Exercise 2

A computational introduction to stochastic differential equations
FTN0332 TN22H006

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How to pass this exercise

This exercise round is concerned with Lectures 4 - 5. To pass this exercise, score $\geq$ **13** points and finish the assignment(s) marked with $\star$. Please submit your assignments in an email sent to `zheng.zhao@it.uu.se` before 13:15, 9 Nov, 2022.

Note

Recall the infinitesimal generator in the lecture note

$$\mathcal{A}\phi(x) \coloneqq (\nabla_x \phi)^\mathsf{T} \, a(x) + \frac{1}{2} \operatorname{tr}\big(\Gamma(x) \, \mathrm{H}_x \phi(x)\big)$$

which we will frequently use in the following assignments.

Assignment 1 (3 points)

Recall the process

$$X(t) = X_0 \exp\bigg(\Big(a - \frac{b^2}{2}\Big) \, t - b \, W(t)\bigg)$$

which solves the SDE

$$\mathrm{d}X(t) = a \, X(t) \, \mathrm{d}t + b \, X(t) \, \mathrm{d}W(t), \quad X(0) = x_0.$$

Let $x_0 = 2$, $a = -3$, and $b = 1$.

- Write down explicitly $\mathbb{E}[\sin(\log(X(t))) \mid X(0) = x_0]$. (Hint: the expectation $\mathbb{E}[\sin(Z)]$ for any Normal $Z \sim \mathrm{N}(\mu, \sigma^2)$ is $\sin(\mu) \exp^{-\sigma^2 \,/\, 2})$.
- Approximate $\mathbb{E}[\sin(\log(X(t))) \mid X(0) = x_0]$ at $t = 1$ by using, for example, Euler–Maruyama or Milstein's method simulation. Compare the approximate result to the theoretical one derived in the above.

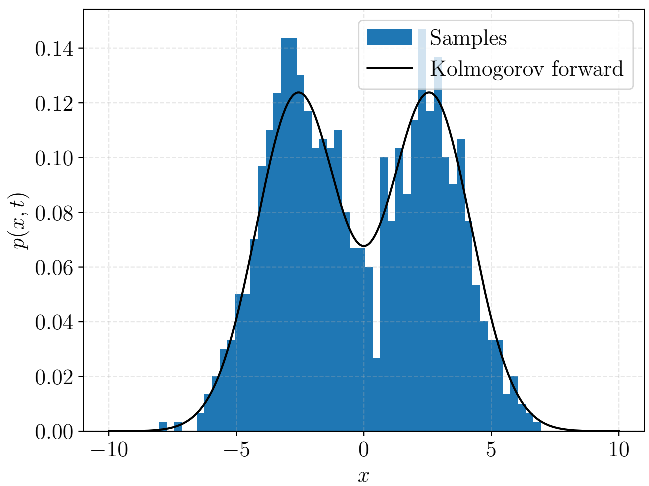

Figure 1: Expected result for Assignment 3.

Assignment 2 (3 points)

Consider an SDE

$$\mathrm{d}X(t) = \tanh(X(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad X(0) = 0.$$

and a function $\phi(x) \coloneqq x^2$.

- Write down $\mathcal{A}^r \phi$ for $r = 0, 1, 2, 3$.
- Approximate $\mathbb{E}[\phi(X(t)) \mid X(0) = x]$ at $t = 5$ by using Euler–Maruyama and compare to the true result by Taylor moment expansion.

Assignment 3 (4 points)

Consider again the hyperbolic tangent SDE defined in Assignment 2, but we set it's initial condition be a Normal $X(0) \sim \mathrm{N}(0, 1)$.

- Simulate 1,000 samples of $X(t)$ at the terminal time (using any scheme) $t = 2$ and plot the histogram of the samples.
- Numerically solve the Kolmogorov forward equation associated with this SDE by the finite difference method. Check if the solution matches the histogram of the samples. You should get a result quite close to Figure 1. Note: when implementing the finite difference method `np.gradient` is your friend. Also note that you might need to try out different spatial and temporal discretisations to get a stable result.

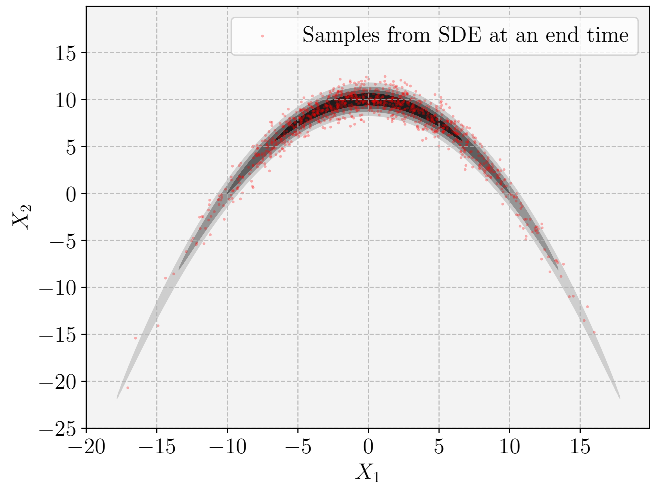

Figure 2: Expected result for Assignment 4.

Assignment 4 (4 points)

Consider a two-dimensional probability density function

$$p(x) = \exp\Big(-\frac{1}{200}\,x_1^2 - \frac{1}{2}(x_2 + b\,x_2^2 - 100\,b)^2\Big), \quad x = \begin{bmatrix}x_1 & x_2\end{bmatrix}^\mathsf{T}.$$

- Plot the the PDF p at Cartesian $x \in [-20, 20] \times [-25, 25]$, using, e.g., `contourf`, `pcolor`, or `imagesc`. The image should look like a crescent.
- Let us use the PDF p to define an SDE

$$\mathrm{d}X(t) = \frac{1}{2}\,\nabla_X \log p(X(t))\,\mathrm{d}t + \mathrm{d}W(t), \quad X(0) = \begin{bmatrix}0 & 0\end{bmatrix}^\mathsf{T}. \tag{1}$$

  Now let t_T be a large enough terminal time, say for example, $t_T = 100$. Use Euler–Maruyama or whatsoever your favourite scheme to simulate $1,000$ samples $X(t_T)$ from the SDE at the terminal time. 2D-scatter-plot the samples and check if they match the PDF p.

  You should get a similar result as in Figure 2.

Assignment 5 (4 points)

Consider an SDE

$$\mathrm{d}X(t) = 4\,(X(t) - \theta\,X(t)^3)\,\mathrm{d}t + \mathrm{d}W(t), \quad X(0) = 1,$$

where θ is a parameter. Set $\theta = 2$ then simulate a trajectory from this SDE at times $t_1 = 10^{-3}, t_2 = 2 \times 10^{-3}, \ldots, t_{10000} = 10$, denoted by $X_{1:10000}$. Now suppose that θ is unknown. Implement the log likelihood function of X at the times, then perform the maximum likelihood estimation to estimate θ with the measurements $X_{1:10000}$. Compared the estimated θ to the true value 2.

Assignment 6 (2 points)

Consider a linear SDE

$$\mathrm{d}X(t) = -\alpha\, X(t)\, \mathrm{d}t + \beta\, \mathrm{d}W(t), \quad X(0) = X_0 \sim \mathrm{N}(m_0, V_0), \quad \alpha > 0.$$

Show that

- $\mathbb{E}[X(t) \mid X(0)] = \exp(-\alpha\, t)\, X_0$,
- and that
$$\mathbb{E}[X(t) \mid X(0)] = \sum_{r=0}^{\infty} \frac{1}{r!} \mathcal{A}^r \phi(X_0)\, t^r,$$
where $\phi(x) \coloneqq x$.

Assignment 7 (2 points)

Let us again use the linear SDE in Assignment 6, and set $\alpha = \beta = 1$.

- Let $m_0 = 4$ and $V_0 = 0$. Sample exactly a trajectory from the SDE on the time interval $[0, 8]$. Do not use any *approximate* sampling scheme (e.g., Euler–Maruyama).
- Let $m_0 = 0$ and $V_0 = \beta^2 \,/\, (2\,\alpha)$ be the stationary mean and variance. Sample a sufficient amount of trajectories and verify that the sample mean and variance (at each t) are numerically close to the stationary mean M_0 and variance V_0.
- (**Bonus +1 point**) Use the sampled trajectories from the previous bullet point question to approximate the cross-covariance $\mathbb{E}[X(t)\, X(s)]$ for some t, s chosen up to you. Verify that it is close to the theoretical result $\beta^2 \,/\, (2\,\alpha) \exp(-\alpha\, |t - s|)$.

Assignment 8 (2 points)

Consider a linear SDE

$$\mathrm{d} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = A \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} \mathrm{d}t + B\, \mathrm{d}W(t), \quad X(0) = X_0 \sim \mathrm{N}(0, V_0),$$

where

$$A \coloneqq \begin{bmatrix} 0 & 1 \\ -3 \,/\, \ell^2 & -2\sqrt{3} \,/\, \ell \end{bmatrix}, \quad B \coloneqq \begin{bmatrix} 0 \\ 2\, (\frac{\sqrt{3}}{\ell})^{3\,/\,2}\, \sigma \end{bmatrix}, \quad \ell = 0.5, \quad \sigma = 1.$$

- Verify that

$$V = \begin{bmatrix} \sigma^2 & 0 \\ 0 & 3 \,/\, \ell^2 \, \sigma^2 \end{bmatrix}$$

solves the Lyapunov equation

$$A\,V + V\,A^{\mathsf{T}} + B\,B^{\mathsf{T}} = 0.$$

- Use `scipy.linalg.solve_continuous_lyapunov` to numerically solve the Lyapunov equation, and verify that it is equal to the theoretical one shown in the previous bullet point.

- (**Bonus +1 point**) Vectorise V then reformulate the Lyapunov equation as a linear system in the vectorised V. Solve the Lyapunov equation by solving the linear system (see, Wikipedia of Lyapunov equation)

Remark: when the SDE starts from the stationary initial condition, X_1 is a Matérn ($\nu = 3/2$) Gaussian process.

Assignment 9 (3 points)

Let us again use the SDE in Assignment 8.

- Use the function `discretise_lti_sde` to compute the conditional mean $\mathbb{E}[X(t_k) \mid X(t_{k-1})]$ and covariance $\mathrm{Cov}[X(t_k) \mid X(t_{k-1})]$ for any times.

- Set the initial covariance V_0 be the stationary one from the Lyapunov equation. Use the conditional mean and covariance functions to sample multiple trajectories from the SDE at time interval $[0, 2]$.

Remark: you may find the function `discretise_lti_sde` in `lec5_linear_sde_mean_cov.ipynb`.