# Complex Analysis Homework 3

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## Question 2

**Question.**
*Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist where $g'(z_0) \neq 0$. Show that*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

*Proof.*
First, I will examine the left hand side of this equation without the limit:

$$\begin{aligned}
\frac{f(z)}{g(z)} &= \frac{f(z) - f(z_0)}{g(z) - g(z_0)} && \text{since } f(z_0) = g(z_0) = 0 \\
&= \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \cdot \frac{\frac{1}{z - z_0}}{\frac{1}{z - z_0}} && \text{since we can multiply by 1} \\
&= \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}}
\end{aligned}$$

Next, I will make use of the following theorem:

**Theorem 1.**
*Suppose that $\lim\limits_{z \to z_0} f(z) = f_0$ and $\lim\limits_{z \to z_0} g(z) = g_0$ with $g_0 \neq 0$, then*

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f_0}{g_0}$$

I will now take the appropriate limit of $\dfrac{f(z)}{g(z)}$ as follows:

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{\frac{f(z) - f(z_0)}{z - z_0}}{\frac{g(z) - g(z_0)}{z - z_0}} && \text{by the above calculation.} \\
&= \frac{\lim\limits_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}{\lim\limits_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0}} && \text{by Theorem 1 and since } \lim_{z \to z_0} \frac{g(z) - g(z_0)}{z - z_0} = g'(z_0) \neq 0 \text{ by assumption.} \\
&= \frac{f'(z_0)}{g'(z_0)} && \text{since we assumed that } f'(z_0) \text{ and } g'(z_0) \text{ exist.}
\end{aligned}$$

Therefore, we have proven the desired statement. □