Complex Analysis Homework 8

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Question 3

Let f be a holomorphic function in $\mathbb{C}$ such that $f(z) \to 0$ as $|z| \to \infty$. Prove that f is identically zero.

Proof.
Since f is entire, we know we can express f as a power series centered at some $z_0 \in \mathbb{C}$. For convenience, choose $z_0 = 0$. Thus,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{for all } z \in \mathbb{C}.$$

From here, assume that f is not a constant function. This means that there exists some set $S = \{n : a_n \neq 0 \text{ and } n \geq 1\}$ that is nonempty. Thus,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + \sum_{n \in S} a_n z^n$$

Using this expression, I will take the limit as $|z| \to \infty$. Since the assumption on f's limit at infinity must hold for all rays that extend to infinity I will choose to take z along the path $\{z : \text{Re}(z) > 0 \text{ and } \text{Im}(z) = 0\}$, the positive real axis. Therefore for $z = x \in \mathbb{R}^+$,

$$\begin{aligned}\lim_{|z| \to \infty} f(z) = \lim_{x \to +\infty} f(x) &= \lim_{x \to +\infty} \left(a_0 + \sum_{n \in S} a_n x^n \right) \\ &= a_0 + \lim_{x \to +\infty} \sum_{n \in S} a_n x^n\end{aligned}$$

Note that due to the way S was defined, all of the coefficients a_n in the limit term must be nonzero. This means that as $x \to \infty$, this limit also goes to ∞ (in modulus, since the coefficients may be complex). However, this contradicts $f(z) \to 0$ as $|z| \to \infty$, so our assumption must have been wrong that f is non-constant. This means that the set S described above must actually be the empty set. Thus, in order for $f(z) \to 0$ as $|z| \to \infty$, we need to impose that a_0 is 0. However, at this point we have concluded that $a_n = 0$ for all $n \in \mathbb{N}_0$, which means f is identically zero. $\square$