Question 4

Let the function f be defined as follows:

$$f(z) = \frac{3z+1}{z^2(z^2-4)}$$

a.) Locate and classify all singular points of f in $\mathbb{C}$

- The singular points occur at $z=0$, $z=2$, and $z=-2$ and they are all poles, as shown below:

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \left|\frac{3z+1}{z^2(z^2-4)}\right| = \left|\frac{1}{-4}\right| \lim_{z\to 0}\left|\frac{1}{z^2}\right| = \infty$$

$$\lim_{z\to 2} |f(z)| = \lim_{z\to 2} \left|\frac{3z+1}{z^2(z+2)(z-2)}\right| = \left|\frac{7}{4\cdot 4}\right| \lim_{z\to 2}\left|\frac{1}{z-2}\right| = \infty$$

$$\lim_{z\to -2} |f(z)| = \lim_{z\to -2} \left|\frac{3z+1}{z^2(z+2)(z-2)}\right| = \left|\frac{-5}{4\cdot(-4)}\right| \lim_{z\to -2}\left|\frac{1}{z+2}\right| = \infty$$

- Furthermore, by looking at the order of these points as zeroes of the denominator, we can see that

$z=0$ is a pole of order 2, $z=2$ and $z=-2$ are poles of order 1

b.) Find the residues at the singular points of f.

- I will use the following formula for residues of poles of order $\leq m$,

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\Big|_{z=z_0}$$

- For $z_0=0$, $m=2$, so

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{1!}\frac{d}{dz}\left(\cancel{z^2}\cdot\frac{3z+1}{\cancel{z^2}(z^2-4)}\right)\Big|_{z=0} = \frac{d}{dz}\left(\frac{3z+1}{z^2-4}\right)\Big|_{z=0}$$

$$= \left(\frac{3(z^2-4)-(3z+1)(2z)}{(z^2-4)^2}\right)\Big|_{z=0} = \frac{3(-4)-(1)(0)}{(-4)^2} = \frac{-12}{16} = \frac{-3}{4} \text{ at } z=0$$

- For $z_0=2$, $m=1$, so

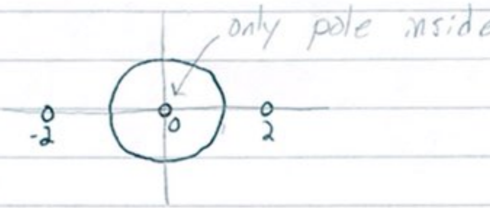
$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{0!}\frac{d^0}{dz^0}\left(\cancel{(z-2)}\frac{3z+1}{z^2(z+2)\cancel{(z-2)}}\right)\Big|_{z=2} = \frac{3(2)+1}{2^2(2+2)} = \frac{7}{16} \text{ at } z=2$$

- For $z_0=-2$, $m=1$, so

$$\operatorname*{Res}_{z=-2} f(z) = \frac{1}{0!}\frac{d^0}{dz^0}\left(\cancel{(z+2)}\frac{3z+1}{z^2\cancel{(z+2)}(z-2)}\right)\Big|_{z=-2} = \frac{3(-2)+1}{(-2)^2(-2-2)} = \frac{-5}{-16} = \frac{5}{16} \text{ at } z=-2$$

c.) Evaluate $\int_\gamma f(z)\,dz$ for γ the positively-oriented circle $|z|=1$

Using Cauchy's Residual Theorem,



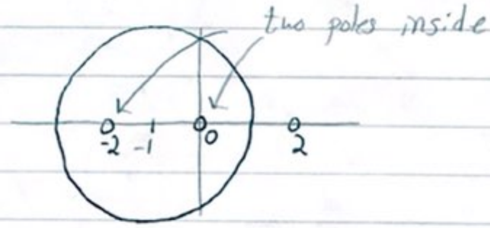
$$\int_\gamma f(z)\,dz = 2\pi i\left(\underset{z=0}{\text{Res}}\, f(z)\right)$$

$$= 2\pi i\left(\frac{-3}{4}\right) \quad \text{by part (b)}$$

$$= \frac{-3\pi i}{2}$$

d.) Same as before, but γ the positively-oriented circle $|z+1| = 2$

Using Cauchy's Residual Theorem,

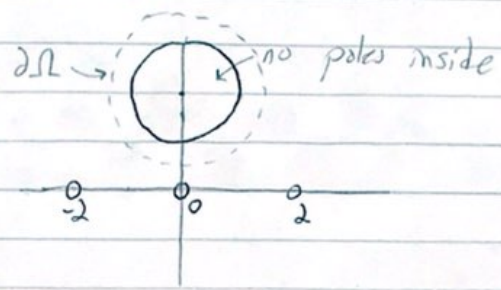


$$\int_\gamma f(z)\,dz = 2\pi i\left(\underset{z=0}{\text{Res}}\, f(z) + \underset{z=-2}{\text{Res}}\, f(z)\right)$$

$$= 2\pi i\left(\frac{-3}{4} + \frac{5}{16}\right) \quad \text{by part (b)}$$

$$= 2\pi i\left(\frac{-12+5}{16}\right) = \frac{-7\pi i}{8}$$

e.) Now γ given by positively-oriented circle $|z-2i|=1$



In this case, we can either use Cauchy's Residual Theorem on an empty summation, or simply the fact that f is holomorphic in $\Omega = D(2i, 1.5)$ and $\gamma^* \subset \Omega$ to use Cauchy's Integral Formula and say

$$\int_\gamma f(z)\,dz = 0$$