Analysis Homework 3

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Question 1

Let the sequence (x_n) be defined recursively as $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2+\sqrt{x_n}}$. Prove that the sequence is convergent.

Proof.
First, I will show that the sequence is bounded. In particular, $x_n \in (0,2)$ for all $n \in \mathbb{N}$. I will prove this by induction:

Base Case: This is clear to see since $x_1 = \sqrt{2} \in (0,2)$.
Inductive Step: Assume that $x_n \in (0,2)$ for some $n \in \mathbb{N}$. Then we have the following inequalities:

$$\begin{aligned}
x_{n+1} &= \sqrt{2+\sqrt{x_n}} \\
&< \sqrt{2+\sqrt{2}} && \text{by inductive hypothesis} \\
&= 1.847\ldots < 2 \\
x_{n+1} &= \sqrt{2+\sqrt{x_n}} \\
&> \sqrt{2+\sqrt{0}} && \text{by inductive hypothesis} \\
&= \sqrt{2} > 0
\end{aligned}$$

Thus, $x_{n+1} \in (0,2)$ and the claim is proven inductively.

Next, I will show that the sequence (x_n) is monotone increasing. To do this, I will prove that $x_n \leq x_{n+1}$ holds by induction:

Base Case: It is clear numerically that $x_1 < x_2$ since $x_1 = \sqrt{2} = 1.414\ldots < 1.785\ldots = \sqrt{2+\sqrt{\sqrt{2}}} = x_2$.
Inductive Step: Assume that for some $n \in \mathbb{N}$, the inequality $x_n < x_{n+1}$ holds true. From this inequality, we can see the following:

$$\begin{aligned}
x_n &< x_{n+1} && \text{by inductive hypothesis} \\
\implies \sqrt{x_n} &< \sqrt{x_{n+1}} \\
\implies 2+\sqrt{x_n} &< 2+\sqrt{x_{n+1}} \\
\implies \sqrt{2+\sqrt{x_n}} &< \sqrt{2+\sqrt{x_{n+1}}} \\
\implies x_{n+1} &< x_{n+2}
\end{aligned}$$

Thus, the sequence is monotone increasing. Therefore, we have shown that (x_n) is both bounded and monotonic and since our metric space here is simply $\mathbb{R}$, we have proven in class that (x_n) must be convergent. Therefore, there exists some $a \in [0,2] \subset \mathbb{R}$ such that $\lim_{n\to\infty} x_n = a$. Finding what this a must be would involve finding the roots to a quartic polynomial which I won't attempt to do. $\square$

Question 2

Let (x_n) and (y_n) be two real sequences such that

$$\left\{\limsup_{n\to\infty} x_n, \limsup_{n\to\infty} y_n\right\} \neq \{+\infty, -\infty\}.$$

Prove that

$$\limsup_{n\to\infty}(x_n+y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

Proof.
Let us fix some $N \in \mathbb{N}$. Then the following inequalities follow simply from the definition of the supremum:

$$x_n \leq \sup\{x_n : n \geq N\} \quad \forall\, n \geq N$$
$$y_n \leq \sup\{y_n : n \geq N\} \quad \forall\, n \geq N$$

These inequalities clearly imply

$$x_n + y_n \leq \sup\{x_n : n \geq N\} + \sup\{y_n : n \geq N\} \quad \forall\, n \geq N$$

Since this holds for all $n \geq N$, then the supremum over all $n \geq N$ of the left side must also be no greater than the right hand side:

$$\sup\{x_n + y_n : n \geq N\} \leq \sup\{x_n : n \geq N\} + \sup\{y_n : n \geq N\}$$

Thus, by taking limits as $N \to \infty$, we get:

$$\lim_{N\to\infty} \sup\{x_n + y_n : n \geq N\} \leq \lim_{N\to\infty} \Big(\sup\{x_n : n \geq N\} + \sup\{y_n : n \geq N\} \Big)$$
$$\implies \limsup_{n\to\infty}(x_n + y_n) \leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n$$

which proves the desired statement. Note the condition that $\{\limsup_{n\to\infty} x_n, \limsup_{n\to\infty} y_n\} \neq \{+\infty, -\infty\}$ guaranteed that we never ran into a case of $\infty - \infty$ which means all calculations I did above were indeed valid as calculations of numbers in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. □

Question 3

Let (x_n) be a sequence of non-negative real numbers such that

$$\sum_{n=1}^{\infty} x_n = +\infty.$$

Prove that

$$\sum_{n=1}^{\infty} \frac{x_n}{1 + x_n} = +\infty$$

Proof.
First, assume that (x_n) is unbounded, i.e. that $\lim_{n\to\infty} x_n = +\infty$. If this is the case, then

$$\lim_{n\to\infty} \left(\frac{x_n}{1 + x_n} \right) = 1 \qquad \text{since the numerator and denominator have the same rate of divergence.}$$

Thus, since $(x_n/(1 + x_n))_n$ does not converge to zero, then it is impossible for the series to converge. On the other hand, if (x_n) is bounded, but the series still diverges, then we have that $x_n \leq M$ for all n and we can say that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{x_n}{1 + x_n} &\geq \sum_{n=1}^{\infty} \frac{x_n}{1 + M} \\ &= \frac{1}{1 + M} \sum_{n=1}^{\infty} x_n \\ &= +\infty \end{aligned}$$

Therefore, the series still diverges when (x_n) is bounded, so it must always diverge. □

Question 4

Let (x_n) be a sequence of non-negative real numbers such that

$$\sum_{n=1}^{\infty} x_n < +\infty.$$

Prove that $\liminf_{n\to\infty} n x_n = 0$.

Proof.
Assume that $\liminf_{n\to\infty} nx_n \neq 0$. Since (x_n) is non-negative, then we must have that $\liminf_{n\to\infty} nx_n > 0$, say equal to some $\delta > 0$. Note that

$$\liminf_{n\to\infty} nx_n = \lim_{N\to\infty} \inf\{nx_n : n \geq N\}$$

Thus, for this limit to be equal to δ we must have that for every ε, there exists some $M \in \mathbb{N}$, such that

$$|\inf\{nx_n : n \geq N\} - \delta| < \varepsilon \quad \forall\, N \geq M$$

Furthermore, since the sequence of infimums must be an increasing sequence as N increases, then we know that δ must be greater than $\inf\{nx_n : n \geq N\}$ for all N. Thus, we can say that

$$\begin{aligned} &\delta - \inf\{nx_n : n \geq N\} < \varepsilon && \forall\, N \geq M \\ \implies\ &\inf\{nx_n : n \geq N\} > \delta - \varepsilon && \forall\, N \geq M \end{aligned}$$

By taking ε to be equal to $\delta/2$, we see that $\inf\{nx_n : n \geq N\}$ is positive for all $N \geq M$. In particular, this means that $nx_n \geq \inf\{nx_n : n \geq M\} =: \mu > 0$ for all $n \geq M$. Thus, we see that $x_n \geq \mu/n$ for all $n \geq M$. However, using this fact we get:

$$\begin{aligned} \sum_{n=1}^{\infty} x_n &= \sum_{n=1}^{M-1} x_n + \sum_{n=M}^{\infty} x_n \\ &\geq \sum_{n=1}^{M-1} x_n + \sum_{n=M}^{\infty} \frac{\mu}{n} \end{aligned}$$

However, note that the far right summation is divergent since it is the tail of a scaled version of the harmonic series. Thus, since the first summation is finite, we can see that $\sum x_n$ diverges. But this is a contradiction to our assumption that $\sum x_n < +\infty$. Therefore, our assumption was wrong and we do indeed have that $\liminf_{n\to\infty} nx_n = 0$. ☐

Question.
Does $\lim_{n\to\infty} nx_n$ *always exist?*

Answer.
No, consider the series

$$x_n = \begin{cases} \frac{1}{n} & \text{for } n \text{ a power of 2} \\ \frac{1}{n^2} & \text{else.} \end{cases}$$

Proof.
In this series above, we have convergence since

$$\begin{aligned} \sum_{n=1}^{\infty} x_n &= \sum_{n \text{ a power of 2}} \frac{1}{n} + \sum_{n \text{ not a power of 2}} \frac{1}{n^2} \\ &= \sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{k=1}^{\infty} \frac{1}{2^{2k}} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

Notice that the two series in the last expression are both convergent, so $\sum x_n$ is convergent as it is bounded by the sum of two convergent series. Furthermore $x_n > 0$ for all n so (x_n) is non-negative meaning the hypotheses of the question are satisfied. However, if we consider the subsequence $(2^k \cdot x_{2^k})$ of (nx_n), then

$$\lim_{k\to\infty} (2^k \cdot x_{2^k}) = \lim_{k\to\infty} 2^k \cdot \frac{1}{2^k} = 1$$

Thus, $\limsup(nx_n) \geq 1$. On the other hand by taking any other subsequence that does not contain powers of 2, say $(n_k \cdot x_{n_k})$ of (nx_n), it is clear to see that

$$\lim_{k\to\infty} (n_k \cdot x_{n_k}) = \lim_{k\to\infty} n_k \cdot \frac{1}{n_k^2} = \lim_{k\to\infty} \frac{1}{n_k} = 0$$

Thus, $\liminf(nx_n) \leq 0$. However, this means that $\liminf(nx_n) \neq \limsup(nx_n)$ which means that $\lim(nx_n)$ does not exist.
☐

Question 5

Let (x_n) and (y_n) be two real sequences such that (x_n) is monotonic and bounded and that $\sum_{n=1}^{\infty} y_n$ is convergent. Prove that

$$\sum_{n=1}^{\infty} x_n y_n$$

is convergent as well.

Proof.
Recall the Dirichlet Test which we proved in class: If the following conditions are met

(a) $\sup\left\{\left|\sum_{n=1}^{N} y_n\right| : N \in \mathbb{N}\right\} < +\infty$

(b) $\forall\, n \in \mathbb{N}, x_n \geq x_{n+1}$

(c) $\lim_{n\to\infty} x_n = 0$

then

$$\sum_{n=1}^{\infty} x_n y_n$$

is convergent. In our case, we have that $\sum y_n$ is convergent. This means that the sequence of partial sums of y_n is convergent. In particular, the sequence of partial sums must have a finite supremum, so condition (a) is met. For condition (b), we are given that (x_n) is monotonic; thus, we can assume that it is monotonic decreasing (otherwise, replace (x_n) with $(-x_n)$ to conclude that $-\sum x_n y_n$ converges which happens if and only if $\sum x_n y_n$ converges). For condition (c) we know that (x_n) is not only monotonic, but also bounded. We have proven that bounded monotonic sequences converge; however, they will likely not converge to 0. Thus, let L be the limit of (x_n) and define the sequence

$$c_n = x_n - L.$$

Then (c_n) is also monotonic decreasing since (x_n) is and in fact $\lim_{n\to\infty} c_n = L - L = 0$. Thus, we can apply the Dirichlet Test to conclude that

$$\sum_{n=1}^{\infty} c_n y_n = \sum_{n=1}^{\infty} x_n y_n - \sum_{n=1}^{\infty} L y_n$$

converges. Furthermore, it is clear to see that the far-right sequence converges as it is simply a scalar multiple of a convergent sequence. Thus, we can rearrange terms to get

$$\sum_{n=1}^{\infty} x_n y_n = \sum_{n=1}^{\infty} c_n y_n + \sum_{n=1}^{\infty} L y_n$$

and conclude that the sequence we are interested in must converge since it is the sum of two convergent sequences. □