College of Engineering

School of Aeronautical and Astronautical Engineering

AAE 567: Introduction To Applied Stochastic Processes

HW2

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Problem 1

[Problem 1 from p.25 of the notes] Let ω and b be vectors in $\mathbb{R}^3$. Recall that the centrifugal force in the rotating frame for a particle with mass m, position b and angular velocity ω is determined by

$$F_{cen} = -m\omega \times (\omega \times b)$$

where $\times$ denotes the cross product. Show that

$$F_{cen} = -m\omega \times (\omega \times b) = m \left\|\omega\right\|^2 P_{\mathcal{H}} b$$

where $P_{\mathcal{H}}$ is the orthogonal projection onto the subspace $\mathcal{H} = \{h \in \mathbb{C}^3 : h \perp \omega\}$ the orthogonal complement of ω. Hint: show that $\omega \times b = A_\omega b$ where A_ω is the skew symmetric matrix defined by

$$A_\omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad and \qquad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

Then show that $-A_\omega^2 = A_\omega^* A_\omega = I - \omega\omega^* = P_{\mathcal{H}}$ when $\|\omega\| = 1$.

Solution:
First of all,

$$\begin{aligned}
F_{cen} &= -m\omega \times (\omega \times b) \\
&= \begin{bmatrix} m\left(\omega_2\left(b_1\,\omega_2 - b_2\,\omega_1\right) + \omega_3\left(b_1\,\omega_3 - b_3\,\omega_1\right)\right) \\ -m\left(\omega_1\left(b_1\,\omega_2 - b_2\,\omega_1\right) - \omega_3\left(b_2\,\omega_3 - b_3\,\omega_2\right)\right) \\ -m\left(\omega_1\left(b_1\,\omega_3 - b_3\,\omega_1\right) + \omega_2\left(b_2\,\omega_3 - b_3\,\omega_2\right)\right) \end{bmatrix} \\
&= \begin{bmatrix} b_1\,m\,{\omega_2}^2 - b_2\,m\,\omega_1\,\omega_2 + b_1\,m\,{\omega_3}^2 - b_3\,m\,\omega_1\,\omega_3 \\ b_2\,m\,{\omega_1}^2 - b_1\,m\,\omega_2\,\omega_1 + b_2\,m\,{\omega_3}^2 - b_3\,m\,\omega_2\,\omega_3 \\ b_3\,m\,{\omega_1}^2 - b_1\,m\,\omega_3\,\omega_1 + b_3\,m\,{\omega_2}^2 - b_2\,m\,\omega_3\,\omega_2 \end{bmatrix} \\
&= \begin{bmatrix} b_1\,m\,({\omega_2}^2 + \omega_3^2) - b_2\,m\,\omega_1\,\omega_2 - b_3\,m\,\omega_1\,\omega_3 \\ b_2\,m\,({\omega_1}^2 + \omega_3^2) - b_1\,m\,\omega_2\,\omega_1 - b_3\,m\,\omega_2\,\omega_3 \\ b_3\,m\,({\omega_1}^2 + \omega_2^2) - b_1\,m\,\omega_3\,\omega_1 - b_2\,m\,\omega_3\,\omega_2 \end{bmatrix}.
\end{aligned}$$

Given A_w

$$\begin{aligned}
\omega \times b &= A_\omega b \\
&= \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \\
&= \begin{bmatrix} b_3\,\omega_2 - b_2\,\omega_3 \\ b_1\,\omega_3 - b_3\,\omega_1 \\ b_2\,\omega_1 - b_1\,\omega_2 \end{bmatrix}
\end{aligned}$$

and

$$-A_\omega^2 = -\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}^2$$
$$= \begin{bmatrix} {\omega_2}^2 + {\omega_3}^2 & -\omega_1\,\omega_2 & -\omega_1\,\omega_3 \\ -\omega_1\,\omega_2 & {\omega_1}^2 + {\omega_3}^2 & -\omega_2\,\omega_3 \\ -\omega_1\,\omega_3 & -\omega_2\,\omega_3 & {\omega_1}^2 + {\omega_2}^2 \end{bmatrix}$$

we know,

$$I - \omega\omega^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 - {\omega_1}^2 & -\omega_1\,\omega_2 & -\omega_1\,\omega_3 \\ -\omega_1\,\omega_2 & 1 - {\omega_2}^2 & -\omega_2\,\omega_3 \\ -\omega_1\,\omega_3 & -\omega_2\,\omega_3 & 1 - {\omega_3}^2 \end{bmatrix}.$$

if $\|\omega\| = 1$, then

$$1 - \omega_1^2 = \omega_2^2 + \omega_3^2$$
$$1 - \omega_2^2 = \omega_1^2 + \omega_3^2$$
$$1 - \omega_3^2 = \omega_1^2 + \omega_2^2$$

Hence,

$$-A_\omega^2 = I - \omega\omega^*$$

and if

$$P_{\mathcal{H}} = I - \omega\omega^*$$

we can compute

$$m\,\|\omega\|^2\,P_{\mathcal{H}}b = mP_{\mathcal{H}}b$$
$$= m \begin{bmatrix} 1 - {\omega_1}^2 & -\omega_1\,\omega_2 & -\omega_1\,\omega_3 \\ -\omega_1\,\omega_2 & 1 - {\omega_2}^2 & -\omega_2\,\omega_3 \\ -\omega_1\,\omega_3 & -\omega_2\,\omega_3 & 1 - {\omega_3}^2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$
$$= \begin{bmatrix} -b_1\,m\,({\omega_1}^2 - 1) - b_2\,m\,\omega_1\,\omega_2 - b_3\,m\,\omega_1\,\omega_3 \\ -b_2\,m\,({\omega_2}^2 - 1) - b_1\,m\,\omega_1\,\omega_2 - b_3\,m\,\omega_2\,\omega_3 \\ -b_3\,m\,({\omega_3}^2 - 1) - b_1\,m\,\omega_1\,\omega_3 - b_2\,m\,\omega_2\,\omega_3 \end{bmatrix}$$
$$= \begin{bmatrix} b_1\,m\,({\omega_2}^2 + \omega_3^2) - b_2\,m\,\omega_1\,\omega_2 - b_3\,m\,\omega_1\,\omega_3 \\ b_2\,m\,({\omega_1}^2 + \omega_3^2) - b_1\,m\,\omega_2\,\omega_1 - b_3\,m\,\omega_2\,\omega_3 \\ b_3\,m\,({\omega_1}^2 + \omega_2^2) - b_1\,m\,\omega_3\,\omega_1 - b_2\,m\,\omega_3\,\omega_2 \end{bmatrix}.$$

Therefore we have proven that

$$F_{cen} = -m\omega \times (\omega \times b) = m\,\|\omega\|^2\,P_{\mathcal{H}}b.$$

q.e.d

Problem 2

[Problem 2 from p.26 of the notes] Consider the matrix T and vector y defined by

$$T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \mathit{and} \qquad y = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Find $\hat{u} \in \mathbb{C}^2$ and d, by hand or without using a computer, to solve the following optimization problem:

$$d = \|y - T\hat{u}\| = inf\left\{\|y - Tu\| : u \in \mathbb{C}^2\right\}.$$

Solution:
Say

$$\hat{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then,

$$\begin{aligned} d &= \|y - T\hat{u}\| \\ &= \left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| \\ &= \left\| \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\| \\ &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0. \end{aligned}$$

Thus, the optimization problem is solved to be

$$\hat{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad d = 0.$$

Problem 3

[Problem 3 from p.25 of the notes] Consider the vector

$$y^* = \begin{bmatrix} 1 & -2 & 3 & 5 & 10 & 8 & 4 \end{bmatrix} = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix}$$

Find an optimal polynomial $p(t) = \sum_0^3 \alpha_j t^j$ of degree at most three and d to solve the following optimization problem

$$d^2 = inf\Big\{\sum_{k=0}^{6} |y_k - p(k)|^2 : p(t) \text{ is a polynomial and deg } p \leq 3\Big\}.$$

Here we are looking for the optimal polynomial of degree at most three such that $p(t)|_{t=k}$ comes "as close as possible" to y_k for $k = 0, 1, 2, ..., 6$. Plot $p(t)$ and the points $\{y_k\}_0^6$ on the same graph over the interval [0,6]. Hint: vander or fliplr(vander) in MATLAB may help.

Solution:
Since the polynomial is less than or equal to the degree of 3,

$$p(\lambda) = \alpha_0 + \alpha_1\lambda + \alpha_2\lambda^2 + \alpha_3\lambda^3 \quad .$$

From this we can define the Vandermonde matrix with the vector $[0, 6]$ as

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \end{bmatrix}.$$

Now we find the coefficients of the polynomial that solve the optimization problem

$$\hat{\alpha}^* = \begin{bmatrix} \alpha_1 & \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}.$$

This is found be computing

$\hat{\alpha} = (V^*V)^{-1}V^*y$

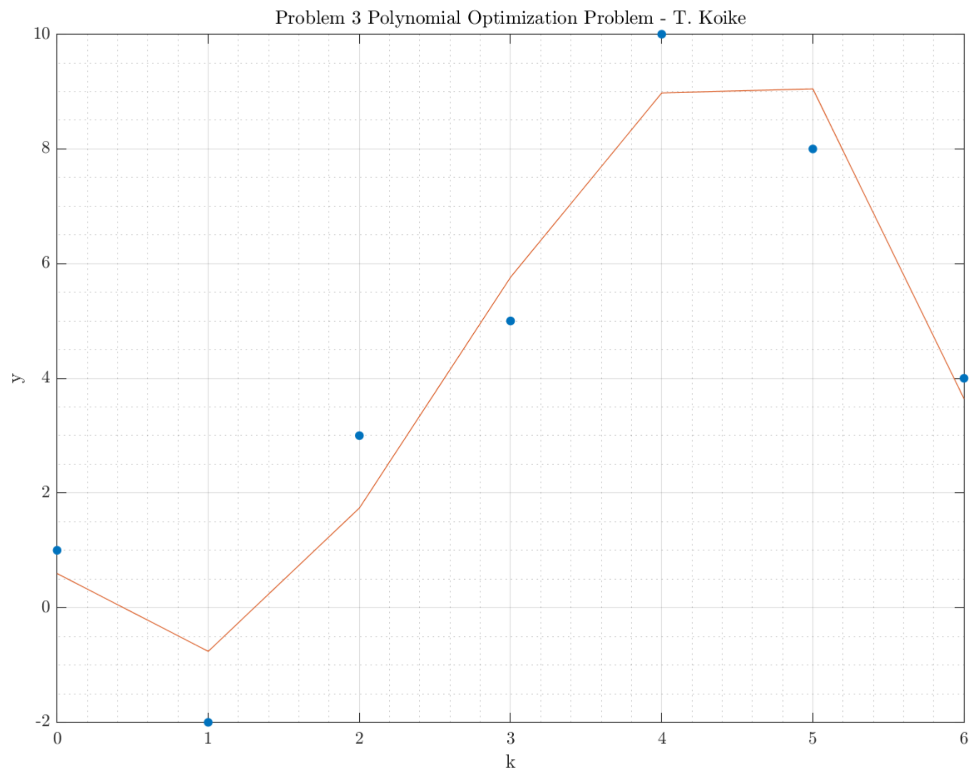
$$= \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \\ 1 & 5 & 25 & 125 \\ 1 & 6 & 36 & 216 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 4 & 9 & 16 & 25 & 36 \\ 0 & 1 & 8 & 27 & 64 & 125 & 216 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \\ 10 \\ 8 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5952 \\ -4.0635 \\ 3.0952 \\ -0.3889 \end{bmatrix}.$$

Then the error becomes

$$d^2 = \|y\|^2 - \langle \hat{\alpha}, V^* y \rangle = 6.1429$$
$$\therefore d = 2.4785 \quad .$$

The plot is as follows



Observing the plot, we can see that the polynomial fit with a degree of 3 is crude and has quite an error.

MATLAB CODE:

```matlab
%% AAE 567 HW3 Problem3

% Housekeeping commands
clear all; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

% Define given vector
y = [1 -2 3 5 10 8 4]';

% Define the Vandermonde matrix
k = [0 1 2 3 4 5 6];
V = fliplr(vander(k));
V = V(:, 1:4);
ahat = pinv(V)*y;
error2 = norm(y)^2 - dot(ahat, V'*y) ;
error = sqrt(error2);

% Plotting
fig = figure("Renderer","painters", "Position",[60 60 900 650]);
    plot(k, y, '.', "MarkerSize", 15)
    hold on; grid on; box on; grid minor;
    plot(k, polyval(flip(ahat)', k), '-')
    title("Problem 3 Polynomial Optimization Problem - T. Koike")
    xlabel('k')
    ylabel('y')
saveas(fig,"hw3_p3_polyopt.png")
```

Problem 4

[Problem 4 from p.25 of the notes] Find the optimal polynomial $p(t) = \sum_{j=0}^{5} \alpha_j t^j$ of degree at most 5 to solve the optimization problem:

$$\int_0^\pi |\sin(t) - p(t)|^2 dt = inf\Big\{\int_0^\pi |\sin(t) - \sum_{j=0}^{5} \alpha_j t^j|^2 dt : \alpha_i \in \mathbb{C}\Big\}.$$

Plot $\sin(t)$ and $p(t)$ on the same graph over the interval $[0, \pi]$.

Solution:
If we have

$$\hat{f} = \sum_{j=0}^{5} \alpha_j t^j \in \mathcal{H}$$

such that

$$\begin{gathered} f = \sin(t) \\ f - \hat{f} \in \mathcal{H} \\ f - \hat{f} \perp 1, \quad f - \hat{f} \perp t, \quad f - \hat{f} \perp t^2 \\ f - \hat{f} \perp t^3, \quad f - \hat{f} \perp t^4, \quad f - \hat{f} \perp t^5 \quad . \end{gathered}$$

Then, we have the following equations from projection theory

$$\begin{aligned} \langle \sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5, 1\rangle &= 0 \\ \langle \sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5, t\rangle &= 0 \\ \langle \sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5, t^2\rangle &= 0 \\ \langle \sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5, t^3\rangle &= 0 \\ \langle \sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5, t^4\rangle &= 0 \\ \langle \sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5, t^5\rangle &= 0 \end{aligned}$$

which can be rewritten as inner products with $\mathcal{L}^2(0, \pi)$

$$\begin{aligned} \langle \sin(t), 1\rangle &= \alpha_0\langle 1, 1\rangle + \alpha_1\langle t, 1\rangle + \alpha_2\langle t^2, 1\rangle + \alpha_3\langle t^3, 1\rangle + \alpha_4\langle t^4, 1\rangle + \alpha_5\langle t^5, 1\rangle \\ \langle \sin(t), t\rangle &= \alpha_0\langle 1, t\rangle + \alpha_1\langle t, t\rangle + \alpha_2\langle t^2, t\rangle + \alpha_3\langle t^3, t\rangle + \alpha_4\langle t^4, t\rangle + \alpha_5\langle t^5, t\rangle \\ \langle \sin(t), t^2\rangle &= \alpha_0\langle 1, t^2\rangle + \alpha_1\langle t, t^2\rangle + \alpha_2\langle t^2, t^2\rangle + \alpha_3\langle t^3, t^2\rangle + \alpha_4\langle t^4, t^2\rangle + \alpha_5\langle t^5, t^2\rangle \\ \langle \sin(t), t^3\rangle &= \alpha_0\langle 1, t^3\rangle + \alpha_1\langle t, t^3\rangle + \alpha_2\langle t^2, t^3\rangle + \alpha_3\langle t^3, t^3\rangle + \alpha_4\langle t^4, t^3\rangle + \alpha_5\langle t^5, t^3\rangle \\ \langle \sin(t), t^4\rangle &= \alpha_0\langle 1, t^4\rangle + \alpha_1\langle t, t^4\rangle + \alpha_2\langle t^2, t^4\rangle + \alpha_3\langle t^3, t^4\rangle + \alpha_4\langle t^4, t^4\rangle + \alpha_5\langle t^5, t^4\rangle \\ \langle \sin(t), t^5\rangle &= \alpha_0\langle 1, t^5\rangle + \alpha_1\langle t, t^5\rangle + \alpha_2\langle t^2, t^5\rangle + \alpha_3\langle t^3, t^5\rangle + \alpha_4\langle t^4, t^5\rangle + \alpha_5\langle t^5, t^5\rangle \end{aligned}$$

We now compute

$$\int_0^\pi \sin(t)dt = 2$$
$$\int_0^\pi t\sin(t)dt = 3.1416$$
$$\int_0^\pi t^2\sin(t)dt = 5.8696$$
$$\int_0^\pi t^3\sin(t)dt = 12.1567$$
$$\int_0^\pi t^4\sin(t)dt = 26.9738$$
$$\int_0^\pi t^5\sin(t)dt = 62.8853 \quad .$$

Since the integral of

$$\int_0^\pi t^n dt = \frac{\pi^{n+1}}{n+1}$$

the Gram matrix becomes

$$G = \begin{bmatrix} 3.1416 & 4.9348 & 10.3354 & 24.3523 & 61.2039 & 160.2315 \\ 4.9348 & 10.3354 & 24.3523 & 61.2039 & 160.2315 & 431.4705 \\ 10.3354 & 24.3523 & 61.2039 & 160.2315 & 431.4705 & 1.1861\text{e}+03 \\ 24.3523 & 61.2039 & 160.2315 & 431.4705 & 1.1861\text{e}+03 & 3.3121\text{e}+03 \\ 61.2039 & 160.2315 & 431.4705 & 1.1861\text{e}+03 & 3.3121\text{e}+03 & 9.3648\text{e}+03 \\ 160.2315 & 431.4705 & 1.1861\text{e}+03 & 3.3121\text{e}+03 & 9.3648\text{e}+03 & 26746 \end{bmatrix}.$$

Thus, to find the coefficients for the optimization problem we solve

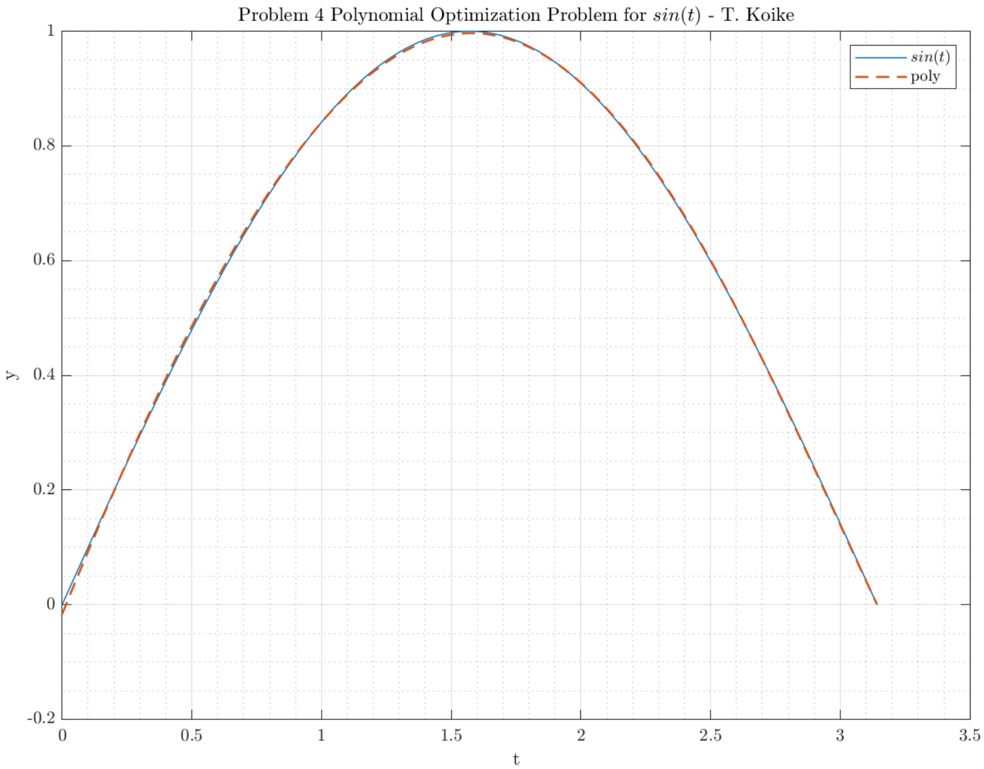
$$G\alpha = \mathbf{y}$$

$$\begin{bmatrix} 3.1416 & 4.9348 & 10.3354 & 24.3523 & 61.2039 & 160.2315 \\ 4.9348 & 10.3354 & 24.3523 & 61.2039 & 160.2315 & 431.4705 \\ 10.3354 & 24.3523 & 61.2039 & 160.2315 & 431.4705 & 1.1861\text{e}+03 \\ 24.3523 & 61.2039 & 160.2315 & 431.4705 & 1.1861\text{e}+03 & 3.3121\text{e}+03 \\ 61.2039 & 160.2315 & 431.4705 & 1.1861\text{e}+03 & 3.3121\text{e}+03 & 9.3648\text{e}+03 \\ 160.2315 & 431.4705 & 1.1861\text{e}+03 & 3.3121\text{e}+03 & 9.3648\text{e}+03 & 26746 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3.1416 \\ 5.8696 \\ 12.1567 \\ 26.9738 \\ 62.8853 \end{bmatrix}$$

Hence,

$$\alpha = \begin{bmatrix} -0.0165 \\ 1.1090 \\ -0.1697 \\ -0.0756 \\ -0.0113 \\ 0.0054 \end{bmatrix}$$

The plot for this optimization is as follows.



Observing the plot, we can see that the polynomial fits the sine wave quite precisely when it is a degree of 5.

The error of this problem turned out to be

$$\int_0^{\pi} \left(\sin(t) - \alpha_0 - \alpha_1 t - \alpha_2 t^2 - \alpha_3 t^3 - \alpha_4 t^4 - \alpha_5 t^5\right) dt = 4.4839\text{e}{-4} \qquad .$$

MATLAB CODE:

```matlab
%% AAE 567 HW3 Problem4

% Housekeeping commands
clear all; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

% Define given vector
syms t
int0 = int(sin(t),0,pi);
int1 = int(t*sin(t),0,pi);
int2 = int(t^2*sin(t),0,pi);
int3 = int(t^3*sin(t),0,pi);
int4 = int(t^4*sin(t),0,pi);
int5 = int(t^5*sin(t),0,pi);
y = [int0; int1; int2; int3; int4; int5];

% The Gram Matrix
G = zeros(6);
for i = 1:6
    for j = 1:6
        G(i,j) = int(t^(j-1)*t^(i-1), 0, pi);
    end
end

% Solve for coefficients
A = G\y;
A = double(A);
%%

% Plotting
t = linspace(0, pi, 10000);
fig = figure("Renderer","painters", "Position",[60 60 900 650]);
    plot(t, sin(t), 'LineWidth', 0.7)
    hold on; grid on; box on; grid minor;
    plot(t, polyval(flip(A)', t), '--', "LineWidth",1.2)
    title("Problem 4 Polynomial Optimization Problem for $sin(t)$ - T. Koike")
    xlabel('t')
    ylabel('y')
    legend('$sin(t)$', 'poly')
saveas(fig,"hw3_p4_sinFitPoly.png")
```

Problem 5

[Problem 5 from p.26 of the notes] In MATLAB let t =linspace(0,π,10000) and $y = \sin(t)$. Find an optimal polynomial $p(t)$ of degree at most 5 and d to solve the following optimization problem:

$$d^2 = inf\left\{\sum_{k=1}^{m} |y(k) - p(t(k))|^2 : p(t) \text{ is a polynomial and deg } p \leq 5\right\} \quad .$$

Plot $\sin(t)$ and $p(t)$ on the same graph over the interval $[0, \pi]$. How does the solution to this problem compare to the solution of Problem 4?

Solution:
Since the polynomial is less than or equal to the degree of 5,

$$p(\lambda) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \alpha_4 t^4 + \alpha_5 t^5 \quad .$$

From this we can define the Vandermonde matrix within the interval of $[0, \pi]$ with points of t =linspace(0,π,10000) for $\sin(t)$. Then we can use the Vandermonde matrix to compute the coefficients for the optimal polynomial. The MATLAB code for this procedure is as follows.

MATLAB CODE:

```
%% AAE 567 HW3 Problem5

% Housekeeping commands
clear all; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

% Define given vector
t = linspace(0, pi, 10000)
y = sin(t);

% Define the Vandermonde matrix
V = fliplr(vander(t));
V = V(:, 1:6);
%%
ahat = pinv(V)*y';
error2 = norm(y)^2 - dot(ahat, V'*y');
error = sqrt(error2);
%%
% Plotting
```

```
fig = figure("Renderer","painters", "Position",[60 60 900 650]);
    plot(t, y, '.', "MarkerSize", 3)
    hold on; grid on; box on; grid minor;
    plot(t, polyval(flip(ahat)', t), '--', 'LineWidth',2.5)
    title("Problem 5 Polynomial Optimization Problem of $\sin (t)$ - T. Koike")
    xlabel('t(k)')
    ylabel('y')
    legend('$sin(t)$', 'poly')
saveas(fig,"hw3_p5_sinDiscretePolyOpt.png")
```

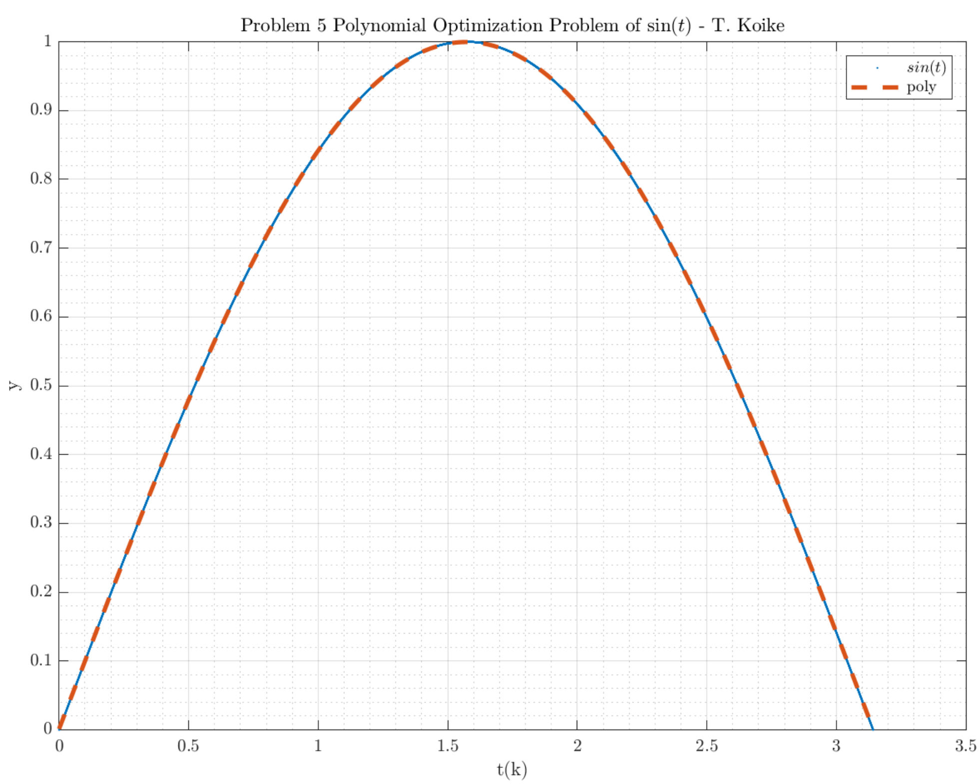
The coefficients for the polynomial becomes

$$\begin{bmatrix}\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5\end{bmatrix} = \begin{bmatrix}0.0013 & 0.9826 & 0.0545 & -0.2338 & 0.0372 & 0.0000\end{bmatrix} \quad .$$

The error becomes

$$d^2 = \|y\|^2 - \langle \hat{\alpha}, V^* y\rangle = 0.0014$$
$$\therefore d = 0.0369 \quad .$$

The plot is as follows



Looking at the plot we can hardly tell any difference from Problem 4, however, if we take a look at the coefficients and error values

	α_0	α_1	α_2	α_3	α_4	α_5	d
Problem 4	-0.0165	1.1090	-0.1697	-0.0756	-0.0013	0.0054	0.00044839
Problem 5	0.0013	0.9826	0.0545	-0.2338	0.0372	0.0000	0.0369

we can tell that the polynomials used for the curve fit is quite different and the optimization of Problem 4 is much preciser than Problem 5.

Problem 6

[Problem 5 from p.26 of the notes] In MATLAB compute $[A, B, C, D]$ = rmodel(40). This creates a stable state space system whose state dimension is 40. (The state space system is randomly generated and gives a different model every time it is implemented.) Let t = linspace(0,10,12000).

Consider the function

$$g(t) = e^{-t}\cos(2t) + e^{\frac{-t}{2}}\sin(t) \quad .$$

Let y be the vector in $\mathbb{C}^{12000}$ and T the matrix mapping $\mathbb{C}^{40}$ into $\mathbb{C}^{12000}$ defined by

$$y = \begin{bmatrix} g(t(1)) \\ g(t(2)) \\ g(t(3)) \\ \vdots \\ g(t(12000)) \end{bmatrix} \qquad and \qquad T = \begin{bmatrix} Ce^{At(1)} \\ Ce^{At(2)} \\ Ce^{At(3)} \\ \vdots \\ Ce^{At(12000)} \end{bmatrix} .$$

Solve the following optimization problem

$$d = \|y - T\hat{x}\| = inf\left\{\|y - Tx\| : x \in \mathbb{C}^{40}\right\} \quad .$$

In other words, find

$$\hat{x} = (T^*T)^{-1}T^*y = \text{pinv}(T)y \quad .$$

The inverse $(T^*T)^{-1}$ may be ill conditioned and one may have to use the pseudo-inverse in MATLAB. Plot $g(t)$ and $T\hat{x}$ on the same graph over [0,10]. Discuss how your solution is an approximation of the solution $\hat{x}_0$ to the corresponding optimization problem :

$$\int_0^{10} |g(t) - Ce^{At}\hat{x}_0|^2 dt = inf\left\{\int_0^{10} |g(t) - Ce^{At}x_0|^2 dt : x_0 \in \mathbb{C}^{40}\right\} \quad .$$

Hint: consider the Riemann integral

$$\int_0^{10} e^{A^*t}C^*Ce^{At}dt \approx \sum e^{A^*t_j}C^*Ce^{At_j}h \qquad \text{and} \qquad \int_0^{10} e^{A^*t}C^*g(t)dt \approx \sum e^{A^*t_j}C^*g(t_j)h$$

Solution:
With the following code we can analyze the results

MATLAB CODE:

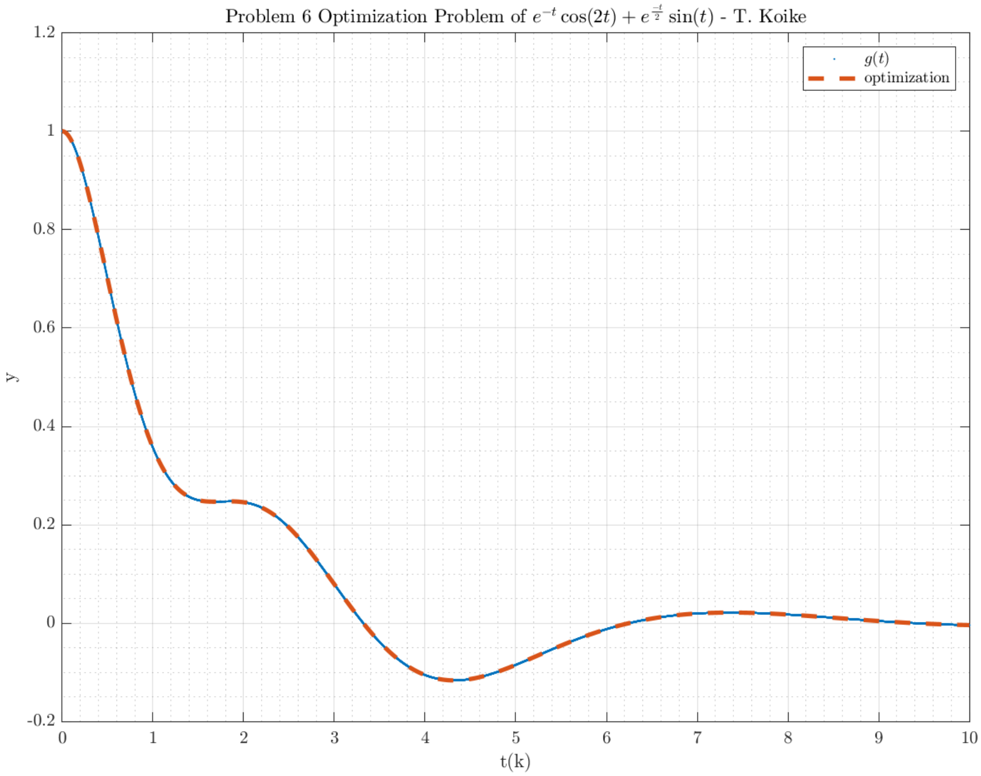
```
%% AAE 567 HW3 Problem6

% Housekeeping commands
```

```
clear all; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

% Define the system
[A,B,C,D] = rmodel(40);

% Define given function
t = linspace(0, 10, 12000)';
y = exp(-t).*cos(2*t) + exp(-t/2).*sin(t);

% Define T
T = zeros(length(y), length(C));
for i = 1:length(T)
    T(i,:) = C * expm(A * t(i));
end

% Compute xhat and the error
xhat = pinv(T)*y;                  % xhat
error = norm(y - T*xhat);          % error

% Plotting
fig = figure("Renderer","painters", "Position",[60 60 900 650]);
    plot(t, y, '.', "MarkerSize", 3)
    hold on; grid on; box on; grid minor;
    plot(t, T * xhat, '--', 'LineWidth',2.5)
    title("Problem 6 Optimization Problem of $e^{-t}\cos(2t) + e^{\frac{-t}{2}}\sin(t)$ - T. Koike")
    xlabel('t(k)')
    ylabel('y')
    legend('$g(t)$', 'optimization')
saveas(fig,"hw3_p6_eqnOpt.png")
```

From MATLAB, we obtain $\hat{x}$ to be a 40 by 1 column vector. Using this value we compute the error d to be

$$d = 2.0822\text{e} - 4$$

where the plot is as follows.



This optimization is an approximation because we have mapped the system matrices to the discrete time data points that are not continuous. Nevertheless the number of points is 12000, there will be points that are left out from being evaluated. In a continuous time system a unique solution would be found by

$$P = \int_0^{10} e^{A^*t}C^*Ce^{At}dt$$

$$P\hat{x} = \int_0^{10} e^{A^*t}C^*g(t)dt$$

$$\hat{x} = P^{-1}\int_0^{10} e^{A^*t}C^*g(t)dt \quad .$$

Whereas, in this problem these values are evaluated as an Riemann Sum of

$$P \approx \sum e^{A^*t_j}C^*Ce^{At_j}h$$

$$P\hat{x} \approx \sum e^{A^*t_j}C^*g(t_j)h$$

$$\hat{x} \approx P^{-1}\sum e^{A^*t_j}C^*g(t_j)h$$

where the increment h is

$$h = \frac{10-0}{12000} = 8.3333\text{e}-4$$

and t are 12000 points between 0 and 10. Furthermore, the function $g(t)$ is mapped to the 12000 points as $y(t)$ and Ce^{At} is mapped to the 12000 points as a matrix T. Hence, we can say that this problem is an approximation.