Complex Analysis Homework 9

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Question 2

Locate and classify the singularities in $\mathbb{C}$ for $f(z)$ equal to the following functions.

(a) $\dfrac{z}{z^2-1}$

- Note that the denominator can be written as $z^2-1=(z-1)(z+1)$. Then it is clear that $z=1$ and $z=-1$ are both isolated singularities of f since $f \in H(D'(1,1))$ and $f \in H(D'(-1,1))$ where $D'(a,r)$ is the punctured disk centered at a with radius r. Furthermore, if we notice that

$$\lim_{z\to 1}|f(z)| = \lim_{z\to 1}\left|\frac{z}{(z+1)(z-1)}\right| = \frac{1}{2}\lim_{z\to 1}\left|\frac{1}{z-1}\right| = \infty \qquad \text{and}$$
$$\lim_{z\to -1}|f(z)| = \lim_{z\to -1}\left|\frac{z}{(z-1)(z+1)}\right| = \frac{1}{2}\lim_{z\to -1}\left|\frac{1}{z+1}\right| = \infty,$$

- then we can deduce that $\boxed{z=1 \text{ and } z=-1 \text{ are both poles for } f(z).}$ In particular, they would both be poles of order 1 since if we consider them as zeroes of the denominator, $g(z)=z^2-1$, then clearly $g(\pm 1)=0$, but $g'(\pm 1)=\pm 2 \neq 0$.

(b) $\tan^2(z)$

- First, recall the definition of $\tan(w)=\frac{\sin(w)}{\cos(w)}$. Thus, $\tan^2(z)=\frac{\sin^2(z)}{\cos^2(z)}$ and we have singularities precisely when $\cos^2(z)=0$. Furthermore, we know that $\cos(z)=0$ if and only if $z=\frac{\pi}{2}+\pi k$ for $k\in\mathbb{Z}$. Let $z_k=\frac{\pi}{2}+\pi k$ for some $k\in\mathbb{Z}$, then we can see

$$\lim_{z\to z_k}|f(z)| = \lim_{z\to z_k}\left|\frac{\sin^2(z)}{\cos^2(z)}\right| = \lim_{z\to z_k}\left|\frac{1}{\cos^2(z)}\right| = \infty$$

- so we can see that these singularities are actually poles. Thus $\boxed{z=\dfrac{\pi}{2}+\pi k \text{ for } k\in\mathbb{Z} \text{ are all poles for } f(z)}$.
We can determine the order of these poles by considering them as zeroes of the denominator and finding the order of them as zeroes. Consider $g(z)=\cos^2(z)$ and let $z_k=\frac{\pi}{2}+\pi k$. We have already seen that $g(z_k)=0$ for all k. Now consider $g'(z)=-2\cos(z)\sin(z)$, we also have that $g'(z_k)=0$ for all k. Going further, we find $g''(z)=2\sin^2(z)-2\cos^2(z)$. However, $g''(z_k)=2\neq 0$ for all k. From this, we conclude that all z_k's are actually poles of order 2.

(c) $\dfrac{z}{1-e^z}$

- Notice at $z=0$, the denominator is equal to $1-e^0=0$ which means $z=0$ is a singularity for f. If we replace e^z with its Taylor Expansion centered at 0, we can observe the following:

$$\begin{aligned}
f(z) = \frac{z}{1-e^z} &= \frac{z}{1-\displaystyle\sum_{n=0}^{\infty}\frac{z^n}{n!}} \\
&= \frac{z}{1-\left(1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\cdots\right)} \\
&= \frac{-z}{z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\cdots} \\
&= -\left(\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}\right)^{-1} \qquad \text{for all } z\neq 0
\end{aligned}$$

- Thus, we have found a suitable extension function for f. Since $f(z)$ is equal to that last series at all points $z \neq 0$, then we can simply extend f to have the value of that series at $z = 0$. This means we're extending f to satisfy $f(0) = -1$ since this series is equal to -1 at 0. Lastly, we can notice that this is the negative reciprocal of a power series which converges in all of $\mathbb{C}$ and is never equal to zero; thus, it is analytic in all of $\mathbb{C}$. This leads us to the conclusion that $\boxed{z = 0 \text{ is a removable singularity for } f}$

(d) e^{-1/z^4}

- Since the function $\frac{1}{z^4}$ is not defined at $z = 0$, we have that $z = 0$ is a singularity for f. I will consider the series expansion of f centered around $z = 0$. To do this, recall that

$$e^w = \sum_{n=0}^{\infty} \frac{w^n}{n!}$$
$$\implies e^{-1/z^4} = \sum_{n=0}^{\infty} \frac{(-1/z^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{4n} n!}$$

- Thus, we can see that $a_{-n} \neq 0$ for infinitely many $n \in \mathbb{N}$ in the Laurent Series expansion of f which indicates that $\boxed{z = 0 \text{ is an essential singularity for } f}$