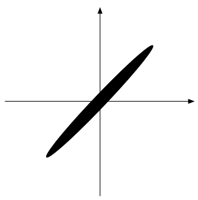
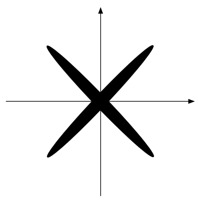
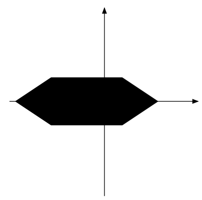
Problem 1: Indicate which of the following are unit balls of valid norms in $\mathbb{R}^2$. Circle "Yes" or "No" for each. For the ones that are "No", indicate which of the three properties of a valid norm are violated (just write this below your choice).



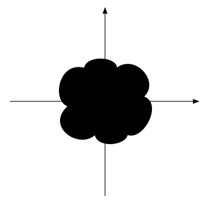
Yes No



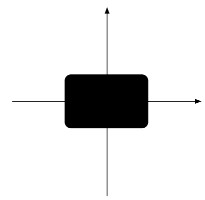
Yes No



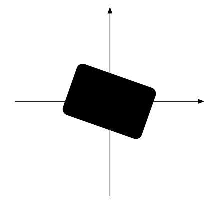
Yes No



Yes No



Yes No



Yes No

Problem 2: Let $\|\boldsymbol{x}\|_{\text{combo}} = \|\boldsymbol{x}\|_1 - \|\boldsymbol{x}\|_\infty$. True or False: $\|\boldsymbol{x}\|_{\text{combo}}$ is a valid norm. **Justify your answer.**

Answer:

Problem 3: Consider the following inner product for $\mathbb{R}^2$, which is valid:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_W = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{W} \boldsymbol{x}, \quad \text{where } \boldsymbol{W} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Find a vector $\boldsymbol{y}$ that is orthogonal to $\boldsymbol{x} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$ (i.e. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_W = 0$).

Answer:

Problem 4: Now suppose that we take

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle_W = \boldsymbol{y}^{\mathrm{T}} \boldsymbol{W} \boldsymbol{x}, \quad \text{where} \quad \boldsymbol{W} = \begin{bmatrix} 3 & 1 \\ 2 & 3 \end{bmatrix}.$$

Is $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_W$ a valid inner product for this choice of $\boldsymbol{W}$? **Justify your answer.**

Answer:

Problem 5: Let $\mathcal{U}$ and $\mathcal{V}$ be linear subspaces of a vector space $\mathcal{S}$.
True or False: The union $\mathcal{U} \cup \mathcal{V}$ is also a linear subspace of $\mathcal{S}$. **Justify your answer.**

Answer:

Problem 6: Set up the linear system of equations whose solution yields the answer to:

$$\underset{a,b,c\in\mathbb{R}}{\text{minimize}} \int_{-1}^{1} |t^3 - at^2 - bt - c|^2 \, dt$$

$$\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Problem 7: Let $f(t) = 13t^9 + 4t^8 - 3t^7 + 11t^6 + t^5 - 9t^4 + 2t^3 - t^2 + t + 7$. Take $\mathcal{E} = \mathrm{Span}\left(\{1, t^2, t^4, t^6, t^8\}\right)$. What is the solution[1] to

$$\underset{\boldsymbol{v}\in\mathcal{E}}{\text{minimize}} \int_{-1}^{1} |v(t) - f(t)|^2 \, dt \ ?$$

Justify your answer.

Answer:

[1]Hint: You do not have to solve a 5×5 system of equations to solve this problem. Think about the orthogonality principle.