Advanced Calc. Homework 11

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November 11, 2020

Notation

For the entirety of this assignment, I will examine power series of the form $\sum a_n x^n$. This series converges for all $|x| < R$ and diverges for all $|x| > R$ with

$$R = \frac{1}{\beta} \qquad \beta = \begin{cases} \limsup \sqrt[n]{|a_n|} & \text{or} \\ \lim \left| \frac{a_{n+1}}{a_n} \right| & \text{if it exists} \end{cases}$$

Furthermore, we define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$.

23.1

For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(a) $\sum n^2 x^n$

- Here, our $a_n = n^2$. Thus,

$$\beta = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(n+1)^2}{n^2} \right| = \lim \frac{n^2 + 2n + 1}{n^2} = \lim \left(1 + \frac{2}{n} + \frac{1}{n^2} \right) = 1$$

- Thus, the $\boxed{\text{Radius of Convergence, } R, \text{ is equal to } 1}$. To find the exact interval of convergence, I will examine the series at $x = \pm 1$. However, in both cases $\lim(n^2)$ and $\lim((-1)^n n^2)$ do not equal zero, so those corresponding series cannot possibly converge. Thus, the $\boxed{\text{Interval of Convergence is } (-1, 1)}$.

(b) $\sum \left(\frac{x}{n} \right)^n$

- Here, our $a_n = \frac{1}{n^n}$. Thus,

$$\beta = \limsup \sqrt[n]{|a_n|} = \limsup \sqrt[n]{\frac{1}{n^n}} = \limsup \frac{1}{n} = 0$$

- Thus, the $\boxed{\text{Radius of Convergence, } R, \text{ is equal to } \infty}$. With $R = \infty$, we are immediately given the interval of convergence is all real numbers. Thus, the $\boxed{\text{Interval of Convergence is } (-\infty, \infty)}$.

(c) $\sum \left(\frac{2^n}{n^2} \right) x^n$

- Here, our $a_n = \left(\frac{2^n}{n^2} \right)$. Thus,

$$\beta = \lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{2^{n+1}}{(n+1)^2} \cdot \frac{n^2}{2^n} \right| = \lim \frac{2n^2}{(n+1)^2} = 2 \lim \frac{n^2}{n^2 + 2n + 1} = 2 \lim \frac{1}{1 + \frac{2}{n} + \frac{1}{n^2}} = 2$$

- Thus, the $\boxed{\text{Radius of Convergence, } R, \text{ is equal to } \frac{1}{2}}$. To find the exact interval of convergence, I will examine the series at $x = \pm\frac{1}{2}$. With $x = \frac{1}{2}$, we have the sum of terms in the form of $\left(\frac{2^n}{n^2} \right) \left(\frac{1}{2} \right)^n = \frac{1}{n^2}$ which we know converges by p-test. For $x = -\frac{1}{2}$, we have the sum of terms in the form of $\left(\frac{2^n}{n^2} \right) \left(-\frac{1}{2} \right)^n = \frac{(-1)^n}{n^2}$ which we know converges by the alternating series test. Thus, the $\boxed{\text{Interval of Convergence is } \left[-\frac{1}{2}, \frac{1}{2} \right]}$.

(d) $\sum \left(\dfrac{n^3}{3^n}\right) x^n$

- Here, our $a_n = \left(\dfrac{n^3}{3^n}\right)$. Thus,

$$\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}\right| = \lim \left|\frac{(n+1)^3}{3n^3}\right| = \frac{1}{3} \lim \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3}$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to } 3}$. To find the exact interval of convergence, I will examine the series at $x = \pm 3$. However, in both cases $\lim(n^3)$ and $\lim((-1)^n n^3)$ do not equal zero, so those corresponding series cannot possibly converge. Thus, the $\boxed{\text{Interval of Convergence is } (-3, 3)}$.

(e) $\sum \left(\dfrac{2^n}{n!}\right) x^n$

- Here, our $a_n = \left(\dfrac{2^n}{n!}\right)$. Thus,

$$\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n}\right| = \lim \frac{2}{n+1} = 0$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to } \infty}$. With $R = \infty$, we are immediately given the interval of convergence is all real numbers. Thus, the $\boxed{\text{Interval of Convergence is } (-\infty, \infty)}$.

(f) $\sum \left(\dfrac{1}{(n+1)^2 \cdot 2^n}\right) x^n$

- Here, our $a_n = \left(\dfrac{1}{(n+1)^2 \cdot 2^n}\right)$. Thus,

$$\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)^2 \cdot 2^n}{(n+2)^2 \cdot 2^{n+1}}\right| = \frac{1}{2} \lim \frac{n^2 + 2n + 1}{n^2 + 4n + 4} = \frac{1}{2} \lim \frac{1 + \frac{2}{n} + \frac{1}{n^2}}{1 + \frac{4}{n} + \frac{4}{n^2}} = \frac{1}{2}$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to } 2}$. To find the exact interval of convergence, I will examine the series at $x = \pm 2$. With $x = 2$, we get the summation of a series with terms of the form $\left(\dfrac{1}{(n+1)^2 \cdot 2^n}\right) \cdot 2^n = \dfrac{1}{(n+1)^2}$ which converges by comparison with $\sum \frac{1}{n^2}$ which converges by the p-series test. With $x = -2$, we get the summation of a series with terms of the form $\left(\dfrac{1}{(n+1)^2 \cdot 2^n}\right) \cdot (-2)^n = \dfrac{(-1)^n}{(n+1)^2}$ which converges by the Alternating Series test. Thus, the $\boxed{\text{Interval of Convergence is } [-2, 2]}$.

(g) $\sum \left(\dfrac{3^n}{n \cdot 4^n}\right) x^n$

- Here, our $a_n = \left(\dfrac{3^n}{n \cdot 4^n}\right)$. Thus,

$$\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{3^{n+1}}{(n+1) \cdot 4^{n+1}} \cdot \frac{n \cdot 4^n}{3^n}\right| = \frac{3}{4} \lim \frac{n}{n+1} = \frac{3}{4} \lim \frac{1}{1 + \frac{1}{n}} = \frac{3}{4}$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to } \frac{4}{3}}$. To find the exact interval of convergence, I will examine the series at $x = \pm\frac{4}{3}$. When $x = \frac{4}{3}$, we get a series with terms of the form $\left(\dfrac{3^n}{n \cdot 4^n}\right)\left(\dfrac{4}{3}\right)^n = \dfrac{1}{n}$ which diverges due to the p series test. However, with $x = -\frac{4}{3}$, we get a series with terms of the form $\left(\dfrac{3^n}{n \cdot 4^n}\right) \cdot \left(-\dfrac{4}{3}\right) = \dfrac{(-1)^n}{n}$ which converges due to the Alternating Series Test. Thus, the $\boxed{\text{Interval of Convergence is } \left[-\frac{4}{3}, \frac{4}{3}\right)}$.

(h) $\sum \left(\dfrac{(-1)^n}{n^2 \cdot 4^n}\right) x^n$

- Here, our $a_n = \left(\dfrac{(-1)^n}{n^2 \cdot 4^n}\right)$. Thus,

$$\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(-1)^{n+1}}{(n+1)^2 \cdot 4^{n+1}} \cdot \frac{n^2 \cdot 4^n}{(-1)^n}\right| = \frac{1}{4} \lim \frac{n^2}{n^2+2n+1} = \frac{1}{4} \lim \frac{1}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{1}{4}$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to 4}}$. To find the exact interval of convergence, I will examine the series at $x = \pm 4$. For $x = 4$, we get a series with terms of the form $\left(\dfrac{(-1)^n}{n^2 \cdot 4^n}\right) \cdot 4^n = \dfrac{(-1)^n}{n^2}$ which converges by the Alternating Series Test. With $x = -4$, we get a series with terms of the form $\left(\dfrac{(-1)^n}{n^2 \cdot 4^n}\right) \cdot (-4)^n = \dfrac{1}{n^2}$ which converges by the p-series test. Thus, the $\boxed{\text{Interval of Convergence is } [-4,4]}$.

23.2

For each of the following power series, find the radius of convergence and determine the exact interval of convergence.

(a) $\sum \sqrt{n} x^n$

- Here, our $a_n = \sqrt{n}$. Thus,

$$\beta = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{\sqrt{n+1}}{\sqrt{n}}\right| = \lim \sqrt{1+\frac{1}{n}} = 1$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to 1}}$. To find the exact interval of convergence, I will examine the series at $x = \pm 1$. For $x = 1$, we get a series with terms of the form $\sqrt{n}$ which diverges since $\lim(\sqrt{n}) \neq 0$. With $x = -1$, we get a series with terms of the form $(-1)^n \sqrt{n}$ which diverges as well since $\lim((-1)^n \sqrt{n}) \neq 0$. Thus, the $\boxed{\text{Interval of Convergence is } (-1,1)}$.

(b) $\sum \left(\dfrac{1}{n\sqrt{n}}\right) x^n$

- Here, our $a_n = \left(\dfrac{1}{n\sqrt{n}}\right)$. Thus,

$$\begin{aligned}
\beta = \limsup \sqrt[n]{|a_n|} = \limsup \sqrt[n]{\frac{1}{n\sqrt{n}}} &= \limsup \frac{1}{n^{\frac{\sqrt{n}}{n}}} = \limsup \frac{1}{n^{1/\sqrt{n}}} \\
&= \limsup n^{-1/\sqrt{n}} \\
&= \limsup e^{\ln(n^{-1/\sqrt{n}})} \\
&= \limsup e^{(-1/\sqrt{n}) \cdot \ln(n)} \\
&= e^{\limsup \left(\frac{-\ln(n)}{\sqrt{n}}\right)} \\
&= e^{\limsup \left(\frac{-\frac{1}{n}}{\frac{1}{2} n^{-\frac{1}{2}}}\right)} && \text{by L'Hopital's Rule} \\
&= e^{\limsup \left(\frac{-2\sqrt{n}}{n}\right)} \\
&= e^{\limsup \left(\frac{-2}{\sqrt{n}}\right)} \\
&= e^0 = 1
\end{aligned}$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to 1}}$. To find the exact interval of convergence, I will examine the series at $x = \pm 1$. For $x = 1$, we get a series with terms of the form $\left(\dfrac{1}{n\sqrt{n}}\right)$ which converges by the Comparison Test with $\frac{1}{n^2}$ for all $n \geq 4$. With $x = -1$, we get a series with terms of the form $\left(\dfrac{(-1)^n}{n\sqrt{n}}\right)$ which converges by the Alternating Series Test. Thus, the $\boxed{\text{Interval of Convergence is } [-1,1]}$.

(c) $\sum x^{n!}$

- Here, our a_n needs to be defined carefully as:

$$a_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

- Thus,

$$\beta = \limsup \sqrt[n]{|a_n|} = \lim_{k\to\infty} \sqrt[k!]{|a_{k!}|} = \lim_{k\to\infty} \sqrt[k!]{1} = 1$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to 1}}$. To find the exact interval of convergence, I will examine the series at $x = \pm 1$. For $x = 1$, we get a series with terms of the form $1^{n!} = 1$ which diverges since $\lim(1) \neq 0$. With $x = -1$, we get a series with terms of the form $(-1)^{n!}$ which diverges as well since $\lim((-1)^{n!}) \neq 0$. Thus, the $\boxed{\text{Interval of Convergence is } (-1, 1)}$.

(d) $\sum \left(\frac{3^n}{\sqrt{n}}\right) x^{2n+1}$

- Again, our a_n needs to be defined carefully as:

$$a_n = \begin{cases} \frac{3^k}{\sqrt{k}} & \text{if } n = 2k+1 \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

- Thus,

$$\begin{aligned} \beta = \limsup \sqrt[n]{|a_n|} = \lim_{k\to\infty} \sqrt[2k+1]{|a_{2k+1}|} &= \lim_{k\to\infty} \left| \frac{a_{2(k+1)+1}}{a_{2k+1}} \right| \\ &= \lim_{k\to\infty} \left| \frac{3^{k+1}}{\sqrt{k+1}} \cdot \frac{\sqrt{k}}{3^k} \right| \\ &= 3 \lim_{k\to\infty} \sqrt{\frac{k}{k+1}} \\ &= 3 \end{aligned}$$

- Thus, the $\boxed{\text{Radius of Convergence, } R\text{, is equal to } \frac{1}{3}}$. To find the exact interval of convergence, I will examine the series at $x = \pm\frac{1}{3}$. For $x = \frac{1}{3}$, we get a series with terms of the form $\left(\frac{3^n}{\sqrt{n}}\right) \cdot \left(\frac{1}{3}\right)^{2n+1} = \frac{1}{\sqrt{n} \cdot 3^{n+1}}$ which converges by a Comparison Test with $\sum \frac{1}{3^n}$ which converges as a geometric series. With $x = -\frac{1}{3}$, we get a series with terms of the form $\left(\frac{3^n}{\sqrt{n}}\right) \cdot \left(-\frac{1}{3}\right)^{2n+1} = \frac{-1}{\sqrt{n} \cdot 3^{n+1}}$ which converges as well by the same Comparison Test. Thus, the $\boxed{\text{Interval of Convergence is } \left[-\frac{1}{3}, \frac{1}{3}\right]}$.