College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 5
Eigenvalues and Vectors of LTI Systems

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Exercise 1

Compute the **eigenvalues** and **eigenvectors** of the matrix

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & 1 \end{pmatrix}$$

From the definition of eigenvalues $\lambda v = Av$ where $v \neq 0$,

$$det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 2 & 3 & 0 \\ -2 & \lambda + 3 & 0 \\ -3 & 5 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 2)\begin{vmatrix} \lambda + 3 & 0 \\ 5 & \lambda - 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 0 \\ -3 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda + 3)(\lambda - 1) + 6(\lambda - 1) = 0$$

$$\lambda^3 - \lambda = 0$$

$$\lambda(\lambda - 1)(\lambda + 1) = 0$$

Thus, the eigenvalues are

$$\boxed{\lambda = -1, 0, 1}$$

When $\lambda = -1$

$$\lambda I - A = \begin{pmatrix} -3 & 3 & 0 \\ -2 & 2 & 0 \\ -3 & 5 & -2 \end{pmatrix}$$

For this matrix conduct a Gaussian Elimination.

Cancel the leading column of row 2: $R_2 \rightarrow R_2 - \frac{2}{3}R_1$

$$\sim \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ -3 & 5 & -2 \end{pmatrix}$$

Cancel the leading column of row 3: $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{pmatrix}$$

Swap the rows 2 and 3: $R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -3 & 3 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Divide row 1 by -3 and row 2 by 2:

$$\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1: $R_1 \rightarrow R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - x_3 & = & 0 \\ x_2 - x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

When $\lambda = 0$

$$\lambda I - A = \begin{pmatrix} -2 & 3 & 0 \\ -2 & 3 & 0 \\ -3 & 5 & -1 \end{pmatrix}$$

For this matrix conduct a Gaussian Elimination.

Swap rows 1 and 3: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -3 & 5 & -1 \\ -2 & 3 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \rightarrow R_2 - \frac{2}{3} R_1$

$$\sim\begin{pmatrix} -3 & 5 & -1 \\ 0 & -1/3 & 2/3 \\ -2 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \to R_3 - \frac{2}{3}R_1$

$$\sim\begin{pmatrix} -3 & 5 & -1 \\ 0 & -1/3 & 2/3 \\ 0 & -1/3 & 2/3 \end{pmatrix}$$

Divide row 1 by -3, multiply row 2 and 3 by -3:

$$\sim\begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}$$

Cancel out row 3 with row 2: $R_3 \to R_3 - R_2$

$$\sim\begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1: $R_1 \to R_1 + \frac{5}{3}R_2$

$$\sim\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - 3x_3 & = & 0 \\ x_2 - 2x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 3/2 \\ 1 \\ 1/2 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}}$$

When $\lambda = 1$

$$\lambda I - A = \begin{pmatrix} -1 & 3 & 0 \\ -2 & 4 & 0 \\ -3 & 5 & 0 \end{pmatrix}$$

For this matrix conduct a Gaussian Elimination.

Swap rows 1 and 3: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ -2 & 4 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \to R_2 - \frac{2}{3}R_1$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ 0 & 2/3 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \to R_3 - \frac{1}{3}R_1$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ 0 & 2/3 & 0 \\ 0 & 4/3 & 0 \end{pmatrix}$$

Cancel row 3: $R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply row 2 by 3/2 and then cancel the second column in row 1:

$$\sim \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Divide the first row by -3

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 & = & 0 \\ x_2 & = & 0 \end{pmatrix}$$

Let $x_3 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

MATLAB Verification

```
% Exercise 1
A = [2,-3,0;2,-3,0;3,-5,1];
[v, d] = eig(A)
```

```
v = 3×3
         0   -0.5774   -0.8018
         0   -0.5774   -0.5345
    1.0000   -0.5774   -0.2673
```

```
d = 3×3
    1.0000         0         0
         0   -1.0000         0
         0         0    0.0000
```

Exercise 2

Compute the **eigenvalues** and **eigenvectors** of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

From the definition of eigenvalues $\lambda v = Av$ where $v \neq 0$,

$$det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\lambda \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\lambda \left(\lambda \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 0 & \lambda \end{vmatrix} \right) - \left(- \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} \right) = 0$$

$$\lambda^4 - 1 = 0$$

Thus, the eigenvalues are

$$\boxed{\lambda = -1, 1, -i, i}$$

When $\lambda = -1$,

$$\lambda I - A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$

Perform Gaussian Elimination on this matrix.

Cancel the leading column of row 4: $\mathrm{R}_4 \rightarrow \mathrm{R}_4 - \mathrm{R}_1$

$$\sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Cancel the second column of row 4: $R_4 \rightarrow R_4 + R_2$

$$\sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

Cancel row 4 with row 3:

$$\sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1 with row 2:

$$\sim \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 or row 2 with row 3:

$$\sim \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 or row 1 with row 3:

$$\sim \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply the matrix with -1

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 + x_4 & = & 0 \\ x_2 - x_4 & = & 0 \\ x_3 + x_4 & = & 0 \end{pmatrix}$$

Let $x_4 = s$ be a free variable. Then,

$$x = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}}$$

When $\lambda = 1$,

$$\lambda I - A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Cancel the leading column of row 4 with row 1: $R_4 \rightarrow R_4 + R_1$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Cancel the leading column of row 4 and column 2 of row 1 with row 2: $R\_4 + R\_2$ & $R\_1 + R\_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Cancel row 4 with row 3:

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1 and 2 with row 3

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - x_4 & = & 0 \\ x_2 - x_4 & = & 0 \\ x_3 - x_4 & = & 0 \end{pmatrix}$$

Let $x_4 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

When $\lambda = i$,

$$\lambda I - A = \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ -1 & 0 & 0 & i \end{pmatrix}$$

Cancel the leading column of row 4: $R_4 \to R_4 - iR_1$

$$\sim \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & i & 0 & i \end{pmatrix}$$

Cancel the leading column of row 4 with row 2: $R_4 \to R_4 - R_2$

$$\sim \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{pmatrix}$$

Cancel row 4 with row 3: $R_4 \to R_4 + iR_3$

$$\sim \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 or row 1 with row 2: $R_1 \to R_1 - iR_2$

$$\sim \begin{pmatrix} i & 0 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 of row 2 with row 3: $R_2 \to R_2 - iR_3$

$$\sim \begin{pmatrix} i & 0 & i & 0 \\ 0 & i & 0 & i \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 of row 1 with row 3: $R_1 \rightarrow R_1 - R_3$

$$\sim \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & i & 0 & i \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply row 1 by $-i$, divide row 2 by i, and multiply row 3 by $-i$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - ix_4 & = & 0 \\ x_2 + x_4 & = & 0 \\ x_3 + ix_4 & = & 0 \end{pmatrix}$$

Let $x_4 = s$ be a free variable. Then,

$$x = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_3 = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}}$$

When $\lambda = -i$,

$$\boxed{v_4 = \overline{v_3} = \begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}}$$

MATLAB Verification

```
% Exercise 2
A = [0,1,0,0;0,0,1,0;0,0,0,1;1,0,0,0]
[v, d] = eig(A)
```

```
v = 4×4 complex
  -0.5000 + 0.0000i   0.0000 - 0.5000i   0.0000 + 0.5000i  -0.5000 + 0.0000i
   0.5000 + 0.0000i   0.5000 + 0.0000i   0.5000 - 0.0000i  -0.5000 + 0.0000i
  -0.5000 + 0.0000i  -0.0000 + 0.5000i  -0.0000 - 0.5000i  -0.5000 + 0.0000i
   0.5000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i
```

```
d = 4×4 complex
  -1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
```

Exercise 3

Determine whether or not the following matrix is nondefective.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

From the definition of eigenvalues $\lambda v = Av$ where $v \neq 0$,

$$det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & -1 & \lambda - 1 \end{vmatrix} = 0$$

$$\lambda \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = 0$$

$$\lambda\big((\lambda)(\lambda - 1) - 1\big) + 1 = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$$(\lambda + 1)^2(\lambda - 1) = 0$$

Thus, the eigenvalues are

$$\boxed{\lambda = -1, -1, 1}$$

We must be careful because -1 is a repeated eigenvalue.

When $\lambda = 1$,

$$\lambda I - A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$$

The reduced echelon form of this matrix is

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Considering $\lambda I - A = 0$, express this with $x = [x_1, x_2, x_3]$,

$$\begin{pmatrix} x_1 - x_3 & = & 0 \\ x_2 - x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$, as a free variable,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_{1,2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$$

When $\lambda = -1$,

$$\lambda I - A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

The reduced echelon form of this matrix is

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Considering $\lambda I - A = 0$, express this with $x = [x_1, x_2, x_3]$,

$$\begin{pmatrix} x_1 - x_3 & = & 0 \\ x_2 + x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$, as a free variable,

$$x = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$\boxed{v_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}}$$

The algebraic multiplicity, $a.m.$ and geometric multiplicity, $g.m.$ are

$$\boxed{a.m. = 3 \quad > \quad g.m. = 2}$$

Thus, matrix A is defective.

MATLAB Verification

```
% Exercise 3
A = [0,1,0; 0,0,1; -1,1,1]
[v, d] = eig(A)

ech1 = rref(real(d(1,1))*eye(3) - A);
ech2 = rref(real(d(3,3))*eye(3) - A);
```

```
v = 3×3 complex
   0.5774 - 0.0000i   0.5774 + 0.0000i  -0.5774 + 0.0000i
   0.5774 - 0.0000i   0.5774 + 0.0000i   0.5774 + 0.0000i
   0.5774 + 0.0000i   0.5774 + 0.0000i  -0.5774 + 0.0000i
```

```
d = 3×3 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 - 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -1.0000 + 0.0000i
```

```
ech1 = 3×3
    1.0000         0   -1.0000
         0    1.0000   -1.0000
         0         0         0
```

```
ech2 = 3×3
     1     0    -1
     0     1     1
     0     0     0
```

Exercise 4

What is the companion matrix whose eigenvalues are $-1, -2, and\ -3$?

From the eigenvalues, we can find the following polynomial equation

$$p(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

$$p(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

Thus, the companion matrix becomes

$$\boxed{C(p) = \begin{pmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{pmatrix}}$$

MATLAB Verification

```
% Exercise 4
u = [1, 6, 11, 6];
C = compan(u)
```

```
C = 3×3
    -6   -11    -6
     1     0     0
     0     1     0
```

Exercise 5

What is the real 2 x 2 companion matrix with eigenvalues $1 + j, 1 - j$?

From the eigenvalues, we can find the following polynomial equation

$$p(\lambda) = (\lambda - 1 + j)(\lambda - 1 - j)$$

$$p(\lambda) = \lambda^2 - (1 + j + 1 - j)\lambda + (1 + j)(1 - j)$$

$$p(\lambda) = \lambda^2 - 2\lambda + 2$$

Thus, the companion matrix becomes

$$\boxed{C(p) = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}}$$

MATLAB Verification

```
% Exercise 5
u = [1, -2, 2];
C = compan(u)
```

```
C = 2×2
     2    -2
     1     0
```

Exercise 6

Suppose A is a real square matrix and the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}$$

are eigenvectors of A corresponding to eigenvalues -1 and 2 and j, respectively. What is the response $x(t)$ of the system $\dot{x} = Ax$ to the following initial conditions.

$$(a)\ \ x(0) = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} \quad (b)\ \ x(0) = \begin{pmatrix} -1 \\ -2 \\ -4 \\ -8 \end{pmatrix} \quad (c)\ \ x(0) = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 9 \end{pmatrix} \quad (d)\ \ x(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

A is a nondefective matrix.

$$x(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} e^{jt} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} e^{-jt}$$

$$x(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} + c_3 e^{jt} + c_4 e^{-jt} \\ -c_1 e^{-t} + 2c_2 e^{2t} + jc_3 e^{jt} - jc_4 e^{-jt} \\ c_1 e^{-t} + 4c_2 e^{2t} - c_3 e^{jt} - c_4 e^{-jt} \\ -c_1 e^{-t} + 8c_2 e^{2t} - jc_3 e^{jt} + jc_4 e^{-jt} \end{pmatrix}.$$

Thus, for each (a)~(d) we only have to find the coefficients $c_1 \sim c_4$ that satisfy

$$x_0 = x(0) = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$$

(a).

We must solve the relation

$$\begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ -1 & 2 & j & -j & -2 \\ 1 & 4 & -1 & -1 & 2 \\ -1 & 8 & -j & j & -2 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$\boxed{x(t) = \begin{pmatrix} 2e^{-t} \\ -2e^{-t} \\ 2e^{-t} \\ -2e^{-t} \end{pmatrix}.}$$

(b).

We must solve the relation

$$\begin{pmatrix} -1 \\ -2 \\ -4 \\ -8 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ -1 & 2 & j & -j & -2 \\ 1 & 4 & -1 & -1 & -4 \\ -1 & 8 & -j & j & -8 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$\boxed{x(t) = \begin{pmatrix} -e^{2t} \\ -2e^{2t} \\ -4e^{2t} \\ -8e^{2t} \end{pmatrix}.}$$

(c).

We must solve the relation

$$\begin{pmatrix} 0 \\ 3 \\ 3 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 2 & j & -j & 3 \\ 1 & 4 & -1 & -1 & 3 \\ -1 & 8 & -j & j & 9 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$\boxed{x(t) = \begin{pmatrix} -e^{-t} + e^{2t} \\ e^{-t} + 2e^{2t} \\ -e^{-t} + 4e^{2t} \\ e^{-t} + 8e^{2t} \end{pmatrix}.}$$

(d).

We must solve the relation

$$\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = c_1\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} + c_2\begin{pmatrix}1\\2\\4\\8\end{pmatrix} + c_3\begin{pmatrix}1\\j\\-1\\-j\end{pmatrix} + c_4\begin{pmatrix}1\\-j\\-1\\j\end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix}1 & 1 & 1 & 1 & 1\\ -1 & 2 & j & -j & 1\\ 1 & 4 & -1 & -1 & -1\\ -1 & 8 & -j & j & -1\end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim\begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0.5-j0.5\\ 0 & 0 & 0 & 1 & 0.5+j0.5\end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix}c_1\\c_2\\c_3\\c_4\end{pmatrix} = \begin{pmatrix}0\\0\\0.5-j0.5\\0.5+j0.5\end{pmatrix}.$$

Hence,

$$x(t) = \begin{pmatrix}(0.5-j0.5)e^{jt} + (0.5+j0.5)e^{-jt}\\ j(0.5-j0.5)e^{jt} - j(0.5+j0.5)e^{-jt}\\ -(0.5-j0.5)e^{jt} - (0.5+j0.5)e^{-jt}\\ -j(0.5-j0.5)e^{jt} + j(0.5+j0.5)e^{-jt}\end{pmatrix}$$

$$x(t) = \begin{pmatrix}(0.5-j0.5)e^{jt} + (0.5+j0.5)e^{-jt}\\ (j0.5+0.5)e^{jt} + (-j0.5+0.5)e^{-jt}\\ (-0.5+j0.5)e^{jt} + (-0.5-j0.5)e^{-jt}\\ (-j0.5-0.5)e^{jt} + (j0.5-0.5)e^{-jt}\end{pmatrix}$$

$$x(t) = \begin{pmatrix}(0.5-j0.5)(cos(t)+jsin(t)) + (0.5+j0.5)(cos(t)-jsin(t))\\ (j0.5+0.5)(cos(t)+jsin(t)) + (-j0.5+0.5)(cos(t)-jsin(t))\\ (-0.5+j0.5)(cos(t)+jsin(t)) + (-0.5-j0.5)(cos(t)-jsin(t))\\ (-j0.5-0.5)(cos(t)+jsin(t)) + (j0.5-0.5)(cos(t)-jsin(t))\end{pmatrix}.$$

Rule out all the imaginary terms

$$x(t) = \begin{pmatrix} 0.5cos(t) + 0.5sin(t) + 0.5cos(t) + 0.5sin(t) \\ 0.5cos(t) - 0.5sin(t) + 0.5cos(t) - 0.5sin(t) \\ -0.5cos(t) - 0.5sin(t) - 0.5cos(t) - 0.5sin(t) \\ -0.5cos(t) + 0.5sin(t) - 0.5cos(t) + 0.5sin(t) \end{pmatrix}$$

$$\boxed{x(t) = \begin{pmatrix} cos(t) + sin(t) \\ cos(t) - sin(t) \\ -cos(t) - sin(t) \\ -cos(t) + sin(t) \end{pmatrix}.}$$

MATLAB Verification

```
% Exercise 6
A1 = [1,1,1,1,2; -1,2,1j,-1j,-2; 1,4,-1,-1,2; -1,8,-1j,1j,-2];
E1 = rref(A1)

A2 = [1,1,1,1,-1; -1,2,1j,-1j,-2; 1,4,-1,-1,-4; -1,8,-1j,1j,-8];
E2 = rref(A2)

A3 = [1,1,1,1,0; -1,2,1j,-1j,3; 1,4,-1,-1,3; -1,8,-1j,1j,9];
E3 = rref(A3)

A4 = [1,1,1,1,1; -1,2,1j,-1j,1; 1,4,-1,-1,-1; -1,8,-1j,1j,-1];
E4 = rref(A4)
```

```
E1 = 4×5
     1     0     0     0     2
     0     1     0     0     0
     0     0     1     0     0
     0     0     0     1     0

E2 = 4×5
     1     0     0     0     0
     0     1     0     0    -1
     0     0     1     0     0
     0     0     0     1     0

E3 = 4×5
     1     0     0     0    -1
     0     1     0     0     1
     0     0     1     0     0
     0     0     0     1     0

E4 = 4×5 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.5000 - 0.5000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.5000 + 0.5000i
```

Exercise 7

Consider a discrete-time LTI system described by $x(k+1) = Ax(k)$.

(a) Suppose that the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of A corresponding to eigenvalues -2 and 3, respectively. What is the response $x(k)$ of the system to the initial condition

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ ?$$

(b) Suppose A is a real matrix and the vector

$$\begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue $2 + 3j$. What is the response $x(k)$ of the system to the initial condition

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \ ?$$

(a).

Assuming that the matrix A is nondefective, we can express the given initial conditions with the linear function with coefficients $c_1 \sim c_4$. This is because we can say that matrix A is 4 x 4 and has 4 linearly independent eigenvectors. We do not know the other two eigenvalues and corresponding eigenvectors but we can still express the following.

$$x(k) = c_1(-2)^k \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 3^k \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \lambda_3^k v_3 + c_4 \lambda_4^k v_4.$$

Then,

$$x(0) = c_1(-2)^0 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 3^0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \lambda_3^0 v_3 + c_4 \lambda_4^0 v_4$$

$$x(0) = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 v_3 + c_4 v_4$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 v_3 + c_4 v_4$$

A combination of $c_1 \sim c_4$ satisfying this relation is

$$\begin{aligned} c_1 &= 0.5 \\ c_2 &= 0.5 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

Thus, the response becomes,

$$x(k) = \frac{(-2)^k}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \frac{3^k}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\boxed{x(k) = \frac{1}{2} \begin{pmatrix} (-2)^k + 3^k \\ -(-2)^k + 3^k \\ (-2)^k + 3^k \\ -(-2)^k + 3^k \end{pmatrix}}$$

(b).

Similar to part (a), assume a nondefective real matrix A to have 4 linearly independent eigenvectors. We have 2 known eigenvalues and eigenvectors, so we can express the following,

$$x(k) = c_1(2+3j)^k \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2(2-3j)^k \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3\lambda_3^k v_3 + c_4\lambda_4^k v_4.$$

Then,

$$x(0) = c_1(2+3j)^0 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2(2-3j)^0 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3\lambda_3^0 v_3 + c_4\lambda_4^0 v_4$$

$$x(0) = c_1 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3 v_3 + c_4 v_4$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3 v_3 + c_4 v_4$$

A combination of $c_1 \sim c_4$ satisfying this relation is

$$\begin{aligned} c_1 &= 0.5 \\ c_2 &= 0.5 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

Thus, the response becomes,

$$x(k) = \frac{(2+3j)^k}{2} \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + \frac{(2-3j)^k}{2} \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}$$

$$\boxed{x(k) = 0.5 \begin{pmatrix} (2+3j)^k + (2-3j)^k \\ j(2+3j)^k - j(2-3j)^k \\ -(2+3j)^k - (2-3j)^k \\ -j(2+3j)^k + j(2-3j)^k \end{pmatrix}}$$

Exercise 8

(You may use MATLAB) Recall the 2-pendulum cart system. Consider the equilibrium configurations defined by

$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

$$E2: \quad (y^e, \theta_1^e, \theta_2^e) = (0, \pi, \pi)$$

Consider state space representations of the linearizations corresponding to the following combinations of parameters and equilibrium conditions:

L1	P1	E1
L2	P1	E2
L3	P2	E1
L4	P2	E2
L5	P3	E1
L6	P3	E2
L7	P4	E1
L8	P4	E2

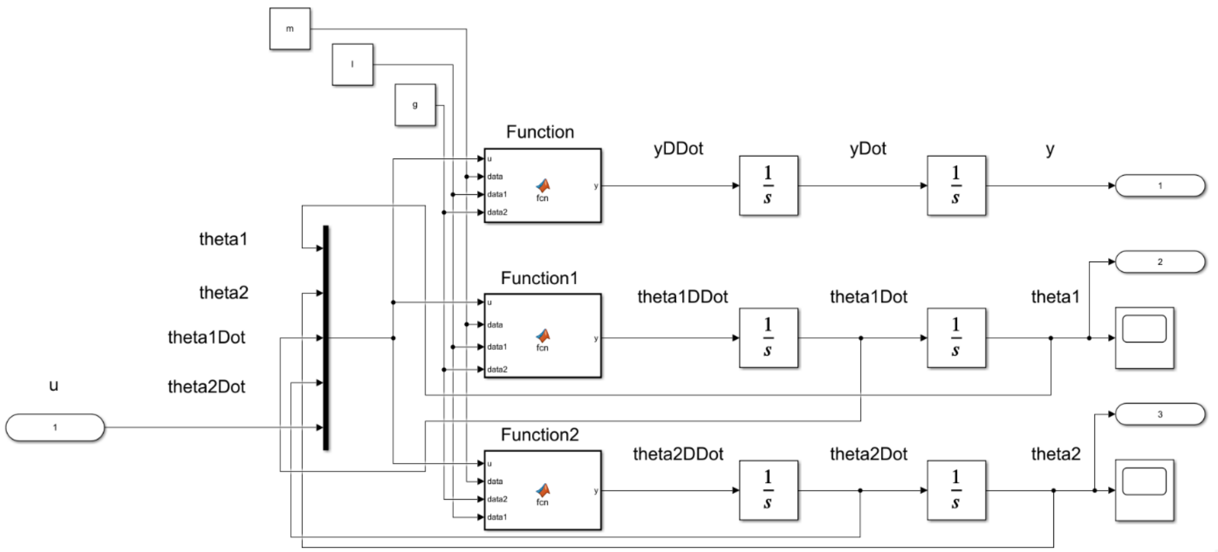
(a) Determine the eigenvalues of all the linearized systems L1-L8.
(b) Compare the behavior of the nonlinear system with that of the linearized system for cases L7 and L8. Illustrate your results with time histories of y, θ_1, and θ_2.

Defined parameter sets:

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

(a).

The Simulink model used for this is shown below,



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

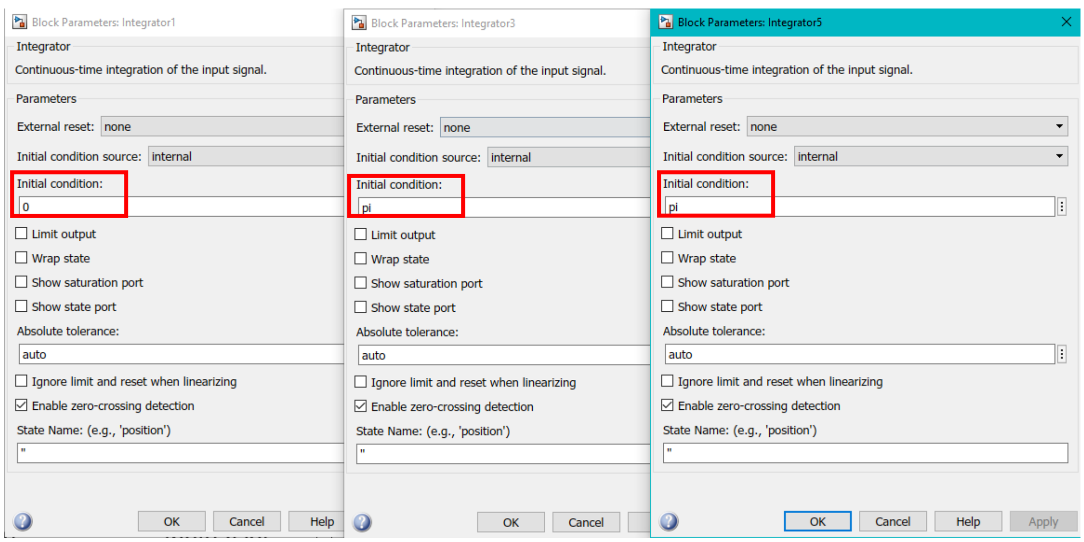
Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



The code to run the linearization and eigenvalue computation is the following

```
% (a)
global m l g ye t1e t2e
param_combo = ["L1","L2","L3","L4","L5","L6","L7","L8"];
for i = 1:numel(param_combo)
    define_params(param_combo(i));
    [A, B, C, D] = linmod('db_pend_cart_lin');
    lin_sys(i).Amat = A;
    lin_sys(i).Bmat = B;
    lin_sys(i).Cmat = C;
    lin_sys(i).Dmat = D;
    sys_ss = ss(A, B, C, D);  % get the state space system
    sys_tf = tf(sys_ss);  % get the transfer function
    lin_sys(i).eigVal = pole(sys_tf);  % get the eigenvalues
end
```

```
function define_params(L)
    % Function to define parameters
    global m l g ye t1e t2e
    if L == "L1"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L2"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; t1e = pi; t2e = pi; % E2
    elseif L == "L3"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L4"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; t1e = pi; t2e = pi; % E2
    elseif L == "L5"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L6"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; t1e = pi; t2e = pi; % E2
    elseif L == "L7"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L8"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; t1e = pi; t2e = pi; % E2
    else
        print('error: did not match any')
    end
end
```

List the state space matrices A, B, C, and D for the linearized system and eigenvalues.

L1:

```
A = 6×6
     0         0         0    1.0000         0         0
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
     0   -0.5000   -0.5000         0         0         0
     0   -1.5000   -0.5000         0         0         0
     0   -0.5000   -1.5000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

L2:

```
A = 6×6
     0         0         0    1.0000         0         0
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
     0   -0.5000   -0.5000         0         0         0
     0    1.5000    0.5000         0         0         0
     0    0.5000    1.5000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1
         0
         0
   -1.4142
   -1.0000
    1.4142
    1.0000
```

L3:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -0.5051   -1.5152         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5051

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
  -0.0000 + 1.4178i
  -0.0000 - 1.4178i
   0.0000 + 1.0025i
   0.0000 - 1.0025i
```

L2:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0    1.5000    0.5000         0         0         0
         0    0.5051    1.5152         0         0         0

B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -0.5051

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1
         0
         0
    1.4178
    1.0025
   -1.4178
   -1.0025
```

L5:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.2500         0         0         0
         0   -1.5000   -0.2500         0         0         0
         0   -0.5000   -1.2500         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
  -0.0000 + 1.3229i
  -0.0000 - 1.3229i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

L6:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.2500         0         0         0
         0    1.5000    0.2500         0         0         0
         0    0.5000    1.2500         0         0         0

B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1
         0
         0
    1.3229
    1.0000
   -1.3229
   -1.0000
```

L7:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -1.0000   -3.0000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    1.0000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.8113i
   0.0000 - 1.8113i
  -0.0000 + 1.1042i
  -0.0000 - 1.1042i
```

L8:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0    1.5000    0.5000         0         0         0
         0    1.0000    3.0000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -1.0000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1
         0
         0
   -1.8113
   -1.1042
    1.8113
    1.1042
```

(b).

For the initial condition,

$$\begin{aligned} y &= y^e + \delta y \\ \theta_1 &= \theta_1^e + \delta\theta_1 \\ \theta_2 &= \theta_2^e + \delta\theta_2 \end{aligned}$$

Thus,

$$\begin{aligned} y_i &= y^e + \delta y_i \\ \theta_{1i} &= \theta_1^e + \delta\theta_{1i} \\ \theta_{2i} &= \theta_2^e + \delta\theta_{2i} \end{aligned}$$

If the ICs defined in HW1 Exercise 7 is used,

	y	θ_1	θ_2	$\dot{y}$	$\dot{\theta}_1$	$\dot{\theta}_2$
IC1	0	-10°	10°	0	0	0
IC2	0	10°	10°	0	0	0
IC3	0	-90°	90°	0	0	0
IC4	0	-90.01°	90°	0	0	0
IC5	0	100°	100°	0	0	0
IC6	0	100.01°	100°	0	0	0
IC7	0	179.99°	0°	0	0	0

Say we use IC1 for the initial condition,

For L7, the initial condition $(y^e, \theta_1^e, \theta_2^e) = (0,0,0)$

$$(\delta y_i, \delta\theta_{1i}, \delta\theta_{2i}) = \left(0, -10° \cdot \frac{\pi}{180°}, 10° \cdot \frac{\pi}{180°}\right) = (0, -0.1745, 0.1745)$$

Then,

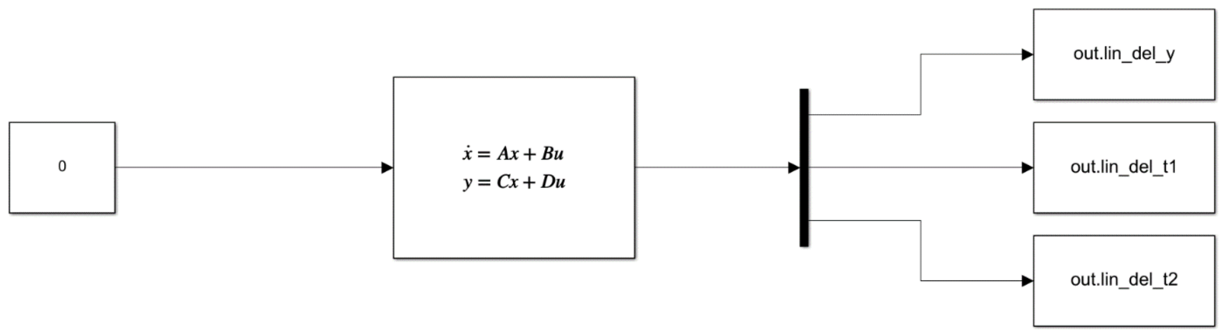
$$(y_i, \theta_{1i}, \theta_{2i}) = (0, -0.1745, 0.1745)$$

We plug these initial conditions to the following models, (*making sure that this we have 3 outputs for the displacement, angle 1, and angle 2.)

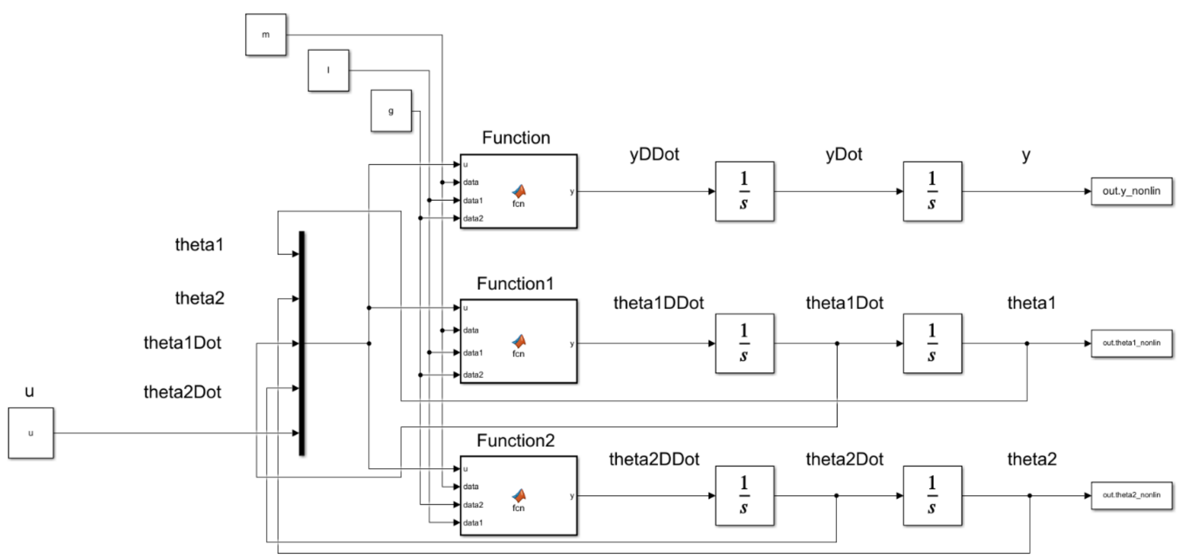
```
C = 3×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
```

```
D = 3×1
     0
     0
     0
```

The linearized state space model:



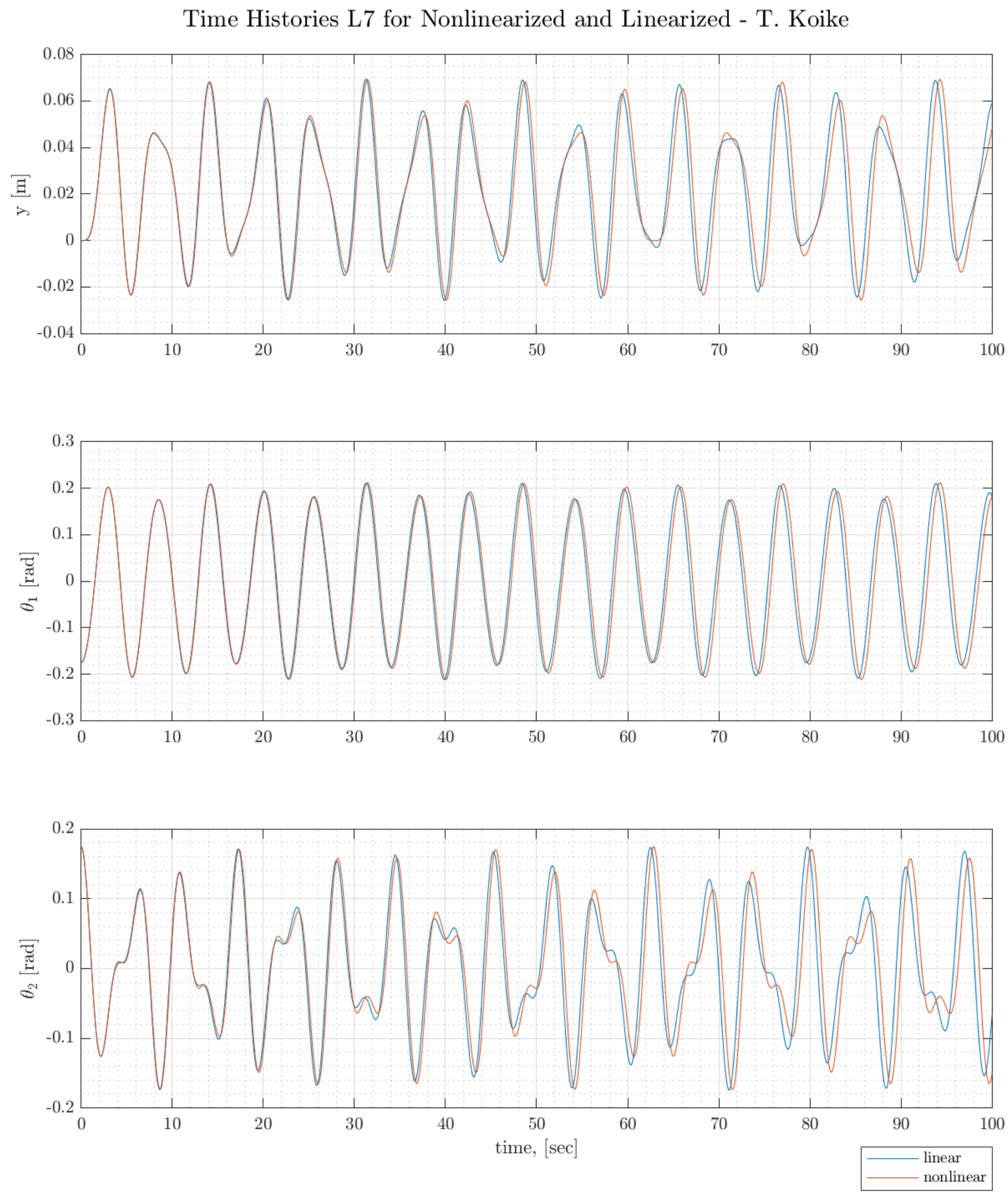
The nonlinear model:



(*The Embedded MATLAB Blocks have the same functions as the previous)

Then the compared response for the two will be plotted out with the following code

```
% L7
% linear
global m l g ye t1e t2e
define_params("L7");
xe = trim('db_pend_cart_lin');
[A, B, C, D] = linmod('db_pend_cart_lin',xe);
del_yi = 0; del_t1i = deg2rad(-10); del_t2i = deg2rad(10);
u = 0; yi = ye + del_yi; t1i = t1e + del_t1i; t2i = t2e + del_t2i;
IC_lin = [del_yi, del_t1i, del_t2i, 0, 0, 0];
sim_lin_res = sim('ss_lin_sys');

time1 = sim_lin_res.lin_del_y.time;
dy = sim_lin_res.lin_del_y.signals.values;
dt1 = sim_lin_res.lin_del_t1.signals.values;
dt2 = sim_lin_res.lin_del_t2.signals.values;

% non-linear
sim_nonlin_res = sim('db_pend_cart_nonlin.slx');

time2 = sim_nonlin_res.tout;
y = sim_nonlin_res.y_nonlin.signals.values;
t1 = sim_nonlin_res.theta1_nonlin.signals.values;
t2 = sim_nonlin_res.theta2_nonlin.signals.values;

% Plotting
fig1 = figure('Renderer',"painters", 'Position', [10 10 900 1000]);
subplot(3,1,1)
hold on; grid on; grid minor; box on;
plot(time1,dy)
plot(time2,y)
ylabel('y [m]')
hold off
subplot(3,1,2)
hold on; grid on; grid minor; box on;
plot(time1,dt1)
plot(time2,t1)
ylabel('$\theta_1$ [rad]')
hold off
subplot(3,1,3)
hold on; grid on; grid minor; box on;
plot(time1,dt2)
plot(time2,t2)
ylabel('$\theta_2$ [rad]')
xlabel('time, [sec]')
h = legend('linear','nonlinear'); set(h, 'Position',  [0.8, 0.05, .1, .025]);
hold off
sgtitle('Time Histories L7 for Nonlinearized and Linearized - T. Koike')
saveas(fig1, 'p8_L7.png')
```



For L8, we use the same initial condition

$$(\delta y_i, \delta\theta_{1i}, \delta\theta_{2i}) = \left(0, -10° \cdot \frac{\pi}{180°}, 10° \cdot \frac{\pi}{180°}\right) = (0, -0.1745, 0.1745)$$

But since,

$$(y^e, \theta_1^e, \theta_2^e) = (0, \pi, \pi)$$

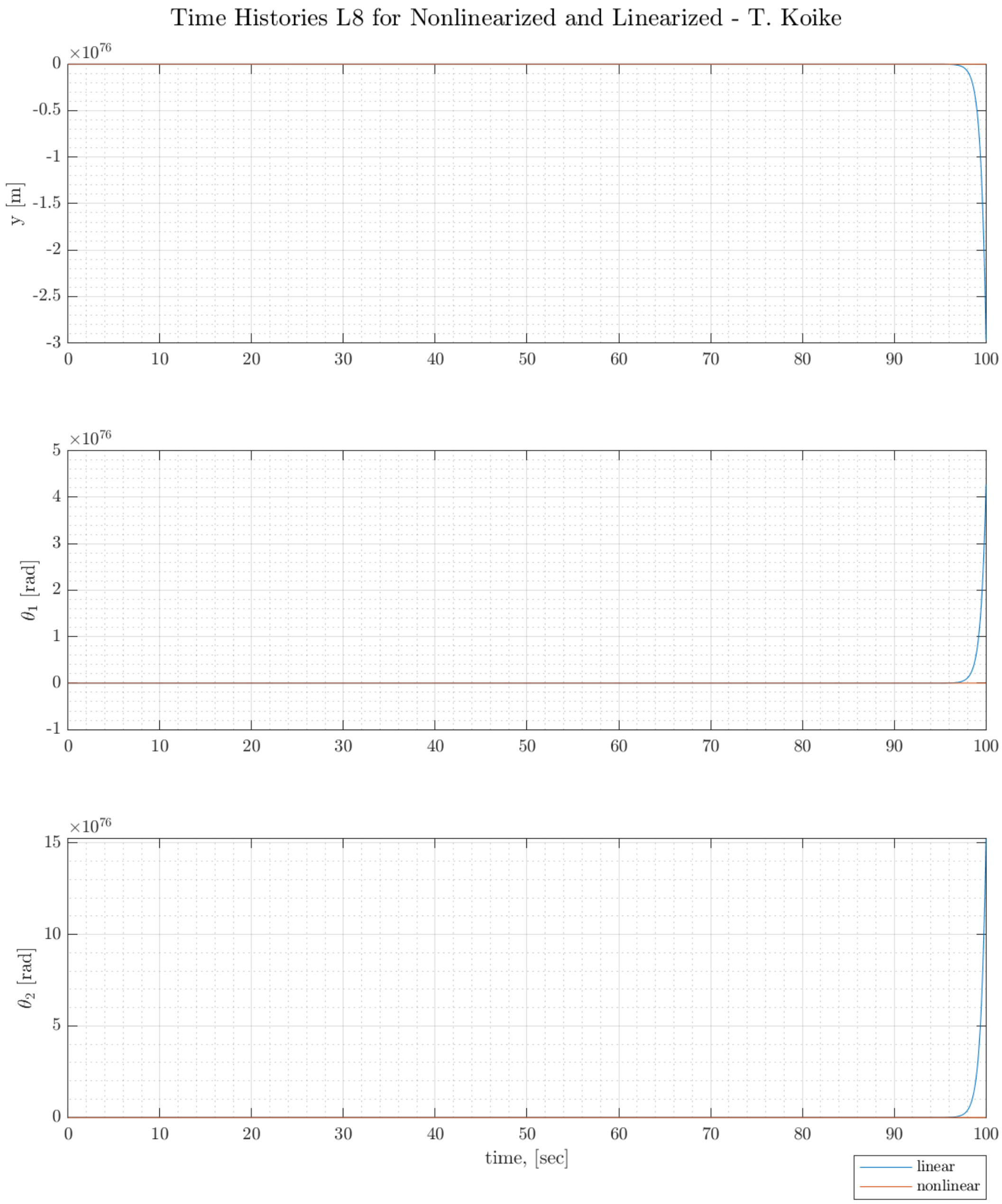
$$(y_i, \theta_{1i}, \theta_{2i}) = \left(0, \pi - 10° \cdot \frac{\pi}{180°}, \pi + 10° \cdot \frac{\pi}{180°}\right)$$

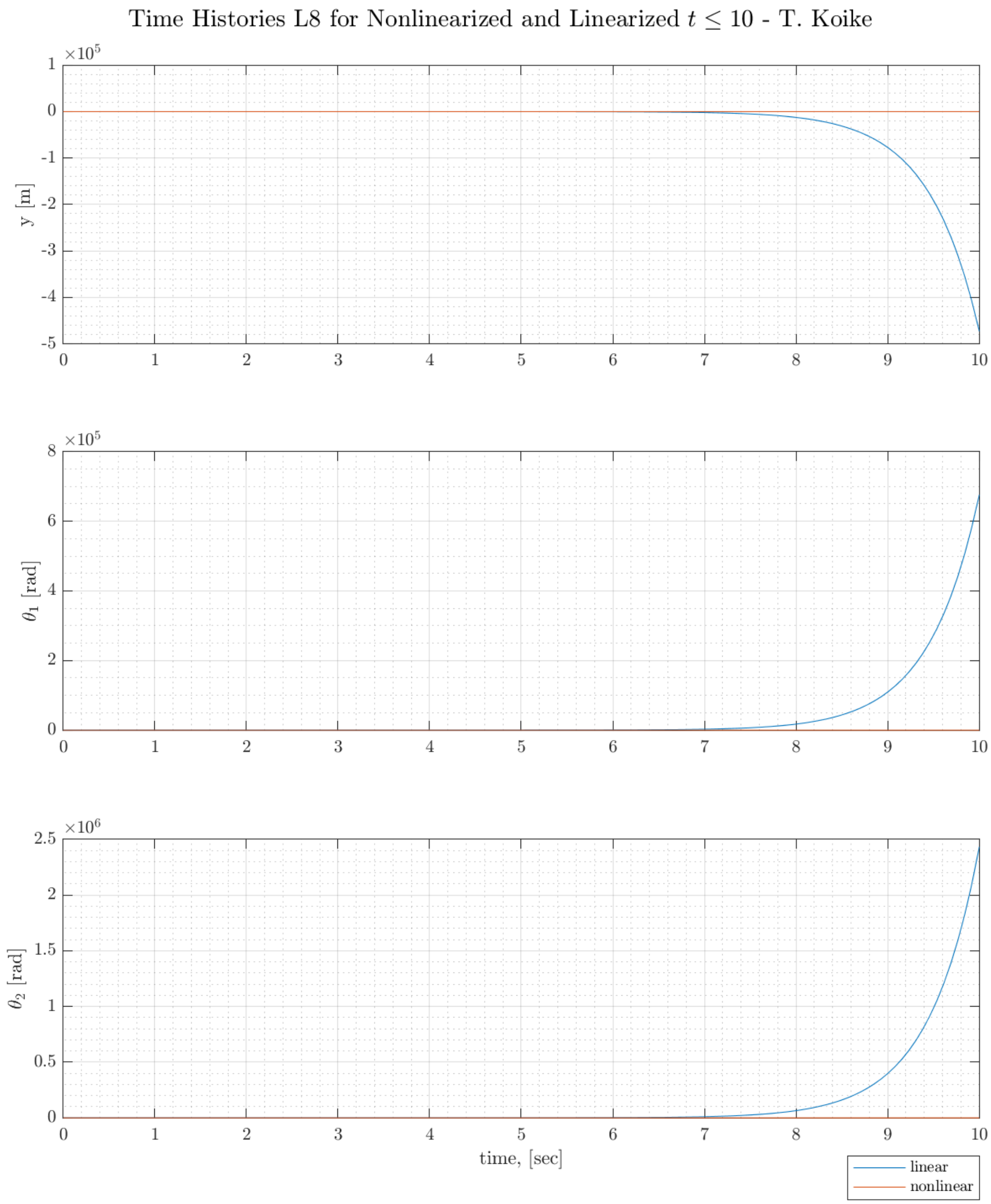
$$(y_i, \theta_{1i}, \theta_{2i}) = (0, 2.9671, 3.3161)$$

We use these as our initial conditions as well as the A, B, C, D matrices for L8 condition. The code we run for this simulation is the following.

```
% L8
% linear
global m l g ye t1e t2e
define_params("L8");
xe = trim('db_pend_cart_lin');
[A, B, C, D] = linmod('db_pend_cart_lin',xe);
del_yi = 0; del_t1i = deg2rad(-10); del_t2i = deg2rad(10);
u = 0; yi = ye + del_yi; t1i = t1e + del_t1i; t2i = t2e + del_t2i;
IC_lin = [del_yi, del_t1i, del_t2i, 0, 0, 0];
sim_lin_res = sim('ss_lin_sys');
time1 = sim_lin_res.lin_del_y.time;
dy = sim_lin_res.lin_del_y.signals.values;
dt1 = sim_lin_res.lin_del_t1.signals.values;
dt2 = sim_lin_res.lin_del_t2.signals.values;
% non-linear
sim_nonlin_res = sim('db_pend_cart_nonlin.slx');
time2 = sim_nonlin_res.tout;
y = sim_nonlin_res.y_nonlin.signals.values;
t1 = sim_nonlin_res.theta1_nonlin.signals.values;
t2 = sim_nonlin_res.theta2_nonlin.signals.values;
% Plotting
fig1 = figure('Renderer',"painters", 'Position', [10 10 900 1000]);
subplot(3,1,1)
hold on; grid on; grid minor; box on;
plot(time1,dy)
plot(time2,y)
ylabel('y [m]')
hold off
subplot(3,1,2)
hold on; grid on; grid minor; box on;
plot(time1,dt1)
plot(time2,t1)
ylabel('$\theta_1$ [rad]')
hold off
subplot(3,1,3)
hold on; grid on; grid minor; box on;
plot(time1,dt2)
plot(time2,t2)
ylabel('$\theta_2$ [rad]')
xlabel('time, [sec]')
h = legend('linear','nonlinear'); set(h, 'Position',  [0.8, 0.05, .1, .025]);
hold off
sgtitle('Time Histories L7 for Nonlinearized and Linearized - T. Koike')
saveas(fig1, 'p8_L8.png')
```

The response becomes




Time Histories L8 for Nonlinearized and Linearized $t \leq 10$ - T. Koike
$\times 10^5$
y [m]
$\times 10^5$
θ_1 [rad]
$\times 10^6$
θ_2 [rad]
time, [sec]
linear
nonlinear

Exercise 9

(You may use MATLAB) This exercise refers to linearizations L7 and L8 of the two pendulum cart system.

(a) For L7 choose an initial state for the linearized system which results in a periodic solution for the linearized system.
(b) For L8 choose an initial state for the linearized system which results in a solution which asymptotically decays to zero for the linearized system.
(c) For L8 choose an initial state for the linearized system which results in a solution whose magnitude grows exponentially for the linearized system.

In each case, simulate both the linearized system and the nonlinear system with initial conditions corresponding to your chosen initial state for the linearized system.

(a).

The A matrix for this condition is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & -0.5 & -0.5 & 0 & 0 & 0 \\ 0 & -1.5 & -0.5 & 0 & 0 & 0 \\ 0 & -1.0 & -3.0 & 0 & 0 & 0 \end{pmatrix}$$

and the corresponding eigenvalues and eigenvectors are

$\boldsymbol{\lambda}$	$\boldsymbol{v}$
0	1 0 0 0 0 0
0	-1.0000 0 0 0.0000 0 0
-0.0000 + 1.8113i	-0.0000 - 0.0893i -0.0000 - 0.1284i -0.0000 - 0.4573i 0.1617 - 0.0000i 0.2326 - 0.0000i

	0.8283 + 0.0000i
-0.0000 - 1.8113i	-0.0000 + 0.0893i -0.0000 + 0.1284i -0.0000 + 0.4573i 0.1617 + 0.0000i 0.2326 + 0.0000i 0.8283 + 0.0000i
0.0000 + 1.1042i	0.0000 - 0.1040i 0.0000 - 0.5782i -0.0000 + 0.3247i 0.1148 + 0.0000i 0.6385 + 0.0000i -0.3585 + 0.0000i
0.0000 - 1.1042i	0.0000 + 0.1040i 0.0000 + 0.5782i -0.0000 - 0.3247i 0.1148 - 0.0000i 0.6385 + 0.0000i -0.3585 - 0.0000i

Thus, the response is characterized by

$$x(t) = \sum_{i}^{6} c_i e^{\lambda_i t} v_i$$

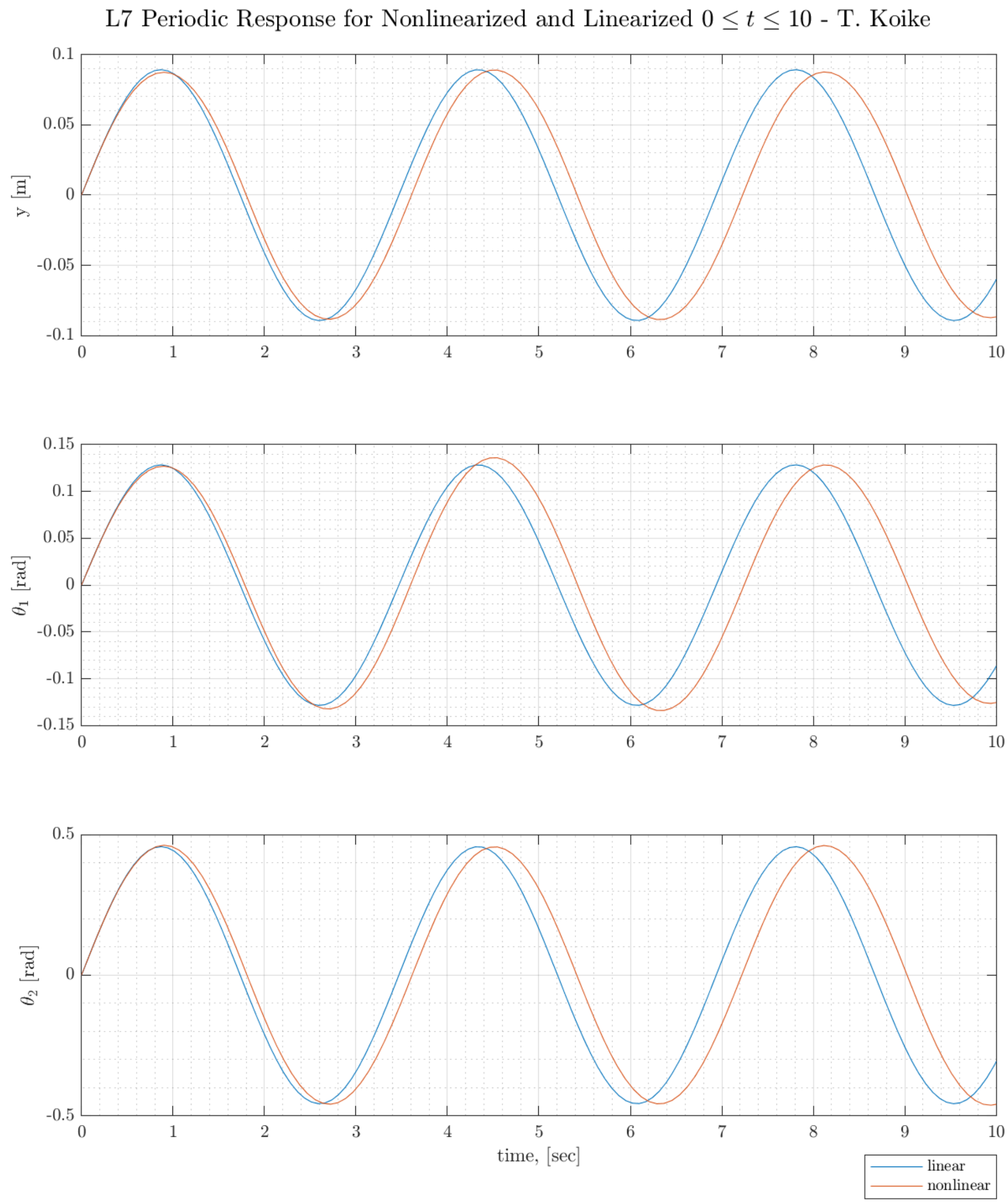
To have a periodic response we want to have c_3 and c_4 to be non-zero and other coefficients to be 0 while the IC to be real.

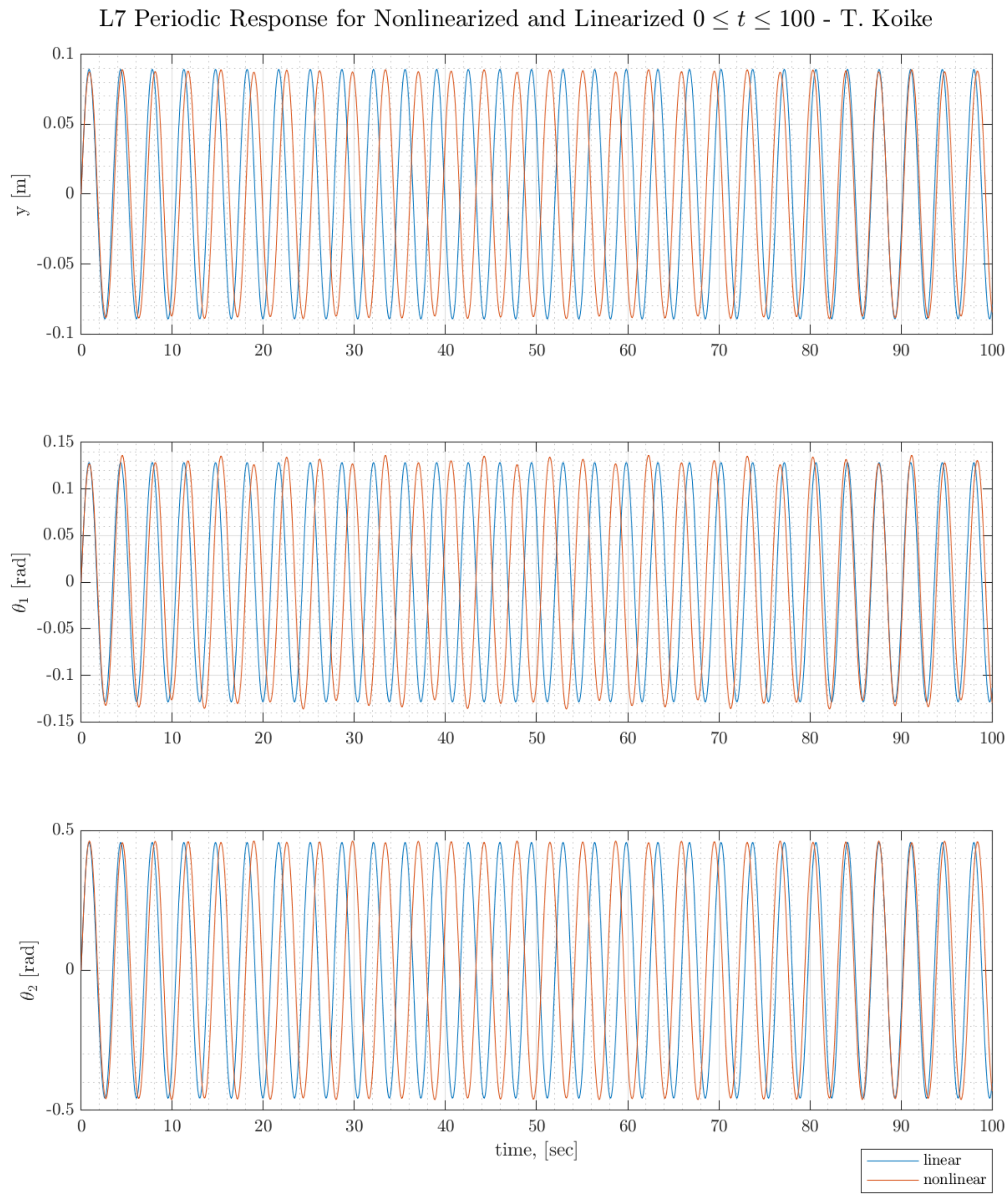
$$x(t) = c_3 e^{1.8113it} v_3 + c_4 e^{-1.8113it}\, v_4$$

This, from the Euler's equation, we can tell is a periodic response. Thus, when $t = 0$, $c_3 = 0.5$, and $c_4 = 0.5$

$$x(0) = 0.5v_3 + 0.5v_4 = 0.5\begin{pmatrix} -0.0893i \\ -0.1284i \\ -0.4573i \\ 0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix} + 0.5\begin{pmatrix} 0.0893i \\ 0.1284i \\ 0.4573i \\ 0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix}$$

Using the models from Exercise 8, we plug the initial condition above to make the behavior of the linearized system to be periodic. The result follows.


L7 Periodic Response for Nonlinearized and Linearized $0 \leq t \leq 10$ - T. Koike
y [m]
θ_1 [rad]
θ_2 [rad]
time, [sec]
linear
nonlinear



$$
x(0) = 0.5v_3 + 0.5v_4 = 0.5\begin{pmatrix} -0.0893i \\ -0.1284i \\ -0.4573i \\ 0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix} + 0.5\begin{pmatrix} 0.0893i \\ 0.1284i \\ 0.4573i \\ 0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix}
$$

```
% L7
% linear
global m l g ye t1e t2e
define_params("L7");
xe = trim('db_pend_cart_lin');
[A, B, C, D] = linmod('db_pend_cart_lin',xe);
[eigVec, eigVal] = eig(A,"vector");
temp = 0.5*(eigVec(:,3) + eigVec(:,4));
% ICs
del_yi = temp(1); del_t1i = temp(2); del_t2i = temp(3);
del_yi_dot = temp(4); del_t1i_dot = temp(5); del_t2i_dot = temp(6);
u = 0; yi = ye + del_yi; t1i = t1e + del_t1i; t2i = t2e + del_t2i;
yi_dot = del_yi_dot; t1i_dot = del_t1i_dot; t2i_dot = del_t2i_dot;

IC_lin = [del_yi, del_t1i, del_t2i, del_yi_dot, del_t1i_dot, del_t2i_dot];
sim_lin_res = sim('ss_lin_sys');
time1 = sim_lin_res.lin_del_y.time;
dy = sim_lin_res.lin_del_y.signals.values;
dt1 = sim_lin_res.lin_del_t1.signals.values;
dt2 = sim_lin_res.lin_del_t2.signals.values;

% non-linear
sim_nonlin_res = sim('db_pend_cart_nonlin.slx');
time2 = sim_nonlin_res.tout;
y = sim_nonlin_res.y_nonlin.signals.values;
t1 = sim_nonlin_res.theta1_nonlin.signals.values;
t2 = sim_nonlin_res.theta2_nonlin.signals.values;

% Plotting
fig1 = figure('Renderer',"painters", 'Position', [10 10 900 1000]);
subplot(3,1,1)
hold on; grid on; grid minor; box on;
plot(time1,dy)
plot(time2,y)
ylabel('y [m]')
hold off
subplot(3,1,2)
hold on; grid on; grid minor; box on;
plot(time1,dt1)
plot(time2,t1)
ylabel('$\theta_1$ [rad]')
hold off
subplot(3,1,3)
hold on; grid on; grid minor; box on;
plot(time1,dt2)
plot(time2,t2)
ylabel('$\theta_2$ [rad]')
xlabel('time, [sec]')
h = legend('linear','nonlinear'); set(h, 'Position',  [0.8, 0.05, .1, .025]);
hold off
sgtitle('L7 Periodic Response for Nonlinearized and Linearized - T. Koike')
saveas(fig1, 'p9_a.png')
```

(b).

Similar to part (a),

The A matrix for this condition is

$$A = \begin{pmatrix} 0 & 0 & 0 & 1.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0 \\ 0 & -0.5 & -0.5 & 0 & 0 & 0 \\ 0 & 1.5 & 0.5 & 0 & 0 & 0 \\ 0 & 1.0 & 3.0 & 0 & 0 & 0 \end{pmatrix}$$

and the corresponding eigenvalues and eigenvectors are

$\boldsymbol{\lambda}$	$\boldsymbol{v}$
0	1 0 0 0 0 0
0	-1.0000 0 0 0.0000 0 0
-1.8113	-0.0893 0.1284 0.4573 0.1617 -0.2326 -0.8283
-1.1042	-0.1040 0.5782 -0.3247 0.1148 -0.6385 0.3585
1.8113	0.0893 -0.1284 -0.4573 0.1617 -0.2326 -0.8283
1.1042	0.1040 -0.5782 0.3247

	0.1148 -0.6385 0.3585

Thus, the response is characterized by

$$x(t) = \sum_{i}^{6} c_i e^{\lambda_i t} v_i$$

To have a asymptotically decaying response we want to have c_3 to be non-zero and other coefficients to be 0 while the IC to be real.

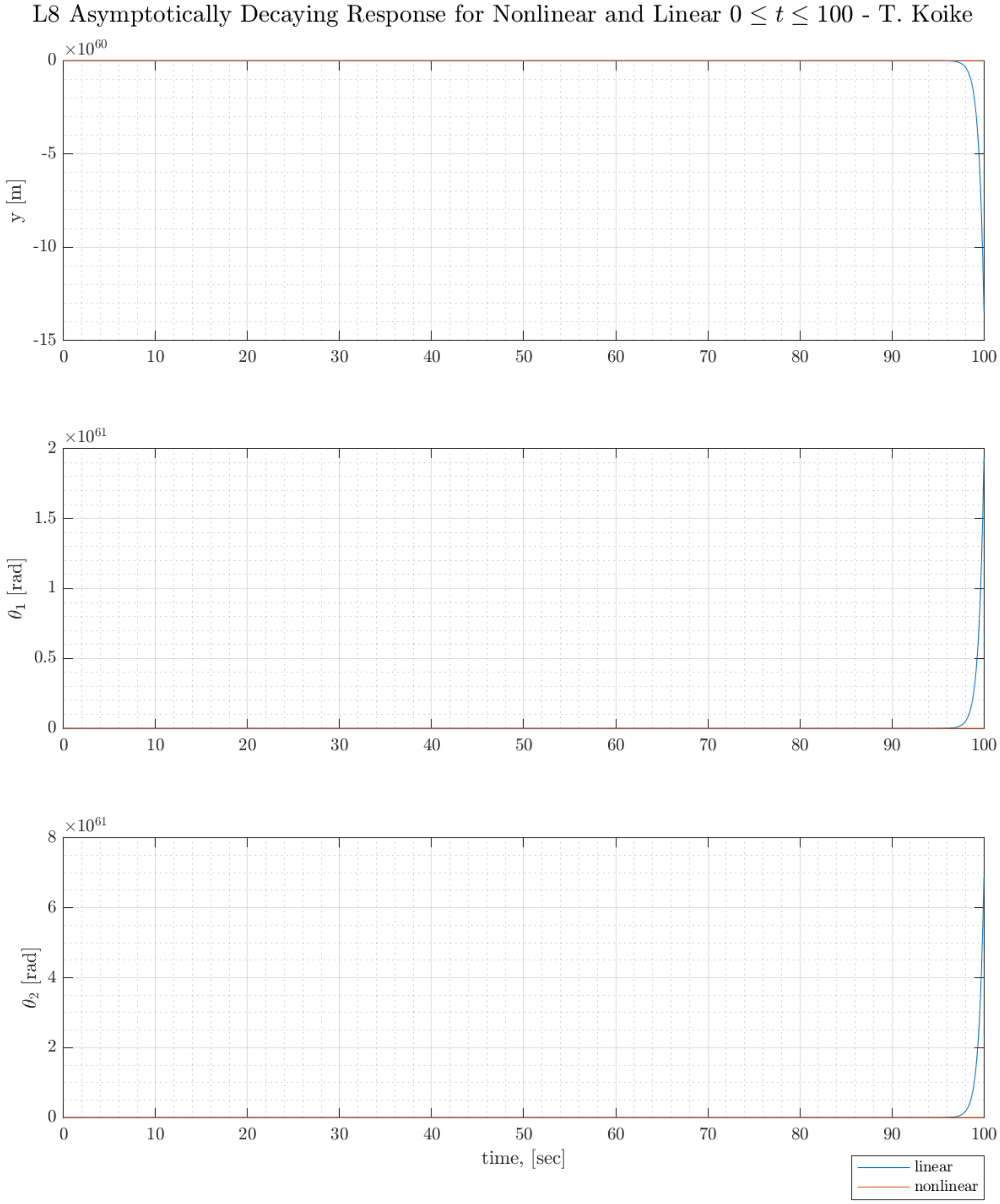
$$x(t) = c_3 e^{-1.8113t} v_3$$

This, from the Euler's equation, we can tell is a periodic response. Thus, when $t = 0,\ c_3 = -1$

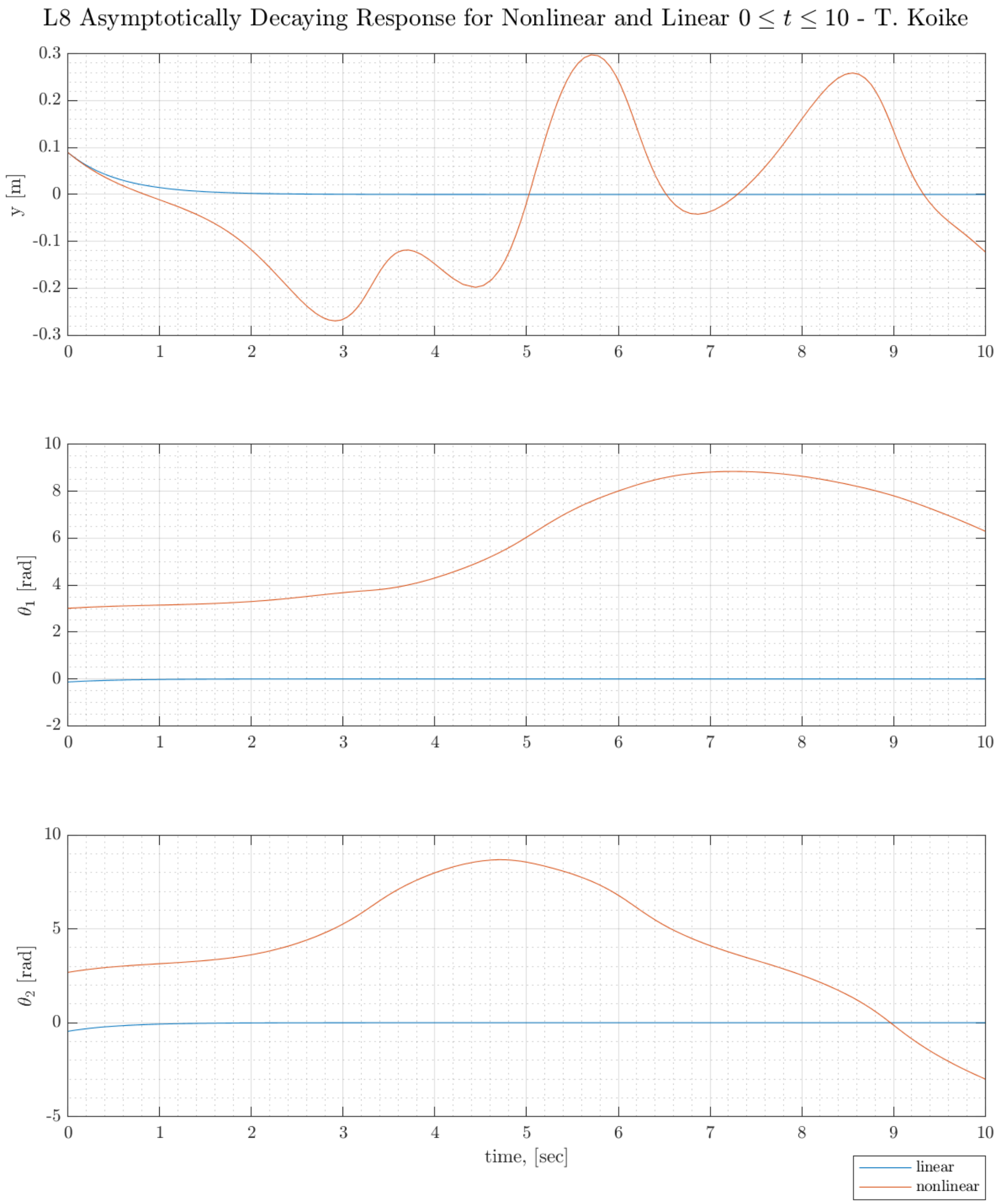
$$x(0) = -v_3 = -\begin{pmatrix} -0.0893 \\ -0.1284 \\ 0.4573 \\ 0.1617 \\ -0.2326 \\ -0.8283 \end{pmatrix} = \begin{pmatrix} 0.0893 \\ 0.1284 \\ -0.4573 \\ -0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix}$$

The 2 plots are shown below for this result

First plot is for time span of 100 seconds



The second plot is for a time span of 10 seconds



The code used is identical to the one in part (a).

Thus, the initial condition selected is

$$\boxed{x(0) = -v_3 = -\begin{pmatrix} -0.0893 \\ -0.1284 \\ 0.4573 \\ 0.1617 \\ -0.2326 \\ -0.8283 \end{pmatrix} = \begin{pmatrix} 0.0893 \\ 0.1284 \\ -0.4573 \\ -0.1617 \\ 0.2326 \\ 0.8283 \end{pmatrix}}$$

(c).

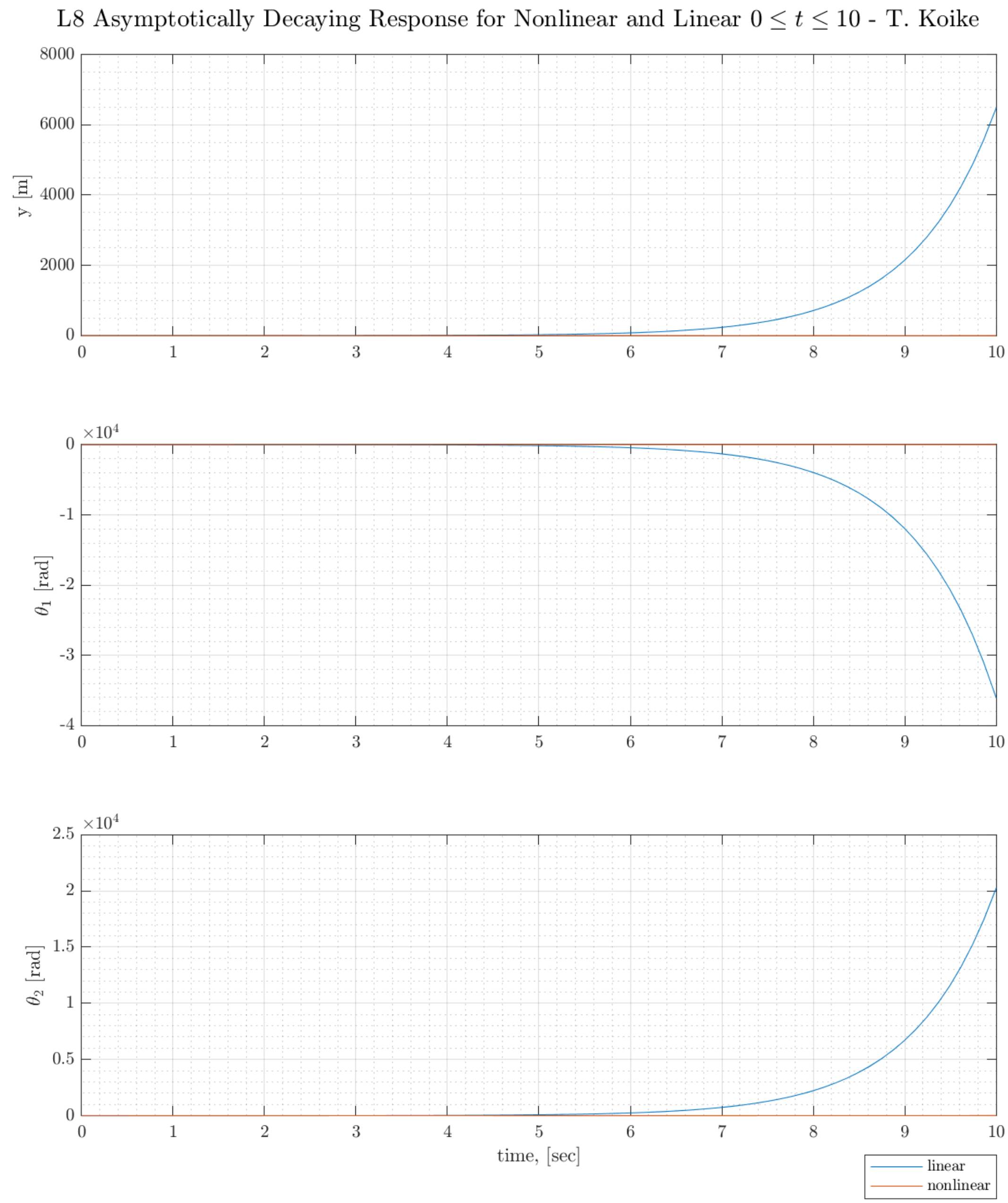
Similar to part (b), to have an asymptotically decaying response we want to have c_6 to be non-zero and other coefficients to be 0 while the IC to be real.

$$x(t) = c_6 e^{1.1042t} v_6$$

This, from the Euler's equation, we can tell is a periodic response. Thus, when $t = 0$, $c_3 = 1$

$$x(0) = v_6 = \begin{pmatrix} 0.1040 \\ -0.5782 \\ 0.3247 \\ 0.1148 \\ -0.6385 \\ -0.3585 \end{pmatrix}$$

The plot is shown on the next page



The code used is identical to that of part (a) and (b).

Hence, the initial condition selected is

$$\boxed{x(0) = v_6 = \begin{pmatrix} 0.1040 \\ -0.5782 \\ 0.3247 \\ 0.1148 \\ -0.6385 \\ -0.3585 \end{pmatrix}}$$