Complex Analysis Homework 6

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Question 5

Determine for which values of z the following series converge absolutely:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{z-1}{z+1}\right)^n$

I will use the Root Test to make these determinations. I will denote $L = \limsup\limits_{n\to\infty} \sqrt[n]{|a_n|}$ for $\{a_n\}$ the sequence being summed in the above infinite series. Note that the series converges absolutely whenever $L < 1$, but must be checked separately whenever $L = 1$.

Answer. (a)

In this case, $a_n = \dfrac{(z+1)^n}{2^n}$, thus

$$\begin{aligned}
\sqrt[n]{|a_n|} &= \sqrt[n]{\left|\frac{(z+1)^n}{2^n}\right|} \\
&= \sqrt[n]{\frac{|(z+1)^n|}{|2^n|}} \\
&= \frac{|z+1|}{2} \\
\implies L = \limsup_{n\to\infty} \sqrt[n]{|a_n|} &= \limsup_{n\to\infty} \frac{|z+1|}{2} = \frac{|z+1|}{2} \\
\implies L < 1 &\iff \frac{|z+1|}{2} < 1 \\
&\iff |z+1| < 2
\end{aligned}$$

Thus, this series converges for all z inside the disk of radius 2 centered at -1. Let's now examine the sum for z along the edge of this disk (when $L = 1$) i.e. z is of the form $z = -1 + 2e^{i\theta}$ for $-\pi < \theta \leq \pi$. Thus, in this case,

$$a_n = \frac{(z+1)^n}{2^n} = \frac{(-1+2e^{i\theta}+1)^n}{2^n} = \frac{2^n e^{in\theta}}{2^n} = e^{in\theta}$$

However, for a series to converge, its underlying sequence must have a limit that converges to 0. In particular, the absolute value of the underlying sequence must converge to 0. However, $|a_n| = |e^{in\theta}| = 1 \implies \lim_{n\to\infty} |a_n| = 1 \neq 0$. Thus, the above series does not converge when z is on the boundary of the disk of radius 2 centered at -1. Therefore,

$$\boxed{\sum_{n=0}^{\infty} \frac{(z+1)^n}{2^n} \text{ converges when } |z+1| < 2 \text{ or, equivalently, when } z \in D(-1,2)}$$

Answer. (b)

In this case, $a_n = \left(\frac{z-1}{z+1}\right)^n$, thus

$$\begin{aligned}
\sqrt[n]{|a_n|} &= \sqrt[n]{\left|\left(\frac{z-1}{z+1}\right)^n\right|} \\
&= \sqrt[n]{\frac{|z-1|^n}{|z+1|^n}} \\
&= \frac{|z-1|}{|z+1|} \\
\implies L = \limsup_{n\to\infty} \sqrt[n]{|a_n|} &= \limsup_{n\to\infty} \frac{|z-1|}{|z+1|} = \frac{|z-1|}{|z+1|} \\
\implies L < 1 &\iff \frac{|z-1|}{|z+1|} < 1 \\
&\iff |z-1| < |z+1| \\
&\iff |z-1|^2 < |z+1|^2 \\
&\iff (x-1)^2 + y^2 < (x+1)^2 + y^2 \qquad \text{for } z = x + iy \\
&\iff (x-1)^2 < (x+1)^2 \\
&\iff x^2 - 2x + 1 < x^2 + 2x + 1 \\
&\iff 0 < 4x \\
&\iff x > 0
\end{aligned}$$

Thus, this series converges for all z with a positive real part. We now need to examine the case where z has a real part equal to 0 (when $L = 1$). In this case, z is of the form $z = iy$. Thus,

$$a_n = \left(\frac{z-1}{z+1}\right)^n = \left(\frac{iy-1}{iy+1}\right)^n$$

$$\implies |a_n| = \left|\left(\frac{iy-1}{iy+1}\right)^n\right| = \left|\frac{iy-1}{iy+1}\right|^n = \left(\frac{|iy-1|}{|iy+1|}\right)^n = \left(\frac{y^2+1}{y^2+1}\right)^n = 1^n = 1$$

Thus, $\lim_{n\to\infty} |a_n| = 1$ which means $\lim_{n\to\infty} a_n \neq 0$, so the series that uses this sequence for its summation term cannot possibly converge. Thus, $\boxed{\sum_{n=0}^{\infty} \left(\frac{z-1}{z+1}\right)^n \text{ converges when } \mathrm{Re}(z) > 0}$