Advanced Calc. Exam 2

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Honor Code

(a) Which of the following constitute Honor Pledge violations? Mark all that apply?

  (i) Unauthorized Collaboration: working together with people when you are not supposed to.
    - Violation

  (ii) Plagiarism: taking someone else's ideas, words or thoughts and using them as your own without any citation, either intentionally or unintentionally.
    - Violation

  (iii) Unauthorized Aid: using something in your academic work you are not allowed to use, such as unauthorized calculators, books or notes in an exam.
    - Violation

  (iv) Falsification: lying about what has occurred.
    - Violation

(b) According to the K-State Honor and Integrity System, which of the following are true?

  (i) A grade of XF can result from a breach of academic honesty. The F indicates failure in the course; the X indicates the reason is an Honor Pledge violation.
    - True

  (ii) After the 1994 cheating incident, the request for an honor system first came from the faculty and the administration.
    - False, it came from the students

  (iii) You may be held accountable for an Honor Pledge violation for posting and/or viewing answers in online sites such as chegg, cramster, study soup, study blue, etc.
    - True

  (iv) If a student has more than one violation during their time at K-State, they're looking at possible suspension or expulsion.
    - True

Question 1

For what values of $p \in \mathbb{R}$ does the series $\displaystyle\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^p}$ converge?

Answer.
I claim that this series $\boxed{\text{only converges for } p > 1}$.

Proof.
First, note that if $p \leq 1$, then we have the following:

$$\left|\frac{(\ln n)^2}{n^p}\right| > \frac{.3}{n^p} \qquad \text{for all } n > 2 \text{ since } \ln(2)^2 \approx .5 \text{ and } \ln(\cdot) \text{ is an increasing function}$$

The .3 was arbitrary here, but was just used to get the proper inequality for all values of the summation. Furthermore, we also know that $\sum \frac{.3}{n^p} = .3 \sum \frac{1}{n^p}$ diverges for all $p \leq 1$ due to Theorem 15.1. Therefore, by the Comparison Test, our given series also diverges for all $p \leq 1$.

Now, considering $p > 1$, I will apply the Integral Test, and consider the following integral:

$$\begin{aligned}
\int_2^{\infty} \frac{(\ln x)^2}{x^p} dx &= \lim_{b\to\infty} \int_2^b \frac{(\ln x)^2}{x^p} dx && (1)\\
&= \lim_{b\to\infty} \left(\left[\frac{(\ln x)^2}{(1-p)x^{p-1}} \right]_{x=2}^{x=b} - \int_2^b \frac{2\ln x}{(1-p)x^p} dx \right) && (2)\\
&= \lim_{b\to\infty} \left(\frac{1}{1-p} \left[\frac{(\ln b)^2}{b^{p-1}} - \frac{(\ln 2)^2}{2^{p-1}} \right] - \frac{2}{1-p} \left[\frac{\ln x}{(1-p)x^{p-1}} \right]_{x=2}^{x=b} + \frac{2}{1-p} \int_2^b \frac{1}{(1-p)x^p} \right) && (3)\\
&= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{(\ln b)^2}{b^{p-1}} - \frac{(\ln 2)^2}{2^{p-1}} - \frac{2}{1-p} \left[\frac{\ln b}{b^{p-1}} - \frac{\ln 2}{2^{p-1}} \right] + \frac{2}{1-p} \left[\frac{1}{(1-p)x^{p-1}} \right]_{x=2}^{x=b} \right) && (4)\\
&= \frac{1}{1-p} \lim_{b\to\infty} \left(\frac{(\ln b)^2}{b^{p-1}} - \frac{(\ln 2)^2}{2^{p-1}} - \frac{2}{1-p} \left[\frac{\ln b}{b^{p-1}} - \frac{\ln 2}{2^{p-1}} \right] + \frac{2}{(1-p)^2} \left[\frac{1}{b^{p-1}} - \frac{1}{2^{p-1}} \right] \right) && (5)
\end{aligned}$$

Where I used integration by parts in lines (2) and (3). In this final limit expression, all powers of $(p-1)$ are strictly positive powers since $p > 1$. Furthermore, we also know that all terms of the form $\frac{\ln b}{b^{p-1}}$ and $\frac{(\ln b)^2}{b^{p-1}}$ are monotonically decreasing *after a certain point* since power terms grow faster than logarithmic terms. Additionally, these terms are strictly positive (i.e. bounded below by 0) as long as $b > 1$. Thus, all the above terms do indeed converge to some real number. What this number is is not important, but let's call it $M \in \mathbb{R}$. Therefore, by the Integral Test, we can say:

$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n^p} \leq \int_2^{\infty} \frac{(\ln x)^2}{x^p} = M \qquad \text{for } p > 1$$

Thus, I have shown that the series does converge for $p > 1$, but does not converge for any $p \leq 1$ which finished the proof. $\square$

Question 2

Consider a function $f : \mathbb{R} \to \mathbb{R}$ such that the sequence $(f(x_n)) \subset \mathbb{R}$ is Cauchy for every Cauchy sequence $(x_n) \subset \mathbb{R}$. Is f necessarily continuous on $\mathbb{R}$?

Answer.
I claim that $\boxed{f \text{ must be continuous on } \mathbb{R}}$.

Proof.
Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Thus, by assumption, we know that $(f(x_n))$ is Cauchy. This means that for some fixed $\varepsilon > 0$, there exists an N_1 such that $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m > N_1$. Additionally, since (x_n) is Cauchy, this means that for $\delta > 0$ fixed, there exists some N_2 such that $|x_n - x_m| < \delta$ for all $n, m > N_2$. Thus, if we fix $m, n > \max\{N_1, N_2\}$, then we can conclude that for $|x_n - x_m| < \delta$, we also have $|f(x_n) - f(x_m)| < \varepsilon$ which means that f is uniformly continuous. In particular, f must be continuous.

$\square$

Question 3

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that the sequence $(f(x_n)) \subset \mathbb{R}$ is Cauchy for every Cauchy sequence $(x_n) \subset \mathbb{R}$. Is f necessarily uniformly continuous on $\mathbb{R}$?

Answer.
I claim that $\boxed{f \text{ may not be uniformly continuous on } \mathbb{R}}$

Proof.
Consider $f(x) = x^2$. Clearly f is continuous and if (x_n) is a Cauchy sequence, then (x_n) is a convergent sequence (say, to x_0) by the completeness of $\mathbb{R}$. Furthermore, $f(x_n) = x_n^2$ converges to x_0^2 by Limit Theorems. Thus, $(f(x_n))$ is a Cauchy sequence. However, x^2 is not uniformly continuous on $\mathbb{R}$ since for $\varepsilon = 1$ and $|x-y| < \delta$, we have $|x^2 - y^2| = |x-y||x+y| > 1$ for $|x+y|$ sufficiently large. Thus, f is not uniformly continuous on $\mathbb{R}$ $\square$

Question 4

Consider an arbitrary function $f : \mathbb{N} \to \mathbb{R}$, is f necessarily continuous on $\mathbb{N}$?

Answer.
I claim that $\boxed{f \text{ must be continuous}}$.

Proof.
I will use the ε–δ definition of continuity. Let $x_0 \in \mathbb{N}$ be arbitrary, and fix $\varepsilon > 0$. Then, if we choose $\delta = 1$, we obtain $|x - x_0| < \delta = 1$. However, since x, x_0 must both be in the domain of f, they must both be natural numbers. Furthermore, the only way for the difference of two natural numbers to be less than 1 is if they are the same number. Thus, $|f(x) - f(x_0)| = |f(x_0) - f(x_0)| = 0 < \varepsilon$. Therefore, $|x - x_0| < 1$ implies $|f(x) - f(x_0)| < \varepsilon$ which means f is continuous. $\square$

Question 5

Given a continuous function $f : \mathbb{N} \to \mathbb{R}$, is f necessarily uniformly continuous on $\mathbb{N}$?

Answer.
I claim that $\boxed{f \text{ must be uniformly continuous}}$.

Proof.
Using the definition of uniform continuity, for $x, y \in \mathbb{N}$ if we choose $\delta = 1$, then we obtain $|x - y| < 1$. However, as discussed in the last problem, this is only possible when $x = y$. Thus, given some fixed $\varepsilon > 0$ and $|x - y| < 1$ we can conclude that $|f(x) - f(y)| = |f(x) - f(x)| = 0 < \varepsilon$. Therefore, f is uniformly continuous. □

Question 6

Let us call a set $B \subset \mathbb{R}$ a *Schumann set* if every sequence $(x_n) \subset B$ has a subsequence that converges to some point in B. Is the set $[0,1] \cup [3,5]$ a Schumann set?

Answer.
I claim that $\boxed{[0,1] \cup [3,5] \text{ is a Schumann set}}$.

Proof.
Let's say $S := [0,1] \cup [3,5]$. First, I claim that any sequence $(x_n) \subset S$ must be bounded. This is clear to see since $|x_n| \leq 5$ and for all $n \in \mathbb{N}$ for any sequence $(x_n) \subset S$. Thus, the Bolzano-Weierstrass Theorem guarantees that every sequence $(x_n) \subset S$ has a convergent subsequence. I will fix some subsequence $(s_n) \subset (x_n) \subset S$ and I claim that $\lim s_n \in S$. This is clear to see by contradiction.

If $\lim s_n < 0$ then we have a contradiction because this tells us that for some some N, $s_n < 0$ for all $n > N$ which contradicts $(s_n) \subset S$.

Likewise, if $1 < \lim s_n < 3$ we get a similar contradiction because this tells us that for some N, $1 < s_n < 3$ for all $n > N$ which again contradicts $(s_n) \subset S$.

Lastly, if $\lim s_n > 5$ we know that for some N, $s_n > 5$ for all $n > N$ which contradicts $(s_n) \subset S$.

Therefore, we must have that $\lim s_n \in S$ which proves the result. $\square$

Question 7

Fix a Schumann set $A \subset \mathbb{R}$ and let $f : A \to \mathbb{R}$ be a continuous function. Is it true that the image of A under f, that is, the set $f(A) := \{y \in \mathbb{R} : y = f(x) \text{ for some } x \in A\}$ is necessarily a Schumann set?

Proof.
Let (x_n) be some sequence in A with a subsequence (s_n) that converges to $s_0 \in A$ (this is possible since A is a Schumann set). Then since $x_n \in A$ for all $n \in \mathbb{N}$, we know that $s_n \in A$ for all n which means $f(x_n) \in f(A)$ for all n and $f(s_n) \in f(A)$ for all n. Thus, for every sequence $(x_n) \subset A$ with a converging subsequence (s_n) we have a corresponding sequence $(f(x_n))$ and its subsequence $(f(s_n))$ both in $f(A)$. Thus, by the continuity of f, we know that $\lim_n f(s_n) = f(s_0)$ and since $s_0 \in A$, $f(s_0) \in f(A)$ which means that $f(A)$ is a Schumann set. $\square$

Question 8

Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}}$ converge?

Answer.
This series $\boxed{\text{does not converge}}$

Proof.
First note the following:

$$\begin{aligned}
\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}} &= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} \\
&= \frac{(n+1)+\sqrt{n}\sqrt{n+1}-\sqrt{n}\sqrt{n+1}-(n)}{\sqrt{n+2}(\sqrt{n+1}+\sqrt{n})} \\
&= \frac{1}{\sqrt{n^2+3n+2}+\sqrt{n^2+2n}} \\
&> \frac{1}{\sqrt{n^2}+\sqrt{n^2}} \\
&= \frac{1}{2n}
\end{aligned}$$

However, we know that $\sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$ diverges by Theorem 15.1. Thus, by the Comparison Test, our original series also diverges. $\square$