Analysis HW 5

Colin Williams

September 29, 2021

Question 1

Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is twice differentiable and such that

(a) f'' is not continuous.

- Consider the function

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

- We can evaluate the derivative for all $x \neq 0$ using a series of chain rules and product rules to get

$$f'(x) = 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right) \qquad x \neq 0$$

When $x = 0$, we can evaluate the derivative by looking at the limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^4 \sin\left(\frac{1}{x}\right)}{x} = \lim_{x\to 0} x^3 \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} \left| x^3 \sin\left(\frac{1}{x}\right) \right| \leq \lim_{x\to 0} |x^3| = 0$$

In particular, we get that the absolute value of our derivative is zero, which means that the derivative itself is $f'(0) = 0$. Comparing this with $f'(x)$ for $x \neq 0$ we see that the function

$$f'(x) = \begin{cases} 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous since $\lim_{x\to 0} x^3 \sin(1/x)$ and $\lim_{x\to 0} x^2 \cos(1/x)$ are both equal to zero.

- We can once again use our rules from calculus to calculate the derivative when $x \neq 0$ to get

$$\begin{aligned} f''(x) &= 12x^2 \sin\left(\frac{1}{x}\right) - 4x \cos\left(\frac{1}{x}\right) - 2x \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) && x \neq 0 \\ &= 12x^2 \sin\left(\frac{1}{x}\right) - 6x \cos\left(\frac{1}{x}\right) - \sin\left(\frac{1}{x}\right) && x \neq 0 \end{aligned}$$

When $x = 0$, we can use the definition of the derivative and calculate the limit as

$$\begin{aligned} \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x\to 0} \frac{4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right)}{x} &= \lim_{x\to 0} 4x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) \\ &\leq \lim_{x\to 0} \left| 4x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) \right| \\ &\leq \lim_{x\to 0} \left| 4x^2 \sin\left(\frac{1}{x}\right) \right| + \lim_{x\to 0} \left| x \cos\left(\frac{1}{x}\right) \right| \\ &\leq \lim_{x\to 0} |4x^2| + \lim_{x\to 0} |x| \\ &= 0 \end{aligned}$$

In particular, the limits before taking absolute values must also be zero meaning the derivative itself is $f''(0) = 0$. Therefore, we can see that f'' exists for all $x \in \mathbb{R}$. Therefore, f is indeed twice differentiable. However, Notice that

$$\lim_{x\to 0} f''(x)$$

does not exist because of the term $\sin(1/x)$. Therefore, f'' is not continuous at $x = 0$, meaning it is not continuous on $\mathbb{R}$.

(b) f'' is continuous, but not differentiable

- Consider the function

$$f(x) = x^2|x|$$

- We can see that f can be piecewise defined as

$$f(x) = \begin{cases} -x^3 & \text{for } x < 0 \\ x^3 & \text{for } x \geq 0 \end{cases}$$

- Since a function can only be differentiable when defined in an open set, we can use this above representation to find the derivative for all $x \neq 0$ by simply using the power rule from Calc I. In this manner, we get

$$f'(x) = \begin{cases} -3x^2 & \text{for } x < 0 \\ 3x^2 & \text{for } x > 0 \end{cases} \qquad = 3x|x| \qquad \forall\, x \neq 0$$

  Also, at the point $x = 0$, we have

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2|x|}{x} = \lim_{x \to 0} x|x| = 0 \implies f'(0) = 0$$

  Thus, the derivative at all other points agrees with the derivative at zero, so we can continuously define $f'(x) = 3x|x|$ for all $x \in \mathbb{R}$.

- Next, I will show that f' is differentiable, i.e. that f is twice differentiable. Note, we have done this example in class (up to a factor of 3). In class we obtained that $g(x) = x|x|$ is differentiable with derivative of $g'(x) = 2|x|$ for all $x \in \mathbb{R}$. Thus, since the constant multiple of a differentiable function is differentiable (with derivative scaled accordingly), then we get that f is twice differentiable with second derivative $f''(x) = 6|x|$ for all $x \in \mathbb{R}$. This function is clearly continuous, but we have shown before that $|x|$ is not differentiable at 0, so f'' is not differentiable at zero either.
- Therefore, f is twice differentiable with a continuous but not differentiable second derivative.

Question 2

Let $f : (a, b) \to \mathbb{R}$ be differentiable and such that $f'(x) > 0$ in (a, b). Prove

(a) f is injective.

(b) $f((a, b))$ is an open interval.

(c) $f^{-1} : f((a, b)) \to \mathbb{R}$ is differentiable.

Proof.

(a) Assume that f is not injective. In other words, there exists some $x_0, y_0 \in (a, b)$ such that $f(x_0) = f(y_0)$ and $x_0 \neq y_0$. WLOG, assume that $x_0 < y_0$. This means that $(x_0, y_0) \subset (a, b)$. Thus, since f is differentiable on (a, b) it must continuous on $[x_0, y_0]$ and differentiable on (x_0, y_0). Therefore, by Rolle's Theorem, we can say that there exists some $c \in (x_0, y_0)$ such that $f'(c) = 0$. This is a contradiction, however, to the fact that $f'(x) > 0$ for all $x \in (a, b)$. Therefore, f must be injective.

(b) I will first show that f is an increasing function on (a, b). Let a_1 and a_2 be two arbitrary points in (a, b) such that $a_1 < a_2$. Then, by the differentiability of f on (a, b), we have that f is continuous on $[a_1, a_2]$ and differentiable on (a_1, a_2). Thus, we can apply the Mean Value Theorem to conclude the existence of a $c \in (a_1, a_2)$ such that

$$f'(c) = \frac{f(a_2) - f(a_1)}{a_2 - a_1}$$

Notice that since $f'(x) > 0$ for all $x \in (a, b)$, we have that $f'(c) > 0$. Also, since $a_1 < a_2$, we have that $a_2 - a_1 > 0$. Thus, for the equality above to hold true, we must have that the numerator is also positive, i.e. $f(a_2) - f(a_1) > 0 \implies f(a_1) < f(a_2)$. Therefore, for any two points in (a, b), f maps the larger input to the larger output, meaning f is monotonically increasing. Furthermore, since f is continuous on (a, b) and (a, b) is connected, then $f((a, b))$ is connected. Thus, since the only connected subsets of $\mathbb{R}$ are intervals, we know that $f((a, b))$ must be an interval. Next, define the following constants in the extended real line:

$$m = \lim_{x \to a^+} f(x) \qquad\qquad M = \lim_{x \to b^-} f(x)$$

if f is defined continuously at its endpoints, then $m = f(a)$ and $M = f(b)$. Regardless, since f is monotonically increasing, we know that m is the infimum and M is the supremum of the interval $f((a, b))$. Furthermore, the infimum and supremum of $f((a,b))$ are never attained. To see this, assume there is a point $x_0 \in (a,b)$ such that $f(x_0) = m$. However, since $a < (a+x_0)/2 < x_0$, we have $f((a+x_0)/2) < f(x_0) = m$ which is a contradiction to m being the infimum (this follows equivalently for M as supremum). Therefore, we have precisely the open interval

$$f((a,b)) = (m, M).$$

(c) Let's first get an intuitive sense for what the derivative should be. Note that $f(f^{-1}(y)) = y$. Thus, by taking derivatives using the chain rule, we get

$$\begin{aligned} f'(f^{-1}(y))\frac{d}{dy}(f^{-1}(y)) &= 1 \\ \implies \frac{d}{dy}(f^{-1}(y)) &= \frac{1}{f'(f^{-1}(y))} \end{aligned}$$

Therefore, if the derivative exists, we would expect it to be of this form. Thus, let $y_0 \in f((a,b))$ with $f(x_0) = y_0$ for unique $x_0 \in (a,b)$. Similarly, for any arbitrary $y \in f((a,b))$, we can find a unique $x \in (a,b)$ such that $f(x) = y$. The uniqueness of both of these follows from the result in part (a). Now, we can examine the limit

$$\begin{aligned} \lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} &= \lim_{y \to y_0} \frac{f^{-1}(f(x)) - f^{-1}(f(x_0))}{f(x) - f(x_0)} \\ &= \lim_{y \to y_0} \frac{x - x_0}{f(x) - f(x_0)} \\ &= \lim_{y \to y_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \end{aligned}$$

Notice as $y \to y_0$, we have that $f(x) \to f(x_0)$ by definition of y and y_0. However, since f is continuous and injective, then $f(x) \to f(x_0) \implies x \to x_0$. Therefore, we can change the bounds of the limit to say

$$\begin{aligned} \lim_{y \to y_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} &= \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} \\ &= \frac{1}{\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}} \\ &= \frac{1}{f'(x_0)} \\ &= \frac{1}{f'(f^{-1}(y_0))} \end{aligned}$$

Note that this is always well defined since $f'(x) > 0$ for all $x \in (a,b)$ so we are never dividing by zero. Therefore, the derivative of f^{-1} exists at y_0 and since that was an arbitrary point of $f((a,b))$, we can say that f^{-1} is differentiable over all of $f((a,b))$.

□

Question 3

Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable. Prove that f is "uniformly differentiable" i.e. that for all $r > 0$, there exists some $s > 0$ such that $\forall\, x, y \in [a,b]$

$$|x - y| < s \implies \left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < r$$

Proof.
Let us fix $r > 0$. Note, we have proven that for any continuous function $g : X \to Y$ where X is compact, that g is uniformly continuous. Thus, since $[a,b]$ is compact and f' is continuous, we can conclude that $f' : [a,b] \to \mathbb{R}$ is uniformly continuous. This means that there exists some $s > 0$ such that

$$|x - y| < s \implies |f'(x) - f'(y)| < r$$

Furthermore, if we consider some $x < y \in [a,b]$, note that since f is differentiable on $[a,b] \supset [x,y]$, we can apply the Mean Value Theorem on (x,y) to conclude that there exists some $c \in (x,y)$ such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

Therefore, assume that $|x - y| < s$. In particular, since $c \in (x, y)$, we have that $|x - c| < s$. Thus, using the previous results with c instead of y, we get

$$\begin{aligned} &|f'(c) - f'(x)| < r \\ \implies &\left| \frac{f(x) - f(y)}{x - y} - f'(x) \right| < r. \end{aligned}$$

□

Question 4

Let $f : [a, b] \to \mathbb{R}$ be thrice differentiable. Prove that there exists some $c, d \in (a, b)$ such that

$$f(b) - f(a) = \frac{f'(a) + f'(b)}{2}(b - a) - \frac{f'''(c)}{12}(b - a)^3 = f'\left(\frac{a + b}{2}\right)(b - a) + \frac{f'''(d)}{24}(b - a)^3$$

Proof.
Define $k \in \mathbb{R}$ as the constant

$$k = \frac{f(b) - f(a) - \frac{f'(a)+f'(b)}{2}(b - a)}{-(b - a)^3}$$

In particular, k satisfies the following equation:

$$f(b) - f(a) = \frac{f'(a) + f'(b)}{2}(b - a) - k(b - a)^3$$

Define the function $g : [a, b] \to \mathbb{R}$ as follows

$$g(t) = f(t) - f(a) - \frac{f'(a) + f'(t)}{2}(t - a) + k(t - a)^3$$

Notice that g must be twice differentiable since f is thrice differentiable, f' is twice differentiable, and polynomials are infinitely differentiable. Thus, let us examine what happens when $t = a$ or $t = b$:

$$\begin{aligned} g(a) &= f(a) - f(a) - \frac{f'(a) + f'(a)}{2}(a - a) + k(a - a)^3 \\ &= 0 \\ g(b) &= f(b) - f(a) - \frac{f'(a) + f'(b)}{2}(b - a) + k(b - a)^3 \\ &= -k(b - a)^3 + k(b - a)^3 \\ &= 0 \end{aligned}$$

Thus, $g(a) = g(b) = 0$, so by Rolle's Theorem, there exists some $c_1 \in (a, b)$ such that $g'(c_1) = 0$. In addition, we get

$$\begin{aligned} g'(t) &= f'(t) - \frac{f''(t)}{2}(t - a) - \frac{f'(a) + f'(t)}{2} + 3k(t - a)^2 \\ \implies g'(a) &= f'(a) - \frac{f''(a)}{2}(a - a) - \frac{f'(a) + f'(a)}{2} + 3k(a - a)^2 \\ &= f'(a) - f'(a) \\ &= 0 \end{aligned}$$

Thus, $g'(a) = g'(c_1) = 0$, so by Rolle's Theorem, there exists some $c_2 \in (a, c_1)$ such that $g''(c_2) = 0$. Using this, we have

$$\begin{aligned} g''(t) &= f''(t) - \frac{f'''(t)}{2}(t - a) - \frac{f''(t)}{2} - \frac{f''(t)}{2} + 6k(t - a) \\ &= 6k(t - a) - \frac{f'''(t)}{2}(t - a) \\ \implies 0 &= 6k(c_2 - a) - \frac{f'''(c_2)}{2}(c_2 - a) && \text{using } t = c_2 \\ \implies \frac{f'''(c_2)}{2}(c_2 - a) &= 6k(c_2 - a) \\ \implies \frac{f'''(c_2)}{12} &= k \end{aligned}$$

Therefore, we have proven the existence of the $c = c_2$ in (a, b) such that the following equality holds:

$$f(b) - f(a) = \frac{f'(a) + f'(b)}{2}(b - a) - \frac{f'''(c)}{12}(b - a)^3$$

For the next equality, let $K \in \mathbb{R}$ be the constant defined as

$$K = \frac{f(b) - f(a) - f'\left(\frac{a+b}{2}\right)(b - a)}{(b - a)^3}$$

In other words, K satisfies the following equation

$$f(b) - f(a) = f'\left(\frac{a + b}{2}\right)(b - a) + K(b - a)^3$$

Furthermore, WLOG, we may assume that $a = -b$, i.e. that our domain is a symmetric closed interval. If this is not already the case define $\tilde{f} : \left[\frac{-(b-a)}{2}, \frac{b-a}{2}\right] \to \mathbb{R}$ as

$$\tilde{f}(x) = f\left(x + \frac{a + b}{2}\right)$$

and notice that $f(a) = \tilde{f}\left(\frac{-(b-a)}{2}\right)$ and $f(b) = \tilde{f}\left(\frac{b-a}{2}\right)$. Furthermore, $f'\left(\frac{a+b}{2}\right) = \tilde{f}'(0)$. Therefore replacing all a with $\frac{-(b-a)}{2}$, all b with $\frac{b-a}{2}$ and f with $\tilde{f}$, we get

$$K = \frac{f(b) - f(a) - f'\left(\frac{a+b}{2}\right)(b - a)}{(b - a)^3} = \frac{\tilde{f}\left(\frac{b-a}{2}\right) - \tilde{f}\left(\frac{-(b-a)}{2}\right) - \tilde{f}'(0)\left(\frac{b-a}{2} + \frac{b-a}{2}\right)}{\left(\frac{b-a}{2} + \frac{b-a}{2}\right)^3} = \frac{\tilde{f}\left(\frac{b-a}{2}\right) - \tilde{f}\left(\frac{-(b-a)}{2}\right) - \tilde{f}'(0)(b - a)}{(b - a)^3}$$

Thus, we see that our problem is equivalent to another problem with symmetric endpoints, so it is fine to assume that $a = -b$, or that our domain is $[-b, b]$ (this is assuming that $b > 0$, but once again can be assumed without loss of generality). With this in mind, we can rephrase our starting expression as

$$\begin{aligned} & f(b) - f(-b) = f'(0)(2b) + K(2b)^3 \\ \implies & f(b) - f(-b) = 2f'(0)b + 8Kb^3 \end{aligned}$$

Next, recall that any function f can be expressed as the sum of an even and an odd function, f_e and f_o respectively. Explicitly, these functions can be written as

$$f_e(x) = \frac{f(x) + f(-x)}{2} \qquad \text{and} \qquad f_o(x) = \frac{f(x) - f(-x)}{2}$$

It is easy to see that they are even and odd respectively and that their sum is simply $f(x)$. Using this representation, we see that $f(b) - f(-b) = 2f_o(b)$. Thus, we get

$$\begin{aligned} & 2f_o(b) = 2f'(0)b + 8Kb^3 \\ \implies & f_o(b) = f'(0)b + 4Kb^3 \end{aligned}$$

We are now in good shape to define the function $h : [-b, b] \to \mathbb{R}$ as

$$h(t) = f_o(t) - f'(0)t - 4Kt^3$$

Notice that h is thrice differentiable since f_o has just as many derivatives as f does (which is three) and polynomials are infinitely differentiable. Thus, if we find distinct zeros of the h, we can apply Rolle's Theorem. Notice these zeros are found at

$$\begin{aligned} h(0) &= f_o(0) - f'(0)(0) - 4K(0)^3 \\ &= f_o(0) \\ &= \frac{f(0) - f(0)}{2} \\ &= 0 \\ h(b) &= f_o(b) - f'(0)b - 4Kb^3 \\ &= f'(0)b + 4Kb^3 - f'(0)b - 4Kb^3 \\ &= 0 \end{aligned}$$

Thus, we have $h(0) = h(b) = 0$ so by Rolle's Theorem, there exists some $d_1 \in (0, b)$ such that $h'(d_1) = 0$. In addition, we have that

$$\begin{aligned}
h'(t) &= f_o'(t) - f'(0) - 12Kt^2 \\
\implies h'(0) &= f_o'(0) - f'(0) - 12K(0)^2 \\
&= \frac{f'(0) + f'(0)}{2} - f'(0) \\
&= f'(0) - f'(0) \\
&= 0
\end{aligned}$$

Thus, we have $h'(0) = h'(d_1) = 0$, so by Rolle's Theorem, there exists some $d_2 \in (0, d_1)$ such that $h''(d_2) = 0$. This implies that

$$\begin{aligned}
h''(t) &= f_o''(t) - 24Kt \\
\implies 0 &= f_o''(d_2) - 24Kd_2 \\
\implies 24Kd_2 &= \frac{f''(d_2) - f''(-d_2)}{2} \\
\implies 24K &= \frac{f''(d_2) - f''(-d_2)}{2d_2} \\
\implies 24K &= \frac{f''(d_2) - f''(-d_2)}{d_2 - (-d_2)}
\end{aligned}$$

However, notice on the RHS of the last equality, we have precisely the form of the Mean Value Theorem. Thus, since f'' is continuous and differentiable on $[-d_2, d_2]$, we can conclude that there exists some $d \in (-d_2, d_2)$ such that

$$\begin{aligned}
f'''(d) &= \frac{f''(d_2) - f''(-d_2)}{d_2 - (-d_2)} \\
\implies 24K &= f'''(d) \\
\implies K &= \frac{f'''(d)}{24}
\end{aligned}$$

Thus, plugging K back into its defining equation, we get

$$f(b) - f(a) = f'\left(\frac{a+b}{2}\right)(b-a) + \frac{f'''(d)}{24}(b-a)^3$$

$\square$