P1: D BP

2020年1月27日 月曜日　　午後0:33

Euler Parameters (Quaternions)

not a minimal set

set 4 variables

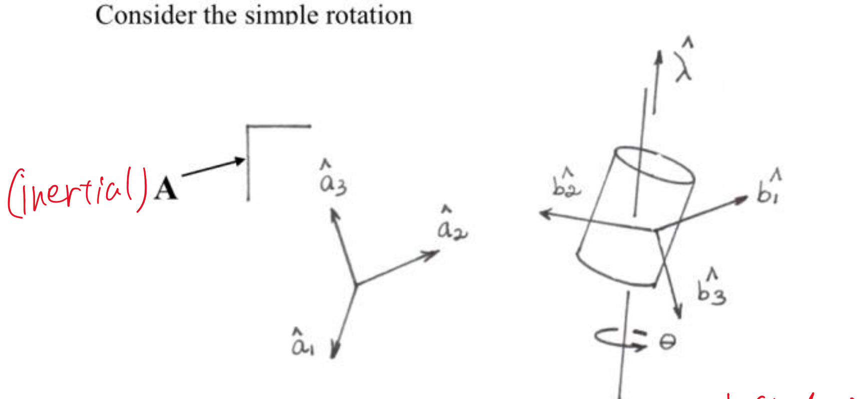
Set of 4 variable with some of same advantages as C_{ij} but only redundant parameter

[Interesting note: used on Skylab and Shuttle
Hughes (p. 29) "computationally superior / best overall choice"
Not in most modern dynamics texts
In Whittaker (1973)]

rigid body → 3 DOF

→ can only have 3 independent eqns.

Consider the simple rotation



4 variables $\underbrace{\varepsilon_1, \varepsilon_2, \varepsilon_3}_{\text{components of a 3-vector}}, \varepsilon_4$ ← scalar

$q = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4] \rightarrow$ tensor 4D quality

$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$

$\bar{\varepsilon} = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3] \triangleq \hat{\lambda} \sin\frac{\theta}{2}$

$\varepsilon_4 \triangleq \cos\frac{\theta}{2}$ euler scalar

Note: $|\varepsilon_i| \le 1$

Observe: 3 DOF and 4 variables

One constraint equation → $\boxed{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1}$

If we know the Euler parameters describing orientation of $\hat{b}_i$ with respect to their original orientations $\hat{a}_i$, can determine the equivalent single rotation $(\hat{\lambda}, \theta)$

$$\hat{\lambda} = \frac{\bar{\varepsilon}}{\sin(\theta/2)} = \frac{\bar{\varepsilon}}{[1-\cos^2\theta/2]^{\frac{1}{2}}} = \frac{\bar{\varepsilon}}{(1-\varepsilon_4^2)^{1/2}}$$

$$\theta = 2\arccos(\varepsilon_4) \qquad \hat{\lambda} = \frac{\varepsilon_1\hat{a}_1 + \varepsilon_2\hat{a}_2 + \varepsilon_3\hat{a}_3}{(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2)^{1/2}}$$

} instantaneous information

What additional quantities might we want to express in terms of Euler parameters?

(1) SRT → Euler parameters expressed as SRT

$$\bar{b} = \bar{a}\cos\theta - \bar{a}\times\hat{\lambda}\sin\theta + \bar{a}\bullet\hat{\lambda}\hat{\lambda}(1-\cos\theta)$$

Note:

$$(\bar{a}\bullet\hat{\lambda})\hat{\lambda} = \cancel{(\hat{\lambda}\bullet\hat{\lambda})}^{1}\bar{a} + \hat{\lambda}\times(\hat{\lambda}\times\bar{a})$$

C A B   A B C A   B   C

$$\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{C}\bullet\bar{A})\bar{B} - (\bar{A}\bullet\bar{B})\bar{C}$$

$$\hat{\lambda}\times(\hat{\lambda}\times\hat{a}) = (\hat{a}\bullet\hat{\lambda})\hat{\lambda} - (\hat{\lambda}\cdot\hat{\lambda})\hat{a}$$

$$\bar{b} = \bar{a}\cos\theta - \bar{a}\times\hat{\lambda}\sin\theta + \bar{a}(1-\cos\theta) + \boxed{\hat{\lambda}\times\left(\hat{\lambda}\times\bar{a}\right)}(1-\cos\theta)$$

$$= -\bar{a}\times\hat{\lambda}\underbrace{\boxed{\sin\theta}}_{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} + \bar{a} + \hat{\lambda}\times\left(\hat{\lambda}\times\bar{a}\right)\boxed{\underbrace{(1-\cos\theta)}_{2\cos^{2}\left(\frac{\theta}{2}\right)}}$$

$$2\left(1-\cos^{2}\left(\frac{\theta}{2}\right)\right)$$

$$= -\bar{a}\times 2\bar{\varepsilon}\cos\left(\frac{\theta}{2}\right) + \bar{a} + \hat{\lambda}\times\left(\hat{\lambda}\times\bar{a}\right)2\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)$$

$$\boxed{\bar{b} = \bar{a} - 2\varepsilon_4\,\bar{a}\times\bar{\varepsilon} + 2\bar{\varepsilon}\times\left(\bar{\varepsilon}\times\bar{a}\right)}$$

SRT

* described SRT in terms of euler parameters

${}^{A}\bar{\varepsilon}^{B}$, ${}^{A}\varepsilon_4^{B}$

(2) direction cosines

Example

$$C_{22} = \cos\theta + \lambda_2^2(1-\cos\theta)$$

$$= \left(2\cos^{2}\left(\frac{\theta}{2}\right)-1\right) + \lambda_2^2\underbrace{\left[1-\left(2\cos^{2}\left(\frac{\theta}{2}\right)-1\right)\right]}_{2\left(1-\cos^{2}\left(\frac{\theta}{2}\right)\right)}$$

$$C_{22} = 2\varepsilon_4^2 - 1 + \lambda_2^2\left(2\sin^{2}\left(\frac{\theta}{2}\right)\right)$$

$$= 2\varepsilon_4^2 - 1 + 2\varepsilon_2^2$$

$$= 2\left(\varepsilon_2^2 + \varepsilon_4^2\right) - 1$$

$$= 2\left(1 - \varepsilon_1^2 - \varepsilon_3^2\right) - 1$$

$$C_{22} = 1 - 2\varepsilon_1^2 - 2\varepsilon_3^2$$

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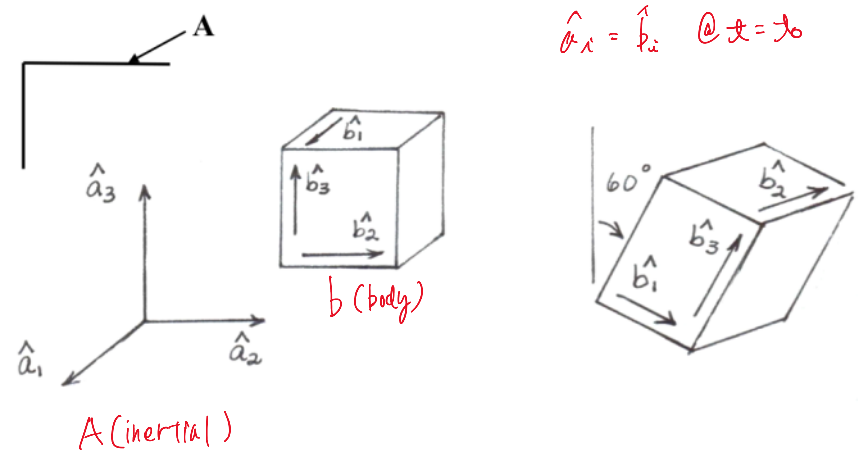
$$
\begin{aligned}
{}^{A}C_{11}^{B} &= 1-2\,{}^{AB}\varepsilon_2^2-2\,{}^{AB}\varepsilon_3^2\\
C_{12} &= 2\left(\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_4\right)\\
C_{13} &= 2\left(\varepsilon_3\varepsilon_1+\varepsilon_2\varepsilon_4\right)\\
C_{21} &= 2\left(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4\right)\\
C_{22} &= 1-2\varepsilon_3^2-2\varepsilon_1^2\\
C_{23} &= 2\left(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4\right)\\
C_{31} &= 2\left(\varepsilon_3\varepsilon_1-\varepsilon_2\varepsilon_4\right)\\
C_{32} &= 2\left(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4\right)\\
C_{33} &= 1-2\varepsilon_1^2-2\varepsilon_2^2
\end{aligned}
$$

9 coupled quadratic equations in 4 unknowns

$$
\boxed{
\begin{aligned}
\varepsilon_1 &= \frac{C_{32}-C_{23}}{4\varepsilon_4}\\
\varepsilon_2 &= \frac{C_{13}-C_{31}}{4\varepsilon_4}\\
\varepsilon_3 &= \frac{C_{21}-C_{12}}{4\varepsilon_4}\\
\varepsilon_4 &= \frac{1}{2}\left(1+C_{11}+C_{22}+C_{33}\right)^{\frac{1}{2}}
\end{aligned}
}
$$

Example Assume that we have knowledge of initial and the (desired) final orientation, Determine the equivalent single rotation

[Might represent, in simplified form, a "real" spacecraft
Problem: know current s/c orientation
assume you want to fire main thrusters to change orbit (main engines, no gimbal); need to reorient vehicle to fire in proper direction (change attitude with attitude control thrusters)]



How to visualize this orientation change:

not unique

$$\text{Sum of two rotations} \begin{cases} 90^\circ \text{ about } \hat{a}_3 & -270^\circ \text{ about } \hat{a}_3 \\ 60^\circ \text{ about } \hat{b}_2 & -300^\circ \text{ about } \hat{b}_2 \end{cases}$$

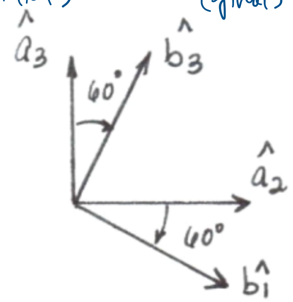
(Great for analysis; great to visualize → will not actually want to implement change this way probably. Is it possible in one burn?)

How do we describe this in terms of a single rotation?

Equivalent single rotation $\left(\hat{\lambda}, \theta\right)$ to do this job?

Note: in this simple problem, possible to construct ${}^{A}C^{B}$ by inspection

(initial) (final)



orthogonality condition

${}^{A}C^{B}$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{a}_1$	0	-1	0
$\hat{a}_2$	$\cos 60^\circ$	0	$\sin 60^\circ$
$\hat{a}_3$	$-\sin 60^\circ$	0	$\cos 60^\circ$

sqr sum = 1

sqr sum = 1

$$\varepsilon_4 = \frac{1}{2}\left(1+C_{11}+C_{22}+C_{33}\right)^{\frac{1}{2}} = \frac{1}{2}\left(1+\cos 60^\circ\right)^{\frac{1}{2}} = 0.6124$$

$$\varepsilon_1 = \frac{C_{32}-C_{23}}{4\varepsilon_4} = \frac{0-\left(\sin 60^\circ\right)}{4\varepsilon_4} = -0.3536$$

$$\varepsilon_2 = \frac{C_{13}-C_{31}}{4\varepsilon_4} = \frac{0-\left(-\sin 60^\circ\right)}{4\varepsilon_4} = 0.3536$$

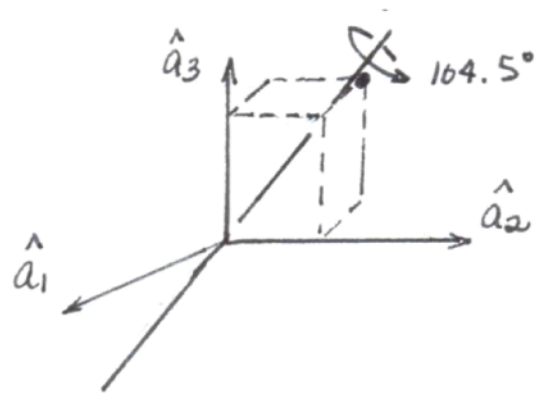
$$\varepsilon_3 = \frac{C_{21}-C_{12}}{4\varepsilon_4} = \frac{\cos 60^\circ-(-1)}{4\varepsilon_4} = 0.6124$$

$$\therefore \bar{\varepsilon} = -0.3536\,\hat{a}_1 + 0.3536\,\hat{a}_2 + 0.6124\,\hat{a}_3$$

$\Downarrow$ can be both $\hat{a}_i$ or $\hat{b}_i$

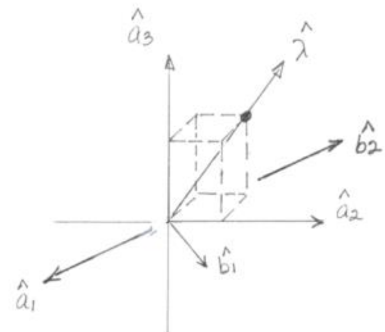
$$\hat{\lambda} = \frac{\varepsilon_1\hat{a}_1+\varepsilon_2\hat{a}_2+\varepsilon_3\hat{a}_3}{\left(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2\right)^{\frac{1}{2}}} = -\frac{1}{\sqrt{5}}\hat{a}_1 + \frac{1}{\sqrt{5}}\hat{a}_2 + \sqrt{\frac{3}{5}}\,\hat{a}_3$$

$$\theta = 2\cos^{-1}\left(\varepsilon_4\right) = 104.5^\circ$$



May or may not be possible (or desirable) to implement this way

Class Example

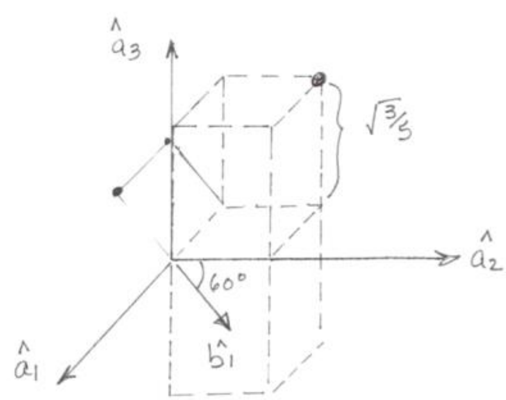


$$\hat{\lambda} = -\frac{1}{\sqrt{5}}\hat{a}_1 + \frac{1}{\sqrt{5}}\hat{a}_2 + \sqrt{\frac{3}{5}}\hat{a}_3 = -\frac{1}{\sqrt{5}}\hat{a}_1 + \hat{\lambda}_p$$

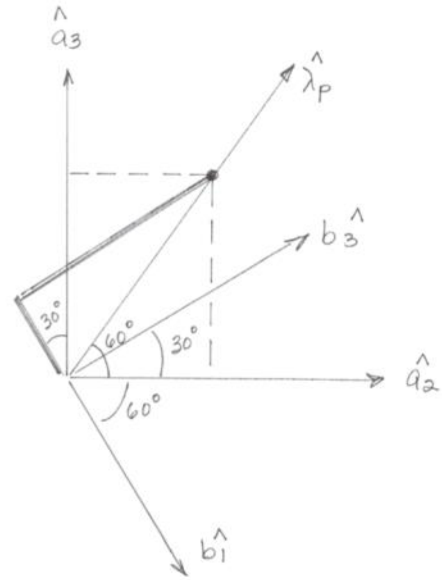
Should possess equal measure numbers in unit vectors $\hat{b}$

Note: $\hat{\lambda} = -\frac{1}{\sqrt{5}}\hat{b}_1 + \frac{1}{\sqrt{5}}\hat{b}_2 + \sqrt{\frac{3}{5}}\hat{b}_3 = \frac{1}{\sqrt{5}}\hat{b}_2 + \hat{\lambda}_p$

Since $\hat{b}_2 = -\hat{a}_1$ really just check $\hat{\lambda}_p$



$$\hat{\lambda}_p = -\frac{1}{\sqrt{5}}\hat{b}_1 + \sqrt{\frac{3}{5}}\hat{b}_3 = \frac{1}{\sqrt{5}}\hat{a}_2 + \sqrt{\frac{3}{5}}\hat{a}_3$$



$$\begin{aligned}\hat{\lambda}_p &= -\frac{1}{\sqrt{5}}\hat{b}_1 + \sqrt{\frac{3}{5}}\hat{b}_3 \\ &= -\frac{1}{\sqrt{5}}\left(\cos 60^\circ\,\hat{a}_2 - \sin 60^\circ\,\hat{a}_3\right) + \sqrt{\frac{3}{5}}\left(\cos 60^\circ\,\hat{a}_3 + \sin 60^\circ\,\hat{a}_2\right) \\ &= \left(-\frac{1}{\sqrt{5}}\cos 60^\circ + \sqrt{\frac{3}{5}}\sin 60^\circ\right)\hat{a}_2 + \left(\frac{1}{\sqrt{5}}\sin 60^\circ + \sqrt{\frac{3}{5}}\cos 60^\circ\right)\hat{a}_3 \\ &= \frac{1}{\sqrt{5}}\hat{a}_2 + \sqrt{\frac{3}{5}}\hat{a}_3 \qquad \checkmark\end{aligned}$$