Analysis Homework 2

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Question 1

Let (X, d) be a metric space, and let (x_n) be a convergent sequence in X. Prove that every subsequence of the sequence (x_n) converges to the same limit.

Proof.
Since (x_n) converges, let x be its limit. Furthermore, let us fix an $r > 0$. Thus, by the convergence, there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we have $d(x, x_n) < r$. Next, let (x_{n_k}) be a subsequence of (x_n). Note that $n_k \geq k$ for all k. Thus, if we consider $k \geq N$ we also have $n_k \geq N$ which means that $d(x, x_{n_k}) < r$ for all $k \geq N$ meaning (x_{n_k}) also converges to x. □

Question 2

Let (X, d) be a metric space, and let (x_n) be a Cauchy sequence in X. Suppose that a subsequence of the sequence (x_n) converges. Prove that the sequence (x_n) converges as well and to the same limit.

Proof.
Let us fix some $r > 0$. Let (x_{n_k}) be the convergent subsequence of (x_n) whose limit is x. Since the sequence converges, we can say that there exists some $N_1 \in \mathbb{N}$ such that $d(x, x_{n_k}) < r/2$ for all $k \geq N_1$. Furthermore, since (x_n) is Cauchy, we know that there exists some $N_2 \in \mathbb{N}$ such that $d(x_n, x_m) < r/2$ for all $n, m \geq N_2$. In particular, since $n_k \geq k$ for all k, we can say that $d(x_n, x_{n_k}) < r/2$ for all $n, k \geq N_2$. Thus, if we let $N := \max\{N_1, N_2\}$, then we can say the following inequalities hold for all $n, k \geq N$:

$$\begin{aligned} d(x, x_n) &\leq d(x, x_{n_k}) + d(x_{n_k}, x_n) \\ &< \frac{r}{2} + \frac{r}{2} \\ &= r \end{aligned}$$

Therefore, we can conclude that (x_n) also converges to the limit x. □

Question 3

Let (X, d) be a complete metric space, and let $Y \subset X$. Prove that (Y, d) is a complete metric space if and only if Y is closed in X.

Proof.
Let us first assume that Y is closed in X. Let $(x_n) \subset Y$ be a Cauchy sequence. Since $Y \subset X$, and X is complete, we have that $x_n \to x \in X$ as $n \to \infty$. Since this sequence (which is a subset of Y) converges to x, then we know that x is a limit point of Y. Thus, since Y is closed, we can say that $x \in Y$ which means the Cauchy sequence (x_n) converges in Y making Y complete.

Next, assume that (Y, d) is complete, but Y is not closed in X. Since Y is not closed, that means that there exists some limit point $x \in Y'$ which is not in Y. Since x is a limit point of Y, we can construct a sequence $(x_n) \subset Y$ which is convergent to x (thus, is a Cauchy sequence). However, we have now constructed a Cauchy sequence in Y which converges to a point not in Y. This is a contradiction to (Y, d) being complete, so our assumption that Y is not closed must have been false. Thus, Y is closed in X. □

Question 4

Let (X, d) be a complete metric space. Prove the following: if $\{F_n\}$ is a collection of non-empty closed bounded subsets of X such that $F_1 \supset F_2 \supset F_3 \supset \ldots$ and

$$\lim_{n\to\infty} \operatorname{diam}(F_n) = 0, \quad \text{then}$$

$$\exists\, x \in X \text{ such that } \bigcap_{n=1}^{\infty} F_n = \{x\}.$$

Proof.
Note that since the diameter of each F_n is tending towards zero, then the diameter of the intersection is also zero. Thus, anything with a diameter of zero is either empty or has exactly one point, so we simply need to prove that the intersection is nonempty.

To do this, I will construct a sequence (x_n) where $x_i \in F_i$ for all $i \in \mathbb{N}$. Note this is possible since each F_i is nonempty. I claim that this sequence is Cauchy. To prove this, I will fix some $N \in \mathbb{N}$ and note that $\operatorname{diam}\{x_n : n \geq N\} = \sup\{d(x_n, x_m) : n, m \geq N\} \leq \operatorname{diam}(F_n) = \sup\{d(x, y) : x, y \in F_n\}$ since $\{x_n : n \geq N\} \subset F_n$. Thus, since the diameters of the F_n go to zero, the diameter of $\{x_n : n \geq N\}$ also goes to zero meaning (x_n) is a Cauchy sequence. Thus, since (X, d) is a complete metric space, we know that (x_n) converges to some $x \in X$.

Furthermore, since each F_n is closed, then we can use the result from the previous question to conclude that (F_n, d) is also a complete metric space. Then, for each F_i, we can consider the tail of (x_n) starting at x_i to be a new sequence which also must be Cauchy and is contained in F_i, thus converges in F_i. Since this is true for all of the subsets, we can finally conclude that (x_n) converges inside of each F_n. Thus, it converges in their intersection, meaning the intersection is nonempty and is in fact equal to $\{x\}$. $\square$