Complex Analysis Homework 5

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Question 3

Evaluate $\displaystyle\int_\gamma f(z)dz$ when:

(a) $f(z)$ is the principal branch

$$z^i = \exp(i\mathrm{Log}(z)), \quad |z| > 0,\ -\pi < \mathrm{Arg}(z) < \pi$$

of the power function z^i and $\gamma(t) = e^{it}$ for $t \in [0, \pi]$.

(b) $f(z)$ is the principal branch

$$z^{a-1} = \exp[(a-1)\mathrm{Log}(z)], \quad |z| > 0,\ -\pi < \mathrm{Arg}(z) < \pi$$

of the function z^{a-1}, where $a \neq 0$ a real number, and γ is the positively oriented circle of radius $R > 0$ about the origin.

Answer. (a)
I will first use the definition of the path integral:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^\pi f(\gamma(t))\gamma'(t)dt \\
&= \int_0^\pi \left(e^{it}\right)^i (ie^{it})dt \\
&= i\int_0^\pi e^{-t}e^{it}dt \\
&= i\int_0^\pi e^{(-1+i)t}dt \\
&= \frac{i}{-1+i}e^{(-1+i)t}\Big|_{t=0}^{t=\pi} \\
&= \frac{i(-1-i)}{(-1+i)(-1-i)}\left[e^{(-1+i)\pi} - e^{(-1+i)0}\right] \\
&= \frac{1-i}{2}\left[e^{-\pi}e^{i\pi} - 1\right] \\
&= \frac{1-i}{2}(-e^{-\pi} - 1) \\
&= \frac{-1+i}{2}(e^{-\pi} + 1)
\end{aligned}$$

Just to be safe, I will calculate it using antiderivatives as well. Note that $\dfrac{d}{dz}\left(\dfrac{1}{1+i}z^{1+i}\right) = z^i$. However, I will need to change the definition of $f(z)$ to use a different branch of the logarithm function. I will use the branch with

$-\frac{\pi}{2} < \operatorname{Arg}(z) < \frac{3\pi}{2}$. Thus,

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_1^{-1} z^i dz \\
&= \left(\frac{1}{1+i} z^{1+i}\right)\Big|_{z=1}^{z=-1} \\
&= \frac{1}{1+i}\left((-1)^{1+i} - 1^{1+i}\right) \\
&= \frac{1}{1+i}\left(\exp((1+i)\log(-1)) - \exp((1+i)\log(1))\right) \\
&= \frac{1}{1+i}\left(\exp((1+i)(\ln(1)+i\pi)) - \exp((1+i)(\ln(1) - i\cdot 0))\right) \\
&= \frac{1}{1+i}\left(e^{i\pi}e^{-\pi} - e^0\right) \\
&= \frac{1-i}{(1+i)(1-i)}\left(-e^{-\pi} - 1\right) \\
&= \frac{-1+i}{2}(e^{-\pi}+1)
\end{aligned}$$

I got the same answer, so I have verified that $\boxed{\int_\gamma f(z)dz = \frac{-1+i}{2}(e^{-\pi}+1)}$

Answer. (b)
I will first use the definition of the path integral with $\gamma(t) = Re^{it},\ t \in [-\pi, \pi]$:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_{-\pi}^{\pi} f(\gamma(t))\gamma'(t)dt \\
&= \int_{-\pi}^{\pi} (Re^{it})^{a-1}(iRe^{it})dt \\
&= iR^a \int_{-\pi}^{\pi} e^{(a-1)it}e^{it}dt \\
&= iR^a \int_{-\pi}^{\pi} e^{ait}dt \\
&= \frac{iR^a}{ai} e^{ait}\Big|_{t=-\pi}^{t=\pi} \\
&= \frac{R^a}{a}\left(e^{\pi ai} - e^{-\pi ai}\right) \\
&= \frac{R^a}{a}\left(\cos(\pi a) + i\sin(\pi a) - \cos(-\pi a) - i\sin(-\pi a)\right) \\
&= \frac{R^a}{a} 2i\sin(\pi a)
\end{aligned}$$

I will check this answer by using antiderivatives as well. Note that $\frac{d}{dz}\left(\frac{1}{a}z^a\right) = z^{a-1}$ for $a \neq 0$. To do this, I will need to consider two separate branch cuts for two different segments of γ. Let $\gamma_1 = Re^{it},\ t \in [-\pi, 0]$, the bottom side of the radius R circle. For this, I will use the branch cut $\log(z) = \ln(|z|) + i\operatorname{Arg}(z)$, $-\frac{3\pi}{2} < \operatorname{Arg}(z) < \frac{\pi}{2}$:

$$\begin{aligned}
\int_{\gamma_1} f(z)dz &= \int_{-R}^{R} z^{a-1}dz \\
&= \frac{1}{a}z^a\Big|_{z=-R}^{z=R} \\
&= \frac{1}{a}\left((R)^a - (-R)^a\right) \\
&= \frac{1}{a}\left(e^{a\operatorname{Log}(R)} - e^{a\operatorname{Log}(-R)}\right) \\
&= \frac{1}{a}\left(e^{a(\ln(R)+i\cdot 0)} - e^{a(\ln(R)-\pi i}\right) \\
&= \frac{1}{a}e^{a\ln(R)}\left(1 - e^{-\pi ai}\right) \\
&= \frac{R^a}{a}e^{-\pi ai}\left(e^{\pi ai} - 1\right)
\end{aligned}$$

Next, I will define $\gamma_2 = Re^{it},\ t \in [0, \pi]$. For this, I will use the branch cut for $\log(z) = \ln(|z|) + i\text{Arg}(z),\ -\frac{\pi}{2} < \text{Arg}(z) < \frac{3\pi}{2}$ and find that:

$$\begin{aligned}
\int_{\gamma_2} f(z)dz &= \int_R^{-R} z^{a-1}dz \\
&= \frac{1}{a}z^a\Big|_{z=R}^{z=-R} \\
&= \frac{1}{a}\left((-R)^a - (R)^a\right) \\
&= \frac{1}{a}\left(e^{a\text{Log}(-R)} - e^{a\text{Log}(R)}\right) \\
&= \frac{1}{a}\left(e^{a(\ln(R)+\pi i)} - e^{a(\ln(R)+i\cdot 0)}\right) \\
&= \frac{1}{a}e^{a\ln(R)}\left(e^{\pi a i} - 1\right) \\
&= \frac{R^a}{a}\left(e^{\pi a i} - 1\right)
\end{aligned}$$

Therefore, we can calculate the total integral as:

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz \\
&= \left[\frac{R^a}{a}e^{-\pi a i}\left(e^{\pi a i} - 1\right)\right] + \left[\frac{R^a}{a}\left(e^{\pi a i} - 1\right)\right] \\
&= \frac{R^a}{a}\left(e^{\pi a i} - 1\right)\left(e^{-\pi a i} + 1\right) \\
&= \frac{R^a}{a}\left(e^{\pi a i} - e^{-\pi a i}\right) \\
&= \frac{R^a}{a}2i\sin(\pi a)
\end{aligned}$$

Thus, the two answers are the same, so $\boxed{\int_\gamma f(z)dz = \frac{R^a}{a}2i\sin(\pi a)}$