Complex Analysis Homework 5

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Question 2

(a) Consider $z_1 = -1 - i$ and $z_2 = -1$. Show that if all powers involved are principal values, then

$$(z_1 z_2)^i \neq z_1^i z_2^i$$

(b) Let c_1, c_2 and $z \neq 0$ be complex numbers. Prove that if all powers involved are principal values, then

$$z^{c_1} z^{c_2} = z^{c_1 + c_2}$$

First, note the definition of the power function for c, w complex numbers with $w \neq 0$:

$$w^c = e^{c \log(w)}$$

If we are using principal values, then we replace $\log(w)$ with $\operatorname{Log}(w) = \ln(r) + i\theta$ for $w = re^{i\theta}$ and $r > 0$ and $-\pi < \theta \leq \pi$.

Answer. (a)
On the one hand, we can calculate $(z_1 z_2)^i$ in the following manner:

$$\begin{aligned}
(z_1 z_2)^i &= [(-1-i)(-i)]^i \\
&= \left[i + i^2\right]^i \\
&= [-1+i]^i \\
&= e^{i\operatorname{Log}(-1+i)} \\
&= e^{i\left(\ln(\sqrt{2}) + i\left(\frac{3\pi}{4}\right)\right)} && \text{since } -1+i = \sqrt{2}e^{i\frac{3\pi}{4}} \\
&= e^{i\ln(\sqrt{2})} e^{-\frac{3\pi}{4}} \\
&= e^{-\frac{3\pi}{4}} \left(\cos(\ln(\sqrt{2})) + i\sin(\ln(\sqrt{2}))\right)
\end{aligned}$$

On the other hand, when we calculate $z_1^i z_2^i$, we get:

$$\begin{aligned}
z_1^i z_2^i &= (-1-i)^i (-i)^i \\
&= e^{i\operatorname{Log}(-1-i)} e^{i\operatorname{Log}(-i)} \\
&= e^{i\left(\ln(\sqrt{2}) + i\left(-\frac{3\pi}{4}\right)\right)} e^{i\left(\ln(1) + i\left(-\frac{\pi}{2}\right)\right)} && \text{since } -1-i = \sqrt{2}e^{-i\frac{3\pi}{4}} \text{ and } -i = e^{-i\frac{\pi}{2}} \\
&= e^{i\ln(\sqrt{2})} e^{\left(\frac{3\pi}{4} + \frac{\pi}{2}\right)} \\
&= e^{\frac{5\pi}{4}} \left(\cos(\ln(\sqrt{2})) + i\sin(\ln(\sqrt{2}))\right)
\end{aligned}$$

However, since $e^{5\pi/4} \neq e^{-3\pi/4}$ these two numbers do not have the same magnitude, so they obviously cannot be equal to one another.

Proof. **(b)**
Let $c_1, c_2, z \in \mathbb{C}$ and let $z \neq 0$. Therefore, we have

$$\begin{aligned}
z^{c_1} z^{c_2} &= e^{c_1 \operatorname{Log}(z)} e^{c_2 \operatorname{Log}(z)} \\
&= e^{c_1 \operatorname{Log}(z) + c_2 \operatorname{Log}(z)} \\
&= e^{(c_1 + c_2)\operatorname{Log}(z)} \\
&= z^{c_1 + c_2}
\end{aligned}$$

Thus, the statement is proven □