Complex Analysis Homework 3

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Question 1

Question.
Define a function f by

$$f(z) = \frac{z}{1+|z|}.$$

(a) Prove that f is continuous on $\mathbb{C}$.

(b) Prove that $f(z_1) = f(z_2)$ implies that $z_1 = z_2$.

(c) Prove that f maps $\mathbb{C}$ onto $D(0,1)$.

Proof. **(a)**
To prove this function is continuous, I will use that we know the following

1. The sum of continuous function is continuous
2. The quotient of continuous function is continuous as long as the denominator is not equal to zero.
3. A function f is continuous at point $z_0 \in \mathbb{C}$ if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever $|z - z_0| < \delta$, we have $|f(z) - f(z_0)| < \varepsilon$.

Thus, to prove that $f(z)$ is continuous, we merely need to show that z, 1, and $|z|$ are continuous and that $1 + |z| \neq 0$. First, since $|z| \geq 0$ for all $z \in \mathbb{C}$, we know that $1 + |z| \geq 1$ for all $z \in \mathbb{C}$; thus, $1 + |z|$ is never equal to zero, so we need not worry about this.

Next, I will show that $g(z) = z$ is a continuous function by examining $|g(z) - g(z_0)|$ for some arbitrary $z_0 \in \mathbb{C}$. In fact, it is as simple as:

$$|g(z) - g(z_0)| = |z - z_0|$$

Thus, $|g(z) - g(z_0)| < \varepsilon$ for any $\varepsilon > 0$ precisely when $|z - z_0| < \delta$ for δ any number $\leq \varepsilon$. Since this holds true for any $z_0 \in \mathbb{C}$, it is clear that $g(z) = z$ satisfies the above definition of continuity for all points in $\mathbb{C}$.

Similarly, I will show that $g(z) = 1$ is a continuous function by looking at $|g(z) - g(z_0)|$ for some arbitrary $z_0 \in \mathbb{C}$. However, it is clear that $|g(z) - g(z_0)| = |1 - 1| = |0| = 0$. Thus, $|g(z) - g(z_0)| < \varepsilon$ for any $\varepsilon > 0$ no matter how we choose δ. Since z_0 can be any point in $\mathbb{C}$, $g(z) = 1$ clearly satisfies the definition of continuity for all points in $\mathbb{C}$.

Lastly, I will show that $g(z) = |z|$ is a continuous function. Again, take z_0 to be some arbitrary point in $\mathbb{C}$, then examine $|g(z) - g(z_0)|$ as follows:

$$\begin{aligned}
|g(z) - g(z_0)| &= ||z| - |z_0|| & \\
&= ||z| - |-z_0|| & \text{since } |w| = |-w| \text{ for all } w \in \mathbb{C} \\
&\leq |z + (-z_0)| & \text{by the "reverse" Triangle Inequality} \\
&= |z - z_0|
\end{aligned}$$

Thus, $|g(z) - g(z_0)| < \varepsilon$ for any $\varepsilon > 0$ precisely when $|z - z_0| < \delta$ for δ any number $\leq \varepsilon$. Since this is true for any $z_0 \in \mathbb{C}$, it is clear that $g(z) = |z|$ satisfies the above definition of a continuous function for all points in $\mathbb{C}$.

Thus, we have shown z, 1, and $|z|$ are all continuous function; therefore, $1 + |z|$ is also a continuous function and since $1 + |z| \neq 0$ for all $z \in \mathbb{C}$, we must have that

$$f(z) = \frac{z}{1+|z|}$$

is a continuous function as well. □

Proof. **(b)**
Assume that two points $z_1, z_2 \in \mathbb{C}$ satisfy $f(z_1) = f(z_2)$. It will be convenient to express z_1 and z_2 in polar coordinates (exponential form), say $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ for $r_1, r_2 > 0$ and θ_1, θ_2 the argument of z_1 and z_2 respectively; thus, $f(r_1 e^{i\theta_1}) = f(r_2 e^{i\theta_2})$. This gives:

$$\begin{aligned}
\frac{r_1 e^{i\theta_1}}{1 + |r_1 e^{i\theta_1}|} &= \frac{r_2 e^{i\theta_2}}{1 + |r_2 e^{i\theta_2}|} \\
\implies \frac{r_1 e^{i\theta_1}}{1 + r_1} &= \frac{r_2 e^{i\theta_2}}{1 + r_2} && \text{since } |re^{i\theta}| = r \\
\implies \frac{r_1}{1 + r_1} e^{i\theta_1} &= \frac{r_2}{1 + r_2} e^{i\theta_2} \\
\implies \frac{r_1}{1 + r_1} = \frac{r_2}{1 + r_2} &\text{ and } \theta_1 = \theta_2 + 2k\pi && \text{by definition of equality of complex numbers in polar form}
\end{aligned}$$

Working with the left equality, we can see that $r_1 + r_1 r_2 = r_2 + r_2 r_1$ by cross multiplying the fractions. Thus, by subtracting $r_1 r_2$, we get $r_1 = r_2$. This, together with $\theta_1 = \theta_2 + 2k\pi$ implies that $z_1 = z_2$ by definition of equality of complex numbers in polar form. Thus, $f(z_1) = f(z_2) \implies z_1 = z_2$ just as desired. □

Proof. **(c)**
We know that for any $z_0 \in D(0, 1)$, we can express z_0 in polar form as $z_0 = r_0 e^{i\theta_0}$ where $0 < r_0 < 1$ and θ_0 is the argument of z_0. From part (b), we showed that for any $z = re^{i\theta}$,

$$f(z) = \frac{r}{1 + r} e^{i\theta} \tag{1}$$

Thus, $f(z) = z_0$ whenever

$$\frac{r}{1 + r} = r_0 \quad \text{and} \quad \theta = \theta_0 + 2k\pi$$

For simplicity, we can choose $\theta = \theta_0$, and then we solve the left equality:

$$\begin{aligned}
\frac{r}{1 + r} &= r_0 \\
\implies r &= r_0(1 + r) \\
\implies r &= r_0 + r \cdot r_0 \\
\implies r - r \cdot r_0 &= r_0 \\
\implies r(1 - r_0) &= r_0 \\
\implies r &= \frac{r_0}{1 - r_0}
\end{aligned}$$

Thus, we have shown that every $z_0 = r_0 e^{i\theta_0} \in D(0, 1)$ has a pre-image given by $z_0^* = \dfrac{r_0}{1 - r_0} e^{i\theta_0}$ such that $f(z_0^*) = z_0$. To complete the picture note that for $f(z)$ defined in (1), the modulus, given by $\dfrac{r}{1 + r}$, is always between 0 and 1 since $0 < r < 1 + r$. Thus, for every $z \in \mathbb{C}$, $f(z) \in D(0, 1)$ and for every $z_0 \in D(0, 1)$, we have shown that there exists a z_0^* such that $f(z_0^*) = z_0$. Therefore, we have shown that f does indeed map every element of $\mathbb{C}$ onto $D(0, 1)$. □