Complex Analysis Homework 2

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Question 4

Question.
Geometrically describe the following subsets of $\mathbb{C}$:

a.) $|z - i| < |z - 1|$

b.) $|z + 2i| \geq 2$

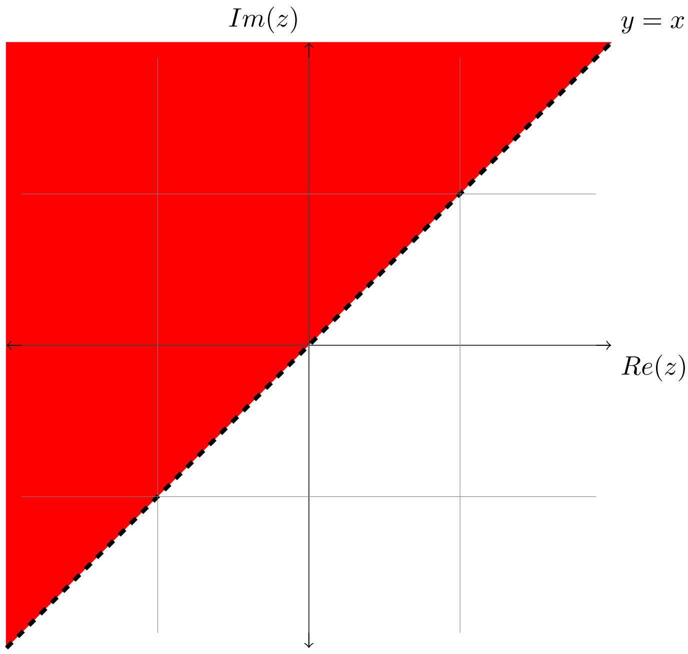
c.) $1 < Re(z) \leq 2$

d.) $Re(z) \neq 0$.

Answer. a.)
If we first rewrite the inequality as $|z - (0 + i)| < |z - (1 + 0i)|$, then geometrically, the above inequality is the set of all points z such that z is closer to $0 + i$ than it is to $1 + 0i$. Next, express z as $z = x + iy$ and notice the distance described is equal when along the line $y = x$; thus, the inequality is satisfied whenever $y > x$. We can verify this statement algebraically by making the substitution $z = x + iy$ into the given inequality:

$$\begin{aligned}
|z - i| &< |z - 1| \\
|x + iy - i| &< |x + iy - 1| \\
|x + i(y - 1)| &< |(x - 1) + iy| \\
|x + i(y - 1)|^2 &< |(x - 1) + iy|^2 \\
x^2 + (y - 1)^2 &< (x - 1)^2 + y^2 \\
x^2 - (x - 1)^2 &< y^2 - (y - 1)^2 \\
x^2 - (x^2 - 2x + 1) &< y^2 - (y^2 - 2y + 1) \\
2x - 1 &< 2y - 1 \\
x &< y
\end{aligned}$$

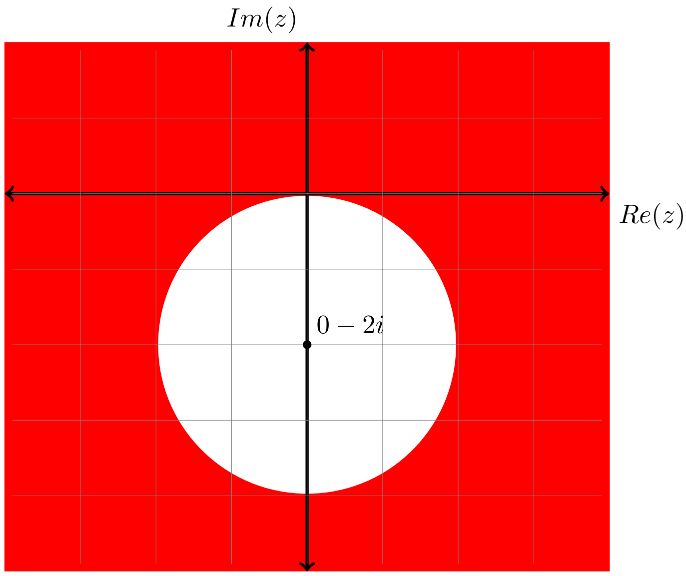
Thus, we arrive at the same conclusion as we did before: all points $z = x + iy$ satisfy the inequality whenever $y > x$. On a graph, this is represented as:



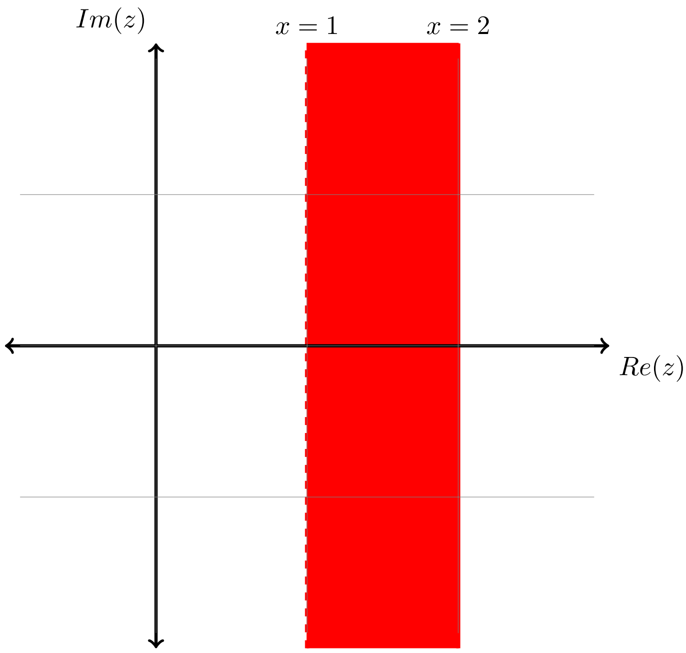
Answer. b.)
First, it may be useful to rewrite this inequality as $|z-(0-2i)| \geq 2$. In this form, it is clear that we are looking for all points z such that the distance between z and $0-2i$ is greater than or equal to a distance of 2. Intuitively, this would be the set of all points on the boundary and outside of a circle of radius 2 centered at $0-2i$. Again, let's verify this algebraically by plugging $z = x+iy$ into the inequality:

$$\begin{aligned}|z+2i| &\geq 2\\|x+iy+2i| &\geq 2\\|x+i(y+2)| &\geq 2\\|x+i(y+2)|^2 &\geq 2^2\\x^2+(y+2)^2 &\geq 2^2\end{aligned}$$

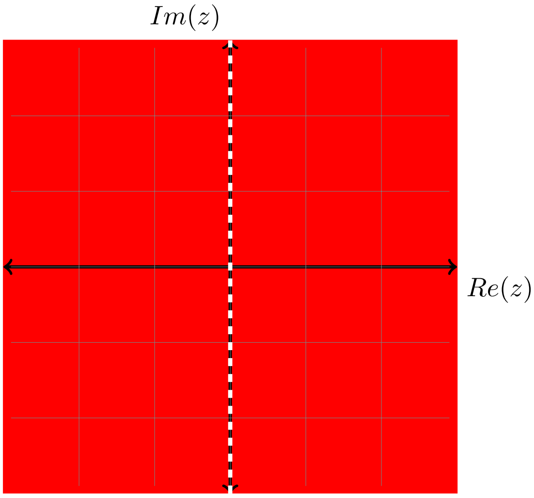
This verifies that the first inequality describes precisely what I claimed: the set of all points z outside of or on the border of the circle of radius 2 centered at $0-2i$. Graphically, this is represented as:



Answer. c.)
If we write z as $z = x+iy$, then it is clear that $Re(z) = x$. Thus, the given inequality is also expressed as $1 < x \leq 2$. Geometrically, this is a vertical strip on the complex plane with the left side of the strip not included at $x = 1$, and the right side of the strip included at $x = 2$. On a graph, this looks like:



Answer. d.)
The above statement represents all points on the complex plane expect those where $Re(z) = 0$. If we write z as $z = x+iy$, then this is equivalent to all points except where $x = 0$. Graphically, this can be represented by:


$Im(z)$
$Re(z)$