Complex Analysis Homework 4

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Question 1

Show in two ways that that the function $f(z) = e^{z^2}$ is entire. What is its derivative $f'(z)$?

Answer.
Method 1
First, note that we can write $f(z)$ as $g(h(z))$ for $g(z) = e^z$ and $h(z) = z^2$. Thus, since we know that the compositions of two entire functions is entire, we merely need to show that $g(z)$ and $h(z)$ are entire. I will start by showing that $g(z)$ is entire.

First, let $z = x + iy$ and write $g(z) = g(x + iy) = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y))$ by Euler's Formula. Thus, we can see that writing $g(z)$ as $u(x, y) + iv(x, y)$ gives us $u(x, y) = e^x \cos(y)$ and $v(x, y) = e^x \sin(y)$. Therefore, we have the following:

$$u_x(x, y) = e^x \cos(y) \qquad u_y(x, y) = -e^x \sin(y) \qquad v_x(x, y) = e^x \sin(y) \qquad v_y(x, y) = e^x \cos(y)$$

So it is clear that the Cauchy-Riemann Equations are satisfied since $u_x = v_y$ and $u_y = -v_x$. Additionally, all of the previous first-order partial derivatives being continuous everywhere is sufficient to prove that $g(z)$ is differentiable everywhere, i.e., is entire.
Next, I will again let $z = x + iy$ and examine $h(z) = h(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy$. Therefore, writing $h(z)$ as $u(x, y) + iv(x, y)$ gives us $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Thus, we have the following:

$$u_x(x, y) = 2x \qquad u_y(x, y) = -2y \qquad v_x(x, y) = 2y \qquad v_y(x, y) = 2x$$

So it is clear that the Cauchy-Riemann Equations are satisfied since $u_x = v_y$ and $u_y = -v_x$. Additionally, all of the previous first-order partial derivatives being continuous everywhere is sufficient to prove that $h(z)$ is differentiable everywhere, i.e., is entire.

Since we have proven that $g(z)$ and $h(z)$ are both entire functions, then we have also proven that $f(z) = g(h(z))$ is an entire function. $\square$

Method 2
To prove that $f(z)$ is entire, I will now look at the Cauchy-Riemann equations on $f(z)$ itself. Again, to do this I will need to consider $z = x+iy$ to give us $f(z) = f(x+iy) = e^{(x+iy)^2} = e^{x^2-y^2+2ixy} = e^{x^2-y^2}e^{i(2xy)} = e^{x^2-y^2}(\cos(2xy)+i\sin(2xy))$. Thus, writing $f(z)$ as $u(x, y) = iv(x, y)$ give us $u(x, y) = e^{x^2-y^2}\cos(2xy)$ and $v(x, y) = e^{x^2-y^2}\sin(2xy)$. Therefore, we have the following:

$$\begin{aligned}
u_x(x, y) &= 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy) = 2e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy)) \\
u_y(x, y) &= -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy) = -2e^{x^2-y^2}(y\cos(2xy) + x\sin(2xy)) \\
v_x(x, y) &= 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy) = 2e^{x^2-y^2}(y\cos(2xy) + x\sin(2xy)) \\
v_y(x, y) &= -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy) = 2e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy))
\end{aligned}$$

So it is clear that the Cauchy-Riemann Equations are satisfied since $u_x = v_y$ and $u_y = -v_x$. Additionally, all of the previous first-order partial derivatives being continuous everywhere is sufficient to prove that $f(z)$ is differentiable everywhere, i.e., is entire. $\square$

Calculating the Derivative

By the previous calculation and the fact that $f'(z) = u_x(x, y) + iv_x(x, y)$, we can easily see that

$$\begin{aligned}
u_x + iv_x &= 2e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy)) + i[2e^{x^2-y^2}(y\cos(2xy) + x\sin(2xy))] \\
&= 2e^{x^2-y^2}(x\cos(2xy) - y\sin(2xy) + iy\cos(2xy) + ix\sin(2xy)) \\
&= 2e^{x^2-y^2}((x+iy)\cos(2xy) + i(x+iy)\sin(2xy)) \\
&= 2(x+iy)e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)) \\
&= 2(x+iy)e^{x^2-y^2}e^{2ixy} \\
&= 2(x+iy)e^{x^2-y^2+2ixy} \\
&= 2(x+iy)e^{(x+iy)^2}
\end{aligned}$$

Thus, using the fact that $z = x + iy$, we get the familiar result that $\boxed{f'(z) = 2ze^{z^2}}$