Lecture: Distributed Algorithms for Consensus & Averaging

Shaoshuai Mou

PURDUE
UNIVERSITY

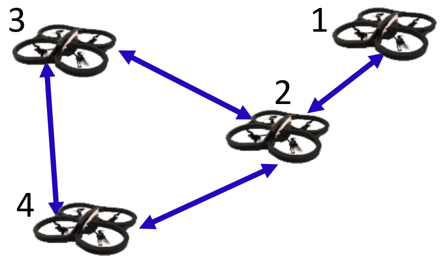
- **Flocking of large swarms**

Speed and heading directions of all birds coordinate to be **the same**.

Consensus: *All agents reach an agreement about the quantity of interest.*

Heading direction/velocity (in UAV-network);
Opinion/decision (in social networks);
clocks/time (in computer networks)

Example 1: Distributed Consensus



$x_i(t) \in \mathbb{R}$: heading direction of UAV *i*.

$\mathcal{N}_i$: the set of UAV *i*'s ***neighbors*** including itself.

calligraphic \mathcal{N}

those agents in agent *i's sensing range*

assume symmetric sensing

There is an edge connecting i and j if and only if i and j are neighbor.

- What are $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$?

$$\mathcal{N}_1 = \{1, 2\} \qquad \mathcal{N}_2 = \{1, 2, 3, 4\} \qquad \mathcal{N}_3 = \{2, 3, 4\} \qquad \mathcal{N}_4 = \{2, 3, 4\}$$

➢ **Problem:**

Develop an iterative update for each UAV's state (i.e. control input)

$$x_i(t+1) = u_i$$

by only using its neighbors states

$$u_i = f_i(x_j(t), j \in \mathcal{N}_i)$$

such that all states converge to reach a ***consensus***, namely

$$x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$$

➢ **Consensus** is the basis for a large number of autonomous agents to work as a cohesive whole, is the key to understand collective behaviors and swarm intelligence.

Consensus-based distributed computation/optimizations.

UAVs' heading direction/velocity; people's opinion/decision variables; computers' clock, …

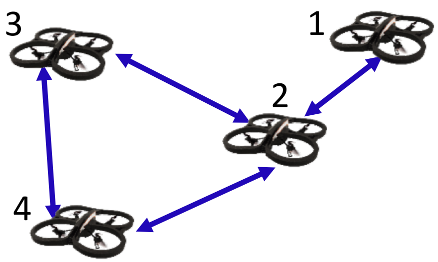
Local Average

- **The Update:** $$\boxed{x_i(t+1) = \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j(t)}$$

d_i :the number of agent *i*'s neighbors.

Try writing out the distributed updates by yourself.

Converge to a consensus??? MATLAB



$$x_1(t+1) = \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t)$$

$$x_2(t+1) = \frac{1}{4}x_1(t) + \frac{1}{4}x_2(t) + \frac{1}{4}x_3(t) + \frac{1}{4}x_4(t)$$

$$x_3(t+1) = \frac{1}{3}x_2(t) + \frac{1}{3}x_3(t) + \frac{1}{3}x_4(t)$$

$$x_4(t+1) = \frac{1}{3}x_2(t) + \frac{1}{3}x_3(t) + \frac{1}{3}x_4(t)$$

- **Compact Form:**

$$x(t+1) = Ax(t) \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$A = ?$$

Try writing out the matrix A by yourself.

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Do all $x_i(t)$ converge to reach a consensus?

Does $x(t)$ ***converge*** *to be constant?*

At the convergence point, is a ***consensus*** *reached?*

$$x(t+1) = Ax(t)$$

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$x_1 = x_2 = x_3 = x_4 = x^*$$

$$x(t) \to x^* \mathbf{1} \qquad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$x(t) = A^t x(0)$$

What ***special structure*** do you observe of this matrix?

MATLAB

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} A^t x(0) = \begin{bmatrix} 1/6 & 1/3 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/4 & 1/4 \\ 1/6 & 1/3 & 1/4 & 1/4 \end{bmatrix} x(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \underbrace{[1/6 \quad 1/3 \quad 1/4 \quad 1/4]\, x(0)}_{x^*}$$

$$\underset{\textbf{\textit{converge}}}{A^t \to} \underset{\textbf{\textit{consensus}}}{A^* = \mathbf{1} q'}$$

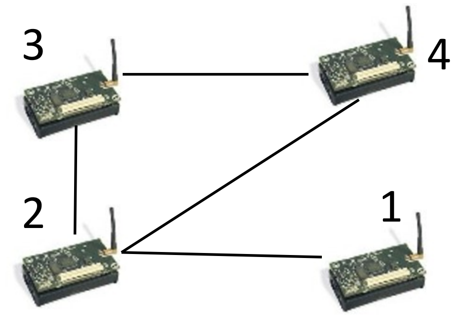
- **A General Update for Consensus:**

Local Weighted Average (convex combination)

$$\boxed{x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} > 0, j \in \mathcal{N}_i \qquad \sum_{j \in \mathcal{N}_i} w_{ij} = 1}$$

w_{ij} : the weight assigned by agent *i* to its neighbor *j*

Example 2: Distributed Consensus for Global Average (**Distributed Averaging**)



$x_i(t) \in \mathbb{R}$: Measurement of sensor *i*.

$\mathcal{N}_i$: the set of sensor *i*'s neighbors including itself.

- What are $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_4$?

$$\mathcal{N}_1 = \{1, 2\} \qquad \mathcal{N}_2 = \{1, 2, 3, 4\} \qquad \mathcal{N}_3 = \{2, 3, 4\} \qquad \mathcal{N}_4 = \{2, 3, 4\}$$

- **Problem:**

Develop an iterative update for each sensor's state (i.e. control input) $x_i(t+1) = u_i$

by only using its neighbors states $u_i = f_i(x_j(t), j \in \mathcal{N}_i)$

such that all states converge to reach

consensus: $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$

and x^* is the ***global average*** of all sensors' initial measurements.

$$x^* = \frac{1}{m}\sum_{i=1}^{m} x_i(0)$$

Distributed Averaging Problem

- Why do we care about the global average?? **robust against white noises**.

$$q + v_i \qquad E[v_i] = 0 \qquad E[\frac{1}{m}\sum_{i=1}^{m} x_i(0)] = q$$

➢ Distributed Averaging:

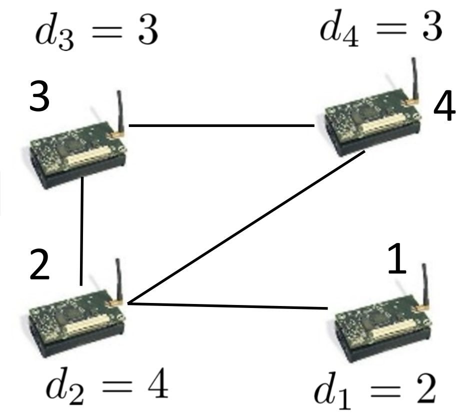
$$x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$$

consensus

$$x^* = \frac{1}{m}\sum_{i=1}^{m} x_i(0)$$

to a specific value

$$d_i = |\mathcal{N}_i|$$



Local weighted average

Metropolis Weights

$$w_{ij} > 0, j \in \mathcal{N}_i \qquad \sum_{j\in\mathcal{N}_i} w_{ij} = 1$$

$$x_i(t+1) = \sum_{j\in\mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} \dfrac{1}{\max\{d_i, d_j\}} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \displaystyle\sum_{j\in\mathcal{N}_i,\ j\neq i} w_{ij} & j = i \end{cases}$$

weight to its neighbor

weight to itself

- Take agent 3 for example: $w_{32} = \dfrac{1}{\max\{d_3, d_2\}} = \dfrac{1}{\max\{3,4\}} = \dfrac{1}{4}$

$$w_{33} = 1 - \frac{1}{4} - \frac{1}{3} = \frac{5}{12} \qquad w_{34} = \frac{1}{\max\{d_3, d_4\}} = \frac{1}{\max\{3,3\}} = \frac{1}{3}$$

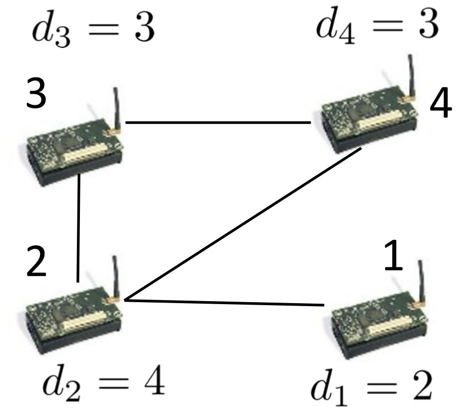
- Do the Metropolis weights satisfy the convex combination requirement?

$$w_{ij} > 0, j \in \mathcal{N}_i \qquad \sum_{j\in\mathcal{N}_i} w_{ij} = 1$$

Try by yourself to prove $w_{ii} > 0$

➢ The Metropolis Update:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} \dfrac{1}{\max\{d_i, d_j\}} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \displaystyle\sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \end{cases}$$



It is **distributed** since only neighbors' information is used.

Try writing out the distributed updates by yourself.

$$x_1(t+1) = \frac{3}{4}x_1(t) + \frac{1}{4}x_2(t)$$

$$x_2(t+1) = \frac{1}{4}x_1(t) + \frac{1}{4}x_2(t) + \frac{1}{4}x_3(t) + \frac{1}{4}x_4(t)$$

$$x_3(t) = \frac{1}{4}x_2(t) + \frac{5}{12}x_3(t) + \frac{1}{3}x_4(t)$$

$$x_4(t) = \frac{1}{4}x_2(t) + \frac{1}{3}x_3(t) + \frac{5}{12}x_4(t)$$

➢ Compact Form: $x(t+1) = Ax(t)$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$A = ?$

Try writing out the matrix A by yourself.

$$A = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Do all $x_i(t)$ converge to reach a consensus, which is the global average?

*Does $x(t)$ **converge** to be constant?*

$$x(t+1) = Ax(t)$$

$$x(t) = A^t x(0)$$

*At the convergence point, is a **consensus** reached?*

$$x_1 = x_2 = x_3 = x_4 = x^*$$

$$x(t) \to x^* \mathbf{1}$$

$$\mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Is the consensus value the global average?

$$x^* = \frac{1}{m} \sum_{i=1}^{m} x_i(0)$$

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} A^t x(0) \overset{?}{=} \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix} x(0) = \mathbf{1} \cdot \frac{1}{4} \cdot \overbrace{\mathbf{1}' x(0)}^{x^*}$$

MATLAB

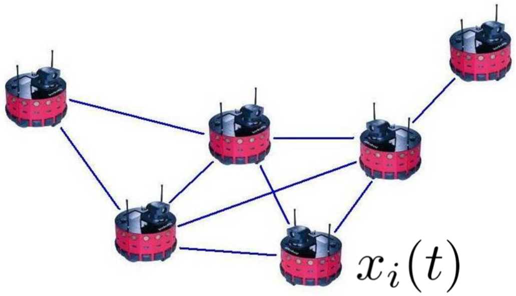
$$A^t \to A^* = \frac{1}{m}\mathbf{1}\mathbf{1}'$$

converge ***Consensus+ Global Average***

Summary

❑ Consensus

$$x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$$



Local Weighted Average

Distributed Algorithm:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$$

$$w_{ij} > 0, j \in \mathcal{N}_i \quad \sum_{j \in \mathcal{N}_i} w_{ij} = 1$$

agent's dynamics: $x_i(t+1) = u_i$

control input: $u_i = f_i(x_j(t), j \in \mathcal{N}_i)$

❑ Consensus for Global Average

$$x_1(t) = x_2(t) = \cdots = x_m(t) = x^* \qquad x^* = \frac{1}{m}\sum_{i=1}^{m} x_i(0)$$

Local Weighted Average with Metropolis weights

Distributed Algorithm:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} \dfrac{1}{\max\{d_i, d_j\}} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \displaystyle\sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \end{cases}$$