# Advanced Calc. Homework 5

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## Introduction

Note, I will use the following definition for the convergence (or limit) of a sequence $(s_n)$ to some real number $s$ if:

$$\forall\, \varepsilon > 0,\ \exists \text{ a number } N \text{ such that } n > N \text{ implies } |s_n - s| < \varepsilon, \tag{1}$$

then the limit exists and is equal to $s$.

## 8.1

Prove the following:

(a) $\lim \dfrac{(-1)^n}{n} = 0$

(b) $\lim \dfrac{1}{n^{\frac{1}{3}}} = 0$

(c) $\lim \frac{2n-1}{3n+2} = \frac{2}{3}$

(d) $\lim \frac{n+6}{n^2-6} = 0$

*Proof.* **(a)**
Using our definition in (1), let $\varepsilon > 0$ be some given number and $(s_n) = \dfrac{(-1)^n}{n}$, then we wish to show $s = 0$. I will first examine $|s_n - s|$:

$$\begin{aligned}
|s_n - s| &= \left|\frac{(-1)^n}{n} - 0\right| \\
&= \left|\frac{(-1)^n}{n}\right| \\
&= \frac{|(-1)^n|}{|n|} \\
&= \frac{1}{n}
\end{aligned}$$

Thus, if we want $|s_n - s| < \varepsilon$, we can make the following algebraic manipulations:

$$\begin{aligned}
&|s_n - s| < \varepsilon \\
\iff\ & \frac{1}{n} < \varepsilon && \text{by the above examination} \\
\iff\ & 1 < n \cdot \varepsilon \\
\iff\ & \frac{1}{\varepsilon} < n
\end{aligned}$$

Therefore, if we choose $N = \dfrac{1}{\varepsilon}$, then it is clear by the above calculations that having $n > N$ implies that $|s_n - s| < \varepsilon$, proving that the limit does, indeed, equal 0. □

*Proof.* **(b)**
Using our definition in (1), let $\varepsilon > 0$ be some given number and $(s_n) = \dfrac{1}{n^{\frac{1}{3}}}$, then we wish to show $s = 0$. I will first

examine $|s_n - s|$:

$$\begin{aligned} |s_n - s| &= \left| \frac{1}{n^{\frac{1}{3}}} - 0 \right| \\ &= \left| \frac{1}{n^{\frac{1}{3}}} \right| \\ &= \frac{|1|}{|n^{\frac{1}{3}}|} \\ &= \frac{1}{n^{\frac{1}{3}}} \end{aligned}$$

Thus, if we want $|s_n - s| < \varepsilon$, we can make the following algebraic manipulations:

$$\begin{aligned} & |s_n - s| < \varepsilon \\ \iff & \frac{1}{n^{\frac{1}{3}}} < \varepsilon && \text{by the above examination} \\ \iff & \frac{1}{n} < \varepsilon^3 \\ \iff & 1 < n \cdot \varepsilon^3 \\ \iff & \frac{1}{\varepsilon^3} < n \end{aligned}$$

Therefore, if we choose $N = \frac{1}{\varepsilon^3}$, then it is clear by the above calculations that having $n > N$ implies that $|s_n - s| < \varepsilon$, proving that the limit does, indeed, equal 0. □

*Proof.* **(c)**
Using our definition in (1), let $\varepsilon > 0$ be some given number and $(s_n) = \frac{2n-1}{3n+2}$, then we wish to show $s = \frac{2}{3}$. I will first examine $|s_n - s|$:

$$\begin{aligned} |s_n - s| &= \left| \frac{2n-1}{3n+2} - \frac{2}{3} \right| \\ &= \left| \frac{3(2n-1)}{3(3n+2)} - \frac{2(3n+2)}{3(3n+2)} \right| \\ &= \left| \frac{(6n-3) - (6n+4)}{9n+6} \right| \\ &= \left| \frac{-7}{9n+6} \right| \\ &= \frac{|-7|}{|9n+6|} \\ &= \frac{7}{9n+6} \end{aligned}$$

Thus, if we want $|s_n - s| < \varepsilon$, we can make the following algebraic manipulations:

$$\begin{aligned} & |s_n - s| < \varepsilon \\ \iff & \frac{7}{9n+6} < \varepsilon && \text{by the above examination} \\ \iff & 7 < \varepsilon(9n+6) \\ \iff & 7 < 9n\varepsilon + 6\varepsilon \\ \iff & 7 - 6\varepsilon < 9n\varepsilon \\ \iff & \frac{7-6\varepsilon}{9\varepsilon} < n \end{aligned}$$

Therefore, if we choose $N = \frac{7-6\varepsilon}{9\varepsilon}$, then it is clear by the above calculations that having $n > N$ implies that $|s_n - s| < \varepsilon$, proving that the limit does, indeed, equal $\frac{2}{3}$. □

*Proof.* **(d)**
Using our definition in (1), let $\varepsilon > 0$ be some given number and $(s_n) = \frac{n+6}{n^2-6}$, then we wish to show $s = 0$. I will first

examine $|s_n - s|$:

$$\begin{aligned}
|s_n - s| &= \left|\frac{n+6}{n^2-6} - 0\right| \\
&= \left|\frac{n+6}{n^2-6}\right| \\
&= \frac{|n+6|}{|n^2-6|} \\
&= \frac{n+6}{n^2-6} && \text{since we can drop } |\cdot| \text{ as long as } n \geq 3 \\
&\leq \frac{7n}{n^2-6} && \text{since } n+6 \leq 7n \ \forall\, n \in \mathbb{N} \\
&\leq \frac{7n}{\frac{1}{3}n^2} && \text{since } n^2 - 6 \geq \frac{1}{3}n^2 \ \forall\, n \geq 3 \in \mathbb{N} \\
&= \frac{21}{n}
\end{aligned}$$

Thus, if we want $|s_n - s| < \varepsilon$, and since we have just shown that $|s_n - s| \leq \frac{21}{n}$, then, if we can show that under some conditions, we have $\frac{21}{n} < \varepsilon$, then we could conclude that $|s_n - s| < \varepsilon$ by the transitivity of "<" under those same conditions. We can do this by making the following algebraic manipulations:

$$\begin{aligned}
|s_n - s| \leq \frac{21}{n} &< \varepsilon \\
\iff 21 &< n \cdot \varepsilon \\
\iff \frac{21}{\varepsilon} &< n
\end{aligned}$$

Since the above calculations were only valid under the condition that $n \geq 3$, then we know that $N$ must be at least 3 as well. Therefore, if we choose $N = \max\left\{3, \frac{21}{\varepsilon}\right\}$, then it is clear by the above calculations that having $n > N$ implies that $\frac{21}{n} < \varepsilon$, which in turn shows that $|s_n - s| < \varepsilon$ since $|s_n - s| \leq \frac{21}{n}$ proving that the limit does, indeed, equal 0. $\square$

## 8.2

Determine the limits of the following sequences, and then prove your claims.

(a) $a_n = \dfrac{n}{n^2+1}$

(b) $b_n = \dfrac{7n-19}{3n+7}$

(c) $c_n = \dfrac{4n+3}{7n-5}$

(d) $d_n = \dfrac{2n+4}{5n+2}$

(e) $s_n = \dfrac{1}{n}\sin(n)$

**Answer. (a)**
I claim that $\boxed{\lim a_n = 0}$

*Proof.*
First, given some $\varepsilon > 0$, I will examine $|a_n - 0|$:

$$\begin{aligned}
|a_n - 0| &= \left|\frac{n}{n^2+1} - 0\right| \\
&= \frac{|n|}{|n^2+1|} \\
&= \frac{n}{n^2+1} \\
&\leq \frac{n}{n^2} && \text{since } n^2 + 1 > n^2 \\
&= \frac{1}{n}
\end{aligned}$$

Thus, since $|a_n - 0| \leq \frac{1}{n}$, then finding a condition where $\frac{1}{n} < \varepsilon$ would be a sufficient condition to show that $|a_n - 0| < \varepsilon$, we will do this by the following algebraic manipulations:

$$\begin{aligned} &\frac{1}{n} < \varepsilon \\ \iff &\frac{1}{\varepsilon} < n \end{aligned}$$

Thus, choosing $N = \frac{1}{\varepsilon}$ guarantees that with $n > N$, then $\frac{1}{n} < \varepsilon$ which in turn guarantees that $|a_n - 0| < \varepsilon$, proving that the limit does, indeed, equal 0. $\square$

**Answer. (b)**

I claim that $\boxed{\lim b_n = \frac{7}{3}}$

*Proof.*

First, given some $\varepsilon > 0$, I will examine $\left|b_n - \frac{7}{3}\right|$:

$$\begin{aligned} \left|b_n - \frac{7}{3}\right| &= \left|\frac{7n-19}{3n+7} - \frac{7}{3}\right| \\ &= \left|\frac{3(7n-19)}{3(3n+7)} - \frac{7(3n+7)}{3(3n+7)}\right| \\ &= \left|\frac{-106}{9n+21}\right| \\ &= \frac{|-106|}{|9n+21|} \\ &= \frac{106}{9n+21} \end{aligned}$$

Thus, to find a condition in which $\left|b_n - \frac{7}{3}\right| < \varepsilon$, we can do the following algebraic manipulations:

$$\begin{aligned} &\left|b_n - \frac{7}{3}\right| < \varepsilon \\ \iff &\frac{106}{9n+21} < \varepsilon && \text{by the above calculations} \\ \iff &106 < 9n\varepsilon + 21\varepsilon \\ \iff &106 - 21\varepsilon < 9n\varepsilon \\ \iff &\frac{106-21\varepsilon}{9\varepsilon} < n \end{aligned}$$

Thus, choosing $N = \frac{106-21\varepsilon}{9\varepsilon}$ guarantees that with $n > N$, then $\left|b_n - \frac{7}{3}\right| < \varepsilon$, proving that the limit does, indeed, equal $\frac{7}{3}$. $\square$

**Answer. (c)**

I claim that $\boxed{\lim c_n = \frac{4}{7}}$

*Proof.*

First, given some $\varepsilon > 0$, I will examine $\left|c_n - \frac{4}{7}\right|$:

$$\begin{aligned} \left|c_n - \frac{4}{7}\right| &= \left|\frac{4n+3}{7n-5} - \frac{4}{7}\right| \\ &= \left|\frac{7(4n+3)}{7(7n-5)} - \frac{4(7n-5)}{7(7n-5)}\right| \\ &= \left|\frac{41}{49n-35}\right| \\ &= \frac{|41|}{|49n-35|} \\ &= \frac{41}{49n-35} && \text{since } 49n - 35 > 0 \ \forall\, n \in \mathbb{N} \end{aligned}$$

Thus, to find a condition in which $\left|c_n - \frac{4}{7}\right| < \varepsilon$, we can do the following algebraic manipulations:

$$\begin{aligned}
&\left|c_n - \frac{4}{7}\right| < \varepsilon \\
\iff &\frac{41}{49n - 35} < \varepsilon && \text{by the above calculations} \\
\iff &41 < 49n\varepsilon - 35\varepsilon \\
\iff &41 + 35\varepsilon < 49n\varepsilon \\
\iff &\frac{41 + 35\varepsilon}{49\varepsilon} < n
\end{aligned}$$

Thus, choosing $N = \dfrac{41 + 35\varepsilon}{49\varepsilon}$ guarantees that with $n > N$, then $\left|c_n - \frac{4}{7}\right| < \varepsilon$, proving that the limit does, indeed, equal $\frac{4}{7}$. □

**Answer. (d)**

I claim that $\boxed{\lim d_n = \frac{2}{5}}$

*Proof.*

First, given some $\varepsilon > 0$, I will examine $\left|d_n - \frac{2}{5}\right|$:

$$\begin{aligned}
\left|d_n - \frac{2}{5}\right| &= \left|\frac{2n+4}{5n+2} - \frac{2}{5}\right| \\
&= \left|\frac{5(2n+4)}{5(5n+2)} - \frac{2(5n+2)}{5(5n+2)}\right| \\
&= \left|\frac{16}{25n+10}\right| \\
&= \frac{|16|}{|25n+10|} \\
&= \frac{16}{25n+10}
\end{aligned}$$

Thus, to find a condition in which $\left|d_n - \frac{2}{5}\right| < \varepsilon$, we can do the following algebraic manipulations:

$$\begin{aligned}
&\left|d_n - \frac{2}{5}\right| < \varepsilon \\
\iff &\frac{16}{25n + 10} < \varepsilon && \text{by the above calculations} \\
\iff &16 < 25n\varepsilon + 10\varepsilon \\
\iff &16 - 10\varepsilon < 25n\varepsilon \\
\iff &\frac{16 - 10\varepsilon}{25\varepsilon} < n
\end{aligned}$$

Thus, choosing $N = \dfrac{16 - 10\varepsilon}{25\varepsilon}$ guarantees that with $n > N$, then $\left|d_n - \frac{2}{5}\right| < \varepsilon$, proving that the limit does, indeed, equal $\frac{2}{5}$. □

**Answer. (e)**

I claim that $\boxed{\lim s_n = 0}$

*Proof.*

First, given some $\varepsilon > 0$, I will examine $|s_n - 0|$:

$$\begin{aligned}
|s_n - 0| &= \left|\frac{1}{n}\sin(n) - 0\right| \\
&= \left|\frac{1}{n}\right| \cdot |\sin(n)| \\
&\leq \frac{1}{n} && \text{since } -1 \leq \sin(n) \leq 1 \ \forall\, n \in \mathbb{N}
\end{aligned}$$

Thus, to find a condition in which $|s_n - 0| < \varepsilon$, it suffices to find a condition in which $\frac{1}{n} < \varepsilon$ since $|s_n - 0| \leq \frac{1}{n}$. We have seen several times that this is equivalent to the condition that $\frac{1}{\varepsilon} < n$.

Thus, choosing $N = \frac{1}{\varepsilon}$ guarantees that with $n > N$, then $|s_n - 0| \leq \frac{1}{n} < \varepsilon$, proving that the limit does, indeed, equal 0. $\square$

## 8.3

Let $(s_n)$ be a sequence of nonnegative real numbers, and suppose $\lim s_n = 0$. Prove $\lim \sqrt{s_n} = 0$. This will complete the proof for Example 5.

*Proof.*
We are given that for every $\varepsilon_1 > 0$, there exists some $N_1 \in \mathbb{R}$ such that $|s_n - 0| < \varepsilon_1$ for every $n > N_1$. From this, we wish to show that for every $\varepsilon_2 > 0$, there exists some $N_2 \in \mathbb{R}$ such that $|\sqrt{s_n} - 0| < \varepsilon_2$ for every $n > N_2$. Let us choose $\varepsilon_1 = \varepsilon_2^2$. Therefore we know the following holds for $n$ greater than or equal to some $N_1$:

$$\begin{aligned}
&|s_n - 0| < \varepsilon_1 && \text{by assumption} \\
\implies &|s_n| < \varepsilon_1 && \\
\implies &s_n < \varepsilon_1 && \text{since the sequence is nonnegative} \\
\implies &s_n < \varepsilon_2^2 && \text{by our assignment of } \varepsilon_1 \\
\implies &\sqrt{s_n} < \varepsilon_2 && \\
\implies &|\sqrt{s_n} - 0| < \varepsilon_2 && \text{since square roots are positive and we can freely subtract 0}
\end{aligned}$$

Thus, we know that $N_2$ exists where the above condition is satisfied. In particular, with $N_1$ given by the choice of $\varepsilon_1 = \varepsilon_2^2$, $N_2 = N_1$, so it obviously exists; thus, $\lim \sqrt{s_n} = 0$. $\square$

## 8.4

Let $(t_n)$ be a bounded sequence, i.e., there exists $M$ such that $|t_n| \leq M$ for all $n$, and let $(s_n)$ be a sequence such that $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

*Proof.*
I will start by noting that for any $\varepsilon_1 > 0$, we know that there exists some $N_1$ such that $|s_n - 0| < \varepsilon$ for all $n > N_1$. In particular, if we are given some $\varepsilon_2 > 0$, then we know that $N_1$ exists for $\varepsilon_1 = \frac{\varepsilon_2}{M}$ From this I will examine $|s_n t_n - 0|$ in the following manner with some $\varepsilon_2$ given:

$$\begin{aligned}
|s_n t_n - 0| &= |s_n t_n| && \\
&= |s_n| \cdot |t_n| && \\
&\leq |s_n| \cdot M && \text{since } |t_n| \leq M \ \forall\, n \in \mathbb{N} \\
&= |s_n - 0| \cdot M && \\
&\leq \varepsilon_1 \cdot M && \text{for all } n > N_1 \text{ by assumption} \\
&= \frac{\varepsilon_2}{M} \cdot M && \text{by our choice of } \varepsilon_1 \\
&= \varepsilon_2
\end{aligned}$$

Thus, we have shown that $|s_n t_n - 0| < \varepsilon_2$ under the condition that $n > N_1$. Therefore, since $\varepsilon_2$ was arbitrarily given to us, we can conclude that $\lim(s_n t_n) = 0$, just as desired. $\square$

## 8.5

(a) Consider three sequences $(a_n), (b_n)$, and $(s_n)$ such that $a_n \leq s_n \leq b_n$ for all $n$ and $\lim a_n = \lim b_n = s$. Prove that $\lim s_n = s$.

(b) Suppose that $(s_n)$ and $(t_n)$ are sequences such that $|s_n| \leq t_n$ for all $n$ and $\lim t_n = 0$. Prove that $\lim s_n = 0$.

*Proof.* **(a)**
What we know:

(1) $a_n \leq s_n$ and $s_n \leq b_n$ for all $n \in \mathbb{N}$

(2) For every $\varepsilon_1 > 0$ there exists some $N_1$ such that $|a_n - s| < \varepsilon_1$ for all $n > N_1$ (since $\lim a_n = s$)

(3) For every $\varepsilon_2 > 0$ there exists some $N_2$ such that $|b_n - s| < \varepsilon_2$ for all $n > N_1$ (since $\lim a_n = s$)

(4) $|x - y| < z$ is equivalent to $y - z < x < y + z$ according to Exercise (3.7)(b)

I will use items (2) and (4) above to conclude that $s - \varepsilon_1 < a_n < s + \varepsilon_1$. Next, I will use items (3) and (4) above to conclude that $s - \varepsilon_2 < b_n < s + \varepsilon_2$. Using these results, I can conclude the following:

$$
\begin{aligned}
s - \varepsilon_1 < a_n \quad \text{and} \quad a_n \leq s_n &\implies s - \varepsilon_1 < s_n \\
b_n < s + \varepsilon_2 \quad \text{and} \quad s_n \leq b_n &\implies s_n < s + \varepsilon_2
\end{aligned}
$$

Since lines (2) and (3) above hold for all $\varepsilon_1$ or $\varepsilon_2$, respectively, I can choose $\varepsilon_1 = \varepsilon_2 = \varepsilon > 0$ and still get the same results above. Therefore, we have that $s - \varepsilon < s_n < s + \varepsilon$. Again, from line (4) above, this shows that $|s_n - s| < \varepsilon$. However, since line (2) only holds for $n > N_1$ and line (3) only holds for $n > N_2$, then we can only validly claim that the last inequality holds whenever $n > N := \max\{N_1, N_2\}$. Therefore, we have, at last, shown that for any given $\varepsilon > 0$, we have $|s_n - s| < \varepsilon$ for all $n > N$; thus, $\lim s_n = s$. □

*Proof.* **(b)**
From Exercise (3.5)(a), we know that $|b| \leq a$ if and only if $-a \leq b \leq a$. From this we can use the assumption that $|s_n| \leq t_n$ for all $n$ to conclude that $-t_n \leq s_n \leq t_n$ for all $n$. Furthermore, we know from the assumption that $\lim t_n = 0$ that for some given $\varepsilon > 0$, that $|t_n - 0| = |t_n| < \varepsilon$ for all $n$ satisfying $n > N$ for some $N$. Using Exercise (3.7)(a), we can see that $-\varepsilon < t_n < \varepsilon$ for $n > N$. Thus,

$$
\begin{aligned}
t_n < \varepsilon &\implies -\varepsilon < -t_n \\
-\varepsilon < -t_n \quad \text{and} \quad -t_n \leq s_n &\implies -\varepsilon < s_n \\
s_n \leq t_n \quad \text{and} \quad t_n < \varepsilon &\implies s_n < \varepsilon \\
-\varepsilon < s_n \quad \text{and} \quad s_n < \varepsilon &\implies |s_n| < \varepsilon \text{ by Exercise (3.7)(a) once again}
\end{aligned}
$$

These above inequalities only hold for $n > N$. Thus, we know that for some given $\varepsilon > 0$, we have that $|s_n| = |s_n - 0| < \varepsilon$ for all $n > N$ for the same $N$ as given above. Thus, we have shown that $\lim s_n = 0$. □

## 8.6

Let $(s_n)$ be a sequence in $\mathbb{R}$

(a) Prove $\lim s_n = 0$ if and only if $\lim |s_n| = 0$.

(b) Observe that if $s_n = (-1)^n$, then $\lim |s_n|$ exists, but $\lim s_n$ does not exist.

*Proof.* **(a)**
$\rightarrow$
Assume that $\lim s_n = 0$, i.e. given some arbitrary $\varepsilon > 0$, there exists some $N$ such that $|s_n - 0| = |s_n| < \varepsilon$ for all $n > N$. Now, I will examine $||s_n| - 0|$:

$$
\begin{aligned}
||s_n| - 0| &= ||s_n|| \\
&= |s_n| \\
&< \varepsilon \qquad \forall\, n > N \text{ by assumption of the the limit of } s_n
\end{aligned}
$$

Thus, $||s_n| - 0| < \varepsilon$ for all $n > N$, so we have shown that $\lim |s_n| = 0$.
$\leftarrow$
Assume that $\lim |s_n| = 0$, i.e. given some arbitrary $\varepsilon > 0$, there exists some $N$ such that $||s_n| - 0| = ||s_n|| = |s_n| < \varepsilon$ for all $n > N$. I will examine $|s_n - 0|$:

$$
\begin{aligned}
|s_n - 0| &= |s_n| \\
&< \varepsilon \qquad \forall\, n > N \text{ by assumption of the the limit of } |s_n|
\end{aligned}
$$

Thus, we have shown that $|s_n - 0| < \varepsilon$ for all $n > N$, so we have shown that $\lim s_n = 0$. □

*Proof.* **(b)**
First, note that $|s_n| = |(-1)^n| = 1$. Thus, $\lim |s_n| = \lim 1 = 1$ since $|1 - 1| = 0 < \varepsilon$ for all $\varepsilon > 0$ no matter what $N$ we choose. Therefore, $\lim |s_n| = 1$.

However, when we look at $\lim s_n = \lim(-1)^n$, we get a different picture. Suppose that $\lim s_n$ does exist, i.e., $\lim s_n = s$ for some $s \in \mathbb{R}$. By the definition of the limit this means that for every $\varepsilon > 0$, there exists some $N$ such that $|s_n - s| < \varepsilon$ for all $n > N$. For convenience, let's limit ourselves to the specific case where $\varepsilon = 1$. This means that $|(-1)^n - s| < 1$ for all $n > N$. Let's break this down into two different cases, we have that $|-1 - s| < 1$ for all odd $n > N$ and $|1 - s| < 1$

for all even $n > N$. This means that the number $s$ must satisfy both of these inequalities. However, if we take these two inequalities to be true we get the following:

$$\begin{aligned} 2 = |1-(-1)| = |(1-s)-(-1-s)| = |(1-s)+-1(-1-s)| &\leq |1-s| + |-1(-1-s)| && \text{by Triangle Inequality} \\ &= |1-s| + |-1-s| \\ &< 1+1 && \text{by the above inequalities} \\ &= 2 \end{aligned}$$

However, this asserts that $2 < 2$, which is ridiculous, so our initial assumption that the limit exists must have been false. Thus, $\lim s_n$ can not exist. ☐

## 8.7

Show that the following sequences do not converge.

(a) $\cos\left(\frac{n\pi}{3}\right)$

(b) $s_n = (-1)^n n$

(c) $\sin\left(\frac{n\pi}{3}\right)$

*Proof.* **(a)**
Assume that the sequence converges (equivalently that the limit of the sequence exists), i.e. $\lim_{n\to\infty}\left(\cos\left(\frac{n\pi}{3}\right)\right) = a$ for some $a \in \mathbb{R}$. This means that for every $\varepsilon > 0$, there exists some $N$ such that $\left|\cos\left(\frac{n\pi}{3}\right) - a\right| < \varepsilon$ for all $n > N$. In particular, if $n > N$ is of the form $n = 6k$ or $n = 6k+3$ for $k \in \mathbb{N}$, and we choose $\varepsilon = 1$, then we get the following 2 inequalities: $|\cos(\frac{6k\pi}{3}) - a| = |1-a| < 1$ and $|\sin(\frac{(6k+3)\pi}{3}) - a| = |-1-a| < 1$. However, in Question 8.6(b) of this homework assignment, we have already shown that the existence of these two inequalities at one time leads to a contradiction asserting that $2 < 2$, which cannot happen. Therefore, our assumption on the existence of the limit must have been false, so this sequence does not converge. ☐

*Proof.* **(b)**
Assume that the sequence converges (equivalently that the limit of the sequence exists), i.e. $\lim(-1)^n n = b$ for some $b \in \mathbb{R}$. This means that for every $\varepsilon > 0$, there exists some $N$ such that $|(-1)^n n - b| < \varepsilon$ for all $n > N$. Let us consider $n > N$ to be even and then also consider $n+2$ which must also be even (Note that $(-1)^n n = n$ for $n$ even). Furthermore, let $\varepsilon = 1$, giving the following inequalities: $|n-b| < 1$ and $|n+2-b| < 1$. From these inequalities, we can get:

$$\begin{aligned} 2 = |(n+2-b)-(n-b)| = |(n+2-b)+-1(n-b)| &\leq |n+2-b| + |-1(n-b)| && \text{by Triangle Inequality} \\ &= |n+2-b| + |n-b| \\ &< 1+1 && \text{by the above inequalities} \\ &= 2 \end{aligned}$$

Therefore, we conclude that $2 < 2$ (uh-oh). This is a contradiction, so our assumption that the limit exists must have been false; therefore, the sequence does not converge. ☐

*Proof.* **(c)**
Assume that the sequence converges (equivalently that the limit of the sequence exists), i.e. $\lim_{n\to\infty}\left(\sin\left(\frac{n\pi}{3}\right)\right) = c$ for some $c \in \mathbb{R}$. This means that for every $\varepsilon > 0$, there exists some $N$ such that $\left|\sin\left(\frac{n\pi}{3}\right) - c\right| < \varepsilon$ for all $n > N$. In particular, I will consider the cases where $n > N$ is of the form $n = 6k+1$ and when $n > N$ is of the form $n = 6k+4$ for $k \in \mathbb{N}$. Furthermore, taking $\varepsilon = \frac{\sqrt{3}}{2}$ gives the following two inequalities: $|\sin(\frac{(6k+1)\pi}{3}) - c| = |\frac{\sqrt{3}}{2} - c| < \frac{\sqrt{3}}{2}$ and $|\sin(\frac{(6k+4)\pi}{3}) - c| = |-\frac{\sqrt{3}}{2} - c| < \frac{\sqrt{3}}{2}$. From these inequalities, we get:

$$\begin{aligned} \sqrt{3} = \left|\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right| &= \left|\left(\frac{\sqrt{3}}{2} - c\right) + -1\left(-\frac{\sqrt{3}}{2} - c\right)\right| \\ &\leq \left|\frac{\sqrt{3}}{2} - c\right| + \left|-1\left(-\frac{\sqrt{3}}{2} - c\right)\right| && \text{by Triangle Inequality} \\ &= \left|\frac{\sqrt{3}}{2} - c\right| + \left|-\frac{\sqrt{3}}{2} - c\right| \\ &< \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} && \text{by the above inequalities} \\ &= \sqrt{3} \end{aligned}$$

This shows that $\sqrt{3} < \sqrt{3}$ which is obviously a contradiction. Therefore, my assumption that the limit exists must have been false; thus, the sequence does not converge. ☐

## 8.9

Let $(s_n)$ be a sequence that converges.

(a) Show that if $s_n \geq a$ for all but finitely many $n$, then $\lim s_n \geq a$.

(b) Show that if $s_n \leq b$ for all but finitely many $n$, then $\lim s_n \leq b$.

(c) Conclude that if all but finitely many $s_n$ belong to $[a, b]$, then $\lim s_n \in [a, b]$.

*Proof.* **(a)**
If $s_n \geq a$ for all but finitely many $n$, then that means there exists some $N_0$ such that $s_n \geq a$ for all $n > N_0$. Let $s = \lim s_n$ and assume that $s < a$. Since the sequence converges, $s \in \mathbb{R}$ and we know that for any $\varepsilon > 0$ there exists some $N$ such that $|s_n - s| < \varepsilon$ for all $n > N$. Let $\varepsilon = a - s$ (note $\varepsilon > 0$ since we assumed $s < a$) and use Exercise (3.7)(b) to conclude that $s - (a - s) < s_n < s + (a - s)$ for all $n > N$ which simplifies to $2s - a < s_n < a$ for all $n > N$. If we make sure our $N$ above satisfies $N > N_0$, then this shows that $s_n < a$ for all $n > N > N_0$. However, one of our conditions on $s_n$ is that it satisfies $s_n \geq a$ for all $n > N_0$, so we have a contradiction. Therefore, our assumption that $s < a$ must have been false, therefore $s \geq a$, just as desired. ☐

*Proof.* **(b)**
If $s_n \leq b$ for all but finitely many $n$, then that means there exists some $N_0$ such that $s_n \leq b$ for all $n > N_0$. Let $s = \lim s_n$ and assume that $s > b$. Since the sequence converges, $s \in \mathbb{R}$ and we know that for any $\varepsilon > 0$ there exists some $N$ such that $|s_n - s| < \varepsilon$ for all $n > N$. Let $\varepsilon = s - b$ (note $\varepsilon > 0$ since we assumed $s > b$) and use Exercise (3.7)(b) to conclude that $s - (s - b) < s_n < s + (s - b)$ for all $n > N$ which simplifies to $b < s_n < 2s - b$ for all $n > N$. If we make sure our $N$ above satisfies $N > N_0$, then this shows that $s_n > b$ for all $n > N > N_0$. However, one of our conditions on $s_n$ is that it satisfies $s_n \leq b$ for all $n > N_0$, so we have a contradiction. Therefore, our assumption that $s > b$ must have been false, therefore $s \leq b$, just as desired. ☐

*Proof.* **(c)**
If $s_n \in [a, b]$ for all but finitely many $n$, then that means there exists some $N_0$ such that $a \leq s_n \leq b$ for all $n > N_0$. Let $s = \lim s_n$, then by part (a) of this question, we know that $a \leq s$. Similarly, by part (b) of this question we can conclude that $s \leq b$. Together, this tells us that $a \leq s \leq b$ or, equivalently, $s \in [a, b]$, just as desired. ☐

## 8.10

Let $(s_n)$ be a convergent sequence, and suppose that $\lim s_n > a$. Prove there exists a number $N$ such that $n > N$ implies $s_n > a$.

*Proof.*
Since $s_n$ converges, we know that $\lim s_n = s$ for some $s \in \mathbb{R}$, i.e. for every $\varepsilon > 0$, there exists some $N$ such that $|s_n - s| < \varepsilon$ for all $n > N$. Since this question supposes that $s > a$, we may choose $\varepsilon = s - a$ (a positive number) and use Exercise (3.7)(b) to conclude that $s - (s - a) < s_n < s + (s - a)$ for all $n > N$ which simplifies to $a < s_n < 2s - a$ for all $n > N$. Thus, the left of these two inequalities shows that there does, indeed, exist some $N$ such that $s_n > a$ for all $n > N$ and that $N$ is precisely the same $N$ that guarantees $|s_n - s| < s - a$ for all $n > N$. ☐