Applied Math HW 4

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Question 1

Let u and v be vectors in $\mathbb{R}^n$, and let A and B be two $n \times n$ matrices.

(a) Find the number of operations (multiplications and divisions) required to compute the scalar product, the norm $||u||_2$ and the rank-one matrix uv^T.

- Let $u = (u_1, u_2, \dots, u_n)$ and $v = (v_1, v_2, \dots, v_n)$. Then, the scalar product is calculated at

$$\langle u, v \rangle = u^T v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Since we are only counting multiplications and divisions, we can clearly see from above that this has one multiplication for each index of the vectors, therefore $\boxed{N_{op}(n) = n.}$

- Recall that $||u||_2 = \sqrt{\langle u, u \rangle}$. We saw from before that the inner product takes n multiplications to complete. If we consider the square root function as an operation, we have $N_{op}(n) = n + 1$, but if we only consider multiplications and divisions, then $\boxed{N_{op}(n) = n.}$

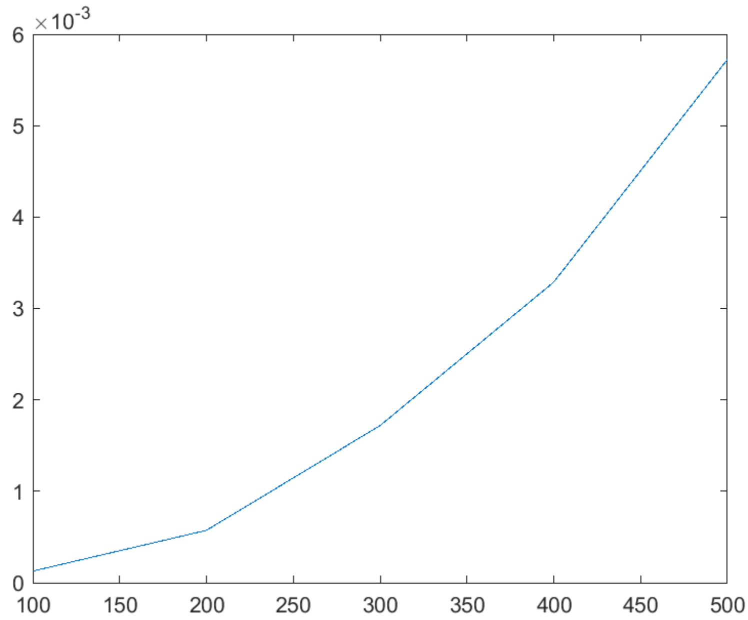
- Notice, the multiplication uv^T is expressed as

$$uv^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 & u_1 v_2 & \cdots & u_1 v_n \\ u_2 v_1 & u_2 v_2 & \cdots & u_2 v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & \cdots & u_n v_n \end{bmatrix}$$

Thus, notice each entry of the resulting $n \times n$ matrix has one multiplication. Therefore, there there are clearly $\boxed{N_{op}(n) = n^2}$ operations involved.

(b) For $n = 100k$ with $k = 1, 2, \dots, 5$, estimate the running time of Matlab (using `tic` and `toc`) for computing the product of two matrices $A =$ `rand(n, n)` and $B =$ `rand(n, n)`. Plot this running time in terms of n.

- Below is my graph of the running time in terms of n. The size of the matrix n is the horizontal axis and the running time in seconds to compute AB is the vertical axis. To prevent inconsistencies from the randomness of the matrices, I did 100 matrix multiplications for each value of n and averaged the computation times. I then took this average running time for a fixed n to be my representative value for the plot which gave much smoother results. Additionally, I found that running my `for` loop gave better results when starting at $k = 5$ and working backwards to $k = 1$ (perhaps because of MATLAB needing to load certain packages on the first use of `tic` and `toc`; thus, a smaller relative error when the first usage is on the longer test-case).



- The values that are represented in this graph are

n	time
100	1.2685×10^{-4}
200	5.7295×10^{-4}
300	1.7192×10^{-3}
400	3.2887×10^{-3}
500	5.7249×10^{-3}

- My code to produce this plot is as follows:

```
N = 100; %Number of times to average each test case
M = 5; %Max value for k
time = zeros(1, M);
n = 100*(1:M);
y = zeros(M, N);

%for k = 1:M
for k = M:-1:1
    C = zeros(n(k), n(k));
    for j = 1:N
        A = rand(n(k), n(k));
        B = rand(n(k), n(k));
        tic;
        C = A*B;
        y(k, j) = toc;
    end
    time(k) = mean(y(k, :));
end


figure;
plot(n, time);
```

(c) Assume that this running time is a polynomial of n, so that for n large enough, $T(n) \approx Cn^s$. In order to find a numerical approximation of the exponent s, plot the logarithm of T in terms of the logarithm of n. Deduce an approximate value of s.

- We have seen in class that the standard algorithm for matrix multiplication takes $O(n^3)$ operations. MATLAB; however, may have a slightly more efficient algorithm, but we should expect it to be similar. Notice the following:

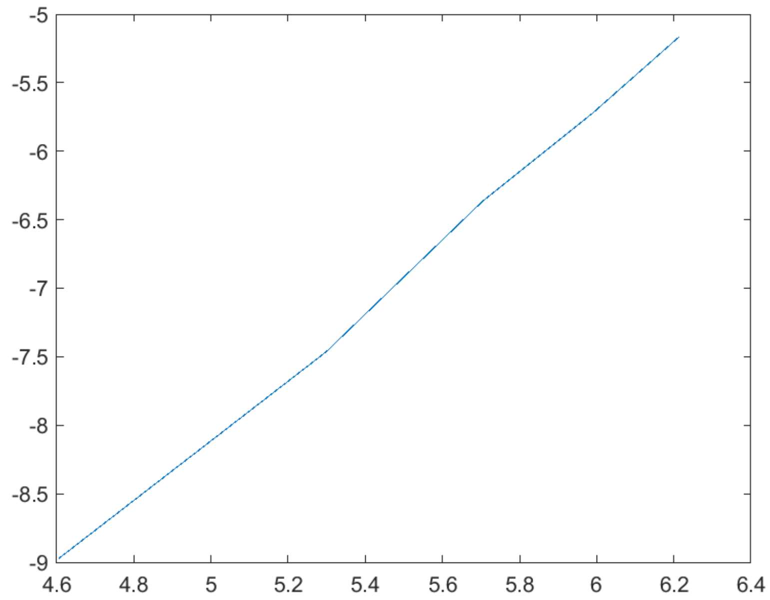
$$
\begin{aligned}
& T(n) \approx Cn^s \\
\implies & \log(T(n)) \approx \log(Cn^s) \\
\implies & \log(T(n)) \approx s\log(n) + \log(C)
\end{aligned}
$$

Therefore, when graphing the logarithm of T versus the logarithm of n, we should expect our s to be the slope of the log-log plot. With this in mind, I appended the following code to the end of the code in the previous section:

```
figure;
w = log(n);
z = log(time);
plot(w, z)

p = polyfit(w, z, 1);
slope = p(1);
```

- The first 4 lines of code produced the following graph:



- This graph has points at the following values:

n	$\ln(n)$	$\ln(\text{time})$
100	4.6052	-8.9725
200	5.2983	-7.4647
300	5.7038	-6.3659
400	5.9915	-5.7173
500	6.2146	-5.1629

- The last two lines of code I added create a linear approximation of this log-log plot and then return the slope of the that line into the variable named `slope`. The output of this was 2.3824 which indicates that $\boxed{s \approx 2.3824}$. This approximation may not be completely accurate due the randomness involved and the fact we are only using 5 test cases with a maximum n of 500. More test cases and a larger set of n's would likely make this estimate more accurate.

Question 2

Let A be an invertible $n \times n$ matrix. Prove the following properties:

(a) $\text{cond}(\alpha A) = \text{cond}(A)$ for all nonzero α.

- Let $\alpha \neq 0$.
- In order to prove the given statement, I will first prove the following property that if A is invertible, then

$$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$$

To show this, simply note that

$$\begin{aligned}(\alpha A)\left(\frac{1}{\alpha}A^{-1}\right) &= \frac{\alpha}{\alpha}AA^{-1} = I \\ \left(\frac{1}{\alpha}A^{-1}\right)(\alpha A) &= \frac{\alpha}{\alpha}A^{-1}A = I\end{aligned}$$

- Therefore, we can compute the condition number:

$$\text{cond}(\alpha A) = ||\alpha A||\ ||(\alpha A)^{-1}|| = |\alpha|\ ||A||\ \left|\left|\frac{1}{\alpha}A^{-1}\right|\right| = \frac{|\alpha|}{|\alpha|}||A||\ ||A^{-1}|| = \text{cond}(A)$$

(b) $\text{cond}_2(AU) = \text{cond}_2(UA) = \text{cond}_2(A)$ for any orthogonal matrix U.

- Recall from HW2, we have proven that $||QAZ||_2 = ||A||_2$ for Q and Z orthogonal matrices. In particular, by taking $Q = I$ and $Z = U$, we get $||AU||_2 = ||A||$ and by taking $Q = U$ and $Z = I$, we get $||UA||_2 = ||A||$. Note, we can also replace A with A^{-1} and U with $U^{-1} = U^T$ (which must be also be orthogonal) and these equalities will still hold true. Therefore, we can compute the condition numbers as:

$$\begin{aligned}\text{cond}_2(AU) &= ||AU||_2||(AU)^{-1}||_2 = ||A||_2||U^TA^{-1}||_2 = ||A||_2||A^{-1}||_2 = \text{cond}_2(A) \\ \text{cond}_2(UA) &= ||UA||_2||(UA)^{-1}||_2 = ||A||_2||A^{-1}U^T||_2 = ||A||_2||A^{-1}||_2 = \text{cond}_2(A)\end{aligned}$$

(c) $n^{-2}\text{cond}_1(A) \leq \text{cond}_\infty(A) \leq n^2\ \text{cond}_1(A)$.

- Recall the following property about matrix norms: $\frac{1}{n}||A||_\infty \leq ||A||_1 \leq n||A||_\infty$. These can also be rephrased as $\frac{1}{n}||A||_1 \leq ||A||_\infty \leq n||A||_1$ which lead to:

$$\begin{aligned}\text{cond}_\infty(A) = ||A||_\infty||A^{-1}||_\infty &\leq \Big(n||A||_1\Big)\Big(n||A^{-1}||_1\Big) = n^2\text{cond}_1(A) \\ \text{cond}_\infty(A) = ||A||_\infty||A^{-1}||_\infty &\geq \left(\frac{1}{n}||A||_1\right)\left(\frac{1}{n}||A^{-1}||_1\right) = \frac{1}{n^2}\text{cond}_1(A)\end{aligned}$$

Putting these inequalities together clearly gives the stated property.

Question 3

Let A be an invertible $n \times n$ matrix and b be a nonzero vector in $\mathbb{R}^n$. Assume that x and $x + \Delta x$ solve

$$Ax = b, \qquad\qquad (A + \Delta A)(x + \Delta x) = b.$$

Prove that the inequality

$$\frac{||\Delta x||}{||x + \Delta x||} \leq \text{cond}(A)\frac{||\Delta A||}{||A||}$$

is optimal in the sense that there are $\Delta A \neq 0$ and $b \neq 0$ such that the equality happens.

Proof.
We have already shown that the inequality holds in class; therefore, I simply need to show that it is an optimal inequality.

By the definition of the operator norm, we know that there exists some vector $x_0 \neq 0 \in \mathbb{R}^n$ such that $||A^{-1}x_0|| = ||A^{-1}||\ ||x_0||$. Define $b = x_0$. So that $Ax = b \implies x = A^{-1}b \implies ||x|| = ||A^{-1}||\ ||b||$. With this in mind, define

$$\Delta A := \frac{b(x + \Delta x)^T}{||x + \Delta x||^2}$$

With this in mind, notice the following properties:

$$\Delta A(x + \Delta x) = b\frac{(x + \Delta x)^T(x + \Delta x)}{||x + \Delta x||^2} = b\frac{||x + \Delta x||^2}{||x + \Delta x||^2} = b \tag{0.1}$$

Furthermore, we have

$$\begin{aligned}||\Delta A|| = \frac{||b(x + \Delta x)^T||}{||x + \Delta x||^2} = \max_{||z||=1}\frac{||b(x + \Delta x)^Tz||}{||x + \Delta x||^2} &= \max_{||z||=1}\frac{|\langle x + \Delta x, z\rangle|\ ||b||}{||x + \Delta x||^2} = \frac{||b||}{||x + \Delta x||} \\ &\implies ||\Delta A||\ ||x + \Delta x|| = ||b||\end{aligned} \tag{0.2}$$

Notice the last equality in the first line of equalities above follows as an inequality with Cauchy-Schwartz and it is attained by taking $z = (x + \Delta x)/||x + \Delta x||$. Therefore, we get

$$\begin{aligned}
&(A + \Delta A)(x + \Delta x) = b \\
\implies &Ax + A\Delta x + \Delta A(x + \Delta x) = b \\
\implies &A\Delta x + \Delta A(x + \Delta x) = 0 && \text{since } Ax = b \\
\implies &-A^{-1}\Delta A(x + \Delta x) = \Delta x
\end{aligned}$$

Therefore, by taking norms of both sides of this equality, we get:

$$\begin{aligned}
||\Delta x|| &= ||A^{-1}\Delta A(x + \Delta x)|| \\
&= ||A^{-1}b|| && \text{by (0.1)} \\
&= ||A^{-1}||\,||b|| && \text{by construction of } b \\
&= ||A^{-1}||\,||\Delta A||\,||x + \Delta x|| && \text{by (0.2)} \\
\implies \frac{||\Delta x||}{||x + \Delta x||} &= ||A^{-1}||\,||\Delta A|| \\
&= ||A||\,||A^{-1}||\frac{||\Delta A||}{||A||} \\
&= \text{cond}(A)\frac{||\Delta A||}{||A||}
\end{aligned}$$

□

Notice I was being slightly deceiving in the preceding proof by using the fact that $y^T y = \langle y, y\rangle = ||y||^2$ which is only true for the vector 2-norm. To make this proof work for the vector p norm in general take

$$\Delta A = \frac{by^T}{||x + \Delta x||_p^2}$$

where y is chosen such that $y^T(x + \Delta x) = ||x + \Delta x||_p^2$. This y will vary depending on p, for example

- $p = 1$. Take $y = ||x + \Delta x||_1(\pm 1, \pm 1, \ldots, \pm 1)^T$ where you choose $+$ or $-$ to match the sign of $x + \Delta x$ in that coordinate.
- $p = \infty$. Take $y = \pm||x + \Delta x||_\infty e_k$ where k is chosen as the coordinate corresponding the largest absolute value of an entry of $x + \Delta x$ and $+$ or $-$ matches the sign of that element.