Complex Analysis Homework 2

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Question 2

Question.
Find all the roots in rectangular coordinates and show them on a graph.

a.) $(-1)^{1/3}$

b.) $8^{1/6}$.

Answer. Some points of note before we begin:

i.) First, notice you can write any $z \in \mathbb{C}$ in rectangular coordinates as $z = x + iy$ or in polar coordinates as $z = r(cos(\theta) + i\sin(\theta))$ for $r = |z|$, $x = r\cos(\theta)$, and $y = r\sin(\theta)$.

ii.) Also, note that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, so $z = re^{i\theta}$

iii.) Lastly, two nonzero complex numbers, $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$ are equal if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2\pi k$ for $k \in \mathbb{Z}$.

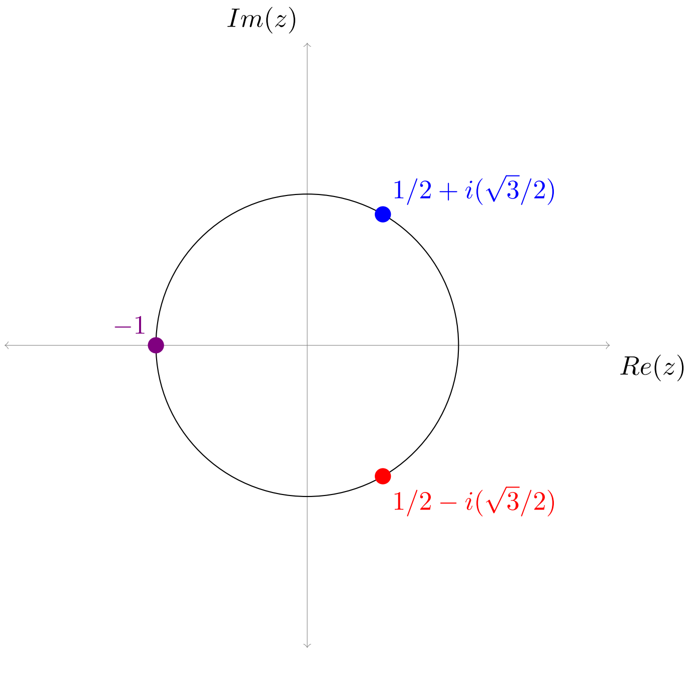
Answer. a.)
If $z_0 = -1$, then $r = |z_0| = \sqrt{(-1)^2 + 0^2} = 1$ and $\cos(\theta) = -1$ and $\sin(\theta) = 0 \implies \theta = -\pi$. Thus, $z_0 = e^{i(-\pi)}$.
We are looking for a $z = re^{i\theta}$ such that $z^3 = z_0$. Or,

$$r^3e^{i3\theta} = e^{i(-\pi)}$$

By *iii.)*, $r^3 = 1$ and $3\theta = -\pi + 2\pi k$.
$\implies r = 1$ and $\theta = -\pi/3 + (2\pi k)/3$.
Since θ is 2π periodic, we only have distinct values for z when $k \in \{0, 1, 2\}$.
Thus, our potential values for z are:

$$\begin{aligned} z &\in \{e^{i(-\pi/3)}, e^{i(-\pi/3+2\pi/3)}, e^{i(-\pi/3+4\pi/3)}\} \\ &= \{e^{i(-\pi/3)}, e^{i(\pi/3)}, e^{i(\pi)}\} \\ &= \{\cos(-\pi/3) + i\sin(-\pi/3), \cos(\pi/3) + i\sin(\pi/3), \cos(\pi) + i\sin(\pi)\} \\ &= \{1/2 + i(-\sqrt{3}/2), 1/2 + i(\sqrt{3}/2), -1 + i(0)\} \\ &= \boxed{\{1/2 - i(\sqrt{3}/2), 1/2 + i(\sqrt{3}/2), -1\}} \end{aligned}$$

If we graph these on the complex plane, they all lie on the circle centered at $(0, 0)$ with radius 1:



Answer. b.)
If $z_0 = 8$, then $r = |z_0| = \sqrt{(8)^2 + 0^2} = 8$ and $\cos(\theta) = 1$ and $\sin(\theta) = 0 \implies \theta = 0$. Thus, $z_0 = 8e^{i0}$.
We are looking for a $z = re^{i\theta}$ such that $z^6 = z_0$. Or,

$$r^6 e^{i6\theta} = 8e^{i0}$$

By *iii.*), $r^6 = 8$ and $6\theta = 0 + 2\pi k$.
$\implies r = \sqrt{2}$ and $\theta = (2\pi k)/6$.
Since θ is 2π periodic, we only have distinct values for z when $k \in \{0, 1, 2, 3, 4, 5\}$.
Thus, our potential values for z are:

$$\begin{aligned} z &\in \{\sqrt{2}e^{i(2\pi 0)/6}, \sqrt{2}e^{i(2\pi)/6}, \sqrt{2}e^{i(4\pi)/6}, \sqrt{2}e^{i(6\pi)/6}, \sqrt{2}e^{i(8\pi)/6}, \sqrt{2}e^{i(10\pi)/6}\} \\ &= \{\sqrt{2}, \sqrt{2}e^{i(\pi/3)}, \sqrt{2}e^{i(2\pi/3)}, \sqrt{2}e^{i(\pi)}, \sqrt{2}e^{i(4\pi/3)}, \sqrt{2}e^{i(5\pi/3)}\} \\ &= \{\sqrt{2}, \sqrt{2}(\cos(\pi/3) + i\sin(\pi/3)), \sqrt{2}(\cos(2\pi/3) + i\sin(2\pi/3)), \\ &\quad \sqrt{2}(\cos(\pi) + i\sin(\pi)), \sqrt{2}(\cos(4\pi/3) + i\sin(4\pi/3)), \sqrt{2}(\cos(5\pi/3) + i\sin(5\pi/3))\} \\ &= \{\sqrt{2}, \sqrt{2}(1/2 + i(\sqrt{3}/2)), \sqrt{2}(-1/2 + i(\sqrt{3}/2)), \sqrt{2}(-1 + i(0)), \sqrt{2}(-1/2 - i(\sqrt{3}/2)), \sqrt{2}(1/2 - i(\sqrt{3}/2))\} \\ &= \boxed{\{\sqrt{2}, \sqrt{2}/2 + i(\sqrt{6}/2), -\sqrt{2}/2 + i(\sqrt{6}/2), -\sqrt{2}, -\sqrt{2}/2 - i(\sqrt{6}/2), \sqrt{2}/2 - i(\sqrt{6}/2)\}} \end{aligned}$$

If we graph these on the complex plane, they all lie on the circle centered at $(0, 0)$ with radius $\sqrt{2}$: