Complex Analysis Homework 9

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Question 3

Find the residues at the singular points in $\mathbb{C}$ for $f(z)$ equal to the following functions

(a) $\dfrac{z}{z^2-1}$

- As seen in the last question, the singular points of f occur at $z=1$ and $z=-1$ and they are both poles of order 1. Thus, using the following formula

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left((z-z_0)^m f(z)\right)\bigg|_{z=z_0} \tag{1}$$

- where z_0 is a pole of f of order at most m for $m \in \mathbb{N}$ [note, I am using the convention that $\frac{d^0}{dz^0}$ represents the "zeroeth" derivative, or the identity operator]. Thus, we have $m=1$ and $z_0=1$ and $z_0=-1$ and we can calculate the following:

$$\begin{aligned}
\operatorname*{Res}_{z=1} f(z) &= \frac{1}{0!}\frac{d^0}{dz^0}\left((z-1)\cdot\frac{z}{(z+1)(z-1)}\right)\bigg|_{z=1} = \frac{1}{1+1} = \frac{1}{2}\\
\operatorname*{Res}_{z=-1} f(z) &= \frac{1}{0!}\frac{d^0}{dz^0}\left((z+1)\cdot\frac{z}{(z+1)(z-1)}\right)\bigg|_{z=-1} = \frac{-1}{-1-1} = \frac{1}{2}
\end{aligned}$$

- Therefore, $\boxed{\operatorname*{Res}_{z=1} f(z) = \operatorname*{Res}_{z=-1} f(z) = \frac{1}{2}}$

(b) $\dfrac{3}{(z-2)^2}$

- It is clear to see that $|f(z)| \to \infty$ as $z \to 2$, so $z=2$ is a singular point and, in fact, a pole for f. Additionally, we can see that $z=2$ is a zero of the denominator of order 2, so it is indeed a pole of order 2. Therefore, by again referring to formula (1) with $m=2$ and $z_0=2$, we can obtain

$$\begin{aligned}
\operatorname*{Res}_{z=2} f(z) &= \frac{1}{1!}\frac{d}{dz}\left((z-2)^2\cdot\frac{3}{(z-2)^2}\right)\bigg|_{z=2}\\
&= \frac{d}{dz}(3)\bigg|_{z=2} = 0
\end{aligned}$$

- Thus, $\boxed{\operatorname*{Res}_{z=2} f(z) = 0}$

(c) $\dfrac{1}{1-z+z^2}$

- This function will have poles whenever the denominator is equal to zero. To find these points, we can use the quadratic formula to obtain:

$$z_{1,2} = \frac{1 \pm \sqrt{1-4(1)(1)}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

- Additionally, by letting $z_1 = \frac{1}{2}\left(1+i\sqrt{3}\right)$ and $z_2 = \frac{1}{2}\left(1-i\sqrt{3}\right)$ we can factor the denominator into $(z-z_1)(z-z_2)$, and we can see that $z=z_1$ and $z=z_2$ are both poles of order 1, so we can use formula (1) with $m=1$

and $z = z_1$ or $z = z_2$ to obtain:

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} f(z) &= \frac{1}{0!}\frac{d^0}{dz^0}\left((z-z_1)\cdot\frac{1}{(z-z_1)(z-z_2)}\right)\Bigg|_{z=z_1} \\
&= \frac{1}{z_1-z_2} = \frac{2}{(1+i\sqrt{3})-(1-i\sqrt{3})} = \frac{1}{i\sqrt{3}} = \frac{-i}{\sqrt{3}} \\
\operatorname*{Res}_{z=z_2} f(z) &= \frac{1}{0!}\frac{d^0}{dz^0}\left((z-z_2)\cdot\frac{1}{(z-z_1)(z-z_2)}\right)\Bigg|_{z=z_2} \\
&= \frac{1}{z_2-z_1} = \frac{2}{(1-i\sqrt{3})-(1+i\sqrt{3})} = \frac{-1}{i\sqrt{3}} = \frac{i}{\sqrt{3}}
\end{aligned}$$

- Thus, $\boxed{\operatorname*{Res}_{z=z_1} f(z) = \frac{-i}{\sqrt{3}} \text{ and } \operatorname*{Res}_{z=z_2} f(z) = \frac{i}{\sqrt{3}}}$

(d) $\tan^2(z)$

- If we let $z_k = \frac{\pi}{2} + \pi k$, then we have seen in the last question that z_k is a pole of order 2 for all $k \in \mathbb{Z}$. I will directly compute the residue by using the integral definition of the residue, i.e.

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\int_\gamma f(z)dz \tag{2}$$

- where γ is a circle around z_0. In our case if we take $\gamma(t) = z_k + e^{it}$ for $t \in [0, 2\pi]$, we obtain:

$$\begin{aligned}
\operatorname*{Res}_{z=z_k} \tan^2(z) &= \frac{1}{2\pi i}\int_\gamma \tan^2(z)dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{\sin^2(z)}{\cos^2(z)}dz \\
&= \frac{1}{2\pi i}\int_\gamma \frac{1-\cos^2(z)}{\cos^2(z)}dz \\
&= \frac{1}{2\pi i}\left(\int_\gamma \sec^2(z)dz - \int_\gamma dz\right)
\end{aligned}$$

- Notice that the second integral is zero since the identity function is entire and γ is closed. Additionally, notice that $\frac{d}{dz}\tan(z) = \sec^2(z)$, and that $\sec^2(z)$ is continuous on some open set containing γ^* (say, for example the annulus of inner radius $\frac{1}{2}$ and outer radius $\frac{3}{2}$ centered at z_k). This is sufficient to show that the first integral is also equal to zero. Thus, $\boxed{\operatorname*{Res}_{z=z_k} f(z) = 0 \text{ for all } k \in \mathbb{Z}}$