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2020年3月6日 金曜日　　午後0:31

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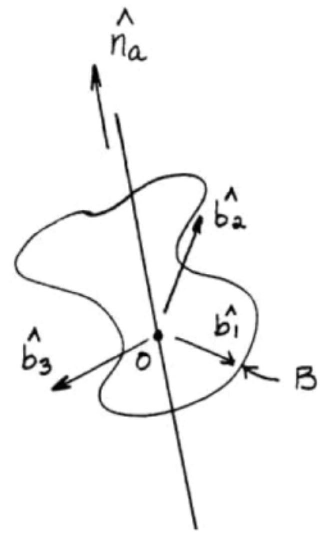
Inertia Ellipsoid

* each pt. has its unique principal direction

For any body B, we can define principal unit vectors $\left(\hat{b}\right)$ for any point O (could be the cm)



$\hat{b}$ principal for pt. O

$$\overline{\overline{I}}^{B/O} = I_1\hat{b}_1\hat{b}_1 + I_2\hat{b}_2\hat{b}_2 + I_3\hat{b}_3\hat{b}_3$$

We can determine the inertia dyadic / inertia matrix for any other vector basis through the similarity transformation

$$\underset{\hat{n}}{[I]} = \underset{\hat{n}\bullet\hat{b}}{[\ell]}\,\underset{\hat{b}}{[I]}\,[\ell]^T$$

shift inertia to different unit vector

$$I = L\,I'\,L^T$$

$$I = C^T\,I'\,C$$

$$I_{ab} = \hat{n}_a \bullet \overline{\overline{I}}^{B/O} \bullet \hat{n}_b$$

specifically to obtain <u>moment of inertia</u> associated with the direction $\hat{n}_a$:

$$\left.I_{aa} = \hat{n}_a \cdot \underset{\hat{b}}{\overline{\overline{I}}^{B/O}} \cdot \hat{n}_a \right\} {}^N C^B$$

Evaluate these dot products using a direction cosine matrix

${}^{N}C^{B}$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{n}_a$	C_{a1}	C_{a2}	C_{a3}
$\hat{n}_b$	C_{b1}	C_{b2}	C_{b3}
$\hat{n}_c$	C_{c1}	C_{c2}	C_{c3}

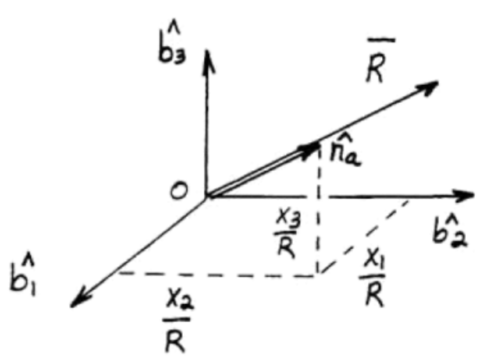
$$I_{aa} = \hat{n}_a \bullet \overline{\overline{I}}^{B/O} \bullet \hat{n}_a$$

$$I_{aa} = I_1 C_{a1}^2 + I_2 C_{a2}^2 + I_3 C_{a3}^2$$

Use this representation to visually assess I_{aa} in relation to I_1, I_2, I_3

Note that I_{aa} can stand in for <u>all</u> possible $\hat{n}_a$ directions

Define some vector R in direction $\hat{n}_a$



$$\overline{R} = R\,\hat{n}_a$$
$$= x_1\hat{b}_1 + x_2\hat{b}_2 + x_3\hat{b}_3$$

$$\hat{n}_a = \frac{x_1}{R}\hat{b}_1 + \frac{x_2}{R}\hat{b}_2 + \frac{x_3}{R}\hat{b}_3$$

$$\hat{n}_a = C_{a1}\hat{b}_1 + C_{a2}\hat{b}_2 + C_{a3}\hat{b}_3$$

$$C_{a1} \leftrightarrow \frac{x_1}{R} \qquad C_{a2} \leftrightarrow \frac{x_2}{R} \qquad C_{a3} \leftrightarrow \frac{x_3}{R}$$

$$I_{aa} = I_1 C_{a1}^2 + I_2 C_{a2}^2 + I_3 C_{a3}^2$$

$$\boxed{I_{aa} = I_1 \frac{x_1^2}{R^2} + I_2 \frac{x_2^2}{R^2} + I_3 \frac{x_3^2}{R^2}}$$

We want to use R to represent how I_{aa} changes over different directions

We want R to be related to I_{aa}; if $\hat{n}_a$ differs in direction, then the value of R adjusts; all values of R taken together creates a surface

Define $\boxed{R = k\, I_{aa}^{-\frac{1}{2}}} = k \frac{1}{\sqrt{I_{aa}}}$

where k is simply a scale factor → arbitrary; raw "size" of R is not significant; it only has **relative** value

$$I_{aa} = \frac{x_1^2}{\left(R\, I_1^{-\frac{1}{2}}\right)^2} + \frac{x_2^2}{\left(R\, I_2^{-\frac{1}{2}}\right)^2} + \frac{x_3^2}{\left(R\, I_3^{-\frac{1}{2}}\right)^2}$$

$k I_{aa}^{-1/2}$

$$I_{aa} = \frac{x_1^2}{\left(k I_{aa}^{-\frac{1}{2}} I_1^{-\frac{1}{2}}\right)^2} + \frac{x_2^2}{\left(k I_{aa}^{-\frac{1}{2}} I_2^{-\frac{1}{2}}\right)^2} + \frac{x_3^2}{\left(k I_{aa}^{-\frac{1}{2}} I_3^{-\frac{1}{2}}\right)^2}$$

divide by I_{aa}

$$\boxed{1 = \frac{x_1^2}{(k I_1^{-1/2})^2} + \frac{x_2^2}{(k I_2^{-1/2})^2} + \frac{x_3^2}{(k I_3^{-1/2})^2}}$$

Equation of an Ellipsoid

with semi-diameters

$$d_1 = k I_1^{-1/2} \quad d_2 = k I_2^{-1/2} \quad d_3 = k I_3^{-1/2}$$

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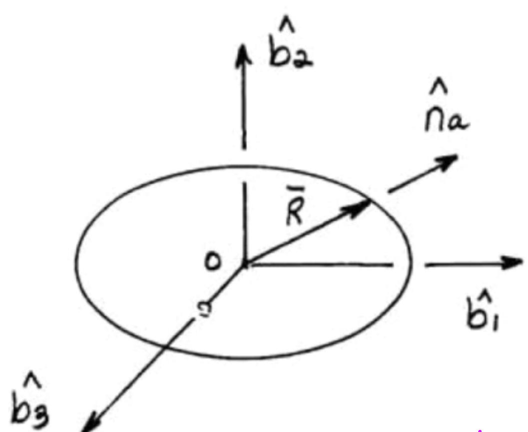


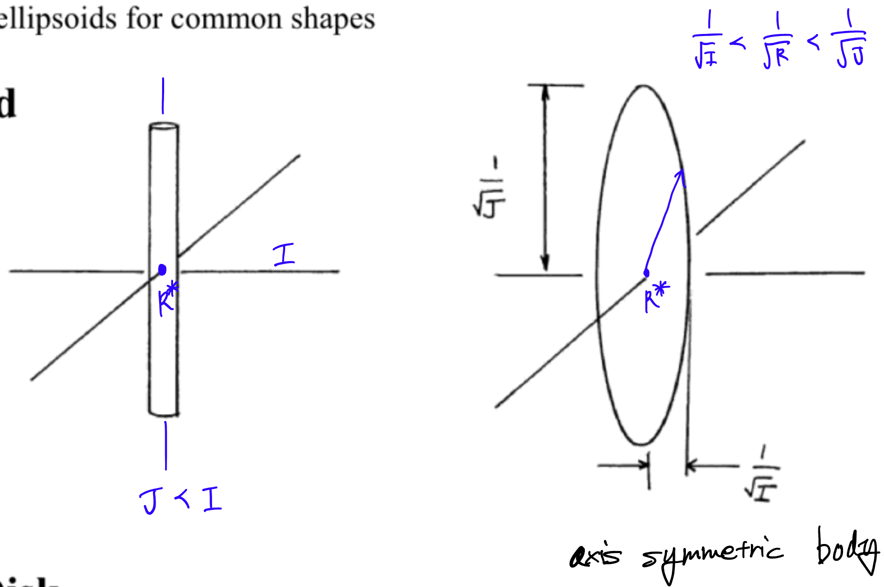
inertia ellipsoid

Determining x_i provides the components of vector $\bar{R}$ in the $\hat{n}_a$ direction

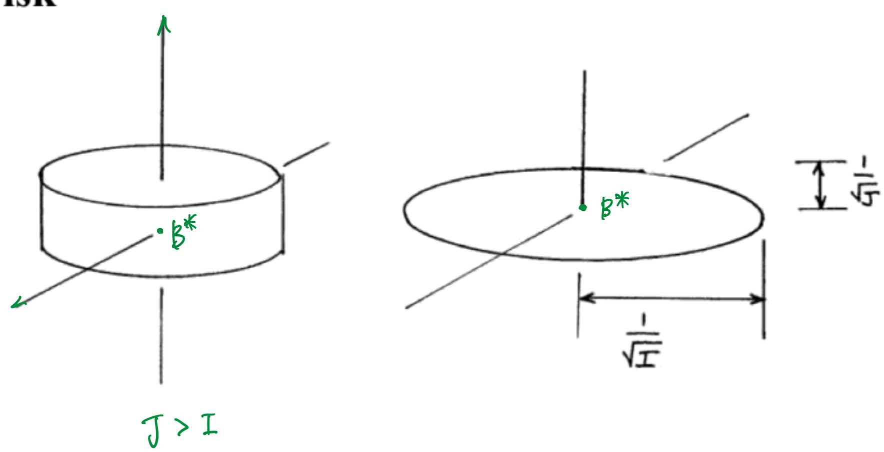
Values of x_1, x_2, x_3 must be such that R values calculated in all directions produce an ellipsoid

Inertia ellipsoids for common shapes

Rod



Disk



Observations:

1. Dynamically every rigid body can be represented by its corresponding inertia ellipsoids.
   In the current EOMs, size and shape have no meaning – only the inertia characteristics. Bodies can be represented and compared on the basis of the ellipsoids. (Body may not be axisymmetric; if inertia ellipsoid is a body of revolution, can be analyzed as such.)

2. Inertia ellipsoid represents inertia properties for one point. Axes (that include the largest and smallest distances to the ellipsoid surface) are parallel to the principal directions.
   ∴ No moment of inertia is smaller than the smallest principal moment; no moment of inertia is larger than the largest principal moment.

3. Spherical inertia ellipsoid through the point are equal $\longrightarrow$ moments of inertia for <u>all</u> lines and $\bar{J}^{(2)} \ldots \bar{J}^{(3)} \ldots \bar{J}^{(\infty)} = 0$

4. Energy/Poinsot ellipsoid concentric with and proportional to the inertia ellipsoid.
   ∴ The major axes are in the same directions and the ratio of semi-diameters is the same

If the inertia characteristics are known, then the principal are known; then, the torque-free solution is known!