AAE 567 quiz Spring 2021:

Write your final answers on the exam.

Open book with Matlab.

Hand in your work.

NAME:

Problem 1. Consider the optimization problem

$$d = \|y - A\widehat{x}\| = \min\{\|y - Ax\| : x \in \mathbb{C}^2\}$$

where the matrix A and vector y are infinite dimensional and given by

$$A = \begin{bmatrix} 1 & 1 \\ a & b \\ a^2 & b^2 \\ a^3 & b^3 \\ \vdots & \vdots \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1 \\ c \\ c^2 \\ c^3 \\ \vdots \end{bmatrix} \quad \text{and} \quad \left\| \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix} \right\|^2 = \sum_{j=0}^{\infty} |f_j|^2$$

Here $a = \frac{1}{2}$ and $b = \frac{2}{3}$ and $c = \frac{3}{4}$. **Hint:** if $|r| < 1$, then $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$

(i) Find an optimal $\widehat{x} \in \mathbb{C}^2$ solving this optimization problem:

$\widehat{x} =$

(ii) The optimal solution $\widehat{x}$ is unique. Circle one: **TRUE** **FALSE**

(iii) Find the error squared

$d^2 =$

Problem 2. The pair $\{C, A\}$ is given by

$$A = \begin{bmatrix} 2 & 6 \\ -2 & -5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

Consider the optimization problem

$$d^2 = \int_0^\infty |26e^{-3t} - Ce^{At}\widehat{x}|^2 dt = \min \left\{ \int_0^\infty |26e^{-3t} - Ce^{At}x|^2 dt : x \in \mathbb{C}^2 \right\}$$

(i) The pair $\{C, A\}$ is observable. Circle one: **TRUE** **FALSE**

(ii) The optimal solution $\widehat{x}$ is unique. Circle one: **TRUE** **FALSE**

(iii) Find $\widehat{x}$ of smallest possible norm which solves this optimization problem:

$\widehat{x} =$

(iv) Find the error squared

$d^2 =$

Problem 3. Let $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ be three independent uniform random variables over the interval $[0, 1]$. Let $\mathbf{y}$ be the random variable defined by

$$\mathbf{y} = \mathbf{x} + \mathbf{u} + \mathbf{v}$$

Let $\mathcal{H}$ be the subspace defined by $\mathcal{H} = \text{span}\{1, \mathbf{y}\}$.

(i) Find the orthogonal projection $\widehat{\mathbf{x}} = P_{\mathcal{H}}\mathbf{x}$, that is, find the constants α and β:

$\widehat{\mathbf{x}} = P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{y} =$

(ii) Find the following error in estimation

$E(\mathbf{x} - \widehat{\mathbf{x}})^2 =$

Problem 4. Let $\mathbf{x}$ and $\mathbf{v}$ be two independent random variables. The density functions for $\mathbf{x}$ and $\mathbf{v}$ are given by

$$f_{\mathbf{x}}(x) = xe^{-x} \quad \text{if } x \geq 0 \quad \text{and} \quad f_{\mathbf{x}}(x) = 0 \quad \text{if } x < 0$$
$$f_{\mathbf{v}}(v) = e^{-v} \quad \text{if } v \geq 0 \quad \text{and} \quad f_{\mathbf{v}}(v) = 0 \quad \text{if } v < 0$$

Assume that the random variable $\mathbf{y} = \mathbf{x} + \mathbf{v}$. Recall that the joint density

$$f_{\mathbf{x},\mathbf{y}}(x, y) = f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x)$$
$$f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x) dx = \int_{0}^{y} f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x) dx \quad (y \geq 0)$$

In this case, $f_{\mathbf{y}}(y) = 0$ for $y < 0$.

(i) Find the following conditional expectation

$\widehat{g}(y) = E(\mathbf{x}|\mathbf{y} = y) =$

(ii) Let $\mathcal{H} = \text{span}\{1, \mathbf{y}\}$. Find α and β such that $P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{y}$.

$P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{y} =$