Complex Analysis Homework 9

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Question 4

Let $\gamma(t) = 2e^{it}$, $t \in [0, 2\pi]$. Use Cauchy's Residue Theorem to evaluate the integrals:

(a) $\displaystyle\int_\gamma z^2 e^{1/z} dz$

- Notice we have a singularity at $z = 0$ since $1/z$ is not defined at 0. Therefore, we can use Cauchy's Residue Theorem to calculate

$$\int_\gamma z^2 e^{1/z} dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(z^2 e^{1/z}\right)$$

- Thus, all that remains is to calculate this residue. To do this, I will find the Laurent Expansion for this function by using the Taylor Expansion for e^w which states that

$$\begin{aligned} e^w &= \sum_{n=0}^{\infty} \frac{w^n}{n!} \\ \implies z^2 e^{1/z} &= z^2 \sum_{n=0}^{\infty} \frac{(1/z)^n}{n!} \\ &= z^2 \sum_{n=0}^{\infty} \frac{1}{z^n n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n-2} n!} \end{aligned}$$

- From this, we can see that the coefficient for the z^{-1} term occurs when $n = 3$. Thus, is equal to $\frac{1}{3!} = \frac{1}{6}$. Using this we obtain that

$$\int_\gamma z^2 e^{1/z} dz = 2\pi i \cdot \operatorname*{Res}_{z=0}\left(z^2 e^{1/z}\right) = \frac{2\pi i}{6} = \boxed{\frac{i\pi}{3}}$$

(b) $\displaystyle\int_\gamma \frac{\sin(z)}{z^2+1} dz$

- Using the same approach as in the last problem, I will first notice that $z = i$ and $z = -i$ are both singularities of this function. In fact, they are both poles of order 1. Thus, I will find the residue of this function at both of those points. Using formula (1) from the last question, we obtain that

$$\begin{aligned} \operatorname*{Res}_{z=i} \frac{\sin(z)}{z^2+1} &= \frac{1}{0!}\frac{d^0}{dz^0}\left((z-i)\frac{\sin(z)}{(z+i)(z-i)}\right)\Bigg|_{z=i} \\ &= \frac{\sin(i)}{2i} = \frac{\frac{e^{i^2} - e^{-i^2}}{2i}}{2i} = \frac{e - e^{-1}}{4} \\ \operatorname*{Res}_{z=-i} \frac{\sin(z)}{z^2+1} &= \frac{1}{0!}\frac{d^0}{dz^0}\left((z-i)\frac{\sin(z)}{(z+i)(z-i)}\right)\Bigg|_{z=-i} \\ &= \frac{\sin(-i)}{-2i} = \frac{\frac{e^{i^2} - e^{-i^2}}{2i}}{2i} = \frac{e - e^{-1}}{4} \end{aligned}$$

- Thus, the integral we are interested in is calculated as following,

$$\int_\gamma \frac{\sin(z)}{z^2+1} dz = 2\pi i\left(\operatorname*{Res}_{z=i}\frac{\sin(z)}{z^2+1} + \operatorname*{Res}_{z=-i}\frac{\sin(z)}{z^2+1}\right) = 2\pi i\left(\frac{e - e^{-1}}{2}\right) = \boxed{2\pi i \sinh(1)}$$