Question 3 Evaluate $\int_\gamma f(z)\,dz$

ⓐ $f(z) = \frac{z^2+3}{z^2}$, $\gamma(t) = e^{-it}$ $t \in [\pi, 2\pi]$ $\Rightarrow \gamma'(t) = -ie^{-it}$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_\pi^{2\pi} \frac{e^{-2it}+3}{e^{-2it}} \cdot (-ie^{-it})\,dt$$

$$= -i\int_\pi^{2\pi} e^{-it} + 3e^{it}\,dt = -i\int_\pi^{2\pi} e^{-it}\,dt - 3i\int_\pi^{2\pi} e^{it}\,dt$$

$$= \frac{-i}{-i} e^{-it}\Big|_\pi^{2\pi} - \frac{3i}{i} e^{it}\Big|_\pi^{2\pi} = \left(e^{-2\pi i} - e^{-\pi i}\right) - 3\left(e^{2\pi i} - e^{\pi i}\right)$$

$$= (1 - -1) - 3(1 - -1) = 2 - 3(2) = 2 - 6 = -4$$

ⓑ $f(z) = \frac{1}{(z-3)^4}$ and γ is circle at center $z=3$ and radius 2

I will parameterize γ by $\gamma(t) = 3 + 2e^{it}$ $t \in [0, 2\pi]$
$\Rightarrow \gamma'(t) = 2ie^{it}$

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt = \int_0^{2\pi} \frac{1}{(3+2e^{it}-3)^4}\left(2ie^{it}\right)dt = 2i\int_0^{2\pi} \frac{1}{(2e^{it})^4} e^{it}\,dt$$

$$= \frac{2i}{2^4}\int_0^{2\pi} e^{-4it} e^{it}\,dt = \frac{i}{2^3}\int_0^{2\pi} e^{-3it}\,dt = \frac{i}{8(-3i)} e^{-3it}\Big|_0^{2\pi}$$

$$= \frac{-1}{24}\left(e^{-6\pi i} - e^0\right) = \frac{-1}{24}(1-1) = \frac{-1}{24}(0) = 0$$