Complex Analysis Homework 8

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November 25, 2020

Question 5

Suppose that z_n is a sequence of distinct points in $D(0,1)$ such that $z_n \to 0$ as $n \to \infty$. Using the Uniqueness Theorem show that:

(a) If f is analytic in $D(0,1)$ and $f(z_n) = \cos(z_n)$ for all $n \in \mathbb{N}$, then $f(z) = \cos(z)$ for all $z \in D(0,1)$.

- Since z_n converges to 0 as $n \to \infty$, that means that the point 0 is a limit point of the set $S = \{z_n : n \in \mathbb{N}\}$. However, this set also happens to be the same set for which $z \in S$ implies that $f(z) = \cos(z)$. Therefore, by the Uniqueness Theorem, since the limit point of S, 0, is inside $D(0,1)$ and both f and $\cos(\cdot)$ are analytic functions, then we can conclude that $f(z) = \cos(z)$ for all $z \in D(0,1)$.

(b) There is no analytic function f defined on $D(0,1)$ such that $f(z_n) = 0$ when n is even and such that $f(z_n) = z_n$ when n is odd.

- Assume that such a function exists. First, consider the set $S_1 = \{z_n : n \in \mathbb{N} \text{ and } n \text{ is odd}\}$. This set still has a limit point of 0 since any subsequence of a convergent sequence has the same limit. Thus, 0 is a limit point of S_1. Furthermore, for any $z \in S_1$ we have that $f(z) = z$ by assumption. Thus, since f and the identify function are both analytic and since $0 \in D(0,1)$, we can use the Uniqueness Theorem to conclude that $f(z) = z$ for all $z \in D(0,1)$. In particular, this implies that $f(z_2) = z_2$ and $f(z_4) = z_4$. Furthermore, by assumption, we know that $f(z_n) = 0$ for all even n which means that $z_2 = z_4 = 0$. However, this contradicts the definition of the sequence which states that z_n is a sequence of distinct points in $D(0,1)$ which in particular means that $z_2 \neq z_4$. This means our assumption of the existence of such an f was false, so no such f exists. $\square$