Advanced Calc. Homework 12

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24.1

Let $f_n(x) = \dfrac{1 + 2\cos^2(nx)}{\sqrt{n}}$. Prove carefully that (f_n) converges uniformly to 0 on $\mathbb{R}$.

Proof.
Let's begin by fixing $\varepsilon > 0$ and examining $|f_n(x) - 0|$ for $x \in \mathbb{R}$:

$$\begin{aligned}
|f_n(x) - 0| = |f_n(x)| &= \left|\frac{1 + 2\cos^2(nx)}{\sqrt{n}}\right| \\
&= \frac{|1 + 2\cos^2(nx)|}{|\sqrt{n}|} \\
&\leq \frac{|1| + |2\cos^2(nx)|}{\sqrt{n}} && \text{by Triangle Inequality} \\
&= \frac{1 + 2|\cos(nx)|^2}{\sqrt{n}} \\
&\leq \frac{1+2}{\sqrt{n}} && \text{since } |\cos(\theta)| \leq 1 \text{ for all } \theta \\
&= \frac{3}{\sqrt{n}}
\end{aligned}$$

Thus, if we set $N := \dfrac{9}{\varepsilon^2}$, we can obtain:

$$\begin{aligned}
|f_n(x) - 0| &\leq \frac{3}{\sqrt{n}} && \text{from above} \\
&< \frac{3}{\sqrt{N}} && \text{for all } n > N \\
&= \frac{3}{\sqrt{9/\varepsilon^2}} \\
&= \frac{3}{3/\varepsilon} = \varepsilon
\end{aligned}$$

Thus, we have shown the existence of N (that does not depend on x) such that $|f_n(x) - 0| < \varepsilon$ for all $x \in \mathbb{R}$ and all $n > N$, proving that (f_n) converges uniformly to 0 on $\mathbb{R}$. $\square$

24.2

For $x \in [0, \infty)$, let $f_n(x) = \dfrac{x}{n}$.

(a) Find $f(x) = \lim f_n(x)$.

- $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \dfrac{x}{n} = 0$. Thus, $f(x) \equiv 0$.

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

- First, note that for $x \in [0,1]$, $f_n(x) = \frac{x}{n} \leq \frac{1}{n}$. Thus, fixing $\varepsilon > 0$ and choosing $N := \frac{1}{\varepsilon}$ yields

$$\begin{aligned} |f_n(x) - 0| = |f_n(x)| = \left|\frac{x}{n}\right| &\leq \left|\frac{1}{n}\right| && \text{as discussed above} \\ &= \frac{1}{n} \\ &< \frac{1}{N} && \text{for all } n > N \\ &= \frac{1}{1/\varepsilon} = \varepsilon \end{aligned}$$

- Therefore, we have shown the existence of N (that does not depend on x) such that $|f_n(x) - 0| < \varepsilon$ for all $x \in [0,1]$ and all $n > N$, proving that $\boxed{(f_n) \text{ converges uniformly to } 0 \text{ on } [0,1]}$.

(c) Determine whether $f_n \to f$ uniformly on $[0,\infty)$.

- Suppose (for contradiction) that $f_n \to f$ uniformly on $[0,\infty)$. This means that taking $\varepsilon = 1$, there exists some N such that $\left|\frac{x}{n}\right| < 1$ for all $n > N$. In particular, we need to have $\left|\frac{x}{N+1}\right| < 1$. However, taking $x = N+2$ yields $\frac{x}{N+1} > 1$ violating the assumption that $f_n \to f$ uniformly. Thus, $\boxed{f_n \text{ does not converge uniformly to } 0 \text{ on } [0,\infty)}$

24.3

For $x \in [0,\infty)$, let $f_n(x) = \frac{1}{1+x^n}$.

(a) Find $f(x) = \lim f_n(x)$.

- First, note that $x^n \to 0$ for $x \in [0,1)$, $x^n \to 1$ for $x = 1$ and $x^n \to \infty$ for $x > 1$. Thus,

$$f(x) = \begin{cases} 1 & \text{for } x \in [0,1) \\ \frac{1}{2} & \text{for } x = 1 \\ 0 & \text{for } x > 1 \end{cases}$$

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$.

- Theorem 24.3 tells us that if $f_n \to f$ uniformly on a set S, then if each f_n is continuous on S, we must have that f is continuous on S as well. However, each f_n is indeed continuous since the numerator, 1, and the denominator, $1 + x^n$, are both continuous functions (and the denominator never equals 0). Additionally, it is clear that f above is not continuous on $[0,1]$ since it has a discontinuity as $x = 1$. Thus, $\boxed{\text{this convergence cannot be uniform}}$ by the contrapositive to Theorem 24.3.

(c) Determine whether $f_n \to f$ uniformly on $[0,\infty)$.

- If we had uniform continuity on $[0,\infty)$, then in particular, we would have uniform continuity on $[0,1]$. However, we showed in the previous question that this is not the case. Thus, $\boxed{\text{this convergence cannot be uniform}}$

24.4

For $x \in [0,\infty)$, let $f_n(x) = \frac{x^n}{1+x^n}$.

(a) Find $f(x) = \lim f_n(x)$.

- Using the same analysis on x^n as we did in the last question, we can see that

$$f(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ \frac{1}{2} & \text{for } x = 1 \\ 1 & \text{for } x > 1 \end{cases}$$

(b) Determine whether $f_n \to f$ uniformly on $[0,1]$.

- We will again use the contrapositive to Theorem 24.3. Note that each f_n is continuous since x^n and $1+x^n$ are both continuous functions and $1 + x^n \neq 0$ for all $x \in [0,1]$. However, f is clearly not continuous on $[0,1]$ since f has a discontinuity at $x = 1$. Thus, $\boxed{\text{this convergence cannot be uniform}}$ by the contrapositive to Theorem 24.3.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

- Since this convergence is not uniform on $[0, 1]$, it cannot be uniformly convergent on $[0, \infty)$, so $\boxed{\text{this convergence cannot be uniform}}$

24.5

For $x \in [0, \infty)$, let $f_n(x) = \dfrac{x^n}{n + x^n}$.

(a) Find $f(x) = \lim f_n(x)$.

- Once again, using the same analysis on x^n, we can see that for $x \leq 1$, $f_n \to 0$. However, for $x > 1$, we need to consider:

$$\lim_{n\to\infty} \frac{x^n}{n + x^n} = \lim_{n\to\infty} \frac{1}{1 + n/x^n} = 1 \qquad \text{since } n/x^n \to 0 \text{ as exponential terms grow faster than linear terms with base} > 1$$

- Thus, we can define f as

$$f(x) = \begin{cases} 0 & \text{for } x \in [0, 1] \\ 1 & \text{for } x > 1 \end{cases}$$

(b) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

- Let's fix $\varepsilon > 0$ and examine $|f_n(x) - f(x)| = |f_n(x) - 0| = |f_n(x)|$ for $x \in [0, 1]$:

$$\begin{aligned}
|f_n(x)| &= \left|\frac{x^n}{n + x^n}\right| \\
&= \frac{|x^n|}{|n + x^n|} \\
&= \frac{x^n}{n + x^n} && \text{since the numerator and denominator are both positive} \\
&\leq \frac{1}{n + x^n} && \text{since } x \leq 1 \\
&\leq \frac{1}{n + 0} && \text{since } x \geq 0 \\
&= \frac{1}{n}
\end{aligned}$$

- Thus, choosing $N := 1/\varepsilon$ yields:

$$\begin{aligned}
|f_n(x)| &\leq \frac{1}{n} && \text{by the above comments} \\
&< \frac{1}{N} && \text{for all } n > N \\
&= \frac{1}{1/\varepsilon} = \varepsilon
\end{aligned}$$

- Therefore, we have shown the existence of N (that does not depend on x) such that $|f_n(x) - 0| < \varepsilon$ for all $x \in [0, 1]$ and all $n > N$, proving that $\boxed{(f_n) \text{ converges uniformly to } 0 \text{ on } [0, 1]}$.

(c) Determine whether $f_n \to f$ uniformly on $[0, \infty)$.

- First, note that each f_n is continuous on $[0, \infty)$ since x^n and $n+x^n$ are both continuous functions and $n+x^n \neq 0$ for all $x \in [0, \infty)$. However, f is not continuous on $[0, \infty)$ since f has different left and right limits at the point $x = 1$, thus $x = 1$ is a discontinuity. Thus, by applying the contrapositive of Theorem 24.3, we can conclude that $\boxed{\text{this convergence is not uniform}}$

24.6

Let $f_n(x) = \left(x - \dfrac{1}{n}\right)^2$ for $x \in [0, 1]$.

(a) Does the sequence (f_n) converge pointwise on the set $[0, 1]$? If so, give the limit function.

- To determine this, lets examine the limit:

$$\begin{aligned}\lim_{n\to\infty}\left(x-\frac{1}{n}\right)^2 &= \lim_{n\to\infty}\left(x^2-\frac{2x}{n}+\frac{1}{n^2}\right)\\ &= \left(x^2-2x(0)+0\right)\\ &= x^2\end{aligned}$$

- Thus, $\boxed{\text{yes, the sequence converges pointwise to } f(x)=x^2}$

(b) Does (f_n) converge uniformly on $[0,1]$? Prove your assertion.

- Let's fix $\varepsilon > 0$ and examine $|f_n(x) - f(x)|$:

$$\begin{aligned}|f_n(x)-f(x)| &= \left|\left(x-\frac{1}{n}\right)^2 - x^2\right|\\ &= \left|x^2-\frac{2x}{n}+\frac{1}{n^2}-x^2\right|\\ &= \left|\frac{1}{n^2}-\frac{2x}{n}\right|\\ &\le \left|\frac{1}{n^2}\right| && \text{since } \frac{2x}{n}\ge 0 \text{ for } x\in[0,1]\\ &= \frac{1}{n^2} && \text{since } \frac{1}{n^2}>0\end{aligned}$$

- Thus, choosing $N := 1/\sqrt{\varepsilon}$ yields:

$$\begin{aligned}|f_n(x)-f(x)| &\le \frac{1}{n^2} && \text{by the above comments}\\ &< \frac{1}{N^2} && \text{for all } n>N\\ &= \frac{1}{(1/\sqrt{\varepsilon})^2}\\ &= \frac{1}{1/\varepsilon} = \varepsilon\end{aligned}$$

- Therefore, we have shown the existence of N (that does not depend on x) such that $|f_n(x) - x^2| < \varepsilon$ for all $x \in [0,1]$ and all $n > N$, proving that $\boxed{(f_n) \text{ converges uniformly to } x^2 \text{ on } [0,1]}$.

25.2

Let $f_n(x) = \dfrac{x^n}{n}$. Show (f_n) is uniformly convergent on $[-1,1]$ and specify the limit function.

Proof.
I claim that the limit function is 0. I will prove this by using the definition of uniform continuity. Let $\varepsilon > 0$ and examine $|f_n(x) - 0| = |f_n(x)|$:

$$\begin{aligned}|f_n(x)| &= \left|\frac{x^n}{n}\right|\\ &= \frac{|x|^n}{n}\\ &\le \frac{1}{n} && \text{since } |x|\le 1 \text{ for all } x\in[-1,1]\end{aligned}$$

Thus, choosing $N := 1/\varepsilon$ yields:

$$\begin{aligned}|f_n(x)| &\le \frac{1}{n} && \text{by the above comments}\\ &< \frac{1}{N} && \text{for all } n>N\\ &= \frac{1}{1/\varepsilon} = \varepsilon\end{aligned}$$

Therefore, we have shown the existence of N (that does not depend on x) such that $|f_n(x) - 0| < \varepsilon$ for all $x \in [-1,1]$ and all $n > N$, proving that $\boxed{(f_n) \text{ converges uniformly to } 0 \text{ on } [-1,1]}$. $\square$

25.3

Let $f_n(x) = \dfrac{n+\cos(x)}{2n+\sin^2(x)}$ for all real numbers x.

(a) Show (f_n) converges uniformly on $\mathbb{R}$.

- I will show this by using the definition of uniform continuity. First, to find our desired limit function, note:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n+\cos(x)}{2n+\sin^2(x)} = \lim_{n\to\infty} \frac{1+\cos(x)/n}{2+\sin^2(x)/n} = \frac{1}{2}$$

- Let $\varepsilon > 0$ and examine $\left|f_n(x) - \frac{1}{2}\right|$:

$$\begin{aligned}
\left|f_n(x) - \frac{1}{2}\right| &= \left|\frac{n+\cos(x)}{2n+\sin^2(x)} - \frac{1}{2}\right| \\
&= \left|\frac{2n+2\cos(x)}{2(2n+\sin^2(x))} - \frac{2n+\sin^2(x)}{2(2n+\sin^2(x))}\right| \\
&= \frac{|2\cos(x) - \sin^2(x)|}{|4n+2\sin^2(x)|} \\
&\le \frac{|2\cos(x)| + |-\sin^2(x)|}{4n+2\sin^2(x)} && \text{by Triangle Inequality} \\
&\le \frac{2+1}{4n+2\sin^2(x)} && \text{since } |\cos(\theta)| \le 1 \text{ and } |\sin(\theta)| \le 1 \text{ for all } \theta \\
&\le \frac{3}{4n} && \text{since } 4n+2\sin^2(x) \ge 4n \text{ for all } x
\end{aligned}$$

- Thus, if we choose $N := 3/(4\varepsilon)$, we obtain:

$$\begin{aligned}
\left|f_n(x) - \frac{1}{2}\right| &\le \frac{3}{4n} && \text{by the above comments} \\
&< \frac{3}{4N} && \text{for all } n > N \\
&= \frac{3}{4\cdot(3/(4\varepsilon))} \\
&= \frac{3}{3/\varepsilon} = \varepsilon
\end{aligned}$$

- Therefore, we have shown the existence of N (that does not depend on x) such that $\left|f_n(x) - \frac{1}{2}\right| < \varepsilon$ for all $x \in \mathbb{R}$ and all $n > N$, proving that $\boxed{(f_n) \text{ converges uniformly to } \frac{1}{2} \text{ on } \mathbb{R}}$.

(b) Calculate $\displaystyle\lim_{n\to\infty} \int_2^7 f_n(x)\,dx$

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$$\begin{aligned}
\lim_{n\to\infty} \int_2^7 f_n(x)\,dx &= \int_2^7 \lim_{n\to\infty} f_n(x)\,dx && \text{by Theorem 25.2 and the uniform convergence of } (f_n) \\
&= \int_2^7 \frac{1}{2}\,dx && \text{since } \frac{1}{2} \text{ is the limit function of } (f_n) \\
&= \frac{x}{2}\Big|_{x=2}^{x=7} \\
&= \frac{7}{2} - \frac{2}{2} = \boxed{\frac{5}{2}}
\end{aligned}$$

25.5

Let (f_n) be a sequence of bounded functions on a set S, and suppose $f_n \to f$ uniformly on S. Prove f is a bounded function on S.

Proof.
Assume that $f_n \to f$ uniformly on S and that each f_n is a bounded function. By the uniform convergence of f_n, we know that (for $\varepsilon = 1$) there exists some N such that for all $n > N$, we have $|f_n(x) - f(x)| < 1$ for all $x \in S$. Thus, in particular this must hold for $n = N+1$. Additionally, since all f_n are bounded functions, we must have that f_{N+1} is bounded, say $|f_{N+1}(x)| \leq M$ for all $x \in S$ and $M > 0$. Thus, we can do the following:

$$\begin{aligned}
&|f_{N+1}(x) - f(x)| < 1 && \\
\iff\ &|f(x) - f_{N+1}(x)| < 1 && \\
\iff\ &f_{N+1}(x) - 1 < f(x) < 1 + f_{N+1}(x) && \text{by Exercise 3.7(b)} \\
\implies\ &-|f_{N+1}(x)| - 1 < f(x) < 1 + |f_{N+1}(x)| && \text{since } a \leq |a| \text{ for all } a \in \mathbb{R} \\
\implies\ &|f(x)| < 1 + |f_{N+1}(x)| && \text{by Exercise 3.7(a)} \\
&\leq 1 + M && \text{by the boundedness of } f_{N+1}
\end{aligned}$$

Thus, we have shown that $|f(x)| < 1 + M$ for all $x \in S$, so this means f is bounded and one possible upper bound is $1 + M$. $\square$

25.6

(a) Show that if $\sum |a_k| < \infty$, then $\sum a_k x^k$ converges uniformly on $[-1, 1]$ to a continuous function.

- First, observe that $|a_k x^k| = |a_k||x|^k \leq |a_k|$ since $x \in [-1, 1]$. Thus, by the Weierstrass M-Test (25.7), the series $\sum a_k x^k$ converges uniformly on $[-1, 1]$. Furthermore, since the function $g_k(x) = a_k x^k$ is a continuous function for all k, then we can use Theorem 25.5 to conclude that $\sum a_k x^k$ represents a continuous function on S.

(b) Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ represent a continuous function on $[-1, 1]$?

- By part (a) of this question, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ will converge uniformly to a continuous function on $[-1, 1]$ if $\displaystyle\sum \left|\frac{1}{n^2}\right|$ is finite. However, by Theorem 15.1, we know that $\displaystyle\sum \frac{1}{n^p}$ converges if and only if $p > 1$. Thus, our desired series converges, so $\boxed{\text{yes, the series represents a continuous function on } [-1, 1]}$

25.7

Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$ converges uniformly on $\mathbb{R}$ to a continuous function.

Proof.
Let (M_n) be a sequence of nonnegative real numbers such that $M_n = \frac{1}{n^2}$. Furthermore, we know from Theorem 15.1 that $\sum M_n$ converges and we know that

$$\left|\frac{1}{n^2} \cos(nx)\right| = \left|\frac{1}{n^2}\right| |\cos(nx)| \leq \frac{1}{n^2} = M_n \qquad \text{for all } x \in \mathbb{R}$$

since $|\cos(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$. Thus, by Weierstrass M-Test (25.7), we can conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$ converges uniformly on $\mathbb{R}$. Therefore, since $g_n(x) = \frac{1}{n^2} \cos(nx)$ is continuous (by the continuity of $\cos(\cdot)$), we can use Theorem 25.5 to finally say that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx)$ represents a continuous function. $\square$

25.8

Show $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of converge 2 and the series converges uniformly to a continuous function on $[-2, 2]$.

Proof.
Recall that the radius of convergence, R of a power series $\sum a_n x^n$ is equal to $\frac{1}{\beta}$ where $\beta = \limsup |a_n|^{1/n}$ or $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right|$

if the limit exists. I will use this second definition of β to find that:

$$\begin{aligned}
\beta = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{n^2 2^n}{(n+1)^2 2^{n+1}}\right| \\
&= \lim_{n\to\infty}\frac{1}{2}\left|\frac{n}{n+1}\right|^2 \\
&= \frac{1}{2}\cdot\left(\lim_{n\to\infty}\frac{n}{n+1}\right)^2 \\
&= \frac{1}{2}\cdot\left(\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}}\right)^2 \\
&= \frac{1}{2}\cdot\left(\frac{1}{1+0}\right)^2 = \frac{1}{2}
\end{aligned}$$

Thus, $R = 2$, just as desired. I will now examine if the series conberges at $x = \pm 2$. For $x = 2$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$ which we know converges by Theorem 15.1. Alternatively, for $x = -2$, we have $\displaystyle\sum_{n=1}^{infty}\frac{(-2)^n}{n^2 2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ which converges by the Alternating Series Test. Therefore, the interval of convergence for this series is indeed $[-2, 2]$. We now need to show that this convergence is uniform and the limiting function is continuous. Note that for $M_n = \frac{1}{n^2}$, we have that

$$\left|\frac{x^n}{n^2 2^n}\right| = \frac{|x|^n}{|n^2 2^n|} \leq \frac{2^n}{n^2 2^n} = \frac{1}{n^2} = M_n \qquad \text{for all } x \in [-2, 2]$$

Thus, since $\sum M_n$ converges (as we have already shown), then we can use the Weierstrass M-Test (25.7) to conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n^2 2^n}$ converges uniformly on $[-2, 2]$. Furthermore, since the function $g_n(x) = \dfrac{x^n}{n^2 2^n}$ is continuous for all n, then we can use Theorem 25.5 to also conclude that the series we are interested in represents a continuous function on $[-2, 2]$. $\square$

25.9

(a) Let $0 < a < 1$. Show the series $\displaystyle\sum_{n=0}^{\infty} x^n$ converges uniformly on $[-a, a]$ to $\dfrac{1}{1-x}$.

- Notice that $|x^n| = |x|^n \leq a^n$ for all $x \in [-a, a]$. Furthermore, $\sum a^n$ converges quite easily by the Root Test (since $\limsup |a^n|^{1/n} = a < 1$). Therefore, the Weierstrass M-Test (25.7) tells us that $\displaystyle\sum_{n=1}^{\infty} x^n$ converges uniformly for all $x \in [-a, a]$. Furthermore, since we know that the convergence is uniform, we can examine the value of this series as follows:

$$\begin{aligned}
&\text{Let } f_k(x) = \sum_{n=0}^{k} x^n \\
&\implies x f_k(x) = \sum_{n=0}^{k} x^{n+1} = \sum_{n=1}^{k+1} x^n \\
&\implies f_k(x) - x f_k(x) = \sum_{n=0}^{k} x^n - \sum_{n=1}^{k+1} x^n = 1 + \sum_{n=1}^{k} x^n - \sum_{n=1}^{k} x^n - x^{k+1} \\
&\implies f_k(x)(1-x) = 1 - x^{k+1} \\
&\implies f_k(x) = \frac{1-x^{k+1}}{1-x} \\
&\implies \sum_{n=0}^{\infty} x^n = \lim_{k\to\infty} f_k(x) = \lim_{k\to\infty}\frac{1-x^{k+1}}{1-x} = \frac{1}{1-x} \qquad \text{since } |x| < 1 \implies x^{k+1} \to 0
\end{aligned}$$

- Thus, the series does indeed converge uniformly on $[-a, a]$ and the limiting function is as desired

(b) Does the series $\displaystyle\sum_{n=0}^{\infty} x^n$ converge uniformly on $(-1, 1)$ to $\dfrac{1}{1-x}$? Explain.

- Exercise 25.5 told us that if a sequence of functions (f_n) on a set S is bounded and $f_n \to f$ uniformly on S, then f must also be bounded on S. Thus, the contrapositive to this statement would say that if (f_n) is a bounded sequence of functions on S such that $f_n \to f$, but f is not bounded on S, then this convergence must not be uniform. However, we can see that $f(x) = \dfrac{1}{1-x}$ is not bounded on $(-1, 1)$ since if we were to claim there exists some $M > 0$ such that $f(x) \leq M$ for all $x \in (-1, 1)$, then we can consider $x_0 = 1 - \frac{1}{2M} \in (-1, 1)$ to get $f(x_0) = \dfrac{1}{1 - (1 - (1/2M))} = \dfrac{1}{1/2M} = 2M > M$. Thus, f cannot possibly be bounded, so Exercise 25.5 tells us that $\boxed{\text{this convergence is not uniform on } (-1, 1).}$