Problem Set 5

AAE 532: Orbital Mechanics
MWF: 11:30-12:20
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Contents

Problem 1

Return to the use of multiple propagators in GMAT. The propagator 'TwoBody' or 'EarthPointMass' is already available (unter the name you have selected for previous assignments). Produce the new propagators: 'EarthMoon', 'EarthSun', 'EarthMoonSun'. Add the different forces to create the new propagators as described in the GMAT Tips document. Add the Moon's orbit (Luna) to the output image. Propagate for 60 days.

(a) Use a start date October 2, 2020 16:00:00. Use the Earth J2000Eq coordinates throughout the simulations. In a Keplerian Coordinate Type, introduce initial conditions such that:

$$r_p = 1.5R_\oplus$$
$$r_a = 200R_\oplus$$
$$\omega = 0^\circ$$
$$i = 30^\circ$$
$$\Omega = 0^\circ$$
$$\theta^* = 0^\circ$$

Explore the 4 propagators (use a different color for each propagated path). Propagate all the trajectories for 60 days. Produce a plot with a view approximately down the Moon Orbit Normal with all four spacecraft. Add views on two other dates: October 7, 2020 and October 11, 2020 at the same time of day. Choose another date in October and add a figure. These simulations all use the relative vector equation of motion for the spacecraft relative to the Earth from Notes Page D2; the perturbations on the right-hand side of the equation vary for each propagator. Does the model make a difference? Is the two-body model adequate for this particular problem? Why or why not? For the trajectory in this anlysis, which relative orbit model would you recommend: two-body, three-body, four-body? Why? Which bodies would you include? What is the impact of the different epoch dates? Why is there such a difference in the paths?

(b) Output some information for each spacecraft at $t = t_f$, the end of the propagation. Determine the following information from the GMAT output: $a, e, r_p, \mathcal{E}, h; r_f, v_f, \theta^*_f, \gamma_f$. Compare the closest approach altitude for all the spacecraft at the end of the simulation. Are any spacecraft in danger of Earth impact? Which perturbation reduced the r_p? Does it occur at all starting epochs?

Problem 1 Solution

(a)

To begin, define the orbit characteristics by navigating to **Resources** → **Spacecraft**

$$e = \frac{r_a - r_p}{r_p + r_a} = 0.985112$$

$$a = \frac{r_p}{1 - e} = 6.42598108e5km$$

Next, navigate to **Resources** → **Propagators** to produce the new propagators.

For example,

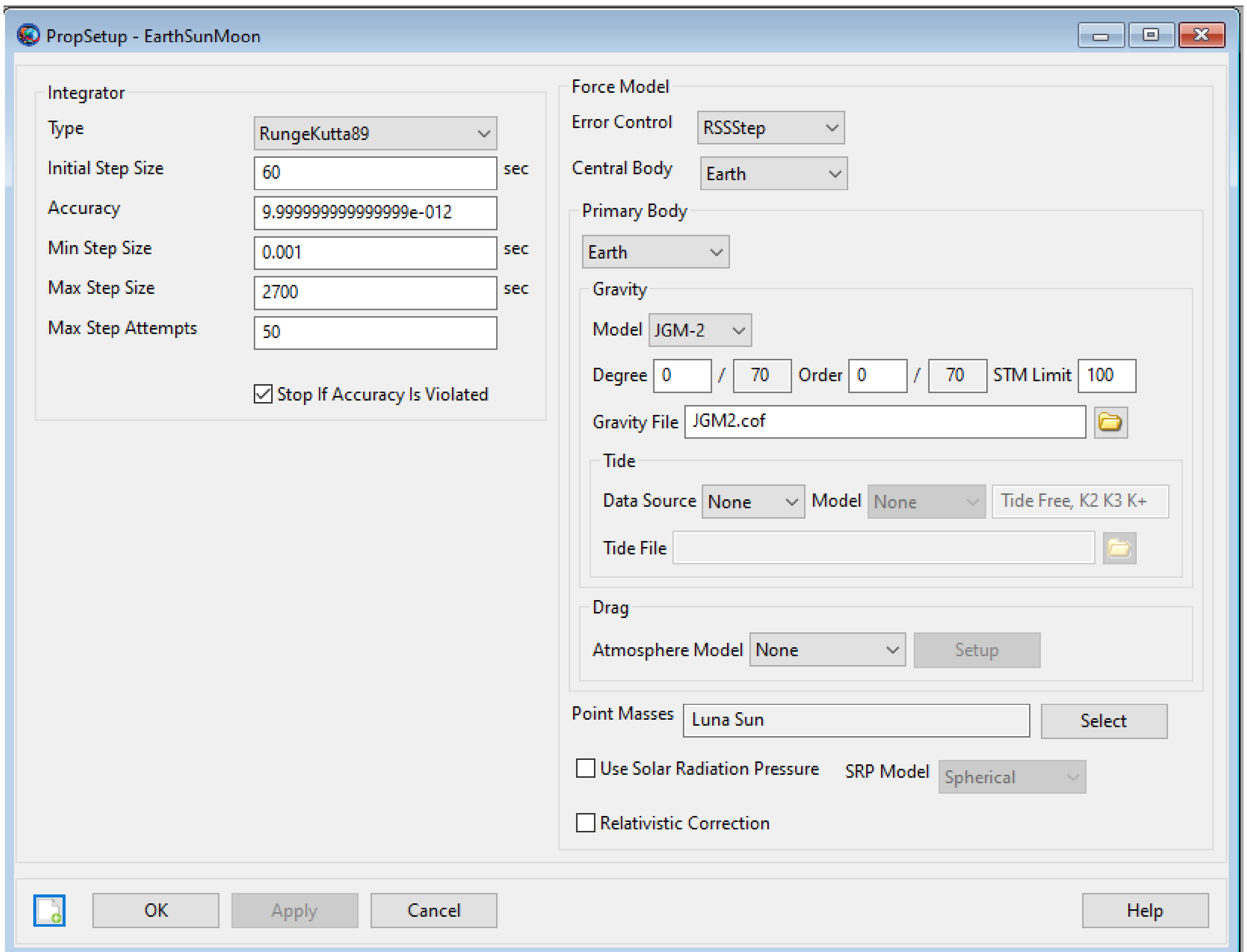

Figure 1: Problem 1: Earth-Moon-Sun Propagator

Each propagator is specified in the name of the spacecraft. Figures 2-5 show the trajectories for each model propagated for various initial epochs.

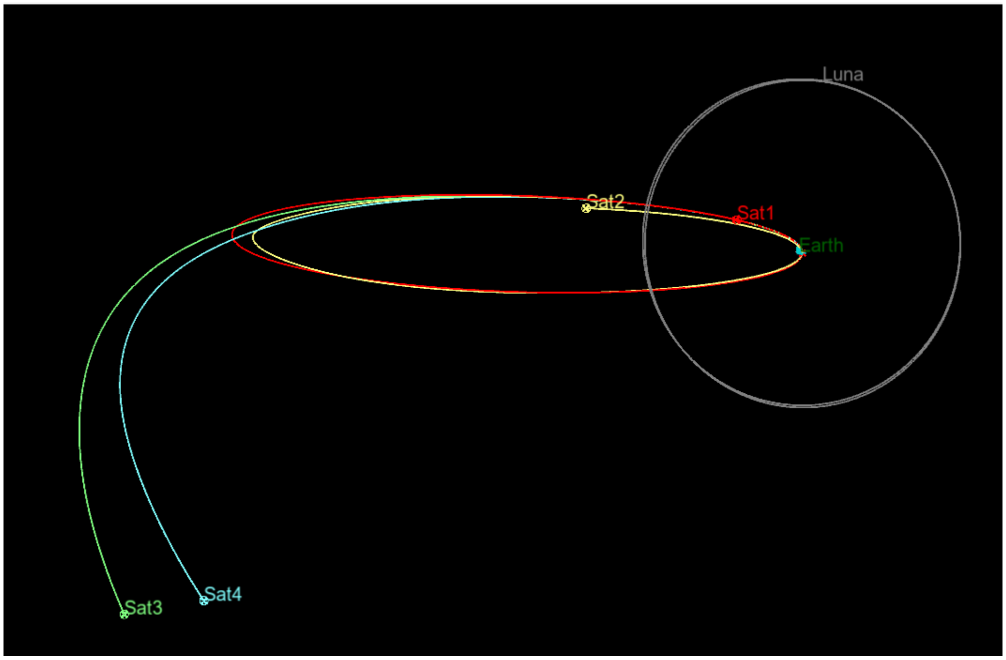

Figure 2: EarthProp, EarthMoonProp, EarthSunProp, EarthMoonSunProp, 10/02/20

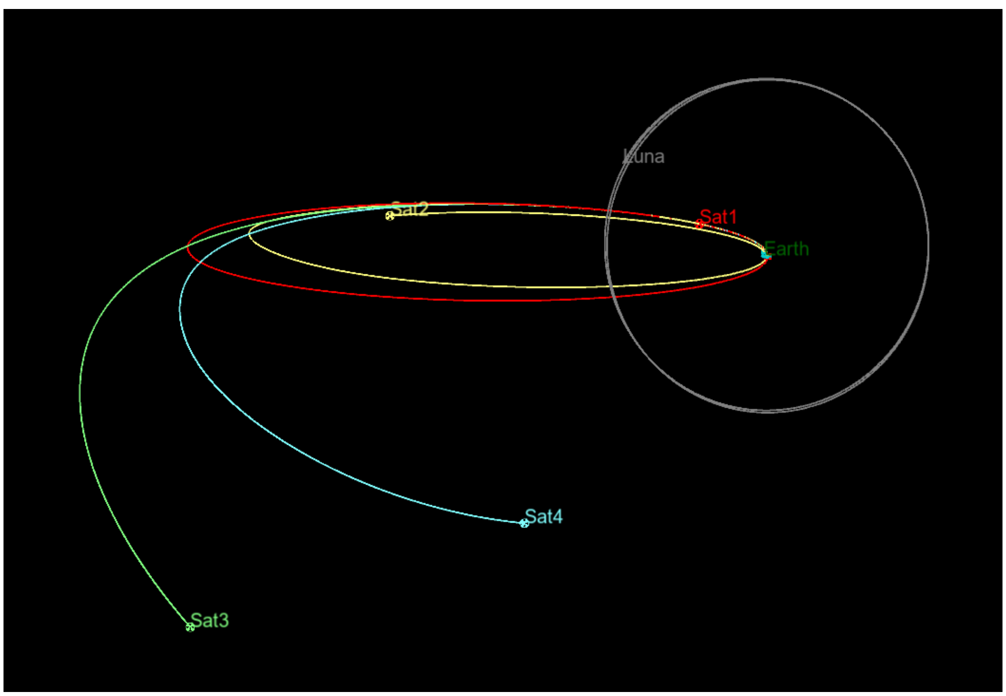

Figure 3: EarthProp, EarthMoonProp, EarthSunProp, EarthMoonSunProp, 10/7/20

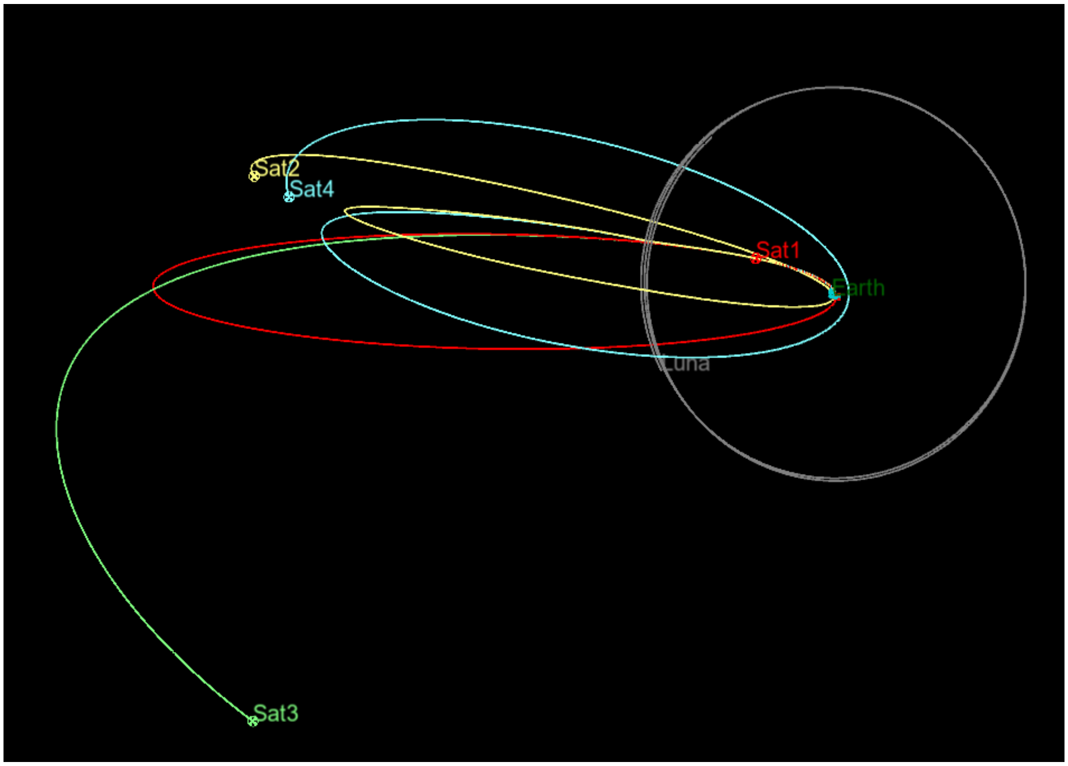

Figure 4: EarthProp, EarthMoonProp, EarthSunProp, EarthMoonSunProp, 10/11/20

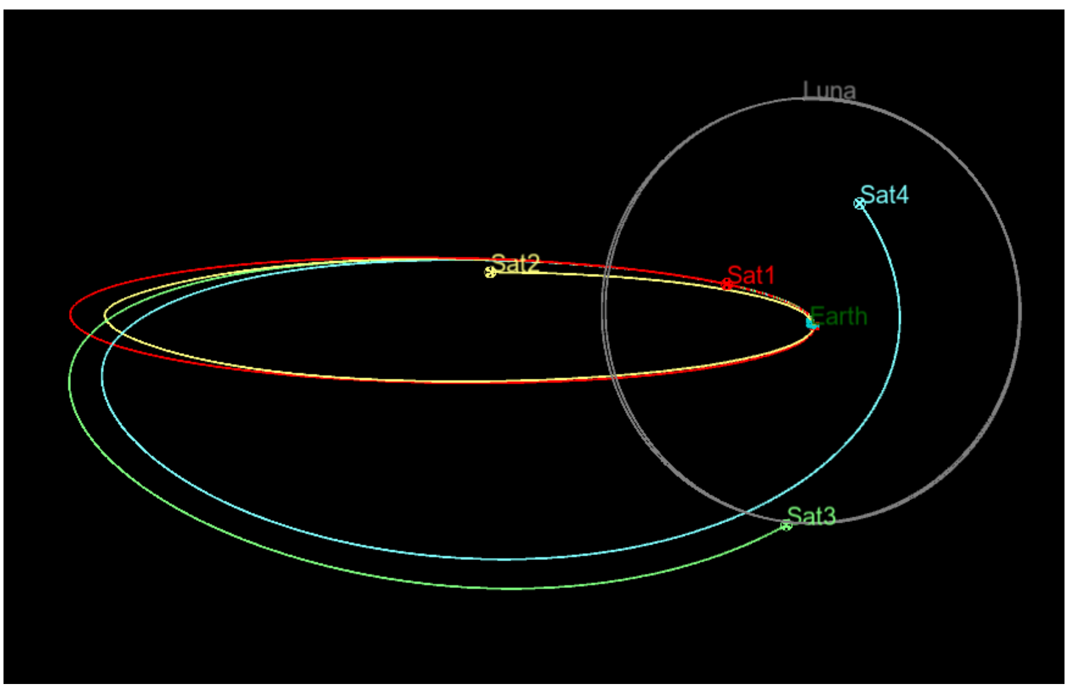

Figure 5: Problem 1: EarthProp, EarthMoonProp, EarthSunProp, EarthMoonSunProp, 10/30/20

As we can observe from the images above, the model definitely makes a difference. Regardless of the epoch, the Earth Point Mass model produces a Keplerian orbit as expected. The other three models deliver non-Keplerian trajectories due to the addition of the perturbing bodies. In general, the two-body model is not adequate for this particular problem. For example, the discrepancies in the four models are evident if we compare the values for r after 60 days for the first epoch date. We find that:

$$r_{\oplus} = 26.15 R_{\oplus}$$

$$r_{\oplus ☾} = 78.83 R_{\oplus}$$

$$r_{\oplus \odot} = 283.15 R_{\oplus}$$

$$r_{\oplus ☾ \odot} = 256.48 R_{\oplus}$$

We observe that in the true system dynamics, the spacecraft is significantly far away from Earth after 60 days. The most accurate description of the physics is the model with the most perturbing forces considered (EarthMoonSunProp). Furthermore, observe the trajectories change based on the epoch dates which tells us their impact is significant. This is to be expected since Moon and the Sun are constantly changing their position relative to Earth thus altering the net perturbing force due to the Moon and the Sun on the spacecraft. Further observations show significant out of plane excursion relative to the Keplerian orbit for both the Earth-Moon and Earth-Sun-Moon models. This is likely due to a Moon passage/flyby where the trajectory is impacted by the Lunar gravity. At a minimum, we would require at least two bodies, namely Earth and Sun to obtain a good approximation of the trajectory accounting for the true system dynamics.

(b)

The GMAT outputs at the end of the propagation for the four epochs dates is shown in Figures 6-9. At the end of the simulation, all four models show the spacecraft at a significant distance away from the Earth. We should specify that the distance at the final time is $RMAG_f$. The closest distance ($RMAG_f$) to the Earth at the end of the simulation is delivered by the Earth Point Mass model for all epochs considered. Now, the closest passing distance relative to the Earth is found by looking at the minimum value of $RMAG$ over the course of the simulation as that is indicative of the spacecraft position as calculated by the four propagators. The closest passing distance relative to Earth is delivered by the Earth-Moon model for Oct 7th and Oct 11th epoch dates. For October 7th, the E-M model delivers $RMAG \approx 6397.76$ km which is $\approx$ 19km from Earth sea level, i.e. the upper troposphere. The spacecraft would be visible! For Oct 11th, $RMAG$ is $\approx$ 5348.16 km. Thus, the spacecraft has crashed into Earth. The perturbation from the Moon reduced the passing distance but it does not occur for all starting epochs. As we concluded before, the epoch date is an important aspect of mission design. We could also look at $RadPer$ and we would see that it is a constant in the Earth point mass model, as per our definitions. However, since the other propagators do not employ conic relationships, the parameter $RadPer$ is not a constant. Depending on the mission objectives, close passage without planned maneuvers to counteract needs to be taken into account in the trajectory design and Earth impact should be avoided. Ultimately, the two-body model does not deliver an accurate approximation of the dynamics for this scenario.

	a	e	ε	r_p	h	r_f	v_f	θ^*	γ_f
E	642598.108639	0.985112	-0.310148	9567.217499	87006.997456	166767.334957	2.039613	154.093604	14.820812
EM	610366.727048	0.986006	-0.326525	8146.256621	82229.921827	502800.720383	0.965645	168.514616	9.750740
ES	1440815.417849	0.320263	-0.138325	9567.217499	717916.148843	1805977.678819	0.405924	152.480273	78.322427
EMS	1210073.423647	0.356565	-0.164701	9567.217499	648854.541344	1635863.989021	0.397398	173.583007	89.987903

Figure 6: GMAT output at the end of the propagation for each model, 10/02/20.

	a	e	ε	h	r_p	r_f	v_f	θ^*	γ_f
E	642598.108640	0.985112	-0.310148	87006.997456	9567.217499	166767.334957	2.039613	154.093604	14.820812
EM	564552.550807	0.985557	-0.353023	80332.774849	6397.763489	840642.264926	0.492216	174.333470	11.194763
ES	1266580.192020	0.280188	-0.157353	682074.208873	9567.217499	1621297.562069	0.420713	181.303927	90.507448
EMS	814052.243853	0.572419	-0.244825	467076.867692	9567.217499	859065.781185	0.662070	230.656846	90.367402

Figure 7: GMAT output at the end of the propagation for each model, 10/07/20.

	a	e	ε	h	r_p	r_f	v_f	θ^*	γ_f
E	642598.108639	0.985112	-0.310148	87006.997456	9567.217499	166767.334958	2.039613	154.093604	14.820812
EM	534366.810277	0.984322	-0.372965	81403.708858	5348.164696	1051771.210430	0.109681	180.923887	135.117752
ES	1153261.813282	0.316566	-0.172814	643135.555288	9567.217499	1454431.003549	0.449989	205.159369	100.682385
EMS	524527.283092	0.916884	-0.379962	182513.123722	9567.217499	995189.238023	0.202809	182.478328	90.069646

Figure 8: GMAT output at the end of the propagation for each model, 10/11/20.

	a	e	ε	h	r_p	r_f	v_f	θ^*	γ_f
E	642598.108639	0.985112	-0.310148	87006.997456	9567.217499	166767.334957	2.039613	154.093604	14.820812
EM	604825.512751	0.985972	-0.329517	81954.596318	7891.591719	570323.059268	0.859518	169.821594	9.624186
ES	821089.698252	0.731294	-0.242726	390200.881479	9567.217499	372529.756441	1.286279	271.987571	125.480351
EMS	705845.836140	0.764250	-0.282357	342081.667660	9567.217499	235466.437739	1.679557	71.160632	86.634096

Figure 9: GMAT output at the end of the propagation for each model, 10/30/20.

Problem 2

A spacecraft is in orbit about mars and is characterized such that $r_p = 1.5R_{♂}$ and $r_a = 6.5_{♂}$. The vehicle is currently located such that $M = -90°$.

(a) Determine the following orbit parameters and spacecraft state information:

$$a, e, p, h, \mathbb{P}, \mathcal{E}, r, v, \theta^*, E, \gamma, (t - t_p)$$

(b) Write $\bar{r}_0, \bar{v}_0$ in terms of components in the directions of $\hat{e}$ and $\hat{p}$.

(c) Determine θ^* after a time equal to 50% of the period, i.e., $\Delta t = 0.5\mathbb{P}$. Use f and g relationship to write $\bar{r}, \bar{v}$ in terms of $\bar{r}_0, \bar{v}_0$. Prove that $f(\theta^* - \theta_0^*), g(\theta^* - \theta_0^*)$ produce the same results as $f(E - E_0), g(E - E_0)$.

(d) Plot the orbit with your Matlab script. By hand, mark on the plot where the spacecraft is currently located by marking $\hat{r}, \hat{\theta}, \bar{r}_0, \theta_0^*$; also sketch the local horizon, $\bar{v}_0$ and γ_0. Do the same at the second location. Identify the arc from t_0 to t?

Problem 2 Solution

(a)

First, we are going to retrieve the constants from Table of Constants:

$$R_{♂} = 3397km$$
$$\mu_{♂} = 42828.314258067km^3/s^2$$

$r_p = 1.5R_{♂}$, $r_a = 6.5R_{♂}$, and at the moment, $M_0 = -90°$. Since $r_p + r_a = a(1-e) + a(1-e) = 2a$,

$$\boxed{a = \frac{r_p + r_a}{2} = \frac{1.5R_{♂} + 6.5R_{♂}}{2} = 4R_{♂} = 1.3588 \times 10^4 km}$$

$$\boxed{e = 1 - \frac{r_p}{a} = 1 - \frac{1.5}{4} \approx 0.625}$$

Since $r_p = \frac{p}{1+e} = \frac{h^2}{\mu_{♂}(1+e)}$,

$$\boxed{h = \sqrt{r_p \mu_{♂}(1+e)} \approx 1.8831 \times 10^4 km^2/s}$$

$$\boxed{\mathbb{P} = \frac{2\pi}{\sqrt{\mu_{♂}}}\sqrt{a^3} \approx 4.8089 \times 10^4 \ seconds \ = 13.358 \ \ hours}$$

$$\boxed{\mathcal{E} = -\frac{\mu_{♂}}{2a} = -1.5759 km^2/s^2}$$

Note that $a, e, p, h, \mathbb{P}, \mathcal{E}$ are constant for a given conic. The rest of the computations depend on *where* exactly you are in the orbit. Note that we can easily retrieve $t - t_p, M$ from θ^*, E with the Kepler's equation:

$$M = E - e \sin E$$

However, the equation is not invertible. If we are given with $t - t_p, M$, we have to solve for E, which is nontrivial. Since there does not exist a closed form of solution to this equation, we can obtain the solution by numerical method. We could use functions in MATLAB like *solve.m, fsolve.m* and etc., or write our own codes to solve for the eccentric anomaly E. While there are many ways to numerically solve the equation, Newton-Raphson method is applied here in the solutions.
As an initial guess, $E_1 = M = -90° = -1/2\pi$ is used. And each step, E of the next step is updated by the following equation:

$$\begin{aligned} E_{n+1} &= E_n - \frac{f(E_n)}{f'(E_n)} \\ f(E) &= E - esinE - M \\ f'(E) &= 1 - ecosE \end{aligned}$$

And continue until $f(E_n) \approx< tol$, where tol is a small user-set number. Within 5 steps of iteration, we get the eccentric anomaly at the initial time:

$$\boxed{E_0 \approx -2.1078\ (rad) \approx -120.77° = 239.23°}$$

now that we have eccentric anomaly, θ^* r and v can be calculated:

$$\boxed{r_0 = a(1 - ecosE_0) = 1.7933 \times 10^4\ km}$$

$$\boxed{v_0 = \sqrt{\frac{2\mu_{♂}}{r} - \frac{\mu_{♂}}{a}} = 1.2746 km/s}$$

$$\boxed{\theta_0^* = 2tan^{-1}(\sqrt{\frac{1+e}{1-e}} tan(\frac{E_0}{2})) = -2.6085(rad) = -149.45° = 210.55°}$$

The flight angle γ can be calculated with h(since $h = rv\cos\gamma$, r, v:

$$\boxed{\gamma_0 = \cos^{-1}(\frac{h}{rv}) = -0.60259\ (rad) = -34.526°}$$

The inverse cosine is double-valued, but since the spacecraft is descending($180° < E < 360°$), we know that $\gamma < 0$.
With the flight angle, $\bar{v_0}$ can be represented as a vector with the radial and transverse component:

$$\bar{v_0} = v_0 \sin\gamma_0 \hat{r} + v_0 \cos\gamma_0 \hat{\theta} = -0.7224\hat{r} + 1.0501\hat{\theta}\ (km/s)$$

We can compute $(t_0 - t_p)$ by:

$$\boxed{(t_0 - t_p) = M_0 \times \sqrt{\frac{a^3}{\mu_{♂}}} = -1.2022 \times 10^4\ s = -3.3395\ hours}$$

The negative value means that it is 3.3395 hours before the satellite reaches the periapsis. If we choose to use positve value, then:

$$(t_0 - t_p) + \mathbb{P} = 3.6067 \times 10^4\ seconds = 10.019\ hours$$

(b)

With the calculated true anomaly, we can acquire $\bar{r_0}$ and $\bar{v_0}$ in the perifocal frame.

$$\boxed{\bar{r_0} = r_0 \cos\theta_0^* \hat{e} + r_0 \sin\theta_0^* \hat{p} = -1.5444 \times 10^4 \hat{e} - 0.9114 \times 10^4 \hat{p}, km}$$

$$\boxed{\bar{v_0} = \frac{\mu_{♂}}{h}(-\sin_0 \theta^* \hat{e} + (e + cos\theta_0^*)\hat{p}) = 1.1559\hat{e} - 0.5372\hat{p}(km/s)}$$

(c)

The mean anomaly at new position, after $\Delta t = 0.75\mathbb{P}$ can be calculated by:

$$M = \sqrt{\frac{\mu_{♂}}{a^3}}(t_0 + \Delta t - t_p) = M_0 + \sqrt{\frac{\mu_{♂}}{a^3}} \cdot 0.5\mathbb{P} = -1/2\pi + \sqrt{\frac{\mu_{♂}}{a^3}} \cdot 0.5 \cdot 2\pi \sqrt{\frac{a^3}{\mu_{♂}}} = 5/6\pi(rad) = 1/2(rad) = 90^\circ$$

With this new mean anomaly, we can acquire new eccentric anomaly with Newton's method as in Problem 2 (a):

$$\boxed{E = 2.1078(rad) = 120.77^\circ}$$

Note that $E = -E_0$ since $M = -M_0$ (think about the symmetry of the orbit, and the Kepler's equation is an odd function w.r.t. E).
And applying the relationship between eccentric anomaly and true anomaly, we get θ^*:

$$\boxed{\theta^* = 2tan^{-1}(\sqrt{\frac{1+e}{1-e}}tan(\frac{E}{2})) = 2.6085(rad) = 149.45^\circ}$$

Now we use Lagrange coefficients to express $\bar{r}, \bar{v}$ with initial vectors, $\bar{r_0}, \bar{v_0}$. There are two ways to compute the coefficients, and we check if they produce the same results.

1) f and g relations: using $\Delta\theta^*$

Using $\Delta\theta^*$, we get general relationships for any conic orbits as below:

$$\bar{r} = f\bar{r_0} + g\bar{v_0} = [1 - \frac{r}{p}(1 - cos(\theta^* - \theta_0^*))]\bar{r_0} + \frac{rr_0}{\sqrt{\mu_{♂}p}}sin(\theta^* - \theta_0^*)\bar{v_0} \tag{1}$$

$$\bar{v} = \dot{f}\bar{r_0} + \dot{g}\bar{v_0} = [\frac{\bar{r_0}\cdot\bar{v_0}}{pr_0}(1 - cos(\theta^* - \theta_0^*)) - \frac{1}{r_0}\sqrt{\frac{\mu_{♂}}{p}}sin(\theta^* - \theta_0^*)]\bar{r_0} + [1 - \frac{r_0}{p}(1 - cos(\theta^* - \theta_0^*))]\bar{v_0} \quad (2)$$

We need to calculate the magnitude of r in advance to calculate $f, g, \dot{f}, \dot{g}$. And values of p, $\theta^* - \theta_0^*$ are calculated as well.

$$r = a(1 - ecos(E)) = 1.7933 \times 10^4 km$$

$$p = a(1 - e^2) = 8.2802 \times 10^3 km$$

$$\theta^* - \theta_0^* = 5.2169(rad) = 298.91°$$

With these, $f, g, \dot{f}, \dot{g}$ are computed as below:

$$f = [1 - \frac{r}{p}(1 - cos(\theta^* - \theta_0^*))] = -0.11884(non - dimensional) \quad (3)$$

$$g = \frac{rr_0}{\sqrt{\mu_{♂} p}}sin(\theta^* - \theta_0^*) = -1.4949 \times 10^4(s) \quad (4)$$

$$\dot{f} = [\frac{\bar{r_0}\cdot\bar{v_0}}{pr_0}(1 - cos(\theta^* - \theta_0^*)) - \frac{1}{r_0}\sqrt{\frac{\mu_{♂}}{p}}sin(\theta^* - \theta_0^*)] = 6.5950 \times 10^{-5}(1/s) \quad (5)$$

$$\dot{g} = [1 - \frac{r_0}{p}(1 - cos(\theta^* - \theta_0^*))] = -0.11884(non - dimensional) \quad (6)$$

Note that units of the values in Equation (3) through Equation (6) are different. f and $\dot{g}$ are non-dimensional, because they relate r_0 to r or v_0 to v). g is in seconds because it converts velocity into position, and $\dot{f}$ is in 1/second because it does the opposite. And putting these values back in Equation (1) and Equation (2), we get $\bar{r}, \bar{v}$ in terms of $\bar{r_0}, \bar{v_0}$:

$$\boxed{\begin{aligned}\bar{r} &= -0.11884\bar{r_0} - 1.4949 \times 10^4(s)\bar{v_0} \quad (7)\\ \bar{v} &= 6.5950 \times 10^{-5}(1/s)\bar{r_0} - 0.11884\bar{v_0} \quad (8)\end{aligned}}$$

2) f and g relations: using ΔE

Only for elliptic orbits, we can use $\Delta E = 4.2156\ (rad) = 241.54°$ instead of $\Delta\theta^*$ and obtain following expressions for $f, g, \dot{f}, \dot{g}$.

$$\bar{r} = f\bar{r_0} + g\bar{v_0} = [1 - \frac{a}{r_0}(1 - cos(E - E_0))]\bar{r_0} + [(t - t_0) - \sqrt{\frac{a^3}{\mu_{♂}}}((E - E_0) - sin(E - E_0))]\bar{v_0} \quad (9)$$

$$\bar{v} = \dot{f}\bar{r_0} + \dot{g}\bar{v_0} = -\frac{\sqrt{\mu_{♂} a}}{rr_0}sin(E - E_0)\bar{r_0} + [1 - \frac{a}{r}(1 - cos(E - E_0))]\bar{v_0} \quad (10)$$

$$f = 1 - \frac{a}{r_0}(1 - cos(E - E_0)) = -0.11884(non - dimensional) \quad (11)$$

$$g = (t - t_0) - \sqrt{\frac{a^3}{\mu_♂}}((E - E_0) - sin(E - E_0)) = -1.4949 \times 10^4(s) \quad (12)$$

$$\dot{f} = -\frac{\sqrt{\mu_♂ a}}{rr_0} sin(E - E_0) = 6.5950 \times 10^{-5}(1/s) \quad (13)$$

$$\dot{g} = [1 - \frac{a}{r}(1 - cos(E - E_0))] = -0.11884(non - dimensional) \quad (14)$$

The values obtained from equation 9 and equation 10 are compared with equation 1 and equation 2. The difference in the computation is very small, thus both methods to calculate $f, g, \dot{f}, \dot{g}$ produce the same result within the represented digits. Using $\Delta\theta^*$ and ΔE to compute Lagrange coefficients result in the same values.

(d)

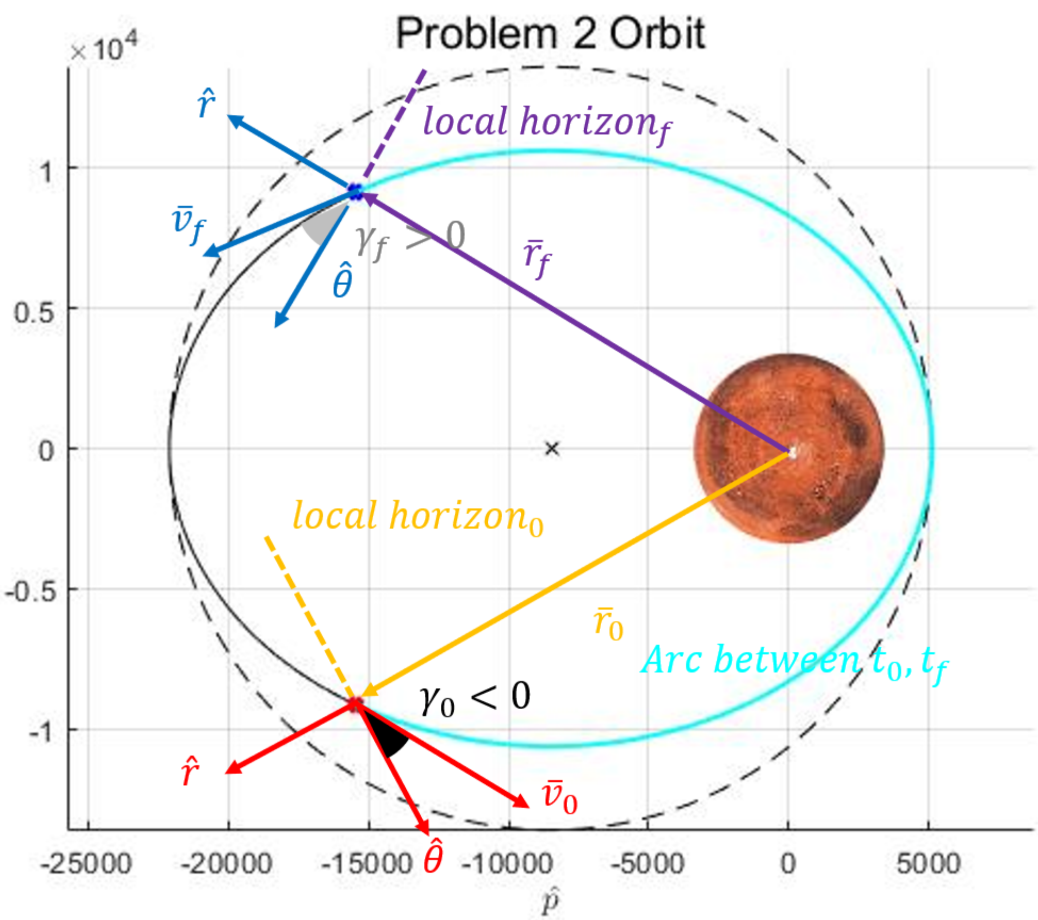

Figure 10: Problem 2 (d) (units: km)

Problem 3

Assume that a vehicle is currently moving in Earth orbit. Assume a two-body relative motion of the spacecraft with respect to Earth. The orbit is described with the following characteristics (relative to an Earth equatorial coordinate system) at the initial time t_1:

$$a = 20R_\oplus$$
$$e = 0.6$$
$$i = 34^\circ$$
$$\Omega = 45^\circ$$
$$\omega = 30^\circ$$
$$\theta = 235^\circ$$

(a) Determine the current state in terms of $\bar{r}, \bar{v}, r, v, \gamma, \theta^*, \mathbb{P}, M, E, (t - t_p)$; write $\bar{r}, \bar{v}$ in terms both rotating orbit unit vectors $(\hat{r}, \hat{\theta}, \hat{h})$, unit vectors $(\hat{n}_x, \hat{n}_y, \hat{n}_z)$ as well as inertial unit vectors $(\hat{x}, \hat{y}, \hat{z})$.

(b) Confirm the general results in GMAT with the conic propagator. Plot the GMAT image viewing down onto the orbit plane.

(c) Use Kepler's equation and determine the values of $\bar{r}, \bar{v}, \theta^*, \gamma$ in exactly 3 days, i.e., time t_2. For this value of $(t_2 - t_1)$, what are the corresponding values of $\theta_2^* - \theta_1^*, E_2 - E_1$. Confirm the result in GMAT.

(d) Plot the orbit in Matlab or GMAT. Mark $\bar{r}, \bar{v}$ at the two times; mark the usual quantities (vectors, local horizon, γ, θ^*) and highlight the arc between the two times.

Problem 3 Solution

(a)

$$\boxed{\begin{aligned}
&\theta^* = \theta - \omega = 205^\circ\\
&a = 20R_\oplus = 1.2756 \times 10^5 km\\
&\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu_\oplus}} = 4.5341 \times 10^5\ s = 5.2479\ days\\
&p = a(1-e^2) = 8.1640 \times 10^4 km\\
&r = \frac{p}{1+e\cos\theta^*} = 1.7895 \times 10^5 km\\
&\mathcal{E} = -\frac{\mu_\oplus}{2a} = -1.5624 km^2/s^2\\
&v = \sqrt{\frac{2\mu_\oplus}{r} - \frac{\mu}{a}} = 1.1533 km/s\\
&\gamma = cos^{-1}(\frac{h}{rv}) = -0.50730(rad) = -29.066^\circ(\because \theta^* > 180^\circ)\\
&E = 2 * tan^{-1}(\sqrt{\frac{1-e}{1+e}}tan(\frac{\theta^*}{2})) = -2.3069(rad) = -132.18^\circ = 227.82^\circ\\
&M = E - esinE = -1.8623(rad) = -106.70^\circ = 253.30^\circ\\
&t - t_p = M\sqrt{\frac{a^3}{\mu_\oplus}} = -1.3438 \times 10^5 s = -1.5554\ days = 3.6925\ days
\end{aligned}}$$

Calculating $\bar{r}, \bar{v}$

Three set of unit vectors are considered:

- Rotating orbit unit vectors($\hat{r}, \hat{\theta}, \hat{h}$): $\hat{r}$ is defined by the direction from the center of Earth to the spacecraft at a given time. And $\hat{h}$ is the direction of $\bar{r} \times \bar{v}$. Then $\hat{\theta} = \hat{h} \times \hat{r}$.

$$\boxed{\begin{aligned}
&\bar{r} = r\hat{r} = 1.7895 \times 10^5 \hat{r}(km) && (15)\\
&\bar{v} = v\sin\gamma\hat{r} + v\cos\gamma\hat{\theta} = -0.5603\hat{r} + 1.0081\hat{\theta}(km/s) && (16)
\end{aligned}}$$

- Inertial 1) Cartesian unit vectors($\hat{x}, \hat{y}, \hat{z}$): These vectors are defined by vernal equinox direction at J2000 epoch ($\hat{x}$), the Earth's rotational axis($\hat{z}$), and $\hat{y} = \hat{z} \times \hat{x}$. The relationship between these unit vectors with the rotating orbit unit vectors($\hat{r}, \hat{\theta}, \hat{h}$) are defined by three angles of rotation. From Cartesian unit vectors, we rotate Ω, i, θ angles with respect to 3-1-3 axis to get the rotating orbit unit vectors. Calculating the rotational matrix, we cat the following table 1 for dot products of unit vectors in both frame.

  $\bar{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z} = r\hat{r}$, using dot products, we can calculate r_x, r_y, r_z.

	$\hat{r}$	$\hat{\theta}$	$\hat{h}$
$\hat{x}$	$c_\Omega c_\theta - s_\Omega c_i s_\theta$	$-c_\Omega s_\theta - s_\Omega c_i c_\theta$	$s_\Omega s_i$
$\hat{y}$	$s_\Omega c_\theta + c_\Omega c_i s_\theta$	$-s_\Omega s_\theta - c_\Omega c_i c_\theta$	$-c_\Omega s_i$
$\hat{z}$	$s_i s_\theta$	$s_i c_\theta$	c_i

Table 1: dot products of unit vectors in two frames

$$r_x = r\hat{r}\cdot\hat{x} = r(\cos\Omega\cos\theta - \sin\Omega\cos i\sin\theta) = 0.1335\times 10^5\ km$$
$$r_y = r\hat{r}\cdot\hat{y} = r(\sin\Omega\cos\theta + \cos\Omega\cos i\sin\theta) = -1.5851\times 10^5\ km$$
$$r_z = r\hat{r}\cdot\hat{z} = r\sin i\sin\theta = -0.8197\times 10^5\ km$$
$$\boxed{\bar{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z} = 0.1335\times 10^5\hat{x} - 1.5851\times 10^5\hat{y} - 0.8197\times 10^5\hat{z}\ (km)}$$

Similarly, using dot products, we can calculate Cartesian components of $\bar{v} = v_x\hat{x}+v_y\hat{y}+v_z\hat{z} = v_r\hat{r} + v_\theta\hat{\theta}$.

$$v_x = v_r\hat{r}\cdot\hat{x} + v_\theta\hat{\theta}\cdot\hat{z} = v_r(\cos\Omega\cos\theta - \sin\Omega\cos i\sin\theta) + v_\theta(-\cos\Omega\sin\theta - \sin\Omega\cos i\cos\theta)$$
$$v_y = v_r\hat{r}\cdot\hat{y} + v_\theta\hat{\theta}\cdot\hat{y} = v_r(\sin\Omega\cos\theta + \cos\Omega\cos i\sin\theta) + v_\theta(-\sin\Omega\sin\theta - \cos\Omega\cos i\cos\theta)$$
$$v_z = v_r\hat{r}\cdot\hat{z} + v_\theta\hat{\theta}\cdot\hat{z} = v_r(\sin i\sin\theta) + v_\theta(\sin i\ cos\theta)$$
$$\boxed{\bar{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} = 0.8810\hat{x} + 0.7412\hat{y} - 0.0667\hat{z}\ \ (km/s)}$$

- Intermediate rotating frame $(\hat{n}_x, \hat{n}_y, \hat{n}_z)$

  This frame is is acquired by rotating the inertial frame $\hat{x}, \hat{y}, \hat{z}$ with respect to $\hat{z}$ by Ω. Thus the direction cosine matrix from the Inertial frame I to the intermediate rotating frame N is:

$${}^IC^N = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\boxed{\underbrace{\bar{r}}_{\hat{n}_x-\hat{n}_y-\hat{n}_z} = {}^IC^R \underbrace{\bar{r}}_{\hat{x}-\hat{y}-\hat{z}} = -1.0264\times 10^5\hat{n}_x - 1.2153\times 10^5\hat{n}_y - 0.8197\times 10^5\hat{n}_z\ (km)}$$
$$\boxed{\underbrace{\bar{v}}_{\hat{n}_x-\hat{n}_y-\hat{n}_z} = {}^IC^R \underbrace{\bar{v}}_{\hat{x}-\hat{y}-\hat{z}} = 1.1471\hat{n}_x - 0.0988\hat{n}_y - 0.0667\hat{n}_z\ (km/s)}$$

(b)

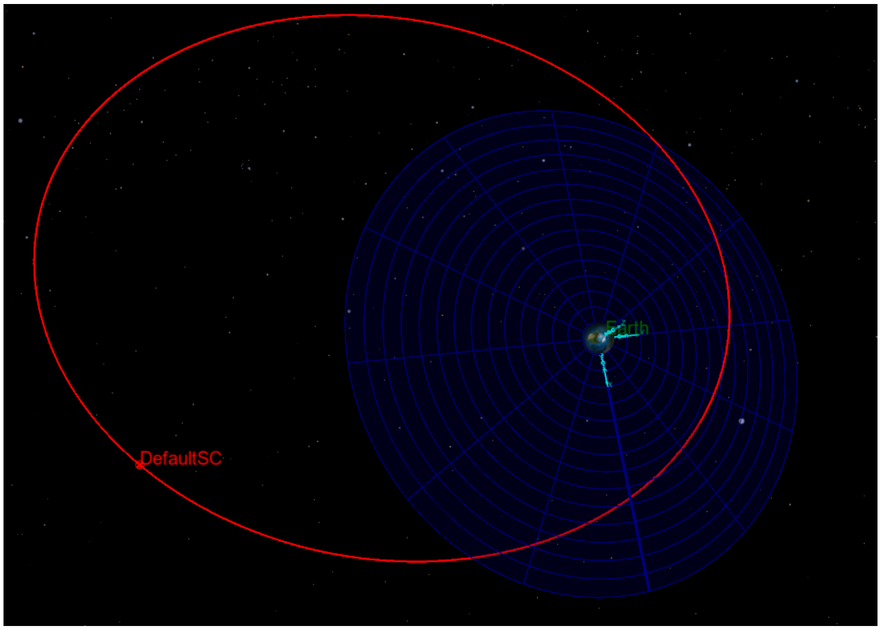

Figure 11: Problem 3 (b) GMAT plot

Orbit parameters	Manual calculation	GMAT result	Match
r_{x1}	1.3557×10^4 km	1.3557×10^4 km	○
r_{y1}	-1.5851×10^5 km	-1.5851×10^5 km	○
r_{z1}	-8.1970×10^4 km	-8.1970×10^4 km	○
v_{x1}	0.8810 km/s	0.8810 km/s	○
v_{y1}	0.7412 km/s	0.7412 km/s	○
v_{z1}	−0.0667 km/s	−0.0667 km/s	○
γ_1	$-29.07°$	$119.07°$	○
θ_1^*	205°	205°	○
E_1	227.82°	227.82°	○
θ_2^*	151.81°	151.81°	○
E_2	26.67°	26.67°	○

Table 2: Comparison between manual calculation and GMAT results: Problem 3

All of the values computed above were confirmed by GMAT results as in Table 2 (note that it lists only a portion of the computed values).

(c)

After 3 days, or $\Delta t = 3 \times 60 \times 60 \times 3 = 259200s$, mean anomaly can be calculated as below:

$$M = \sqrt{\frac{\mu_\oplus}{a^3}}(t - t_p) = \sqrt{\frac{\mu_\oplus}{a^3}}(-134387 + 259200) = 1.7296(rad) = 99.098^\circ$$

Using Kepler's Equation, eccentric anomaly is calculated:

$$E = 2.2108(rad) = 126.67^\circ$$

With this, true anomaly is calculated:

$$\boxed{\theta^* = 2tan^{-1}(\sqrt{\frac{1+e}{1-e}}tan(\frac{E}{2})) = 2.6496(rad) = 151.81^\circ}$$

$$r = \frac{p}{1 + ecos\theta^*} = 1.7327 \times 10^5 \ km$$

And the magnitude of velocity and the gamma angle:

$$v = \sqrt{\frac{2\mu_\oplus}{r} - \frac{\mu}{a}} = 1.2149km/s$$

$$\gamma = cos^{-1}(\frac{h}{rv}) = 0.5416(rad) = 31.029^\circ(\because \theta^* < 180^\circ)$$

Using the equations used in (a), the vector $\bar{r}$ can be expressed in three ways as below:

$$\boxed{\bar{r} = 1.7327 \times 10^5 \hat{r}(km)}$$

Likewise, the velocity vector is:

$$\boxed{\bar{v} = 0.6263\hat{r} + 1.0411\hat{\theta}(km/s)}$$

$$\boxed{\begin{aligned} &\theta_2^* - \theta_1^* = -53.19^\circ = 306.81^\circ \\ &E_2 - E_1 = 258.85^\circ \end{aligned}}$$

Note that even though $\theta_2^* < \theta_1^*$, we would want a positive angle for $\theta_2^* - \theta_1^*$ since it is the location at a later time. Same holds for the eccentric anomaly.

(d)

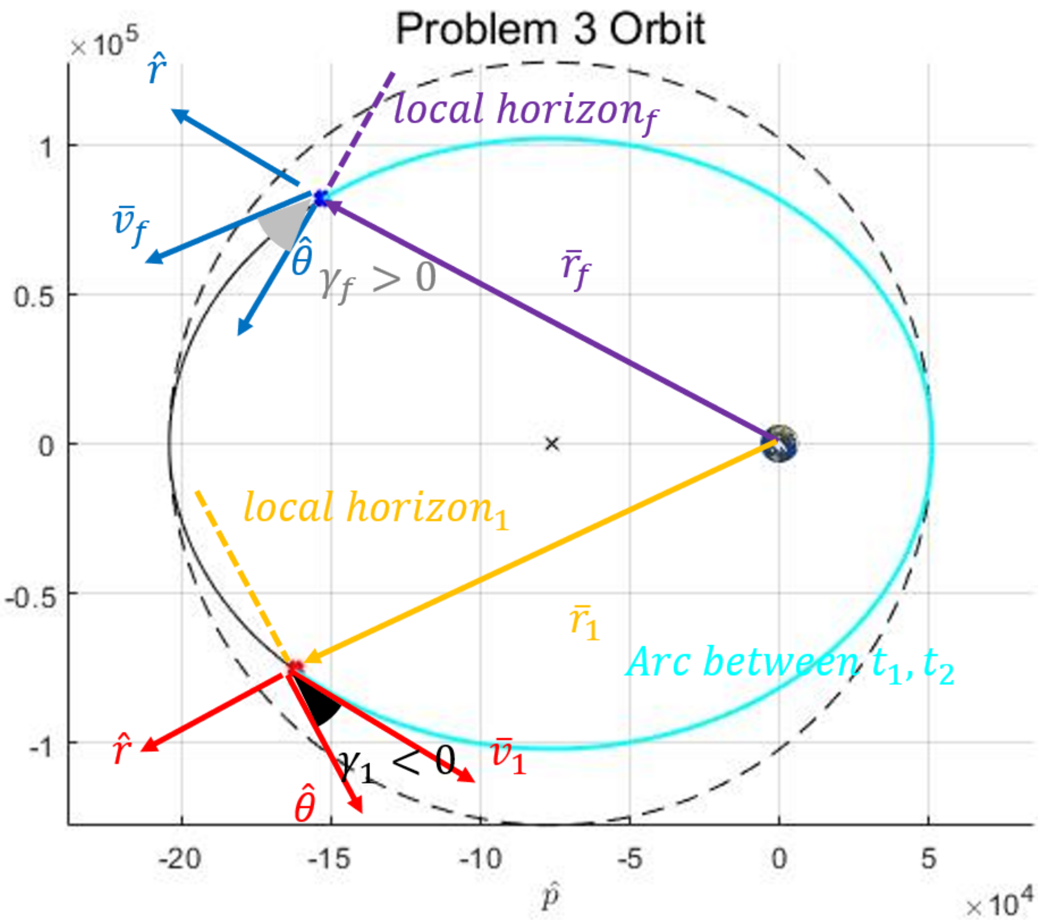

Figure 12: Problem 3(d): Orbit sketch (units: km)

Problem 4

A vehicle is moving in some Earth orbit; assume a two-body model. At a certain time, the following information is given:

$$\bar{r}_1 = 0.15R_\oplus \hat{x} - 1.44R_\oplus \hat{y} - 0.65R_\oplus \hat{z}$$
$$\bar{v}_1 = 6.62\hat{x} + 2.70\hat{y} - 1.56\hat{z} km/s$$

(a) Determine $a, e, i, \omega, \Omega, \gamma, \theta^*, M, E, (t - t_p)$. Are you sure it is an ellipse? Why? What quantity do you check to assess the type of conic?

(b) Sketch the orbit in the orbit plane: add $r, v, \theta^*, \lambda, l.h., \omega, \hat{n}_x$.

(c) Sketch the orbit in 3D (or a section of the orbit) to mark the following quantities: $\Omega, i, \hat{h}, AN, DN$, direction of mottion. Is periapsis above or below the fundamental plane? How do you know? What is θ^* at the AN? DN?

Problem 4 Solution

(a)

$$r \approx 1.46R_\oplus = 1.01 \times 10^4 km$$

Similarly, the magnitude of $\bar{v}$ is:

$$v = 7.31 km/s$$

With these two values, the energy of the orbit is calculated:

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu_\oplus}{r} = -12.6 km^2/s^2$$

Since the energy of the orbit is negative, this orbit is ellipse. We can also use the sign of a and the compare magnitude of the velocity with $\sqrt{2}v_c$ to check if it an ellipse.
Since $\mathcal{E} = -\frac{\mu_\oplus}{2a}$,

$$\boxed{a = \frac{\mu_\oplus}{2\mathcal{E}} = 15811.24 km}$$

Since $\mathcal{E} = \frac{v^2}{2} - \frac{\mu_\oplus}{r} = \frac{v_p^2}{2} - \frac{\mu_\oplus}{r_p} = \frac{h^2}{2r_p^2} - \frac{\mu_\oplus}{r_p} = \frac{\mu_\oplus^2(1+e)^2}{2h^2} - \frac{\mu_\oplus^2(1+e)}{h^2} = -\frac{\mu_\oplus^2}{2h^2}(1-e^2)$, $e = \sqrt{1 + \frac{2h^2\mathcal{E}}{\mu_\oplus^2}}$.

$$\bar{h} = \bar{r} \times \bar{v} = 2.552147\hat{x} - 2.595264\hat{y} + 6.338465\hat{z}(km^2/s)$$

$$h = 73092.40 km^2/s$$

$$\boxed{e = \sqrt{1 + \frac{2h^2\mathcal{E}}{\mu_\oplus^2}} = 0.39}$$

With a and e, we can check whether there is collision by calculating r_p

$$r_p = a(1-e) = 9640km > R_\oplus$$

So there is no collision.
And from conic equation,

$$\boxed{\theta^* = \pm cos^{-1}(\frac{1}{e}(\frac{h^2}{\mu_\oplus r} - 1)) = -33.843^\circ = 326.16^\circ}$$

$\dot{r} = \bar{v} \cdot \bar{r} = -11997.28 < 0$, we choose a negative true anomaly since it is in the descending leg.

$$\boxed{\gamma = -cos{-1}(\frac{h}{rv}) = -9.321325^\circ}$$

With θ^*, we can compute eccentric anomaly:

$$\boxed{E = 2tan^{-1}(\sqrt{\frac{1-e}{1+e}}tan(\frac{\theta^*}{2})) = -22.78^\circ = 337.22^\circ}$$

With E, we can compute mean anomaly:

$$\boxed{M = E - esinE = -0.2465095(rad) = -14.12^\circ = 345.88^\circ}$$

Since $M = \sqrt{\frac{\mu_\oplus}{a^3}}(t - t_p)$,

$$\boxed{(t - t_p) = M\sqrt{\frac{a^3}{\mu_\oplus}} = -776.27s}$$

negative time means it is 776.27 seconds before reaching the periapsis.
Now, looking at the other orbital elements(i, Ω, ω), we should calculate the direction of angular momentum($\hat{h}$ and transverse direction(θ).

$$\hat{r} = \bar{r}/r = 0.09451738\hat{x} - 0.9073668\hat{y} - 0.4095752\hat{z}$$

$$\hat{h} = \frac{\bar{h}}{h} = 0.3491673\hat{x} - 0.3550662\hat{y} + 0.8671852\hat{z}$$

$$\hat{\theta} = \hat{h} \times \hat{r} = 0.9322814\hat{x} + 0.2249744\hat{y} - 0.2832629\hat{z}$$

With these three unit vectors of rotating orbit unit vectors, we can calculate i, Ω, θ by using the dot products listed in Table 1.

$$\boxed{i = cos^{-1}(\hat{h} \cdot \hat{z}) = cos^{-1}0.8671852 = 29.87^\circ}$$

Note that there is no quadrant ambiguity since $0 \leq i \leq 180^\circ$. Going on to compute Ω, use $\sin\Omega \sin i = \hat{h} \cdot \hat{x}$ and $-\cos\Omega \sin i = \hat{h} \cdot \hat{y}$

$$\Omega = sin^{-1}(\frac{\hat{h} \cdot \hat{x}}{\sin i}) = 44.52° \; or \; 135.48°$$
$$\Omega = \cos^{-1}(-\frac{\hat{h} \cdot \hat{y}}{\sin i}) = \pm 44.52° \Rightarrow$$
$$\boxed{\Omega = 44.52°}$$

Next is θ, and we can use $\sin i \sin\theta = \hat{r} \cdot \hat{z}$, $\sin i \cos\theta = \hat{\theta} \cdot \hat{z}$.

$$\theta = sin^{-1}(\frac{\hat{r} \cdot \hat{z}}{\sin i}) = -55.33221° or - 124.66779°$$
$$\theta = cos^{-1}(\frac{\hat{\theta} \cdot \hat{z}}{\sin i}) = \pm 124.66779° \Rightarrow$$
$$\boxed{\theta = 235.3322° = -124.6678°}$$

The quadrant ambiguity is solved by applying two equations when computing Ω, θ.
By definition:

$$\boxed{\omega = \theta - \theta^* = 235.3322 - 326.1572 = -90.8250° = 269.1750°}$$

(b)

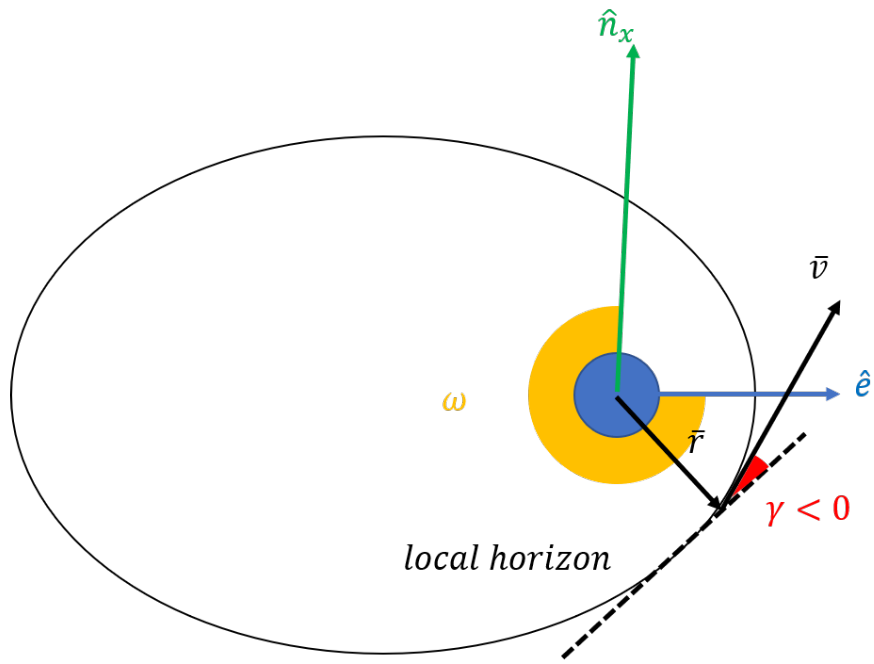

Figure 13: Problem 4 (b): Orbit plane sketch

Note that the sketch is not to scale, and eccentricity is not drawn to it's exact value. It is just a sketch!

(c)

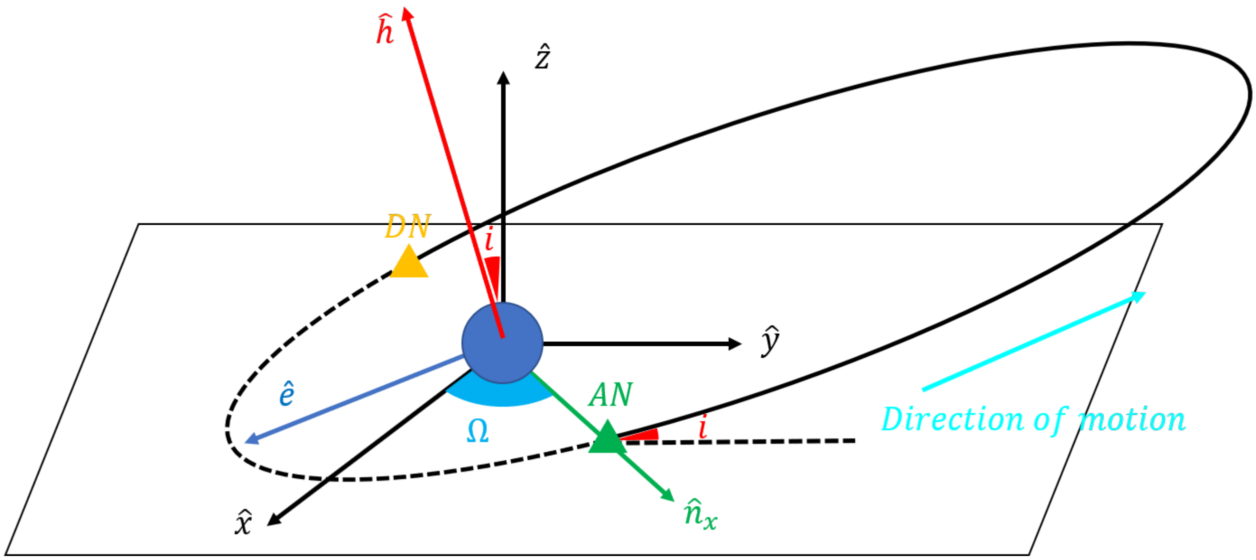

Figure 14: Problem 4 (c): 3D Orbit sketch

We know that the periapsis is below the fundamental plane from the value of ω. Since $180° < \omega < 360°$, the periapsis is below the fundamental plane.
We could decide the true anomaly at the ascending node and descending node using the Table 1. We know at these points, the $\hat{z}$ component of the position vector should be zero:

$$\begin{aligned}
&\hat{r} \cdot \hat{z} = \sin i \sin \theta = 0 \Rightarrow \\
&\sin \theta = 0 \Rightarrow \\
&\theta = 0°, 180°
\end{aligned}$$

And using the definition of θ, we have:

$$\boxed{\theta^* = \theta - \omega = 90.8250° \ (AN), -89, 175° (DN)}$$

Note that from the sketch, we know that true anomaly at the ascending node should be between 0, 180°, and vice versa.