Problem Set 4

AAE 532: Orbital Mechanics
MWF: 11:30-12:20
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October 2, 2020

Contents

Problem 1

At some initial time, t_0, a spacecraft is in orbit about the Earth. Its orbit is characterized by $e = 0.6$ and $p = 6R_{\oplus}$. It is currently located at the point in the orbit such that $\theta_0^* = 90°$.

(a) Determine the following orbit parameters and spacecraft state information: $a, r_p, r_a, period, \mathcal{E}, r_0, v_0, E_0, \gamma_0$ [Always list all angles in degrees.] Compare v at this location with $\sqrt{2}v_c$. Should $v < or > \sqrt{2}v_c$? Can your v value be correct? How do you know?

(b) Write $\bar{r}_0$ and $\bar{v}_0$ in terms of components in the direction of $\hat{e}$ and $\hat{p}$.

(c) Determine the time t_0 relative to periapsis, i.e., at $\theta_0^* = 90°$; also determine the final time t_f relative to periapsis, i.e., when $E_f = 225°$. What is the time-of-flight (TOF), i.e., $(t_f - t_0)$ as well as $\Delta\theta^*$ and ΔE?

(d) Plot the entire orbit in MATLAB. (Do not use polar plots; compute $\hat{e}$ and $\hat{p}$ components along the path.) By hand, on the plot, mark the location of the satellite at t_0 and t_f: at each location, indicate $r, \theta^*, \bar{v}, E, \gamma$; also, skecth the local horizon. Indicate the arc used between t_0 and t_f

Problem 1 Solution

(a)

First plug in numbers for the Earth radius:

$$p = 6R_{\oplus} = 3.8268 \times 10^4 km$$

Now we can use the relationship between the semi-major axis and semi-latus rectum to fine a:

$$a = \frac{p}{1-e^2} = 5.9795 \times 10^4 km$$

Continuing for perigee/apogee:

$$r_p = \frac{p}{1+e} = 2.3918 \times 10^4 km$$
$$r_a = \frac{p}{1-e} = 9.5672 \times 10^4 km$$

Period:

$$\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu_{\oplus}}} = 1.4551 \times 10^5 seconds = 1.6842 days = 40.421 hours$$

Energy:

$$\mathcal{E} = -\frac{\mu_{\oplus}}{2a} = -3.3330 km^2/s^2$$

Position and velocity (magnitude)

$$
\begin{aligned}
r_0 &= \frac{p}{1+e\cos\theta^*} = p = 3.8268\times 10^4 km \\
v_0 &= \sqrt{2\mathcal{E} + 2\frac{\mu_\oplus}{r_0}} = 3.7637 km/s \\
&\text{Or alternatively} \\
v_0 &= \sqrt{\mu_\oplus(\frac{2}{r_0} - \frac{1}{a})} = 3.7637 km/s
\end{aligned}
$$

Eccentric anomaly:

$$E_0 = 2\arctan(\sqrt{\frac{1-e}{1+e}}\tan(\frac{\theta^*}{2})) = 0.92730 rad = 53.130°$$

For the flight path angle, we can use the definition of the angular momentum:

$$
\begin{aligned}
\vec{h} &= \vec{r}\times\vec{v} &(1)\\
&= \ r\hat{r}\times[v sin\gamma\hat{r} + v cos\gamma\hat{\theta}] &(2)\\
&= rv\cos\gamma[\hat{r}\times\hat{\theta}] &(3)\\
&= rv\cos\gamma\hat{h} \Rightarrow &(4)\\
h &= rv\cos\gamma &(5)
\end{aligned}
$$

We can compute the angular momentum from the given semi-latus rectum and then the flight path angle.

$$
\begin{aligned}
h &= \sqrt{p\mu} = 1.2350\times 10^5 km^2/s \\
\gamma_0 &= \arccos(\frac{h}{r_0 v_0}) = 0.54042 rad = 30.964°
\end{aligned}
$$

Note that we choose positive value for the flight path angle, since we are in the ascending leg of the orbit ($0 < \theta^* < 180°$)
Now, let's compute the circular velocity at this radius:

$$
\begin{aligned}
v_c &= \frac{\mu}{r_0} = \sqrt{\frac{\mu}{r_0}} = 3.2273 km/s \\
\sqrt{2}v_c &= 4.5642 km/s \\
v_0 &< \sqrt{2}v_c
\end{aligned}
$$

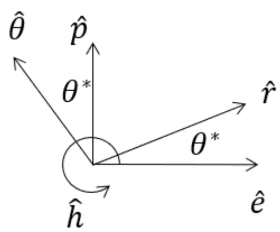
v_0 should be smaller than $\sqrt{2}v_c$. Otherwise, it implies that the energy is $\mathcal{E} = \frac{v^2}{2} - \frac{\mu_\oplus}{r} > 0$. So it is no longer a closed orbit around the Earth. One thing we can check to see if v values is correct is to compare that value with $\sqrt{2}v_c$. If it is ellipse, it should not be bigger than $\sqrt{2}v_c$ at any point along the orbit.

(b)

Position/velocity vectors in $\hat{r}-\hat{\theta}-\hat{h}$ frame are readily available:

$$\bar{r} = 3.8268 \times 10^4 \hat{r}\ km$$
$$\bar{v} = v\sin\gamma_0 \hat{r} + v\cos\gamma_0 \hat{\theta} = 1.9364\hat{r} + 3.2274\hat{\theta}\ km/s$$

Now, we can compute a direction cosine matrix or DCM to relate the coordinate frames $\hat{r}-\hat{\theta}-\hat{h}$ to $\hat{e}-\hat{p}-\hat{h}$



	$\hat{r}$	$\hat{\theta}$	$\hat{h}$
$\hat{e}$	$\cos\theta^*$	$-\sin\theta^*$	0
$\hat{p}$	$\sin\theta^*$	$\cos\theta^*$	0
$\hat{h}$	0	0	1

Figure 1: Constructing a direction cosine matrix

Now, we simply multiply the above DCM by the column vector representation of the spacecraft's position and velocity:

$$\begin{bmatrix} r_{\hat{e}} \\ r_{\hat{p}} \\ r_{\hat{h}} \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.8268 \times 10^4 \\ 0 \\ 0 \end{bmatrix} = \boxed{\underset{\hat{e}-\hat{p}-\hat{h}}{\vec{r}} = 3.8268 \times 10^4\ \hat{p}\ \text{km}} \tag{6}$$

$$\begin{bmatrix} v_{\hat{e}} \\ v_{\hat{p}} \\ v_{\hat{h}} \end{bmatrix} = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ & 0 \\ \sin 90^\circ & \cos 90^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.9364 \\ 3.2274 \\ 0 \end{bmatrix} = \boxed{\underset{\hat{e}-\hat{p}-\hat{h}}{\vec{v}} = -3.2274\ \hat{e} + 1.9364\ \hat{p}\ \text{km/s}} \tag{7}$$

For the given configuration at the initial time, we can see that $\hat{r} = \hat{p}, \hat{\theta} = -\hat{e}$.
Now moving on to the final time, we can compute the true anomaly with:

$$\boxed{\theta_f^* = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E_f}{2}\right)\right) = -2.7332(rad) = -156.60^\circ = 203.40^\circ}$$

Then we can compute the radius and the magnitude of the velocity at this true anomaly:

$$\boxed{\begin{aligned} r_f &= \frac{p}{1+\cos\theta_f^*} = 8.5163 \times 10^4\ km \\ v_f &= \sqrt{\mu_\oplus\left(\frac{2}{r_f} - \frac{1}{a}\right)} = 1.6415 km/s \\ \gamma_f &= -\arccos\left(\frac{h}{r_f v_f}\right) = -27.938^\circ \end{aligned}}$$

Note that we choose negative flight path angle since we are in the descending leg of the orbit. And similarly, we can represent the position and velocity vectors in the $\hat{r} - \hat{\theta} - \hat{h}$ frame:

$$\bar{r}_f = r_f \hat{r} = 8.5163 \times 10^4 \hat{r}\ km$$
$$\bar{v}_f = v_f \sin\gamma_f \hat{r} + v_f \cos\gamma_f \hat{\theta} = -0.7691\hat{r} + 1.4502\hat{\theta}\ km/s$$

$$\begin{bmatrix} r_{\hat{e}} \\ r_{\hat{p}} \\ r_{\hat{h}} \end{bmatrix} = \begin{bmatrix} \cos\theta_f^* & -\sin\theta_f^* & 0 \\ \sin\theta_f^* & \cos\theta_f^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 8.5164 \times 10^4 \\ 0 \\ 0 \end{bmatrix} = \boxed{\underset{\hat{e}-\hat{p}-\hat{h}}{\vec{r}_f} = -7.8158 \times 10^4\ \hat{p} - 3.3825 \times 10^4 \hat{e}\ \text{km}} \tag{8}$$

$$\begin{bmatrix} v_{\hat{e}} \\ v_{\hat{p}} \\ v_{\hat{h}} \end{bmatrix} = \begin{bmatrix} \cos\theta_f^* & -\sin\theta_f^* & 0 \\ \sin\theta_f^* & \cos\theta_f^* & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.7691 \\ 1.4502 \\ 0 \end{bmatrix} = \boxed{\underset{\hat{e}-\hat{p}-\hat{h}}{\vec{v}_f} = 1.2818\ \hat{e} - 1.0255\ \hat{p}\ \text{km/s}} \tag{9}$$

(c)

To determine t_0, we need to compute M_0, or the mean anomaly:

$$M_0 = E_0 - e\sin E_0 = 0.4473 rad = 25.628^\circ$$
$$M_0 = nt_0$$

The mean motion is:

$$n = \sqrt{\frac{\mu}{a^3}} = 4.3179 \times 10^{-5} rad/s = (2.4740 \times 10^3)^\circ/s$$

Then t_0:

$$t_0 - t_p = \frac{M_0}{n} = 1.0359 \times 10^4 (seconds) = 2.8775(hours) = 0.11990(days)$$

Then we can repeat the same process for the final eccentric anomaly:

$$M_f = E_f - e\sin E_f = 4.3512 rad = 249.31^\circ$$
$$t_f - t_p = \frac{M_f}{n} = 1.0077 \times 10^5 (seconds) = 27.992(hours) = 1.1663(days)$$
$$\theta_f^* = 2\arctan\left(\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E_f}{2}\right)\right) = -2.7332(rad) = -156.60^\circ = 203.40^\circ$$

$$\boxed{\begin{aligned} &TOF = t_f - t_0 = 9.0414 \times 10^4 (seconds) = 25.115(hours) = 1.0465(days) \\ &\Delta\theta^* = 113.40^\circ \\ &\Delta E = 171.87^\circ \end{aligned}}$$

(d)

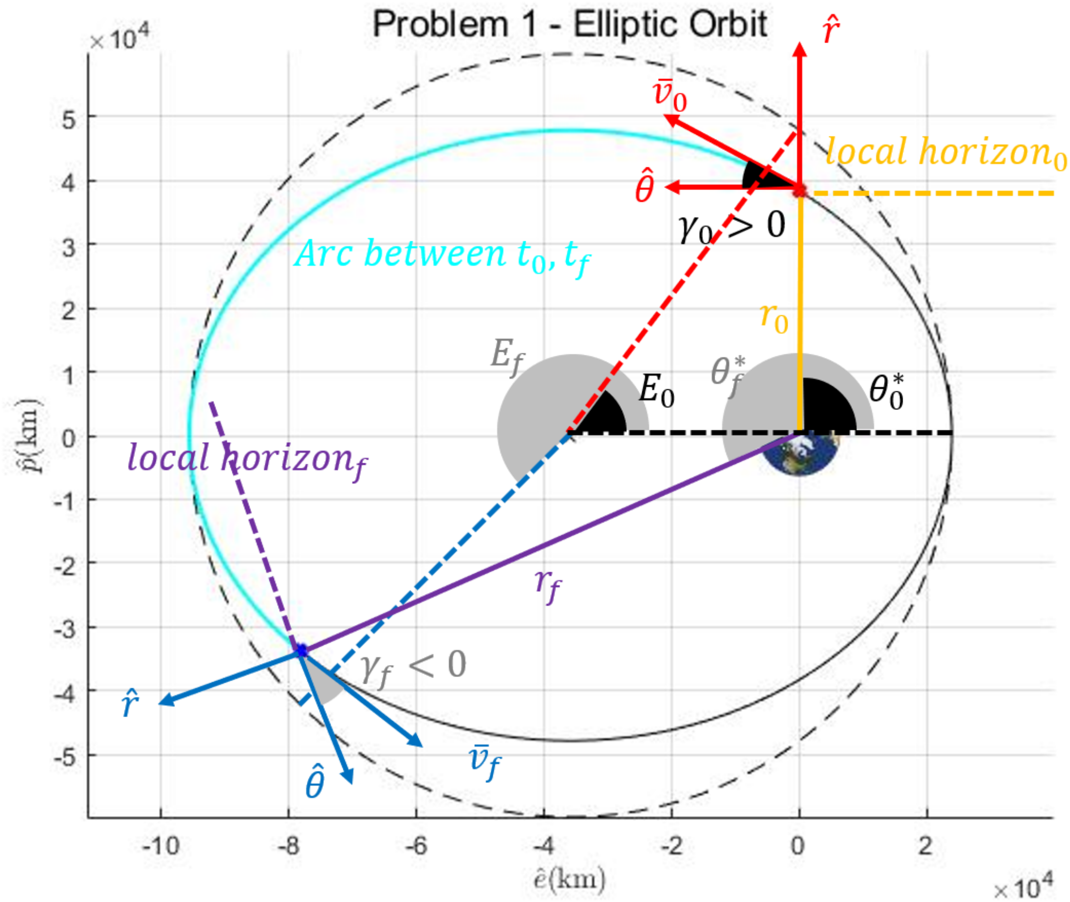


Figure 2: Problem 1: Elliptic Orbit

Problem 2

Return to Problem 1 and confirm your results in GMAT. Use October 2, 2020 as the start date.

(a) What initial state can be input to GMAT for Sat1? Can you locate the rest of the quantities that were requested in Problem 1(a) and 1(b); do they confirm that your computations are correct? Can you determine the ime difference $t_f - t_0$? Compare your Matlab plot and the GMAT plot. Is your GMAT plot consistent with your MATLAB plot?

(b) Also print out the data from GMAT. (You can submit output from the file generated in the propagate window under the Mission Sequence. Cut-and-paste the sections with the required data into a Word document. Highlight the requested quantities. You can also create a Report file; you may not want to include the entire file but, again cut-and-paste.)

(c) Add an X-Y plot to the output. Plot speed as a function of elapsed time in seconds. Print the plot. Mark your time that you computed in Problem 1. Does the max velocity location in your plot correlate to the periapsis time in the GMAT plot?

Problem 2 Solution

(a)

We can use the following initial values for GMAT simulation:

$$\begin{aligned}
&\text{State Type: Keplerian}\\
&SMA = 5.979 \times 10^4 km\\
&ECC = 0.6\\
&INC = 0^\circ\\
&RAAN = 0^\circ\\
&AOP = 0^\circ\\
&TA = 90^\circ
\end{aligned}$$

Note that we can arbitrary values for INC, RAAN, AOP, since they will not change the shape of the orbit. And we can propatate until $E_f = 225^\circ$. (Tip: we can employ a stopping condition when the eccentric anomaly reaches a certain value.) Then for output, we can check for the quantities that we were interested in Problem 1(a) and Problem 1(b). Indeed, the value from the GMAT (we can export elapsed time) coincides with the computed value:

$$t_f - t_0 = 1.0465 days$$

The plot from GMAT is consistent with the MATLAB plot.

(b)

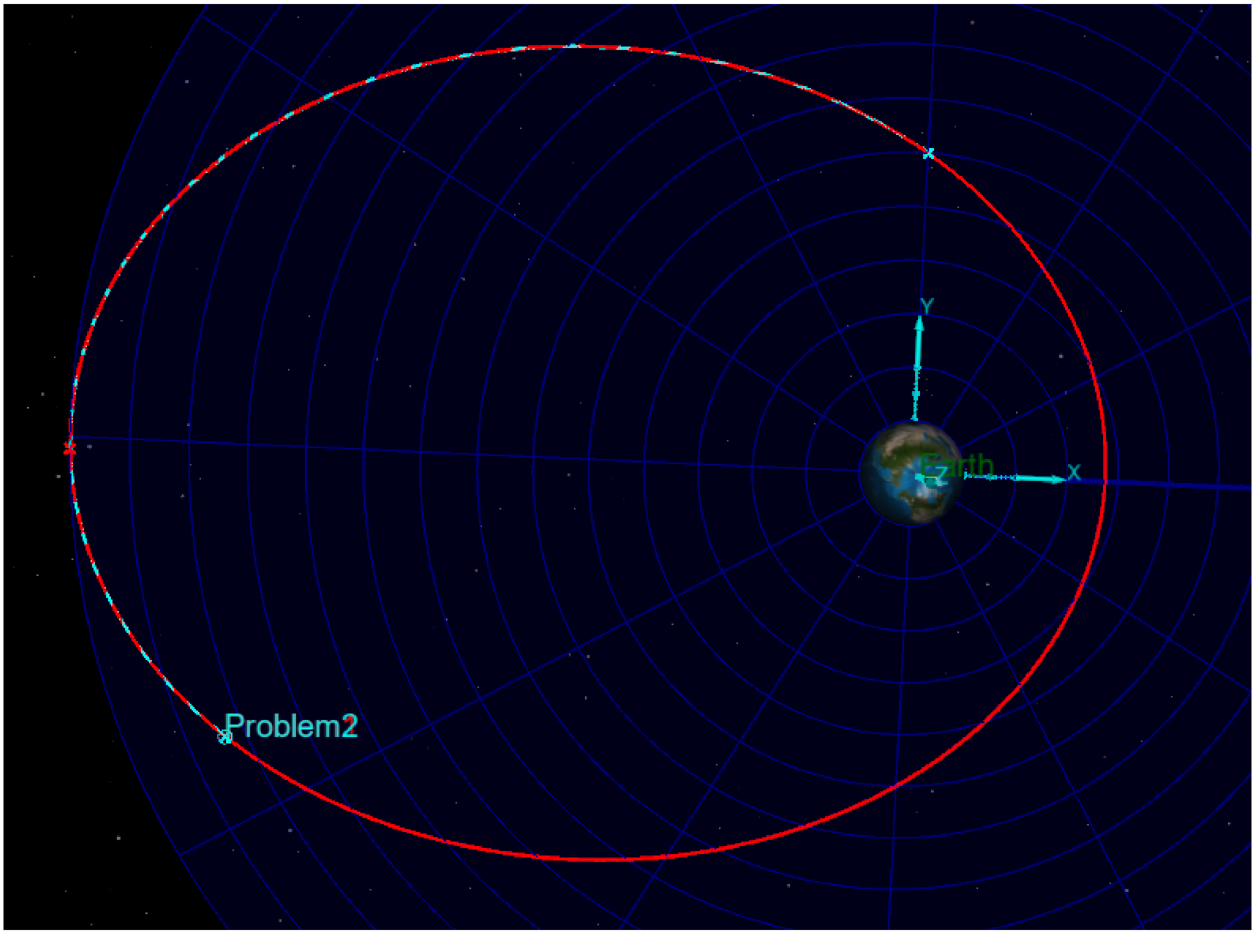


Figure 3: Problem 2: GMAT Plot (in OpenFrames)

Problem2.ElapsedDays	Problem2.Earth.TA	Problem2.Earth.EA	Problem2.Earth.MA	Problem2.Earth.Energy	Problem2.Earth.HMAG	Problem2.Earth.OrbitPeriod	
0	89.99999999999999	53.13010235415597	25.62812818787646	-3.33305674470038	123506.9539368657	145515.2765446653	t_0
1.046454380906653	203.4018390250728	225.0000000156248	249.3085405585084	-3.333056744698095	123506.9539368719	145515.276544815	t_f

Problem2.Earth.RadApo	Problem2.Earth.RadPer	Problem2.Earth.RMAG	Problem2.Earth.SemilatusRectum	Problem2.Earth.SMA	Problem2.EarthMJ2000Eq.FPA	Problem2.ElapsedSecs	
95672.04450000002	23918.011125	38268.8178	38268.81779999998	59795.02781250001	59.03624346792648	0	t_0
95672.04450008433	23918.01112499772	85163.90959412759	38268.81780000383	59795.02781254102	117.938352736089	90413.65851033479	t_f

Figure 4: GMAT Report

Also, the data retrieved from the GMAT report file match the computed values.
Figure 4 shows a portion of the report file relevant to the computations in Problem 1. We can see that all the parameters, $t_f - t_0, \theta_0^*, \theta_f^*, E_0, E_f, M_0, M_f, \mathcal{E}, h, \mathbb{P}, r_p, r_a, r_0, r_f, a, \gamma_0$ match. Note that the flight path angle in GMAT (noted as FPA) shows β (to be discussed later in class), not γ. So we could retrieve the γ angle by subtracting from $90°$.

(c)

Now, we proceed to make an X-Y Plot portraying orbital speed as a function to elapsed time in seconds. In the Figure 5, x-axis is the time elapsed from thet initial time, which is assumed as t_0. Then it is propagated for one period of the orbit to see the fluctuation of speed during this period.

The smallest speed corresponds to apoapsis, when our spacecraft is the farthest distance away from the Earth. Periapsis corresponds to the location on the plot with the highest speed. If the time difference between the peak and trough of the following plot is computed by visual inspection, we see that there is a time difference of 73000 seconds (very nearly half the Orbital Period). They should exactly be half a period apart, however since GMAT numerically integrates the orbital path, there is no guarantee that the computed time-step is exactly on the apses points. Therefore, there will be a small discrepancy in this calculation.

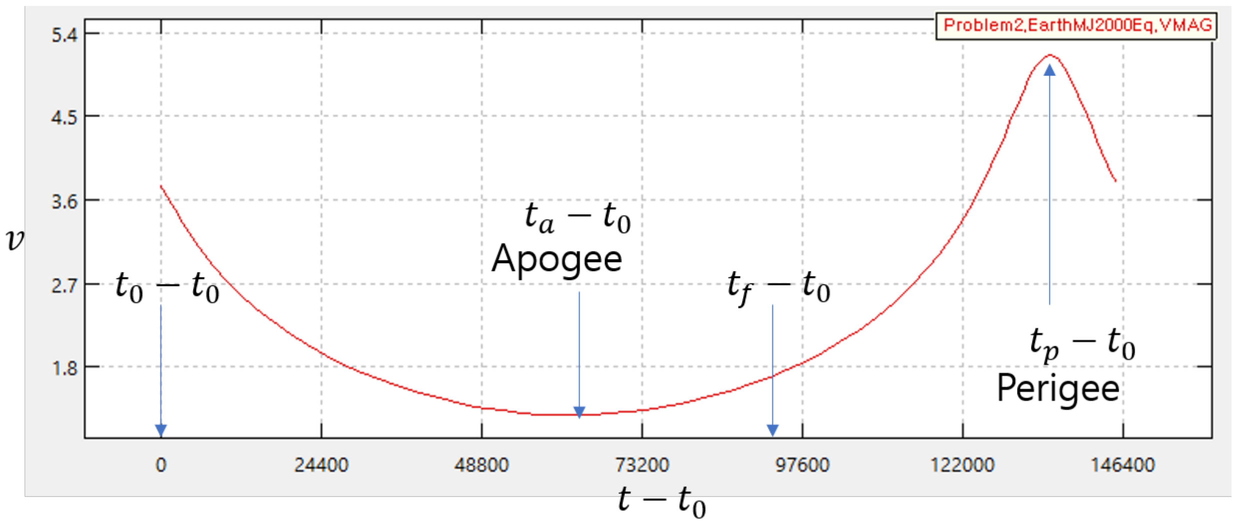


Figure 5: X-Y Plot from GMAT

Problem 3

To investigate the requirements for departure, assume that a spacecraft is departing the vicinity of the Earth along a parabolic path. Consider the spacecraft to be located at perigee on the parabola.

(a) A circular parking orbit about the Earth may be defined at 225 km altitude. At a perigee altitude of 225 km on the parabola, compare the escape velocity on the parabola with the relative velocity in a circular orbit with the same altitude.
To shift from the circular orbit to the escape trajectory, what % increase in velocity is required?

(b) Compute the velocity along the parabola as it departs the vicinity of the Earth, that is, at the following distances: r = 2R⊕, 10R⊕, 75R⊕, 200R⊕, 800R⊕ and an additional distance of your choice.
Determine the true anomaly θ^* that corresponds to each distance. Also include the time since periapsis at each distance (in days).

(c) In the MATLAB script from the first problem, plot the parabola corresponding for altitude 225 km between -140°< θ^* < 140°. Mark on the plot: r, v, γ at θ^* = -120°; also, sketch the l.h. (local horizon). Also add the directrix. Compare θ^* and γ, is there a pattern?

(d) At r = 75R⊕ , is it reasonable to model the problem as a two-body (Earth and spacecraft)?

Problem 3 Solution

(a) Let us begin by evaluating the energy equation to find a relationship between circular velocity and escape velocity. For a parabola, recall that:

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu_\oplus}{r} = 0$$
$$v = \sqrt{\frac{2\mu_\oplus}{r}}$$
$$v = \sqrt{2} * v_{circ}$$

We can compute the circular at the given altitude and the velocity on the parabola (escape velocity) as:

$$v_{circ} = \sqrt{\frac{\mu_\oplus}{r_p}}$$
$$v_{circ} = 7.7695 km/s$$

$$v_{esc} = \sqrt{2} * v_{circ}$$
$$v_{esc} = 10.9877 km/s$$

We have found the circular velocity to be less than the escape velocity. Thus, to shift from the circular orbit to the escape trajectory, a 29.29 % increase in velocity is required.

(b) To compute the velocity along the parabola for the requested distances, we can use the escape velocity expression derived in part (a). In addition, we can use Barker's equation to obtain the time since periapsis at each distance. Note that the velocity vectors can be easily solved with using the flight path angles in (c), but here we are interested in the magnitude of the velocities, or how small they become as the distances grow.

We obtain the following:
1. r = 2R⊕, v = 7.9054 km/s, $\theta^* = 87.9784°$, t = 0.0176 days
2. r = 10R⊕, v = 3.5354 km/s, $\theta^* = 142.4616°$, t = 0.1591 days
3. r = 75R⊕, v = 1.2909 km/s, $\theta^* = 166.5056°$, t = 2.9178 days
4. r = 200R⊕, v = 0.7905 km/s, $\theta^* = 171.7483°$, t = 12.5471 days
5. r = 800R⊕, v = 0.3953 km/s, $\theta^* = 175.8768°$, t = 99.7997 days
6. r = 100,000R⊕, v = 0.0354 km/s, $\theta^* = 179.6313°$, t = 1.3921e5 days
7. r = 1E10R⊕, v = 1.1180E-14 km/s, $\theta^* = 179.9988°$, t = 1.2060E12 years

We observe that we cannot obtain exactly zero velocity relative to the Earth since we cannot be at an infinite distance away. From (7), we can see that achieving a nearly "zero" velocity would take 1.2E12 years! The same observation can be made for θ^*. It is never *exactly* 180°because it would mean that we are at an infinite distance away from the Earth.

(c) The plot of the parabola corresponding to an altitude 225 km between -140°$< \theta^* <$ 140°with requested quantities marked is shown in Figure 6 on Page 13. In addition, if we compute γ at each of the distances given above, we would observe that $\theta^* = 2^*\gamma$. At $\theta^* = 180°$, $\gamma = 90°$, thus the velocity is parallel to $\hat{\theta}$ direction.

(d) At r = 75R⊕, we can model the system using the 2-body model if we assume that the Earth and s/c are isolated. However, at r = 75R⊕, the s/c would be ≈ 478,350km away! This distance is beyond the lunar vicinity and far away enough that the trajectory designer must account for the gravitational influences of other bodies, i.e. Moon and/or Sun.

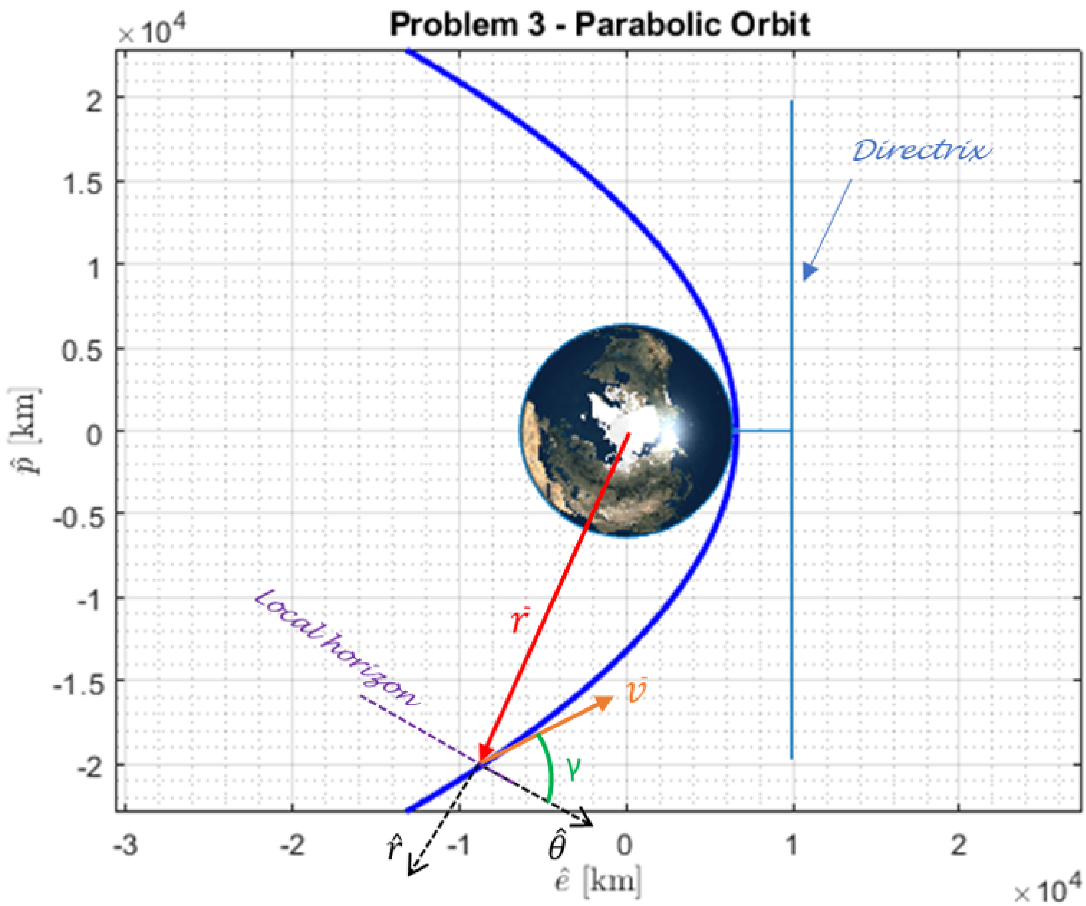


Figure 6: Problem 3: Parabolic orbit with marked quantities.

Problem 4

As part of the new lunar initiative, an unmanned probe is approaching the Moon on a hyperbola. The hyperbola is defined such that |a| = 7050 km and the passage altitude is 800 km altitude. At the “current” time, the probe is located at θ^* = -60°.

(a) Determine the following additional orbital characteristics: r_p, v_p, b, h, δ, v_∞, $\mathcal{E}$. Determine the following quantities at the current time: r, v, γ, H, time till perilune.

(b) Use your Matlab script and plot the hyperbola between θ^* = ± 110°. Mark the probe at θ^* = -60°and label b, aim point, $\delta/2$, v, γ, r, θ^*. (Always include the local horizon!)

(c) Determine r, v, γ at θ^* = 100°; Add this information to the plot.

Problem 4 Solution

(a) To begin, let's first compute the orbital energy, which we know is positive for a hyperbola:

$$\mathcal{E} = \frac{\mu_☾}{2|a|}$$
$$\mathcal{E} = 0.3477 km^2/s^2$$

The requested orbital quantities are as follows:

$$r_p = Altitude + R_☾ = 2.5374 \times 10^3 km$$
$$v_p = \sqrt{\mu_☾ * (\frac{1}{|a|} + \frac{2}{r_p})} = 2.1354 km/s$$
$$v_\infty = \sqrt{2 * \mathcal{E}} = 0.8339 km/s$$
$$e = \frac{r_p}{|a|} + 1 = 1.3600$$
$$p = |a|(e^2 - 1) = 5.9902 \times 10^3 km$$
$$h = \sqrt{\mu_☾ * p} = 5.4184 \times 10^3 km^2/s$$
$$\delta = 2 * sin^{-1}(\frac{1}{e}) = 94.6719°$$

We expected the flyby angle to be large due to the close pass at the Moon. In later classes we will see that we can use a planet's gravity field to gain equivalent ΔV (a function of δ and v_∞) without using on-board thrusters!

The θ^*_∞ can be computed using the eccentricity of the conic section as:

$$\theta^*_\infty = cos^{-1}(\frac{-1}{e}) = 137.3360°$$

We can use the geometric relations of the hyperbola to compute the aim point as follows:

$$b = (r_p + |a|) * \sin(180° - \theta^*_\infty) = 6.4970 \times 10^3 km$$

At the current time, the requested quantities for the probe are:

$$r = \frac{p}{1 + e * cos(-60°)} = 3.5644 \times 10^3 km$$
$$v = \sqrt{\mu_☾ * (\frac{1}{|a|} + \frac{2}{r})} = 1.8565 km/s$$
$$\gamma = cos^{-1}(\frac{h}{r * v}) = -35.0321°$$

Note that the flight path angle should be negative, since we are in the descending part of the hyperbola ($\theta^* < 0$)
We can use Kepler's equation to find the hyperbolic anomaly and time until perilune:

$$H = cosh^{-1}(\frac{\frac{r}{|a|} + 1}{e}) = -0.4588$$

Here, $H < 0$ since $\theta^* < 0$. To determine t_0, we need to compute MA, or the mean anomaly:

$$N = e * sinh(H) - H = -0.18735$$

Then, time until perilune is:

$$(t - t_p) = \frac{N}{\sqrt{\frac{\mu_☾}{|a|^3}}} = -1.5838 \times 10^3 s$$

(b)

The hyperbolic trajectory with the marked quantities is:

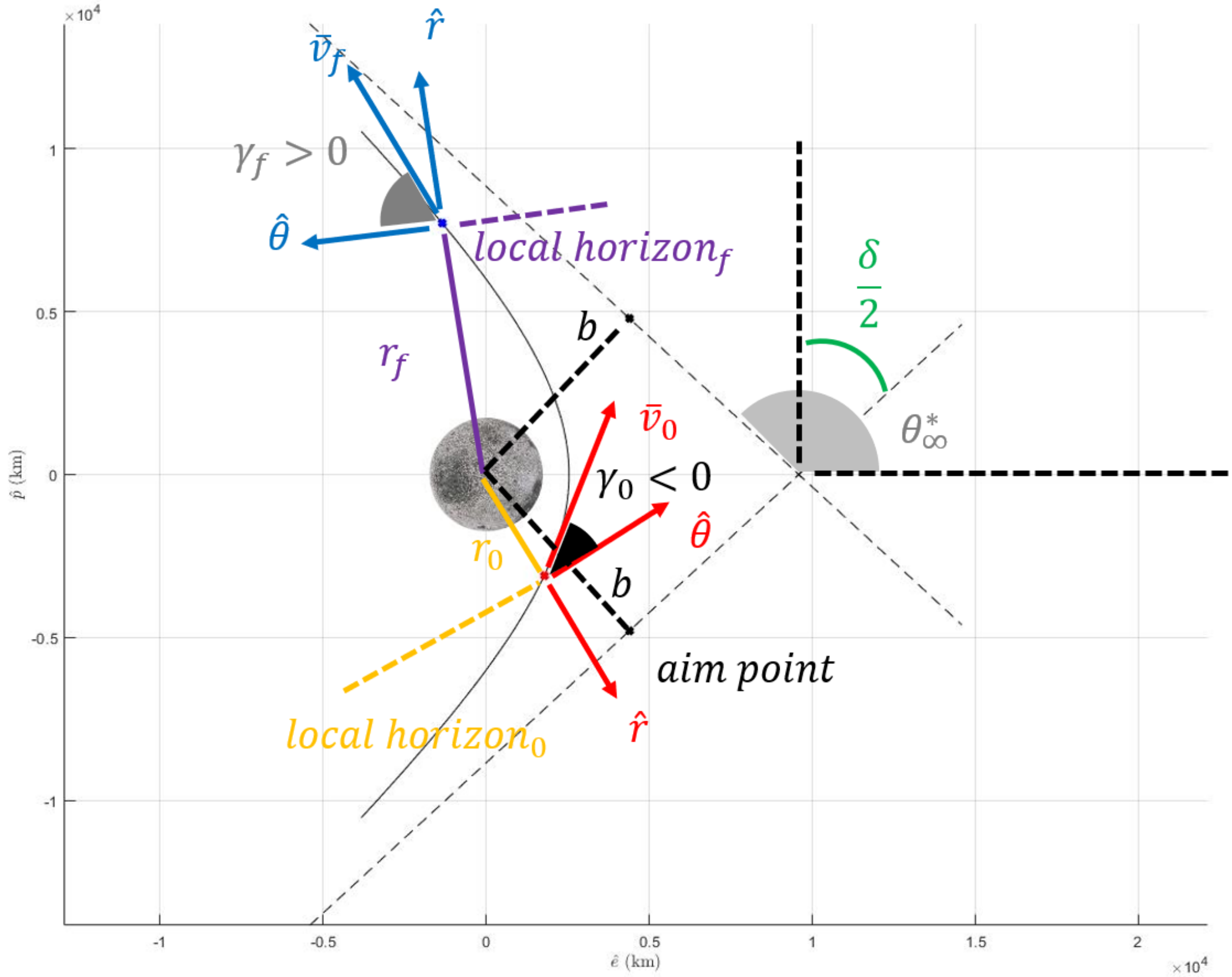


Figure 7: Problem 4: Hyperbolic orbit with marked quantities.

Figure 7 shows the orbit for this problem. Note that the subscript 0 corresponds to values for $\theta^* = -60°$, and subscript f corresponds to values for $\theta^* = 100°$.

(c)

At $\theta^* = 100°$, r, v, γ are:

$$r = \frac{p}{1 + e * cos(100°)} = 7.8393 \times 10^3 km$$

$$v = \sqrt{\mu_{☾} * (\frac{1}{|a|} + \frac{2}{r})} = 1.3951 km/s$$

$$\gamma = cos^{-1}(\frac{h}{r * v}) = 60.301^{\circ}$$

Here, the flight path angle is positive since $\theta^* > 0$