Analysis HW 8

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October 26, 2021

Question 1

Let $f \in \mathbf{L}(\mathbb{R}^n, \mathbb{R}^m)$. Define the kernel of f as $\ker(f) = f^{-1}(0)$. Prove that f is injective if and only if, $\ker(f) = \{0\}$.

Proof.
First, assume that f is injective. Assume also, for the sake of contradiction, that there exists some nonzero $x \in \mathbb{R}^n$ such that $f(x) = 0$. In other words, $x \in \ker(f)$. Notice that since x is not the zero vector, then $x \neq 2x$, so by injectivity of f, $f(2x) \neq f(x) = 0$. On the other hand, using the linearity of f, we have

$$f(2x) = f(x + x) = f(x) + f(x) = 0 + 0 = 0$$

which is a contradiction to our previous expression that $f(2x) \neq 0$. Therefore, our assumption was wrong, and the only vector in $\ker(f)$ is the zero vector.

Next, assume that $\ker(f) = \{0\}$. Next, let $x, y \in \mathbb{R}^n$ such that $x \neq y$. Equivalently, $x - y \neq 0$. In particular, $x - y \notin \ker(f)$ so that $f(x - y) \neq 0$. However, by linearity of f, $f(x - y) = f(x) - f(y) \neq 0$. Equivalently, we get $f(x) \neq f(y)$, meaning that f is injective. □

Question 2

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + y^2}, & x^2 + y^2 \neq 0 \\ 0, & x = y = 0. \end{cases}$$

Prove that $D_1 f(0, 0) = D_2 f(0, 0) = 0$, but f is not continuous at $(0, 0)$.

Proof.
First, calculating $D_1 f(0, 0)$:

$$D_1 f(0,0) = \lim_{t \to 0} \frac{f((0,0) + (t,0)) - f((0,0))}{t} = \lim_{t \to 0} \frac{f((t,0))}{t} = \lim_{t \to 0} \frac{\frac{t \cdot 0}{t^2 + 0^2}}{t} = \lim_{t \to 0} \frac{0}{t^3} = 0$$

The calculation for $D_2 f(0, 0)$ would be identical since f is symmetric in x and y. Therefore, $D_1 f(0, 0) = D_2 f(0, 0) = 0$. Next, I will show that f is not continuous at $(0, 0)$. To do this, I will show that $\lim_{(x,y) \to (0,0)} f(x, y) \neq 0$. Let $v_t = (t, t)$, then consider

$$\lim_{t \to 0} f(v_t) = \lim_{t \to 0} \frac{t^2}{t^2 + t^2} = \lim_{t \to 0} \frac{t^2}{2t^2} = \lim_{t \to 0} \frac{1}{2} = \frac{1}{2} \neq 0$$

Thus, the limit $\lim_{(x,y) \to (0,0)} f(x, y)$, if it even exists, can certainly not be equal to zero since it is not equal to zero for this path through the origin. Thus, f is not continuous at the point $(0, 0)$. □

Question 3

Let X be an open subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^m$ be such that $D_1 f, D_2 f, \ldots, D_n f$ are defined and bounded in X. Prove that f is continuous.

Proof.
First, I will fix $x_0 = (x_0^1, x_0^2, \ldots, x_0^n) \in X$. Then, for $x = (x^1, x^2, \ldots, x^n)$, define the sequence $x_i = x_{i-1} + (x^i - x_0^i) e_i$ for $i = 1, 2, \ldots, n$. Notice $x_1 = (x^1, x_0^2, x_0^3, \ldots, x_0^n)$, $x_2 = (x^1, x^2, x_0^3, \ldots, x_0^n)$, …, $x_n = x$.

Next, for the intervals $I_i = [-|x^i - x_0^i|, |x_i - x_0^i|]$ for $i = 1, 2, \ldots, n$, define the following functions: $\gamma_i : I_i \to X$ as $\gamma_i(t) = x_{i-1} + te_i$ for $i = 1, 2, \ldots, n$. Then, examine the function $(f \circ \gamma_i) : I_i \to \mathbb{R}^m$ and notice

$$(f \circ \gamma_i)'(t) = \lim_{h \to 0} \frac{(f \circ \gamma)(t+h) - (f \circ \gamma)(t)}{h} = \lim_{h \to 0} \frac{f(x_{i-1} + (t+h)e_i) - f(x_{i-1} + te_i)}{h} = D_i f(x_{i-1} + te_i)$$

Since $D_i f$ exists, then we can see that $f \circ \gamma_i$ is indeed differentiable and by extension must be continuous on I_i. Therefore, using the mean value theorem over the interval $[0, x^i - x_0^i]$ (or on $[0, x_0^i - x^i]$, whichever is nonempty), we get that there exists some $t_i \in I_i^\circ$ (in particular in one of the open half-intervals described above) such that

$$\begin{aligned}
\frac{(f \circ \gamma_i)(x^i - x_0^i) - (f \circ \gamma_i)(0)}{x^i - x_0^i} &= (f \circ \gamma_i)'(t_1) \\
\implies \frac{f(x_{i-1} + (x^i - x_0^i)e_i) - f(x_{i-1})}{x^i - x_0^i} &= D_i f(x_{i-1} + t_i e_i) \\
\implies \frac{f(x_i) - f(x_{i-1})}{x^i - x_0^i} &= D_i f(x_{i-1} + t_i e_i) \\
\implies f(x_i) - f(x_{i-1}) &= D_i f(x_{i-1} + t_i e_i)(x^i - x_0^i)
\end{aligned}$$

Next, recall that each $D_i f$ is bounded. Therefore, there exists some $M_i \in \mathbb{R}$ such that $|D_i f(x)| \le M_i$ for all $x \in X$. Let $M := \max\{M_i\}_{i=1}^n$ which gives

$$|f(x_i) - f(x_{i-1})| = |D_i f(x_{i-1} + t_i e_i)(x^i - x_0^i)| \le M_i |x^i - x_0^i| \le M |x^i - x_0^i|$$

With all of this in place, let $r > 0$ be given and let $|x - x_0| < r/(M\sqrt{n})$ Notice, we have the following:

$$\begin{aligned}
|f(x) - f(x_0)| &= |f(x_n) - f(x_0)| \\
&= |f(x_n) - f(x_{n-1}) + f(x_{n-1}) - f(x_{n-2}) + \cdots - f(x_1) + f(x_1) - f(x_0)| \\
&\le |f(x_n) - f(x_{n-1})| + |f(x_{n-1}) - f(x_{n-2})| + \cdots + |f(x_1) - f(x_0)| \\
&\le M\Big(|x^n - x_0^n| + |x^{n-1} - x_0^{n-1}| + \cdots + |x^1 - x_0^1|\Big) \\
&= M\Big\langle x - x_0, (1, 1, \ldots, 1)\Big\rangle \\
&\le M|x - x_0|\,|(1, 1, \ldots, 1)| && \text{by Cauchy-Schwarz} \\
&= M\sqrt{n}|x - x_0| \\
&< M\sqrt{n}\frac{r}{M\sqrt{n}} \\
&= r
\end{aligned}$$

Thus, for every $r > 0$, we can find some s (which is $r/(M\sqrt{n})$) such that $|x - x_0| < s$ implies that $|f(x) - f(x_0)| < r$ which means that f is continuous at x_0. Since x_0 was arbitrary, we know that f is continuous in all of X. $\square$

Question 4

Let X be an open subset of $\mathbb{R}^n$, and let $f : X \to \mathbb{R}$ be differentiable. Suppose that $x_0 \in X$ is a point of maximum of f (that is, $\forall\, x \in X,\ f(x_0) \ge f(x)$). Prove that $f'(x_0) = 0$.

Proof.
Let X be an open subset of $\mathbb{R}^n$ and let $x_0 = (x_0^1, x_0^2, \ldots, x_0^n) \in X$ be a point of maximum of the differentiable function $f : X \to \mathbb{R}$. Notice that since X is open, there is an open ball around x_0 contained in X. In particular, for each coordinate of x_0, there is some r_i such that for all $t \in I_i := (-r_i, r_i)$ we have $x_0 + te_i \in X$. With this in mind, define $g_i : I_i \to \mathbb{R}$ such that

$$g_i(t) = f(x_0 + te_i)$$

Notice that since f attains its maximum at x_0, then g_i attains its maximum at $t = 0$. Therefore, we can use the one-dimensional version of Fermat's Theorem for maximum points to say that $g_i'(0) = 0$. Notice that g_i' exists because f itself is differentiable. Therefore, we get

$$\begin{aligned}
g_i'(0) &= \lim_{t \to 0} \frac{g_i(t) - g_i(0)}{t} \\
\implies 0 &= \lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t} \\
\implies 0 &= D_i f(x_0)
\end{aligned}$$

Therefore, each partial derivative of f at x_0 is equal to zero. Then, since the partial derivatives fully determine the form of the derivative of a differentiable function, we get that for any arbitrary $v = (v^1, v^2, \dots, v^n) \in \mathbb{R}^n$

$$f'(x_0)v = \sum_{k=1}^{n} v^k D_k f(x_0) = \sum_{k=1}^{n} v^k (0) = 0$$

Thus, since $f'(x_0)$ maps any vector in $\mathbb{R}^n$ to the point $0 \in \mathbb{R}$, we can conclude that $f'(x_0)$ must be identically zero. $\square$