Complex Analysis Homework 7

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Question 2

Let $p(z)$ be a polynomial of degree $k > 0$. Prove that the series $\sum_{n=0}^{\infty} p(n)z^n$ has a radius of convergence equal to 1 and that there exists a polynomial $q(z)$ of degree k such that

$$\sum_{n=0}^{\infty} p(n)z^n = q(z)(1-z)^{-(k+1)}, \quad |z| < 1.$$

Proof. Recall that the radius of convergence of a power series $\sum a_n z^n$ is $R = \dfrac{1}{\limsup \sqrt[n]{|a_n|}}$. Also, note that if the limit $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists, then it is equal to $\limsup \sqrt[n]{|a_n|}$. Thus, if the limit exists, then $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$. In our case here, $a_n = p(n)$. Since $p(n)$ is a polynomial of degree k say that $p(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k$. Therefore, we have the following:

$$\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{p(n)}{p(n+1)}\right| \\
&= \lim_{n\to\infty}\left|\frac{a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k}{a_0 + a_1(n+1) + a_2(n+1)^2 + \cdots + a_k(n+1)^k}\right| \\
&= \lim_{n\to\infty}\left|\frac{1/n^k}{1/n^k}\cdot\frac{a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k}{a_0 + a_1(n+1) + a_2(n+1)^2 + \cdots + a_k(n+1)^k}\right| \\
&= \lim_{n\to\infty}\left|\frac{a_0 n^{-k} + a_1 n^{1-k} + a_2 n^{2-k} + \cdots + a_k}{a_0 n^{-k} + a_1\cdot\frac{n+1}{n^k} + a_2\cdot\frac{(n+1)^2}{n^k} + \cdots + a_k\left(\frac{n+1}{n}\right)^k}\right| \\
&= \frac{\lim_{n\to\infty}\left|a_0 n^{-k} + a_1 n^{1-k} + a_2 n^{2-k} + \cdots + a_k\right|}{\lim_{n\to\infty}\left|a_0 n^{-k} + a_1\cdot\dfrac{n+1}{n^k} + a_2\cdot\dfrac{(n+1)^2}{n^k} + \cdots + a_k\left(\dfrac{n+1}{n}\right)^k\right|} \\
&= \frac{|a_0(0) + a_1(0) + a_2(0) + \cdots + a_k|}{|a_0(0) + a_1(0) + a_2(0) + \cdots + a_k(1)^k|} \\
&= \frac{|a_k|}{|a_k|} = 1
\end{aligned}$$

This finishes the proof that the radius of convergence is 1. I will now use an inductive proof on k to show the existence of the polynomial q. However, before I begin, I will make the following remarks

Lemma 1. For $k \in \mathbb{N}$, we have:

(a) For $f(z) = \dfrac{1}{1-z}$, we have that $f^{(k)}(z) = \dfrac{k!}{(1-z)^{k+1}}$.

(b) For $f(z) = z^n$, we have that $f^{(k)}(z) = n(n-1)(n-2)\cdots(n-(k-1))z^{n-k}$

Proof.
(a) is easily shown for $k = 1$ and then inductively shown for $k+1$ by differentiating the last equation to get $f^{(k+1)} = \dfrac{(k+1)!}{(1-z)^{k+2}}$. (b) is also very easy to see for $k = 1$ and then for $k+1$ we have $f^{(k+1)}(z) = n(n-1)(n-2)\cdots(n-(k-1))(n-k)z^{n-(k+1)}$. This proves both of these results for all $k \in \mathbb{N}$. Note in (b) if $k > n$, then at some point in our product, we multiply by $(n-n) = 0$ which makes the whole function zero. This is fine, however, since any derivative of a polynomial of higher order than the polynomial itself, must be zero. $\square$

Lemma 2.
For any $k \in \mathbb{N}$, we can rewrite n^k as $n^k = n(n-1)(n-2)\cdots(n-(k-1)) + b(n)$ for $b(n)$ some polynomial of order $k-1$.

Proof.
I will again show this inductively. It is obvious for $k=1$, so I will start with $k=2$ to show $n^k = n^2 = n(n-1)+n$. Thus, $b(n) = n$ is a polynomial of order $k-1 = 1$. For $k+1$ we have $n^{k+1} = (n-k)(n^k)+kn^k = (n-k)[n(n-1)(n-2)\cdots(n-(k-1))+\tilde{b}(n)]+kn^k$ by the inductive Hypothesis. In turn this is equal to $n(n-1)(n-2)\cdots(n-(k-1))(n-k)+\tilde{b}(n)(n-k)+kn^k$ where $b(n) = \tilde{b}(n)(n-k)+kn^k$ is our polynomial of degree $(k+1)-1 = k$ since $\tilde{b}(n)$ was a polynomial of degree $k-1$ by the inductive hypothesis. Thus, we have shown this lemma to be true for all $k \in \mathbb{N}$. □

I will now proceed with the inductive proof for the existence of the polynomial q:
<u>Base Case: $k = 1$</u>
For $k=1$, we have the following:

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(n)z^n &= \sum_{n=0}^{\infty}(a_0 + a_1 n)z^n \\
&= a_0 \sum_{n=0}^{\infty} z^n + a_1 \sum_{n=0}^{\infty} nz^n \\
&= a_0 \sum_{n=0}^{\infty} z^n + a_1 z \sum_{n=0}^{\infty} nz^{n-1} \\
&= a_0 \sum_{n=0}^{\infty} z^n + a_1 z \sum_{n=0}^{\infty} \frac{d}{dz}(z^n) && \text{by Lemma 1} \\
&= a_0 \sum_{n=0}^{\infty} z^n + a_1 z \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n\right) && \text{by Lemma 1(b)} \\
&= a_0 \left(\frac{1}{1-z}\right) + a_1 z \frac{d}{dz}\left(\frac{1}{1-z}\right) \\
&= \left(\frac{a_0}{1-z}\right) + a_1 z \left(\frac{1}{(1-z)^2}\right) && \text{by Lemma 1(a)} \\
&= \frac{a_0(1-z) + a_1 z}{(1-z)^2} \\
&= \frac{(a_1 - a_0)z + a_0}{(1-z)^2}
\end{aligned}
$$

Thus, we have shown that for $k=1$, our polynomial q of order 1 is equal to $(a_1 - a_0)z + a_0$.
<u>Inductive step: Assume true for some natural number k</u>
We are now assuming the statement is true for some natural number k and we will be examining the statement for $k+1$. I will first bring in the following notation, any function written below in the form $t_m(n)$ will be a polynomial in n of order m. For example, $s_7(w)$ is a polynomial of order 7 in the variable w. Therefore, we now have the following:

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(n)z^n &= \sum_{n=0}^{\infty}(a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k + a_{k+1}n^{k+1})z^n \\
&= \sum_{n=0}^{\infty}(a_0 + a_1 n + a_2 n^2 + \cdots + a_k n^k)z^n + \sum_{n=0}^{\infty} a_{k+1}n^{k+1}z^n \\
&= q_k(z)\frac{1}{(1-z)^{k+1}} + a_{k+1}\sum_{n=0}^{\infty} n^{k+1}z^n && \text{by Inductive Hypothesis} \\
&= q_k(z)\frac{1}{(1-z)^{k+1}} + a_{k+1}\sum_{n=0}^{\infty}[n(n-1)(n-2)\cdots(n-(k-1))(n-k) + b_k(n)]z^n && \text{by Lemma 2} \\
&= q_k(z)\frac{1}{(1-z)^{k+1}} + a_{k+1}\sum_{n=0}^{\infty}[n(n-1)(n-2)\cdots(n-k)]z^n + a_{k+1}\sum_{n=0}^{\infty} b_k(n)z^n \\
&= \frac{q_k(z)}{(1-z)^{k+1}} + \frac{a_{k+1}\tilde{q}_k(z)}{(1-z)^{k+1}} + a_{k+1}z^{k+1}\sum_{n=0}^{\infty}[n(n-1)(n-2)\cdots(n-k)]z^{n-(k+1)} && \text{by the Inductive Hypothesis}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{q_k(z) + a_{k+1}\tilde{q_k}(z)}{(1-z)^{k+1}} + a_{k+1}z^{k+1}\sum_{n=0}^{\infty}\frac{d^{k+1}}{dz^{k+1}}(z^n) && \text{by Lemma 1(b)}\\
&= \frac{q_k(z) + a_{k+1}\tilde{q_k}(z)}{(1-z)^{k+1}} + a_{k+1}z^{k+1}\frac{d^{k+1}}{dz^{k+1}}\left(\sum_{n=0}^{\infty} z^n\right)\\
&= \frac{q_k(z) + a_{k+1}\tilde{q_k}(z)}{(1-z)^{k+1}} + a_{k+1}z^{k+1}\frac{d^{k+1}}{dz^{k+1}}\left(\frac{1}{1-z}\right)\\
&= \frac{q_k(z) + a_{k+1}\tilde{q_k}(z)}{(1-z)^{k+1}} + a_{k+1}z^{k+1}\left(\frac{(k+1)!}{(1-z)^{k+2}}\right) && \text{by Lemma 1(a)}\\
&= \frac{[q_k(z) + a_{k+1}\tilde{q_k}(z)][1-z] + a_{k+1}z^{k+1}(k+1)!}{(1-z)^{k+2}}
\end{aligned}
$$

Thus, we can see our q exists for the case of $k+1$ as well. In fact, we have that $q_{k+1} = [q_k(z) + a_{k+1}\tilde{q_k}(z)][1-z] + a_{k+1}z^{k+1}(k+1)!$. Where the q_k and $\tilde{q_k}$ came from the inductive hypothesis of the existence of q for a lower degree problem. Thus, by the principle of mathematical induction, we have shown that q exists for all values of $k \in \mathbb{N}$ which means the statement is true for all polynomials p. Also note that in this process, I used only the geometric series $\displaystyle\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and its derivatives which all converge for all $|z| < 1$. Thus, this would have been another approach to prove that the radius of convergence for the initial series is also equal to 1. $\square$