Complex Analysis Homework 4

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September 25, 2020

Question 4

Use the estimation theorem to obtain the following upper bounds:

(a) $\left|\int_\gamma \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}$
where γ is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ in the first quadrant

(b) $\left|\int_{\gamma(0;R)} \frac{z-1}{z+1} dz\right| \leq \frac{2\pi R(R+1)}{R-1}$
where $\gamma(0; R)$ denotes the circular contour with center 0 and radius $R > 1$.

In order to provide these estimates, I will use the result that given some path $\gamma : [a, b] \to \mathbb{C}$ and some continuous function $f : \gamma^* \to \mathbb{C}$. Suppose there exists some $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \gamma^*$, then

$$\left|\int_\gamma f(z)dz\right| \leq M \cdot L(\gamma) \tag{1}$$

for $L(\gamma)$ the length of the path γ.

Answer. (a)
First, we need to find our upper bound M for $|f(z)| = \left|\frac{1}{z^2-1}\right|$ with $z \in \gamma^*$ the image of $\gamma = 2e^{it}, t \in [0, \frac{\pi}{2}]$:

$$\begin{aligned}
|f(z)| &= \left|\frac{1}{z^2-1}\right| \\
&= \frac{1}{|z^2-1|} \\
&\leq \frac{1}{\big||z^2| - |-1|\big|} && \text{by reverse triangle inequality} \\
&= \frac{1}{\big||z|^2 - 1\big|} \\
&= \frac{1}{|2^2-1|} && \text{since } z \in \gamma^* \text{ has modulus 2} \\
&= \frac{1}{3}
\end{aligned}$$

Therefore we will define our chosen upper bound M to be $\frac{1}{3}$. Next, we will calculate $L(\gamma)$. First, note the Circumference of a circle is given by the formula $C = 2r\pi$ for r the radius of the circle. Thus, since our γ is one-fourth of the circumference of the circle with radius 2, we get that $L(\gamma) = \frac{2(2)\pi}{4} = \pi$. Therefore,

$$\boxed{\left|\int_\gamma \frac{dz}{z^2-1}\right| \leq \frac{\pi}{3}}$$

Answer. (b)

First, we need to find our upper bound M for $|f(z)| = \left|\frac{z-1}{z+1}\right|$ with $z \in \gamma^*$ the image of $\gamma(0;R) = Re^{it}, t \in [0, 2\pi]$:

$$\begin{aligned}
|f(z)| = \left|\frac{z-1}{z+1}\right| &= \frac{|z-1|}{|z+1|} \\
&\leq \frac{|z| + |-1|}{|z+1|} && \text{by the triangle inequality} \\
&\leq \frac{|z|+1}{||z|-|1||} && \text{by the reverse triangle inequality} \\
&= \frac{R+1}{|R-1|} && \text{since } z \in \gamma^* \text{ has modulus } R \\
&= \frac{R+1}{R-1} && \text{since } R > 1
\end{aligned}$$

Thus, we can define our upper bound M to be this above fraction. Furthermore, since γ is simply a circle with radius R, we know that $L(\gamma) = 2\pi R$. Therefore,

$$\boxed{\left|\int_\gamma \frac{z-1}{z+1} dz\right| \leq \frac{2\pi R(R+1)}{R-1}}$$