Complex Analysis Homework 2

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Question 3

Question. *a.)*
Prove that $\lim_{z\to 0} f(z)$ *exists and is equal to 0 for each of the following choices of f(z):*

i.) $\frac{|z|^2}{z}$

ii.) $\frac{(Re(z))(Im(z))}{|z|}$

For the following proofs, we want to show that $\forall \varepsilon > 0$, we can find a $\delta > 0$ such that:

$$|f(z) - 0| < \varepsilon \quad \text{whenever} \quad 0 < |z - 0| < \delta \tag{1}$$

Proof. **i.)**
In this case, $f(z) = \frac{|z|^2}{z}$, so

$$\begin{aligned} |f(z) - 0| = |f(z)| &= \left| \frac{|z|^2}{z} \right| \\ &= \frac{||z|^2|}{|z|} \\ &= \frac{|z||z|}{|z|} \\ &= \frac{|z|}{1} = |z|. \end{aligned}$$

Therefore, $|f(z) - 0| < \varepsilon$ if and only if $|z| < \varepsilon$. So, it is clear that if we choose $\delta \leq \varepsilon$, then the condition (1) is satisfied and the given limit converges to 0. □

Proof. **ii.)**
In this case, $f(z) = \frac{(Re(z))(Im(z))}{|z|}$, so

$$\begin{aligned} |f(z) - 0| = |f(z)| &= \left| \frac{(Re(z))(Im(z))}{|z|} \right| \\ &= \frac{|Re(z)||Im(z)|}{|z|} \\ &\leq \frac{|z||z|}{|z|} \quad \text{since} \quad |Re(z)| \leq |z| \quad \text{and} \quad |Im(z)| \leq |z| \\ &= \frac{|z|}{1} = |z| \end{aligned}$$

Since $|f(z) - 0| \leq |z|$, then if $|z| < \varepsilon$, we know that $|f(z) - 0| < \varepsilon$. Therefore, it is clear that if we choose $\delta \leq \varepsilon$, then the condition (1) is satisfied and the given limit converges to 0. □

Question. *b.)*
Prove, by letting z *approach* 0 *along suitable rays, that*

$$\lim_{z\to 0} \frac{\bar{z}}{|z|}$$

does not exist.

Proof.
First, let z be a purely-Real, positive, number, i.e. let z lie on the line $Im(z) = 0$. This means $z = x$ for $x \in \mathbb{R}^+$. Applying this to the limit gives:

$$\begin{aligned}\lim_{z \to 0} \frac{\bar{z}}{|z|} &= \lim_{x \to 0^+} \frac{\bar{x}}{|x|} \\ &= \lim_{x \to 0^+} \frac{x}{x} \\ &= \lim_{x \to 0^+} 1 = 1.\end{aligned}$$

Thus, along this ray, the limit converges to 1.
Now, let z be a purely-Imaginary number with $Im(z) > 0$, i.e. z lies on the line $Re(z) = 0$. This means $z = iy$ for $y \in \mathbb{R}^+$. Applying this to the limit gives:

$$\begin{aligned}\lim_{z \to 0} \frac{\bar{z}}{|z|} &= \lim_{y \to 0^+} \frac{\overline{iy}}{|iy|} \\ &= \lim_{y \to 0^+} \frac{-iy}{\sqrt{0^2 + y^2}} \\ &= \lim_{y \to 0^+} \frac{-iy}{|y|} \\ &= \lim_{y \to 0^+} \frac{-iy}{y} \\ &= \lim_{y \to 0^+} -i = -i\end{aligned}$$

Thus, along this ray, the limit converges to $-i$.
Since $1 \neq -i$, this limit does not exist. $\square$