College of Engineering
School of Aerospace Engineering

AE6230: Structural Dynamics

Problem Set 1

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I Problem One

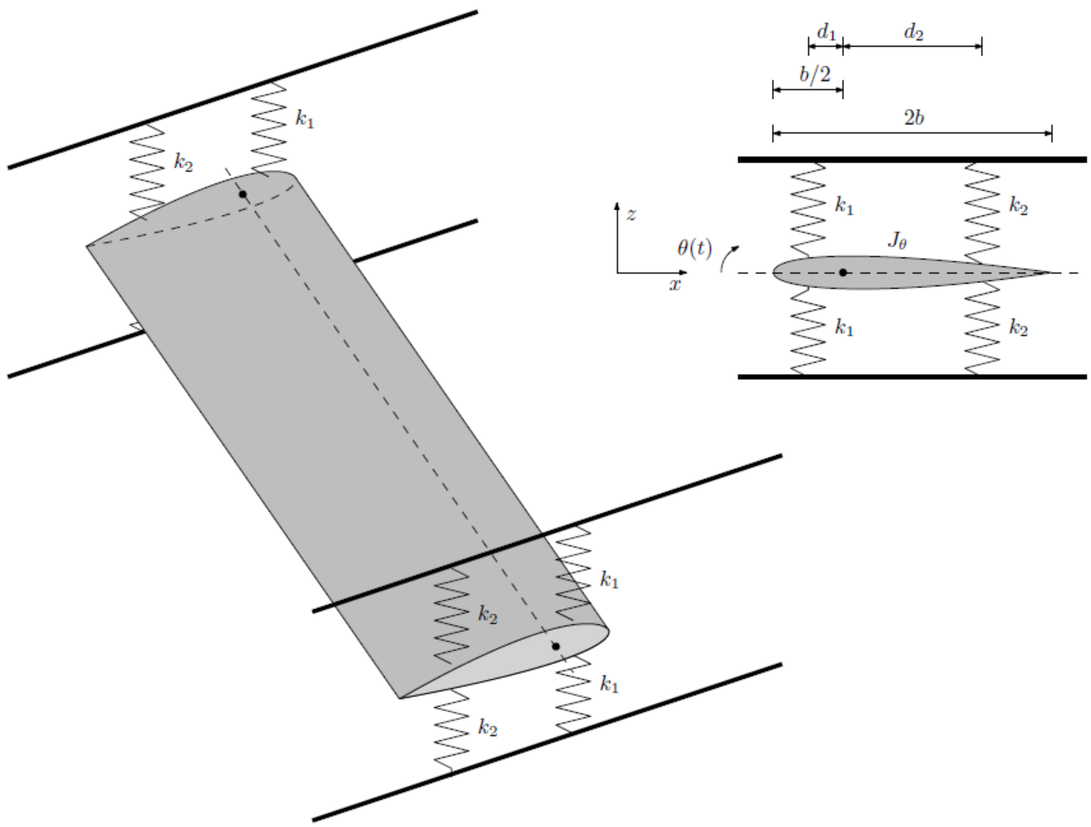

Figure 1: Schematic of wind-tunnel wing model.

Consider a uniform rigid wing mounted in a wind-tunnel test section (Fig. 1). The wing can pitch about the quarter-chord axis and the pitch motion is restrained by four springs on each end (near the wind-tunnel walls). The front springs have spring constant k_1 and are attached to the wing upper and lower surfaces at a distance d_1 ahead of the quarter chord (toward the leading edge); the rear springs have spring constant k_2 and connect to the wing at a distance d_2 downstream of the quarter chord (toward the trailing edge). The wing moment of inertia about the pitch axis is denoted by J_θ. Assuming the pitch angle θ as the degree of freedom (see the convention in Fig. 1) and neglecting the wing self-weight, answer the following questions:

1. After drawing the free-body diagram for the system:
   (a) Derive the equation of motion for studying its free vibrations.
   (b) Determine the natural frequency ω_n and evaluate it for the parameters in Table 1.
2. Modify the attachment point of either the front or rear springs to increase ω_n by 15%.
3. Assuming that four linear viscous dampers c_1 are added to the initial system (one for each front spring):
   (a) Find the minimum value of c_1 such that any free response satisfies

$$\delta = \ln \frac{x(t_1)}{x(t_2)} \geq 0.2, \tag{I.1}$$

   where δ is the logarithmic decrement and t_1 and $t_2 = t_1 + T$ are two consecutive oscillation peaks.
   (b) Evaluate the frequency of the damped motion ω_d and compare it with ω_n.
4. Considering the system with the viscous dampers

(a) Determine the free response for the initial conditions $\theta_0 = 5$ deg and $\dot{\theta}_0 = 0$.
(b) Plot the free response for $t \in [0, 5]$ seconds.

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Spring constant of the front springs	k_1	25 N/m
Spring constant of the rear springs	k_2	$0.75k_1$
Half chord	b	0.10 m
Distance of the front springs from the pitch axis	d_1	$b/4$
Distance of the rear springs from the pitch axis	d_2	b
Moment of inertia about the pitch axis	J_θ	0.0004 kg·m^2

Solution:
$\boxed{1}$ The free body diagram in the $x - z$ plane is as follows.

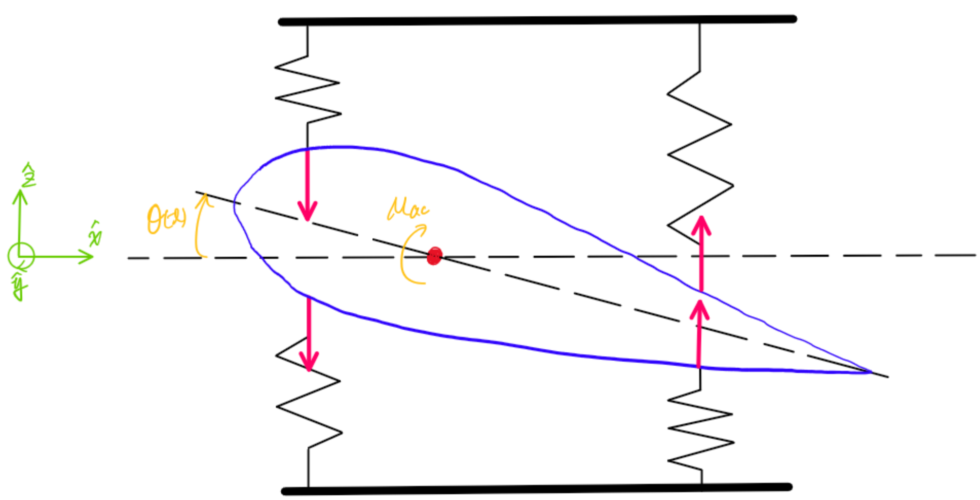

Figure 2: The free body diagram of problem 1.

(a) Since the airfoil is fixed by the quarter-chord axis it does not have any translation motion in the x, y, or z directions. However, it is capable of rotating about the y-axis. In the diagram, the subscript T and B denote the top and bottom forces respectively. These forces are exerted by the springs and since we can assume a small angle approximation for this system we can find the displacement of each springs allowing you to know each of the forces:

$$F_{1T} = F_{1B} = k_1 d_1 \sin\theta \approx k_1 d_1 \theta$$
$$F_{2T} = F_{2B} = k_2 d_2 \sin\theta \approx k_2 d_2 \theta.$$

Now while keeping in mind that there are 8 springs in total, from Euler's equation we can find that the equation of motion is

$$J_\theta \ddot{\theta} = \sum \mathbf{M} = M_{ac} - 2d_1 F_{1T} - 2d_1 F_{1B} - 2d_2 F_{2T} - 2d_2 F_{2B}.$$

Now since this is a free vibration problem $M_{ac} = 0$, and therefore the EOM is

$$J_\theta \ddot{\theta} = -4(k_1 d_1^2 + k_2 d_2^2)\theta. \tag{I.2}$$

(b) If we reorganize the equation I.2, we get

$$\ddot{\theta} + \frac{4(k_1 d_1^2 + k_2 d_2^2)}{J_\theta}\theta = 0$$

Hence, the natural frequency becomes

$$\boxed{\omega_n = \sqrt{\frac{4(k_1 d_1^2 + k_2 d_2^2)}{J_\theta}}.} \tag{I.3}$$

If we evaluate I.3 with the parameters in Table 1, we get

$$\boxed{\omega_n = 45.06939 \text{ rad/s} .}$$

$\boxed{2}$ To change the length of d_1 to increase the natural frequency by 15% the following equation should be solved

$$k_1 d_{1,new}^2 + k_2 d_2^2 = 1.15^2(k_1 d_1^2 + k_2 d_2^2)$$
$$d_{1,new} = \sqrt{\frac{1.15^2(k_1 d_1^2 + k_2 d_2^2) - k_2 d_2^2}{k_1}}$$
$$\therefore d_{1,new} = 0.056968 = 0.569676b.$$

However, since $d_{1,new} > b/2$ this is not feasible. Thus we change d_2 to

$$d_{2,new} = \sqrt{\frac{1.15^2(k_1 d_1^2 + k_2 d_2^2) - k_1 d_1^2}{k_2}}$$
$$\boxed{\therefore d_{2,new} = 0.116163 = 1.161626b.}$$

Since $d_{2,new} < 1.5b$, this is feasible and is the answer.

$\boxed{3}$ (a) Because the displacement for the front springs were $d_1\theta$, this means that the rate of displacement is the derivative of this expression and is $d_1\dot{\theta}$. Then including this in the EOM we have

$$\ddot{\theta} + \frac{4c_1 d_1}{J_\theta}\dot{\theta} + \frac{4(k_1 d_1^2 + k_2 d_2^2)}{J_\theta}\theta = 0, \tag{I.4}$$

then the damping coefficient becomes

$$\zeta = \frac{4c_1 d_1}{2J_\theta \omega_n} = \frac{c_1 d_1}{\sqrt{J_\theta(k_1 d_1^2 + k_2 d_2^2)}} = \frac{c_1 d_1}{\gamma}.$$

For a damping free response we know that the solution for this ODE takes the form of

$$x(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \phi).$$

Hence,

$$\frac{x(t_1)}{x(t_2)} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+T)}} = e^{\zeta\omega_n T},$$

and since $T = \frac{1}{\omega_d} = \frac{1}{\omega_n\sqrt{1-\zeta^2}}$ this becomes

$$\exp\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right).$$

Thus,

$$\delta = \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{c_1 d_1}{\sqrt{\gamma^2 - c_1^2 d_1^2}}.$$

Now if we solve

$$\begin{aligned}
\frac{c_1 d_1}{\sqrt{\gamma^2 - c_1^2 d_1^2}} &\geq 0.2 \\
c_1^2 d_1^2 &\geq 0.04(\gamma^2 - c_1^2 d_1^2) \\
c_1 &\geq \sqrt{\frac{0.04}{1.04}}\frac{\gamma}{d_1} = \sqrt{\frac{0.04 J_\theta(k_1 d_1^2 + k_2 d_2^2)}{1.04 d_1^2}}
\end{aligned}$$

Hence, the value for the minimum c_1 value is

$$\boxed{c_{1,min} = \sqrt{\frac{0.04 J_\theta(k_1 d_1^2 + k_2 d_2^2)}{1.04 d_1^2}} = 0.000637 \text{ kg/s}\ .}$$

(b) If we use this minimum c_1 value we have

$$\zeta = \frac{c_{1,min} d_1}{\gamma} = 0.1961$$

and

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 44.1942.$$

The difference between the damped frequency and the natural freqency is

$$\boxed{\frac{\omega_n - \omega_d}{\omega_n} \times 100 = 1.9419\%.}$$

This is a small change in the frequency.

$\boxed{4}$ (a) Since we know that for a $0 < \zeta < 1$ damped free vibration system the general solution is

$$x(t) = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{\zeta\omega_n x_0 + v_0}{\omega_d}\sin\omega_d t\right) \tag{I.5}$$

If we plug the values evaluated in the previous questions into this solution we have the following result

$$\boxed{\theta(t) = e^{-8.8388t}\left(0.0873\cos(44.1942t) + 0.0175\sin(44.1942t)\right).}$$

(b) The response is as follows. The code is in the Appendix IV.

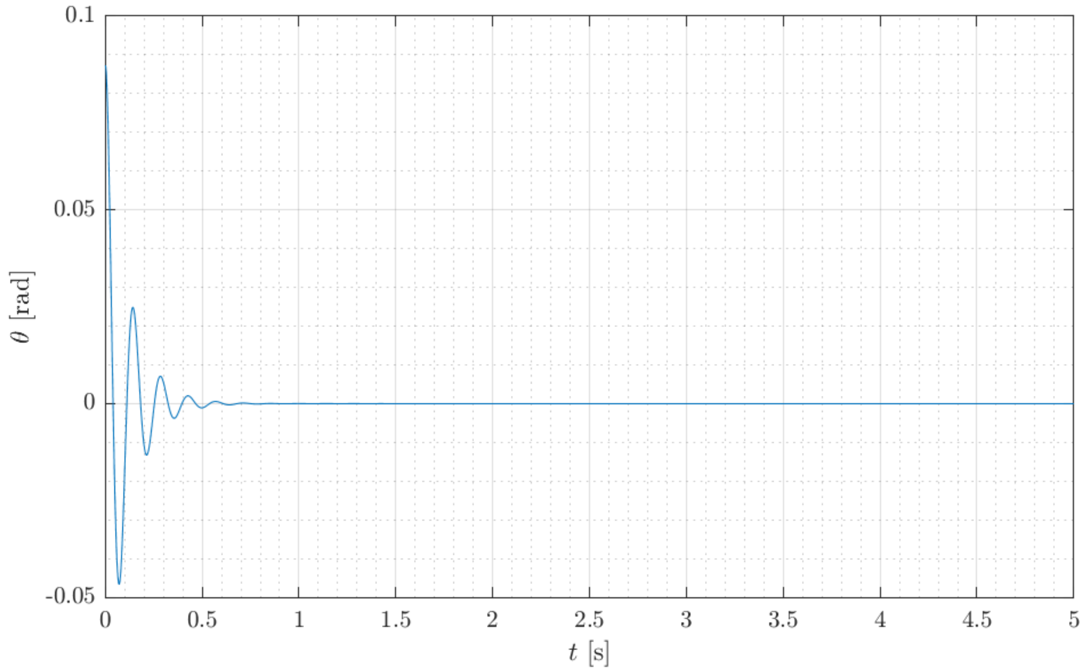

Figure 3: Free response of the system with given initial conditions.

II Problem Two

Consider the system in Problem 1 but now with the parameters in Table 2. The system is excited by a moment

$$M(t) = M_0 \sin \omega t \tag{II.1}$$

about the pitch axis, with zero initial conditions. Answer the following questions:

1. Plot the magnitude $|H(i\omega)|$ and phase lag $\phi(\omega)$ of the frequency response for $\omega/\omega_n \in [0, 4]$.
2. Using the complex response method:
   (a) Determine the steady-state forced response.
   (b) Plot the steady-state force response for $t \in [10T, 10T + 0.5]$ seconds where T is the period of the excitation.
3. Using the time-domain method
   (a) Determine the steady-state forced response.
   (b) Plot the steady-state force response for $t \in [10T, 10T + 0.5]$ seconds where T is the period of the excitation.
   (c) Determine the complete forced response including the transient phase.
   (d) Plot the complete forced response and the transient terms for $t \in [0, 4]$ seconds.

Table 2: Parameter values for Problem 2.

Parameter	Symbol	Value
Moment of inertia about pitch axis	J_θ	0.0004 kg·m^2
Natural frequency	ω_n	50 rad/s
Viscous damping factor	ζ	0.04
Excitation amplitude	M_0	0.1 N·m
Excitation frequency	ω	$0.5\omega_n$

Solution:
$\boxed{1}$ We know that from the given parameters the ODE can be written in the following form

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{M_0}{J_\theta}\sin\omega t.$$

From this we have

$$|\mathcal{H}(i\omega)| = \frac{1}{J_\theta}\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{-0.5} = \frac{1}{4(k_1 d_1^2 + k_2 d_2^2)}\left[(1 - r^2)^2 + 4\zeta^2 r^2\right]^{-0.5},$$

and

$$\phi(r) = \arctan\left(\frac{2\zeta r}{1 - r^2}\right)$$

where $r = \omega_n/\omega$. If we plot this for the given range we have the plots as follows.

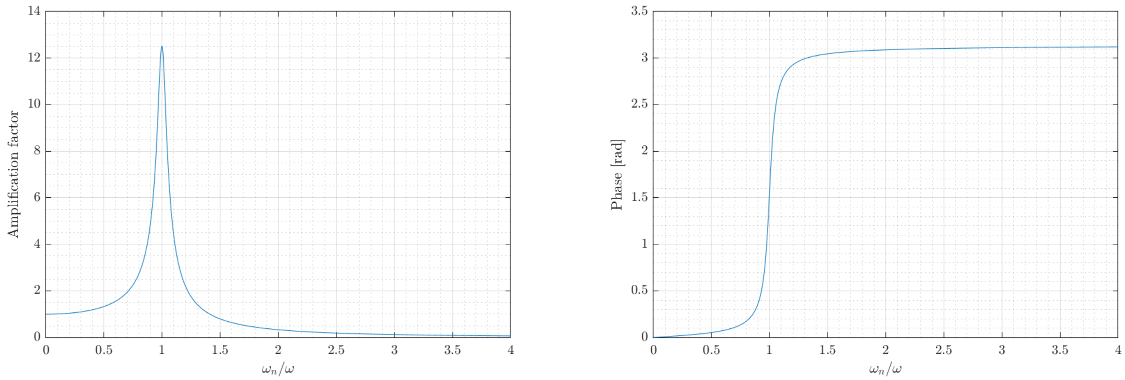

Figure 4: Amplification factor (right) and the phase (left) for the system depicted in harmonic excitation system depicted in problem 1 for the given parameters.

$\boxed{2}$ (a) From our class notes we know that for a SDOF harmonic excitation the steady state response is in the following form

$$\theta(t) = M_0|\mathcal{H}(i\omega)|\sin(\omega t - \phi)$$

$$\boxed{\therefore \theta(t) = \frac{M_0}{4(k_1 d_1^2 + k_2 d_2^2)}\left[(1-r^2)^2 + 4\zeta^2 r^2\right]^{-0.5}\sin\left(\omega t - \arctan\left(\frac{2\zeta r}{1-r^2}\right)\right).}$$

where $r = \omega_n/\omega$.

(b) Since we already have computed the amplification factor and the phase in the previous problem so we can plot this response easily. The plot is as follows.

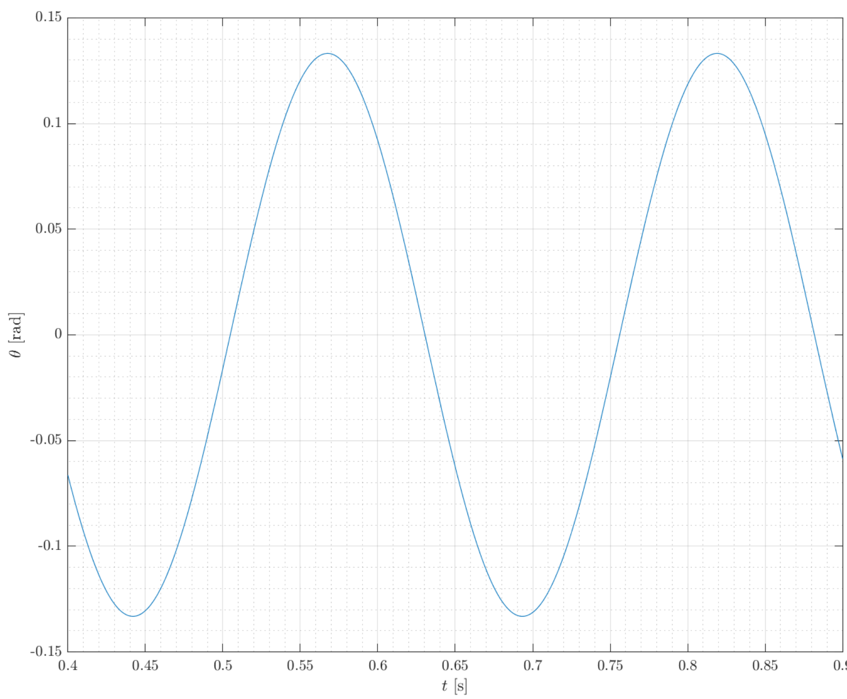

Figure 5: Steady state response of the harmonic excitation.

$\boxed{3}$ (a) The steady state response using the time-domain method takes the form as follows.

$$\theta_p(t) = B_1 \cos\omega t + B_2 \sin\omega t,$$

where

$$B_1 = \frac{M_0}{J_\theta}\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2) + (2\zeta\omega_n\omega)^2}, \qquad B_2 = -\frac{M_0}{J_\theta}\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2) + (2\zeta\omega_n\omega)^2}.$$

Hence

$$\boxed{\theta_p(t) = \frac{M_0}{J_\theta}\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2) + (2\zeta\omega_n\omega)^2}\sin\omega t - \frac{M_0}{J_\theta}\frac{2\zeta\omega_n\omega}{(\omega_n^2 - \omega^2) + (2\zeta\omega_n\omega)^2}\cos\omega t}$$

(b) The plot is as follows

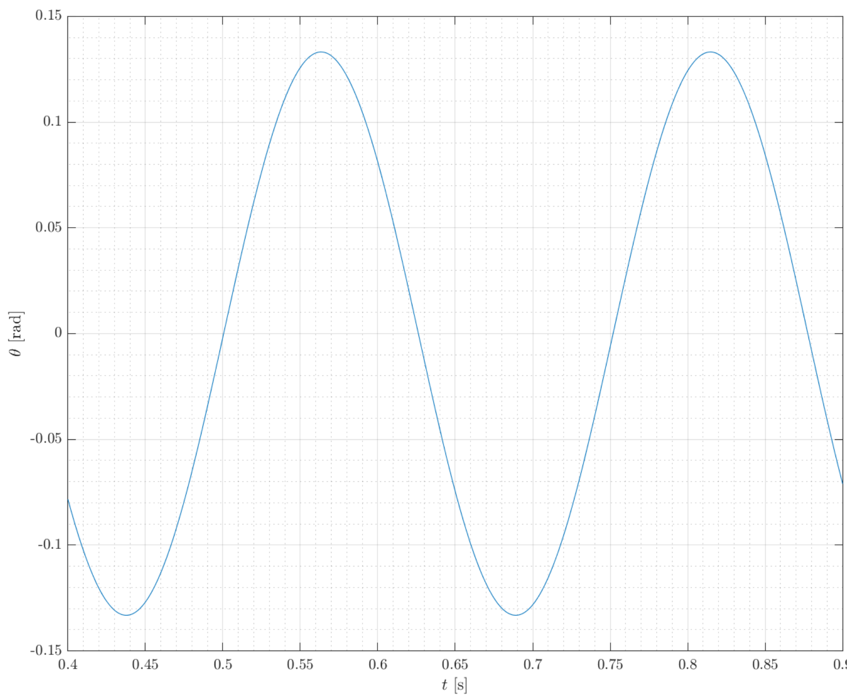

Figure 6: Steady state response of the harmonic excitation.

(c) The transient response is

$$x_h(t) = e^{-\zeta\omega_n t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t).$$

From the zero initial conditions we can find the complete forced response. We will use MATLAB to do this task the Code is in the Appendix.

$$\boxed{\theta(t) = 0.1330\sin(25t) - 0.0071\cos(25t) + \mathrm{e}^{-2t}(0.0071\cos(49.9600t) - 0.0662\sin(49.9600t))}$$

(d) The response is as follows

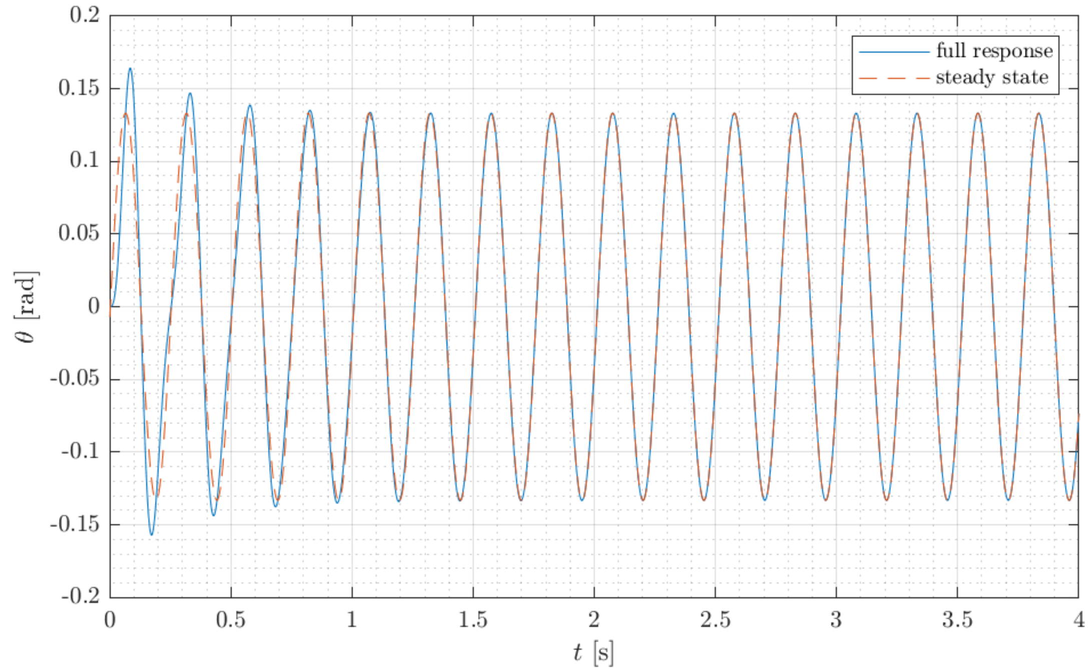

Figure 7: Full response of the harmonic excitation.

III Problem Three

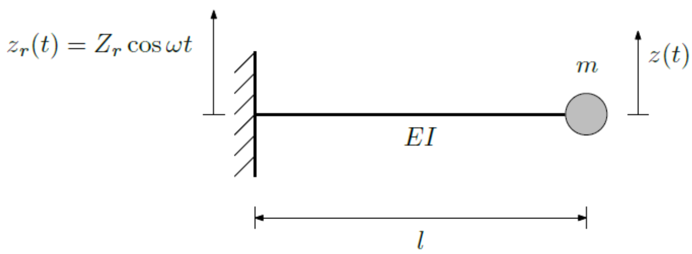

Figure 8: Schematic of a cantilevered beam in bending with a tip mass subject to harmonic motion of its root.

Consider the massless uniform isotropic cantilevered beam in bending in Fig. 8 with a tip mass. The beam root undergoes harmonic motion

$$z_r(t) = Z_r \cos \omega t. \tag{III.1}$$

Very slight damping is present in the system such that, after a transient phase, the tip mass motion $z(t)$ contains only the excitation frequency ω. The impact of such slight damping on the amplitude and phase of $z(t)$ is assumed to be negligible. Answer the following questions:

1. After showing the free-body diagram, derive the equation of motion for the tip mass.
2. Considering the parameters in Table 3, determine:
   (a) The natural frequency ω_n.
   (b) The maximum excitation frequency $\omega < \omega_n$ such that $|z(t) - z_e| \leq 1.1|Z_r|$ where z_e is the equilibrium at the tip mass.

Table 3: Parameter values for Problem 3.

Parameter	Symbol	Value
Beam length	l	0.5m
Beam bending stiffness	EI	5 N·m^2
Tip mass	m	0.5 kg

Solution:
$\boxed{1}$ The free body diagram is as follows.

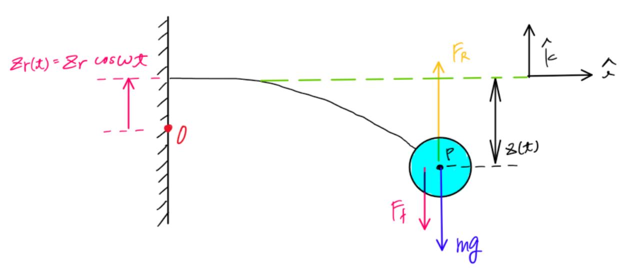

Figure 9: Free body diagram of problem 3.

In the diagram, the forces are: F_R, restorating force; F_f, fictitious force from the root motion; $F_g = mg$, gravity force from the point mass. Then the total force on the point mass becomes

$$\sum \mathbf{F}_z = (F_R - F_f - F_g)\hat{\mathbf{z}}$$

Now for the massless cantilever beam with a point mass at the end the stiffness coefficient is $k = \frac{3EI}{l^3}$ and so the forces are

$$F_R = kz, \quad F_f = m\ddot{z}_r, \quad F_g = mg$$

Thus we have

$$\begin{gathered}
m\ddot{z}(-\hat{\mathbf{z}}) = (\sum F_z = kz - m\ddot{z}_r - mg)\hat{\mathbf{z}} \\
m\ddot{z} + kz = m\ddot{z}_r + mg \\
\ddot{z} + \frac{k}{m}z = \ddot{z}_r + g \\
\boxed{\ddot{z} + \frac{3EI}{ml^3} = -Z_r\omega^2 \cos\omega t + g}
\end{gathered}$$

$\boxed{2}$ (a) Now the natural frequency is

$$\boxed{\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3EI}{ml^3}} = 15.4919 \text{ rad/s}}$$

(b) We can solve the ODE we derived in problem 1 using matlab and the code is in the appendix IV. The equation is

$$z(t) = \frac{Z_r\omega^2}{\omega_n^2 - \omega^2}(\cos\omega_n t - \cos\omega t) + \frac{g}{\omega_n^2}(1 - \cos\omega_n t)$$

Now since the second term $(1 - \cos\omega_n t)$ corresponds to the equilibrium of z_e, $z(t)$ is only the amplitude on the first term, and therefore,

$$\begin{gathered}
\frac{Z_r\omega^2}{\omega_n^2 - \omega^2} \leq 1.1Z_r \\
\omega^2 \leq 1.1(\omega_n^2 - \omega^2) \\
\boxed{\omega \leq \sqrt{\frac{1.1}{2.1}}\omega_n = 11.2122 \text{ rad/s} .}
\end{gathered}$$

IV Appendix

Problem 1: MATLAB Code

```
% AE6230 HW1 Problem 1
% Author: Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% Parameters

k1 = 25;
k2 = 0.75*k1;
b = 0.1;
d1 = b/4;
d2 = b;
J = 0.0004;

%% P1(b)

wn = sqrt(4*(k1*d1^2 + k2*d2^2)/J);
fprintf("wn: %f \n",wn);

%% P2

d1new = sqrt( (1.15^2 * (k1*d1^2 + k2*d2^2) - k2*d2^2) / k1 );
fprintf("d1new: %f = %f*b \n", d1new, d1new/b);

d2new = sqrt( (1.15^2 * (k1*d1^2 + k2*d2^2) - k1*d1^2) / k2 );
fprintf("d2new: %f = %f*b \n", d2new, d2new/b);

%% P3

gamma = J * (k1*d1^2 + k2*d2^2);
c1min = sqrt(0.04/1.04) * gamma / d1;
fprintf("c1min: %f \n", c1min)

zeta = c1min * d1 / gamma;
fprintf("zeta: %f \n", zeta);

wd = wn * sqrt(1 - zeta^2);
fprintf("wd: %f \n",wd);

w_diff = (wn - wd) / wn * 100;
fprintf("w_diff: %f \n", w_diff);

%% P4

theta0 = deg2rad(5);
thetadot0 = 0;
```

```
A = zeta*wn
B = (zeta*wn*theta0 + thetadot0)/wd

% Plot the response
tspan = 0:0.001:5;
res = exp(-A*tspan) .* (theta0*cos(wd*tspan) + B*sin(wd*tspan));

fig = figure(Renderer="painters", Position=[60 60 700 400]);
    plot(tspan, res)
    xlabel("$t$ [s]")
    ylabel("$\theta$ [rad]")
    grid on; grid minor; box on;
saveas(fig, "p4_response.png");
```

Problem 2: MATLAB Code

```
% AE6230 HW1 Problem 2
% Author: Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% Parameters

wn = 50;
z = 0.04;
M0 = 0.1;
w = 0.5*wn;
J = 0.0004;

%% P1

r = 0:0.001:4;
k = wn^2*J;
Hiw = 1/k * ((1 - r.^2).^2 + 4*z^2*r.^2).^(-0.5);
phi = atan2(2*z*r, 1-r.^2);

% Plot the results
fig = figure(Renderer="painters", Position=[60 60 600 400]);
    plot(r, Hiw)
    xlabel("$\omega_n/\omega$")
    ylabel("Amplification factor")
    grid on; grid minor; box on;
saveas(fig, "p2_1_ampfac.png");

fig = figure(Renderer="painters", Position=[60 60 600 400]);
    plot(r, phi)
    xlabel("$\omega_n/\omega$")
    ylabel("Phase [rad]")
    grid on; grid minor; box on;
```

```
saveas(fig, "p2_1_phase.png");

%% P2

T = 1/w;
tspan = 10*T:0.001:10*T+0.5;
r = w/wn;
k = wn^2*J;
Hiw = 1/k * ((1 - r.^2).^2 + 4*z^2*r.^2).^(-0.5);
phi = atan2(2*z*r,1-r.^2);
res1 = M0 * Hiw * sin(w * tspan - phi);

fig = figure(Renderer="painters", Position=[60 60 900 700]);
    plot(tspan, res1)
    xlabel("$t$ [s]")
    ylabel("$\theta$ [rad]")
    grid on; grid minor; box on;
saveas(fig, "p2_2b_response.png");

%% P3

den = (wn^2 - w^2)^2 + (4*z^2*wn^2*w^2);
B1 = M0/J * (wn^2 - w^2) / den;
B2 = -M0/J * (2*z*wn*w) / den;

% B = M0/J * ((wn^2-w^2)^2 + (2*z*wn*w)^2)^(-0.5);
% phi = atan2(2*z*wn*w, wn^2-w^2);

% B1 = M0*Hiw*cos(phi)
% B2 = -M0*Hiw*sin(phi)

res2 = B1 * sin(w * tspan) + B2 * cos(w * tspan);
% res2 = B .* sin(w * tspan - phi);


fig = figure(Renderer="painters", Position=[60 60 900 700]);
    plot(tspan, res2)
    xlabel("$t$ [s]")
    ylabel("$\theta$ [rad]")
    grid on; grid minor; box on;
saveas(fig, "p2_3b_response.png");

%% P3

syms t real
syms theta(t)
syms zeta omega_n omega M_0 J_theta real
assume(zeta<1 & zeta>0)

thetadot = diff(theta,t);
thetaddot = diff(thetadot,t);
ode = thetaddot + 2*zeta*omega_n*thetadot + omega_n^2*theta == M_0/J_theta * sin(omega*t);
cond1 = theta(0) == 0;
cond2 = thetadot(0) == 0;
```

```
cond = [cond1 cond2];
thetaSol(t) = dsolve(ode,cond);
thetaSol(t) = simplify(thetaSol)
thetaSol(t) = subs(thetaSol, [omega_n, omega, M_0, J_theta, zeta], [wn,w,M0,J,z])
thetaSol(t) = simplify(expand(thetaSol))
thetaSol(t) = collect(thetaSol, exp(-2*t))

%% Plot
tspan = 0:0.001:4;

fig = figure(Renderer="painters", Position=[60 60 700 400]);
    plot(tspan, thetaSol(tspan))
    xlabel("$t$ [s]")
    ylabel("$\theta$ [rad]")
    grid on; grid minor; box on; hold on;
    plot(tspan, B1 * sin(w * tspan) + B2 * cos(w * tspan), '--')
    hold off; legend(["full response", "steady state"])
saveas(fig, "p2_3d_fullresponse.png");
```

Problem 3: MATLAB Code

```
% AE6230 HW1 Problem 3
% Author: Tomoki Koike

%% Housekeeping commands


clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% ODE solve

syms z(t)
syms omega_n Z_r omega g real
zdot = diff(z,t);
zddot = diff(zdot, t);
ode = zddot + omega_n^2*z == -Z_r*omega^2*cos(omega*t) + g;
cond1 = z(0) == 0;
cond2 = zdot(0) == 0;
cond = [cond1, cond2];
zsol(t) = dsolve(ode,cond)
simplify(zsol)
```