Complex Analysis Homework 8

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Question 4

Let f be holomorphic in $\mathbb{C}$.

(a) Prove that if $|f(z)| > M$ in $\mathbb{C}$, then f is constant.

- Consider the function $g = 1/f$. Then, since $|f(z)| > M$ for all $z \in \mathbb{C}$, we know that $f(z) \neq 0$ for all $z \in \mathbb{C}$ which means that g is also holomorphic in $\mathbb{C}$. Furthermore, we can see that g is bounded since:

$$|g(z)| = \left|\frac{1}{f(z)}\right| = \frac{1}{|f(z)|} < \frac{1}{M} \qquad \text{for all } z \in \mathbb{C}$$

- Thus, Liouville's Theorem tells us that g must be a constant function since it is bounded and entire. Therefore, if g is constant, say $g(z) = C$ for all $z \in \mathbb{C}$, then we can say that $f(z) = 1/g(z) = 1/C$ for all $z \in \mathbb{C}$, so f is constant as well. $\square$

(b) Prove that if e^f is bounded in $\mathbb{C}$, then f is constant.

- Recall that $g(z) = e^z$ is an entire function, thus, since f is also entire, then the function $g \circ f = e^f$ is also entire. Thus, since we're assuming e^f is bounded, then we can apply Liouville's Theorem to say that e^f is constant. Thus, since the derivative of a constant is 0, we know that

$$\frac{d}{dz}\left(e^{f(z)}\right) = e^{f(z)} f'(z) = 0 \qquad \text{for all } z \in \mathbb{C}$$

This last equality implies that $f'(z) = 0$ for all $z \in \mathbb{C}$ since the exponential function never maps to 0. Thus, f must also be a constant function since its derivative is zero in all of $\mathbb{C}$.