AAE 364: Controls Systems Analysis

HW7: Root Locus Analysis

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Problem 1

Consider the unity feedback system shown in Figure 1:

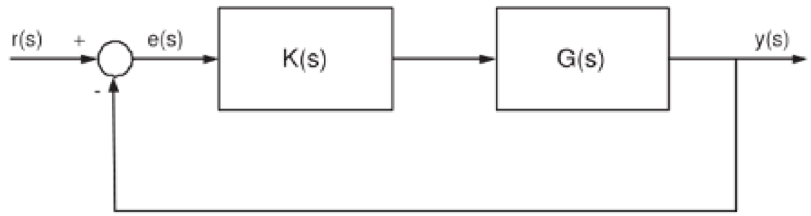


Figure 1: A unity feedback system.

Plot the root locus for the system with

1. $$K(s) = k, \ \ G(s) = \frac{1}{s(s+2)(s^2+4s+5)}$$

This is a feedback sys., so

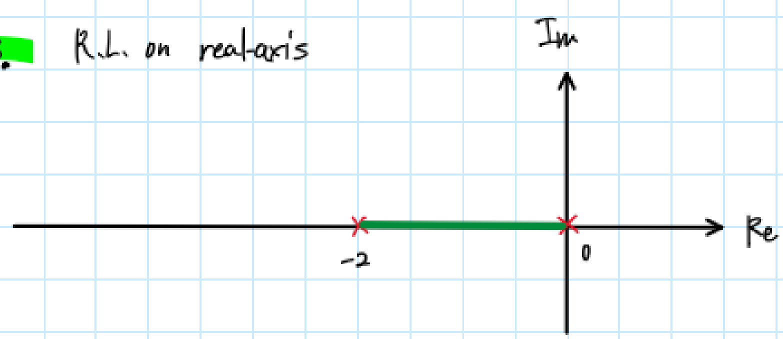
$$CE := 1 + KL(s) = 1 + k\frac{1}{s(s+2)(s^2+4s+5)} = 0$$

rule 1. Poles & zeros

- no zeros $\longrightarrow$ $m = 0$
- from $s(s+2)(s^2+4s+5) = s(s+2)(s+2+j)(s+2-j) = 0$
  poles equal $\quad s = 0, -2, -2 \pm j \longrightarrow n = 4$

rule 2. symmetry exists

rule 3. R.L. on real-axis



rule 4. Asymptotes

$$\theta_a = \frac{180° + 360° l}{n-m} = 45° + 90° l \qquad l = 0, 1, 2, 3$$

$$\therefore \theta_a = 45°, 135°, 225°, 315°$$

$$\sigma_a = \frac{\sum p_i - \sum z_i}{n-m} = \frac{0 - 2 - 2 + j - 2 - j}{4} = -\frac{3}{2}$$

rule 5. Break-in/away points

$$\frac{d}{ds}\left[-\frac{1}{L(s)}\right] = 0 \longrightarrow \frac{d}{ds}\left[-s(s+2)(s^2+4s+5)\right] = 0$$

$$-\frac{d}{ds}(s^4+6s^3+13s^2+10s) = 0$$

$$-4s^3-18s^2-26s-10 = 0$$

roots $\Rightarrow \hat{s}_1 = -0.6018$, $\hat{s}_2 = -1.9491 \pm 0.5958j$

break-in point

rule 6. Angle of departure

$\angle L(s^d) = -180° \longrightarrow s^d$ is a point near $-2+j$

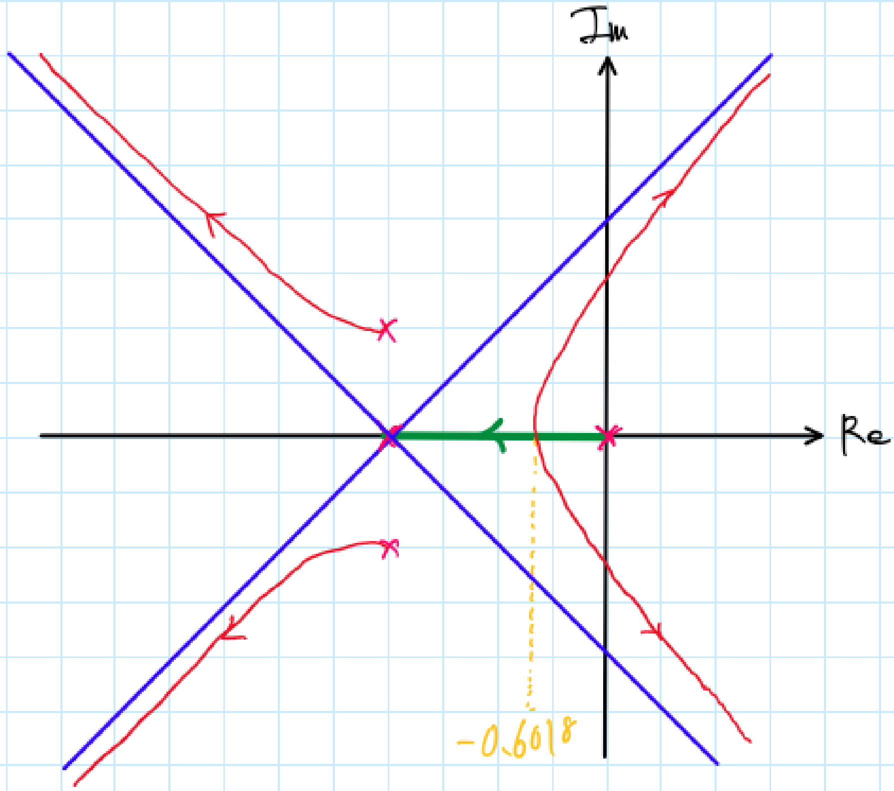
$$\angle L(s^d) = 0 - \theta_d - \arg(-2+j-0) - \arg[-2+j-(-2-j)] - \arg[-2+j-(-2)] = 180°$$

$$\Longleftrightarrow \quad \theta_d = -153.4349°$$

rule 7. Intersection of RL w/ $j\omega$-axis

$$1 + \hat{k}L(j\hat{\omega}) = 0$$

$$1 + \hat{k}\frac{1}{j\hat{\omega}(j\hat{\omega}+2)(-\hat{\omega}^2+4j\hat{\omega}+5)} = 0$$



2.

$$K(s) = k, \;\; G(s) = \frac{s^2+6s+10}{s^2+2s+10}$$

$$CE := 1 + kL(s) = 1 + k\frac{s^2+6s+10}{s^2+2s+10} = 0$$

rule 1 Poles & zeros

- Poles : $s^2+2s+10=0 \longrightarrow s = -1 \pm 3j \longrightarrow n = 2$
- zeros : $s^2+6s+10=0 \longrightarrow s = -3 \pm j \longrightarrow m = 2$

rule 2. Symmetry

rule 3 R.L. on real axis $\rightarrow$ none

rule4 Asymptotes $\therefore n-m=0 \longrightarrow$ none

rule5 Breaks-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = \frac{d}{ds}\left(-\frac{s^2+2s+10}{s^2+6s+10}\right) = 0$$

$$\frac{(2s+2)(s^2+6s+10)-(s^2+2s+10)(2s+6)}{(s^2+6s+10)^2} = 0$$

$$s^2 = \sqrt{-10}$$

not a break-in/away ✗ $\quad s = \sqrt{10}\ j$

rule 6 Angle of Departure

point s^d is right close to $-1+3j$

$$\angle L(s^d) = -180^\circ = \arg[-1+3j-(-3+j)] + \arg[-1+3j-(-3-j)]$$
$$- \theta_d - \arg[-1+3j-(-1-3j)]$$

$$\theta_d = 180^\circ + 45^\circ + \arctan(2) - 90^\circ$$

$$\theta_d = 198.4349^\circ$$

Angle of Arrival

set point s^d is right close to $-3+j$

$$\angle L(s^d) = -180^\circ = \theta_d + \arg[-3+j-(-3-j)]$$
$$-\arg[-3+j-(-1+3j)] - \arg[-3+j-(-1-3j)]$$

$$\theta_d = -288.4349^\circ$$

rule 7 Intersection of R.L. w/ ωj

$$1 + \hat{k} L(\hat{\omega} j) = 0$$

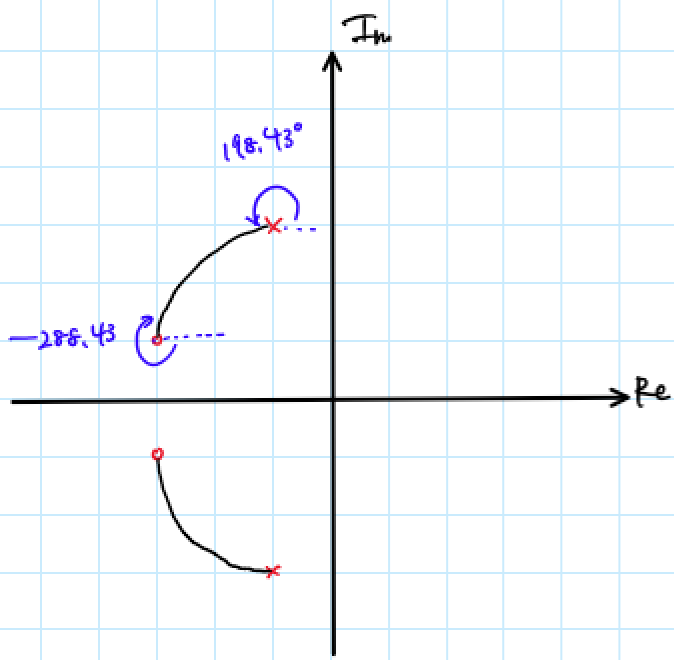
$$1 + \hat{k}\,\frac{(\hat{\omega} j)^2 + 6\hat{\omega} j + 10}{(\hat{\omega} j)^2 + 2\hat{\omega} j + 10} = 0$$

$$\hat{k} = -\frac{-\hat{\omega}^2 + 2\hat{\omega} j + 10}{-\hat{\omega}^2 + 6\hat{\omega} j + 10}$$

$\mathrm{Re}(\mathrm{LHS}) = \mathrm{Re}(\mathrm{RHS}) \iff \hat{k} = 10 - \hat{\omega}^2$

$\mathrm{Im}(\mathrm{LHS}) = \mathrm{Im}(\mathrm{RHS}) \iff 0 = -2\hat{\omega}$

$\hat{k} = 0$, $\hat{\omega} = 0$ → no intersect



3.

$$K(s) = k, \quad G(s) = \frac{s+9}{s(s^2+4s+11)}$$

$$CE := 1 + kL(s) = 1 + k\frac{s+9}{s(s^2+4s+11)}$$

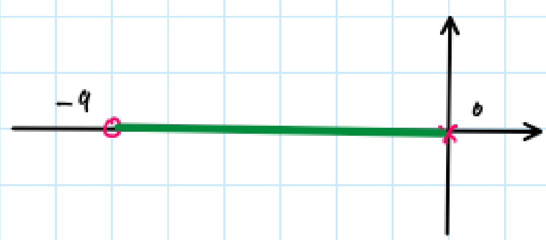
Rule 1 Poles and zeros

- Poles: $s(s^2+4s+11) = 0 \longrightarrow s = 0,\ -2 \pm 2.6458j \longrightarrow n = 3$
- zeros: $s+9 = 0 \longrightarrow s = -9 \longrightarrow m = 1$

$n - m = 2$

Rule 2 Symmetry

Rule 3. R.L on real-axis



Rule 4 Asymptote

$$\theta_a = \frac{180° + 360° l}{n-m} \qquad l = 0, 1$$

$$= 90°,\ 270°$$

$$\sigma_a = \frac{\Sigma p_i - \Sigma z_i}{n-m} = \frac{0 - 2 + \sqrt{7}j - 2 - \sqrt{7}j - (-9)}{2}$$

$$= 2.5$$

Rule 5 Break-in/away point

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = \frac{d}{ds}\left(\frac{s^3+4s^2+11s}{s+9}\right) = 0$$

$$\frac{(s+9)(3s^2+8s+11)-s^3-4s^2-11s}{(s+9)^2}=0$$

$$-2\hat{s}^3-31\hat{s}^2-72\hat{s}-99=0$$

$$\hat{s}_1=-13.0284 \qquad \hat{s}_2=-1.2358+1.5704j$$

Rule 6 Angle of departure

s^d is close to $-2+\sqrt{7}j$

$$\angle L(s^d)=-180^\circ=\arg[-2+\sqrt{7}j-(-9)]-\theta_d-\arg(-2+\sqrt{7}j)$$
$$-\arg[-2+\sqrt{7}j-(-2-\sqrt{7}j)]$$

$$\theta_d=-16.3819^\circ$$

Rule 7 Intersection of RL w/ Im-axis

$$CE:\ 1+\hat{k}L(j\hat{\omega})=0$$

$$1+\hat{k}\frac{j\hat{\omega}+9}{j\hat{\omega}(-\hat{\omega}^2+4j\hat{\omega}+11)}=0$$

$$j\hat{\omega}(-\hat{\omega}^2+4j\hat{\omega}+11)+\hat{k}(j\hat{\omega}+9)=0$$

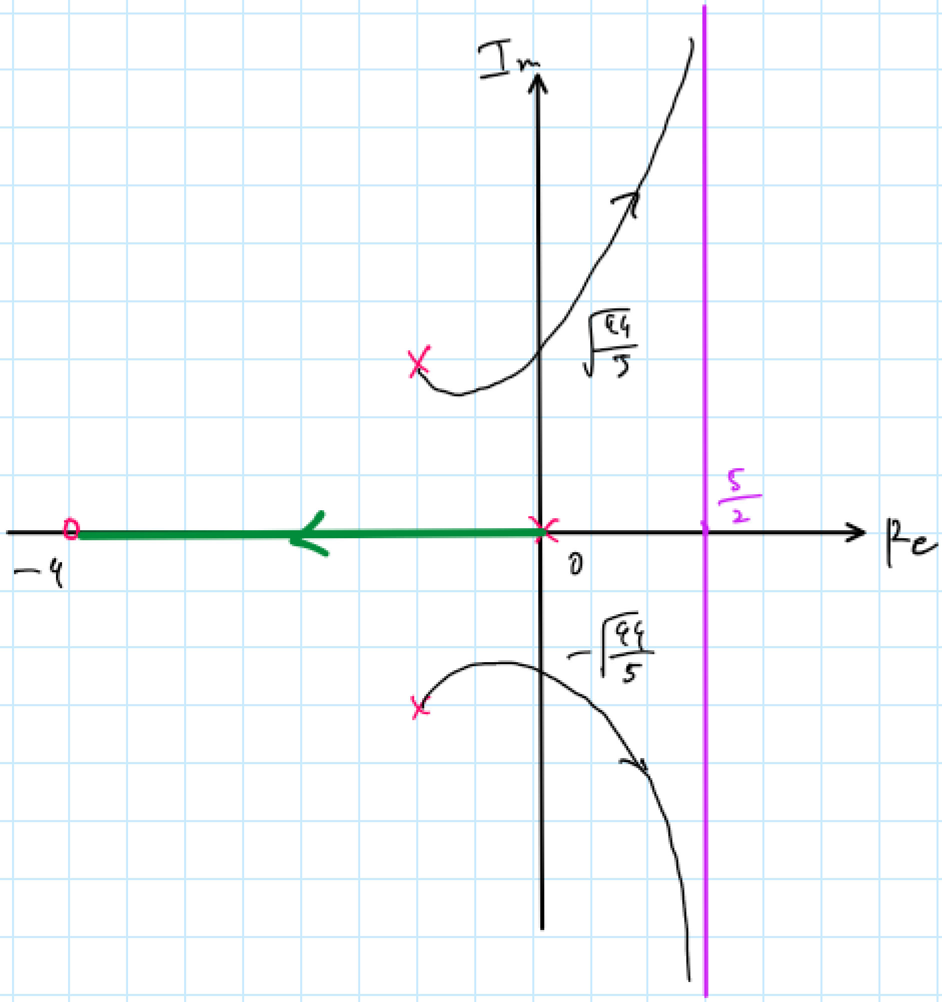
$$\mathrm{Im}(LHS)=\mathrm{Im}(RHS)$$

$$-\hat{\omega}^3+11\hat{\omega}+k\hat{\omega}=0$$

$$\mathrm{Re}(LHS)=\mathrm{Re}(RHS)$$

$$-4\hat{\omega}^2+9k=0$$

$$\hat{\omega} = \pm\sqrt{\frac{99}{5}} \qquad \hat{k} = \frac{49}{5}$$



B–6–8. Consider a unity-feedback control system with the following feedforward transfer function:

$$G(s) = \frac{K}{s(s^2 + 4s + 8)}$$

Plot the root loci for the system. If the value of gain K is set equal to 2, where are the closed-loop poles located?

$$CE := 1 + KL(s) = 1 + K\frac{1}{s(s^2+4s+8)} = 0$$

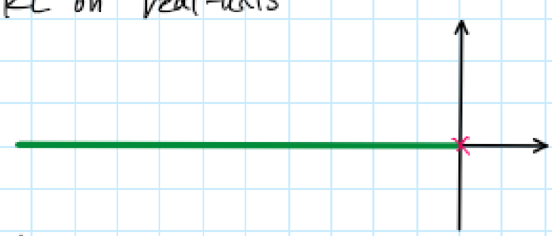
Rule1 Poles and zeros

Poles: $s(s^2+4s+8) = 0 \rightarrow s = 0, -2 \pm 2i \rightarrow n = 3$

zeros: 0 $\quad n - m = 3$

Rule2 Symmetry

Rule3 RL on real-axis



Rule4 Asymptotes

$$\theta_a = \frac{180° + 360° \ell}{n-m} \quad \ell = 0, 1, 2$$

$$= 60°, 180°, 300°$$

$$\sigma_a = -1.3333$$

Rule5 Break-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0 \longrightarrow \frac{d}{ds}\left(-s(s^2+4s+8)\right) = 0$$

$$3s^2 + 8s + 8 = 0$$

$$\hat{s}_1 = -1.3333 \pm 0.9428j$$

Tomol

Rule6 Angle of departure

s^d close to $-2+2j$

$$\angle L(s^d) = 0 - \theta_d - \arg[-2+2j-(-2-2j)]$$
$$-\arg[-2+2j] = -180^\circ$$

$$\theta_d = -45^\circ$$

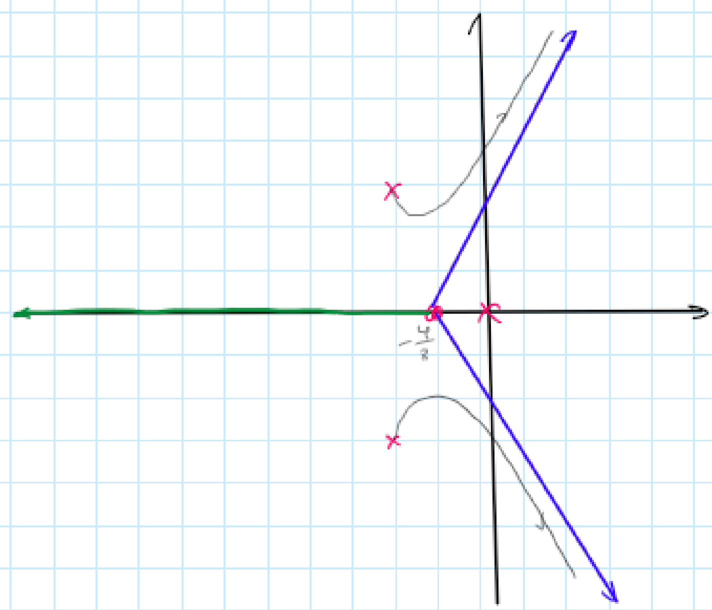
Rule 7 Intersection of R.L. with $j\omega$-axis

$$CE := 1 + \hat{k}L(j\hat{\omega}) = 0$$

$$\hat{k} = j\hat{\omega}^3 + 4\hat{\omega}^2 - 8j\hat{\omega}$$

Im: $0 = \hat{\omega}^3 - 8\hat{\omega}$ $\quad\quad \hat{\omega} = \pm 2\sqrt{2}$

Re: $\hat{k} = 4\hat{\omega}^2$ $\quad\quad k = 32$



$$G(s) = \frac{\frac{2}{s(s^2+4s+8)}}{1+\frac{2}{s(s^2+4s+8)}} = \frac{2}{s^3+4s^2+8s+2}$$

Tomok

$$\boxed{\text{poles: } -0.2887,\ -1.8557 \pm 1.8669j}$$

B–6–11. Consider the system shown in Figure 6–101. Plot the root loci with MATLAB. Locate the closed-loop poles when the gain K is set equal to 2.

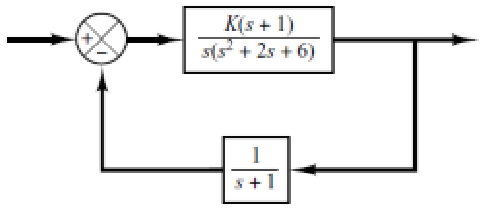
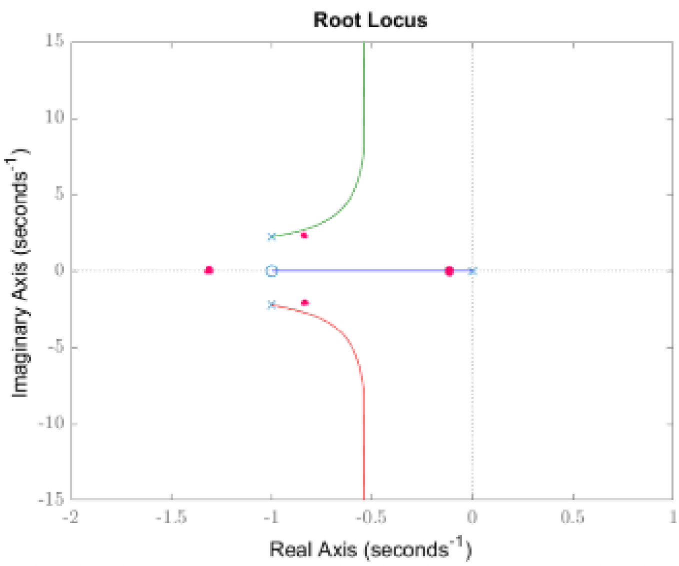


Figure 6–101
Control system.



• are the locations

CL $K=2$

$$G_c(s) = \frac{\frac{2(s+1)}{s(s^2+2s+6)}}{1+\left(\frac{2(s+1)}{s(s^2+2s+6)}\right)\left(\frac{1}{s+1}\right)}$$

$$= \frac{2(s+1)^2}{s(s^2+2s+6)(s+1)+2(s+1)}$$

$$= \frac{2(s+1)^2}{s^4+3s^3+8s^2+8s+1}$$

Tomoki

$\boxed{\text{poles: } -0.1449,\ -1.2355,\ -0.8098 \pm 2.2203j}$

Problem 3: Spacecraft

Consider the unity-feedback system in Figure 1 with the plant $G(s)$ representing the spacecraft attitude dynamics shown in Figure 2:

$$\frac{\theta(s)}{T_c(s)} = \frac{0.036(s+25)}{s^2(s^2+0.04s+1)} \tag{1}$$

Sketch the root loci of the unity-feedback system, with $K(s) = K$, as K varies from 0 to $+\infty$ (as accurately as you can)

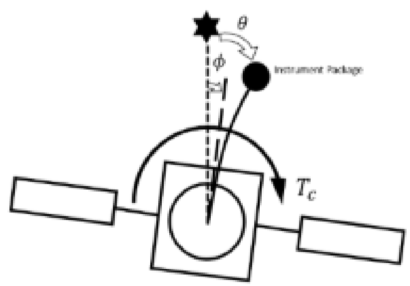


Figure 2: Two-body Model of Satellite

$$CE := 1 + KL(s) = 1 + K\frac{0.036(s+25)}{s^2(s^2+0.04s+1)}$$

rule 1 Poles & zeros

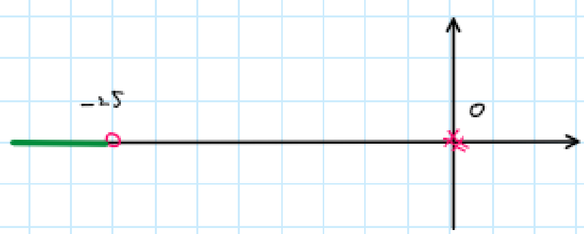
zeros: $s+25=0 \longrightarrow s=-25 \longrightarrow m=1$

poles: $s^2(s^2+0.04s+1) \longrightarrow s=0,\ -0.02\pm 0.9998j \rightarrow n=4$

$n-m=3$

rule 2 symmetry

rule 3



rule 4 Asymptotes

$$\theta_a = \frac{180^\circ + 360^\circ l}{n-m} \qquad l = 0, 1, 2$$

$$= 60^\circ,\ 180^\circ,\ 300^\circ$$

Tomc

$$\sigma_a = \frac{\Sigma p_i - \Sigma z_i}{n-m} = 8.32$$

rule 5 Break-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$$\frac{d}{ds}\left(-\frac{s^2(s^2+0.04s+1)}{0.036(s+25)}\right) = 0$$

$$\frac{d}{ds}\left(-\frac{s^4+0.04s^3+s^2}{0.036s+0.9}\right) = 0$$

$$(0.036s+0.9)(4s^3+0.12s^2+2s) - 0.036(s^4+0.04s^3+s^2) = 0$$

$\hat{s}_1 = 0$ $\quad \hat{s}_2 = -33.350$ $\quad \hat{s}_3 = -0.0125 \pm 0.7070j$

break away $\quad$ break in

rule 6 Angle of departure

s^d is close $-0.02+0.9998j$

$$\angle L(s^d) = -180^\circ = \arg[-0.02+0.9998j-(-25)] - \theta_d$$
$$-\arg[-0.02+0.9998j-(-0.02-0.9998j)]$$
$$-\arg[-0.02+0.9998j]$$

$$\theta_d = -90^\circ$$

rule 7 Intersection of R.L w/ Im axis

$$1+\hat{k}L(\hat{\omega}j) = 0$$

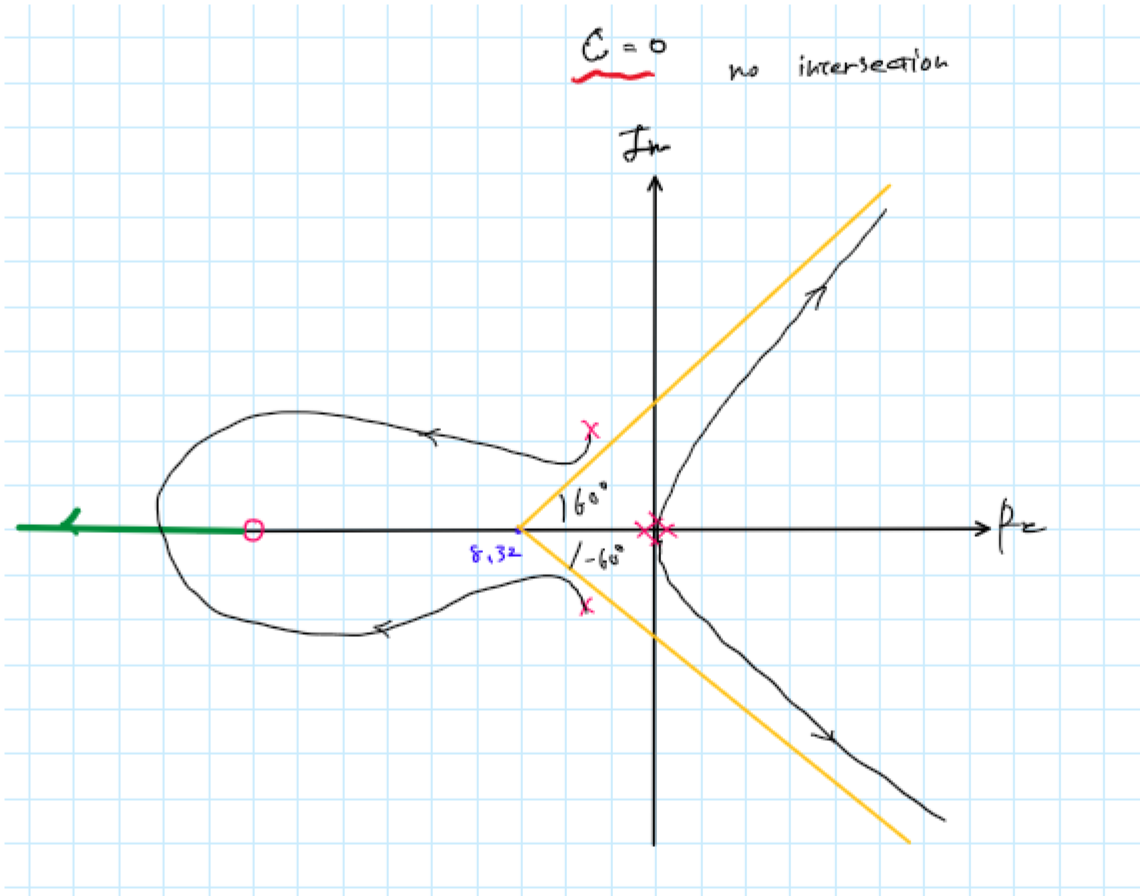
$$1+\hat{k}\frac{0.036(j\hat{\omega}+25)}{-\hat{\omega}^2(-\hat{\omega}^2+0.04\hat{\omega}j+1)} = 0$$

$$\hat{\omega}^4 - 0.04j\hat{\omega}^3 - \hat{\omega}^2 + 0.036\hat{k}j\hat{\omega} + 0.9\hat{k} = 0$$

$$\begin{cases} Im: & -0.04\hat{\omega}^3 + 0.036\hat{k}\hat{\omega} = 0 \\ Re: & \hat{\omega}^4 - \hat{\omega}^2 + 0.9\hat{k} = 0 \end{cases}$$

$$\begin{cases} Im - & 0.036\hat{k} = 0.04\hat{\omega}^2 \\ Re - & \hat{\omega}^4 - \hat{\omega}^2 + \hat{\omega}^2 = 0 \end{cases}$$

Tom


C = 0
no intersection
Im
Re
60°
-60°
8.32

Appendix

AAE364 HW7 MATLAB CODE

```
clear all; close all; clc;
```

```
fdir = 'C:\Users\Tomo\Desktop\studies\2020-
Spring\AAE364\matlab\matlab_output\hw7';
set(groot, 'defaulttextinterpreter',"latex");
set(groot, 'defaultAxesTickLabelInterpreter',"latex");
set(groot, 'defaultLegendInterpreter',"latex");
```

P1

```
% 1
num = [1];
den = conv([1 0],[1 2]);
den = conv(den,[1 4 5])
poles = roots(den)
zrs = roots(num)
[angs,sigma] = RL_asymptote(zrs,poles)
bi_pt = break_in_away_pt(num,den)
theta_d = departure_angle_calc(zrs, poles, poles(2), "deg", "poles")
[k,w] = intersection_IM_axis(num,den)
L = tf(num, den)
fig1 = figure(1);
rlocus(L)
saveas(fig1, fullfile(fdir, 'p1_1.png'));
```

```
% 2
num = [1 6 10];
den = [1 2 10];
poles = roots(den)
zrs = roots(num)
[angs,sigma] = RL_asymptote(zrs,poles)
bi_pt = break_in_away_pt(num,den)
theta_d = departure_angle_calc(zrs, poles, poles(1), "deg","poles")
theta_d_arr = departure_angle_calc(zrs,poles,zrs(1),"deg","zeros")
[k,w] = intersection_IM_axis(num,den)
```

```
L = tf(num,den);
fig2 = figure(2);
rlocus(L)
saveas(fig2, fullfile(fdir, 'p1_2.png'));
```

```
% 3
num = [1 9];
den = conv([1 0],[1 4 11])
poles = roots(den)
zrs = roots(num)
[angs,sigma] = RL_asymptote(zrs,poles)
bi_pt = break_in_away_pt(num,den)
theta_d = departure_angle_calc(zrs, poles, poles(2), "deg","poles")
[k,w] = intersection_IM_axis(num,den)
L = tf([1 9], [1 4 11 0]);
fig3 = figure(3);
rlocus(L)
saveas(fig3, fullfile(fdir, 'p1_3.png'));
```

P2

```
% B-6-8
num = [1];
den = conv([1 0],[1 4 8])
poles = roots(den)
zrs = roots(num)
[angs,sigma] = RL_asymptote(zrs,poles)
bi_pt = break_in_away_pt(num,den)
theta_d = departure_angle_calc(zrs, poles, poles(2), "deg","poles")
[k,w] = intersection_IM_axis(num,den)
L = tf([0 1], [1 4 8 0]);
fig4 = figure(4);
rlocus(L)
saveas(fig4, fullfile(fdir, 'Figure4.png'));
```

```
% B-6-11
num = [1 1];
den = conv([1 0],[1 2 6])
poles = roots(den)
zrs = roots(num)
[angs,sigma] = RL_asymptote(zrs,poles)
```

```
bi_pt = break_in_away_pt(num,den)
theta_d = departure_angle_calc(zrs, poles, poles(2), "deg","poles")
[k,w] = intersection_IM_axis(num,den)
L = tf([1 1], [1 2 6 0]);
fig5 = figure(5);
rlocus(L)
saveas(fig5, fullfile(fdir, 'Figure5.png'));
```

P3

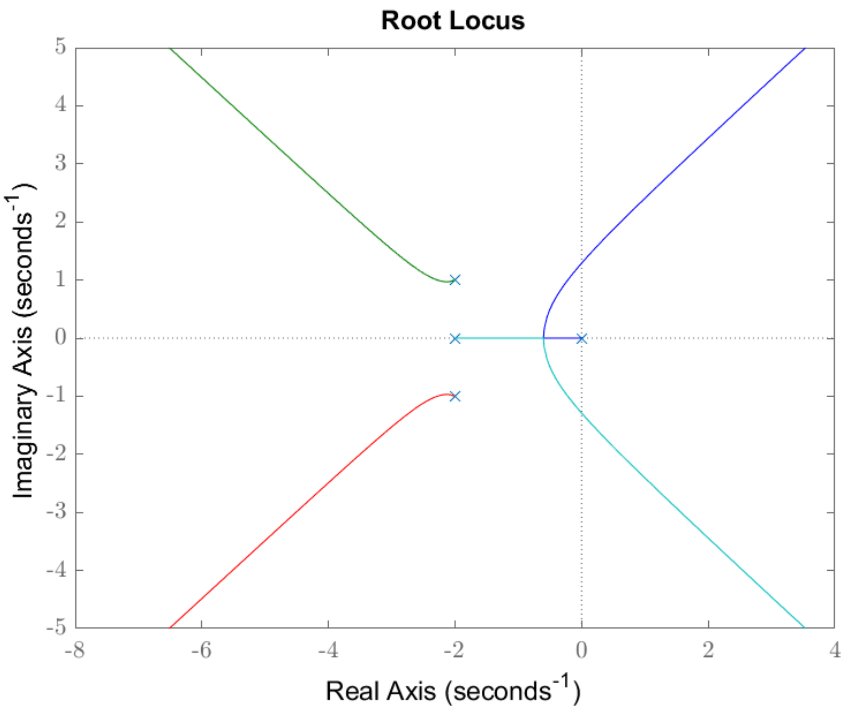
```
num = 0.036*[1 25];
den = conv([1 0 0],[1 0.04 1])
poles = roots(den)
zrs = roots(num)
[angs,sigma] = RL_asymptote(zrs,poles)
bi_pt = break_in_away_pt(num,den)
theta_d = departure_angle_calc(zrs, poles, poles(3), "deg","poles")
[k,w] = intersection_IM_axis(num,den)
L = tf(num, den);
fig6 = figure(6);
rlocus(L)
saveas(fig6, fullfile(fdir, 'Figure6.png'));


function theta = departure_angle_calc(zrs, poles, obj, angle_type, obj_type)
    %{
        zrs: the zrs of the transfer function
        poles: the poles of the transfer function
        obj: the aimed pole to find the departure angle
    %}
    if obj_type == "poles"
        idx = find(poles==obj);
        poles(idx) = [];
    else
        idx = find(zrs==obj);
        zrs(idx) = [];
    end

    theta = 0;
    if not(isempty(zrs))
        for i = 1:length(zrs)
            theta = theta + angle(obj - zrs(i));
        end
    end
```

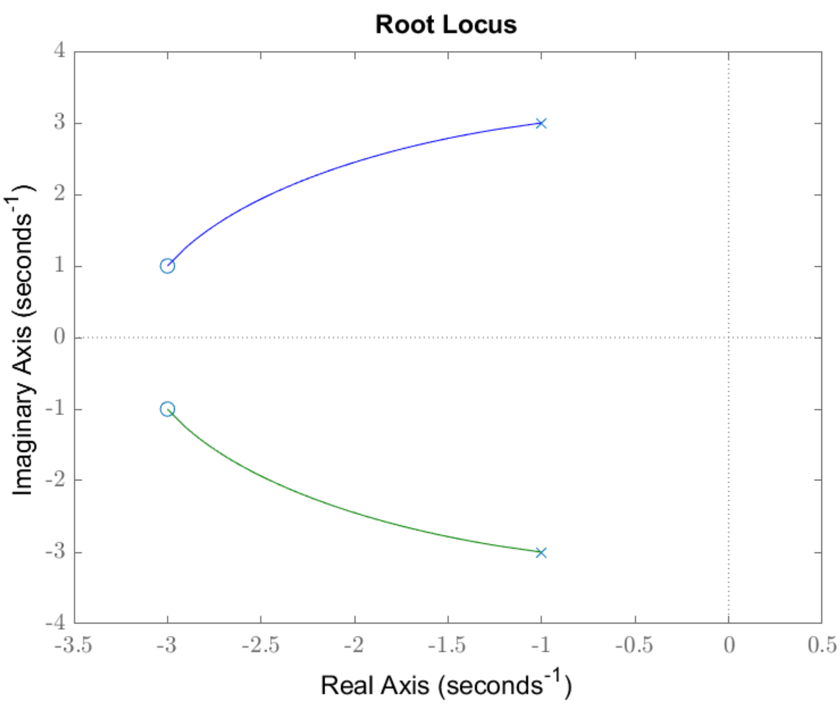
```
    for i = 1:length(poles)
        theta = theta - angle(obj - poles(i));
    end

    if obj_type == "poles"
        theta = theta + deg2rad(180);
    else
        theta = -deg2rad(180) - theta;
    end

    if angle_type == "deg"
        theta = rad2deg(theta);
    end
end

function rts = break_in_away_pt(num,den)
    [q, d] = polyder(-den,num)
    rts = roots(q)
    rts = rts(rts==real(rts));
end

function [angs, sigma] = RL_asymptote(zrs, poles)
    n = length(poles)
    m = length(zrs)
    angs = zeros([1,n-m]);
    for i = 0:(n-m)-1
        angs(i+1) = (180 + 360*i)/(n - m);
    end
    sigma = (sum(poles) - sum(zrs))/(n - m);
end

function [K, W] = intersection_IM_axis(num, den)
    syms k w
    n = length(den);
    f1 = 0;
    f2 = 0;
    if rem(n,2) == 1
        for i = 1:2:n
            if rem(i-1,4) == 0
                f1 = f1 + den(i)*w^(n-i);
            else
                f1 = f1 + den(i)*w^(n-i)*(-1);
            end
        end
        for i = 2:2:n-1
```

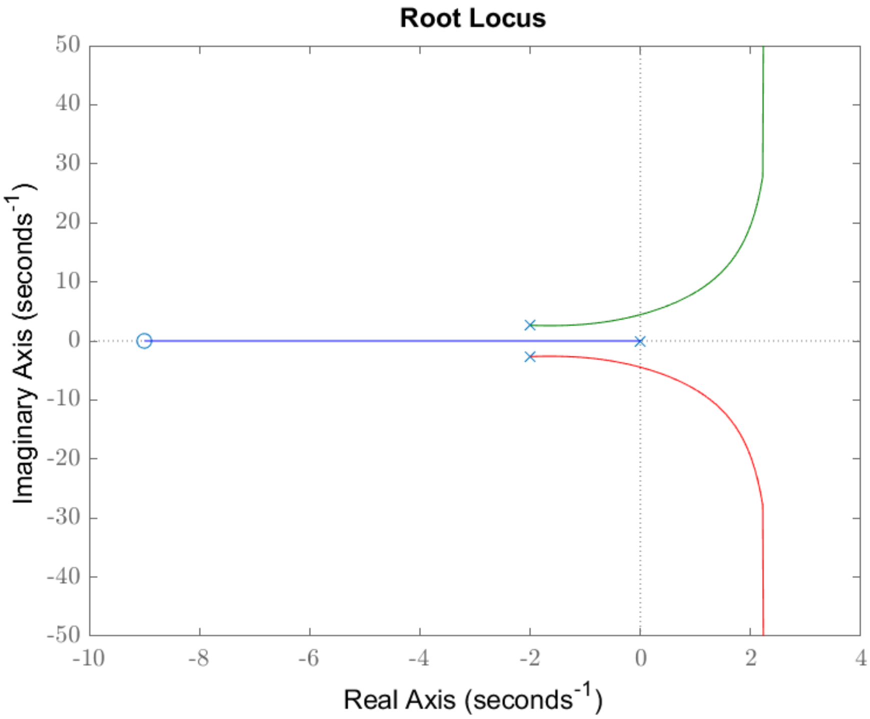
```
            if rem(i-1,4) == 1
                f2 = f2 + den(i)*w^(n-i);
            else
                f2 = f2 + den(i)*w^(n-i)*(-1);
            end
        end
    elseif rem(n,2) == 0
        for i = 1:2:n-1
            if rem(i-1,4) == 0
                f1 = f1 + den(i)*w^(n-i);
            else
                f1 = f1 + den(i)*w^(n-i)*(-1);
            end
        end
        for i = 2:2:n
            if rem(i-1,4) == 1
                f2 = f2 + den(i)*w^(n-i);
            else
                f2 = f2 + den(i)*w^(n-i)*(-1);
            end
        end
    end
    n = length(num);
    p1 = 0;
    p2 = 0;
    if rem(n,2) == 1
        for i = 1:2:n
               if rem(i-1,4) == 0
                   p1 = p1 + num(i)*w^(n-i);
               else
                   p1 = p1 + num(i)*w^(n-i)*(-1);
               end
        end
        for i = 2:2:n-1
               if rem(i-1,4) == 1
                   p2 = p2 + num(i)*w^(n-i);
               else
                   p2 = p2 + num(i)*w^(n-i)*(-1);
               end
        end
    elseif rem(n,2) == 0
        for i = 1:2:n-1
            if rem(i-1,4) == 0
                p1 = p1 + num(i)*w^(n-i);
            else
```

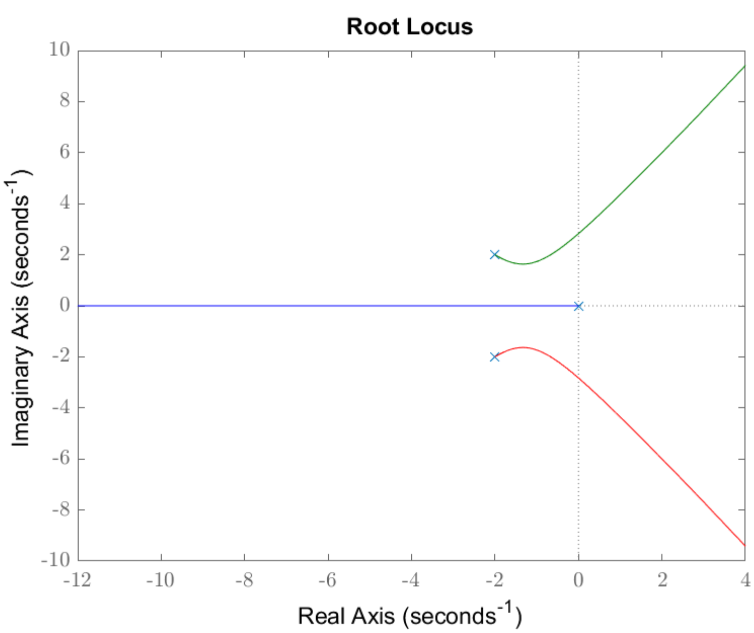
```
                p1 = p1 + num(i)*w^(n-i)*(-1);
            end
        end
        for i = 2:2:n
            if rem(i-1,4) == 1
                p2 = p2 + num(i)*w^(n-i);
            else
                p2 = p2 + num(i)*w^(n-i)*(-1);
            end
        end
    end
    eqn1 = k*p1 == f1
    eqn2 = k*p2 == f2
    a = vpasolve([eqn1 eqn2], [k w]);
    K = double(a.k);
    W = double(a.w);
end
```

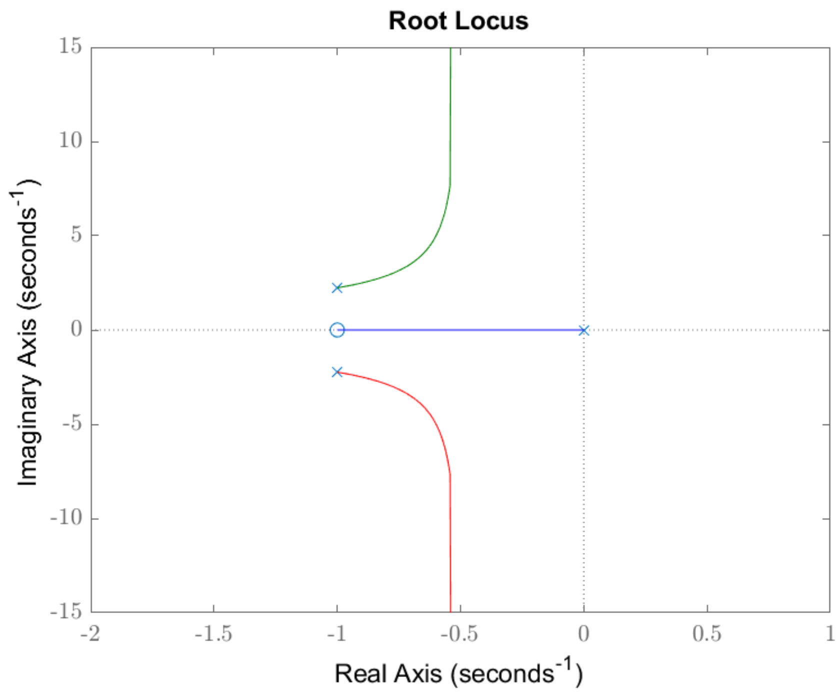
MATLAB PLOTS

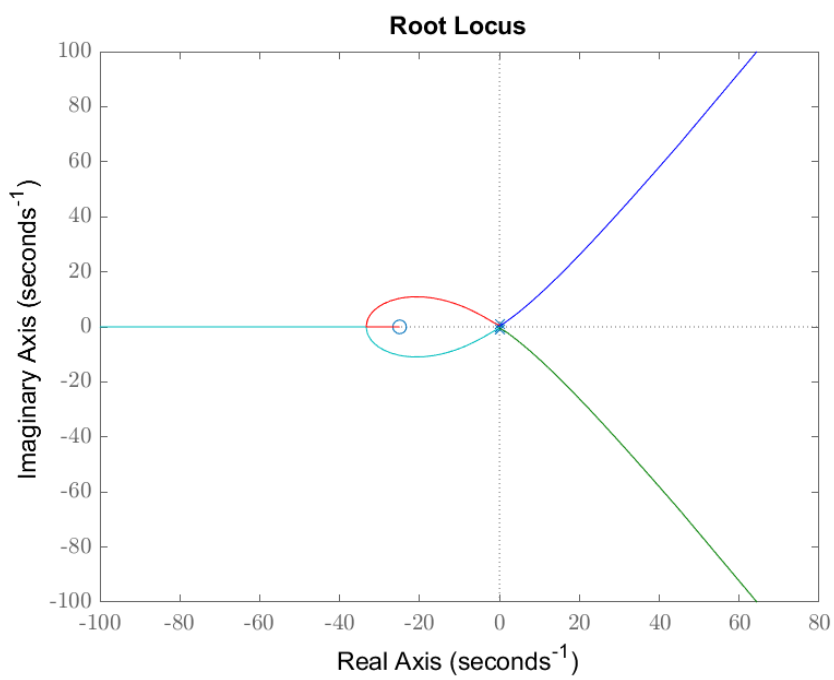





Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
50
40
30
20
10
0
-10
-20
-30
-40
-50
-10
-8
-6
-4
-2
0
2
4


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
10
8
6
4
2
0
-2
-4
-6
-8
-10
-12
-10
-8
-6
-4
-2
0
2
4


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
15
10
5
0
-5
-10
-15
-2
-1.5
-1
-0.5
0
0.5
1


Root Locus
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
100
80
60
40
20
0
-20
-40
-60
-80
-100
-100
-80
-60
-40
-20
0
20
40
60
80