Complex Analysis Homework 7

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Question 3

Use the definition to show that the sequence of functions $f_n(z) = \dfrac{1}{nz}$ is pointwise convergent, but not uniformly convergent, to $f(z) = 0$ on the domain $\Omega = D(0,1)\backslash\{0\}$.

Proof. I will first show that $f_n(z)$ is pointwise convergent. Let $z \in D(0,1)\backslash\{0\}$ and $\varepsilon > 0$ be fixed. Let us examine $|f_n(z) - f(z)| = |f_n(z)|$:

$$\begin{aligned} |f_n(z)| &= \left|\frac{1}{nz}\right| \\ &= \frac{1}{n|z|} \end{aligned}$$

Thus, if we define $N_\varepsilon(z) := \frac{1}{\varepsilon|z|}$, then for $n > N_\varepsilon(z)$ we have the following:

$$\begin{aligned} |f_n(z) - f(z)| = |f_n(z)| &= \frac{1}{n|z|} \\ &< \frac{1}{N_\varepsilon(z)|z|} \\ &= \frac{1}{|z|/(\varepsilon|z|)} \\ &= \varepsilon \end{aligned}$$

This shows that f_n is pointwise convergent to $f(z) = 0$. However, if we look at our choice for $N_\varepsilon(z)$, we see that it has no upper bound because as $|z| \to 0$, then $N_\varepsilon(z) \to \infty$. Thus, it is impossible to find an N_ε that does not depend on the specific point z, so f_n does not converge uniformly to $f(z) = 0$. □