Complex Analysis Homework 6

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Question 1

With the aid of the equalities (for $z = x + iy$),

$$|\sin(z)|^2 = \sin^2(x) + \sinh^2(y), \qquad |\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$$

show that

(a) $|\sinh(y)| \leq |\sin(z)| \leq \cosh(y)$

(b) $|\sinh(y)| \leq |\cos(z)| \leq \cosh(y)$

Answer. (a)
It is clear from the first equality that $\sinh^2(y) = |\sin(z)|^2 - \sin^2(x)$. Since $\sin(x) \in [-1, 1] \implies \sin^2(x) \in [0, 1]$. Thus, $\sin^2(x) \geq 0$ for all $x \in \mathbb{R} \implies -\sin^2(x) \leq 0$ for all $x \in \mathbb{R}$. Thus, using the equality I just found, we get

$$\begin{aligned}
\sinh^2(y) &= |\sin(z)|^2 - \sin^2(x) \\
&\leq |\sin(z)|^2 - 0 \\
&= |\sin(z)|^2 \\
\implies |\sinh(y)| &\leq |\sin(z)| \quad \text{the first of the two inequalities we need to show}
\end{aligned}$$

Likewise, if we use the fact that $\sinh^2(y) = \cosh^2(y) - 1$, then we get from the first equality that $|\sin(z)|^2 = \sin^2(x) + \cosh^2(y) - 1$. Furthermore, using the same analysis on $\sin(x)$ as above, we can determine that $\sin^2(x) \leq 1$ for all $x \in \mathbb{R}$. Thus, putting this together, we get:

$$\begin{aligned}
|\sin(z)|^2 &= \sin^2(x) + \cosh^2(y) - 1 \\
&\leq 1 + \cosh^2(y) - 1 \\
&= \cosh^2(y) \\
\implies |\sin(z)| &\leq \cosh(y) \quad \text{the second of the two inequalities we need to show}
\end{aligned}$$

Thus, we have now shown that $|\sinh(y)| \leq |\sin(z)| \leq \cosh(y)$, exactly as desired.

Answer. (b)
It is clear from the second equality that $\sinh^2(y) = |\cos(z)|^2 - \cos^2(x)$. Since $\cos(x) \in [-1, 1] \implies \cos^2(x) \in [0, 1]$. Thus, $\cos^2(x) \geq 0$ for all $x \in \mathbb{R} \implies -\cos^2(x) \leq 0$ for all $x \in \mathbb{R}$. Thus, using the equality I just found, we get

$$\begin{aligned}
\sinh^2(y) &= |\cos(z)|^2 - \cos^2(x) \\
&\leq |\cos(z)|^2 - 0 \\
&= |\cos(z)|^2 \\
\implies |\sinh(y)| &\leq |\cos(z)| \quad \text{the first of the two inequalities we need to show}
\end{aligned}$$

Likewise, if we use the fact that $\sinh^2(y) = \cosh^2(y) - 1$, then we get from the second equality that $|\cos(z)|^2 = \cos^2(x) + \cosh^2(y) - 1$. Furthermore, using the same analysis on $\cos(x)$ as above, we can determine that $\cos^2(x) \leq 1$ for all $x \in \mathbb{R}$. Thus, putting this together, we get:

$$\begin{aligned}
|\cos(z)|^2 &= \cos^2(x) + \cosh^2(y) - 1 \\
&\leq 1 + \cosh^2(y) - 1 \\
&= \cosh^2(y) \\
\implies |\cos(z)| &\leq \cosh(y) \quad \text{the second of the two inequalities we need to show}
\end{aligned}$$

Thus, we have now shown that $|\sinh(y)| \leq |\cos(z)| \leq \cosh(y)$, exactly as desired.