Question 2

At which points $z = x + iy$ is $f(z) = z|z|$ differentiable?

First, note $f(z) = f(x+iy) = (x+iy)\sqrt{x^2+y^2} = x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$

Thus, for $f(z) = u(x,y) + iv(x,y)$ $u(x,y) = x\sqrt{x^2+y^2}$ and $v(x,y) = y\sqrt{x^2+y^2}$

$\Rightarrow u_x = \sqrt{x^2+y^2} + x\left(\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x\right) = (x^2+y^2)^{-1/2}(x^2+y^2+x^2) = (x^2+y^2)^{-1/2}(2x^2+y^2)$

$\Rightarrow u_y = x\left(\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y\right) = xy(x^2+y^2)^{-1/2}$

$\Rightarrow v_x = y\left(\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x\right) = xy(x^2+y^2)^{-1/2}$

$\Rightarrow v_y = \sqrt{x^2+y^2} + y\left(\frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y\right) = (x^2+y^2)^{-1/2}(x^2+y^2+y^2) = (x^2+y^2)^{-1/2}(x^2+2y^2)$

Thus, the Cauchy-Riemann Equations are satisfied when

$$u_x = \frac{2x^2+y^2}{\sqrt{x^2+y^2}} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}} = v_y \quad \text{and} \quad u_y = \frac{xy}{\sqrt{x^2+y^2}} = \frac{-xy}{\sqrt{x^2+y^2}} = -v_x$$

These are not continuous when $x^2+y^2=0$ (i.e. at the origin), so exclude that point which simplifies the equations to

$$x^2 = y^2 \quad \text{and} \quad xy = -xy$$

The right equation only has a solution when x or y is 0, however that is only satisfied in the left equation when $x=y=0$. Thus, C-R is not satisfied anywhere (except possibly at the origin).

Therefore, I must check for differentiability at $z=0$:

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = \lim_{z\to 0} \frac{z|z|}{z} = \lim_{z\to 0} |z| = 0$$

Thus, $f(z) = z|z|$ is differentiable only at $z=0$ with derivative equal to 0.