ESE 541 Week 13 & 14 Practice Problems - Solutions

Problem 1. Find the eigenvalues and eigenvectors for this 2x2 matrix:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$$

We start by finding the eigenvalues. First calculate $|A - \lambda I| = 0$, where A is the matrix we are interested in computing eigenvalues/eigenvectors for, and I is the 2 by 2 identity matrix:

$$\begin{vmatrix} \text{-6-}\lambda & 3 \\ 4 & \text{5-}\lambda \end{vmatrix} = 0$$

Given a 2 by 2 matrix (like the one below), we can find its determinant through $ad - bc$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Calculating that determinant on our above matrix gets:

$$(-6 - \lambda)(5 - \lambda) - 3 \times 4 = 0$$

Lastly, solving for λ yields:

$$\lambda = -7 \quad or \quad 6$$

There are two possible eigenvalues. Now that we know the eigenvalues, we can find their corresponding eigenvectors:

$$Av = \lambda v$$

Put in the values we know:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

After multiplying we get these two equations:

$$-6x + 3y = 6x$$

$$4x + 5y = 6y$$

Either equation reveals that y = 4x. We can pick any x, y that satisfy the equation, so the eigenvector is any non-zero multiple of this:

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

For eigenvalue -7:

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -7 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$-6x + 3y = -7x$$

$$4x + 5y = -7y$$

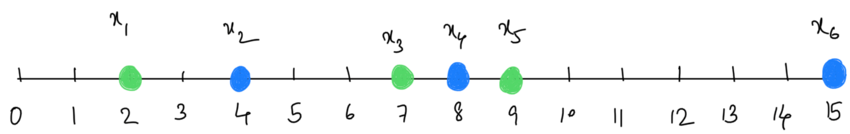
so the eigenvector is any non-zero multiple of this:

$$\begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

For another example, we suggest you check out this video, where are steps we go through above are explained in more detail.

Problem 2. In this problem, you will run a couple iterations of the K-means algorithm in a simple one-dimensional dataset, refer to the figure below.

Consider the one-dimensional dataset $D = \{x_1 = 2, x_2 = 4, x_3 = 7, x_4 = 8, x_5 = 9, x_6 = 15\}$. We would like to group these points into 2 clusters represented by the colors Green and Blue. To cluster this points, we will use the K-means algorithm with the following initial colors: x_1, x_3 and x_5 are initially Green; x_2, x_4 and x_6 are initially Blue.



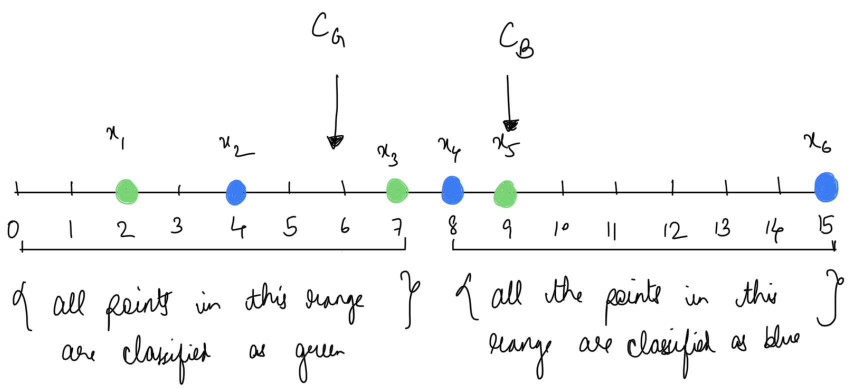
Answer the following questions:

A. What is the value of the Green and Blue centroid in the initially colored configuration?

B. Run a first iteration of the K-means algorithm and recompute the location of the centroids. What is the value of the new Green and Blue centroids after the first iteration?

C. Run a second iteration of the K-means algorithm and recompute the location of the centroids. Have the centroids changed?

Compute the centroids by taking the mean of the green and blue points.



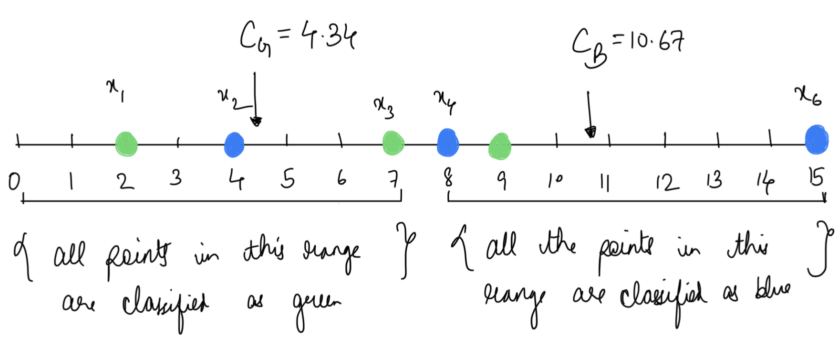
For green centroid,

$$C_G = \frac{x_1 + x_3 + x_5}{3}$$
$$C_G = \frac{2 + 7 + 9}{3}$$
$$C_G = 6$$

Similarly for blue centroid,

$$C_B = \frac{x_2 + x_4 + x_6}{3}$$
$$C_B = \frac{4 + 8 + 15}{3}$$
$$C_B = 9$$

For the first iteration, since the centroids have been calculated, the points will be reclassified as x_1, x_2 and x_3 are now Green; x_4, x_5 and x_6 are now Blue.



Hence for green centroid,

$$C_G = \frac{x_1 + x_2 + x_3}{3}$$

$$C_G = \frac{2 + 4 + 7}{3}$$

$$C_G = \frac{13}{3}$$

$$C_G = 4.34$$

Similarly for blue centroid,

$$C_B = \frac{x_4 + x_5 + x_6}{3}$$

$$C_B = \frac{8 + 9 + 15}{3}$$

$$C_B = \frac{32}{3}$$

$$C_B = 10.67$$

For the second iteration, since the centroids have been recalculated, the points will be reclassified. However, the points remain as they were in the last part: as x_1, x_2 and x_3 Green; x_4, x_5 and x_6 are Blue. Hence, the centroids will not change after the second iteration.