Advanced Calc. Exam 1

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Question 1

Prove that $P(n) = 8^n - 3^n$ is divisible by 5 for every $n \in \mathbb{N}$.

Proof.
Base case $n = 1$:
For $n = 1$, we have $P(n) = P(1) = 8^1 - 3^1 = 8 - 3 = 5$ and 5 is clearly divisible by 5, so the above statement holds for $n = 1$.

Inductive Step:
First, note that for a number to be divisible by 5, that is equivalent to saying that number is an integer multiple of 5. From this, assume that $P(n)$ is divisible by 5 for some $n = k \in \mathbb{N}$, i.e. $P(k) = 8^k - 3^k = 5m$ for some integer m, this is the Inductive Hypothesis. Next, I will examine whether $P(k+1)$ must also be divisible by 5 in the following manner:

$$\begin{aligned}
P(k+1) &= 8^{k+1} - 3^{k+1} && \text{by the definition of } P(n) \\
&= 8 \cdot 8^k - 3 \cdot 3^k && \text{by exponent properties} \\
&= 8 \cdot (8^k - 3^k + 3^k) - 3 \cdot 3^k && \text{by adding and subtracting } 3^k \\
&= 8 \cdot (8^k - 3^k) + 8 \cdot 3^k - 3 \cdot 3^k && \text{by distributing the 8} \\
&= 8 \cdot (8^k - 3^k) + (8 - 3) \cdot 3^k && \text{by factoring} \\
&= 8 \cdot (5m) + 5 \cdot 3^k && \text{by using the Inductive Hypothesis} \\
&= 5(8m + 3^k) && \text{by factoring.}
\end{aligned}$$

Thus, we have shown that $P(k+1) = 5\widetilde{m}$ for $\widetilde{m} = 8m + 3^k \in \mathbb{Z}$. Therefore, we have shown that if $P(k)$ is divisible by 5 for any $k \in \mathbb{N}$, then we must have that $P(k+1)$ is divisible by 5 as well. Additionally, since we have shown that $P(1)$ is divisible by 5, then by the Principle of Mathematical Induction, we can conclude that $P(n)$ is divisible by 5 for all $n \in \mathbb{N}$, exactly what we wanted to show. $\square$

Question 2

Is $\sqrt[3]{5-\sqrt{5}}$ a rational number?

Answer.
First, note that if we let $x_0 = \sqrt[3]{5-\sqrt{5}}$, we can find a polynomial with x_0 as one of its roots by doing the following algebraic manipulations:

$$\begin{aligned}
x_0 &= \sqrt[3]{5-\sqrt{5}} \\
\implies x_0^3 &= 5-\sqrt{5} && \text{by cubing both sides} \\
\implies x_0^3 - 5 &= -\sqrt{5} && \text{by subracting 5 from both sides} \\
\implies (x_0^3-5)^2 &= 5 && \text{by squaring both sides} \\
\implies x_0^6 - 10x_0^3 + 25 &= 5 && \text{by expanding the binomial} \\
\implies x_0^6 - 10x_0^3 + 20 &= 0 && \text{by subracting 5 from both sides}
\end{aligned}$$

Thus, we can see that $x_0 = \sqrt[3]{5-\sqrt{5}}$ is a root of the polynomial $f(x) = x^6 - 10x^3 + 20$. Therefore, we can now apply the "Rational Zeros Theorem" to the polynomial f with $n = 6$, $c_6 = 1$, $c_5 = c_4 = 0$, $c_3 = -10$, $c_2 = c_1 = 0$, and $c_0 = 20$. The Rational Zeros Theorem states that if f has any rational zeros, r, then they must be of the form $r = \frac{c}{d}$ where c divides $c_0 = 20$ and d divides $c_6 = 1$. From this we know that $c \in \{\pm1, \pm2, \pm4, \pm5, \pm10, \pm20\}$ and $d \in \{\pm1\}$ which implies that $r \in \{\pm1, \pm2, \pm4, \pm5, \pm10, \pm20\}$. Now, let's evaluate f at all of the possible values for r:

$$\begin{aligned}
f(1) &= (1)^6 - 10(1)^3 + 20 && = 11 \\
f(-1) &= (-1)^6 - 10(-1)^3 + 20 && = 31 \\
f(2) &= (2)^6 - 10(2)^3 + 20 && = 4 \\
f(-2) &= (-2)^6 - 10(-2)^3 + 20 && = 164 \\
f(4) &= (4)^6 - 10(4)^3 + 20 && = 3476 \\
f(-4) &= (-4)^6 - 10(-4)^3 + 20 && = 4756 \\
f(5) &= (5)^6 - 10(5)^3 + 20 && = 14395 \\
f(-5) &= (-5)^6 - 10(-5)^3 + 20 && = 16895 \\
f(10) &= (10)^6 - 10(10)^3 + 20 && = 990020 \\
f(-10) &= (-10)^6 - 10(-10)^3 + 20 && = 1010020 \\
f(20) &= (20)^6 - 10(20)^3 + 20 && = 63920020 \\
f(-20) &= (-20)^6 - 10(-20)^3 + 20 && = 64080020
\end{aligned}$$

Clearly, none of these values are 0, so none of the possible values for r are actually zeros of f. Thus, f has no rational zeros, so since $x_0 = \sqrt[3]{5-\sqrt{5}}$ *is* a root of f, it cannot be rational. Therefore, $\boxed{\sqrt[3]{5-\sqrt{5}} \text{ is } \underline{\text{not}} \text{ a rational number}}$.

Question 3

Consider the set $X := \{\square, \triangle\}$ with operation "+" defined as follows:

$$\square + \square = \square \qquad \square + \triangle = \triangle \qquad \triangle + \square = \triangle \qquad \triangle + \triangle = \square$$

Show that $(X, +)$ satisfies A1 - A4 and find which element of X plays the role of "0".

Answer.
<u>A1</u>
I will check that $a + (b + c) = (a + b) + c$ for all $a, b, c \in X$:

$$\begin{array}{rcll}
\square + (\square + \square) = \square + \square = \square & \text{and} & (\square + \square) + \square = \square + \square = \square & \checkmark \\
\square + (\square + \triangle) = \square + \triangle = \triangle & \text{and} & (\square + \square) + \triangle = \square + \triangle = \triangle & \checkmark \\
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\end{array}$$

Thus, A1 is true for all possible choices of $a, b, c \in X$, so A1 is satisfied.
<u>A2</u>
I will check that $a + b = b + a$ for all $a, b \in X$. First, note that for $a = b$, this is obviously true as the left and right sides of the equation would be the exact same expressions. Thus, I will consider all cases where $a \neq b$:

$$\begin{array}{rcll}
\square + \triangle = \triangle & \text{and} & \triangle + \square = \triangle & \checkmark \\
\triangle + \square = \triangle & \text{and} & \square + \triangle = \triangle & \checkmark
\end{array}$$

Therefore, $a + b = b + a$ for all $a, b \in X$ both when $a = b$ and when $a \neq b$, so A2 is satisfied.
<u>A3</u>
I will check that $a + 0 = a$ for all $a \in X$. First, note that "0" in this context is given by $\square$, I will show this in the following:

$$\begin{array}{ll}
\square + \square = \square & \checkmark \\
\triangle + \square = \triangle & \checkmark
\end{array}$$

Thus, $a + \square = a$ for all $a \in X$, so $\boxed{\text{"0"} = \square}$ and A3 is satisfied.
<u>A4</u>
I will check that for every $a \in X$, there exists an element $-a \in X$ such that $a + (-a) = 0$. In this context, since "0"$= \square$, we are trying to show that this $-a$ satisfies $a + (-a) = \square$. I will try to find this $-a$ for every element in X:

$$\begin{array}{lll}
\text{for } a = \square,\ -a = \square \text{ since} & \square + \square = \square & \checkmark \\
\text{for } a = \triangle,\ -a = \triangle \text{ since} & \triangle + \triangle = \square & \checkmark
\end{array}$$

Thus, $-a$ exists for every $a \in X$, so A4 is satisfied.

Question 4

Let $S, T \subset \mathbb{R}$ be nonempty bounded sets. Provide a proof or counterexample for the following:

$$\sup(S) \leq \sup(T) \implies S \subset T \tag{1}$$

Answer.
I will provide a counterexample to show that the previous statement is false. Let $S = (2, 5)$ and let $T = (6, 10)$, then it is easy to see that $\sup(S) = 5$ and $\sup(T) = 10$. Thus, the condition for (1) is satisfied as $\sup(S) = 5 \leq 10 = \sup(T)$. However, the implication is not true because, for example, $\pi \in S$ but $\pi \notin T$, so it is clear that $S \not\subset T$. Therefore, $\boxed{\text{this counterexample shows that (1) is false}}$.

Question 5

Fix an arbitrary $a \in \mathbb{R}$ and define the set $S := \{r \in \mathbb{Q} : r < a\}$. Show that $\sup(S) = a$.

Answer.
Since $\sup(S)$ is defined as the least upper bound of S, we first need to check that a is indeed an upper bound of S. However, due to how S is defined, every $r \in S$ satisfies $r < a$, so in particular $r \leq a$ for all $r \in S$; thus, a is an upper bound by the definition of Upper Bound.

Next, I will show that a is actually the Least Upper Bound. To do this, assume that there exists a smaller upper bound for S, i.e. let $b \in \mathbb{R}$ be an upper bound for S with $b < a$. However, by the Density of $\mathbb{Q}$, we know that for any two $a, b \in \mathbb{R}$ with $b < a$ there exists some $r \in \mathbb{Q}$ such that $b < r < a$. This implies that there is an $r_0 \in S$ that is greater than b meaning b cannot actually be an upper bound. Thus, our assumption of the existence of a smaller upper bound must have been false, so we know that a is not only an upper bound of S, but, in fact, the Least Upper Bound of S. Equivalently, we have shown that $\sup(S) = a$, just as desired.