Complex Analysis Final

Question 1

Suppose that the function f is analytic in a domain $\Omega \subset \mathbb{C}$. Prove that f is constant if $Re(f)$ is constant.

Proof

Assume that $Re(f)$ is constant, i.e. $Re(f(z)) = C \in \mathbb{C}$ $\forall z \in \mathbb{C}$.

Consider the function $g(z) = e^{f(z)}$, and its modulus:

$$|g(z)| = |e^{f(z)}| = |e^{Re(f(z)) + i\,Im(f(z))}| = |e^{Re(f(z))}||e^{i\,Im(f(z))}| = e^{c}$$

Thus, $g(z)$ has the same modulus for every point $z \in \mathbb{C}$
In particular, $g(z)$ is bounded (by e^c) within Ω and attains its maximum. Thus, since f is analytic in Ω, then g is also analytic in Ω by the analyticity of the exponential function. Therefore by the Maximum Modulus Theorem, g is a constant function in Ω. This means that $g'(z) = 0$ $\forall z \in \Omega$. Alternatively, $g'(z) = f'(z)\,e^{f(z)}$ which implies that $f'(z)\,e^{f(z)} = 0$ $\forall z \in \Omega$.

Since the exponential function never attains zero, we can conclude that $f'(z) \equiv 0$. However, having a derivative of zero in a domain implies that f is constant in that domain, so we have proven the statement. ∎