Complex Analysis Homework 8

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Question 1

Let $\gamma(t) = 2e^{it}, t \in [0, 2\pi]$. Evaluate the following integrals:

(a) $\displaystyle\int_\gamma \frac{z^3+5}{z-i}\,dz$

- We cannot use Cauchy's Integral Theorem because this function is discontinuous at $z = i \in D(0,2)$. However, we can use Cauchy's Integral Formula, which tells us that

$$f(z_0) = \frac{1}{2\pi i}\int_{\gamma(a,r)} \frac{f(z)}{z-z_0}\,dz \qquad \text{for all } z_0 \in D(a,r)$$

- Using this with $f(z) = z^3 + 5$ and $z_0 = i$, we get

$$\begin{aligned}\int_\gamma \frac{z^3+5}{z-i}\,dz &= 2\pi i(i^3+5)\\ &= \boxed{2\pi + 10\pi i}\end{aligned}$$

(b) $\displaystyle\int_\gamma \frac{\sin(z)}{z^2+1}\,dz$

- Recall that

$$\frac{1}{z^2+1} = \left(\frac{1}{z-i} - \frac{1}{z+i}\right)\cdot\frac{1}{2i}$$

- by using partial fractions. Thus, the integral becomes

$$\begin{aligned}\int_\gamma \frac{\sin(z)}{z^2+1}\,dz &= \int_\gamma \sin(z)\left(\frac{1}{z-i} - \frac{1}{z+i}\right)\cdot\frac{1}{2i}\,dz\\ &= \frac{1}{2i}\left(\int_\gamma \frac{\sin(z)}{z-i}\,dz - \int_\gamma \frac{\sin(z)}{z+i}\,dz\right)\end{aligned}$$

- I can evaluate each of these integrals separately by using Cauchy's Integral Formula as stated in the previous part of this question with $f(z) = \sin(z)$, $z_0 = i$ in the first integral, and $z_0 = -i$ in the second integral. Thus, we get

$$\begin{aligned}\int_\gamma \frac{\sin(z)}{z^2+1}\,dz &= \frac{1}{2i}\left[2\pi i(\sin(i)) - 2\pi i(\sin(-i))\right]\\ &= \pi[\sin(i) - \sin(-i)]\\ &= \pi\left[\frac{e^{i\cdot i} - e^{-i\cdot i}}{2i} - \frac{e^{i\cdot(-i)} - e^{-i\cdot(-i)}}{2i}\right] && \text{by the definition of the complex sine function}\\ &= \pi\left[\frac{e^{-1}-e}{2i} - \frac{e-e^{-1}}{2i}\right]\\ &= \frac{\pi}{i}\left[e^{-1}-e\right]\\ &= \boxed{\pi i\left[e-e^{-1}\right]}\end{aligned}$$