6 - 1 Introduction to Eeletromagnetism

① Theoretical foundation

Maxwell's equations (公理)

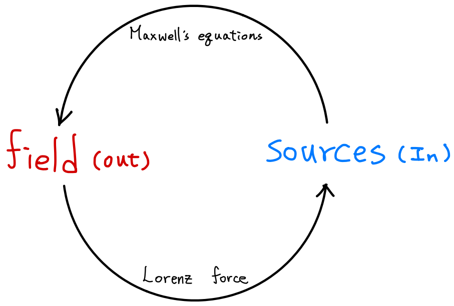
$\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$ — ⓐ

$\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0$ — ⓑ

$\vec{\nabla}\times\vec{E}(\vec{r},t) + \frac{\partial}{\partial t}\vec{B}(\vec{r},t) = 0$ — ⓒ

$\vec{\nabla}\times\vec{B}(\vec{r},t) - \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \mu_0\vec{J}(\vec{r},t)$ — ⓓ

[Physical mechanism][Field] = [Sources]



Lorenz force (公理)

$\vec{F}(\vec{r},t) = q\left[\vec{E}(\vec{r},t) + \vec{V}(t)\times\vec{B}(\vec{r},t)\right]$ where $\vec{V}(t) = \frac{d}{dt}\vec{r}(t)$, $\vec{r}(t)$ = the position of q at time "t"

② Inhomogeneous linearity

If $(\vec{E}_1,\vec{B}_1) \leftarrow (\rho_1,\vec{J}_1)$ and $(\vec{E}_2,\vec{B}_2) \leftarrow (\rho_2,\vec{J}_2)$

Implies $(c_1\vec{E}_1 + c_2\vec{E}_2,\ c_1\vec{B}_1 + c_2\vec{B}_2) \leftarrow (c_1\rho_1\ c_2\rho_2,\ c_1\vec{J}_1 + c_2\vec{J}_2)$ where $c_1, c_2 \in \mathbb{R}$

③ Uniqueness of definite solution

The definite solution of the Maxwell's equations is unique (Helmholz theory)

EX: $\frac{d}{dx}y(x) = x$, $\underline{y(x=0) = 1}$ Additional condition

The general solution is $y(x) = \frac{1}{2}x^2 + C$. $C = 1$ is the definite solution

④ Electromagnetic wave equation

ⓐ, ⓑ of the Maxwell's equation are initial conditions of ⓒ, ⓓ from the mathematical viewpoint

$\vec{\nabla}\times ⓒ \Rightarrow \vec{\nabla}\times\left[\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B}\right] = 0$

$= \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} + \vec{\nabla}\times\frac{\partial}{\partial t}\vec{B}$

$= \frac{1}{\varepsilon_0}\vec{\nabla}\rho - \nabla^2\vec{E} + \frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E} + \mu_0\vec{J}\right) = 0 \Rightarrow \left(\nabla^2 - \mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\right)\vec{E} = \mu_0\frac{\partial}{\partial t}\vec{J} + \frac{1}{\varepsilon_0}\vec{\nabla}\rho$

The same reason $(\nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2})\vec{B} = -\mu_0 \vec{\nabla}\times\vec{J}$

If $\rho=0$, $\vec{J}=0$ $\Rightarrow (\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix} = 0$ ※ $\because \varepsilon_0\mu_0 = \frac{1}{c^2}$ ∴ 電磁波在真空中波速是光速

＊ 源產生場到無源區的波動方程

⑤ $\vec{\nabla}\times\vec{E} + \frac{\partial}{\partial t}\vec{B} = 0$ 附帶資訊

$\Rightarrow$ 若靜磁則 $\vec{\nabla}\times\vec{E} = 0$ 靜電場

若靜電則 $\frac{\partial}{\partial t}\vec{B} = 0$ 靜磁場

∴ 靜電 $\longleftrightarrow$ 靜磁

＊ 這種情況下

$$\begin{cases}\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \\ \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = 0 \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J}\end{cases}$$

電磁無關？ x

⑥ 局域電荷守恆（規範轉換不變性）＊見相對論

$\frac{\partial}{\partial t}$ ⓐ $\Rightarrow \frac{\partial}{\partial t}\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\rho$

L.H.S. $\vec{\nabla}\cdot\frac{1}{\varepsilon_0\mu_0}(\vec{\nabla}\times\vec{B} - \mu_0\vec{J}) = -\frac{1}{\varepsilon_0}\vec{\nabla}\cdot\vec{J}$ $\therefore \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\rho = -\frac{1}{\varepsilon_0}\vec{\nabla}\cdot\vec{J} \Rightarrow \frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{J} = 0$ （守恆方程）

[證明] 將上式選取一個区域做体積分

$$\int_\Omega d^3r\,\frac{\partial}{\partial t}\rho = -\int_\Omega d^3r\,\vec{\nabla}\cdot\vec{J} \Rightarrow \frac{d}{dt}\int_\Omega d^3r\,\frac{\partial}{\partial t}\rho = \frac{d}{dt}Q_\Omega = -\int_{\partial\Omega} d\vec{A}\cdot\vec{J}$$

If $\frac{d}{dt}Q_\Omega > 0 \Rightarrow \int_{\partial\Omega} d\vec{A}\cdot\vec{J} < 0$ $\vec{J}$ 方向流入区域 #

若電荷密度不隨時間改變 $\Rightarrow \frac{\partial}{\partial t}\rho = 0 \longleftrightarrow \vec{\nabla}\cdot\vec{J} = 0$ （穩恆電流）

⑦ Maxwell's equations 中電和磁不完全對稱

$\vec{\nabla}\cdot\vec{E}(\vec{r},t) = \frac{1}{\varepsilon_0}\rho(\vec{r},t)$ $\vec{\nabla}\times\vec{B}(\vec{r},t) - \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}(\vec{r},t) = \mu_0\vec{J}(\vec{r},t)$

$\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0$ （沒磁荷） $\vec{\nabla}\times\vec{E}(\vec{r},t) + \frac{\partial}{\partial t}\vec{B}(\vec{r},t) = 0$ （沒磁流）

⑧ Maxwell's equations 中 ⓐ, ⓑ 是初始條件

[證明] 考慮 $\frac{\partial}{\partial t}$ ⓐ $\Rightarrow \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E} - \frac{1}{\varepsilon_0}\rho) = \vec{\nabla}\cdot\frac{\partial}{\partial t}\vec{E} - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\rho$

$$\Rightarrow \vec{\nabla}\cdot\frac{1}{\mu_0\varepsilon_0}(\vec{\nabla}\times\vec{B} - \mu_0\vec{J}) - \frac{1}{\varepsilon_0}\frac{\partial}{\partial t}\rho$$

$$\Rightarrow -\vec{\nabla}\cdot\vec{J} - \frac{\partial}{\partial t}\rho = 0$$

$\therefore \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E} - \frac{1}{\varepsilon_0}\rho) = 0$ $\therefore \vec{\nabla}\cdot\vec{E} - \frac{1}{\varepsilon_0}\rho = f(\vec{r})$ 但 ⓐ 式 $\vec{\nabla}\cdot\vec{E} - \frac{1}{\varepsilon_0}\rho = f(\vec{r},t) = 0$

$\therefore f(\vec{r}) = 0$ ⓑ 式同理 #

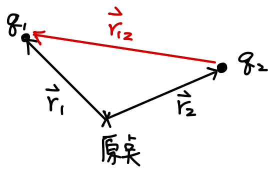
⑨ 古典電磁學僅非量子且合乎相對論

		非量子	量子
質點	非相對論	古典力學	量子力學（薛丁格方程）
	相對論	相對論力學	相對論量子力學（狄拉克方程）
場	非相對論	X	X
	相對論	古典電磁學 廣義相對論	量子電動力學 QED 量子色動力學 QCD 電弱理論

7-1 靜電學的理論基礎

① 庫侖靜電定律

$$\begin{cases} \vec{F}_{12} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \quad \text{(2作用在1的力；1相對2)} \\ \text{可線性疊加原則} \end{cases}$$



＊ 1. 只適用點電荷

2. 只適用於靜電情況（q_1, q_2 相對靜止）

3. 靜電力為聯心力 ⇒ 保守力（可定義 U 且 $\vec{F} = -\vec{\nabla}U$）

4. 靜電力的大小和距離平方成反比

（3.、4.）必然滿足球殼定理

5. 平等性 ⇒ 互易性

§ 電場 $\vec{E}$

[定義] $\vec{E}(\vec{r},t) = \lim\limits_{q\to 0}\dfrac{1}{q}\vec{F}_e(\vec{r},t)$ （數學上的定義）

（q：測試電荷）

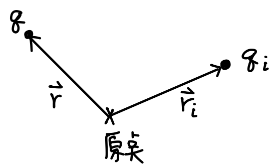
1. 電力線輔助描述 $\vec{E}(\vec{r},t)$ 在任一特定 t 的空間分布

2. 靜電情況下的電力線（即靜電場）只有直的聚散成分 ⇒ $\vec{\nabla}\times\vec{E} = 0$

§ 多質點系統下的靜電力

$$\vec{F}_q(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\sum_i^N \frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i)$$

$$\Rightarrow \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i^N \frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i) \quad \text{— (i)}$$



② 非質點系統的靜電場

從 (i) 式變成積分 ⇒ $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \dfrac{\rho(\vec{r}')\,d^3\vec{r}'}{|\vec{r}-\vec{r}'|}(\vec{r}-\vec{r}')$

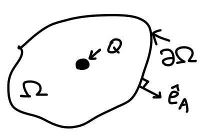
（$\vec{E}(\vec{r})$：場點；$\rho(\vec{r}')$：體電荷密度；$\vec{r}'$：源點）

＊所有源點相加產生的場點

③ 高斯定律的積分形式

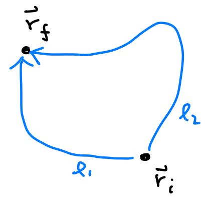
$$\begin{cases} \displaystyle\int_S d\vec{A}\cdot\vec{E}(\vec{r}) = \frac{1}{\varepsilon_0}Q_\Omega \quad \text{其中 } S = \partial\Omega \quad \text{— (i)} \\ \displaystyle\oint_L d\vec{r}\cdot\vec{E}(\vec{r}) = 0 \quad \text{其中 L 為任一封閉曲線} \quad \text{— (ii)} \end{cases}$$

（$d\vec{A}$：外法線方向；S：任意封閉曲面）



[推導]

(ii) $\because \vec{F}_e(\vec{r}) =$ 保守力 $\therefore \int_{\vec{r}_i}^{\vec{r}_f} d\vec{r}\cdot\vec{F}(\vec{r}) = W$ 和路徑無關

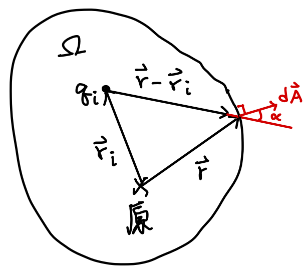


$\therefore$ 走路徑 $L = \ell_1 + \ell_2^-$ 有 $\int_{\ell_1, \vec{r}_i}^{\vec{r}_f} d\vec{r}\cdot\vec{F}_e = \int_{\ell_2, \vec{r}_i}^{\vec{r}_f} d\vec{r}\cdot\vec{F}_e \Rightarrow \oint_L d\vec{r}\cdot\vec{F}_e = 0$

$\Rightarrow \oint_L d\vec{r}\cdot\vec{E} = 0$

(i) $\because \vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i^N \frac{q_i}{|\vec{r}-\vec{r}_i|^3}(\vec{r}-\vec{r}_i)$ $\therefore \int_S d\vec{A}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\underbrace{\sum_{i=1}^N}_{\text{求和收斂(有限)}} q_i \int_S d\vec{A}\cdot\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3}$ 然後考慮 Ω 區域

1. 若 q_i 在 Ω 中：



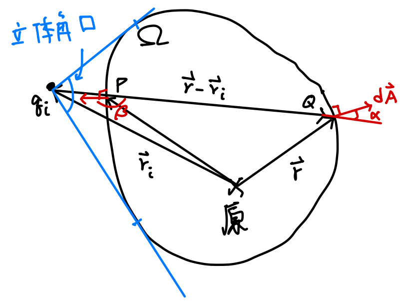
$$\frac{1}{4\pi\varepsilon_0}\sum_i q_i \int \underbrace{dA\cdot\cos\alpha}_{\text{球面上}} \frac{R}{R^3} = \frac{1}{4\pi\varepsilon_0}\sum_i q_i \iint R^2\sin\theta\, d\theta\, d\phi\, \frac{R}{R^3}$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_i q_i \iint \cancel{R^2}\, d(\cos\theta)\, \frac{\cancel{R}}{\cancel{R^3}}\, d\phi$$

(負號吸收)

$$= \frac{1}{4\pi\varepsilon_0}\sum_i q_i \underbrace{\int_0^{2\pi} d\phi \int_\pi^0 d\cos\theta}_{4\pi} = \frac{1}{\varepsilon_0} Q_\Omega$$

2. 若在 Ω 外：



$\begin{cases} P \text{ 點的 } d\vec{A} \text{ 和 } \vec{r}-\vec{r}_i \text{ 夾鈍角為負值} \\ Q \text{ 點的 } d\vec{A} \text{ 和 } \vec{r}-\vec{r}_i \text{ 夾銳角為正值} \end{cases}$

$\Rightarrow$ P 和 Q 兩兩對消 $(-\square q_i + \square q_i)$

$\therefore q_i \int_S d\vec{A}\cdot\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} = 0$

總的來說 $\Rightarrow \int_S d\vec{A}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_i q_i \int d\vec{A}\cdot\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|^3} = \frac{1}{4\pi\varepsilon_0}\sum_i \begin{cases} 4\pi q_i & \text{if } q_i \in \Omega \\ 0 & \text{if } q_i \notin \Omega \end{cases}$

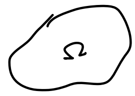
$= \frac{1}{\varepsilon_0} Q_\Omega$ #

④ 高斯定律的微分形式

$\begin{cases} \vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \\ \vec{\nabla}\times\vec{E} = 0 \end{cases}$ (全空間)

[推導] 在②中 $\vec{E}(\vec{r})$ 可表示為

在Ω內有靜電荷



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\vec{\imath}}{\imath^3}\rho(\vec{r}')$$

$$\underset{\text{對 }\vec{r}}{\vec{\nabla}}\times\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\vec{\nabla}\times\int_{\Omega} d^3r' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\int_{\Omega} d^3r' \rho(\vec{r}')\,\overset{0}{\cancel{\vec{\nabla}\times\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}} = 0$$

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\vec{\nabla}\cdot\int_{\Omega} d^3r' \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}\rho(\vec{r}') = \frac{1}{4\pi\varepsilon_0}\int_{\Omega} d^3r' \rho(\vec{r}')\,\overset{4\pi\delta(\vec{r}-\vec{r}')}{\cancel{\vec{\nabla}\cdot\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}}} = \frac{1}{\varepsilon_0}\begin{cases}\rho(\vec{r}) & ;\ \Omega\text{ 包含 }\vec{r}=\vec{r}' \\ 0 & ;\ \Omega\text{ 不包含 }\vec{r}=\vec{r}'\end{cases}$$

$$= \frac{1}{\varepsilon_0}\rho(\vec{r})\ ;\ \forall\,\vec{r}\ \#$$

⑤ 電位 $V(\vec{r})$

$$\begin{cases} V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} \\ \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r}) \end{cases}$$

暗示 $\vec{\nabla}\times\vec{E} = 0$

[推導]

1. 在②中 $\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}') = \frac{1}{4\pi\varepsilon_0}\int d^3r' \rho(\vec{r}')\left(-\vec{\nabla}\frac{1}{|\vec{r}-\vec{r}'|}\right) = -\vec{\nabla}\left[\frac{1}{4\pi\varepsilon_0}\int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}\right]\ \#$

2. 在④中：

$$\begin{cases}\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho \\ \vec{\nabla}\times\vec{E} = 0\end{cases}$$

由 Helmholtz 定理有 $\vec{E}(\vec{r}) = \frac{-1}{4\pi}\vec{\nabla}\int d^3r' \frac{\overset{\frac{1}{\varepsilon_0}\rho(\vec{r}')}{\cancel{\vec{\nabla}'\cdot\vec{E}(\vec{r}')}}}{|\vec{r}-\vec{r}'|} + \frac{1}{4\pi}\vec{\nabla}\times\int d^3r' \frac{\overset{0}{\cancel{\vec{\nabla}'\times\vec{E}(\vec{r}')}}}{|\vec{r}-\vec{r}'|}$

∴ 旋度為 0 的場一定可以寫成某個純量場的負梯度 #

* Helmholtz 定理要求場要在 $r\to\infty$ 時以 $\frac{1}{r^{1+\alpha}}$ $(\alpha>0)$ 才成立

⑥ 帕松方程

$$\begin{cases}\nabla^2 V(\vec{r}) = -\frac{1}{\varepsilon_0}\rho(\vec{r}) \\ \vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})\end{cases}$$

若 $\rho(\vec{r})=0$ Laplace 方程

[推導]

由 $\vec{\nabla}\cdot\vec{E} = \frac{1}{\varepsilon_0}\rho(\vec{r})$ 和 $\vec{E} = -\vec{\nabla}V(\vec{r})$ 有 $\vec{\nabla}\cdot(-\vec{\nabla}V(\vec{r})) = \frac{1}{\varepsilon_0}\rho(\vec{r}) \Rightarrow \nabla^2 V(\vec{r}) = \frac{-1}{\varepsilon_0}\rho(\vec{r})\ \#$