AAE 567 Spring 2018 Homework 1 Solutions

Feb 9, 2018

1.5.2 Problem 1

Let ω and b be vectors in $\mathbb{R}^3$. Recall that the centrifugal force in the rotating frame for a particle with mass m position b and angular velocity ω is given by $F_{cen} = -m\omega \times (\omega \times b)$ where $\times$ denotes the cross product. Show that

$$F_{cen} = -m\omega \times (\omega \times b) = m\|\omega\|^2 P_{\mathcal{H}} b$$

where $P_{\mathcal{H}}$ is the orthogonal projection onto the subspace $\mathcal{H} = \{h \in \mathbb{C}^3 : h \perp \omega\}$ the orthogonal complement of ω. Hint: show that $\omega \times b = A_\omega b$ where A_ω is the skew symmetric matrix defined by

$$A_\omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \quad \text{and} \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

Then show that $-A_\omega^2 = A_\omega^* A_\omega = P_{\mathcal{H}}$ when $\|\omega\| = 1$.

Solution. Notice that

$$A_\omega^* A_\omega = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} = \begin{bmatrix} \omega_2^2 + \omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 \\ -\omega_2\omega_1 & \omega_1^2 + \omega_3^2 & -\omega_2\omega_3 \\ -\omega_3\omega_1 & -\omega_3\omega_2 & \omega_1^2 + \omega_2^2 \end{bmatrix}$$

If $\|\omega\| = 1$, then $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$. In this case, we have

$$A_\omega^* A_\omega = \begin{bmatrix} 1 - \omega_1^2 & -\omega_1\omega_2 & -\omega_1\omega_3 \\ -\omega_2\omega_1 & 1 - \omega_2^2 & -\omega_2\omega_3 \\ -\omega_3\omega_1 & -\omega_3\omega_2 & 1 - \omega_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}$$

This readily implies that

$$A_\omega^* A_\omega = I - \omega\omega^* \qquad (\text{when } \|\omega\| = 1) \tag{1}$$

However, $\omega\omega^* = P_{\mathcal{H}^\perp}$ the orthogonal projection onto $\mathcal{H}^\perp = \text{span}\ \{\omega\}$ when $\|\omega\| = 1$. To see this observe that for any f in $\mathbb{C}^3$, we have $\omega\omega^* f \in \mathcal{H}^\perp$ and $f - \omega\omega^* f \perp \omega$. Hence $\omega\omega^* = P_{\mathcal{H}^\perp}$ when $\|\omega\| = 1$. This with (1) readily implies that

$$A_\omega^* A_\omega = I - \omega\omega^* = I - P_{\mathcal{H}^\perp} = P_{\mathcal{H}} \qquad (\text{when } \|\omega\| = 1) \tag{2}$$

Here we used the fact that for any subspace $\mathcal{H}$ in a Hilbert space, $P_{\mathcal{H}} = I - P_{\mathcal{H}^\perp}$.

Without loss of generality, let us assume that $\omega \neq 0$. Since ω and $\frac{\omega}{\|\omega\|}$ span the same one dimensional subspace, we have

$$A_\omega^* A_\omega = \|\omega\|^2 \left(A_{\frac{\omega}{\|\omega\|}}\right)^* A_{\frac{\omega}{\|\omega\|}} = \|\omega\|^2 P_{\mathcal{H}} \tag{3}$$

In other words,

$$F_{cen} = -m\omega \times (\omega \times b) = mA_\omega^* A_\omega b = m\|\omega\|^2 P_{\mathcal{H}} b.$$

1.5.2 Exercise 2

We are given

$$T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

and wish to find, without using a computer, $\hat{u} \in \mathbb{C}^2$ such that

$$d = \|y - T\hat{u}\| = \inf \left\{ \|y - Tu\| : u \in \mathbb{C}^2 \right\}.$$

If T is one to one, then by theorem 1.5.1 we have $\hat{u} = (T^*T)^{-1}T^*y$. It is noted that T is one to one if and only if its columns are linearly independent, or equivalently, T^*T is strictly positive. Clearly, T is one to one. Hence $\hat{u} = (T^*T)^{-1}T^*y$.

To compute $\hat{u} = (T^*T)^{-1}T^*y$, observe that

$$T^* = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \text{ and } T^*T = \begin{bmatrix} 6 & 5 \\ 5 & 6 \end{bmatrix}.$$

Using this we obtain

$$(T^*T)^{-1} = \frac{1}{11}\begin{bmatrix} 6 & -5 \\ -5 & 6 \end{bmatrix}$$
$$(T^*T)^{-1}T^* = \frac{1}{11}\begin{bmatrix} 1 & 7 & -4 \\ 1 & -4 & 7 \end{bmatrix}$$
$$\hat{u} = (T^*T)^{-1}T^*y = \frac{1}{11}\begin{bmatrix} 1 \\ 12 \end{bmatrix}.$$

The error d is given by $\|y - T\hat{u}\|$. We have

$$T\hat{u} = \frac{1}{11}\begin{bmatrix} 13 \\ 14 \\ 25 \end{bmatrix} \text{ and } y - T\hat{u} = \frac{1}{11}\begin{bmatrix} 9 \\ -3 \\ -3 \end{bmatrix}$$

with

$$d^2 = \frac{9}{11} \text{ and } d = \frac{3}{\sqrt{11}} \approx 0.9045.$$

Finally, it is noted that $\widehat{u} = T^{-r}y$ where $T^{-r}y$ is the Moore-Penrose pseudo inverse of T. In Matlab T^{-r} is pinv(T).

The following matlab code solves this problem:

```
%1.5.2 2
T = [1, 1; 2, 1; 1, 2];  %given
y = [2; 1; 2]; %given
uhat = inv(T' *T)*T'*y %optimal solution or
uhat = pinv(T)*y; %optimal solution
d = norm(y-T*uhat); %error
```

1.5.2 Exercise 3

We are given the data

$$y^* = \begin{bmatrix} 1 & -2 & 3 & 5 & 10 & 8 & 4 \end{bmatrix} = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix}.$$

We want to find the polynomial $\sum_0^3 \alpha_j t^j$ of degree at most three that solves the optimization problem

$$d^2 = \inf\left\{\sum_{k=0}^{6} |y_k - p(k)|^2 : p(t) \text{ is a polynomial with degree } \leq 3\right\}.$$

To compute the optimal solution we will use the Vandermonde technique in section 1.5.1 of the notes. We begin by noting that the λ vector from that section is given by $\lambda = [0\ 1\ 2\ 3\ 4\ 5\ 6]$. We then construct the Vandermonde matrix from equation (5.7):

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \end{bmatrix}.$$

Because $\{\lambda_j\}_0^6$ are distinct, the Vandermonde is guaranteed to be one to one. One can also directly verify that V is one to one.

Let $p(t) = \sum_0^3 \alpha_j t^j$ be the polynomial of degree at most 3 formed by $\{\alpha_j\}_0^3$. We use equation (5.8) to note that for a vector $\alpha = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^{tr}$, we have

$$V\alpha = \begin{bmatrix} p(0) & p(1) & p(2) & p(3) & p(4) & p(5) & p(6) \end{bmatrix}^{tr}.$$

The optimal solution is given by

$$\hat{\alpha} = (V^*V)^{-1}V^*y = V^{-r}y \approx [0.5952\ \ -4.0635\ 3.0952\ \ -0.3889]^*.$$

(Recall that V^{-r} is the Moore-Penrose pseudo inverse.) In other words, the unique optimal polynomial is given by

$$p(t) = 0.5952 - 4.0635t + 3.0952t^2 - 0.3889t^3.$$

The error d is determined by $d^2 = \sum_{k=0}^6 |y_k - p(k)|^2 = \|y - V\hat{\alpha}\|^2$. We obtain

$$d^2 \approx 6.1429 \text{ and } d \approx 2.4785$$

The plot of y and $p(t)$ is included in Figure 1

Matlab code for this problem:

```
y = [1 -2 3 5 10 8 4]'; %given
lambda = [0 1 2 3 4 5 6]; %given
v = fliplr(vander(lambda)); %set up vandermonde matrix
v = v(:,1:4); %take only the columns we need
alpha = pinv(v)*y; %solve for coefficients
```

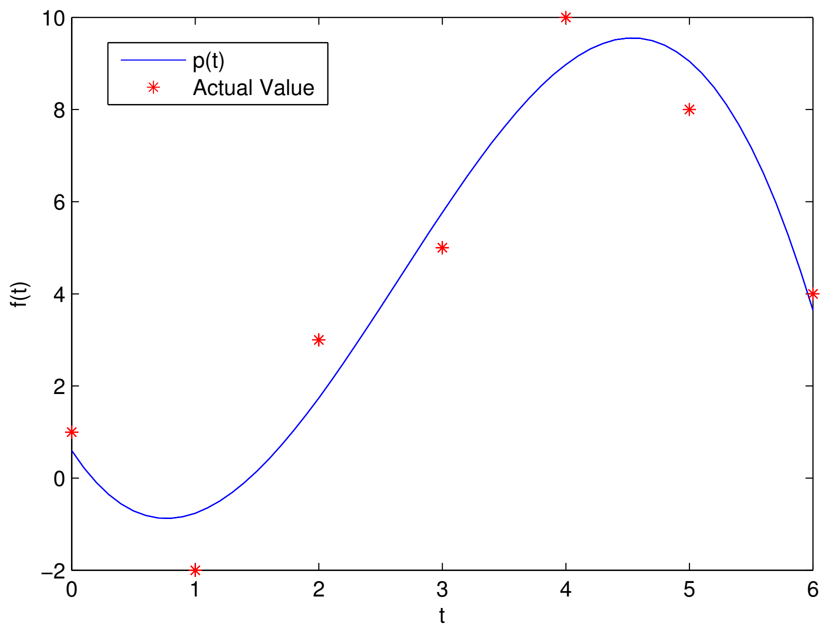

Figure 1: Plot for $t = 0$ to 6 of y and the fit $p(x)$ for 1.5.2 Exercise 3

```
d = norm(y - v*alpha)
dsq = d^2

t = 0:.1:6;
v2 = fliplr(vander(t));
v2 = v2(:,1:4);
res=v2*alpha;
plot(t,res)
hold on
plot(lambda,y,'r*')
xlabel('t')
ylabel('f(t)')
legend('p(t)','Actual Value')
```

1.5.2 Exercise 4

We wish to find the optimal polynomial $p(t) = \sum_{j=0}^{5} \alpha_j t^j$ which solves the problem

$$\int_0^\pi |\sin(t) - p(t)|^2 dt = \inf \left\{ \int_0^\pi |\sin(t) - \sum_{j=0}^{5} \alpha_j t^j|^2 dt : \alpha_i \in \mathbb{C} \right\}$$

We are looking for the optimal polynomial of degree at most 5 which approximates the function $f(t) = \sin(t)$ over the interval $0 \leq t \leq \pi$. Let us use the basis $f_i(t) = t^{i-1}$ for $i = 1, 2, \cdots, 6$ for the polynomials of degree at most 6.

This problem parallels the example in the notes from section 1.3.1. We begin by constructing the Gram matrix, G, which is 6-by-6 in this case because we have a 5th degree polynomial (0th through 5th order terms). The components of G are given by $G_{ij} = (t^{i-1}, t^{j-1}) = \int_0^\pi t^{j+i-2} dt$. We have

$$G \approx \begin{bmatrix} 3.14 & 4.93 & 10.34 & 24.35 & 61.20 & 160.23 \\ 4.93 & 10.34 & 24.35 & 61.20 & 160.23 & 431.47 \\ 10.34 & 24.35 & 61.20 & 160.23 & 431.47 & 1186.1 \\ 24.35 & 61.20 & 160.23 & 431.47 & 1186.1 & 3312.1 \\ 61.20 & 160.23 & 431.47 & 1186.1 & 3312.1 & 9364.8 \\ 160.23 & 431.47 & 1186.1 & 3312.1 & 9364.8 & 26746 \end{bmatrix}.$$

We also assemble the vector $\{(f, f_i)\}_1^6$ made up of inner products of the function to be approximated and each of the polynomial orders, where the i^{th} component is given by $(\sin(t), t^{i-1}) = \int_0^\pi t^{i-1}\sin(t)dt$. We will call this vector b to be concise. We obtain

$$b = \begin{bmatrix} 2 & 3.1416 & 5.8696 & 12.1567 & 26.9738 & 62.8853 \end{bmatrix}$$

We then use equation (3.6) from theorem 1.3.2 in the notes to obtain the coefficients of the polynomial:

$$\alpha = bG^{-1} = \begin{bmatrix} 0.0013 & 0.9826 & 0.0545 & -0.2338 & 0.0372 & 0 \end{bmatrix}.$$

Finally, it is noted that the coefficient for the 5th order term is 0. Adding t^5 to the set of functions we can project onto does not improve the result.

Now that we have our polynomial approximation, we plot both the original function and the approximation over the interval $[0, \pi]$. Figure 2 shows plots of both $\sin(t)$ and $p(t)$ over the interval. From this we see that they are very close in value. Figure 3 shows the difference between $\sin(t)$ and $p(t)$, which gives a better idea of what the differences are between the two functions over the given interval.

The Matlab code to solve this problem is:

```
%1.5.2 4 - see section 1.3.1 for the method to solve this
G = zeros(6,6); %initialize G matrix
fg = zeros(1,6); % initialize vector of inner products of f and g
syms t; %make t a symbol so we can integrate over it
for i=1:6
    fg(i) = int(t^(i-1)*sin(t),0,pi); %populate the inner products
    for j=1:6
        G(i,j) = int(t^(i+j-2),0,pi); %inner products for G
    end
end
alpha = fg/G; %coefficients for the polynomial
t = 0:.01:pi;
plot(t,sin(t));
V = fliplr(vander(t));
V = V(:,1:6);
hold on
plot(t,V*alpha','r')
xlabel('time')
ylabel('function value')
```

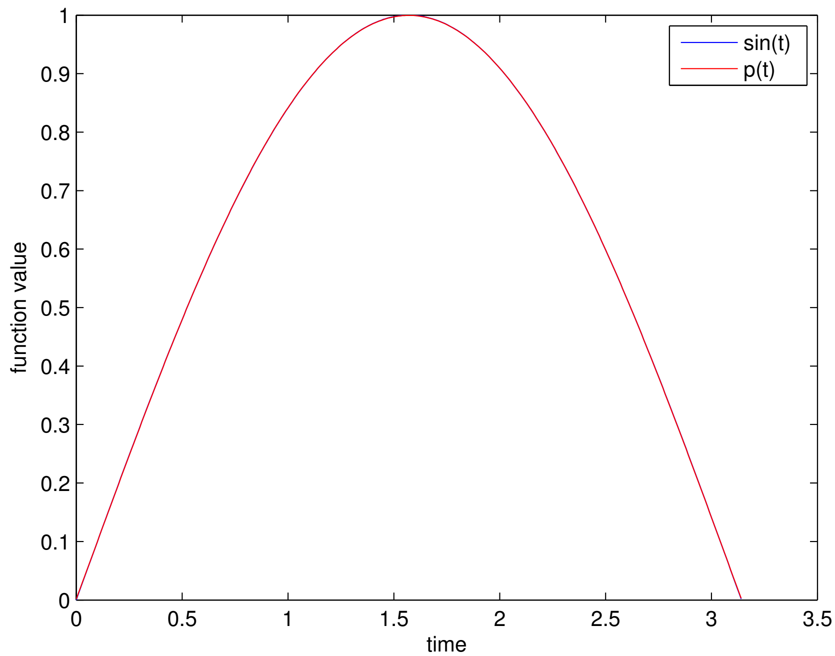

Figure 2: Plot for $t = 0$ to π of $\sin(t)$ and the fit $p(x)$ for 1.5.2 Exercise 4

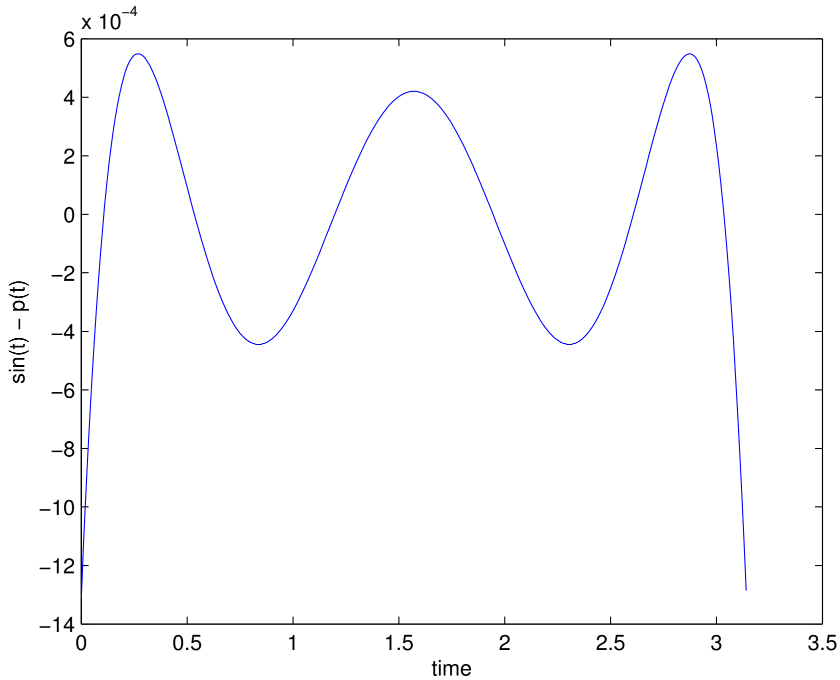

Figure 3: Plot for $t = 0$ to π of $\sin(t) - p(t)$
for exercise 4

```
legend('sin(t)','p(t)')

figure()
plot(t,sin(t)'-V*alpha')
xlabel('time')
ylabel('sin(t) - p(t)')
```

1.5.2 Exercise 5

Let $t = \text{linspace}(0, \pi, 10000)$ and $y = \sin(t)$. As in section 1.5.1, our solution for the optimal coefficients is $\alpha = (V^*V)^{-1}V^*y = V^{-r}y$ where V mapping $\mathbb{C}^6$ into $\mathbb{C}^{10000}$ is the Vandermonde matrix defined in (5.7) for $\lambda = t$ with t as defined above. We obtain

$$\alpha \approx \begin{bmatrix} 0.0013 & 0.9826 & 0.0545 & -0.2338 & 0.0372 & 0 \end{bmatrix}$$

To the precision we have presented the numbers, these coefficients are identical to the coefficients found in the previous problem (exercise 4).

The error d is given by $d^2 = \|y\|^2 - ((V^*V)^{-1}V^*y, V^*y)$. We obtain

$$d \approx 0.0369$$

Figures 4 and 5 depict plots of the functions $\sin(t)$ and $p(t)$, and the difference between the two functions, respectively. We note that these look the same as the plots in the previous problem because this problem is a discrete approximation of the continuous problem.

The following Matlab script solves the problem:

```
%1.5.2 5
t = linspace(0,pi,10000); %given
y = sin(t)'; %given (needs to be transposed for the dimensions to work)
v = fliplr(vander(t)); %put the vandermonde matrix in the right order
v = v(:,1:6); %take ony the columns we need
alpha = v\y; %solve for coefficients
d = norm(sin(t)'-v*alpha);
plot(t,sin(t))
hold on
plot(t,v*alpha,'r')
xlabel('time')
ylabel('function values')
legend('sin(t)','p(t)')
```

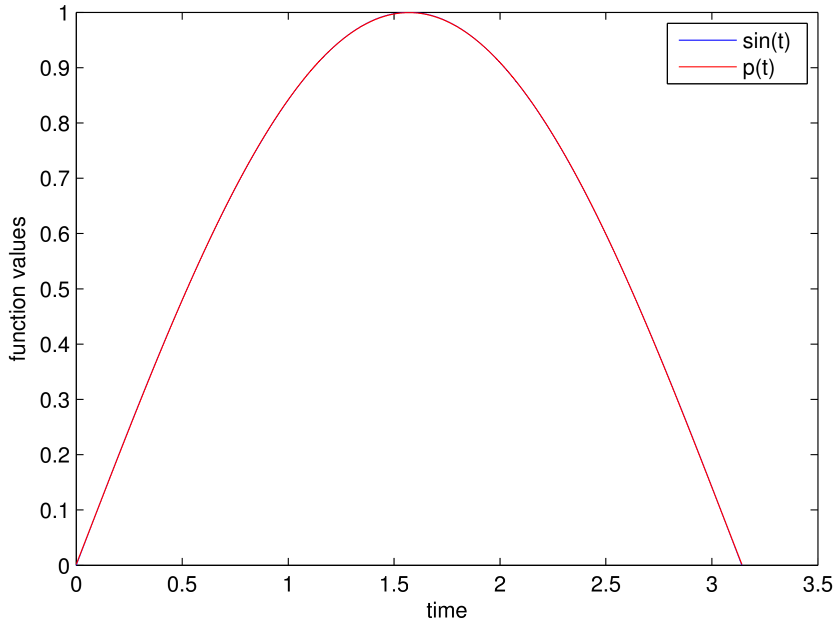

Figure 4: Plot for $t = 0$ to π of $\sin(t)$ and the fit $p(x)$ for 1.5.2 Exercise 5

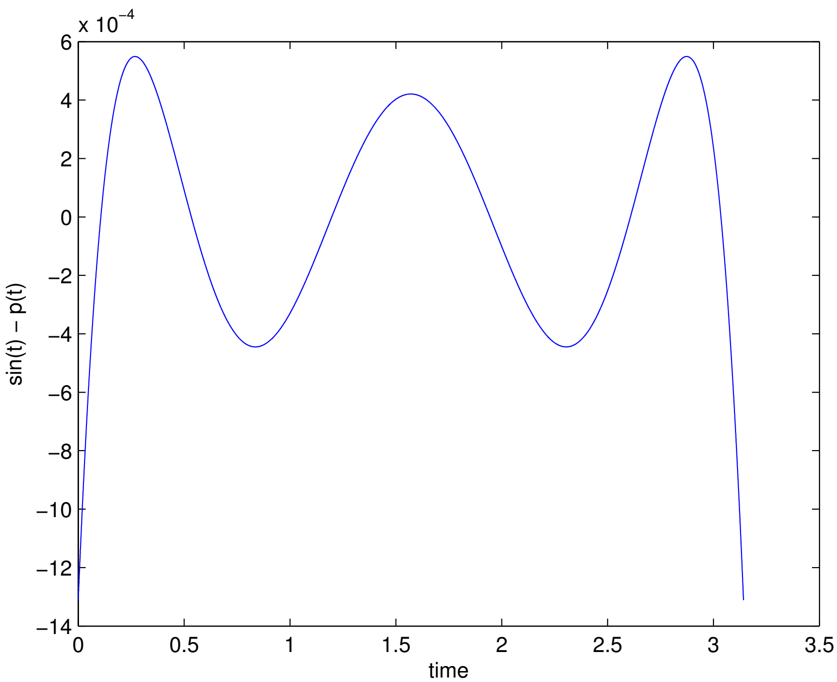

Figure 5: Plot for $t = 0$ to π of $\sin(t) - p(t)$ for 1.5.2 Exercise 5

```
figure()
plot(t,sin(t)'-v*alpha)
xlabel('time')
ylabel('sin(t) - p(t)')
```

1.5.2 Exercise 6

Because of the way this problem is set up, there is no unique answer, but there is a correct method to obtain the answer. We simply apply theorem 1.5.1 part (iii) to obtain $\hat{u} = (T^*T)^{-1}T^*y$ where $(T^*T)^{-1}T^*$ is the pseudo-inverse of T. We also have the error $d = \|y - T\hat{u}\|$. Plots will vary due to the random system matrices.

We have $\|y - Tx\|^2 = \sum_{j=1}^{12000} |y_j - (Tx)_j|^2 = \sum_{j=1}^{12000} |y_j - Ce^{At_j}x|^2$, which (for t_j on the interval $[0, 10]$) is the Riemann sum approximating $\int_0^{10} |y - Ce^{At}x|^2 dt$. The previous few problems hinted at this result.

The following Matlab script solves the problem:

```
%1.5.2 6
[A,B,C,D] = rmodel(40); %given
t = linspace(0,10,12000)'; %given (need transpose for correct dimension)
y = exp(-t).*cos(2*t)+ exp(-t/2).*sin(t); %given
T=[]; %initialize T matrix, then populate as given
for t = linspace(0,10,12000) %there is probably a more efficient way to do this
    T = [T; C*expm(A*t)];
end
x_hat = T\y; %find optimal solution
d = norm(y - T*x_hat);
```