Complex Analysis Homework 6

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Question 2

Solve the equation $\cos(z) = 2$.

Answer.
I will start by using the identity that for $z = x + iy$, $\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$. From this we get,

$$\begin{aligned} &\cos(z) = 2 \\ \iff &\cos(x)\cosh(y) - i\sin(x)\sinh(y) = 2 \\ \iff &\begin{cases} \cos(x)\cosh(y) = 2 \\ \sin(x)\sinh(y) = 0 \end{cases} \quad \text{AND} \end{aligned}$$

I will proceed by examining the second of these equalities:

$$\begin{aligned} &\sin(x)\sinh(y) = 0 \\ \iff &\begin{cases} \sin(x) = 0 \\ \sinh(y) = 0 \end{cases} \quad \text{OR} \\ \iff &\begin{cases} x = k\pi, k \in \mathbb{Z} \\ y = 0 \end{cases} \quad \text{OR} \end{aligned}$$

Let's first consider what would happen if we allow $y = 0$. This would cause our other equation, $\cos(x)\cosh(y) = 2$ to be equivalent to $\cos(x)\cosh(0) = \cos(x) = 2$. However, $\cos(x) = 2$ has no solutions in $\mathbb{R}$ so this is not possible. Therefore, we can only have the possibility that $x = k\pi, k \in \mathbb{Z}$. From this, our other equation becomes $\cos(k\pi)\cosh(y) = (-1)^k\cosh(y) = 2$. This is equivalent to saying $\cosh(y) = 2 \cdot (-1)^k$. However, we know that $\cosh(y) > 0$ for all $y \in \mathbb{R}$, so we must have that $(-1)^k$ is positive, i.e. that k is even, say $k = 2n$ for $n \in \mathbb{Z}$. Thus, our solution is $x = 2n\pi$ and $y = \cosh^{-1}(2)$, i.e. $\boxed{z = 2n\pi + i\cosh^{-1}(2) \text{ for } n \in \mathbb{Z}}$