Advanced Calc. Homework 4

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September 23, 2020

7.1

Write out the first five terms of the following sequences:

(a) $s_n = \dfrac{1}{3n+1}$

- $\dfrac{1}{3(1)+1} = \boxed{\dfrac{1}{4}}, \quad \dfrac{1}{3(2)+1} = \boxed{\dfrac{1}{7}}, \quad \dfrac{1}{3(3)+1} = \boxed{\dfrac{1}{10}}, \quad \dfrac{1}{3(4)+1} = \boxed{\dfrac{1}{13}}, \quad \dfrac{1}{3(5)+1} = \boxed{\dfrac{1}{16}}$

(b) $b_n = \dfrac{3n+1}{4n-1}$

- $\dfrac{3(1)+1}{4(1)-1} = \boxed{\dfrac{4}{3}}, \quad \dfrac{3(2)+1}{4(2)-1} = \dfrac{7}{7} = \boxed{1}, \quad \dfrac{3(3)+1}{4(3)-1} = \boxed{\dfrac{10}{11}}, \quad \dfrac{3(4)+1}{4(4)-1} = \boxed{\dfrac{13}{15}}, \quad \dfrac{3(5)+1}{4(5)-1} = \boxed{\dfrac{16}{19}}$

(c) $c_n = \dfrac{n}{3^n}$

- $\dfrac{1}{3^1} = \boxed{\dfrac{1}{3}}, \quad \dfrac{2}{3^2} = \boxed{\dfrac{2}{9}}, \quad \dfrac{3}{3^3} = \dfrac{3}{27} = \boxed{\dfrac{1}{9}}, \quad \dfrac{4}{3^4} = \boxed{\dfrac{4}{81}}, \quad \dfrac{5}{3^5} = \boxed{\dfrac{5}{243}}$

(d) $\sin\left(\dfrac{n\pi}{4}\right)$

- $\sin\left(\dfrac{(1)\pi}{4}\right) = \boxed{\dfrac{\sqrt{2}}{2}}, \quad \sin\left(\dfrac{(2)\pi}{4}\right) = \boxed{1}, \quad \sin\left(\dfrac{(3)\pi}{4}\right) = \boxed{\dfrac{\sqrt{2}}{2}}, \quad \sin\left(\dfrac{(4)\pi}{4}\right) = \boxed{0}, \quad \sin\left(\dfrac{(5)\pi}{4}\right) = \boxed{\dfrac{-\sqrt{2}}{2}}$

7.2

For each of sequences in the last question, determine (without formal proof) whether it converges and, if it converges, give its limit.

(a) Since the denominator is growing without bound and the numerator is constant, $\boxed{\text{this sequence converges to } 0}$

(b) Since the sequence can be rewritten in the following way:

$$b_n = \frac{3 + \frac{1}{n}}{4 - \frac{1}{n}}$$

and both of the $\dfrac{1}{n}$ terms are going to zero, then $\boxed{b_n \text{ must converge to } \dfrac{3}{4}}$

(c) Since 3^n grows much faster than n does, the denominator is becoming much larger than the numerator, so $\boxed{\text{this sequence converges to } 0}$

(d) Since the $\sin(\cdot)$ function is 2π periodic, we know that $\sin\left(\dfrac{n\pi}{4}\right)$ must repeat every 8 values of n and these 8 values of n do not give the same result, so $\boxed{\text{this sequence does not converge}}$

7.3

For each sequence below, determine (without formal proof) whether it converges and, if it converges, give its limit.

(a) $a_n = \dfrac{n}{n+1}$

- Note this can be rewritten as $a_n = 1 - \dfrac{1}{n+1}$, and the second term goes to 0, so $\boxed{a_n \text{ converges to } 1}$

(b) $b_n = \dfrac{n^2+3}{n^2-3}$

- Note this can be rewritten as $b_n = \dfrac{1+\frac{3}{n^2}}{1-\frac{3}{n^2}}$ and both of the $\dfrac{3}{n^2}$ terms go to zero, so $\boxed{b_n \text{ converges to } 1}$

(c) $c_n = 2^{-n}$

- This is equivalent to $c_n = \dfrac{1}{2^n}$ and 2^n grows without bound, so $\boxed{c_n \text{ converges to } 0}$

(d) $t_n = 1 + \dfrac{2}{n}$

- $\dfrac{2}{n}$ goes to zero as n gets large, so $\boxed{t_n \text{ converges to } 1}$

(e) $x_n = 73 + (-1)^n$

- This oscillates between equaling 72 and 74, so $\boxed{\text{this sequence does not converge}}$

(f) $s_n = (2)^{\frac{1}{n}}$

- $\dfrac{1}{n}$ goes to zero as n gets large and $2^0 = 1$, so $\boxed{s_n \text{ converges to } 1}$

(g) $y_n = n!$

- The factorial function grows without bound, so $\boxed{\text{this sequence does not converge}}$

(h) $d_n = (-1)^n n$

- In absolute value, $|d_n| = n$ and this grows without bound, but any convergent series' absolute value must be bounded, so $\boxed{\text{this sequence does not converge}}$

(i) $\dfrac{(-1)^n}{n}$

- If n is even, then $\dfrac{1}{n}$ converges to 0 and if n is odd, then $\dfrac{-1}{n}$ converges to 0, so $\boxed{\text{this sequence converges to } 0}$

(j) $\dfrac{7n^3+8n}{2n^3-3}$

- This fraction can be rewritten as $\dfrac{7+\frac{8}{n^2}}{2-\frac{3}{n^3}}$ and the second terms in the numerator and the denominator both go to zero, so overall $\boxed{\text{this sequence converges to } \dfrac{7}{2}}$

(k) $\dfrac{9n^2-18}{6n+18}$

- The numerator grows faster than the denominator, so $\boxed{\text{this sequence does not converge}}$

(l) $\sin\left(\dfrac{n\pi}{2}\right)$

- Since the $\sin(\cdot)$ function is 2π periodic, we know that $\sin\left(\dfrac{n\pi}{2}\right)$ must repeat every 4 values of n, but these 4 values of n do not give the same result, so $\boxed{\text{this sequence does not converge}}$

(m) $\sin(n\pi)$

- For every $n \in \mathbb{N}$, $\sin(n\pi) = 0$, so $\boxed{\text{this sequence converges to } 0}$

(n) $\sin\left(\dfrac{2n\pi}{3}\right)$

- Since the $\sin(\cdot)$ function is 2π periodic, we know that $\sin\left(\dfrac{2n\pi}{3}\right)$ must repeat every 3 values of n, but these 3 values of n do not give the same result, so $\boxed{\text{this sequence does not converge}}$

(o) $\dfrac{1}{n}\sin(n)$

- Since $\sin(n)$ is always between -1 and 1, we know that n in the denominator will eventually be much larger since it grows without bound, so $\boxed{\text{this sequence converges to 0}}$

(p) $\dfrac{2^{n+1}+5}{2^n-7}$

- We can rewrite this fraction as $\dfrac{2+\frac{5}{2^n}}{1-\frac{7}{2^n}}$ and both of the second terms in the numerator and the denominator go to zero, so $\boxed{\text{this sequence converges to 2}}$

(q) $\dfrac{3^n}{n!}$

- Eventually, $n!$ is bigger than 3^n and then it continues to grow faster, so $\boxed{\text{this sequence converges to 0}}$

(r) $\left(1+\dfrac{1}{n}\right)^2$

- The stuff in parenthesis tend towards 1 since $\dfrac{1}{n}$ goes to 0. Since $1^2=1$, we can say $\boxed{\text{this sequence converges to 1}}$

(s) $\dfrac{4n^2+3}{3n^2-2}$

- We can rewrite this fraction as $\dfrac{4+\frac{3}{n^2}}{3-\frac{2}{n^2}}$ and both of the second terms in the numerator and the denominator go to 0, so we can say $\boxed{\text{this sequence converges to } \dfrac{4}{3}}$

(t) $\dfrac{6n+4}{9n^2+7}$

- The denominator grows faster than the numerator, so we can say $\boxed{\text{this sequence converges to 0}}$

7.4

Give examples of the following:

(a) A sequence (x_n) of irrational numbers having a limit $\lim(x_n)$ that is a rational number.

- Let $x_n = \dfrac{\sqrt{17}}{n}$, then the denominator of x_n is growing without bound and the numerator is constant, so $\lim(x_n)=0$, a rational number, but all x_i's in the sequence are irrational due to the irrationality of $\sqrt{17}$

(b) A sequence (r_n) of rational numbers having a limit $\lim(r_n)$ that is an irrational number.

- Let r_n be a sequence whose nth term is the first n decimal places of $\frac{1}{\sqrt{2}}$. i.e. $r_1 = 0.7, r_2 = 0.70, r_3 = 0.707, r_4 = 0.7071, \ldots$ then it is clear that $\lim(r_n) = \frac{1}{\sqrt{2}}$, an irrational number. However, each r_i is a rational number since it is a finite decimal expansion, which must be rational.

7.5

Determine the following limits. No proofs are required but show the relevant algebra.

(a) $\lim(s_n)$ where $s_n = \sqrt{n^2+1} - n$

- First, make the following algebraic manipulations:

$$\begin{aligned}
s_n &= \sqrt{n^2+1} - n \\
&= \left(\sqrt{n^2+1} - n\right) \cdot \frac{\sqrt{n^2+1}+n}{\sqrt{n^2+1}+n} \\
&= \frac{(n^2+1) + n\sqrt{n^2+1} - n\sqrt{n^2+1} - n^2}{\sqrt{n^2+1}+n} \\
&= \frac{n^2+1-n^2}{\sqrt{n^2+1}+n} \\
&= \frac{1}{\sqrt{n^2+1}+n}
\end{aligned}$$

- Thus, it is clear that s_n in this form has a denominator that is growing without bound and a constant numerator; thus, $\boxed{\lim(s_n) = 0}$.

(b) $\lim(\sqrt{n^2+n} - n)$

- I will make a similar algebraic manipulation with this expression as well:

$$\begin{aligned}
\sqrt{n^2+n} - n &= \left(\sqrt{n^2+n} - n\right) \cdot \frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \frac{(n^2+n) + n\sqrt{n^2+n} - n\sqrt{n^2+n} - n^2}{\sqrt{n^2+n}+n} \\
&= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n} \\
&= \frac{n}{\sqrt{n^2+n}+n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \frac{1}{\sqrt{\frac{1}{n^2}(n^2+n)}+1} \\
&= \frac{1}{\sqrt{1+\frac{1}{n}}+1}
\end{aligned}$$

- When this expression is written in this manner, we can see that all terms are constant except the $\dfrac{1}{n}$ term, which goes to zero. Thus, $\boxed{\lim(\sqrt{n^2+n} - n) = \dfrac{1}{\sqrt{1+0}+1} = \dfrac{1}{2}}$

(c) $\lim(\sqrt{4n^2+n} - 2n)$

- I will once again do the following algebraic manipulations:

$$\begin{aligned}
\sqrt{4n^2+n}-2n &= \left(\sqrt{4n^2+n}-2n\right)\cdot\frac{\sqrt{4n^2+n}+2n}{\sqrt{4n^2+n}+2n} \\
&= \frac{(4n^2+n)+2n\sqrt{4n^2+n}-2n\sqrt{4n^2+n}-4n^2}{\sqrt{4n^2+n}+2n} \\
&= \frac{4n^2+n-4n^2}{\sqrt{4n^2+n}+2n} \\
&= \frac{n}{\sqrt{4n^2+n}+2n} \\
&= \frac{n}{\sqrt{4n^2+n}+2n}\cdot\frac{\frac{1}{n}}{\frac{1}{n}} \\
&= \frac{1}{\sqrt{\frac{1}{n^2}(4n^2+n)}+2} \\
&= \frac{1}{\sqrt{4+\frac{1}{n}}+2}
\end{aligned}$$

- When this expression is written in this manner, we can see that all terms are constant except the $\dfrac{1}{n}$ term, which goes to zero. Thus, $\boxed{\lim(\sqrt{4n^2+n}-2n)=\dfrac{1}{\sqrt{4+0}+2}=\dfrac{1}{4}}$