Analysis HW 9

Colin Williams

November 4, 2021

Question 1

Let $X \subset \mathbb{R}^n$ be open. Let $x_0 \in X$ and let $f : X \to \mathbb{R}^m$ be differentiable at x_0. Let $Y \subset \mathbb{R}^m$ be open and such that $f(x_0) \in Y$. Finally, let $g : Y \to \mathbb{R}^k$ be differentiable at $f(x_0)$. Prove that $D_j(g \circ f)(x_0) = \sum_{i=1}^m (D_i g)(f(x_0))(D_j f^i)(x_0)$, where $f^i(x)$ is the i-th component of $f(x)$.

Proof.
Recall from class, we have shown that since f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $g \circ f$ is differentiable at x_0 with derivative $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$. Using this with the definition of the partial derivative, we get:

$$D_j(g \circ f)(x_0) = (g \circ f)'(x_0)e_j = g'(f(x_0))f'(x_0)e_j = g'(f(x_0))D_j f(x_0)$$

Then, denoting $f(x) = (f^1(x), f^2(x), \cdots, f^m(x))$, we see that

$$\begin{aligned} f(x) &= \sum_{i=1}^m f^i(x)e_i \\ \implies D_j f(x_0) &= D_j\left(\sum_{i=1}^m f^i(x_0)e_i\right) \qquad \text{This may be bad (unusual) notation?} \\ &= \sum_{i=1}^m D_j f^i(x_0)e_i \\ &= \Big(D_j f^1(x_0), D_j f^2(x_0), \cdots, D_j f^m(x_0)\Big) \end{aligned}$$

Similarly, since g is differentiable at $f(x_0)$, we can decompose g into the linear combination of its partial derivatives. Let $v \in Y \subset \mathbb{R}^m$, then if $v = (v^1, v^2, \ldots, v^m)$, we get:

$$\begin{aligned} g'(y)v &= \sum_{i=1}^m v^i D_i g(y) \\ \implies g'(f(x_0))v &= \sum_{i=1}^m v^i (D_i g)(f(x_0)) \\ \implies g'(f(x_0))D_j f(x_0) &= \sum_{i=1}^m D_j f^i(x_0)(D_i g)(f(x_0)) \\ &= \sum_{i=1}^m (D_i g)(f(x_0))(D_j f^i(x_0)) \end{aligned}$$

Therefore, we have the desired equality. □

Question 2

Let $X \subset \mathbb{R}^n$ be open, and let $f : X \to \mathbb{R}^n$ be continuously differentiable and such that $f'(x)$ is invertible for every $x \in X$. Prove that $f(X)$ is open in $\mathbb{R}^n$.

Proof.
The hypotheses of the Inverse Function Theorem are clearly satisfied, so we can use that result to say that for every $x \in X$, there exists some open neighborhood of x, say $U_x \subset X$, such that $V_x = f(U_x) \subset f(X)$ is also open. Let us construct these V_x's for each $x \in X$. By definition, we know that every $y_0 \in f(X)$ can be expressed as $y_0 = f(x_0)$ for some $x_0 \in X$. In particular this means that $y_0 \in V_{x_0}$. Therefore, we have

$$f(X) = \bigcup_{x \in X} V_x$$

Where the inclusion $\subset$ follows from the discussion above and the inclusion $\supset$ follows since each V_x is a subset of $f(X)$, so their union must also be a subset. Lastly since each V_x is open and the arbitrary union of open sets is open, then we can conclude that $f(X)$ is also open. $\square$

Question 3

Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(0) = 0$ and $f(x) = x + 2x^2 \sin(1/x)$ if $x \neq 0$. Prove that f is differentiable and $f'(0) = 1$, but f is NOT injective in any neighborhood of $x = 0$. Why does the Inverse Function Theorem not apply here?

Proof.
It is easy to see that f is differentiable at all nonzero inputs since the functions $\{x, x^2, \sin(x), 1/x\}$ are all differentiable for $x \neq 0$, so their sum/product/composition is also differentiable for $x \neq 0$. Therefore, I will use the definition of the derivative to determine the differentiability at the point $x = 0$:

$$\begin{aligned}\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{x + 2x^2 \sin(1/x)}{x} \\ &= \lim_{x\to 0} 1 + 2x \sin(1/x)\end{aligned}$$

Since $\sin(\cdot)$ is bounded and $2x \to 0$ as $x \to 0$, then we see that $2x \sin(1/x) \to 0$ as $x \to 0$. Thus, the limit above exists and in fact $f'(0) = 1$.

To see the lack of injectivity, let us first look at the derivative of the function for non-zero x-values:

$$f'(x) = 1 + 4x \sin(1/x) - 2 \cos(1/x)$$

Consider the point $x_k = \frac{1}{2k\pi}$ for $k \in \mathbb{Z}\backslash\{0\}$ which gives

$$f'(x_k) = 1 + \frac{4}{2k\pi} \sin(2k\pi) - 2 \cos(2k\pi) = 1 - 2 = -1$$

Next, consider the point $x_n = \frac{1}{(2n+1)\pi}$ for $n \in \mathbb{Z}$ which gives

$$f'(x_n) = 1 + \frac{4}{(2n+1)\pi} \sin((2n+1)\pi) - 2 \cos((2n+1)\pi) = 1 + 2 = 3$$

Therefore, given some neighborhood of the point 0, say $U_r(0)$, we can choose k and n large enough [say $|k| > \frac{1}{2\pi r}$ and $|n| > \frac{1}{2\pi r} - \frac{1}{2}$] such that $x_k, x_n \in U_r(0)$. In particular, since $U_r(0)$ is connected, then the open interval connecting x_k and x_n is contained inside of $U_r(0)$. In particular, since $f'(x_k) < 0$ and $f'(x_n) > 0$, using the Intermediate Value Theorem on f' over the interval connecting x_k and x_n, we can conclude that there exists some point $x_0 \in U_r(0)$ such that $f'(x_0) = 0$. Furthermore, this point x_0 must be a local maximum or minimum (not an inflection point) since the derivative switches signs on either side of that point.

WLOG, assume that x_0 is a point of maximum (proof is nearly identical with a minimum), then there is an ε-neighborhood of x_0 such that $x_1 := x_0 - \varepsilon/2$ and $x_2 := x_0 + \varepsilon/2$ satisfy $f(x_1) < f(x_0)$ and $f(x_2) < f(x_0)$. If $f(x_1) = f(x_2)$ we are done and f is not injective. If not, assume that $f(x_1) < f(x_2) < f(x_0)$. Then, by intermediate value theorem, there exists some $\xi \in (x_1, x_0)$ such that $f(\xi) = f(x_2)$. If $f(x_2) < f(x_1) < f(x_0)$, then by IVT, there exists some $\zeta \in (x_2, x_0)$ such that $f(\zeta) = f(x_1)$. In either case, f is not injective. $\square$

Answer.
The reason that the Inverse Function Theorem does not apply is because our function f needs to be continuously differentiable. It is, in fact, differentiable, but the derivative is not continuous. Namely, the $\cos(1/x)$ term in the derivative (for $x \neq 0$) does not have a limit as $x \to 0$ (although the derivative at zero is defined and equal to 1). Thus, the function is not continuously differentiable, so we cannot apply the Inverse Function Theorem.

Question 4

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (e^x \cos(y), e^x \sin(y))$. Prove that f is continuously differentiable and that $f'(x, y)$ is invertible at every point $(x, y) \in \mathbb{R}^2$. Is f injective? Surjective?

Proof.

Let us first calculate the partial derivatives of f:

$$\begin{aligned}
D_1 f(x,y) &= \lim_{t\to 0} \frac{f(x+t,y)-f(x,y)}{t} \\
&= \lim_{t\to 0} \left(\frac{e^{x+t}\cos(y) - e^x\cos(y)}{t}, \frac{e^{x+t}\sin(y)-e^x\sin(y)}{t} \right) \\
&= \lim_{t\to 0} \frac{e^t - 1}{t}(e^x\cos(y), e^x\sin(y)) \\
&= \frac{d}{dx}[e^x]_{x=0}(e^x\cos(y), e^x\sin(y)) \\
&= (e^x\cos(y), e^x\sin(y)) \\
D_2 f(x,y) &= \lim_{t\to 0} \frac{f(x,y+t)-f(x,y)}{t} \\
&= \lim_{t\to 0} \left(\frac{e^x\cos(y+t)-e^x\cos(y)}{t}, \frac{e^x\sin(y+t)-e^x\sin(y)}{t} \right) \\
&= e^x \left(\lim_{t\to 0} \frac{\cos(y+t)-\cos(y)}{t}, \lim_{t\to 0}\frac{\sin(y+t)-\sin(y)}{t} \right) \\
&= e^x \left(\frac{d}{dy}[\cos(y)], \frac{d}{dy}[\sin(y)] \right) \\
&= (-e^x\sin(y), e^x\cos(y))
\end{aligned}$$

Notice each component function is the product of continuous functions, so in fact each partial derivative of f is continuous. Therefore, since continuous partial derivatives imply continuous differentiability, then we have that f is continuously differentiable. Furthermore, if we have a vector $v = (\alpha, \beta) \in \mathbb{R}^2$, then we see that

$$\begin{aligned}
f'(x,y)v = \sum_{i=1}^{2} D_i f(x,y)v^i &= \alpha(e^x\cos(y), e^x\sin(y)) + \beta(-e^x\sin(y), e^x\cos(y)) \\
&= e^x\Big(\alpha\cos(y) - \beta\sin(y), \alpha\sin(y)+\beta\cos(y)\Big)
\end{aligned}$$

If I wish to have v in the kernel of $f'(x,y)$, then this previous quantity must be the zero vector. Since $e^x \neq 0$ for all $x \in \mathbb{R}$, this gives us

$$\alpha\cos(y) - \beta\sin(y) = 0 \qquad\qquad \alpha\sin(y)+\beta\cos(y) = 0$$

If we take $\alpha = 0$, then we are left with $\beta\sin(y) = 0$ and $\beta\cos(y) = 0$. However, $\sin(\cdot)$ and $\cos(\cdot)$ are never zero at the same input. Therefore, for both of these to hold, we must have that $\beta = 0$ as well. Additionally, if we first assume that $\beta = 0$, then we can similarly conclude that $\alpha = 0$. Therefore, assume that $\alpha, \beta \neq 0$ which gives:

$$\frac{\alpha}{\beta} = \frac{\sin(y)}{\cos(y)} \qquad\qquad \frac{\alpha}{\beta} = -\frac{\cos(y)}{\sin(y)}$$

$$\begin{aligned}
&\implies \frac{\sin(y)}{\cos(y)} = \frac{-\cos(y)}{\sin(y)} \\
&\implies \sin^2(y) = -\cos^2(y) \\
&\implies \sin^2(y)+\cos^2(y) = 0
\end{aligned}$$

However, this last line is a contradiction to the Pythagorean identity for $\sin(\cdot)$ and $\cos(\cdot)$, so we have a contradiction. Therefore, it is not possible for α or β to be non-zero (in particular, the issue arose from assuming β was non-zero which means we must have $\beta = 0$, but we already showed that this means $\alpha = 0$). Thus, if $v \in \ker(f'(x,y))$, then we must have that $v = (0,0)$. In other words, $\ker(f'(x,y)) = \{(0,0)\}$ which means that $f'(x,y)$ is invertible for all $(x,y) \in \mathbb{R}^2$. $\square$

Answer.
First, note that f is not injective since

$$f(0,0) = f(0,2\pi) = (1,0)$$

but $(0,0) \neq (0,2\pi)$. Additionally, f is not surjective since there is no pair (x,y) such that $f(x,y) = (0,0)$. If there were, then

$$\begin{aligned}
&(e^x\cos(y), e^x\sin(y)) = (0,0) \\
\implies &e^x(\cos(y),\sin(y)) = (0,0) \\
\implies &(\cos(y),\sin(y)) = (0,0)
\end{aligned}$$

However, if we take the Euclidean norm of both sides of this last "equality" we see that the norm of the left is one and the norm of the right is zero. Therefore, we cannot find such a y that makes this equality hold true, so no pair of points maps to $(0, 0)$ which means that f is not surjective.

If I'm not mistaken, this is an example of the Inverse Function Theorem holding only in a local sense and not a global sense. If we restrict our subset, U, of our domain from the Inverse Function Theorem to be any open set where the y-coordinates are bounded to an interval of length less than 2π then there are indeed no inconsistencies with the Inverse Function Theorem.