a finite set. Consequently, each finite set S is compact by the same construction as above, and we can conclude that the compact subsets are precisely the finite subsets of X.

Question 4

Consider the metric space (X, d), where $X = \mathbb{Q}$, $d(x,y) = |x - y|$. Prove that the subset $Y = \{x \in X \mid 2 < x^2 < 3\}$ is closed and bounded, but not compact.

Proof.
First, it is clear that Y is bounded since $Y \subset \{x \in X \mid x^2 < 4\} = U_2(0)$. Next, I will use the following two theorems that we have proven in class:

Theorem 1.
Let (A, d) be a metric space where $B \subseteq A$ and let $C \subseteq B$, then C is open in $B \iff C = B \cap G$ and G is open in A.

Theorem 2.
Let (A, d) be a metric space where $B \subseteq A$, then B is closed if and only if B^c is open.

First, I will show that $Y^c = \{x \in X \mid x^2 \leq 2 \text{ or } x^2 \geq 3\}$ is an open set. I will do this by using Theorem 1 with $A = \mathbb{R}$, $B = \mathbb{Q}$ ($= X$), $C = Y^c$, and $G = \{x \in \mathbb{R} \mid x^2 < 2 \text{ or } x^2 > 3\}$. To verify that the hypotheses of the Theorem hold, I simply need to realize that G is open in $\mathbb{R}$ since it is the union of three open sets in $\mathbb{R}$: namely, $(-\infty, -\sqrt{3}), (-\sqrt{2}, \sqrt{2})$, and $(\sqrt{3}, \infty)$. Note that since $\sqrt{2}$ and $\sqrt{3}$ are not rational, then Y^c is equivalent to $Y^c = \{x \in X \mid x^2 < 2 \text{ or } x^2 > 3\}$. Thus, $Y^c = \mathbb{Q} \cap G$, so by Theorem 1, Y^c is open in $\mathbb{Q}$.

Now, by using Theorem 2, I can say that Y must be closed in $\mathbb{Q}$ since we have established that Y^c is open in $\mathbb{Q}$.

Finally, I need to show that Y is not compact. To do this, I will find a cover for Y that does not admit a finite subcover. Consider the intervals:

$$U_n = \left(\sqrt{3} + \frac{1}{n}(\sqrt{2} - \sqrt{3}), \sqrt{3} + \frac{1}{n+1}(\sqrt{2} - \sqrt{3})\right)$$
$$V_n = \left(-\sqrt{2} + \frac{1}{n}(\sqrt{2} - \sqrt{3}), -\sqrt{2} + \frac{1}{n+1}(\sqrt{2} - \sqrt{3})\right)$$

Notice that U_n simply partitions the interval $(\sqrt{2}, \sqrt{3})$ into distinct intervals where the nth interval is of length $(\sqrt{2} - \sqrt{3})(1/(n+1) - 1/n)$. Similarly, V_n covers the interval $(-\sqrt{3}, -\sqrt{2})$. Furthermore, we can see that each of these individual intervals is open in $\mathbb{Q}$ since they are equal to the same (open) interval in $\mathbb{R}$ intersected with $\mathbb{Q}$ which makes them each open. Thus, the set $\{W_n\}_{n=1}^{\infty}$ for $W_n = U_n \cup V_n$ is an open cover for Y. However, there does not exist any finite subcover because $W_i \cap W_j = \emptyset$ for all $i \neq j$. Thus, removing any W_i from the set no longer makes it a cover, since the elements in that set is not included in any other sets. Thus, Y is not compact. $\square$