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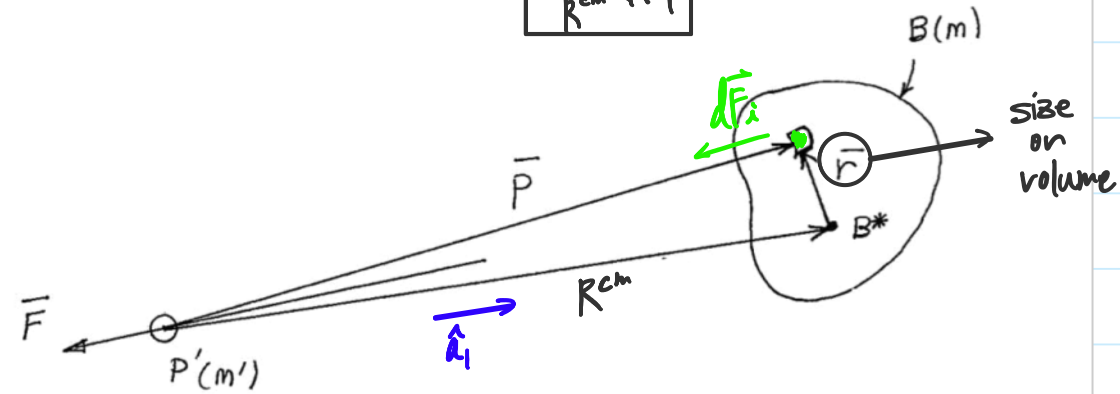
Gravity Expansion

reasonable assumption

$\boxed{\frac{r}{R^{cm}} \ll 1}$ if true $\rightarrow$ can approx



For $|\bar{r}| << |\bar{R}|$ useful form for general expression for $\bar{F}$

$$\boxed{\bar{F} = -G m' \int \bar{p}(\bar{p} \bullet \bar{p})^{-\frac{3}{2}} \upsilon \, d\tau}$$ exact gravity force

$\bar{p} = \bar{R} + \bar{r}$

$$\bar{F} = -G m' \int (\bar{R} + \bar{r}) \underbrace{\left[(\bar{R} + \bar{r}) \bullet (\bar{R} + \bar{r})\right]^{-\frac{3}{2}}}_{(R^2 + 2\bar{R}\cdot\bar{r} + r^2)^{-\frac{3}{2}}} \upsilon \, d\tau$$

$$\bar{F} = -G m' \int R \left(\frac{\bar{R}}{R} + \frac{\bar{r}}{R} \right) \frac{1}{R^3} \left(\frac{R^2}{R^2} + \frac{2\bar{R} \bullet \bar{r}}{R^2} + \frac{r^2}{R^2} \right)^{-\frac{3}{2}} \upsilon \, d\tau$$

Define $\frac{\bar{R}}{R} = \hat{a}_1$ $\qquad \frac{\bar{r}}{R} = \bar{q}$ (small magnitude)

$$\bar{F} = -\frac{G\,m'}{R^2}\int(\hat{a}_1 + \bar{q})\underbrace{\left(1 + 2\hat{a}_1 \bullet \bar{q} + q^2\right)^{-\frac{3}{2}}}_{\text{use binomial expansion}}\upsilon\, d\tau$$

R not necessarily constant but not involved in integration

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots\ldots..$$

brackets

$$\bar{F} = -\frac{G\,m'}{R^2}\int(\hat{a}_1 + \bar{q})\left[1 - \frac{3}{2}\left(2\hat{a}_1 \bullet \bar{q} + q^2\right) + q^2 + \frac{15}{8}\left((4\hat{a}_1 \bullet \bar{q})^2 + 4\hat{a}_1 \bullet \bar{q}\, q^2 + q^4\right) + \ldots\ldots..\right]\upsilon\, d\tau$$

Neglect terms of order q^3 and higher $\longrightarrow$ keep terms to second order!

3 2

$$\bar{F} = -\frac{G\,m'}{R^2}\int\Big\{\hat{a}_1\left[1 - \frac{3}{2}\left(2\hat{a}_1 \bullet \bar{q} + q^2\right) + \frac{15}{2}(\hat{a}_1 \bullet \bar{q})^2 + \ldots..\right. + \bar{q}\left[1 - 3\hat{a}_1 \bullet \bar{q} + \ldots\ldots\right]\Big\}\upsilon\, d\tau$$

1

Observe:

Term 1 $\int \bar{q}\,\upsilon\, d\tau \Rightarrow$ zero

def. of CM from basepoint

$$\bar{q} = \frac{\bar{r}}{R} \Longrightarrow \frac{1}{R}\int \bar{r}\,\upsilon\, d\tau = \text{zero}$$

from CM

$\hat{a}_1 = \frac{\bar{R}}{R}$

Term (2) $\int \hat{a}_1 \hat{a}_1 \bullet \bar{q} \ \upsilon \, d\tau$

$\hat{a}_1$ not involved in integration

$\hat{a}_1 \cdot \hat{a}_1 \cdot \cancel{\int \bar{q} \upsilon d\tau} = \text{zero}$ 0

Term (3) $\int \hat{a}_1 \ \upsilon \, d\tau$

$\hat{a}_1$ not involved in integrate

$\hat{a}_1 \int \upsilon d\tau = \hat{a}_1 \int dm = m \, \hat{a}_1$

total mass

$\Downarrow$

Substitute back $\bar{q} = \frac{\bar{r}}{R}$

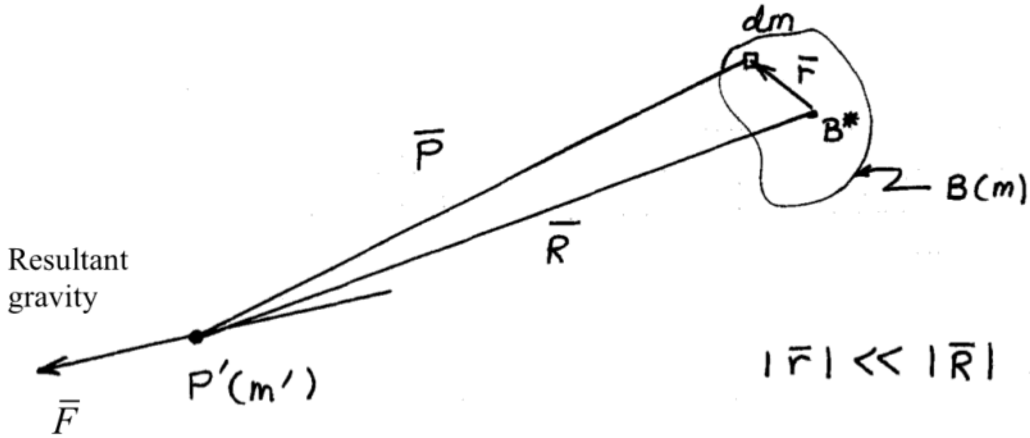
total mass

$$\bar{F} = -\frac{G\,m'\,m}{R^2}\hat{a}_1 + \frac{3}{2}\frac{G\,m'}{R^2}\left[\hat{a}_1 \int \left(\frac{r}{R}\right)^2 \upsilon\, d\tau - 5\hat{a}_1 \int \underbrace{\left(\hat{a}_1 \bullet \frac{\bar{r}}{R}\right)^2}_{} \upsilon\, d\tau + 2\int \frac{\bar{r}}{R}\hat{a}_1 \bullet \frac{\bar{r}}{R} \upsilon\, d\tau\right] + \ldots\ldots$$

Note: $\left(\hat{a}_1 \bullet \frac{\bar{r}}{R}\right)^2 = \left(\hat{a}_1 \bullet \frac{\bar{r}}{R}\right)\left(\hat{a}_1 \bullet \frac{\bar{r}}{R}\right) = \frac{\hat{a}_1 \bullet \bar{r}\,\bar{r} \bullet \hat{a}_1}{R^2}$

inertia moment is an important piece of gravity force

$$\bar{F} = -\frac{G\,m'\,m}{R^2}\hat{a}_1 + \underbrace{\frac{3}{2}\frac{G\,m'}{R^4}\left[\hat{a}_1 \int r^2 \upsilon\, d\tau - 5\hat{a}_1\hat{a}_1 \bullet \int \bar{r}\,\bar{r}\, \upsilon\, d\tau \bullet \hat{a}_1 + 2\int \bar{r}\,\bar{r}\; \upsilon\, d\tau \bullet \hat{a}_1\right]}_{\text{related to inertia integrals}} + \ldots\ldots$$



$$\bar{F} = -\frac{Gm'm}{R^2}\hat{a}_1 + \frac{3}{2}\frac{Gm'}{R^4}\Big[\hat{a}_1 \int r^2\, \upsilon\, d\tau - 5\hat{a}_1\hat{a}_1 \bullet \int \bar{r}\,\bar{r}\, \upsilon\, d\tau \bullet \hat{a}_1 + 2\int \bar{r}\,\bar{r}\, \upsilon\, d\tau \bullet \hat{a}_1\Big] + \ldots$$

Can relate integrals to inertia properties of *B*

$$\bar{\bar{I}}^{B/B^*} = \int \nu \left(\bar{\bar{U}} r^2 - \bar{r}\,\bar{r}\right) d\tau$$

$\hat{u}_1\hat{n}_1 + \hat{u}_2\hat{n}_2 + \hat{u}_3\hat{n}_3$

Observations:

1. Note: $tr\left(\bar{\bar{I}}\right) = I_{11} + I_{22} + I_{33}$

   For any vector basis $\hat{n}_i$

   $$tr\left(\bar{\bar{I}}\right) = \hat{n}_1 \bullet \bar{\bar{I}} \bullet \hat{n}_1 + \hat{n}_2 \bullet \bar{\bar{I}} \bullet \hat{n}_2 + \hat{n}_3 \bullet \bar{\bar{I}} \bullet \hat{n}_3$$

   $$\begin{aligned} tr\left(\bar{\bar{I}}\right) = \; & \hat{n}_1 \bullet \int \upsilon\left(\bar{\bar{U}} r^2 - \bar{r}\,\bar{r}\right) d\tau \bullet \hat{n}_1 \\ & + \hat{n}_2 \bullet \int \upsilon\left(\bar{\bar{U}} r^2 - \bar{r}\,\bar{r}\right) d\tau \bullet \hat{n}_2 \\ & + \hat{n}_3 \bullet \int \upsilon\left(\bar{\bar{U}} r^2 - \bar{r}\,\bar{r}\right) d\tau \bullet \hat{n}_3 \end{aligned}$$

   $$\begin{aligned} tr\left(\bar{\bar{I}}\right) = \int \Big[& \hat{n}_1 \bullet \bar{\bar{U}} r^2 \bullet \hat{n}_1 + \hat{n}_2 \bullet \bar{\bar{U}} r^2 \bullet \hat{n}_2 + \hat{n}_3 \bullet \bar{\bar{U}} r^2 \bullet \hat{n}_3 \\ & - \left(\hat{n}_1 \bullet \bar{r}\,\bar{r} \bullet \hat{n}_1 + \hat{n}_2 \bullet \bar{r}\,\bar{r} \bullet \hat{n}_2 + \hat{n}_3 \bullet \bar{r}\,\bar{r} \bullet \hat{n}_3\right) \Big] \upsilon\, d\tau \end{aligned}$$

   $\hat{n}_1\hat{n}_1 + \hat{n}_2\hat{n}_2 + \hat{n}_3\hat{n}_3$

   But $\quad \hat{n}_i \bullet \bar{\bar{U}} r^2 \bullet \hat{n}_i = 3r^2$

   $$tr\left(\bar{\bar{I}}\right) = \int 3r^2 - \left(r_1^2 + r_2^2 + r_3^2\right) \Big] \upsilon\, d\tau$$

   ➔ $$\boxed{tr\left(\bar{\bar{I}}\right) = \int 2r^2 \upsilon\, d\tau}$$

   ➔ $$\boxed{\int r^2 \upsilon\, d\tau = \frac{tr\left(\bar{\bar{I}}\right)}{2}}$$

2. $\bar{\bar{I}} = \int \left[\bar{\bar{U}} r^2 - \bar{r}\,\bar{r} \right] \upsilon \, d\tau$

$$\bar{\bar{I}} = \bar{\bar{U}} \int r^2 \, \upsilon \, d\tau - \underbrace{\int \bar{r}\,\bar{r} \, \upsilon \, d\tau}_{\text{other integral in } \bar{F}}$$

OR

$$\int \bar{r}\,\bar{r} \, \upsilon \, d\tau = \bar{\bar{U}} \boxed{\int r^2 \, \upsilon \, d\tau} - \bar{\bar{I}}$$

$$\boxed{\int \bar{r}\,\bar{r} \, \upsilon \, d\tau = \frac{\bar{\bar{U}} \, \mathrm{tr}(\bar{\bar{I}})}{2} - \bar{\bar{I}}}$$

Now return to gravity force!

$$\bar{F} = -\frac{G\,m'm}{R^2}\hat{a}_1 + \frac{3}{2}\frac{G\,m'}{R^4}\Big[\hat{a}_1 \int r^2\, \upsilon\, d\tau - 5\hat{a}_1\hat{a}_1 \bullet \int \bar{r}\,\bar{r}\,\upsilon\, d\tau \bullet \hat{a}_1 +$$
$$+ 2\int \bar{r}\,\bar{r}\ \ \upsilon\, d\tau \bullet \hat{a}_1\Big] + \ldots\ldots$$

Substitute for integrals

$$\bar{F} = -\frac{G\,m'm}{R^2}\hat{a}_1 + \frac{3}{2}\frac{G\,m'}{R^4}\Big[\hat{a}_1 \frac{tr\left(\bar{\bar{I}}\right)}{2} - 5\hat{a}_1\hat{a}_1 \bullet \left(\frac{\bar{\bar{U}}\,tr\left(\bar{\bar{I}}\right)}{2} - \bar{\bar{I}}\right) \bullet \hat{a}_1$$
$$+ 2\left(\frac{\bar{\bar{U}}\,tr\left(\bar{\bar{I}}\right)}{2} - \bar{\bar{I}}\right) \bullet \hat{a}_1\Big] + \ldots\ldots$$

$$\bar{F} = -\frac{G\,m'm}{R^2}\hat{a}_1 + \frac{3}{2}\frac{G\,m'}{R^4}\left[\hat{a}_1 \frac{tr\left(\bar{\bar{I}}\right)}{2} - 5a_1 \bullet \left(\frac{tr\left(\bar{\bar{I}}\right)}{2}\right) + 5\hat{a}_1\hat{a}_1 \bullet \bar{\bar{I}} \bullet \hat{a}_1\right.$$
$$\left. + \left(tr\left(\bar{\bar{I}}\right)\hat{a}_1 - 2\bar{\bar{I}} \bullet \hat{a}_1\right)\right] + \ldots\ldots$$

$$\boxed{\bar{F} = -\frac{Gm'm}{R^2}\hat{a}_1 - \frac{3}{2}\frac{Gm'}{R^4}\left[tr(\bar{\bar{I}}) - 5\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1\right]\hat{a}_1 - \frac{3\,Gm'}{R^4}\bar{\bar{I}} \cdot \hat{a}_1 + \ldots}$$

can write in form

$$\bar{F} = -\frac{G m' m}{R^2}\left(\hat{a}_1 + \sum_{i=2}^{\infty} \bar{f}^{(i)} \right)$$

where $\bar{f}^{(i)}$ are a collection of terms of degree i in $\frac{r}{R}$

$$\bar{f}^{(2)} = \frac{1}{mR^2}\left\{ \frac{3}{2}\left[tr(\bar{\bar{I}}) - 5\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1 \right]\hat{a}_1 + 3\bar{\bar{I}} \cdot \hat{a}_1 \right\}$$

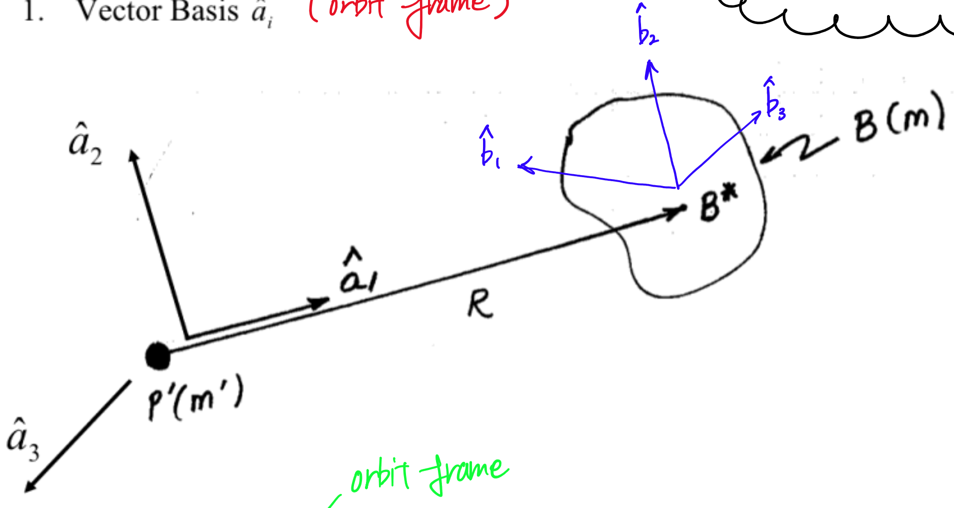
Useful approximations:

1. Particle term
2. 2nd order effects

To clarify the significance of $\bar{f}^{(2)}$, write it out in component format in different vector bases

spacecraft in orbit around earth

1. Vector Basis $\hat{a}_i$ (orbit frame)



orbit frame

Let $I_{ij} = \hat{a}_i \bullet \bar{\bar{I}} \bullet \hat{a}_j \qquad (j,k=1,2,3)$

$$\bar{f}^{(2)} = \frac{1}{mR^2}\left\{\frac{3}{2}\left[tr\left(\bar{\bar{I}}\right) - 5\,\hat{a}_1 \bullet \bar{\bar{I}} \bullet \hat{a}_1\right]\hat{a}_1 + 3\bar{\bar{I}} \bullet \hat{a}_1\right\}$$

$$\bar{f}^{(2)} = \frac{1}{mR^2}\left\{\frac{3}{2}\left[I_{11} + I_{22} + I_{33} - 5I_{11}\right]\hat{a}_1 + 3\left(I_{11}\hat{a}_1 + I_{21}\hat{a}_2 + I_{31}\hat{a}_3\right)\right\}$$

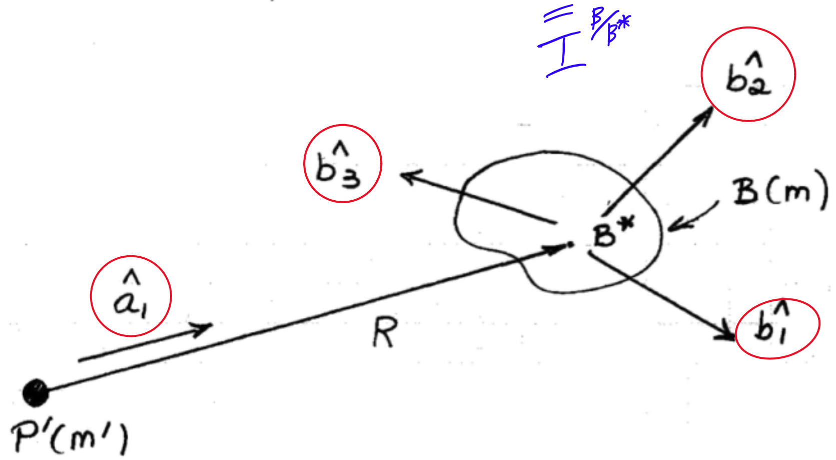
$$\bar{f}^{(2)} = \frac{3}{mR^2}\left\{\frac{1}{2}\left[I_{22} - I_{33} - 2I_{11}\right]\hat{a}_1 + I_{21}\hat{a}_2 + I_{31}\hat{a}_3\right\}$$

1) $\hat{a}_i$ (orbit frame) is used for evaluation

2) I are all time dep. → rates of change for $\dot{I}$?

2. Vector Basis $\hat{b}_i$ ← Body-fixed

$\overline{\overline{I}}^{B/B^*}$



Let $\hat{b}_i$ be parallel to central principal axes

$$\boxed{I_j = \hat{b}_j \bullet \overline{\overline{I}}^{B/B^*} \bullet \hat{b}_j}$$

↳ introduce kinematic variables into dynamic model

$C_{ij} = \hat{a}_i \bullet \hat{b}_j$ →

${}^{A}C^{B}$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{a}_1$	C_{11}	C_{12}	C_{13}
$\hat{a}_2$	C_{21}	C_{22}	C_{23}
$\hat{a}_3$	C_{31}	C_{32}	C_{33}

orientation must be incorporated in the gravity expression

↳ changes gravity torque

$$\overline{f}^{(2)} = \frac{1}{mR^2}\left\{\frac{3}{2}\left[tr\left(\overline{\overline{I}}\right) - 5\,\hat{a}_1 \bullet \overline{\overline{I}} \bullet \hat{a}_1\right]\hat{a}_1 + 3\overline{\overline{I}} \bullet \hat{a}_1\right\}$$

↳ I's are constant ! b/c in $\hat{b}_i$

$tr(\bar{\bar{I}}) = I_1 + I_2 + I_3$ (invariance) always the same

$$\hat{a}_1 \bullet \bar{\bar{I}} \bullet \hat{a}_1 = \hat{a}_1 \bullet \left[\hat{b}_1 I_1 \hat{b}_1 + \hat{b}_2 I_2 \hat{b}_2 + \hat{b}_3 I_3 \hat{b}_3 \right] \bullet \hat{a}_1$$

$$= I C_{11}^2 + I_2 C_{12}^2 + I_3 C_{13}^2$$

$$\bar{f}^{(2)} = \frac{1}{mR^2} \left\{ \frac{3}{2} \left[I_1 + I_2 + I_3 - 5\left(I_1 C_{11}^2 + I_2 C_{12}^2 + I_3 C_{13}^2 \right) \right] \hat{a}_1 + \right.$$
$$\left. + 3\left(I_1 C_{11} \hat{b}_1 + I_2 C_{12} \hat{b}_2 + I_3 C_{13} \hat{b}_3 \right) \right\}$$

$$\boxed{C_{13}\hat{a}_1 + C_{23}\hat{a}_2 + C_{33}\hat{a}_3}$$

$$\boxed{C_{12}\hat{a}_1 + C_{22}\hat{a}_2 + C_{32}\hat{a}_3}$$

$$\boxed{C_{11}\hat{a}_1 + C_{21}\hat{a}_2 + C_{31}\hat{a}_3}$$

$$\boxed{\begin{aligned} \bar{f}^{(2)} = \frac{3}{mR^2} \Big\{ & \frac{1}{2} \left[I_1 \left(1 - 3C_{11}^2 \right) + I_2 \left(1 - 3C_{12}^2 \right) + I_3 \left(1 - 3C_{13}^2 \right) \right] \hat{a}_1 + \\ & + \left[I_1 C_{21} C_{11} + I_2 C_{22} C_{12} + I_3 C_{23} C_{13} \right] \hat{a}_2 \\ & + \left[I_1 C_{31} C_{11} + I_2 C_{32} C_{12} + I_3 C_{33} C_{13} \right] \hat{a}_3 \Big\} \end{aligned}}$$

Note: I's here are CONSTANT

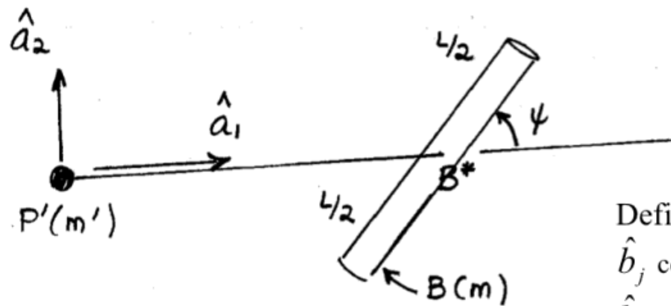
1) Inertia elements are in terms of $\hat{b}_i$

2) Inertia elements are constant

Example

body w/ same inertia characteristic

Return to "rod" satellite but consider a more general orientation (restricted to 2D)



$$\bar{\bar{I}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{mL^2}{12} & 0 \\ 0 & 0 & \frac{mL^2}{12} \end{bmatrix}$$

Define:

$\hat{b}_j$ central principal axes

$\hat{b}_j = \hat{a}_j \quad @\, \psi = 0^\circ$

$$\bar{\bar{I}}^{B/B^*} = \frac{mL^2}{12}\left(\hat{b}_2\hat{b}_2 + \hat{b}_3\hat{b}_3\right)$$

${}^AC^B$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{a}_1$	c_ψ	$-s_\psi$	0
$\hat{a}_2$	s_ψ	c_ψ	0
$\hat{a}_3$	0	0	1

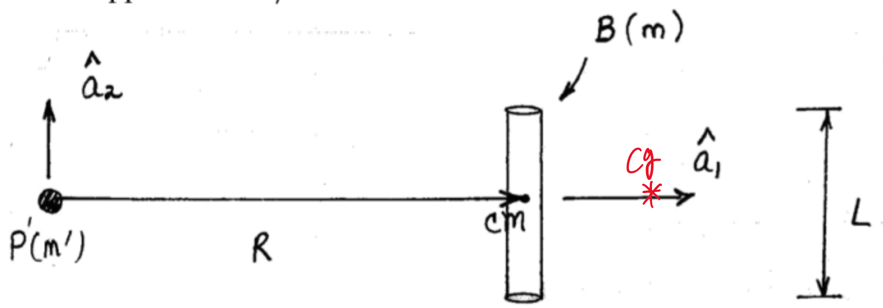
$$\bar{f}^{(2)} = \frac{3}{mR^2}\left\{ \frac{1}{2}\left[0 + \frac{mL^2}{12}\left(1 - 3s_\psi^2\right) + \frac{mL^2}{12}(1-0)\right]\hat{a}_1 + \left[0 + \frac{mL^2}{12}c_\psi(-s_\psi) + 0\right]\hat{a}_2 + 0\hat{a}_3 \right\}$$

$$\bar{F} = -\frac{Gm'm}{R^2}\left\{\hat{a}_1 + \overbrace{\frac{L^2}{8R^2}\left(2-3s_\psi^2\right)\hat{a}_1 - \frac{L^2}{8R^2}s_{2\psi}\hat{a}_2}^{\bar{f}^{(2)}}\right\}$$

particle @ center of mass

add effect due to rod shape. positive or negative depending on attitude.

What happens when $\psi = 90^\circ$?



$\psi = 90^\circ$

$$\bar{F} = -\frac{Gm'm}{R^2}\left\{\hat{a}_1 + \frac{L^2}{8R^2}(-1)\hat{a}_1\right\}$$

$$\bar{F} = -\frac{Gm'm}{R^2}\left(1-\frac{L^2}{8R^2}\right)\hat{a}_1$$

cm ≠ cg : force is less than if all the mass is @ cm

Recall cg location

$$R^{cg} = \left[\frac{Gm'm}{\left|\bar{F}\right|}\right]^{\frac{1}{2}} \longrightarrow \left|\bar{F}\right| = \frac{Gm'm}{R^2}\left(\frac{8R^2-L^2}{8R^2}\right)$$

$$R^{cg} = \left[\frac{8R^4}{\left(8R^2 - L^2\right)} \right]^{\frac{1}{2}}$$

If $L > R$ assumptions associated with approximation violated

$R = 4L$ (assumption)

$$R^{cg} = \left(\frac{128}{127}\right)^{1/2} R$$

Note: $\psi = 0^\circ \longrightarrow$ orientation stable