4.) Show that, for $z \in \mathbb{C}$,

$$|z| \le |Re(z)| + |Im(z)| \le \sqrt{2}|z|$$

Proof

let $z = x+iy \in \mathbb{C}$, then $|z| = \sqrt{x^2+y^2}$, $|Re(z)| = |x|$, $|Im(z)| = |y|$

Using this the first inequality becomes:

$$\sqrt{x^2+y^2} \le |x|+|y| \iff x^2+y^2 \le (|x|+|y|)^2$$

I will start with the L.H.S. and attempt to derive the inequality:

$$x^2+y^2 \le x^2+y^2+2|x||y| = (|x|+|y|)^2$$
$$\iff \sqrt{x^2+y^2} \le |x|+|y|$$

Thus, the first inequality holds. Equality holds whenever x or y is 0. e.g. $|5+0i| = |5| = |5|+|0|$ ✓

$|0+2i| = |2| = |0|+|2|$ ✓

In a similar manner, the second inequality becomes:

$$|x|+|y| \le \sqrt{2}\sqrt{x^2+y^2} \iff (|x|+|y|)^2 \le 2(x^2+y^2)$$

Again, I will start with the LHS and attempt to derive the inequality:

$$\begin{aligned}(|x|+|y|)^2 = x^2+y^2+2|x||y| &= x^2+y^2+2|x||y|+x^2+y^2-x^2-y^2 \\ &= 2(x^2+y^2)-(x^2+y^2-2|x||y|) \\ &= 2(x^2+y^2)-(|x|-|y|)^2 \\ &\le 2(x^2+y^2)\end{aligned}$$

$$\iff |x|+|y| \le \sqrt{2}\sqrt{x^2+y^2}$$

Thus, the second inequality holds as well. Equality holds here whenever $x=y$.

e.g. take $z=2+2i \Rightarrow |x|+|y| = 2+2 = 4$ ✓

$\sqrt{2}\sqrt{x^2+y^2} = \sqrt{2}\sqrt{2^2+2^2} = \sqrt{2}\cdot 2\sqrt{2} = 4$