Complex Analysis Homework 7

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Question 1

Write down an expansion of the form $\sum_{n=0}^{\infty} a_n z^n$ for the following:

(a) $\dfrac{1}{(1+z)^3}$,

(b) ze^{z^2}.

In each case, specify where the expansion is valid.

Answer. (a)
Examine the following function:

$$f(z) := \frac{1}{1+z}$$

By taking derivatives, we can see that,

$$f'(z) = \frac{-1}{(1+z)^2}$$
$$f''(z) = \frac{2}{(1+z)^3}$$

Also, note that by using the fact that $\dfrac{1}{1-w} = \sum_{n=0}^{\infty} w^n$ for all $|w| < 1$. Thus,

$$f(z) = \frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{for all } |z| < 1$$

The function we are interested in is $\dfrac{1}{(1+z)^3} = \dfrac{1}{2} f''(z)$ for f defined above. Therefore,

$$\begin{aligned}
\frac{1}{(1+z)^3} &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{1+z}\right) = \frac{1}{2}\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}(-1)^n z^n\right) && \text{by the examination above} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\frac{d^2}{dz^2}\left((-1)^n z^n\right) \\
&= \frac{1}{2}\sum_{n=1}^{\infty}\frac{d}{dz}\left(n(-1)^n z^{n-1}\right) \\
&= \frac{1}{2}\sum_{n=2}^{\infty}(-1)^n n(n-1) z^{n-2} \\
&= \sum_{n=0}^{\infty}\frac{(-1)^{n+2}}{2}(n+2)(n+1) z^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}(n+1)(n+2) z^n
\end{aligned}$$

The first series was valid whenever $|z| < 1$ and since the derivative of a power series has the same radius of convergence, our final series $\boxed{\text{converges for } |z| < 1}$ and for $\boxed{a_n = \dfrac{(-1)^n}{2}(n+1)(n+2)}$.

Answer. (b)
Recall the series expansion for e^w:

$$e^w = \sum_{n=0}^{\infty} \frac{1}{n!} w^n \quad \text{for all } z \in \mathbb{C}$$

Thus, we can also find an expansion for $f(z) := e^{z^2}$:

$$f(z) = e^{z^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (z^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n}$$

Additionally, we can note that $f'(z) = 2ze^{z^2}$. Thus, the function we are interested in is $ze^{z^2} = \frac{1}{2} f'(z)$, so we can find it's expansion by doing the following:

$$\begin{aligned} ze^{z^2} = \frac{1}{2} \frac{d}{dz} \left(e^{z^2} \right) &= \frac{1}{2} \frac{d}{dz} \left(\sum_{n=0}^{\infty} \frac{1}{n!} z^{2n} \right) && \text{by the examination above} \\ &= \frac{1}{2} \sum_{n=0}^{\infty} \frac{d}{dz} \left(\frac{1}{n!} z^{2n} \right) \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{2n}{n!} z^{2n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{2n-1} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n+1} \end{aligned}$$

Since the first series was valid for all $z \in \mathbb{C}$ and the fact that the derivative of a power series has the same radius of convergence, we know that this final series $\boxed{\text{converges for all } z \in \mathbb{C}}$. Note this isn't quite the form we want for this question, so I will make the following definition:

$$a_n = \begin{cases} \frac{1}{k!} & \text{if } n \text{ is an odd number of the form } n = 2k+1 \text{ for } k \in \mathbb{N} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Thus, the above series is in the desired form of $\sum_{n=0}^{\infty} a_n z^n$ with the same convergence as discussed above.