Advanced Calc. Homework 10

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November 5, 2020

18.5

(a) Let f and g be continuous functions on $[a, b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one $x_0 \in [a, b]$.

(b) Show Example 1 can be viewed as a special case of part (a).

Proof. **(a)**
If $f(a) = g(a)$ or $f(b) = g(b)$, then the result follows immediately, so assume $f(a) \neq g(a)$ and $f(b) \neq g(b)$. Let $h(x) = f(x) - g(x)$, then by Theorem 17.3/Theorem 17.4(i) h is also a continuous function. We can see that $h(a) = f(a) - g(a) > 0$ and $h(b) = f(b) - g(b) < 0$. Additionally $a < b$ by the definition of the interval $[a, b]$. Therefore, by the Intermediate Value Theorem, there exists some $x_0 \in (a, b)$ such that $h(x_0) = 0$. Combining this with the fact that we could possibly have $f(a) = g(a)$ or $f(b) = g(b)$, then we get that $f(x_0) = g(x_0)$ for some $x_0 \in [a, b]$. □

Answer. (b)
In Example 1, we showed that for a continuous function $f : [0, 1] \to [0, 1]$, then $f(x_0) = x_0$ for some $x_0 \in [0, 1]$. This is equivalent to part (a) of this question for $[a, b] = [0, 1]$ and $g(x) = x$ since $f(a) = f(0) \geq g(0) = 0$ and $f(b) = f(1) \leq g(1) = 1$ due to f being contained in $[0, 1]$.

18.6

Prove $x = \cos(x)$ for some $x \in \left(0, \frac{\pi}{2}\right)$.

Proof.
Note that if we let $f(x) = \cos(x)$ and $g(x) = x$, then we can see that $f(0) = 1 \geq 0 = g(0)$ and $f\left(\frac{\pi}{2}\right) = 0 \leq \frac{\pi}{2} = g\left(\frac{\pi}{2}\right)$. Thus, from Exercise 18.5(a) (and the fact that f and g are both continuous), we can conclude that there exists some $x_0 \in \left[0, \frac{\pi}{2}\right]$ such that $f(x_0) = g(x_0)$. However, we have just seen that $f(0) \neq g(0)$ and $f\left(\frac{\pi}{2}\right) \neq g\left(\frac{\pi}{2}\right)$, so that means there does indeed exist some $x \in \left(0, \frac{\pi}{2}\right)$ such that $x = \cos(x)$. □

18.7

Prove $xe^x = 2$ for some x in $(0, 1)$.

Proof.
Let $f(x) = xe^x$, then f is a continuous function by the continuity of e^x and x and Theorem 17.4(ii). Furthermore, note that $f(0) = 0e^0 = 0 < 2$ and $f(1) = 1e^1 \approx 2.718 > 2$. Thus, by the Intermediate Value Theorem, there exists some $x_0 \in (0, 1)$ such that $f(x_0) = 2$, proving the statement. □

18.8

Suppose f is a real-valued continuous function on $\mathbb{R}$ and $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove there exists x between a and b such that $f(x) = 0$.

Proof.
Let I be defined as follows:

$$I := \begin{cases} [a, b] & \text{if } a < b \\ [b, a] & \text{if } b < a \end{cases}$$

Furthermore, without loss of generality, assume that $f(a) > f(b)$ (if not, we can simply reverse the roles of a and b and note that I is defined appropriately to allow this. Also see that we cannot have $f(a) = f(b)$ as this would cause $f(a)f(b) = f(a)^2 \geq 0$). It is clear that $f(a) \neq 0$ and $f(b) \neq 0$ since the opposite of this would cause $f(a)f(b) = 0$ which is not true. Therefore, since $f(a)f(b) < 0$, we must have that the sign of $f(a)$ differs from the sign of $f(b)$ because if they were the same, then $f(a)f(b) > 0$ which is not true. Thus, since $f(a) > f(b)$ and they have different signs, we can conclude that $f(a) > 0$ and $f(b) < 0$. Then, by the continuity of f, we can apply the Intermediate Value Theorem to conclude that there exists some $x \in I$ such that $f(x) = 0$. □

18.9

Prove that a polynomial function f of odd degree has at least one real root.

Proof.
Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be an arbitrary polynomial with $n \in \mathbb{N}$ odd. Without loss of generality, assume that $a_n = 1$ (if not, simply consider the rest of this proof with the function $\left(\frac{1}{a_n}\right) f$). Thus, we are considering $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. This is equivalent to

$$f(x) = x^n \left(1 + \frac{a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{x^n}\right) \tag{1}$$

Taking limits as x goes to $\pm\infty$ leaves the bracketed term irrelevant since

$$\begin{aligned}\lim_{x\to\pm\infty} 1 + \frac{a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{x^n} &= \lim_{x\to\pm\infty} 1 + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \\ &= 1 + 0 + \cdots + 0 + 0 \\ &= 1.\end{aligned}$$

Thus, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^n = +\infty$ and $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^n = -\infty$ since n is odd and any negative number to an odd power is negative (since $p^{2k+1} = p^{2k} \cdot p = c \cdot p < 0$ for c some positive number). In particular, this means that there exists some $a < 0$ such that $f(a) < 0$ and there exists some $b > 0$ such that $f(b) > 0$ which means we can use the Intermediate Value Theorem to indicate that there exists some $x_0 \in (a, b)$ such that $f(x_0) = 0$, making x_0 a root for f. $\square$

18.10

Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist $x, y \in [0, 2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.

Proof.
Consider $g(x) = f(x+1) - f(x)$ on $[0, 1]$. From this, we have that $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = f(0) - f(1) = -g(0)$. The important relationship here is that $g(1) = -g(0)$. If $g(0) = 0$, then clearly we have $f(1) = f(0)$ or if $g(1) = 0$, then we have that $f(2) = f(1)$ which both satisfy the condition we which to show. Otherwise, $g(0)$ and $g(1)$ have different signs. Thus the point 0 lies in between $g(0)$ and $g(1)$ which indicates (along with the possibility that $g(0) = 0$ or $g(1) = 0$) that there exists some $x_0 \in [0, 1]$ such that $g(x_0) = 0 \iff f(x_0 + 1) = f(x_0)$ which clearly satisfies the statement we are trying to prove with $x = x_0$ and $y = x_0 + 1$. $\square$

19.1

Which of the following continuous functions are uniformly continuous on the specified set? Justify your answers by using any theorems.

(a) $f(x) = x^{17}\sin(x) - e^x \cos(3x)$ on $[0, \pi]$,

- This is $\boxed{\text{uniformly continuous}}$ since we know that $x^{17}, \sin(x), e^x$, and $\cos(3x)$ are all continuous functions (thus, their sum and product is continuous). Therefore, since f is continuous on a closed interval, $[0, \pi]$, Theorem 19.2 assures us that f is uniformly continuous on $[0, \pi]$.

(b) $f(x) = x^3$ on $[0, 1]$,

- This is $\boxed{\text{uniformly continuous}}$ since we know that x^3 is continuous. Thus, f is continuous on $[0, 1]$, so Theorem 19.2 assures us that f is uniformly continuous on $[0, 1]$.

(c) $f(x) = x^3$ on $(0, 1)$,

- If we make the extension

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in (0,1) \\ 0 & \text{for } x = 0 \\ 1 & \text{for } x = 1 \end{cases}$$

Then $\tilde{f}$ is continuous on $[0, 1]$ since for any sequence $(s_n) \subset (0, 1)$ converging to 0, we have $\tilde{f}(s_n) = f(s_n) = s_n^3$ which converges to $\tilde{f}(0) = 0$ and for any sequence $(t_n) \subset (0, 1)$ converging to 1, we have $\tilde{f}(t_n) = f(t_n) = t_n^3$ which converges to $\tilde{f}(1) = 1$ and clearly $\tilde{f}$ is continuous in $(0, 1)$ by the continuity of x^3. Thus, $\tilde{f}$ is continuous on $[0, 1]$, so it must be uniformly continuous on $[0, 1]$ by Theorem 19.2. Furthermore, this means that f must be $\boxed{\text{uniformly continuous}}$ on $(0, 1)$.

(d) $f(x) = x^3$ on $\mathbb{R}$,

- Assume that f is uniformly continuous on $\mathbb{R}$. Then by Definition 19.1, we know that that there exists some $\delta > 0$ such that $x, y \in \mathbb{R}$ and $|x - y| < \delta$ implies that $|x^3 - y^3| < 1$. In this case, choose $y = x + \frac{\delta}{2}$ which gives the following:

$$
\begin{aligned}
|x_3 - y^3| = |x-y||x^2+xy+y^2| &= \left|x - \left(x + \frac{\delta}{2}\right)\right| \left|x^2 + x\left(x+\frac{\delta}{2}\right) + \left(x + \frac{\delta}{2}\right)^2\right| \\
&= \frac{\delta}{2}\left|3x^2 + \frac{3\delta}{2}x + \frac{\delta}{4}\right| \\
&> \frac{\delta}{2}\left|\frac{2}{\delta} + \frac{3\delta}{2}\sqrt{\frac{2}{3\delta}} + \frac{\delta}{4}\right| && \text{whenever } x > \sqrt{\frac{2}{3\delta}} \\
&> \frac{\delta}{2}\left|\frac{2}{\delta}\right| && \text{since } \frac{3\delta}{2}\sqrt{\frac{2}{3\delta}} + \frac{\delta}{4} > 0 \\
&= 1
\end{aligned}
$$

Thus, we have shown that for $y = x + \frac{\delta}{2}$ and $x > \sqrt{\frac{2}{3\delta}}$ that $|f(x) - f(y)| > 1$ which means that f is $\boxed{\text{not uniformly continuous}}$.

(e) $f(x) = \frac{1}{x^3}$ on $(0, 1]$,

- Theorem 19.4 tells us that if f is uniformly continuous on a set S and (s_n) is a Cauchy sequence in S, then $(f(s_n))$ is a Cauchy sequence. However, taking $(s_n) = \frac{1}{n}$ gives us $s_n \in (0, 1]$ for all n and (s_n) is a Cauchy sequence. However, $f(s_n) = \frac{1}{(1/n)^3} = n^3$ which is not a Cauchy sequence since it is not convergent; thus, f is $\boxed{\text{not uniformly continuous}}$.

(f) $f(x) = \sin\left(\frac{1}{x^2}\right)$ on $(0, 1]$,

- Let's once again use Theorem 19.4. In this case, take $(s_n) = \sqrt{\frac{2}{\pi + 2n\pi}}$. It is easy to see that $s_n \in (0, 1]$ for all n and it is easy to see that (s_n) is Cauchy since it is a convergent sequence. However, $f(s_n) = \sin\left(\frac{1}{\sqrt{\frac{2}{\pi+2n\pi}}^2}\right) = \sin\left(\frac{\pi}{2} + n\pi\right)$. We can see that $f(s_n)$ is not Cauchy since the subsequence with odd indices converges to -1 and the subsequence with even indices convergest o 1. Thus, $f(s_n)$ is not convergent, so it is not Cauchy which implies that f is $\boxed{\text{not uniformly convergent}}$.

(g) $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ on $(0, 1]$.

- Let's make the following extension of f:

$$
\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in (0,1] \\ 0 & \text{for } x = 0 \end{cases}
$$

I will first show that $\tilde{f}$ is continuous in its domain, $[0, 1]$. Note for all $x \in (0, 1]$, $\tilde{f}$ is clearly continuous since x^2, $\sin(x)$, and $\frac{1}{x}$ are all continuous when $x \neq 0$. Thus, the only point of contention is $x = 0$, so I will examine the continuity of $\tilde{f}$ at $x = 0$. Let $\varepsilon > 0$ be fixed and consider $|\tilde{f}(x) - \tilde{f}(0)| = |\tilde{f}(x)| = \left|x^2 \sin\left(\frac{1}{x}\right)\right| \leq |x^2|$. Thus choosing $\delta = \sqrt{\varepsilon}$ we get $|x - 0| = |x| < \sqrt{\varepsilon}$ which implies that $|\tilde{f}(x) - \tilde{f}(0)| \leq |x^2| < \left|\sqrt{\varepsilon}^2\right| = \varepsilon$. Thus, $\tilde{f}$ is continuous at $x = 0$, so $\tilde{f}$ is continuous in all of $[0, 1]$, so Theorem 19.2 tells us that $\tilde{f}$ is uniformly continuous on $[0, 1]$ which further means that f is $\boxed{\text{uniformly continuous}}$ on $(0, 1]$.

19.2

Prove each of the following functions is uniformly continuous on the indicated set by directly verifying the ε–δ property in Definition 19.1.

(a) $f(x) = 3x + 11$ on $\mathbb{R}$,

- Let $\varepsilon > 0$ be fixed and let $x, y \in \mathbb{R}$. Let's first consider the following:

$$
|f(x) - f(y)| = |3x + 11 - (3y + 11)| = 3|x - y|
$$

Thus, choosing $\delta = \frac{\varepsilon}{3}$ gives us $|x - y| < \frac{\varepsilon}{3}$. Therefore, we have $|f(x) - f(y)| = 3|x - y| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon$ satisfying the definition of uniform continuity.

(b) $f(x) = x^2$ on $[0, 3]$,

- Let $\varepsilon > 0$ be fixed and let $x, y \in [0, 3]$. Let's first consider the following:

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y|$$

Note that since $x, y \in [0, 3]$, $|x + y| \leq 6$ for any x, y. Thus, choosing $\delta = \frac{\varepsilon}{6}$ gives $|x - y| < \frac{\varepsilon}{6}$ which in turn leads to $|f(x) - f(y)| = |x - y||x + y| \leq |x - y| \cdot 6 < \frac{\varepsilon}{6} \cdot 6 = \varepsilon$, satisfying the definition of uniform continuity.

(c) $f(x) = \frac{1}{x}$ on $\left[\frac{1}{2}, \infty\right)$.

- Let $\varepsilon > 0$ be fixed and let $x, y \in \left[\frac{1}{2}, \infty\right)$. Let's first consider the following:

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| = \frac{|x - y|}{|xy|}$$

Note that since $x, y \geq \frac{1}{2}$, $|xy| \geq \frac{1}{4} \implies \frac{1}{|xy|} \leq 4$. With this in mind, choose $\delta = \frac{\varepsilon}{4}$ which gives $|x - y| < \frac{\varepsilon}{4}$. Using this with the estimate above we get $|f(x) - f(y)| = \frac{|x-y|}{|xy|} \leq 4|x - y| < 4 \cdot \frac{\varepsilon}{4} = \varepsilon$, satisfying the definition of uniform continuity.

19.5

Which of the following continuous functions is uniformly continuous on the specified set? Justify your answers, using appropriate theorems.

(a) $\tan(x)$ on $\left[0, \frac{\pi}{4}\right]$,

- Since $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and $\sin(x)$ and $\cos(x)$ are both continuous in $\left[0, \frac{\pi}{4}\right]$ and $\cos(x) \neq 0$ for any $x \in \left[0, \frac{\pi}{4}\right]$, then we can apply Theorem 17.4(iii) to conclude that $\tan(x)$ is continuous on $\left[0, \frac{\pi}{4}\right]$. From this, we can use Theorem 19.2 to conclude that $\tan(x)$ is $\boxed{\text{uniformly continuous}}$ on $\left[0, \frac{\pi}{4}\right]$.

(b) $\tan(x)$ on $\left[0, \frac{\pi}{2}\right)$,

- According to Exercise 19.4(a), if a function is uniformly continuous on a bounded set S, then that function is also bounded on S. In our case here $S = \left[0, \frac{\pi}{2}\right)$ is our bounded set. However, $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty$, which indicates that $\tan(x)$ is not bounded on S, thus $\tan(x)$ is $\boxed{\text{not uniformly continuous}}$.

(c) $\frac{1}{x} \sin^2(x)$ on $(0, \pi]$,

- In Example 9 on Page 149, they defined

$$\tilde{h}(x) = \begin{cases} \frac{\sin(x)}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

where the domain of $\tilde{h}$ was $\mathbb{R}$. They find that that $\tilde{h}$ is uniformly continuous on this domain. If instead, we restrict the domain to $[0, \pi]$ then it would still be a uniformly continuous function. Thus, define the following function:

$$f(x) = \begin{cases} \frac{\sin^2(x)}{x} & \text{for } x \in (0, \pi] \\ 0 & \text{for } x = 0 \end{cases}$$

It is clear that $f(x) = \sin(x)\tilde{h}(x)$ for $\tilde{h}$ restricted to the domain $[0, \pi]$. Thus, by the continuity of $\sin(x)$ and the Example 9 determining that $\tilde{h}$ was also continuous, we can conclude that f is continuous on $[0, \pi]$. Using Theorem 19.5 therefore tells us that $\frac{\sin^2(x)}{x}$ on $(0, \pi]$ is a $\boxed{\text{uniformly continuous}}$ function.

(d) $\frac{1}{x-3}$ on $(0, 3)$,

- Using exercise 19.4(a), we can note that this function is not bounded on $(0, 3)$, specifically as x approaches 3, so it must be $\boxed{\text{not uniformly continuous}}$.

(e) $\frac{1}{x-3}$ on $(3, \infty)$,

- Using the same reasoning we did for part (d) of this question, we can note that this function is not bounded on $(3, \infty)$, once again where x approaches 3, so Exercise 19.4(a) tells us that this function must be $\boxed{\text{not uniformly continuous}}$

(f) $\frac{1}{x-3}$ on $(4, \infty)$.

- Recall the derivative of $\frac{1}{x-3}$ is $\frac{-1}{(x-3)^2}$. Furthermore, note that this derivative is bounded on $(4, \infty)$ since $\left|\frac{-1}{(x-3)^2}\right| < 1$ for all $x > 4$. Thus, by using Theorem 19.6, we can say that this function is $\boxed{\text{uniformly continuous}}$.

19.6

(a) Let $f(x) = \sqrt{x}$ for $x \geq 0$. Show f' is unbounded on $(0,1]$ but f is nevertheless uniformly continuous on $(0,1]$. Compare with Theorem 19.6.

- By using the power rule, $f'(x) = \frac{1}{2\sqrt{x}}$. This is unbounded on $(0,1]$ because for any $M > 0$, we can find a $\delta > 0$ such that $|x| < \delta$ implies that $|f'(x)| > M$. Let $\delta = \frac{1}{4M^2}$. This yields, $|f'(x)| = \left|\frac{1}{2\sqrt{x}}\right| > \left|\frac{1}{2\sqrt{1/(4M^2)}}\right| = \left|\frac{2M}{2}\right| = M$, which shows f' is unbounded. Despite this, f is still uniformly continuous on $(0,1]$. We can show this by noting that $\sqrt{x}$ is continuous on $[0,1]$ thus, by Theorem 19.2, $\sqrt{x}$ is uniformly continuous on $[0,1]$ and by Theorem 19.5 this shows that f is uniformly continuous on $(0,1]$. This shows that the converse to Theorem 19.6 is not true – just because a function has an unbounded derivative does not mean it must be not uniformly continuous.

(b) Show f is uniformly continuous on $[1,\infty)$.

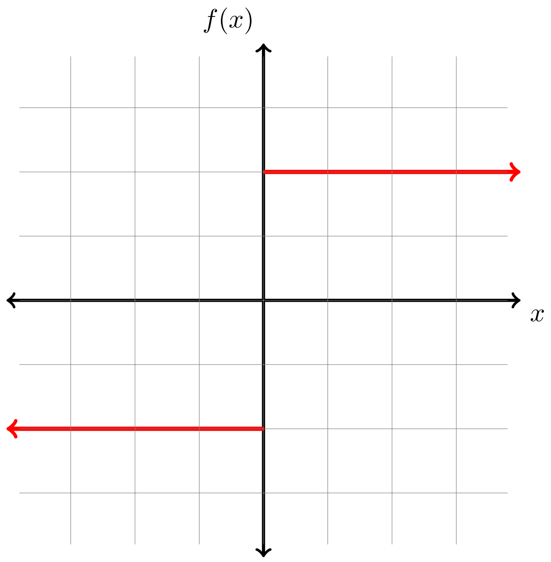
- Let $\varepsilon > 0$ be fixed and let $x, y \in [1,\infty)$, then let's examine the following:

$$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x-y}{\sqrt{x}+\sqrt{y}}\right| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$$

  Note that $\sqrt{x} + \sqrt{y} \geq 2$ for all $x, y \geq 1$ which means that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}$ for all $x, y \geq 1$. This motivates the choice of $\delta = 2\varepsilon$, i.e. $|x - y| < 2\varepsilon$. From this, we get $|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2}|x-y| < \frac{1}{2}(2\varepsilon) = \varepsilon$, proving that f is uniformly continuous on $[1,\infty)$.

20.1

Sketch the function $f(x) = \dfrac{x}{|x|}$. Determine, by inspection, the limits: $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to-\infty} f(x)$, and $\lim_{x\to 0} f(x)$ if they exist, or indicate when they do not exist.

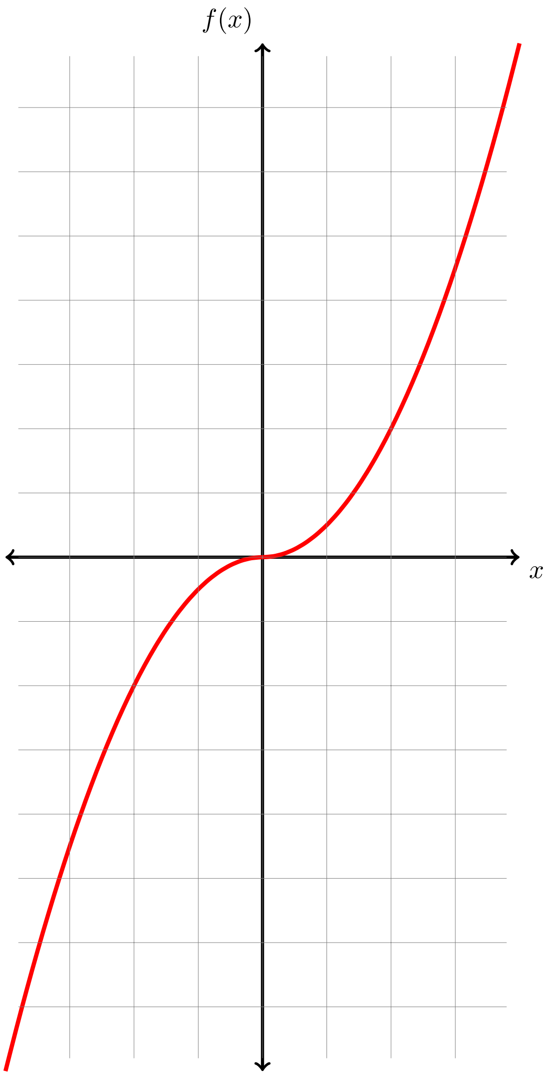


From this picture, we can conclude the following:

- $\lim_{x\to\infty} f(x) = 1$
- $\lim_{x\to 0^+} f(x) = 1$
- $\lim_{x\to 0^-} f(x) = -1$
- $\lim_{x\to-\infty} f(x) = -1$
- $\lim_{x\to 0} f(x)$ does not exist.

20.2

Sketch the function $f(x) = \dfrac{x^3}{|x|}$. Determine, by inspection, the limits: $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to-\infty} f(x)$, and $\lim_{x\to 0} f(x)$ if they exist, or indicate when they do not exist.

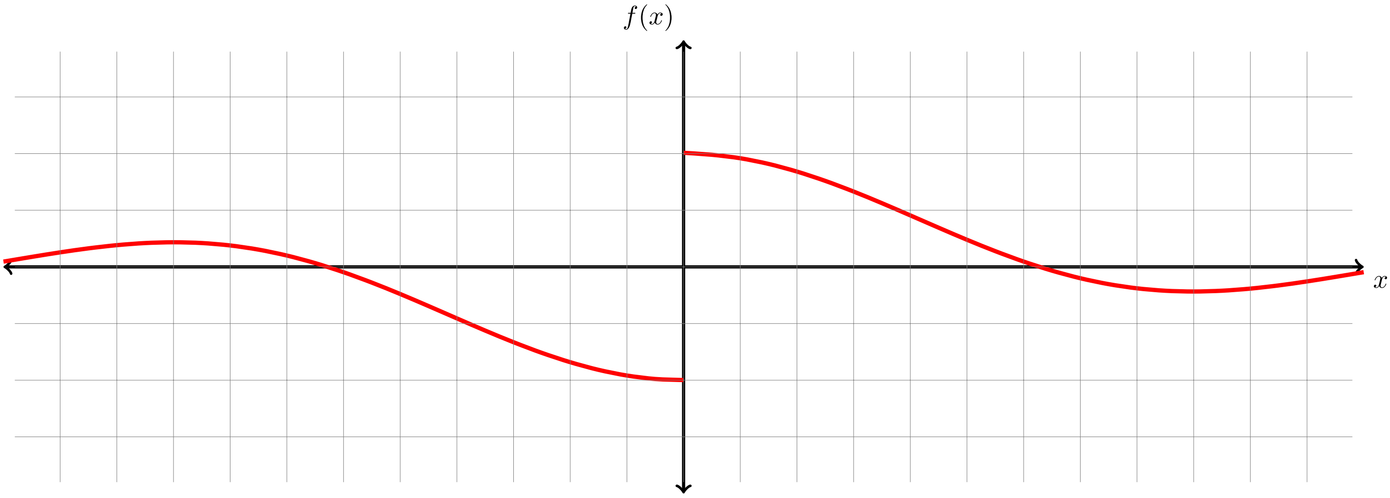


From this picture, we can conclude the following:

- $\lim_{x\to\infty} f(x) = \infty$
- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 0^-} f(x) = 0$
- $\lim_{x\to -\infty} f(x) = -\infty$
- $\lim_{x\to 0} f(x) = 0$

20.3

Sketch the function $f(x) = \dfrac{\sin(x)}{|x|}$. Determine, by inspection, the limits: $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to -\infty} f(x)$, and $\lim_{x\to 0} f(x)$ if they exist, or indicate when they do not exist.

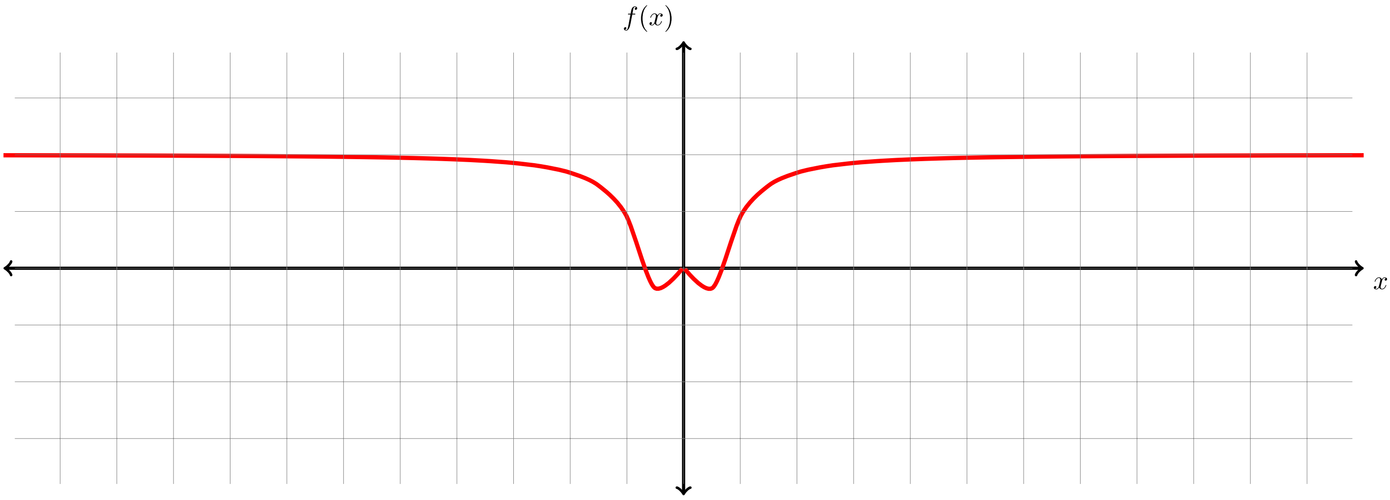


From this picture, we can conclude the following:

- $\lim_{x\to\infty} f(x) = 0$
- $\lim_{x\to 0^+} f(x) = 1$
- $\lim_{x\to 0^-} f(x) = -1$
- $\lim_{x\to -\infty} f(x) = 0$
- $\lim_{x\to 0} f(x)$ does not exist.

20.4

Sketch the function $f(x) = x\sin\left(\frac{1}{x}\right)$. Determine, by inspection, the limits: $\lim_{x\to\infty} f(x)$, $\lim_{x\to 0^+} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to -\infty} f(x)$, and $\lim_{x\to 0} f(x)$ if they exist, or indicate when they do not exist.



From this picture, we can conclude the following:

- $\lim_{x\to\infty} f(x) = 1$
- $\lim_{x\to 0^+} f(x) = 0$
- $\lim_{x\to 0^-} f(x) = 0$
- $\lim_{x\to -\infty} f(x) = -1$
- $\lim_{x\to 0} f(x) = 0$