Complex Analysis Homework 4

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Question 5

Evaluate

$$\int_\gamma ze^{z^2}dz$$

where $\gamma(t) = i + e^{it}, t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Answer.
First, let $f(z) = ze^{z^2}$, then note that if f has a primitive (i.e. if there exists some function F such that $F' = f$), then we can evaluate this integral using a familiar analog from real-valued integrals. Namely,

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

where a and b are defined as the real numbers that specify $\gamma's$ interval of definition. In other words, γ is defined on $[a, b]$. Motivated by the analogous real-valued primitive, I will claim that $F(z) = \frac{1}{2}e^{z^2}$

Proof of claim
I have already shown in Question 1 of this homework assignment that the derivative of e^{z^2} exists and is equal to $2ze^{z^2}$. Therefore, by the property of scalar multiplication on derivatives, we can say that $F'(z) = \frac{1}{2} \cdot \frac{d}{dz}(e^{z^2}) = \frac{1}{2}(2ze^{z^2}) = ze^{z^2}$. Thus, F is, indeed, the primitive of f. $\square$

With this in place, calculating the above integral is fairly straightforward:

$$\begin{aligned}
\int_\gamma ze^{z^2}dz &= F\left(\gamma\left(\frac{\pi}{2}\right)\right) - F\left(\gamma\left(-\frac{\pi}{2}\right)\right) \\
&= F\left(i + e^{i\frac{\pi}{2}}\right) - F\left(i + e^{i(-\frac{\pi}{2})}\right) \\
&= F(2i) - F(0) \\
&= \frac{1}{2}e^{(2i)^2} - \frac{1}{2}e^{0^2} \\
&= \frac{1}{2}(e^{-4} - 1) \\
\implies & \boxed{\int_\gamma ze^{z^2} = \frac{1 - e^4}{2e^4}}
\end{aligned}$$