Lecture: Distributed Algorithms for General Consensus

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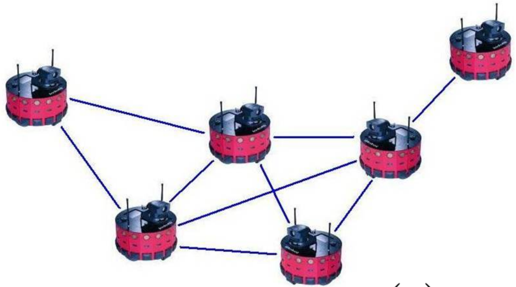
Review of Consensus Algorithms



$x_i(t)$

❑ Consensus:

just a constant

- **Objective:** $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$

- **Update:**

Local Weighted Average (convex combination)

$$\boxed{x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \quad \sum_{j=1}^{m} w_{ij} = 1}$$

agent's dynamics: $x_i(t+1) = u_i$

control input: $u_i = f_i(x_j(t), j \in \mathcal{N}_i)$

distributed

w_{ij} : the weight assigned by agent *i* to agent *j*

- **Compact Form:** $x(t+1) = Ax(t)$

$A \in \mathbb{R}^{m \times m}$ with entries: $A_{ij} = \begin{cases} w_{ij} & \text{if } j \in \mathcal{N}_i \\ 0 & \text{otherwise} \end{cases}$

$$A\mathbf{1} = \mathbf{1} \qquad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

- **Analysis:**

$q \in \mathbb{R}^m$ (unknown)

$$\boxed{A^t \rightarrow \mathbf{1}\, q'}$$

Consensus is reached.

$$x(t) \rightarrow \mathbf{1} \underbrace{q'\, x(0)}_{x^*} = \begin{bmatrix} x^* \\ x^* \\ \vdots \\ x^* \end{bmatrix}$$

❑ Consensus to the Global Average

- **Objective:** $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$ $\quad x^* = \frac{1}{m}\sum_{i=1}^{m} x_i(0) = \frac{1}{m}\mathbf{1}'x(0)$ $\quad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

  consensus   to a specific value (the global average)

- **Distributed Update 1:** *Local weighted average*   Metropolis Weights

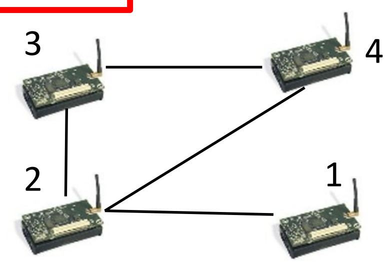
$$w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \qquad \sum_{j=1}^{m} w_{ij} = 1$$

$$\boxed{x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} \min\{\frac{1}{d_i}, \frac{1}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \end{cases}}$$

- **Distributed Update 2:** Periodic Gossiping.

  (1,2), (3,4), (2,3), (2,4)   (1,2), (3,4), (2,3), (2,4), ….

$$x(4) = M_{24}M_{23}M_{34}M_{12}x(0) \qquad x(4(t+1)) = Ax(4t)$$



- **Analysis:** $\boxed{A^t \to \frac{1}{m}\mathbf{1}\mathbf{1}'}$ ⟹ Consensus to the global average is reached.

$$x(t) \to \mathbf{1}\,\underbrace{\frac{1}{m}\mathbf{1}'\,x(0)}_{}^{x^*} = \begin{bmatrix} x^* \\ x^* \\ \vdots \\ x^* \end{bmatrix}$$

Consensus to a more general value

to a specific convex combination

$$\gamma_i > 0, \ \sum_{i=1}^{m} \gamma_i = 1$$

consensus

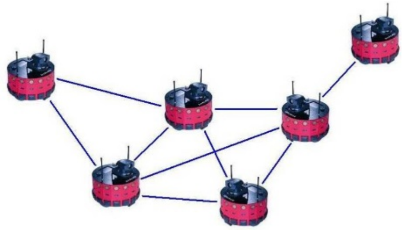
- **Objective:** $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$ $\quad x^* = \sum_{i=1}^{m} \gamma_i x_i(0) = \gamma' x(0)$

$$\gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_m \end{bmatrix}, \ \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Why do we care about such specific convex combination?

When we fuse data in a large sensor network, accurate (inaccurate) measurements should be with higher (lower) weights.

The global average $\frac{1}{m}\sum_{i=1}^{m} x_i(0)$ give equal weights to all sensors' measurements. (works for heterogeneous networks)



Let each sensor i be additionally assigned with a parameter γ_i

(The larger γ_i is, the more accurate the sensor is)

- For consensus, we need

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \qquad \sum_{j=1}^{m} w_{ij} = 1$$

- For consensus to the global average, we **further** need Metroplolis weights:

$$w_{ij} = \begin{cases} \min\{\frac{1}{d_i}, \ \frac{1}{d_j}\} & j \in \mathcal{N}_i, \ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i, \ j \neq i} w_{ij} & j = i \end{cases}$$

- How to achieve $\sum_{i=1}^{m} \gamma_i x_i(0)$?

Any ideas?

How to choose w_{ij} to achieve a specific convex combination $\gamma' x(0)$?

- **A Distributed Update to achieve $\gamma' x(0)$:**

Consensus: $$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) \qquad \sum_{j \in \mathcal{N}_i} w_{ij} = 1 \qquad w_{ij} > 0, j \in \mathcal{N}_i$$

bi-directional Networks

Furthermore:
Corless' weights

$$w_{ij} = \begin{cases} \frac{1}{\gamma_i} \min\{\frac{\gamma_i}{d_i}, \frac{\gamma_j}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \end{cases} \qquad d_i = |\mathcal{N}_i|$$

❖ *L. Coduti, M. Corless. A decentralized algorithm for assigning the weight parameters in a general synchronous consensus network. IEEE Conference on Decision and Control. 2012.*

$$x_1(t+1) = (1 - \frac{0.05}{0.44}) x_1(t) + \frac{1}{0.44} \cdot \min\{0.05, 0.22\} \cdot x_2(t)$$

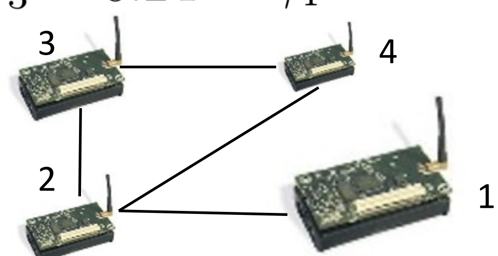
$$x_2(t+1) = \frac{1}{0.2} \cdot 0.05 \cdot x_1(t) + (1 - \frac{0.14}{0.2}) x_2(t) + \frac{1}{0.2} \cdot 0.05 \cdot x_3(t) + \frac{1}{0.2} \cdot 0.04 \cdot x_4(t)$$

$$x_3(t+1) = \frac{1}{0.24} \cdot 0.05 \cdot x_2(t) + (1 - \frac{1}{0.24} \cdot 0.09) x_3(t) + \frac{1}{0.24} \cdot 0.04 \cdot x_4(t)$$

$$x_4(t+1) = \frac{1}{0.12} \cdot 0.04 \cdot x_2(t) + \frac{1}{0.12} \cdot 0.04 \cdot x_3(t) + (1 - \frac{0.08}{0.12}) x_4(t)$$

$$\frac{\gamma_3}{d_3} = \frac{0.24}{3} = 0.08 \qquad \frac{\gamma_4}{d_4} = \frac{0.12}{3} = 0.04$$

$$\gamma_3 = 0.24 \qquad \gamma_4 = 0.12$$



$$\gamma_2 = 0.2 \qquad \gamma_1 = 0.44$$

$$\frac{\gamma_2}{d_2} = \frac{0.2}{4} = 0.05 \qquad \frac{\gamma_1}{d_1} = \frac{0.44}{2} = 0.22$$

- **Compact Form:**

$$x(t+1) = Ax(t)$$

$$A = \begin{bmatrix} \frac{0.39}{0.44} & \frac{0.05}{0.44} & 0 & 0 \\ \frac{0.05}{0.2} & \frac{0.06}{0.2} & \frac{0.05}{0.2} & \frac{0.04}{0.2} \\ 0 & \frac{0.05}{0.24} & \frac{0.15}{0.24} & \frac{0.04}{0.24} \\ 0 & \frac{0.04}{0.12} & \frac{0.04}{0.12} & \frac{0.04}{0.12} \end{bmatrix}$$

Observations on A

All entries are non-negative.

Row sum is 1. $A\mathbf{1} = \mathbf{1}$ $\quad \mathbf{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$\gamma' A = \gamma'$ $\quad \gamma = \begin{bmatrix} 0.44 \\ 0.2 \\ 0.24 \\ 0.12 \end{bmatrix}$

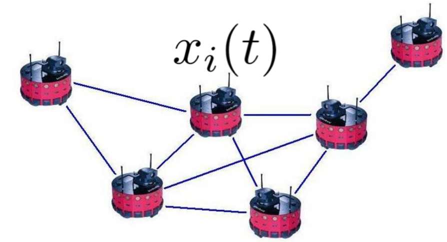
- **Analysis:**

$$\boxed{A^t \rightarrow \mathbf{1}\,\gamma'}$$

Consensus to a specific convex combination $\sum_{i=1}^{m} \gamma_i x_i(0)$ is reached.

$$x(t) \rightarrow \mathbf{1}\underbrace{\gamma' x(0)}_{x^*} = \begin{bmatrix} x^* \\ x^* \\ \vdots \\ x^* \end{bmatrix}$$

Summary of Distributed Consensus



agent's dynamics: $x_i(t+1) = u_i$

distributed control: $u_i = f_i(x_j(t), j \in \mathcal{N}_i)$

✓ **Objective:** $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$

✓ **Update:** $x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$

w_{ij} : the weight assigned by agent *i* to agent *j*

Consensus Goals	Choices of Weights
x^* is an unknown constant	$w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \quad \sum_{j=1}^{m} w_{ij} = 1$
x^* is the global average $\frac{1}{m}\sum_{i=1}^{m} x_i(0)$	$w_{ij} = \begin{cases} \min\{\frac{1}{d_i}, \frac{1}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, & \text{otherwise.} \end{cases}$
x^* is a specific convex combination $\sum_{i=1}^{m} \gamma_i x_i(0)$	$w_{ij} = \begin{cases} \frac{1}{\gamma_i}\min\{\frac{\gamma_i}{d_i}, \frac{\gamma_j}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, & \text{otherwise.} \end{cases}$

✓ **Analysis:** $x(t+1) = Ax(t) \quad A \in \mathbb{R}^{m \times m}$ with entries $A_{ij} = w_{ij}$

$$x(t) \to A^t x(0)$$

Stochastic Matrices

Consensus

$$A_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

Consensus for global average

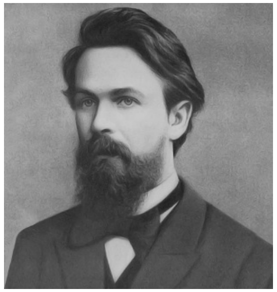
$$A_2 = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Consensus for convex combination

$$A = \begin{bmatrix} \frac{0.39}{0.44} & \frac{0.05}{0.44} & 0 & 0 \\ \frac{0.05}{0.2} & \frac{0.06}{0.2} & \frac{0.05}{0.2} & \frac{0.04}{0.2} \\ 0 & \frac{0.05}{0.24} & \frac{0.15}{0.24} & \frac{0.04}{0.24} \\ 0 & \frac{0.04}{0.12} & \frac{0.04}{0.12} & \frac{0.04}{0.12} \end{bmatrix}$$

What do you observe something in common among these matrices??

- *All entries are **non-negative***
- *All matrices are **row stochastic***. $A\mathbf{1} = \mathbf{1}$



Andrey Markov

- A row stochastic matrix P describes a **Markov chain** with the *ij*th entry as the transition probability, namely, probability of moving from state *i* to *j* in one-time step.

$$P = \begin{bmatrix} P_{1,1} & P_{1,2} & \dots & P_{1,j} & \dots & P_{1,S} \\ P_{2,1} & P_{2,2} & \dots & P_{2,j} & \dots & P_{2,S} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{i,1} & P_{i,2} & \dots & P_{i,j} & \dots & P_{i,S} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ P_{S,1} & P_{S,2} & \dots & P_{S,j} & \dots & P_{S,S} \end{bmatrix}.$$

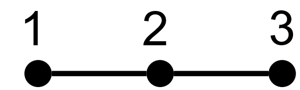
$$\sum_{j=1}^{n} P_{ij} = 1$$

Long-time probability distribution

Stationary probability vector is a row vector such that $\pi P = \pi$

- A_2 *is **doubly stochastic***. $A\mathbf{1} = \mathbf{1}, \quad \mathbf{1}' A = \mathbf{1}'$

Summary by Exercise



Write a distributed update for:

- Consensus:

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$$

- Consensus to global average:

  - Distributed Update with Metropolis Weights:

$$w_{ij} = \begin{cases} \min\{\frac{1}{d_i},\ \frac{1}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, \ \text{otherwise.} & \end{cases}$$

  - Periodic Gossiping:

➢ Consensus to $\gamma' x(0)$ $\quad \gamma = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/4 \end{bmatrix}$

- Distributed Update with Metropolis Weights:

$$w_{ij} = \begin{cases} \frac{1}{\gamma_i} \min\{\frac{\gamma_i}{d_i}, \frac{\gamma_j}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, \text{ otherwise.} \end{cases}$$