Problem 3 { 30 pts}. Solve the following optimization problem

$$\begin{aligned} d^2 &= \inf\left\{\int_0^\infty \left|e^{-3t} - ae^{-t} - be^{-2t}\right|^2 dt : a \in \mathbb{C} \text{ and } b \in \mathbb{C}\right\} \\ &= \int_0^\infty \left|e^{-3t} - \alpha e^{-t} - \beta e^{-2t}\right|^2 dt \end{aligned}$$

In other words, find α, β and d^2.

$$\alpha = -\frac{3}{10} \quad \text{and} \quad \beta = \frac{6}{5} \quad \text{and} \quad d^2 = \frac{1}{600}$$

The inner product is

$$(f, g) = \int_0^\infty f(t)\overline{g(t)}dt$$

In particular, for $\Re(\lambda) < 0$ and $\Re(\mu) < 0$, we have

$$(e^{\lambda t}, e^{\mu t}) = \int_0^\infty e^{\lambda t} e^{\overline{\mu} t} = -\frac{1}{\lambda + \overline{\mu}}$$

Recall that

$$\begin{bmatrix}\alpha & \beta\end{bmatrix} = \begin{bmatrix}(e^{-t}, e^{-3t}) & (e^{-2t}, e^{-3t})\end{bmatrix} G^{-1}$$

where G is the Gram matrix formed by $\{e^{-t}, e^{-2t}\}$. In fact, the Gram matrix is

$$G = \begin{bmatrix}(e^{-t}, e^{-t}) & (e^{-t}, e^{-2t}) \\ (e^{-2t}, e^{-t}) & (e^{-2t}, e^{-2t})\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4}\end{bmatrix}$$

$$G^{-1} = \begin{bmatrix}18 & -24 \\ -24 & 36\end{bmatrix}$$

$$\begin{bmatrix}(e^{-t}, e^{-3t}) & (e^{-2t}, e^{-3t})\end{bmatrix} = \begin{bmatrix}\frac{1}{4} & \frac{1}{5}\end{bmatrix}$$

This readily implies that

$$\begin{bmatrix} \alpha & \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 18 & -24 \\ -24 & 36 \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} & \frac{6}{5} \end{bmatrix}$$

Finally,

$$\begin{aligned} d^2 &= \|e^{-3t}\|^2 - \begin{bmatrix} (e^{-t}, e^{-3t}) & (e^{-2t}, e^{-3t}) \end{bmatrix} G^{-1} \begin{bmatrix} (e^{-t}, e^{-3t}) \\ (e^{-2t}, e^{-3t}) \end{bmatrix} \\ &= \frac{1}{6} - \begin{bmatrix} -\frac{3}{10} & \frac{6}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \end{bmatrix} = \frac{1}{600} \end{aligned}$$