Complex Analysis Homework 6

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Question 3

Let f be analytic in a domain G. Let g be defined by $g(z) = e^{f(z)}$. Prove that if g is constant, then f is constant.

Proof.
First, if we let $h(z) = e^z$ (which we know to be analytic everywhere, in particular analytic in G), then it is clear that $g(z) = (h \circ f)(z) = h(f(z))$. Thus, since g is the composition of two functions who are analytic in G, then g must also be analytic in G.

Thus, for every point $z \in G$, $g'(z) = h'(f(z)) \cdot f'(z) = e^{f(z)} f'(z)$ by the chain rule. Furthermore, from the assumption that g is constant, we know that $g'(z) = 0$ for all z. Thus, we have that $e^{f(z)} f'(z) = 0$ for all $z \in G$. Additionally, since we know that $e^w \neq 0$ for all $w \in \mathbb{C}$, we can conclude that $f'(z) = 0$ for all $z \in G$. Finally, from the Theorem at the beginning of Section 25 (on page 73), we can conclude that since $f'(z) = 0$ for all $z \in G$, then $f(z)$ must be constant throughout G.

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