Introduction to the course

A computational introduction to
stochastic differential equations

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Content

Content

Teaching team

In this year's course we have

Zheng Zhao
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Mohamed Abdalmoaty
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Muhammad Emzir
KFUPM

Cagatay Yildiz
University of Tübingen

Nathanael Bosch
University of Tübingen

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Prerequisites

The course is PhD level (based on Uppsala University definition), but Master students are welcome as well.

Preferably you know

- Linear algebra.
- Real analysis.
- Probability theory.
- Ordinary differential equations.
- Python NumPy, even better, JAX.

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Arrangements

The base/essential courses are worth 6 credits.

You can upgrade to 9 credits by doing some additional works. We detail these later.

The course grade is pass or fail.

Please complain if you think the course works are too much and deserve more credits!

Arrangements

The course consists of lectures, exercises, and project work.

To get 6 credits, you need to:

- Actively participate lectures 1-10.
- Pass the three exercises. Each exercise has a passing rule.
- Pass the project work.

Depending on the number of students remain, you may do the project work in groups.

ARRANGEMENTS

To get 9 credits, you need to:

- Fulfil the requirements for the 6 credits.
- Actively participate the six seminar lectures.
- Select four from the six seminar lectures, then pass their exercises/writing assignments.

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Essential courses

Lecture 2

Stochastic differential/integral equations. 21 Oct, 2022. Room 101132, Ångström. Time is 13:15-15:00. Unless otherwise stated, we will omit the time.

This lecture explains what a stochastic differential equation (SDE) is, in particular, the meaning of $\mathrm{d}W(t)$ in

$$\mathrm{d}X(t) = a\big(X(t)\big)\,\mathrm{d}t + b\big(X(t)\big)\,\mathrm{d}W(t).$$

We also explain the celebrated Itô's formula and the Markov property which makes SDEs data-scalable models.

Essential courses

Lecture 3

Numerical solution to stochastic differential equation. 24 Oct, 2022. Room 101127, Ångström.

This lecture shows practical methods to simulate SDEs. For example, Euler–Maruyama and Milstein's method.

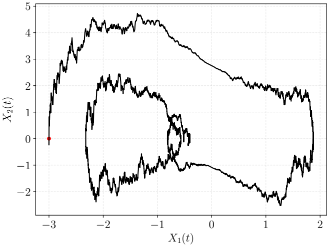


Figure: A simulation of a 2D Duffing–van der Pol SDE.

Lecture 4

Statistical properties of SDE solutions. 28 Oct, 2022. Room 101142, Ångström.

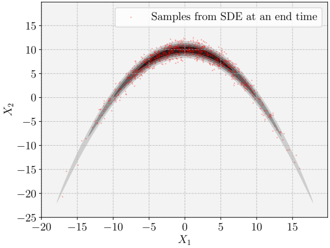
Solutions of SDEs are continuous-time Markov processes. In many occasions, we are interested in the statistics of SDEs.

Essential courses

Lecture 4

This lecture is concerned with:

- Compute $\mathbb{E}[\phi(X(t)) \mid X(s)]$ for any function ϕ of interests.
- Probability density function of the solution (i.e., Kolmogorov forward equation/Fokker–Planck, see the index figure in the github repository).
- Stationary solution to the Kolmogorov forward equation.
- Parameter estimation.



Essential courses

Lecture 5

Linear SDEs and Gaussian processes. 31 Oct, 2022. Room 101146, Ångström.

In this lecture we study a fundamental class of SDEs: linear SDEs. We show that they govern (Markov) Gaussian processes which are widely used in, for instance, machine learning, geostatistics, and fluid mechanics. We then show how to compute their mean and covariance functions.

Essential courses

Exercise 1

2 November, 2022. Room 101127, Ångström.

Deliver the solutions to the exercises related to Lectures 2 - 3. You have around 12 days to do the exercises.

Recall to complain if you need more time!

Essential courses

Lectures 6 - 7

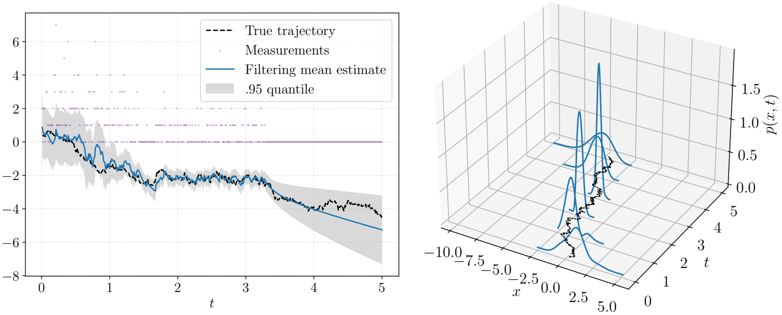
Filtering and smoothing I & II. 4 November, 2022, Room 101127, and 7 Nov, 2022, Room 101150, respectively.

Imagine that you have a latent process and a bunch of observation/data of the process. To estimate the latent process from the data is a filtering problem.

For example, Kalman filtering.

Essential courses

Lectures 6 - 7



Essential courses

Lecturs 6 - 7

The filtering has a great number of applications.

- The Apollo program.
- Object tracking, navigation, and control.
- Time series analysis.
- Regression, interpolation, and extrapolation.
- ...

Essential courses

Exercise 2

9 November, 2022. Room 101127, Ångström.

Deliver the solutions to the exercises related to Lectures 4 - 5. You have around 11 days to do the exercises.

Recall to complain if you need more time!

Essential courses

Lecturs 8

SDE system identification by Mohamed Abdalmoaty. 11 November, 2022. Room 101142, Ångström.

This lectures shows how to estimate the unknown parameters in the system (i.e., SDE) with discrete-time measurements.

Essential courses

Exercise 3
18 November, 2022. Room 101127, Ångström.

Deliver the solutions to the exercises related to Lectures 6 - 8. You have around 7-14 days to do the exercises.

Recall to complain if you need more time!

Essential courses

Project presentation

16 December, 2022. Room 101142, Ångström.

The project work is a paper (`\documentclass[10pt,a4paper]{article}`, at least 4 pages) that targets at a research/application/theory topic. The aim of the project work is to show that you can do computations with SDEs and you can also apply SDEs to model/solve your research problems.

Essential courses

In principle you should come up with your own project topic.

Project topic examples

- (Computational biology) Use stochastic SIR models to describe infectious diseases (e.g., COVID-19)
- (Quantitative finance) Analyse interest rates with financial SDE models (e.g., Cox–Ingersoll–Ross and Chan–Karolyi–Longstaff–Sanders)
- (Scientific computing) Solve PDEs with Feynman–Kac formula
- (Machine learning) Solve Gaussian process regression with SDEs
- (Machine learning) Diffusion-based generative models
- (Statistics) Exact SDE sampling (see, e.g., Beskos, 2005)
- (Statistics) Metropolis adjusted Langevin algorithm

See more in the course homepage.

Essential courses

If we count from Lecture 4, then approximately you have 1 month + 19 days to prepare the project work.

Depending on the number of students remain, you may do the project work in groups.

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Seminar courses

Lecture 1

Continuous-time filtering by Muhammad Emzir. 17 November, 2022, Zoom.

This course is an extension of the filtering and smoothing lectures, and we deal with continuous-time measurements. We talk about the celebrated Zakai and Kushner equations, as well as projection filtering.

Seminar courses

Lecture 2

SDEs and Markov Chain and Monte Carlo by Cagatay Yildiz. 22 November, 2022, 09:15 - 11:00. Room 4101 Ångström.

Essentially how to construct MCMC kernels with SDEs. We will put more details prior to the seminar.

Seminar courses

Lecture 3

Probabilistic numerics for ordinary differential equations by Nathanael Bosch. 22 November, 2022, 13:15 - 15:00. Room 2001 Ångström.

This lecture shows how to solve ordinary differential equations numerically with uncertainty quantification. We will put more details prior to the seminar.

Seminar courses

Lecture 4

TBD. 28 November. Room 101127 Ångström.

The topic of this lecture is flexible. Possible options:

- SDEs in statistical signal processing
- Faster computation with JAX which is a Python library that supports JIT compilation and auto-diff
- Normalising flows.

We will issue a vote.

Seminar courses

Lecture 5

TBD by Roland Hostettler. 2 December. Room 101127 Ångström.

Topic TBD.

Seminar courses

Lecture 6

Constructions of Brownian motion and Itô integral. 5 December. Room 101127 Ångström.

The essential lectures are concerned with computations of Brownian motion and stochastic integrals. This seminar lecture shows what exactly the Brownian motion and stochastic integrals are, and how they are defined and constructed.

Content

Reading materials

Basic

- Hui-Hsiung Kuo. Introduction to stochastic integration. Springer, 2006.
- Simo Särkkä and Arno Solin. Applied stochastic differential equations. Cambridge University Press, 2019.
- Gabriel Lord, Catherine Powell, and Tony Shardlow. An introduction to computational stochastic PDEs. Cambridge University Press, 2014.

More theoretical

- Ioannis Karatzas and Steve E. Shreve. Brownian motion and stochastic calculus. Springer, 2nd edition, 1991.
- Jean-François Le Gall. Brownian motion, martingales, and stochastic calculus. Springer, 2013.
- René Schilling and Lothar Partzsch. Brownian motion. De Gruyter, 2012.

Tack!