Analysis Theorems

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1 Topology Review

Theorem 1.1.
Y is closed if and only if Y^c is open.

Theorem 1.2.

1. *The complement of arbitrary unions is the arbitrary intersection of complements.*
2. *The arbitrary union of open sets is open and the finite intersection of open sets is is open.*
3. *The arbitrary intersection of closed sets is closed and the finite union of closed sets is closed.*

Theorem 1.3.
Let $Y \subset X$, then $\overline{Y}$ is closed and if $Z \subset X$ is closed with $Y \subset Z$, then $\overline{Y} \subset Z$.

Proposition 1.1.
Every compact set is closed.

Proposition 1.2.
If Y is bounded, then $\forall\, x \in X$, there exists $r_x > 0$ such that $Y \subset U_{r_x}(x)$.

Proposition 1.3.
Y is bounded if and only if $diam(Y) < +\infty$.

Proposition 1.4.
$diam(\overline{Y}) = diam(Y)$

Proposition 1.5.
Every compact set is bounded.

Theorem 1.4 (Heine-Borel Theorem)**.**
If (X, d) is $\mathbb{R}^n$ with the standard Euclidean metric, then $Y \subset X$ is compact if and only if Y is closed and bounded.

Theorem 1.5 (Relativity)**.** *Let $Y \subset X$, then*

1. *Let $Z \subset Y$, then Z is open in Y if and only if $Z = Y \cap G$ where G is open in X*
2. *Let $Z \subset Y$, then Z is compact in Y if and only if Z is compact in X*

2 Convergence

Proposition 2.1.
If (x_n) is convergent, then (x_n) is bounded.

Proposition 2.2.
$\{x_n\}' \subset (x_n)^*$

Lemma 2.1.
Let $S \subset X$, suppose $S' \neq \emptyset$, let $x \in S'$, then $\forall\, r > 0$, $U_r(x) \cap S$ is infinite

Proposition 2.3.
The set $(x_n)^*$ of subsequential limits of (x_n) is closed in X.

Proposition 2.4.
Every Cauchy Sequence is bounded.

Proposition 2.5.
Every convergent sequence is Cauchy

Proposition 2.6.
If (X, d) is a complete metric space, and $Y \subset X$, then (Y, d) is complete if and only if Y is closed in X.

Theorem 2.1.
If X is compact, then $\forall\, (x_n) \subset X$, $(x_n)^* \neq \emptyset$. In other words, every sequence in a compact metric space has a convergent subsequence.

Theorem 2.2.
Every compact metric space (X, d) is complete.

Lemma 2.2.
Suppose $Y \subset X$, then Y is compact if and only if Y is closed in X.

Lemma 2.3.
Let $\{K_n\}_{n\in\mathbb{N}}$ be a countable family of compact non-empty subsets of X such that $K_1 \supset K_2 \supset K_3 \supset \cdots$ and diam$(K_n) \to 0$ as $n \to \infty$, then there exists some $x \in X$ such that

$$\bigcap_{n=1}^{\infty} K_n = \{x\}.$$

Theorem 2.3.
Every monotonic sequence in $\mathbb{R}$ is convergent (possible to $\pm\infty$).

Corollary 2.1.
$\emptyset \neq (x_n)^* \subset [-\infty, \infty]$ for every sequence in $\mathbb{R}$.

Proposition 2.7.
$\sup(x_n)^*, \inf(x_n)^* \in (x_n)^*$

Theorem 2.4.

$$\limsup_{n\to\infty} x_n = \sup(x_n)^* \qquad \text{and} \qquad \liminf_{n\to\infty} x_n = \inf(x_n)^*$$

Corollary 2.2.
The following are equivalent:

1. $\lim_{n\to\infty} x_n = x$
2. $(x_n)^* = \{x\}$
3. $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$

Theorem 2.5 (Comparison Test)**.**
If $\forall\, n \in \mathbb{N}$, $0 \leq x_n \leq y_n$, then

$$\sum_{n=1}^{\infty} x_n \leq \sum_{n=1}^{\infty} y_n$$

Theorem 2.6 (Root Test)**.**
If we define

$$r := \limsup_{n\to\infty} \left(\sqrt[n]{x_n} \right)$$

Then we have

$$\sum_{n=1}^{\infty} x_n \begin{cases} < +\infty & \text{if } r < 1 \\ ? & \text{if } r = 1 \\ = +\infty & \text{if } r > 1 \end{cases}$$

Proposition 2.8 (Root vs. Ratio)**.**
Suppose for all $n \in \mathbb{N}$, $x_n > 0$, then

$$\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \liminf_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \sqrt[n]{x_n} \leq \limsup_{n\to\infty} \frac{x_{n+1}}{x_n}$$

Theorem 2.7 (Dirichlet Test)**.**
Suppose the following:

1. $\sup \left\{ \left| \sum_{n=1}^{N} y_n \right| : N \in \mathbb{N} \right\} < +\infty$
2. $\forall\, n \in \mathbb{N}, x_n \geq x_{n+1}$
3. $\lim_{n\to\infty} x_n = 0$

Then, $\sum_{n=1}^{\infty} x_n y_n$ *is convergent.*

Theorem 2.8 (Rearrangement Theorem)**.**
If we have a bijective map $k \mapsto n_k$, and

$$S = \sum_{n=1}^{\infty} x_n \qquad\qquad S' = \sum_{k=1}^{\infty} x_{n_k}.$$

Then,

$$\sum_{n=1}^{\infty} |x_n| < +\infty \implies \sum_{k=1}^{\infty} |x_{n_k}| < +\infty \qquad \text{and} \qquad S = S'$$

Theorem 2.9.
If we have convergence in the previous theorem, but not absolute convergence, then we can have S' converge to whatever value we want.

3 Continuity

Proposition 3.1.
If $f : X \to Y$ *is continuous at* a *and* $g : Y \to Z$ *is continuous at* $y = f(a)$*, then* $g \circ f : X \to Z$ *is continuous at* $x = a$*.*

Proposition 3.2.
f *is continuous if and only if* $\forall\, G \subset Y$ *which are open in* Y*, then* $f^{-1}(G)$ *is also open in* X*.*

Theorem 3.1.
If X *is compact and* f *is continuous, then* $f(X)$ *is compact.*

Corollary 3.1.
If $f : X \to Y$ *for* X *and* f *as above and* $Y = \mathbb{R}$*, then* f *attains its maximum and minimum on* X*.*

Theorem 3.2.
If X *is compact metric space,* $f : X \to Y$ *is a continuous bijection, then* Y *is compact and* $f^{-1} : Y \to X$ *is also continuous.*

Theorem 3.3.
If X *is compact and* f *is continuous, then* $f : X \to Y$ *is uniformly continuous.*

Theorem 3.4.
If (X, d) *is a connected metric space and* $f : X \to Y$ *is continuous, then* $f(X)$ *is connected.*

Corollary 3.2.
If X *and* f *are as above, then* $f(X)$ *is an interval.*

4 Differentiability

Theorem 4.1 (Fermat)**.**
If $f : (a, b) \to \mathbb{R}$ and $c \in (a, b)$ such that $f(c) = \max\{f(a, b)\}$ or $f(c) = \min\{f(a, b)\}$, then if $f'(c)$ exists, we must have $f'(c) = 0$.

Theorem 4.2 (Rolle's Theorem)**.**
Suppose $f : [a, b] \to \mathbb{R}$ is continuous and that $f : (a, b) \to \mathbb{R}$ is differentiable such that $f(a) = f(b)$, then there exists some $c \in (a, b)$ such that $f'(c) = 0$.

Theorem 4.3 (Lagrange's Mean Value Theorem)**.**
Assume that $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, then there exists some $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Corollary 4.1.
If $f : (a, b) \to \mathbb{R}$ is differentiable and $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.

Theorem 4.4.
Suppose $f : U_r(x_0) \to \mathbb{R}$ is continuous and that $f : \mathring{U}_r(x_0) \to \mathbb{R}$ is differentiable and $\lim_{x\to x_0} f'(x)$ exists, then $f'(x_0) = \lim_{x\to x_0} f'(x)$

Theorem 4.5 (Generalized MVT)**.**
Let $f, g : [a, b] \to \mathbb{R}$ be continuous and $f, g : (a, b) \to \mathbb{R}$ be differentiable. Assume $g'(x) \neq 0$ for all $x \in (a, b)$, then

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \qquad \text{for some } c \in (a, b)$$

Theorem 4.6 (Taylor's Formula)**.**
If $f : U_r(x_0) \to \mathbb{R}$ is $(n + 1)$-times differentiable, then

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}(x - x_0)^{n+1}$$

for some c between x_0 and x.

5 Sequences of Functions

Theorem 5.1.
A sequence of functions (f_n) is uniformly convergent if and only if it is Cauchy.

Theorem 5.2. *If (f_n) is a sequence of continuous functions which is Cauchy, then $\lim_{n\to\infty} f_n$ is also continuous.*

Theorem 5.3. *Suppose the following:*

1. *(X, d) is compact*
2. *$(f_n) \subset C(X)$*
3. *$f = \lim_{n\to\infty} f_n$ is continuous*
4. *For all $x \in X$, $f_1(x) \geq f_2(x) \geq \cdots$*

Then, $f_n \to f$ uniformly as $n \to \infty$.

Theorem 5.4 (Weierstrass Theorem)**.**
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $\forall\, r > 0$, there exists some polynomial $P(x)$ such that

$$||f - P|| = \sup\{|f(x) - P(x)| : x \in [a, b]\} < r$$

Theorem 5.5. *Let $X = (a, b)$ and suppose $f_n : X \to \mathbb{R}$ is such that*

1. *For all $n \in \mathbb{N}$, f_n is differentiable.*
2. *(f_n') is a Cauchy Sequence.*
3. *There exists an $x \in X$ such that $(f_n(x))$ is convergent.*

Then, (f_n) is a Cauchy sequence and $\lim_{n\to\infty} f_n(x)$ is differentiable with derivative $\frac{d}{dx}[\lim_{n\to\infty} f_n(x)] = \lim_{n\to\infty} f_n'(x)$

Proposition 5.1 (Homework Assignment)**.**
If (X, d) is compact and $(f_n) \subset C(X)$ with

$$\sum_{n=1}^{\infty} |f_n(x)| \in C(X)$$

Then, $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent.

Theorem 5.6 (Arzela-Ascoli)**.**
Let (X, d) be compact and let $f_n : X \to \mathbb{R}$ be an equicontinuous, pointwise bounded sequence. Then f_n is uniformly bounded and there is a subsequence f_{n_k} that converges uniformly.

Theorem 5.7 (Adaptation of Arzela-Ascoli)**.**
Let (X, d) be compact and suppose $f_n : X \to \mathbb{R}$ is equicontinuous and pointwise convergent, then f_n is uniformly convergent and $\lim_{n\to\infty} f_n(x)$ is uniformly continuous on X.

6 Multivariable Functions

Theorem 6.1.
Let $X \subset \mathbb{R}^n$ be open, then $f : X \to \mathbb{R}^m$ is continuously differentiable if and only if $D_1 f, D_2 f, \dots, D_n f : X \to \mathbb{R}^m$ are all continuous.

Theorem 6.2 (Chain Rule)**.**
Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be open. Let $f : X \to \mathbb{R}^m$ be differentiable at some $x_0 \in X$. Suppose $y_0 = f(x_0) \in Y$ and that $g : Y \to \mathbb{R}^k$ is differentiable at y_0. Then $g \circ f : X \to \mathbb{R}^k$ is differentiable at x_0 with $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$

Lemma 6.1.
Let $I_n \in L(\mathbb{R}^n, \mathbb{R}^n)$ be such that $I_n x = x \; \forall \; x \in \mathbb{R}^n$. Let $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ be such that $||A|| < 1$. Then $I_n - A$ is invertible and

$$||(I - A)^{-1}|| \leq \frac{1}{1 - ||A||}$$

Lemma 6.2.
Let $\gamma : GL(\mathbb{R}^n) \to GL(\mathbb{R}^n)$. Then $\gamma(A) = A^{-1}$ is continuous and $GL(\mathbb{R}^n)$ is open in the space $L(\mathbb{R}^n, \mathbb{R}^n)$.

Note that $GL(X)$ represents the invertible linear transformations from $X \to X$

Lemma 6.3 (Contraction Principle)**.**
Let (X, d) be a complete metric space. Suppose that $\varphi : X \to X$ is a (strict) contraction, i.e. $\exists \; C \in (0, 1)$ such that $\forall \; x, y \in X, d(\varphi(x), \varphi(y)) \leq Cd(x, y)$. Then, there exists a unique fixed point $x \in X$ such that $\varphi(x) = x$.

Lemma 6.4.
Let $X \subset \mathbb{R}^n$ be open and convex. Let $f : X \to \mathbb{R}^m$ be differentiable. Suppose $\exists \; M > 0$ such that $||f'(x)|| \leq M$ for all $x \in X$. Then, $\forall \; a, b \in X, |f(b) - f(a)| \leq M|b - a|$

Theorem 6.3 (Inverse Function Theorem)**.**
Let $X \subset \mathbb{R}^n$ be open and $f : X \to \mathbb{R}^n$ be continuously differentiable with $x_0 \in X$ and $f'(x_0) \in GL(\mathbb{R}^n)$. Then, there exists an open $U \subset \mathbb{R}^n$ with $x_0 \in U \subset X$ and such that

1. *$f : U \to \mathbb{R}^n$ is injective.*
2. *$V = f(U)$ is open in $\mathbb{R}^n$.*
3. *$f^{-1} : V \to \mathbb{R}^n$ is continuously differentiable.*
4. *$(f^{-1})'(f(x)) = (f^{-1}(x))^{-1}$ for all $x \in U$.*

Theorem 6.4 (Implicit Function Theorem)**.**
Let $X \subset \mathbb{R}^{n+m}$ be open and $f : X \to \mathbb{R}^n$ be continuously differentiable. Denote $[f'(x, y)](\alpha, \beta) = [f'(x, y)](\alpha, 0) + [f'(x, y)](0, \beta) = f'_x(x, y)\alpha + f'_y(x, y)\beta$ where $f'_x(x, y) \in L(\mathbb{R}^n, \mathbb{R}^n)$ and $f'_y(x, y) \in L(\mathbb{R}^m, \mathbb{R}^n$.

Suppose $(x_0, y_0) \in X$ such that $f'_x(x_0, y_0) \in GL(\mathbb{R}^n)$. Then, there exists open sets U, V with $U \subset X$ and $V \subset \mathbb{R}^m$ such that $(x_0, y_0) \in U, y_0 \in V$.

Assume $f(x_0, y_0) = 0$, then there exists a unique $g : V \to \mathbb{R}^n$ which is continuously differentiable such that $\forall \; y \in V$, $(g(y), y) \in U$ and $f(g(y), y) = 0$ for $y \in V$ and $g(y_0) = x_0$. Furthermore,

$$g'(y_0) = -f'_x(x_0, y_0)^{-1} f'_y(x_0, y_0)$$

Proposition 6.1.
Let $X \subset \mathbb{R}^2$ be open. Let $f : X \to \mathbb{R}$ be such that $D_{21} f$ is defined and in X. Let $(x_0, y_0) \in X$ and let $a, b > 0$ be such that $[x_0, x_0 + a] \times [y_0, y_0 + b] \subset X$, then $\exists \; x_1 \in (x_0, x_0 + a), y_1 \in (y_0, y_0 + b)$ such that

$$f(x_0 + a, y_0 + b) - f(x_0, y_0 + b) - f(x_0 + a, y_0) + f(x_0, y_0) = abD_21f(x_1, y_1)$$

Theorem 6.5.
Let $X \subset \mathbb{R}^2$ be open, and $f : X \to \mathbb{R}$ be such that $D_{21} f, D_2 f$ are defined in X. Let $(x_0, y_0) \in X$ be such that $D_{21} f$ is continuous at (x_0, y_0), then $D_{12} f(x_0, y_0)$ exists and $D_{12} f(x_0, y_0) = D_{21} f(x_0, y_0)$

Corollary 6.1 (Taylor's Linearization)**.**
The following linearization holds in a sufficiently small neighborhood:

$$f(x_0 + a, y_0 + b) = f(x_0, y_0) + D_1 f(x_0, y_0)a + D_2 f(x_0, y_0)b + \frac{D_{11} f(x_0, y_0)}{2}a^2 + \frac{D_{22} f(x_0, y_0)}{2}b^2 + D_{12} f(x_0, y_0)ab + o(a^2 + b^2)$$

Theorem 6.6 (HW 8 #1)**.**
Let $f \in \mathbf{L}(\mathbb{R}^n, \mathbb{R}^m)$. Then, f is injective if and only if $\ker(f) = \{0\}$.

Theorem 6.7 (HW 8 #3).
Let X be an open subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}^m$ be such that $D_1 f, D_2 f, \ldots, D_n f$ are defined and bounded in X. Then, f is continuous.

Theorem 6.8 (HW 8 #4).
Let X be an open subset of $\mathbb{R}^n$ and let $f : X \to \mathbb{R}$ be differentiable. Suppose that $x_0 \in X$ is a point of maximum of f, then $f'(x_0) = 0$.

Theorem 6.9 (HW 9 #1).
Let $X \subset \mathbb{R}^n$ be open. Let $x_0 \in X$ and let $f : X \to \mathbb{R}^m$ be differentiable at x_0. Let $Y \subset \mathbb{R}^m$ be open and such that $f(x_0) \in Y$. Finally, let $g : Y \to \mathbb{R}^k$ be differentiable at $f(x_0)$. Then,

$$D_j(g \circ f)(x_0) = \sum_{i=1}^{m} (D_i g)(f(x_0))(D_j f^i)(x_0)$$

where $f^i(x)$ is the i-th component of $f(x)$.

Theorem 6.10 (HW 9 #2).
Let $X \subset \mathbb{R}^n$ be open and let $f : X \to \mathbb{R}^n$ be continuously differentiable and such that $f'(x)$ is invertible for every $x \in X$. Then $f(X)$ is open in $\mathbb{R}^n$.

Theorem 6.11 (HW 10 #1).
Let $X \subset \mathbb{R}^n$ be an open connected set. Let $f : X \to \mathbb{R}^n$ be differentiable and such that $f'(x) = 0$ for all $x \in X$. Then, f is a constant function.

Theorem 6.12 (HW 11 #2).
If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, then f is not injective.

7 Integration

Proposition 7.1.
Let $\{Y_\alpha\} \subset 2^X$, then there is a unique Σ that is a minimal σ-algebra such that $\{Y_\alpha\} \subset \Sigma$.

This Σ is called the σ-completion of $\{Y_\alpha\}$

Lemma 7.1.
If $\mu : \Sigma \to [0, +\infty]$ is a positive measure on X, then the following properties hold:

1. $\mu(\emptyset) = 0$
2. *If $Y_1, Y_2, \ldots, Y_N \in \Sigma$ with $m \neq n \implies Y_m \cap Y_n = \emptyset$, then*
$$\mu\left(\bigcup_{n=1}^{N} Y_n\right) = \sum_{n=1}^{N} \mu(Y_n)$$
3. *If $Y_1, Y_2 \in \Sigma$ are such that $Y_1 \subset Y_2$, then $\mu(Y_1) \leq \mu(Y_2)$.*
4. *If $\{Y_n : n \in \mathbb{N}\}$ are nested subsets where $Y_1 \subset Y_2 \subset Y_3 \subset \cdots$, then*
$$\mu\left(\bigcup_{n=1}^{N} Y_n\right) = \lim_{n\to\infty} \mu(Y_n)$$

Proposition 7.2.
Let (X, Σ, μ) be a measure space and let $(Y, d_Y), (Z, d_Z)$ be metric spaces. Let $f : X \to Y$ be measurable and let $g : Y \to Z$ be continuous, then $g \circ f : X \to Z$ is measurable.

Theorem 7.1.
Suppose (X, Σ, μ) is a measure space. Let $f, g : X \to \mathbb{R}$ be measurable and consider $F : \mathbb{R}^2 \to \mathbb{R}$ some continuous function. Then $F(f(x), g(x))$ is measurable

Proposition 7.3.
Let $(f_n(x))$ be a sequence functions such that $f_n : X \to [-\infty, +\infty]$ is measurable for all $n \in \mathbb{N}$. Then, all of the following are measurable:

- $a(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$
- $b(x) = \inf\{f_n(x) : n \in \mathbb{N}\}$
- $c(x) = \limsup_{n\to\infty} f_n(x)$
- $d(x) = \liminf_{n\to\infty} f_n(x)$

Proposition 7.4.
Let $f : X \to [0, +\infty]$ be measurable, then $\exists$ simple, measurable functions $s_n : X \to \mathbb{R}$ such that $\forall\, x \in X$
$$0 \leq s_1(x) \leq s_2(x) \leq s_3(x) \leq \cdots \leq f(x) = \lim_{n\to\infty} s_n(x) = \sup_{n\in\mathbb{N}} s_n(x)$$

Proposition 7.5.
$$\int_X \sum_{n=1}^{N} s_n d\mu = \sum_{n=1}^{N} \int_X s_n d\mu$$

Theorem 7.2.
Let $f : X \to [0, +\infty]$ be measurable. Define $\nu_f : \Sigma \to [0, +\infty]$ by
$$\nu_f(Y) = \int_Y f d\mu$$
then ν_f is a positive measure.

Theorem 7.3 (Monotone Convergence Theorem)**.**
Suppose (f_n) is a sequence of functions such that for all $n \in \mathbb{N}$, $f_n : X \to [0, +\infty]$ is measurable and
$$f_1 \leq f_2 \leq f_3 \leq \cdots \leq f = \lim_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} f_n$$
Then,
$$\int_X f d\mu = \lim_{n\to\infty} \int_X f_n d\mu$$

Lemma 7.2 (Fatou Lemma).
Let $f_n : X \to [0, +\infty]$ be measurable, then

$$\int_X \liminf_{n\to\infty} f_n d\mu \le \liminf_{n\to\infty} \int_X f_n d\mu$$

Proposition 7.6 (Additivity).
Let $f, g : X \to \mathbb{R}$ be in $\mathcal{L}(\mu)$. Then, $f + g \in \mathcal{L}(\mu)$ with

$$\int_X f + g d\mu = \int_X f d\mu + \int_X g d\mu$$

Proposition 7.7 (Triangle Inequality).

$$\left|\int_X f d\mu\right| \le \int_X |f| d\mu$$

Theorem 7.4 (Dominated Convergence).
Suppose $f_n : X \to [-\infty, +\infty]$ are measurable functions such that

1. *$\exists\, g \in \mathcal{L}(\mu)$ such that $|f_n| \le g$ for all $n \in \mathbb{N}$.*
2. *$f_n \to f$ pointwise as $n \to \infty$.*

Then, $f_n \in \mathcal{L}(\mu), f \in \mathcal{L}(\mu)$ and

$$\int_X f_n d\mu \to \int_X f d\mu \qquad as\ n \to \infty$$

Lemma 7.3.
For $X \subset \mathbb{R}$ and (X, Σ, μ) a measure space, define the Outer Measure as

$$\mu^*(X) = \inf\{\sum_{n=1}^{\infty} \mu(G_n) : G_n \in \Sigma_0, G_n open, X \subset \bigcup_{n=1}^{\infty} G_n\}$$

where Σ_0 is the collection of unions of intervals (open, closed, or half-open) of $\mathbb{R}$. Then $\mu^*(X) = \mu(X)$ for all $X \in \Sigma_0$.

Theorem 7.5.
Let $(\mathbb{R}, \Sigma, \mu)$ be the Lebesgue Measure Space. Let $f : [a, b] \to \mathbb{R}$ be bounded. Then f is Riemann Integrable if and only if $\mu(\{x \in [a, b] : f \text{ is discontinuous at } x\}) = 0$. In this case, $f \in \mathcal{L}(\mu)$ and $\int_a^b f dx = \int_{[a,b]} f d\mu$ where the first is the Riemann Integral and the second is the Lebesgue Integral.

Theorem 7.6 (HW 12 #1).
Let Y_n be a sequence in Σ such that $Y_1 \supset Y_2 \supset Y_3 \supset \cdots$. If $\mu(Y_1) < +\infty$, then

$$\mu\left(\bigcap_{n\in\mathbb{N}} Y_n\right) = \lim_{n\to\infty} \mu(Y_n)$$

Theorem 7.7 (HW 12 #3).
If $f, g : X \to [-\infty, +\infty]$ are two measurable functions, then $\{x : f(x) = g(x)\}$ is a measurable set.