Final Exam
AE 6511 Optimal Guidance and Control

Instructions

1. This is a **open-book, open-notes** exam
2. You will need to upload the exam to Canvas by **5:00pm ET on December 9, 2021**
3. **No collaboration of any kind** between students is allowed
4. Include all intermediate steps for full credit. Box your answer and **state the solution clearly**
5. **Points will be subtracted for sloppiness**
6. Total number of points is 100

Good luck!

Student Agreement

I certify that I have read and understand the above ground rules for the exam. I also understand that any violations of these rules or those of the Georgia Tech Honor Code will be treated as a violation of the Honor Code.

Name and Signature ______________________________

1. (**10pts**) Consider the problem of maximizing the range of a rocket plane in horizontal flight. The equations of motion of the rocket are

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= v \\ m\frac{\mathrm{d}v}{\mathrm{d}t} &= c\beta - D \\ \frac{\mathrm{d}m}{\mathrm{d}t} &= -\beta \end{aligned}$$

where c is a positive constant, D is the drag which depends on speed v and lift L as follows

$$D = Av^2 + BL^2, \qquad A \text{ and } B = \text{const.} > 0$$

and the lift L is adjusted to balance the weight mg. The thrust depends on the rate of fuel consumption by

$$T = -c\frac{\mathrm{d}m}{\mathrm{d}t}$$

We wish to determine how the thrust T must be varied in order to maximize the range $x(t_1) - x(0)$. Deduce the extremals for this problem. (Hint: Use mass as the independent variable and find the optimal velocity profile as a function of the mass).

2. (**10pts**) Consider the linear dynamic system that is self-adjoint, that is,

$$\dot{x} = Fx + u, \qquad F = -F^{\mathsf{T}}$$

Determine $u(t)$ subject to the constraint $u(t) \in U = \{u : \|u\|^2 = 1\}$ such that $x(t_\mathrm{f}) = 0$, where t_f is a minimum. Find the feedback solution; i.e., express $u(t)$ explicitly as a function of x and t.

3. (**10pts**) Consider the minimum fuel problem with cost

$$J(u) = \int_0^{t_\mathrm{f}} |u(t)|\,\mathrm{d}t$$

with the scalar dynamics $\dot{x}(t) = -ax(t) + u(t)$, the control constraint

$$|u(t)| \le u_{\max}$$

and boundary conditions $x(0) = x_0$ and $x(t_\mathrm{f}) = 0$. Here, $a > 0$ and x_0, t_f fixed.

   (a) Determine the set of initial conditions x_0 for which the endpoint constraint $x(t_\mathrm{f}) = 0$ can be satisfied. Your answer should be in the form of an inequality.

   (b) Use the minimum principle to determine the optimal control, including an explicit expression for the switch times. (Hint: The optimal control is a “coast-burn” strategy.)

   (c) Show that the optimal control can be written as a time-varying switching feedback control law $u(t) = \phi(x(t), t)$. Sketch some optimal trajectories to demonstrate the control law.

4. (**10pts**) Consider the problem

$$\min J(u) = \tfrac{1}{2} \int_0^T x_1^2 \, \mathrm{d}t$$

subject to the constraints

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= -u \end{aligned}$$

where

$$|u| \leq 1$$

and with boundary conditions

$$x_1(0) = 0, \qquad x_2(0) = -0.5, \qquad x_1(T) = x_2(T) = 0$$

and the final time T is free.

(a) What is the optimal trajectory?

(b) Can we reach the origin using only bang-bang control?

(c) Compare the cost of the optimal strategy versus the pure bang-bang strategy.

(d) What is the terminal time for the two control strategies?

5. (**10pts**) Find the path of minimum time connecting two points on the surface of the Earth through a tunnel in the Earth. The tunnel is assumed to be evacuated, and friction is negligible. The only force acting on the particle is gravity. Note that the gravitational force per unit mass inside the Earth is directed radially toward the center of the Earth and increases *linearly* with the radius from zero at the center. (Hint: use spherical coordinates).

6. (**10pts**) Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u_1(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= u_2(t) \end{aligned}$$

with initial conditions

$$x_1(0) = 1, \quad x_2(0) = 0, \quad x_3(0) = -1, \quad x_4(0) = 0$$

terminal conditions

$$x_1(t_\mathrm{f}) = x_3(t_\mathrm{f}), \quad x_2(t_\mathrm{f}) = x_4(t_\mathrm{f})$$

with the final time free, and performance index

$$J(u_1, u_2) = \tfrac{1}{2}(x_2(t_\mathrm{f}) - 2)^2 + \tfrac{1}{2} \int_0^{t_\mathrm{f}} (u_1^2(t) + u_2^2(t)) \, \mathrm{d}t$$

Find the optimal control for this problem. Plot the trajectories of the system.

7. (**40pts**) Consider the problem of the longitudinal flight of a hypersonic vehicle over a spherical Earth. The equations of motion are given by

$$\begin{aligned}
\dot{r} &= v \sin\gamma \\
\dot{\phi} &= \frac{v}{r}\cos\gamma \\
\dot{v} &= -\frac{D}{m} - \frac{\mu \sin\gamma}{r^2} \\
\dot{\gamma} &= \frac{L}{mv} - \frac{\mu\cos\gamma}{r^2 v} + \frac{v\cos\gamma}{r}
\end{aligned}$$

where r is the distance from center of the Earth, ϕ is the longitude angle, v is the velocity, and γ is the flight-path angle. The control is the angle of attack α. The lift and drag forces are given by

$$L = \frac{1}{2}\rho(r)v^2 S C_L$$

where $C_L = a_0 + a_1\alpha$, and

$$D = \frac{1}{2}\rho(r)v^2 S C_D$$

where $C_D = b_0 + b_1\alpha + b_2\alpha^2$ where α is in deg, and where we assume an exponential model of the density

$$\rho(r) = \rho_0 \exp\left(-\frac{r - r_0}{H_s}\right)$$

We wish to maximize the final value of the longitude, that is,

$$\max\ \mathcal{J} = -\phi(t_\mathrm{f})$$

where the final time t_f is unspecified. The boundary conditions for this problem are given below

State	Initial Value	Final Value	Units
h	121.9	30.48	km
ϕ	-25	28.61	deg
v	7,626	908.15	m/s
γ	-1.25	$[-6, 0]$	deg

The vehicle is subject to path constraint as follows: :

$$\Lambda = k_\lambda \sqrt{\rho}\, v^3 \leq \Lambda_{\max} \qquad \text{(Heating)}$$

$$q = \frac{1}{2}\rho v^2 \leq q_{\max} \qquad \text{(Dynamic Pressure)}$$

$$n = \frac{\sqrt{L^2 + D^2}}{m} \leq n_{\max} \qquad \text{(Normal Load)}$$

The vehicle parameters and the aerodynamic and atmospheric model parameters are given in the tables below.

(a) Using a code of your choice, compute the optimal control and the corresponding optimal trajectory. For better numerical accuracy, you may want to non-dimensionalize your equations using as length scale r_0 the final altitude, and the time scale as $\tau = \sqrt{r_0/g}$.

Parameter	Value	Units
S	149.3881	m^2
m	38,000	kg
$\Lambda_{\max}$	4×10^5	W/m^2
$q_{\max}$	14,500	kg/ms^2
$n_{\max}$	$5g_0$	m/s^2
k_λ	9.4369×10^{-5}	$kg^{0.5}/m^{1.5}$

Parameter	Value	Units
a_0	-0.20704	-
a_1	0.029244	-
b_0	0.07854	-
b_1	-0.61592×10^{-2}	-
b_2	0.621408×10^{-3}	-

Parameter	Value	Units
μ	3.986×10^{14}	m^3/s^2
ρ_0	1.225	kg/m^3
H_s	7,254.24	m
r_0	6,371	km

(b) Plot the optimal control $\alpha(t)$ and the optimal trajectory $x(t) = (r(t), \phi(t), v(t), \gamma(t)))$ for $0 \leq t \leq t_{\mathrm{f}}$.

(c) Plot the time history of the Hamiltonian along with history of the co-states.

(d) Develop a neighboring guidance scheme to augment the nominal optimal control you computed from part a).

(e) Demonstrate the benefits of the neighboring optimal guidance scheme by applying a small constant vertical wind disturbance acting on the vehicle at $t = 1,000$ sec. Plot the open-loop and closed-loop trajectories of the vehicle subject to perturbations for different values of the magnitude of the wind strength.

(f) Evaluate your neighboring guidance scheme on parametric variations of the nominal values of the aerodynamic parameters a_0, a_1, b_0, b_1, b_2 by introducing a perturbation to their nominal values. Which of these parameters has the largest effect on the resulting trajectories? Plot several trajectories ($\sim 10-20$) with $\pm10\%$ random variations in these parameters from their nominal values above, and compare the resulting trajectories with and without the neighboring guidance scheme.