Complex Analysis Homework 8

Colin Williams

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Question 2

Let $\gamma(0,r)$ be the circle centered at 0 with radius r, taken counter-clockwise.

(a) Let $a, b \in \mathbb{C}$ with $|a|, |b| \neq 1$. Evaluate the following and distinguish different cases:

$$\int_{\gamma(0,1)} \left(\frac{z-b}{z-a}\right)^2 \, dz.$$

- The first case I will consider is when $a = b$.
- From this, we can simply use the definition of the path integral to get

$$\begin{aligned}
\int_{\gamma(0,1)} \left(\frac{z-b}{z-a}\right)^2 \, dz &= \int_{\gamma(0,1)} \left(\frac{z-a}{z-a}\right)^2 \, dz \\
&= \int_{\gamma(0,1)} 1^2 \, dz \\
&= \int_0^{2\pi} ie^{it} \, dt \\
&= \frac{i}{i} e^{it} \Big|_{t=0}^{t=2\pi} \\
&= e^{2\pi i} - e^0 = 1 - 1 = 0.
\end{aligned}$$

- The next case I will consider is when $|a| < 1$ and $a \neq b$.
- In this case, we have $a \in D(0,1)$ and we can recall Cauchy's Integral Formula (for derivatives) that states:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} \, dz \qquad \text{for } z_0 \in D(c,r),\ \gamma(t) = c + re^{it},\ f \in H(\Omega),\ \overline{D}(c,r) \subset \Omega$$

- Therefore, by splitting up the square into the numerator and denominator of the fraction separately, we can see that our "n" is equal to 1, our "z_0" is equal to a, and our "$f(z)$" is equal to $(z-b)^2$ which is holomorphic everywhere. Thus, using this formula, we get:

$$\begin{aligned}
\int_{\gamma(0,1)} \frac{(z-b)^2}{(z-a)^2} \, dz &= \frac{2\pi i}{1!} f'(a) \\
&= 2\pi i \left[(z-b)^2\right]'_{z=a} \\
&= 4\pi i(a-b)
\end{aligned}$$

- The last case I will consider is when $|a| > 1$ and $a \neq b$.
- In this case, consider the open convex set $\Omega = D(0, |a| - \varepsilon)$ where $\varepsilon > 0$ is fixed and chosen small enough such that $|a| - \varepsilon > 1$. Thus, with $f(z) := \left(\frac{z-b}{z-a}\right)^2$, we know that $f(z)$ is holomorphic in Ω as $a \notin \Omega$ and we know that $\gamma(0,1)^* \subset \Omega$. Thus, by Cauchy's Integral Theorem for Convex Sets, we can conclude that

$$\int_{\gamma(0,1)} \left(\frac{z-b}{z-a}\right)^2 \, dz = 0$$

- Finally, notice that our case of $a = b$ was not a special case, as we get the same result of 0 in either of the previous formulas. Thus, we can summarize as:

$$\boxed{\int_{\gamma(0,1)} \left(\frac{z-b}{z-a}\right)^2 \, dz = \begin{cases} 4\pi i(a-b) & \text{when } |a| < 1 \\ 0 & \text{when } |a| > 1 \end{cases}}$$

(b) Let n be a positive integer and evaluate

$$\int_{\gamma(0,2)} z^{-n} \cos(z)\, dz.$$

- Once again, recall Cauchy's Integral Formula (for derivatives) that states:

$$f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\, dz \qquad \text{for } z_0 \in D(c,r),\ \gamma(t) = c + re^{it},\ f \in H(\Omega),\ \overline{D}(c,r) \subset \Omega$$

- Thus, in our case, we can see that our "k" is equal to $n-1$, our "z_0" is equal to 0, and our "$f(z)$" is equal to $\cos(z)$. Thus, we can calculate our integral as:

$$\int_{\gamma(0,2)} \frac{\cos(z)}{z^n}\, dz = \frac{2\pi i}{(n-1)!} f^{(n-1)}(0)$$

- Note that for the derivatives of $f(z) = \cos(z)$ we have the following relations:

$$f^{(k)}(z) = \begin{cases} \cos(z) & \text{if } k = 4m \\ -\sin(z) & \text{if } k = 4m+1 \\ -\cos(z) & \text{if } k = 4m+2 \\ \sin(z) & \text{if } k = 4m+3 \end{cases} \qquad \text{for } m \in \mathbb{N}_0$$

- Furthermore, note that $\cos(0) = 1$ and $\sin(0) = 0$, so our given integral is zero whenever $n-1$ is odd (i.e. when n is even). Additionally, if $n-1$ is even (i.e. when n is odd of the form $2p+1$ for $p \in \mathbb{N}_0$), then we get $f^{(n-1)}(0) = f^{(2p)}(0) = (-1)^p$. Thus, we can summarize our integral as

$$\boxed{\int_{\gamma(0,2)} z^{-n} \cos(z)\, dz = \begin{cases} \dfrac{2\pi i(-1)^p}{(2p)!} & \text{whenever } n \text{ is odd of the form } n = 2p+1, p \in \mathbb{N}_0 \\ 0 & \text{whenever } n \text{ is even} \end{cases}}$$