Analysis HW 11

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Question 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2$. Find all points that have no neighborhood where the equation $f(x, y) = 0$ can be solved either for x in terms of y or for y in terms of x.

Answer.
First, consider the partial derivatives of f:

$$D_1 f(x, y) = 6x^2 + 6y^2 - 6x = 6[x(x - 1) + y^2]$$
$$D_2 f(x, y) = 12xy + 6y = 6y[2x + 1]$$

To see when we cannot have solve for x in terms of y, we need to see when the Implicit Function Theorem does not apply, i.e. when $D_1 f(x, y)$ is not invertible; or in the one dimensional case, when $D_1 f(x, y) = 0$. This happens when:

$$6[x(x - 1) + y^2] = 0$$
$$x(1 - x) = y^2$$

To find the problematic points satisfying this relation and satisfying $f(x, y) = 0$ we can substitute all instances of y^2 with $x(1 - x)$ in the equation $f(x, y) = 0$:

$$[f(x, y) = 0]_{D_1 f(x,y)=0}$$
$$\implies 2x^3 + 6x^2(1 - x) - 3x^2 + 3x(1 - x) = 0$$
$$\implies 2x^3 - 6x^3 + 6x^2 - 3x^2 - 3x^2 + 3x = 0$$
$$\implies 3x = 4x^3$$
$$\implies x = 0, \quad \text{or} \quad \frac{3}{4} = x^2$$
$$\implies x = 0, \quad \text{or} \quad x = \frac{\sqrt{3}}{2} \quad \text{or} \quad x = \frac{-\sqrt{3}}{2}$$

It is impossible to have $x = -\sqrt{3}/2$ as this would make $y^2 = x(1 - x)$ a negative number. However, for the other two cases, we have

$$y = \pm\sqrt{0(1 - 0)} \qquad\qquad y = \pm\sqrt{\frac{\sqrt{3}}{2}\left(1 - \frac{\sqrt{3}}{2}\right)} = \pm\sqrt{\frac{\sqrt{3}}{2} - \frac{3}{4}}$$
$$= 0 \qquad\qquad \approx \pm 0.340625$$

Thus, the points where we cannot solve for x in terms of y are precisely the points

$$(0, 0) \qquad \left(\frac{\sqrt{3}}{2}, 0.340625\right) \qquad \left(\frac{\sqrt{3}}{2}, -0.340625\right)$$

To analyze the situation in the other direction, let us look at when $D_2 f(x, y) = 0$. This happens when:

$$6y[2x + 1] = 0$$
$$\implies y = 0 \quad \text{or} \quad x = -\frac{1}{2}$$

Using these values into the equation $f(x,y)=0$, we get:

$$\begin{aligned} f(x,0) &= 0 \\ \implies 2x^3 - 3x^2 &= 0 \\ \implies x^2(2x-3) &= 0 \\ \implies x = 0 \quad \text{or} \quad x &= \frac{3}{2} \\ f(-1/2, y) &= 0 \\ \implies -\frac{1}{4} - 3y^2 - \frac{3}{4} + 3y^2 &= 0 \\ \implies -1 &= 0 \text{ (not good)} \end{aligned}$$

We see when $x=-1/2$, we don't have any solutions at all for y. However, for $y=0$, we do have two solutions for x. Both of these solutions will correspond to points where we cannot solve for y in terms of x. These points are:

$$(0,0) \qquad \left(\frac{3}{2}, 0\right)$$

Together, all of the problematic points are:

$$(0,0) \qquad \left(\frac{\sqrt{3}}{2}, 0.340625\right) \qquad \left(\frac{\sqrt{3}}{2}, -0.340625\right) \qquad \left(\frac{3}{2}, 0\right)$$

Question 2

Let $f:\mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Prove that f is not injective. Hint: if $f'(x,y) \neq 0$ at some point, you may use the Implicit Function Theorem.

Proof.
If $f'(x,y)=0$ for all $(x,y)\in\mathbb{R}^2$, then f is a constant function and trivially not injective as all inputs map to the same output. Thus, assume that there exists some point $(x_0,y_0)\in\mathbb{R}^2$ such that $f'(x_0,y_0)\neq 0$. In particular, this means that either $D_1 f(x_0,y_0)\neq 0$ or that $D_2 f(x_0,y_0)\neq 0$. WLOG, assume that $D_1 f(x_0,y_0)\neq 0$ (the proof is symmetric for the other case). Furthermore, assume that $f(x_0,y_0)=0$ (otherwise, consider the function $g:\mathbb{R}^2\to\mathbb{R}$ defined as $g(x,y)=f(x,y)-f(x_0,y_0)$ which clearly satisfies $g(x_0,y_0)=0$, g is also continuously differentiable with $D_1 f(x_0,y_0)=D_1 g(x_0,y_0)\neq 0$, and g is injective iff f is injective).

Since the hypotheses of the Implicit Function Theorem are satisfied near the point (x_0,y_0), there is an open set $V\subset\mathbb{R}$ containing the point y_0 such that there exists a unique function $h:V\to\mathbb{R}$ which is continuously differentiable with $h(y_0)=x_0$ and such that $f(h(y),y)=0$ for all $y\in V$. Since this set V is an open set, it is in fact not a singleton (since singletons are not open sets in the usual topology on $\mathbb{R}$). Therefore, there are distinct points $y_1\neq y_2\in V$ which satisfy $f(h(y_1),y_1)=f(h(y_2),y_2)=0$. However, since $y_1\neq y_2$, then in particular, the points $(h(y_1),y_1)\neq(h(y_2),y_2)$. Therefore, f maps two distinct points to the same image, meaning f is indeed not injective. □

Question 3

Let X be a neighborhood of some point (x_0,y_0) in $\mathbb{R}^2$ and let $f:X\to\mathbb{R}$ be twice continuously differentiable. Derive the formula: as $(x,y)\to(x_0,y_0)$,

$$\begin{aligned} f(x,y) = f(x_0,y_0) + [D_1 f(x_0,y_0)](x-x_0) + [D_2 f(x_0,y_0)](y-y_0) + \frac{1}{2}[D_{11} f(x_0,y_0)](x-x_0)^2 + \\ + [D_{21} f(x_0,y_0)](x-x_0)(y-y_0) + \frac{1}{2}[D_{22} f(x_0,y_0)](y-y_0)^2 + o[(x-x_0)^2+(y-y_0)^2] \end{aligned}$$

Proof.
First, recall the identity

$$f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0) = [D_{21} f(x_1,y_1)](x-x_0)(y-y_0). \tag{0.1}$$

To get the desired equality, I will use Taylor's formula on $f(x,y_0)$ as a function of x and again on $f(x_0,y)$ as a function of y. Denote $g(x)=f(x,y_0)$ and $h(y)=f(x_0,y)$ and note that $g'(x_0)=D_1 f(x_0,y_0), g''(x_0)=D_{11} f(x_0,y_0), h'(y_0)=D_2 f(x_0,y_0)$, and $h''(y_0)=D_{22} f(x_0,y_0)$. Therefore, using the Taylor expansion of g near the point x_0, we obtain:

$$g(x) = g(x_0) + g'(x_0)(x-x_0) + \frac{g''(x_0)}{2}(x-x_0)^2 + o[(x-x_0)^2]$$

Similarly for h near the point y_0, we obtain:

$$h(y) = h(y_0) + h'(y_0)(y - y_0) + \frac{h''(y_0)}{2}(y - y_0)^2 + o[(y - y_0)^2]$$

Therefore, continuing from equation (0.1), we get:

$$\begin{aligned}
f(x, y) &= f(x, y_0) + f(x_0, y) - f(x_0, y_0) + [D_{21} f(x_1, y_1)](x - x_0)(y - y_0) \\
&= g(x) + h(y) - g(x_0) + [D_{21} f(x_1, y_1)](x - x_0)(y - y_0) \\
&= g(x_0) + g'(x_0)(x - x_0) + \frac{g''(x_0)}{2}(x - x_0)^2 + o[(x - x_0)^2] + h(y_0) + h'(y_0)(y - y_0) \\
&\qquad + \frac{h''(y_0)}{2}(y - y_0)^2 + o[(y - y_0)^2] - g(x_0) + [D_{21} f(x_1, y_1)](x - x_0)(y - y_0) \\
&= f(x_0, y_0) + [D_1 f(x_0, y_0)](x - x_0) + [D_2 f(x_0, y_0)](y - y_0) + \frac{1}{2}[D_{11} f(x_0, y_0)](x - x_0)^2 \\
&\qquad + [D_{21} f(x_0, y_0)](x - x_0)(y - y_0) + \frac{1}{2}[D_{22} f(x_0, y_0)](y - y_0)^2 + o[(x - x_0)^2 + (y - y_0)^2]
\end{aligned}$$

Where the last step was obtained simply by canceling out the $g(x_0)$ and the $-g(x_0)$ and then replacing g and h with their definitions in terms of f. Furthermore, this is indeed the equality we wished to show. The only non-trivial bit of work done above was when writing $o[(x-x_0)^2]+o[(y-y_0)^2] = o[(x-x_0)^2+(y-y_0)^2]$. This is easy to show if we denote $a(x) = o[(x - x_0)^2]$ and $b(y) = o[(y - y_0)^2]$ as $x \to x_0$, then by defining $c(x, y) = a(x) + b(y)$, we see that

$$\begin{aligned}
\left| \lim_{(x,y)\to(x_0,y_0)} \frac{c(x, y)}{(x - x_0)^2 + (y - y_0)^2} \right| &= \lim_{(x,y)\to(x_0,y_0)} \frac{|a(x) + b(y)|}{|(x - x_0)^2 + (y - y_0)^2|} \\
&\leq \lim_{(x,y)\to(x_0,y_0)} \frac{|a(x)|}{|(x - x_0)^2 + (y - y_0)^2|} + \lim_{(x,y)\to(x_0,y_0)} \frac{|b(x)|}{|(x - x_0)^2 + (y - y_0)^2|} \\
&= \lim_{x\to x_0} \frac{|a(x)|}{|x - x_0|^2} + \lim_{y\to y_0} \frac{|b(x)|}{|y - y_0|^2} \\
&= 0
\end{aligned}$$

where the last equality follows since $a(x) = o[(x-x_0)^2]$ and $b(x) = o[(y-y_0)^2]$. Therefore, $c(x, y) = o[(x-x_0)^2+(y-y_0)^2]$. Using this result cleans up my previous work and gives us the equality that we wanted (at least in a neighborhood where $(x, y) \to (x_0, y_0)$ that is). □

Question 4

Let X be an uncountable set. Let Σ be the collection of all subsets Y of X such that either Y or Y^c is at most countable (i.e. countable or finite). For $Y \in \Sigma$, let $\mu(Y) = 1$ whenever Y is uncountable, and $\mu(Y) = 0$ otherwise. Prove that Σ is a σ-algebra and μ is a positive measure on Σ.

Proof.
First I will verify that Σ is a σ-algebra:

- $X \in \Sigma$ since $X^c = \emptyset$ is a finite set (with zero elements). Therefore, X is a subset of X with a complement that is at most countable, so $X \in \Sigma$.
- Assume that $Y \in \Sigma$. Then, it is trivial that $Y^c \in \Sigma$ since it requires that either Y^c or $(Y^c)^c = Y$ is at most countable. However, that requirement is already satisfied by the fact that $Y \in \Sigma$.
- Let $\{Y_n : n \in \mathbb{N}\} \subset \Sigma$. Then, consider

  $$Y = \bigcup_{n=1}^{\infty} Y_n$$

  If each Y_n is at most countable, then Y is the countable union of (at most) countable sets, so Y is also (at most) countable; therefore, $Y \in \Sigma$. Otherwise, there is some $Y_k \subset Y$ which is uncountable. However, we must have that Y_k^c is indeed at most countable since $Y_k \in \Sigma$. Then, if we consider Y^c which is expressed as

  $$Y^c = \left(\bigcup_{n=1}^{\infty} Y_n \right)^c = \bigcap_{n=1}^{\infty} Y_n^c$$

  We see that $Y^c \subset Y_k^c$ and since Y_k^c is at most countable, then we must also have that Y^c is at most countable. Therefore, we see that either Y or Y^c must be at most countable, so in fact $Y \in \Sigma$.

The previous bullet points all illustrate that Σ is indeed a σ-algebra. Next, I will show that μ is a positive measure on Σ. First, it is clear that μ maps into non-negative extended real numbers and is not identically positive infinity. Next, let $\{Y_n : n \in \mathbb{N}\} \subset \Sigma$ be a collection of disjoint sets in Σ. First, assume that each Y_n is at most countable, then $\mu(Y_n) = 0$ for all n. Furthermore, denoting Y as the union of all Y_n, we see (just as in the verification of Σ being a σ-algebra) that Y must also be at most countable. Therefore, $\mu(Y) = 0$ so we get:

$$\sum_{n=1}^{\infty} \mu(Y_n) = \sum_{n=1}^{\infty} 0 = 0 = \mu(Y) = \mu\left(\bigcup_{n=1}^{\infty} Y_n\right)$$

Next, assume that not every Y_n is at most countable. I claim that there can only be a single Y_n which is uncountable. If not assume there are some Y_i and Y_j which are both uncountable. Recall that each Y_n is disjoint so in fact $Y_i \cap Y_j = \emptyset$, or alternatively $Y_i \subset Y_j^c$. However, since $Y_j \in \Sigma$ and Y_j is uncountable, then we must have that Y_j^c is at most countable. Therefore, we get that Y_i which is uncountable is a subset of Y_j^c which is at most countable – which is a contradiction. Therefore, there cannot exist more than one uncountable set Y_n.

With this in place, say that Y_j is our ONE uncountable set. Then Y must clearly be uncountable as well since $Y_j \subset Y$. Therefore, $\mu(Y_j) = \mu(Y) = 1$ and $\mu(Y_n) = 0$ for all $n \neq j$. Using this, we have

$$\sum_{n=1}^{\infty} \mu(Y_n) = \mu(Y_j) + \sum_{\substack{n=1 \\ n \neq j}}^{\infty} \mu(Y_n) = 1 + \sum_{\substack{n=1 \\ n \neq j}}^{\infty} 0 = 1 = \mu(Y) = \mu\left(\bigcup_{n=1}^{\infty} Y_n\right)$$

Therefore, μ is indeed a positive measure on Σ. $\square$