Lecture: Jordan Form and Convergence Lemma

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Jordan form of a matrix $M \in \mathbb{R}^{n\times n}$.

$$M = TJT^{-1}$$

Jordan blocks

$$J = \begin{bmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m \end{bmatrix} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{\kappa_i \times \kappa_i} \quad [\lambda], \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \quad \ldots$$

- An eigenvalue is **simple** if it has single Jordan block of size 1. $\kappa_i = 1$
- If all eigenvalues are simple, the matrix is called ***diagonalizable***.
- All symmetric matrices are diagonalizable.

Application into convergence analysis

$$M^t = TJ^tT^{-1} = T \begin{bmatrix} J_1^t & 0 & \cdots & 0 \\ 0 & J_2^t & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & J_m^t \end{bmatrix} T^{-1}$$

- Convergence of Jordan blocks J_i^t

$$\begin{bmatrix} \lambda_i & 1 & \cdots & 0 \\ 0 & \lambda_i & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & \lambda_i \end{bmatrix}_{\kappa_i \times \kappa_i}^{t} = \begin{bmatrix} \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \binom{t}{2}\lambda_i^{t-2} & \cdots & \binom{t}{\kappa_i-1}\lambda_i^{t-\kappa_i+1} \\ 0 & \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} & \cdots & \binom{t}{n-2}\lambda_i^{t-\kappa_i+2} \\ & & \ddots & \ddots & \vdots \\ & & & \lambda_i^t & \binom{t}{1}\lambda_i^{t-1} \\ & & & & \lambda_i^t \end{bmatrix}$$

	$\lvert\lambda_i\rvert < 1$	$\lvert\lambda_i\rvert > 1$	$\lvert\lambda_i\rvert = 1$		
			$\kappa_i \geq 2$	$\kappa_i = 1$ ($J_i^t = \lambda_i^t$)	
				$\lambda_i \neq 1$	$\lambda_i = 1$
$\lim_{t\to\infty} J_i^t =$	0	∞	∞	oscillation	1

$$\lim_{t\to\infty} M^t = \lim_{t\to\infty} TJ^tT^{-1} = \begin{cases} \mathbf{0} & \text{if the spectral radius of M is strictly less than 1} \\ \textbf{constant} & \text{if 1 is the largest eigenvalue in magnitude with Jordan block size 1.} \\ \text{not converge.} & \text{otherwise} \end{cases}$$

Convergence of Linear Time-Invariant System $x(t+1) = Ax(t)$

$$\lim_{t\to\infty} A^t = \begin{cases} 0 & \text{if the spectral radius of M is strictly less than 1} \\ A^* & \text{if } \begin{cases} \bullet \text{ 1 is a simple eigenvalue;} \\ \bullet \text{ 1 is the largest eigenvalue in magnitude.} \\ \bullet \text{ all the other eigenvalues are with magnitude strictly less than 1} \end{cases} \\ \text{not converge.} & \text{otherwise} \end{cases}$$

Further Analysis under these eigenvalue conditions.

Consensus

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\text{eig }(\mathrm{A}) = \{1, 0.5643, 0.1477, 0\}$$

Consensus for global average

$$A = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

$$\text{eig }(\mathrm{A}) = \{1, 0.833, 0.75, 0\}$$

Consensus for convex combination

$$A = \begin{bmatrix} \frac{0.39}{0.44} & \frac{0.05}{0.44} & 0 & 0 \\ \frac{0.05}{0.2} & \frac{0.06}{0.2} & \frac{0.05}{0.2} & \frac{0.04}{0.2} \\ 0 & \frac{0.05}{0.24} & \frac{0.15}{0.24} & \frac{0.04}{0.24} \\ 0 & \frac{0.04}{0.12} & \frac{0.04}{0.12} & \frac{0.04}{0.12} \end{bmatrix}$$

$$\text{eig }(\mathrm{A}) = \{1, 0.847, -0.019, 0.267\}$$

Under consensus algorithm, one has $x(t) \to A^* x(0)$

What is A^* ? Will the consensus be reached?

Lemma (Convergence):

Suppose $A \in \mathbb{R}^{n\times n}$ is such that

- 1 is a simple eigenvalue
- 1 is the largest eigenvalue in magnitude.
- all the other eigenvalues are with magnitude strictly less than 1

Then $A^t \to vw'$ as fast as $|\lambda_2|^t \to 0$

where v, w are right and left eigenvectors of *A* corresponding to 1 and $w'v = 1$

λ_2 denotes the 2nd largest eigenvalue of A in magnitude.

Proof: Since 1 is an **simple** eigenvalue, and **strictly larger** than other eigenvalues,

then the Jordan form of A is

$$A = T\begin{bmatrix}1 & 0\\ 0 & B\end{bmatrix}T^{-1}$$

where the spectral radius of B is less than 1.

Write $T^{-1} = \begin{bmatrix}w'\\ \bar{T}_w\end{bmatrix}$ $\quad T = \begin{bmatrix}v & \bar{T}_v\end{bmatrix}$

Since $T^{-1}T = I$ $\quad w'v = 1$

$$AT = T\begin{bmatrix}1 & 0\\ 0 & B\end{bmatrix} \qquad Av = v$$

$$T^{-1}A = \begin{bmatrix}1 & 0\\ 0 & B\end{bmatrix}T^{-1} \qquad w'A = w'$$

$$\lim_{t\to\infty} A^t = T\lim_{t\to\infty}\begin{bmatrix}1 & 0\\ 0 & B^t\end{bmatrix}T^{-1}$$

$$= T\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}T^{-1} = vw'$$

$B^t \to 0$ as fast as $\rho(B)^t \to 0$,namely $|\lambda_2|^t \to 0$

Analysis of Distributed Consensus

$$x(t+1) = Ax(t)$$

$$\lim_{t\to\infty} A^t = vw' \qquad Av = v \qquad w'A = w' \qquad w'v = 1$$

- **Consensus:** A is *row stochastic*, $A\mathbf{1} = \mathbf{1}$ $\quad v = \mathbf{1}$ $\quad$ Choose w such that $w'\mathbf{1} = 1$

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\lim_{t\to\infty} A^t = \mathbf{1}w'$$

$$\lim_{t\to\infty} x(t) = \mathbf{1}w'x(0) = (w'x(0))\mathbf{1}$$

- **Consensus for Global Average** $\frac{1}{n}\mathbf{1}'x(0)$: $\quad$ A is *doubly stochastic*, $A\mathbf{1} = \mathbf{1},\ \mathbf{1}'A = \mathbf{1}'$

$$A = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Choose v, w as $v = \mathbf{1}, w = \frac{1}{n}\mathbf{1}$

$$\lim_{t\to\infty} A^t = \frac{1}{n}\mathbf{1}\mathbf{1}'$$

global average

$$\lim_{t\to\infty} x(t) = \frac{1}{n}\mathbf{1}\mathbf{1}'x(0) = (\frac{1}{n}\sum_{i=1}^{n} x_i(0))\mathbf{1}$$

- **Consensus for convex combination** $\gamma'x(0)$ **:**

$$A = \begin{bmatrix} \frac{0.39}{0.44} & \frac{0.05}{0.44} & 0 & 0 \\ \frac{0.05}{0.2} & \frac{0.06}{0.2} & \frac{0.05}{0.2} & \frac{0.04}{0.2} \\ 0 & \frac{0.05}{0.24} & \frac{0.15}{0.24} & \frac{0.04}{0.24} \\ 0 & \frac{0.04}{0.12} & \frac{0.04}{0.12} & \frac{0.04}{0.12} \end{bmatrix}$$

$$A\mathbf{1} = \mathbf{1} \qquad \gamma'A = \gamma' \qquad \gamma'\mathbf{1} = 1$$

$$\lim_{t\to\infty} A^t = \mathbf{1}\gamma' \qquad \lim_{t\to\infty} x(t) = \mathbf{1}\gamma'x(0)$$

Convergence to Consensus $\lim_{t \to \infty} A^t = \mathbf{1} w'$

Eigenvalue/Eigenvector Properties *of A*

$A\mathbf{1} = \mathbf{1}$

1 is the largest eigenvalue in magnitude. $|\lambda(A)| \leq 1$

1 is a simple eigenvalue

all the other eigenvalues are with magnitude strictly less than 1

Gershgorin Circle Theorem

A is row stochastic

Analysis on Non-Negative Matrices
Perron-Frobenius Theorem

- One additionally requires $w = \frac{1}{n}\mathbf{1}$ for consensus to global average $\frac{1}{n}\mathbf{1}' x(0)$

- One additionally requires $w = \gamma$ for consensus to $\gamma' x(0)$