Analysis HW 4

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Question 1

Let (X, d_X) and (Y, d_Y) be metric spaces where $f : X \to Y$ is continuous and $y \in Y$. Prove that the set $\{x \in X : f(x) = y\}$ is closed in X.

Proof.
Note that the set we are interested in is simply $f^{-1}(\{y\})$. Recall that $f^{-1}(U^c) = f^{-1}(U)^c$ for all $U \subset Y$. Thus, $f^{-1}(\{y\})^c = f^{-1}(\{y\}^c) = f^{-1}(Y \backslash \{y\})$. Next, note that $Y \backslash \{y\}$ is an open set. To see this, let $y_0 \in Y \backslash \{y\}$. Let $r := d_Y(y_0, y)/2$. Then it is clear that

$$\begin{aligned} U_r(y_0) = \{x \in Y \mid d(x, y_0) < r\} \not\ni \{y\} \\ \implies U_r(y_0) \subset Y \backslash \{y\} \end{aligned}$$

This means that y_0 is an interior point of $Y \backslash \{y\}$ and this can be done for all $y_0 \in Y \backslash \{y\}$, so $Y \backslash \{y\} \subset (Y \backslash \{y\})^\circ$ meaning the set is open. Furthermore, since f is continuous, we know that f^{-1} maps opens sets in Y to open sets in X. In other words, $f^{-1}(Y \backslash \{y\})$ is open. By the equalities shown above, this means that $f^{-1}(\{y\})^c$ is open, or equivalently, that $f^{-1}(\{y\})$ is closed. Therefore, we have proven that the given set is closed. $\square$

Question 2

Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $Z = \mathbb{R}^{n+m}$ each equipped with the appropriate Euclidean metric. Let $K \subset X$ be compact and let $f : K \to Y$ be continuous. Prove that the graph of f, $\Gamma(f) = \{(x, f(x)) : x \in K\} \subset Z$ is compact.

Proof.
Since f is continuous and K is compact, then we know that $f(K) \subset Y$ is compact. In particular, since $X = \mathbb{R}^n$ and since $Y = \mathbb{R}^m$, then we know that K and $f(K)$ are both closed and bounded. Thus, our goal is simply to show that $\Gamma(f) \subset Z = \mathbb{R}^{n+m}$ is closed and bounded.

Since K and $f(K)$ are both bounded, they both have finite diameters, say $\text{diam}(K) = r$ and $\text{diam}(f(K)) = \rho$. Since we are using the Euclidean metric, this means that $[(w_1 - x_1)^2 + (w_2 - x_2)^2 + \cdots + (w_n - x_n)^2]^{1/2} < r$ for all $w, x \in K$ with $w = (w_1, w_2, \ldots, w_n)$ and $x = (x_1, x_2, \ldots, x_n)$. Similarly, $[(y_1 - z_1)^2 + (y_2 - z_2)^2 + \cdots + (y_m - z_m)^2]^{1/2} < \rho$ for all $y, z \in f(K)$ with $y = (y_1, y_2, \ldots, y_m)$ and $z = (z_1, z_2, \ldots, z_n)$. Thus, let $(w, y), (x, z)$ be two arbitrary elements of $\Gamma(f)$. By examining their distance, we have

$$\begin{aligned} |(w, y) - (x, z)| &= \sqrt{(w_1 - x_1)^2 + (w_2 - x_2)^2 + \cdots + (w_n - x_n)^2 + (y_1 - z_1)^2 + (y_2 - z_2)^2 + \cdots + (y_m - z_m)^2} \\ &< \sqrt{r^2 + \rho^2} \end{aligned}$$

Thus, the diameter of $\Gamma(f)$ is at most $\sqrt{r^2 + \rho^2}$, so $\Gamma(f)$ must also be bounded. Next, I will show that $\Gamma(f)$ is closed.

Let (x_n, y_n) be a sequence in $\Gamma(f)$ that converges to the point $(x, y) \in Z$. Since the sequence is contained in $\Gamma(f)$, then the point (x, y) is either in $\Gamma(f)$ or in $\Gamma(f)'$, so that $(x, y) \in \overline{\Gamma(f)}$. Furthermore, since $\{(x_n, y_n)\} \subset \Gamma(f)$, then the sequence $(x_n) \subset K$ and $(y_n) \subset f(K)$. In particular, by construction of $\Gamma(f)$, $y_n = f(x_n)$. Thus, since K is closed, $x \in K$ and since $f(K)$ is closed, then $y \in f(K)$. Thus, since $y_n = f(x_n)$ and $y_n \to y$ and $x_n \to x$, then by continuity of f, we have $f(x_n) \to f(x) = y$. Thus, we have $(x, y) = (x, f(x)) \in \cancel{K \times f(K) =} \Gamma(f)$. Therefore, $\Gamma(f)$ contains its limit points, so it is closed.

We have now shown that $\Gamma(f)$ is both closed and bounded. Since $\Gamma(f) \subset Z = \mathbb{R}^{n+m}$, we have that $\Gamma(f)$ is compact. $\square$

Question 3

Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces and let $f : X \to Y$ and $g : Y \to Z$ be uniformly continuous. Prove that $g \circ f : X \to Z$ is uniformly continuous as well.

Proof.
Fix an $r > 0$, then since g is uniformly continuous, we know there exists some $s_0 > 0$ such that for all $y, z \in Y$ such that $d_Y(y, z) < s_0$ we have $d_Z(g(y), g(z)) < r$.

Next, fix some $w, x \in X$. Then, since f is uniformly continuous, we know that there exists some $s > 0$ such that $d_X(w, x) < s$ implies $d_Y(f(w), f(x)) < s_0$.

Thus, since $f(w), f(x) \in Y$ and satisfy $d_Y(f(w), f(x)) < s_0$, then we can conclude that $d_Z(g(f(w)), g(f(x))) < r$. In total, we have concluded the following: for all $r > 0$, there exist some $s > 0$ such that for all $w, x \in X$, the inequality $d_X(w, x) < s$ implies that $d_Z(g \circ f(w), g \circ f(x)) < r$ which means that $g \circ f$ is uniformly continuous. $\square$

Question 4

Let $(X, d_X(x, y)) = (\mathbb{R}^n, |x - y|)$ and (Y, d_Y) be a complete metric space. Let $B \subset X$ be bounded and let $f : B \to Y$ be uniformly continuous. Prove that $f(B)$ is bounded.

Proof.
My idea is to show that the extension of f given as $g : \overline{B} \to Y$ is continuous, so that $g(\overline{B})$ is compact (since $\overline{B} \subset \mathbb{R}^n$ is closed and B bounded implies $\overline{B}$ bounded). We have proven in class that all compact spaces are bounded, so this would show that $g(\overline{B})$ is bounded. Thus, since $f(B) \subset g(\overline{B})$, then $f(B)$ is bounded.

More formally, define $g : \overline{B} \to Y$ as

$$g(x) = \begin{cases} f(x) & \text{if } x \in B \\ \lim_{n\to\infty} f(x_n) & \text{if } x \in B' \text{ and } x = \lim_{n\to\infty} x_n \text{ for } x_n \in B \end{cases}$$

Thus, it is clear that g is continuous on B since f is assumed to be (uniformly) continuous on B. Therefore, I simply need to show that g is continuous at each point $x \in B'$. Let $x \in B'$, then there is a sequence (x_n) in B such that $\lim_{n\to\infty} x_n = x$. In particular, since the sequence is convergent, it must be Cauchy. The first thing I will prove is that the sequence $g(x_n) = f(x_n)$ is also Cauchy. Since f is uniformly continuous, then for every $r > 0$, we can find some $s > 0$ such that for any $x, y \in B$ with $|x - y| < s$, we have $d_Y(f(x), f(y)) < r$. Thus, since (x_n) is Cauchy, then there exists some N such that $|x_n - x_m| < s$ for all $n, m \geq N$. Therefore, if we fix some $r > 0$, then there exists some N (the same as from before) such that $d_Y(f(x_n), f(x_m)) < r$ for all $n, m \geq N$ since $|x_n - x_m| < s$ for all $n, m \geq N$. Thus, $f(x_n)$ is indeed a Cauchy sequence.

- Note if we only had continuity and not uniform continuity, then we may have a different s for each $x \in B$. Therefore, saying that $|x_n - x_m| < s$ only guarantees that $d_Y(f(x_n), f(x_m)) < r$ for those fixed n and m, not necessarily all $n, m \geq N$.

Since we have that $f(x_n)$ is Cauchy, then we know that the sequence $(f(x_n))$ converges in Y since Y is a complete metric space. This means that g is well defined since for every $x \in B'$, $\lim_{n\to\infty} f(x_n)$ exists and we know that if limits exist, then they are unique. Now I will show that g is continuous at x. In fact, it may be more direct to show g is uniformly continuous on B'.

Let $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$ be two points in B'. Let us fix some $r > 0$, then we have the following facts we can work with:

(a) As we have shown, $f(x_n)$ and $f(y_n)$ should both converge to $f(x)$ and $f(y)$ respectively. In particular, this means we can find an N_1 and an N_2 such that $d_Y(f(x_n), f(x)) < r/3$ for all $n \geq N_1$ and that $d_Y(f(y_n), f(y)) < r/3$ for all $n \geq N_2$.

(b) Since f is uniformly continuous, then there exists some $s > 0$ such that if $|x_n - y_n| < 3s$, then $d_Y(f(x_n), f(y_n)) < r/3$.

(c) Since $x_n \to x$, then there exists some N_3 such that $|x_n - x| < s$ for all $n \geq N_3$. Similarly, there exists some N_4 such that $|y_n - y| < s$ for all $n \geq N_4$.

(d) If we choose x and y close enough so that $|x - y| < s$, then using (c), we get that for all $n \geq \max\{N_3, N_4\}$,

$$|x_n - y_n| = |x_n - x + x - y + y - y_n| \leq |x_n - x| + |x - y| + |y - y_n| < s + s + s = 3s$$

Therefore, taking $n \geq \max\{N_1, N_2, N_3, N_4\}$, then for all $x, y \in B'$ such that $|x - y| < s$, we have

$$\begin{aligned} d_Y(f(x), f(y)) &\leq d_Y(f(x), f(x_n)) + d_Y(f(x_n), f(y_n)) + d_Y(f(y_n), f(y)) && \text{by Triangle Inequality} \\ &< r/3 + d_Y(f(x_n), f(y_n)) + r/3 && \text{by (a)} \\ &< r/3 + r/3 + r/3 && \text{by (b) since we have (d)} \\ &= r \end{aligned}$$

In total, this proves that for any $r > 0$, there exists some $s > 0$ such that for any $x, y \in B'$ such that $|x - y| < s$ it implies that $d_Y(f(x), f(y)) < r$. Therefore, g (which is the extension of f to $\overline{B}$) is uniformly continuous on B'. We already knew g was uniformly continuous on B, therefore g is uniformly continuous on $\overline{B} = B' \cup B$. Therefore, as it was discussed at the beginning of the proof, this means that $f(B)$ is bounded. $\square$