Complex Analysis Homework 9

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Question 1

Find the Laurent series representation for the function $f(z) = z^3 \sin\left(\frac{1}{z}\right)$ and specify the region in which the representation is valid.

Answer.
I could use the contour integral definition for the coefficients of the Laurent Series, but it would likely be easier to calculate the Taylor Series Expansion of $\sin(w)$, since $\sin(\cdot)$ is an analytic function, and apply that appropriately. Note that the Taylor series for $\sin(\cdot)$ centered at 0 is:

$$\sin(w) = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(0)}{k!} w^k \quad \forall\, w \in \mathbb{C}$$

To calculate this explicitly, note the following about the derivatives of $\sin(w)$:

$$\sin^{(k)}(w) = \begin{cases} \sin(w) & \text{for k = 4m} \\ \cos(w) & \text{for k = 4m + 1} \\ -\sin(w) & \text{for k = 4m + 2} \\ -\cos(w) & \text{for k = 4m + 3} \end{cases} \qquad \text{for } m \in \mathbb{N}_0$$

Thus, since $\sin(0) = 0$ and $\cos(0) = 1$, we can see that we only have nonzero coefficients in the Taylor Series whenever k is an odd integer. Thus, assume that $k = 2n + 1$ for $n \in \mathbb{N}_0$, then $\sin^{(k)}(0) = (-1)^n$. Thus, we have

$$\sin(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1} \quad \forall\, w \in \mathbb{C} \tag{1}$$

Therefore, by substituting $\frac{1}{z} := w$, we and multiplying the expression by z^3, we can obtain:

$$\begin{aligned} f(z) = z^3 \sin\left(\frac{1}{z}\right) &= z^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z}\right)^{2n+1} \\ &= z^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n+1}(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n-2}(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2-2n} \end{aligned}$$

This last series represents our Laurent Series with coefficients $a_k = 0$ for all odd k and all $k > 2$ and given as the coefficient in the series for all even $k \leq 2$. Note that in (1) it was valid for all $w \in \mathbb{C}$. However, at $z = 0$, we have $w = \frac{1}{0} \notin \mathbb{C}$. This is the only problematic point, so this would be the only point we exclude from consideration. Thus, this Laurent Expansion above is valid for all z such that $|z| > 0$.