P3: Q

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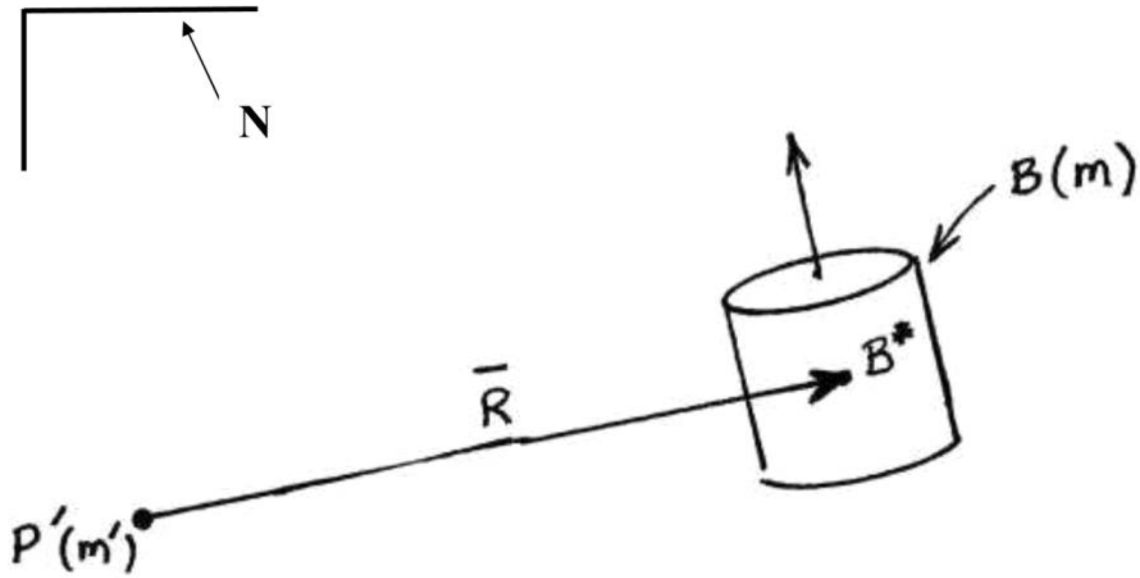
Gravitational Moment on an Axisymmetric Body in Circular Orbit



Problem: axisymmetric body B (m) ; particle P′ (m')
Only force/torque is gravity

Assumptions:

$$\bar{F} = -\frac{\mu m}{R^2}\hat{a}_1 \qquad (\mu = G m')$$

mass of attracting body

gravitational parameter

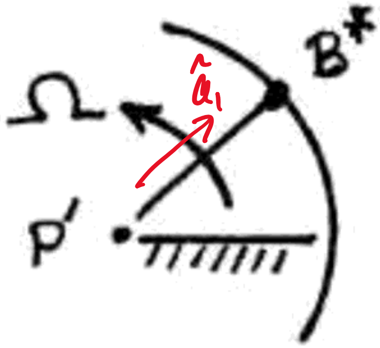
$$\bar{M} = \frac{3\mu}{R^3}\hat{a}_1 \times \bar{\bar{I}}^{B/B^*} \bullet \hat{a}_1$$

Note: this ($\bar{F} + \bar{M}$) implies assumption

① translational motion does not depend on rotational motion

motion

rotational motion <u>does</u> depend on translational motion



Consider motions such that B^* moves on a circular orbit of radius R with orbital angular velocity Ω

circ orbit rate?

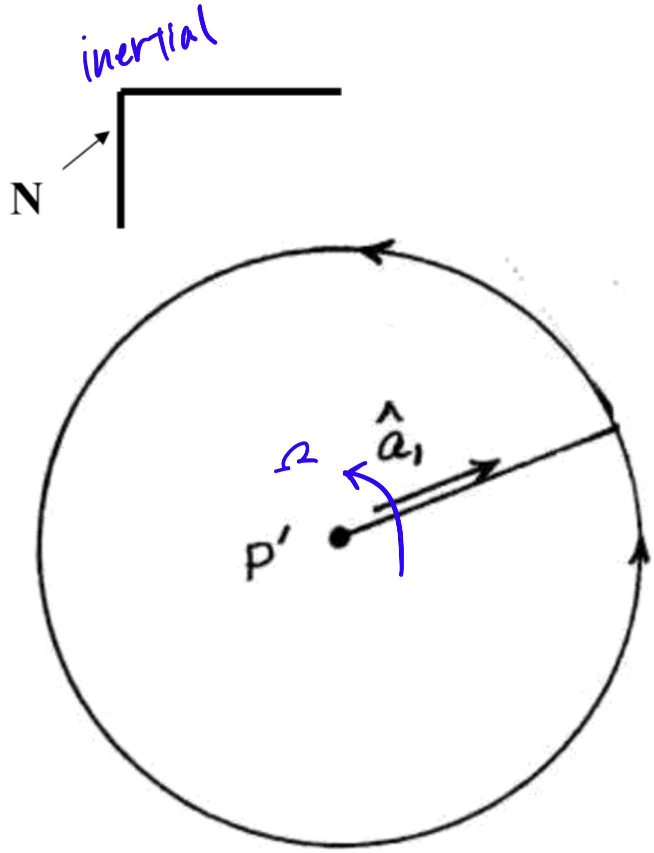
Angular rate Ω is particularly useful because it is constant:

1. characteristic angular velocity $\omega^* = \frac{\omega}{\Omega}$

2. characteristic independent variable

$$\nu = \text{\# of orbits} = \text{\# of revolutions}$$

$$\nu = \frac{\Omega t}{2\pi} \text{ (only true for circ orbit)}$$



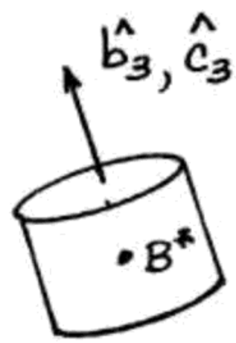
Definitions:

$\hat{a}$: orbit-fixed

${}^{N}\overline{\omega}^{A} = \Omega\, \hat{a}_3$ orbit normal direc

$\hat{a}_2$

$\dfrac{{}^{N}d\hat{a}_1}{dt} = \Omega\, \hat{a}_2$



$\hat{c}$: non-physical frame

$\hat{c}_3 = \hat{b}_3$

choose 's'

${}^{C}\overline{\omega}^{B} = s\hat{c}_3$

$\hat{b}$: body-fixed

${}^{N}\overline{\omega}^{B} = \omega_i \hat{c}_i$

$I = \hat{c}_1 \bullet \overline{\overline{I}}^{B/B^*} \bullet \hat{c}_1 = \hat{c}_2 \bullet \overline{\overline{I}}^{B/B^*} \bullet \hat{c}_2$

$J = \hat{c}_3 \bullet \overline{\overline{I}}^{B/B^*} \bullet \hat{c}_3$

Determine diff eqns — rotational motion { dyn, kin   Q4

Analysis:

Dynamic Differential Equations

$$\bar{M}^{B^*} = \frac{{}^N d\, {}^N \bar{H}^{B/B^*}}{dt}$$

(Could use Lagrangian, Hamiltonian or other method (like Kane) –Newtonian is a convenient choice for this problem)

Begin with kinematics on RHS ⟵ same as torque-free

$${}^N\bar{\omega}^B = \omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3 = {}^N\bar{\omega}^C + {}^C\bar{\omega}^B$$

$${}^N\bar{H}^{B/B^*} = \bar{\bar{I}}^{B/B^*} \cdot {}^N\bar{\omega}^B$$

$$\frac{{}^N d\, {}^N\bar{H}}{dt} = \frac{{}^C d\, {}^N\bar{H}}{dt} + \boxed{{}^N\bar{\omega}^C} \times \bar{H}$$

correct application of BKE

$$\left({}^N\bar{\omega}^B - {}^C\bar{\omega}^B \right)$$

$$\frac{{}^N d\, {}^N\bar{H}}{dt} = \left[I\left(\dot{\omega}_1 + s\omega_2\right) + \left(J - I\right)\omega_2\omega_3 \right]\hat{c}_1 +$$
$$+ \left[I\left(\dot{\omega}_2 - s\omega_1\right) - \left(J - I\right)\omega_1\omega_3 \right]\hat{c}_2 +$$
$$+ J\dot{\omega}_3\, \hat{c}_3$$

Ω

Ω^2 (mean motion)

Now depart torque-free because LHS $\neq 0$

$$\bar{M}^{B^*} = \frac{3\mu}{R^3}\hat{a}_1 \times \bar{\bar{I}} \bullet \hat{a}_1 = 3\boxed{\Omega^2}\hat{a}_1 \times \bar{\bar{I}} \bullet \hat{a}_1$$

defined in $\hat{c}$'s

Define

${}^AC^C$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	C_{11}	C_{12}	C_{13}
$\hat{a}_2$	x	x	x
$\hat{a}_3$	x	x	x

$\hat{c}$'s

$$\bar{M}^{B^*} = 3\Omega^2\left[(I_3 - I_2)C_{12}{}^{AC}C_{13}\,\hat{c}_1 + (I_1 - I_3)C_{13}C_{11}\,\hat{c}_2 + \right.$$
$$\left. + (I_2 - I_1)C_{11}C_{12}\,\hat{c}_3\right]$$

$$\bar{M}^{B^*} = 3\Omega^2\left[(J-I)\,C_{12}C_{13}\,\hat{c}_1 + (I-J)\,{}^{AC}C_{13}C_{11}\,\hat{c}_2\right]$$

no component of $\bar{M}$ about of symmetry - does that make sense?

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$\hat{c}$'s

Differential Equations

$$\boxed{\begin{aligned}
\hat{c}_1 &: \quad I(\dot{\omega}_1 + s\omega_2) + (J-I)\,\omega_2\omega_3 = 3\Omega^2 (J-I)\, {}^{A}C^{C}\; C_{12} C_{13} \\
\hat{c}_2 &: \quad I(\dot{\omega}_2 - s\omega_1) - (J-I)\,\omega_1^{N}\omega_3^{B} = 3\Omega^2 (I-J)\, C_{13} C_{11} \\
\hat{c}_3 &: \quad J\dot{\omega}_3 = 0
\end{aligned}}$$

vector basis $\hat{c}$

independent variable: t

dependent variables: $\omega_1, \omega_2, \omega_3, \; C_{11}, \; C_{12}, \; C_{13}$

(+ Ω if orbit not circ)

differential equations are functions of the kinematic variables
which set of kinematic variables will we want to use?

Assume kinematic variables are Euler parameters (quaternions)

$$C_{11} = 1 - 2\varepsilon_2^2 - 2\varepsilon_3^2$$
$$C_{12} = 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4)$$
$$C_{13} = 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)$$

$$I(\dot{\omega}_1 + s\omega_2) + (J-I)\omega_2\omega_3 = 12\Omega^2 (J-I)(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4)(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)$$
$$I(\dot{\omega}_2 - s\omega_1) - (J-I)\omega_1\omega_3 = 6\Omega^2 (I-J)(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)(1 - 2\varepsilon_2^2 - 2\varepsilon_3^2$$
$$J\dot{\omega}_3 = 0$$

Careful → which ε are these? ${}^{A}\varepsilon^{C}$

We do not get off as easily as for the torque-free case

$\boxed{\omega_3 \quad \text{constant}}$

Does this make sense?

Body attracted by a particle at P'; $\hat{c}_3$ and R form a plane; grav force attracts equally on each side of plane so will cancel and produce <u>no moment about axis of symmetry</u>

The other components ω_1, ω_2 will be functions of time

$$\dot{\omega}_1 = -s\omega_2 + \left(1 - \frac{J}{I}\right)\left[\omega_2\omega_3 - 12\Omega^2\left(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4\right)\left(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4\right)\right]$$

$$\dot{\omega}_2 = s\omega_1 - \left(1 - \frac{J}{I}\right)\left[\omega_1\omega_3 - 6\Omega^2\left(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4\right)\left(1 - 2\varepsilon_2^2 - 2\varepsilon_3^2\right)\right]$$

Two equations (coupled, nonlinear) $\Longrightarrow$ 6 unknown dependent variables

Cannot be solved unless

1. ε : known as function of time
2. Additional DE's for kin. variables

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Also require the solution for ${}^A\varepsilon_i^C = {}^A\varepsilon_i^C(t)$

Diff Eqn's that govern ${}^A\dot{\varepsilon}_i^C$

Differential equations that govern

Kinematic differential equations for Euler parameters?

$$\dot{\varepsilon} = \frac{1}{2}\,\omega\,E^T$$

$${}^A\dot{\underline{\varepsilon}}^C = \frac{1}{2}\,{}^A\omega^C\,\underline{E}^T$$

written in terms of $\hat{c}$

$$\begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

Note that $\omega_1, \omega_2, \omega_3$ are measure numbers for ${}^N\overline{\omega}^B$ as written in $\hat{c}$!!

$${}^N\overline{\omega}^B = {}^N\overline{\omega}^A + {}^A\overline{\omega}^C + {}^C\overline{\omega}^B$$

$${}^A\overline{\omega}^C = \underbrace{{}^N\overline{\omega}^B}_{\omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3} - {}^N\overline{\omega}^A - \underbrace{{}^C\overline{\omega}^B}_{s\hat{C}_3}$$

$${}^{N}\bar{\omega}^{A} = \Omega \hat{a}_3$$

${}^{A}C^{C}$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	x	x	x
$\hat{a}_2$	x	x	x
$\hat{a}_3$	x	x	x

$${}^{N}\bar{\omega}^{A} = \Omega\left(C_{31}\hat{c}_1 + C_{32}\hat{c}_2 + C_{33}\hat{c}_3\right)$$

$${}^{N}\bar{\omega}^{A} = \Omega\left[2\left(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4\right)\hat{c}_1 + 2\left(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4\right)\hat{c}_2 + \left(1 - 2\varepsilon_1^2 - 2\varepsilon_2^2\right)\hat{c}_3\right]$$

$$\Longrightarrow \quad {}^{A}\bar{\omega}^{C} \bullet \hat{c}_1 = \omega_1 - 2\Omega\left(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4\right)$$

$${}^{A}\bar{\omega}^{C} \bullet \hat{c}_2 = \omega_2 - 2\Omega\left(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4\right)$$

$${}^{A}\bar{\omega}^{C} \bullet \hat{c}_3 = \omega_3 - \Omega\left(1 - 2\varepsilon_1^2 - 2\varepsilon_2^3\right) - s$$

$${}^{A}\dot{\varepsilon}^{C} = \frac{1}{2}\, {}^{A}\omega^{C}\ E^{T}$$

written in terms of $\hat{c}$

$${}^{A}E^{C} = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

$$2\,\dot{\varepsilon}_1 = \varepsilon_2\left(\omega_3 - s + \Omega\right) - \varepsilon_3\omega_2 + \varepsilon_4\omega_1$$

$$2\,\dot{\varepsilon}_2 = \varepsilon_3\omega_1 + \varepsilon_4\omega_2 - \varepsilon_1\left(\omega_3 - s + \Omega\right)$$

$$2\,\dot{\varepsilon}_3 = \varepsilon_4\left(\omega_3 - s - \Omega\right) + \varepsilon_1\omega_2 - \varepsilon_2\omega_1$$

$$2\,\dot{\varepsilon}_4 = -\varepsilon_1\omega_1 - \varepsilon_2\omega_2 - \varepsilon_3\left(\omega_3 - s - \Omega\right)$$

If $\omega_i \neq \text{constant}$ these equations are coupled and nonlinear

In the torque-free problem, ω_i are constant and expressions for ε_i are analytically available.
What about now? no

Now solve kinematic and dynamic differential equations simultaneously

7 diff eqns $\longrightarrow$ coupled
Nonlinear
NOT solvable analytically