Complex Analysis Homework 5

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Question 4

Evaluate

$$\int_\gamma z^m \overline{z}^n dz$$

where m and n are integers and γ is the unit circle $|z| = 1$ taken counter-clockwise

Answer.
I will start by parameterizing γ by $\gamma(t) = e^{it},\ t \in [0, 2\pi]$. Next, note that $\overline{e^{it}} = \overline{\cos(t) + i\sin(t)} = \cos(t) - i\sin(t) = \cos(-t) + i\sin(-t) = e^{-it}$. Thus, using the definition of the path integral:

$$\begin{aligned}
\int_\gamma z^m \overline{z}^n dz &= \int_0^{2\pi} (e^{it})^m (e^{-it})^n (ie^{it}) dt && (1)\\
&= i\int_0^{2\pi} e^{(m-n+1)it} dt && (2)\\
&= \frac{i}{i(m-n+1)} e^{(m-n+1)it}\Big|_{t=0}^{t=2\pi} && (3)\\
&= \frac{1}{m-n+1}\left(e^{2\pi i(m-n+1)} - 1\right) && (4)\\
&= 0 \text{ since } m-n+1 \in \mathbb{Z} && (5)
\end{aligned}$$

However, this above calculation is only valid whenever $m-n+1 \neq 0$ i.e. when $m \neq n-1$. When $m = n-1$, we have the integral in line (2) reducing to $i\int_0^{2\pi} dt = 2\pi i$. Thus, $\boxed{\int_\gamma z^m \overline{z}^n dz = 0 \text{ when } m \neq n-1}$ and $\boxed{\int_\gamma z^m \overline{z}^n dz = 2\pi i \text{ when } m = n-1}$