A-AE 567 quiz Spring 2021:

Write your final answers on the exam.

Open book with Matlab.

Hand in your work.

NAME:

Problem 1. Consider the optimization problem

$$d = \|y - A\widehat{x}\| = \min\{\|y - Ax\| : x \in \mathbb{C}^2\}$$

where the matrix A and vector y are infinite dimensional and given by

$$A = \begin{bmatrix} 1 & 1 \\ a & b \\ a^2 & b^2 \\ a^3 & b^3 \\ \vdots & \vdots \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 1 \\ c \\ c^2 \\ c^3 \\ \vdots \end{bmatrix} \quad \text{and} \quad \left\| \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \end{bmatrix} \right\|^2 = \sum_{j=0}^{\infty} |f_j|^2$$

Here $a = \frac{1}{2}$ and $b = \frac{2}{3}$ and $c = \frac{3}{4}$. **Hint:** if $|r| < 1$, then $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$

(i) Find an optimal $\widehat{x} \in \mathbb{C}^2$ solving this optimization problem:

$$\widehat{x} = \begin{bmatrix} -\frac{4}{5} \\ \frac{16}{9} \end{bmatrix} = \begin{bmatrix} -0.8 \\ 1.7778 \end{bmatrix}$$

(ii) The optimal solution $\widehat{x}$ is unique. $\boxed{\text{TRUE}}$

(iii) Find the error squared

$$d^2 = \frac{16}{1575} = 0.0102$$

Finally, it is noted that $d = 0.1008$.

Solution for Problem 1. Since the columns of A are linearly independent the optimal solution $\widehat{x}$ is unique and $\widehat{x} = (A^*A)^{-1}A^*y$. Moreover,

$$A^*A = \begin{bmatrix} \frac{1}{1-a^2} & \frac{1}{1-ab} \\ \frac{1}{1-ab} & \frac{1}{1-b^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\frac{1}{4}} & \frac{1}{1-\frac{1}{3}} \\ \frac{1}{1-\frac{1}{3}} & \frac{1}{1-\frac{4}{9}} \end{bmatrix} = \begin{bmatrix} \frac{4}{3} & \frac{3}{2} \\ \frac{3}{2} & \frac{9}{5} \end{bmatrix}$$

Because A^*A is strictly positive, this also shows that the optimal solution $\widehat{x}$ is unique. Notice that

$$(A^*A)^{-1} = \begin{bmatrix} 12 & -10 \\ -10 & \frac{80}{9} \end{bmatrix}$$

Furthermore,

$$A^*y = \begin{bmatrix} \frac{1}{1-ac} \\ \frac{1}{1-bc} \end{bmatrix} = \begin{bmatrix} \frac{1}{1-\frac{3}{8}} \\ \frac{1}{1-\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} \frac{8}{5} \\ 2 \end{bmatrix}$$

Finally,

$$\widehat{x} = (A^*A)^{-1}A^*y = \begin{bmatrix} 12 & -10 \\ -10 & \frac{80}{9} \end{bmatrix} \begin{bmatrix} \frac{8}{5} \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ \frac{16}{9} \end{bmatrix}$$

The error is given by

$$\begin{aligned} d^2 &= \|y - A\widehat{x}\|^2 = \|y\|^2 - \|A\widehat{x}\|^2 = \frac{1}{1-c^2} - (A^*A\widehat{x}, \widehat{x}) \\ &= \frac{1}{1-\frac{9}{16}} - \widehat{x}^* A^*A(A^*A)^{-1}A^*y = \frac{16}{7} - \widehat{x}^*A^*y \\ &= \frac{16}{7} - \begin{bmatrix} -\frac{4}{5} & \frac{16}{9} \end{bmatrix} \begin{bmatrix} \frac{8}{5} \\ 2 \end{bmatrix} = \frac{16}{7} - \frac{512}{225} = \frac{16}{1575} \end{aligned}$$

In other words, $d^2 = \frac{16}{1575}$.

Problem 2. The pair $\{C, A\}$ is given by

$$A = \begin{bmatrix} 2 & 6 \\ -2 & -5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

Consider the optimization problem

$$d^2 = \int_0^\infty |26e^{-3t} - Ce^{At}\widehat{x}|^2 dt = \min \left\{ \int_0^\infty |26e^{-3t} - Ce^{At}x|^2 dt : x \in \mathbb{C}^2 \right\}$$

(i) The pair $\{C, A\}$ is observable. FALSE

(ii) The optimal solution $\widehat{x}$ is unique. FALSE

(iii) Find $\widehat{x}$ of smallest possible norm which solves this optimization problem:

$$\widehat{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(iv) Find the error squared

$$d^2 = \frac{169}{6} = 28.1667$$

Finally, it is noted that $d = \frac{13}{\sqrt{6}} \approx 5.3072$.

Solution for Problem 2. Notice that

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}$$

has rank one. So the pair $\{C, A\}$ is not observable. Because $\{C, A\}$ is not observable, an optimal solution to the corresponding optimization problem is not unique.

The optimal solution $\widehat{x}$ with smallest norm is given by

$$\begin{aligned} \widehat{x} &= P^{-r} \int_0^\infty e^{A^* t} C^* 26 e^{-3t} dt \\ P &= \int_0^\infty e^{A^* t} C^* C e^{At} dt \\ 0 &= A^* P + PA + C^* C \end{aligned}$$

Here P^{-r} is the Moore-Penrose inverse of P, that is, $P^{-r} = \text{pinv}(P)$ in Matlab. Furthermore, $P = \text{lyap}(A', C'C)$ in Matlab. Using Matlab

$$P = \begin{bmatrix} 2 & 3 \\ 3 & \frac{9}{2} \end{bmatrix} \quad \text{and} \quad P^{-r} = \frac{1}{169} \begin{bmatrix} 8 & 12 \\ 12 & 18 \end{bmatrix}$$

Notice that $P \geq 0$ and singular. This also shows that $\{C, A\}$ is not observable. Using the, syms t, command in Matlab

$$\widehat{x} = \text{pinv}(P) * \text{int}(\text{expm}(A' * t) * C' * 26 * \exp(-3 * t), 0, \text{inf}) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Finally, the squared error is given by

$$\begin{aligned} d^2 &= \|26e^{-3t}\|^2 - \widehat{x}^* P \widehat{x} = 26^2 \int_0^\infty |e^{-3t}|^2 dt - \widehat{x}^* \begin{bmatrix} 2 & 3 \\ 3 & \frac{9}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \frac{26^2}{6} - \begin{bmatrix} 2 & 3 \end{bmatrix} \begin{bmatrix} 13 \\ \frac{39}{2} \end{bmatrix} = \frac{26^2}{6} - \frac{169}{2} = \frac{169}{6} \end{aligned}$$

Therefore $d^2 = \frac{169}{6}$.

Problem 3. Let $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ be three independent uniform random variables over the interval $[0,1]$. Let $\mathbf{y}$ be the random variable defined by

$$\mathbf{y} = \mathbf{x} + \mathbf{u} + \mathbf{v}$$

Let $\mathcal{H}$ be the subspace defined by $\mathcal{H} = \text{span}\{1, \mathbf{y}\}$.

(i) Find the orthogonal projection $\widehat{\mathbf{x}} = P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{y}$:

$$\widehat{\mathbf{x}} = P_{\mathcal{H}}\mathbf{x} = \frac{\mathbf{y}}{3}$$

Note $\alpha = 0$ and $\beta = \frac{1}{3}$.

(ii) Find the following error in estimation

$$E(\mathbf{x} - \widehat{\mathbf{x}})^2 = \frac{1}{18}$$

In particular, $d = \frac{1}{3\sqrt{2}} = 0.2357$.

Solution for Problem 3. It turns out that $E(\mathbf{x}|\mathbf{y}) = \frac{\mathbf{y}}{3}$ whenever $\mathbf{y} = \mathbf{x} + \mathbf{u} + \mathbf{v}$ and $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ are three independent (finite variance) random variables with the same density. Hence $P_{\mathcal{H}}\mathbf{x} = \frac{\mathbf{y}}{3}$. Lets assume that one is not aware of this result and directly show that $P_{\mathcal{H}}\mathbf{x} = \frac{\mathbf{y}}{3}$.

For the general case, let us assume that $\mathbf{y} = \mathbf{x} + \mathbf{u} + \mathbf{v}$ where $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ are three independent random variables with the same density. Because $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$ all have the same density we can set

$$\mu = E\mathbf{x} = E\mathbf{u} = E\mathbf{v} \quad \text{and} \quad \xi = E\mathbf{x}^2 = E\mathbf{u}^2 = E\mathbf{v}^2 \quad \text{and} \quad g = \begin{bmatrix} 1 \\ \mathbf{y} \end{bmatrix}$$

Then $\widehat{\mathbf{x}} = P_{\mathcal{H}}\mathbf{x} = R_{\mathbf{x}g}R_g^{-1}g$. Notice that

$$\begin{aligned} R_{\mathbf{x}g} &= \begin{bmatrix} E\mathbf{x} & E(\mathbf{x}\mathbf{y}) \end{bmatrix} = \begin{bmatrix} \mu & E\big(\mathbf{x}(\mathbf{x}+\mathbf{u}+\mathbf{v})\big) \end{bmatrix} \\ &= \begin{bmatrix} \mu & E\mathbf{x}^2 + E\mathbf{x}\mathbf{u} + E\mathbf{x}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \mu & \xi + E\mathbf{x}E\mathbf{u} + E\mathbf{x}E\mathbf{v} \end{bmatrix} \\ &= \begin{bmatrix} \mu & \xi + \mu^2 + \mu^2 \end{bmatrix} \end{aligned}$$

In other words,

$$R_{\mathbf{x}g} = \begin{bmatrix} \mu & \xi + 2\mu^2 \end{bmatrix}$$

Now observe that

$$\begin{aligned} E\mathbf{y}^2 &= E\big(\mathbf{x}+\mathbf{u}+\mathbf{v}\big)^2 = E\mathbf{x}^2 + E\mathbf{u}^2 + E\mathbf{v}^2 + 2E\mathbf{x}\mathbf{u} + 2E\mathbf{x}\mathbf{v} + 2E\mathbf{u}\mathbf{v} \\ &= 3\xi + 2E\mathbf{x}E\mathbf{u} + 2E\mathbf{x}E\mathbf{v} + 2E\mathbf{u}E\mathbf{v} = 3\xi + 6\mu^2 \end{aligned}$$

In other words, $E\mathbf{y}^2 = 3\xi + 6\mu^2$. Notice that

$$R_g = \begin{bmatrix} E1 & E\mathbf{y} \\ E\mathbf{y} & E\mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} 1 & 3\mu \\ 3\mu & 3\xi + 6\mu^2 \end{bmatrix} \quad \text{and} \quad R_g^{-1} = \frac{1}{3(\xi - \mu^2)} \begin{bmatrix} 3\xi + 6\mu^2 & -3\mu \\ -3\mu & 1 \end{bmatrix}$$

Using this we have

$$
\begin{aligned}
\widehat{\mathbf{x}} = R_{\mathbf{x}g}R_g^{-1}g &= \frac{1}{3(\xi-\mu^2)}\begin{bmatrix}\mu & \xi+2\mu^2\end{bmatrix}\begin{bmatrix}3\xi+6\mu^2 & -3\mu\\ -3\mu & 1\end{bmatrix}g\\
&= \frac{1}{3(\xi-\mu^2)}\begin{bmatrix}0 & \xi-\mu^2\end{bmatrix}\begin{bmatrix}1\\ \mathbf{y}\end{bmatrix} = \frac{\mathbf{y}}{3}
\end{aligned}
$$

Finally, it is noted that $\xi-\mu^2 = E\mathbf{x}^2-(E\mathbf{x})^2 = E(\mathbf{x}-\mu)^2$, which is precisely the variance of $\mathbf{x}$. Hence $\xi-\mu^2 \neq 0$ unless $\mathbf{x}$ is a constant. So without loss of generality, we can assume that $\mathbf{x}$ is not a constant. The squared error is given by

$$
E(\mathbf{x}-\widehat{\mathbf{x}})^2 = E\mathbf{x}^2 - E\widehat{\mathbf{x}}^2 = \xi - \frac{1}{9}E\mathbf{y}^2 = \xi - \frac{3(\xi+2\mu^2)}{9} = \frac{2(\xi-\mu^2)}{3}
$$

Therefore

$$
d^2 = \frac{2(\xi-\mu^2)}{3} = \frac{2(E\mathbf{x}^2-\mu^2)}{3} = \frac{2E(\mathbf{x}-\mu)^2}{3}
$$

In our case, $\mathbf{x}$ is uniform over the interval $[0,1]$. Hence

$$
\mu = E\mathbf{x} = \frac{1}{2} \quad \text{and} \quad \xi = E\mathbf{x}^2 = \int_0^1 x^2dx = \frac{1}{3}
$$

Therefore

$$
d^2 = \frac{2(\frac{1}{3}-\frac{1}{4})}{3} = \frac{\frac{2}{12}}{3} = \frac{1}{18}
$$

Problem 4. Let $\mathbf{x}$ and $\mathbf{v}$ be two independent random variables. The density functions for $\mathbf{x}$ and $\mathbf{v}$ are given by

$$f_{\mathbf{x}}(x) = xe^{-x} \quad \text{if } x \geq 0 \quad \text{and} \quad f_{\mathbf{x}}(x) = 0 \quad \text{if } x < 0$$
$$f_{\mathbf{v}}(v) = e^{-v} \quad \text{if } v \geq 0 \quad \text{and} \quad f_{\mathbf{v}}(v) = 0 \quad \text{if } v < 0$$

Assume that the random variable $\mathbf{y} = \mathbf{x} + \mathbf{v}$. Recall that the joint density

$$f_{\mathbf{x},\mathbf{y}}(x, y) = f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x)$$
$$f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x) dx = \int_{0}^{y} f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x) dx \quad (y \geq 0)$$

In this case, $f_{\mathbf{y}}(y) = 0$ for $y < 0$.

(i) Find the following conditional expectation

$$\widehat{g}(y) = E(\mathbf{x}|\mathbf{y} = y) = \frac{2y}{3}$$

(ii) Let $\mathcal{H} = \text{span}\{1, \mathbf{y}\}$. Find α and β such that $P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{y}$.

$$P_{\mathcal{H}}\mathbf{x} = \frac{2\mathbf{y}}{3}$$

In particular, $\alpha = 0$ and $\beta = \frac{2}{3}$.

Solution for Problem 4. Notice that for $y \geq 0$, we have

$$f_{\mathbf{y}}(y) = \int_0^y f_{\mathbf{x}}(x) f_{\mathbf{v}}(y-x)dx = \int_0^y xe^{-x}e^{-(y-x)}dx = e^{-y}\int_0^y xdx = \frac{y^2e^{-y}}{2}$$

In other words,

$$\begin{aligned} f_{\mathbf{y}}(y) &= \frac{y^2e^{-y}}{2} \qquad && \text{if } y \geq 0 \\ &= 0 && \text{otherwise} \end{aligned}$$

Moreover,

$$f_{\mathbf{x}|\mathbf{y}}(x|y) = \frac{f_{\mathbf{x},\mathbf{y}}(x,y)}{f_{\mathbf{y}}(y)} = \frac{2xe^{-x}e^{-(y-x)}}{y^2e^{-y}} = \frac{2x}{y^2} \qquad \text{if } 0 < x \leq y$$

To be specific

$$\begin{aligned} f_{\mathbf{x}|\mathbf{y}}(x|y) &= \frac{2x}{y^2} \qquad && \text{if } 0 < x \leq y \\ &= 0 && \text{otherwise} \end{aligned}$$

(Notice that for fixed $y > 0$, we have that $f_{\mathbf{x}|\mathbf{y}}(x|y)$ is indeed a density function in x.)

Hence

$$E(\mathbf{x}|\mathbf{y} = y) = \int_{-\infty}^{\infty} x f_{\mathbf{x}|\mathbf{y}}(x|y)dx = \frac{2}{y^2}\int_0^y x^2dx = \frac{2y}{3}.$$

In other words, $E(\mathbf{x}|\mathbf{y} = y) = \frac{2y}{3}$. Recall that $E(\mathbf{x}|\mathbf{y}) = P_{\mathcal{G}}\mathbf{x}$ where $\mathcal{G}$ is the subspace formed by all function of $\mathbf{y}$ with finite variance. Clearly, $\mathcal{H} = \text{span}\{1, \mathbf{y}\}$ is a subspace of $\mathcal{G}$. Since $P_{\mathcal{G}}\mathbf{x} = E(\mathbf{x}|\mathbf{y}) = \frac{2\mathbf{y}}{3}$ is a vector in $\mathcal{H}$ and $\mathcal{H} \subset \mathcal{G}$, it follows that

$$P_{\mathcal{H}}\mathbf{x} = P_{\mathcal{H}}P_{\mathcal{G}}\mathbf{x} = P_{\mathcal{H}}\frac{2\mathbf{y}}{3} = \frac{2\mathbf{y}}{3}$$

In other words,

$$\alpha + \beta\mathbf{y} = P_{\mathcal{H}}\mathbf{x} = \frac{2\mathbf{y}}{3}$$

that is, $\alpha = 0$ and $\beta = \frac{2}{3}$. One can also verify this directly.