ESE 541 Week 1 Practice Problems - Solutions

Problem 1. For the learning problems described below, answer the following questions: (i) is the problem a regression or a classification? (ii) is the problem supervised or unsupervised?, (iii) if available, what is the number of features p and the number of data points N?

1. Consider the problem of predicting the electric energy consumption in a city at 9 PM tomorrow using the values of the energy consumption at 9 PM on the previous 21 days. To train your predictor, you have access to historical data of the past 10 years (ignoring leap days). Hint: Your answer about N should take into account that the initial 21 days of this 10-year period cannot be used as outputs in your training set.

(i) regression
(ii) supervised
(iii) $p = 21, N = (365 \times 10) - 21 = 3629$

2. Consider the problem of detecting a failure in a power grid; in particular, you want to find out whether a particular transmission line has failed or not at a given time. To make a decision, you use voltage measurements in 10 different nodes of the power grid and use voltage fluctuations to decided if the transmission line is operational or if it failed. These voltages are measured using a device able to sample these voltages at a temporal resolution of 600Hz. Your algorithm needs to make a decision about failures using the voltage measurements corresponding to the last 5 seconds. You have access to historical data containing 100 examples of time series with and without line failures.

(i) classification
(ii) supervised
(iii) $p = 10 \times 600 \times 5 = 30000, N = 100$.

Problem 2. For each one of the random variables below, plot their pdf (or pmf), and compute their expectations, as well as their variances:

1. The outcome of the flip of a fair coin, where 'Head'=1 and 'Tail'=0.

Applying the definition of expectation, we have

$$\mathbb{E}(X) = \frac{1}{2} \times 0 + \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \frac{1}{2} \times 0^2 + \frac{1}{2} \times 1^2 - [\mathbb{E}(X)]^2 = \frac{1}{4}$$

2. The sum of three fair coins flipped simultaneously.

Let X_1, X_2, and X_2 be the random variables denoting the outcomes of each coin. Since all coins are fair, we have that

$$\mathbb{E}(X_1 + X_2 + X_3) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) = 3 \times \frac{1}{2} = 3/2$$

Also, the variance of the sum of independent r.v.s is the sum of their variances,

$$\mathrm{Var}(X_1 + X_2 + X_3) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \mathrm{Var}(X_3) = 3 \times \frac{1}{4} = \frac{3}{4}$$

3. The outcome of flipping a fair die with 4 faces.

We call the random outcome by D. The possible outcomes are the integers between 1 and 4. These outcomes have the same probability of 1/4. Hence,

$$\mathbb{E}(D) = \sum_{i=1}^{4} d_i p_i = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + \ldots + 4 \times \frac{1}{4} = \frac{10}{4} = \frac{5}{2}$$

$$\text{Var}(D) = \mathbb{E}(D^2) - [\mathbb{E}(D)]^2 = 1^2 \times \frac{1}{4} + 2^2 \times \frac{1}{4} + \ldots + 4^2 \times \frac{1}{4} - \left(\frac{5}{2}\right)^2 = \frac{5}{4}$$

4. The sum of a fair die with four faces and a coin ('Head'=1 and 'Tail'=0) thrown simultaneously (and independently).

Let D be the random variable denoting the outcome of the die roll. Let X be the random variable denoting the outcome of the coin flip. Since these two random variables are independent, we can use the linearity of expectation to calculate the expected value of the sum and the variance sum law to find the variance of the sum.

$$\mathbb{E}(D + X) = \mathbb{E}(D) + \mathbb{E}(X) = \frac{5}{2} + \frac{1}{2} = 3$$

$$\text{Var}(D + X) = \text{Var}(D) + \text{Var}(X) = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}$$

5. A uniform (real) number in the range $[-1, 1]$.

Let us denote by U the random variable uniformly distributed from -1 to 1; hence, $f_U(u) = 1/2$ for $u \in [-1, 1]$, 0 otherwise. Therefore,

$$\mathbb{E}(U) = \int_{-1}^{1} u f_U(u) du = 0$$

$$\begin{aligned}
\mathrm{Var}(U) &= \mathbb{E}(U^2) - [\mathbb{E}(U)]^2 \\
&= \int_{-\infty}^{\infty} u^2 f_U(u) du - [\mathbb{E}(U)]^2 \\
&= \int_{-1}^{1} u^2 \left(\frac{1}{2}\right) du - 0^2 \\
&= \frac{1}{3}
\end{aligned}$$

6. The sum of a fair coin and a real number chosen uniformly at random in the range $[-1, 1]$.

Let X be the random variable denoting the outcome of the fair coin. Let U be the uniform random variable. Since these two random variables are independent, we can use the linearity of expectation to calculate the expected value of the sum and the variance sum law to find the variance of the sum.

$$\mathbb{E}(X + U) = \mathbb{E}(X) + \mathbb{E}(U) = \frac{1}{2} + 0 = \frac{1}{2}.$$

$$Var(X + U) = Var(X) + Var(U) = \frac{1}{4} + \frac{1}{3} = \frac{7}{12}$$

Problem 3. Throw a fair dice with 6 faces and denote its outcome by the random variable $D \in \{1, 2, \ldots, 6\}$. Then, draw a random value U uniformly at random from the interval $[0, d]$, where d is the value of the dice you previously threw. Answer the following questions:

1. Compute the marginal probabilities $\Pr(D = d)$ for $d \in \{1, 2, \ldots, 6\}$ and $\Pr(U \in [0, 1])$. Hint: you might want to use the total probability theorem; namely, $\Pr(U \in [0, 1]) = \sum_{d=1}^{6} \Pr(U \in [0, 1] | D = d) \Pr(D = d)$.

We have that $\Pr(D = d) = 1/6$ for all d. Also, we have that $\Pr(U \in [0, 1] | D = d) = \frac{1}{d}$. Hence,

$$\begin{aligned}
\Pr(U \in [0, 1]) &= \sum_{d=1}^{6} \Pr(U \in [0, 1] | D = d) \Pr(D = d) \\
&= \sum_{d=1}^{6} \frac{1}{d} \frac{1}{6} = \frac{49}{120}.
\end{aligned}$$

2. Compute the marginal expectation $\mathbb{E}(U)$. Hint: you might want to apply the following identity, $\mathbb{E}(U) = \sum_{d=1}^{6} \mathbb{E}(U | D = d) \, p_D(d)$.

Since $\mathbb{E}(U | D = d) = d/2$, we have that

$$\mathbb{E}(U) = \sum_{d=1}^{6} \mathbb{E}(U | D = d) \frac{1}{6} = \frac{1}{6} \sum_{d=1}^{6} \frac{d}{2} = \frac{21}{12}$$