Complex Analysis Homework 2

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Question 1

Question.
Show the following equations hold by writing the factors on the left in exponential form first:

a.) $(\sqrt{3}+i)^6 = -64$

b.) $(1+\sqrt{3}i)^{-10} = 2^{-11}(-1+\sqrt{3}i)$.

First, note that for $z \in \mathbb{C}$ expressed as $z = x + iy$ in rectangular form is equivalent to $z = r(cos(\theta) + i\sin(\theta))$ for $r = |z|$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$ in polar form.
Also, note that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, so $z = re^{i\theta}$

Proof. **a.)**
If $z = \sqrt{3}+i$, then, $r = |z| = \sqrt{(\sqrt{3})^2 + (1)^2} = 2$ and $\cos(\theta) = \sqrt{3}/2$ and $\sin(\theta) = 1/2 \implies \theta = \pi/6$
Thus,

$$\begin{aligned}
z = re^{i\theta} &= 2e^{i(\pi/6)} \\
z^6 &= (2e^{i(\pi/6)})^6 \\
&= 2^6 e^{i\pi} \\
&= 64(\cos(\pi) + i\sin(\pi)) \\
&= 64(-1 + i(0)) \\
z^6 = (\sqrt{3}+i)^6 &= -64
\end{aligned}$$

which is our desired equality. □

Proof. **b.)**
If $z = 1+\sqrt{3}i$, then, $r = |z| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$ and $\cos(\theta) = 1/2$ and $\sin(\theta) = \sqrt{3}/2 \implies \theta = \pi/3$
Thus,

$$\begin{aligned}
z = re^{i\theta} &= 2e^{i(\pi/3)} \\
z^{-10} &= (2e^{i(\pi/3)})^{-10} \\
&= 2^{-10}e^{i(-10\pi/3)} \\
&= 2^{-10}e^{i(-9\pi/3)}e^{i(-\pi/3)} \\
&= 2^{-10}e^{i(-3\pi)}e^{i(-\pi/3)} \\
&= 2^{-10}e^{i\pi}e^{i(-\pi/3)} \\
&= 2^{-10}(\cos(\pi) + i\sin(\pi))(\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})) \\
&= 2^{-10}(-1 + i(0))(\frac{1}{2} - i\frac{\sqrt{3}}{2}) \\
&= 2^{-10}(-1)(\frac{1}{2})(1 - \sqrt{3}i) \\
z^{-10} = (1+\sqrt{3}i)^{-10} &= 2^{-11}(-1+\sqrt{3}i)
\end{aligned}$$

which is our desired equality. □