AAE 339: Aerospace Propulsion

Homework 3: Non-isentropic Analysis with Rayleigh and Fanno Flows

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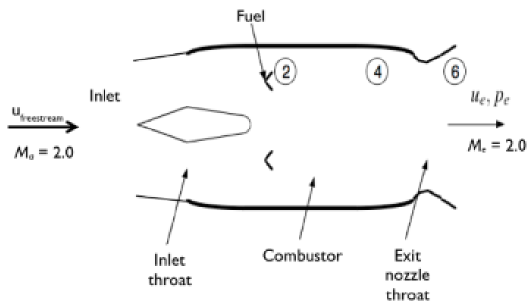
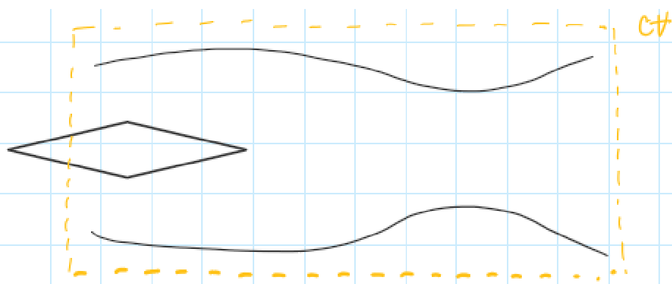
Problem 1

A ramjet is a simple device for supersonic flight. In its simplest configuration, it could look like the cylindrical device shown below (*not scaled*). The supersonic air flow is ingested through a round inlet. A centerbody is used to form a converging-diverging flowpath to decelerate the flow to Mach 1 at the throat of the inlet, and then further decelerate it in a subsonic diffuser to a Mach number about $M = 0.3$ or so, prior to its entry to the combustor where fuel is added and burnt to provide an increase in stagnation temperature. Typical designs use this combustor entry Mach number ($M_2 \sim 0.3$) so that Rayleigh losses are not so high, and to ensure combustion is efficient. Finally the combustion products are accelerated through a converging-diverging nozzle to a supersonic velocity u_e and pressure at the exit plane p_e. Because the flow through the exit nozzle is supersonic, the pressure at the nozzle exit plane p_e is generally not equal to the ambient pressure p_a, rather it is defined by the expansion ratio of the nozzle $\varepsilon = A_e/A_t$ and the stagnation pressure at the nozzle inlet, p_{04}. The sketch is poor, assume the flow is 1D at the exit.



Consider this simple configuration in supersonic flight at sea level ($T_a = 20$ °C, $p_a = 0.1$ MPa). The flight Mach number $M_a = 2.0$; stagnation temperature rise in combustor $\Delta T_0 = 800$ K; and Mach number at the nozzle exit $M_e = M_6 = 2.0$. First, consider the "ideal ramjet," where there are no stagnation pressure losses due to inlet shocks, friction, or Rayleigh loss in the combustor.

a) Determine the properties p and p_0, and T and T_0 at the following locations: in the freestream (a); at the throat of the inlet where $M_t = 1.0$; at the entry to the combustor (2), at the nozzle throat (subscript t), and the exit plane (6 or e).



assume this to be isentropic / adiabatic $\Longleftrightarrow$ no $\dot{Q}$ & $\dot{W}$

$\Longrightarrow$ p_0 = const. T_0 changes only through combustor $\Delta T_0 = 800$ K

1D-uniform flow $\gamma = 1.4$ = const. $R = R_{air} = 287.05 \frac{J}{kg \cdot K}$

(∞) @ free stream = inlet

for the freestream properties, using isentropic relations we know that

$$\boxed{\begin{aligned} &M_\infty = M_a = 2.0, \quad p_\infty = p_a = 0.1\,MPa, \quad T_\infty = T_a = 293.15\,K \\ &p_{0,\infty} = \left(1+\frac{\gamma-1}{2}M_\infty^2\right)^{\gamma/\gamma-1} p_\infty = \left(1+0.2\cdot 2.0^2\right)^{1.4/0.4}(0.1\times10^6\,Pa) = 782445\,Pa \\ &T_{0,\infty} = \left(1+\frac{\gamma-1}{2}M_\infty^2\right)T_a = \left(1+0.4\cdot 2.0^2\right)(293.15\,K) = 527.67\,K \end{aligned}}$$

(1) @ inlet throat

$$M_1 = M_t = 1.0 = u_1 / \sqrt{\gamma R T_1}$$

$$\boxed{\begin{aligned} p_{0,1} &= p_{0,\infty} = 782445\,Pa \\ T_{0,1} &= T_{0,\infty} = 527.67\,K \end{aligned}}$$

now use p_0 & T_0 to find p_1 & T_1

$$p_1 = p_0 \Big/ \left(1+\frac{\gamma-1}{2}M_1^2\right)^{\gamma/\gamma-1} = (782445\,Pa) \Big/ \left(1+\frac{1.4-1}{2}\cdot 1.0^2\right)^{\frac{1.4}{1.4-1}}$$

$$\boxed{p_1 = 413351\,Pa}$$

so $\left(\frac{T_0}{T}\right)^{\gamma/\gamma-1} = \frac{p_0}{p} \iff \frac{T_0}{T} = \left(\frac{p_0}{p}\right)^{\gamma-1/\gamma} \iff T = \left(\frac{p}{p_0}\right)^{\gamma-1/\gamma} T_0$

$$\therefore T_1 = \left(\frac{p_1}{p_0}\right)^{\gamma-1/\gamma} T_0 = \left(\frac{413351\,Pa}{782445\,Pa}\right)^{0.4/1.4}\cdot(527.67\,K) = 439.72\,K \qquad \boxed{T_1 = 439.72\,K}$$

(2) @ entry to combustor $\Rightarrow M_2 = 0.3$

$$\boxed{\begin{aligned} P_{0,2} &= P_{0,1} = 782445 \text{ Pa} \\ T_{0,2} &= T_{0,1} = 527.67 \text{ K} \end{aligned}}$$

$$P_2 = P_{0,2} \Big/ \left(1 + \frac{\gamma-1}{2} M_2^2\right)^{\gamma/\gamma-1} = (782445 \text{ Pa}) \Big/ \left(1 + \frac{0.4}{2} \cdot 0.3^2\right)^{1.4/0.4}$$

$$\therefore \boxed{P_2 = 735083 \text{ Pa}}$$

$$T_2 = \left(\frac{P_2}{P_0}\right)^{\gamma-1/\gamma} T_0 = \left(\frac{735083 \text{ Pa}}{782445 \text{ Pa}}\right)^{0.4/1.4} (527.67 \text{ K}) = 518.34 \text{ K} \qquad \boxed{T_2 = 518.34 \text{ K}}$$

(4) @ nozzle throat $\quad M_4 = 1$

$P_0 = \text{const.} \qquad \therefore \boxed{P_{0,4} = 782445 \text{ Pa}}$

and T_0 is only changed by ΔT_0 @ combustor

$$\therefore T_{0,4} = T_{0,1} + \Delta T_0 = 527.67 \text{ K} + 800 \text{ K} = 1327.67 \text{ K}$$

$$\boxed{T_{0,4} = 1327.67 \text{ K}}$$

then,

$$T_4 = T_0 \Big/ \left(1 + \frac{\gamma-1}{2} M_4^2\right) = 1327.67 \text{ K} \Big/ \left(1 + \frac{0.4}{2} \cdot 0.3^2\right) = 1304.19 \text{ K}$$

$$\boxed{T_4 = 1106.39 \text{ K}}$$

$$P_4 = 782445 \text{ Pa} \Big/ \left(1 + \frac{\gamma-1}{2} M_4^2\right)^{\frac{\gamma}{\gamma-1}} = 413351 \text{ Pa} \qquad \boxed{P_4 = 413351 \text{ Pa}}$$

(6) @ exit $\quad M_6 = M_e = 2.0$

$$\boxed{P_{0,6} = 782445 \text{ Pa}}$$

$$P_6 = P_{0,6} \Big/ \left(1 + \frac{\gamma-1}{2} M_6^2\right)^{\gamma/\gamma-1} = (782445 \text{ Pa}) \Big/ \left(1 + 0.2 \cdot 2.0^2\right)^{1.4/0.4}$$

$$\boxed{P_6 = 100\,000 \text{ Pa}}$$

$$\boxed{T_{0,6} = T_{0,4} = 1327.67 \text{ K}}$$

$$T_6 = \left(\frac{P_6}{P_{0,6}}\right)^{\gamma-1/\gamma} T_{0,6} = \left(\frac{100000 \text{ Pa}}{782445 \text{ Pa}}\right)^{0.4/1.4} (1327.67 \text{ K})$$

$$\boxed{T_6 = 737.59}$$

b) On the basis of a flow $\dot{m}$ = 1.0 kg/s, calculate the area and velocity at the following places: inlet entry, nozzle throat, and nozzle exit. Neglect the added flow of fuel in the combustor, in the next section we will see it is about 5% of the air flow.

from mass conservation for cv (control volume)

$$\cancel{\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall} + \int_{cs} \rho \bar{u}\cdot\bar{n}\, dA = 0$$

$$-\dot{m}_{in} + \dot{m}_{out} = 0 \iff \dot{m}_{in} = \dot{m}_{out} = \dot{m} = 1.0 \text{ kg/s}$$

then using the equation

$$\frac{\dot{m}}{A} = P_0\sqrt{\frac{\gamma}{RT_0}}\cdot M\left(\frac{1}{1+\frac{\gamma-1}{2}M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\iff A = \frac{\dot{m}}{P_0 M}\sqrt{\frac{RT_0}{\gamma}}\left(\frac{1}{1+\frac{\gamma-1}{2}M^2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

thus,

$$A_i = \frac{\dot{m}}{P_{0,\infty}M_\infty}\sqrt{\frac{RT_{0,\infty}}{\gamma}}\left(\frac{1}{1+\frac{\gamma-1}{2}M_\infty^2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$A_i = \frac{(1.0 \text{ kg/s})}{(782445 \text{ Pa})(2.0)}\sqrt{\frac{(287.05 \frac{J}{kg K})(527.67 K)}{1.4}}\left(\frac{1}{1+0.2\cdot 2.0^2}\right)^{-\frac{2.4}{0.8}}$$

$$\boxed{A_i = 1.2258\times10^{-3} \text{ m}^2}$$

similarly $A_4 = A_{nozzle,throat}$

$$A_4 = \frac{\dot{m}}{P_{0,4}M_4}\sqrt{\frac{RT_{0,4}}{\gamma}}\left(\frac{1}{1+\frac{\gamma-1}{2}M_4^2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\boxed{A_4 = 2.3446\times10^{-3} \text{ m}^2}$$

$A_6 = A_{nozzle,exit}$

$$A_6 = \frac{\dot{m}}{P_{0,6}M_6}\sqrt{\frac{RT_{0,6}}{\gamma}}\left(\frac{1}{1+\frac{\gamma-1}{2}M_6^2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\boxed{A_6 = 1.9442\times10^{-3} \text{ m}^2}$$

velocities

$$u_i = u_\infty = M_i \sqrt{\gamma R T_{0\infty}}$$

$$u_i = (2)\left[(1.4)\left(287.05\,\frac{J}{kgK}\right)(293.15K)\right]^{1/2}$$

$$\boxed{u_i = 686.23 \text{ m/s}}$$

similarly

$$u_4 = M_4 \sqrt{\gamma R T_{04}}$$

$$\boxed{u_4 = 666.90 \text{ m/s}}$$

$$u_6 = M_6 \sqrt{\gamma R T_{06}}$$

$$\boxed{u_6 = 1087.68 \text{ m/s}}$$

c) Use the general thrust equation derived in HW 1 to calculate the generated thrust. Set the control surfaces upstream of the inlet where the flow is one-dimensional and well-defined, and at the nozzle exit plane.

from HW1 we know the thrust equation is

$$T = \dot{m}_a (u_e - u_a) + (P_e - P_a) A_e$$

$$\dot{m}_a = \dot{m} = 1.0\ kg/s$$

$$u_e = M_6 \sqrt{\frac{\gamma R T_{0,6}}{1 + \frac{\gamma - 1}{2} M_6^2}} = (2.0) \sqrt{\frac{(1.4)(287.05 \frac{J}{kg K})(1327.67 K)}{1 + 0.2 \cdot 2.0^2}}$$

$$u_e = 1088.9\ m/s$$

similarly

$$u_a = M_\infty \sqrt{\frac{\gamma R T_{0,\infty}}{1 + \frac{\gamma - 1}{2} M_\infty^2}} = 686.46\ m/s$$

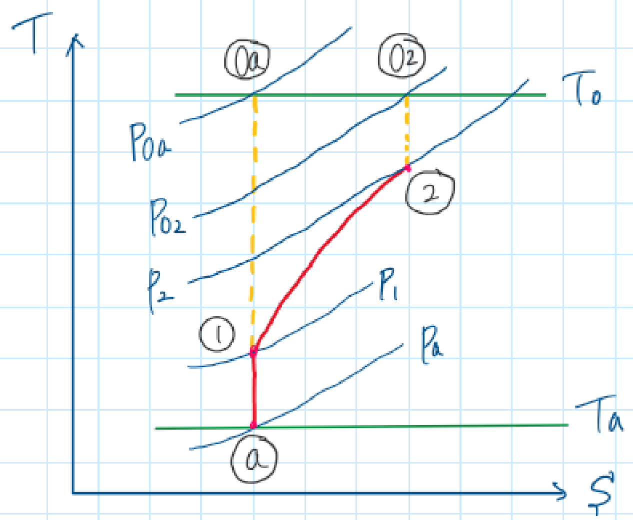
$$\therefore F_{th} = (1.0\ kg/s)(1088.9\ m/s - 686.46\ m/s) + \cancel{(78244.5 Pa - 78244.5 Pa)} A_6 \rightarrow 0$$

$$\boxed{F_{th} = 402.44\ N}$$

d) Now make the problem more realistic by including representative losses in stagnation pressure: a 10% loss in the inlet as the flow is decelerated from freestream conditions to $M = 0.3$ at the inlet to the combustor ($p_{02}/p_{0a} = 0.9$); and a 5% loss in stagnation pressure in the combustor due to heat addition (the Rayleigh loss, you will calculate it later). With these losses, calculate thrust and compare to the answer in part c.

You may assume that the specific heat and molecular weight of the flow are constant ($c_p = 1.0$ kJ/kg-K, $\gamma = 1.4$, $MW = 29$ g/g-mole = 29 kg/k-mole).

Draw the T-s diagram for elucidation



P_{0a} remains unchanged as $P_{0a} = P_{0,\infty} = 782445\ Pa$

However, $P_{02}' = 0.9\, P_{0a} = 0.9\,(782445\ Pa) = 704201\ Pa$

and $P_{0,4}' = 0.95\, P_{02}' = 668991\ Pa \implies P_{06}' = P_{04}'$

$T_{0a} = T_{0,\infty} = 527.67\ K$

$M_a = 2.0$

now because T_{06} remains unchanged from problem (a)

$T_{06} = 1327.67 K$

thus

$u_a = 686.46$ m/s (unchanged)

$u_6 = 1088.9$ m/s (unchanged)

$F'_{Th} = \dot{m}(u_6 - u_a) + (P'_{06} - P'_{0a}) A_6$

$F'_{Th} = (1.0 \text{ kg/s})(1088.9 \text{ m/s} - 686.46 \text{ m/s})$

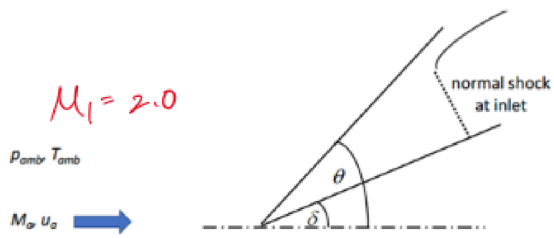
$+ (668991 \text{ Pa} - 782445 \text{ Pa})(1.9442 \times 10^{-3} \text{ m}^2)$

$\boxed{F'_{Th} = 181.56 \text{ N}}$

Compared to the "ideal ramjet" there was a 220.88 N (54.9 %) loss in thrust due To the pressure losses

Problem 2

Let's look a bit more carefully at stagnation pressure losses at the supersonic inlet. One configuration might include a spike to induce an oblique shock, followed by a normal shock at the inlet as shown in this figure. As shown here, the ramjet uses a 2D inlet with a ramp to generate one unattached oblique shock wave. The turning angle of the 2D ramp δ = 14.7°, which yields a wave angle θ = 45°. A terminal normal shock is developed at the cowl lip. Use the VaTech on-line calculator to calculate the properties across the oblique shock and the normal shock. Compare this loss to the case where a single normal shock would be used at the inlet.



Use VaTech calculator

(1) For single normal shock configuration

$M_1 = 2.0$, $\gamma = 1.4$

$\rightarrow M_2 = 0.57735$, $\boxed{P_{02}/P_{01} = 0.72087}$, $P_1/P_{02} = 0.17729$

$P_2/P_1 = 4.5$, $\rho_2/\rho_1 = 2.6666$, $T_2/T_1 = 1.6875$

(2) For oblique shock then normal shock configuration

$M_1 = 2.0$, $\gamma = 1.4$, $\delta = 14.7°$

→ $\boxed{M_2 = 1.45812}$, $\theta = 45°$, $P_2/P_1 = 2.1619$

$\rho_2/\rho_1 = 1.7118$, $T_2/T_1 = 1.2630$

$\boxed{P_{02}/P_{01} = 0.95495}$, $M_{1n} = 1.4128$, $M_{2n} = 0.73439$

for normal shock this is the mach #

→ $M_3 = 0.71645$, $\boxed{P_{03}/P_{02} = 0.94250}$, $P_1/P_{02} = 0.30699$

$P_3/P_1 = 2.31338$, $\rho_3/\rho_1 = 1.7901$, $T_3/T_2 = 1.2925$

$$\frac{P_{03}}{P_{01}} = \frac{P_{03}}{P_{02}} \cdot \frac{P_{02}}{P_{01}} = (0.94250)(0.95495)$$

$\boxed{P_{03}/P_{01} = 0.90004}$

$\boxed{T_3/T_1 \approx 1.6324 \text{ lower than single normal shock}}$

thus, the oblique shock and normal shock configuration has

$90.004\% - 72.087\% = 17.917\%$

less decrease in stagnation pressure

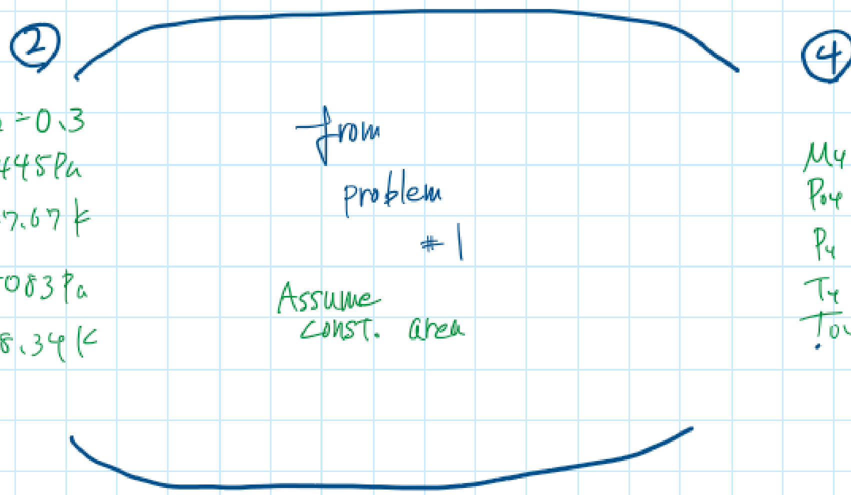
the oblique-normal shock has higher $M = 0.71645$ than single normal shock $M = 0.57735$

Problem 3

Consider Rayleigh loss for the ideal ramjet. Calculate p_0, p, and M at the combustor exit/nozzle inlet (station 4), and calculate the loss in stagnation pressure due to heat addition. Thermal choking (accelerating the flow to $M = 1$ by heat addition) is sometimes considered as an alternative to a mechanical throat. For the ideal ramjet in Problem 1, what stagnation temperature T_0^* would produce a thermally-choked flow at the exit of the combustor? Calculate p_0^*.

$\gamma = 1.4$

<1>



$M_2 = 0.3$
$P_{02} = 782445\,Pa$
$T_{02} = 527.67\,K$
$P_2 = 735083\,Pa$
$T_2 = 518.34\,K$

$M_4 = ?$
$P_{04} = ?$
$P_4 = ?$
$T_4 = ?$
$T_{04} = 1327.67\,K$

$$\frac{T_{04}}{T^*} = \frac{T_{04}}{T_{02}} \cdot \frac{T_{02}}{T^*}$$

$$= \left(\frac{1327.67K}{527.67K}\right)\left(\underbrace{0.34686}_{\text{from Rayleigh table @ } M = 0.3}\right) = 0.87273$$

Interpolate using $\frac{T_{04}}{T^*} = 0.87273$

$$M_4 = \frac{0.66 - 0.65}{0.87709 - 0.86833}\left(0.87273 - 0.86833\right) + 0.65$$

$$\boxed{M_4 = 0.65502}$$

now

$$\frac{P_{04}}{p^*} = \frac{1.05502 - 1.05820}{0.66 - 0.65}\left(0.65502 - 0.65\right) + 1.05820$$

$$\frac{P_{04}}{p^*} = 1.05660$$

then

$$P_{04} = \left(\frac{P_{04}}{P_0^*}\right)\left(\frac{P_0^*}{P_{02}}\right) P_{02} = 1.05660 \cdot \left(\frac{1}{\underbrace{1.1985}_{@\,M=0.2}}\right) \cdot 782445 \text{ Pa}$$

$$\boxed{P_{04} = 689805 \text{ Pa}}$$

also

$$\frac{P_4}{p^*} = \frac{1.4908 - 1.5080}{0.66 - 0.65}\left(0.65502 - 0.65\right) + 1.5080$$

$$\frac{P_4}{p^*} = 1.4994$$

then

$$P_4 = \left(\frac{P_4}{p^*}\right)\left(\frac{p^*}{P_2}\right) P_2 = 1.4994 \left(\frac{1}{2.1314}\right) 735083 \text{ Pa}$$

$$\boxed{P_4 = 517117 \text{ Pa}}$$

therefore the stagnation pressure loss is

$$P_{02} - P_{04} = 782445 \text{ Pa} - 689805 \text{ Pa}$$

$$= \boxed{92640 \text{ Pa}}$$

<ii> now we assume $M_4 = 1$

@ $M_4 = 1$ from Rayleigh table $\frac{T_{04}}{T^*} = 1.000$

since $T_{04} = 1327.67\,K$

$$T^* = \frac{1327.67\,K}{1.0000} = \boxed{1327.67\,K}$$

and

$P_0^* = $ const. for any M #

thus,

$$\frac{P_{02}}{P_0^*} = 1.0000$$

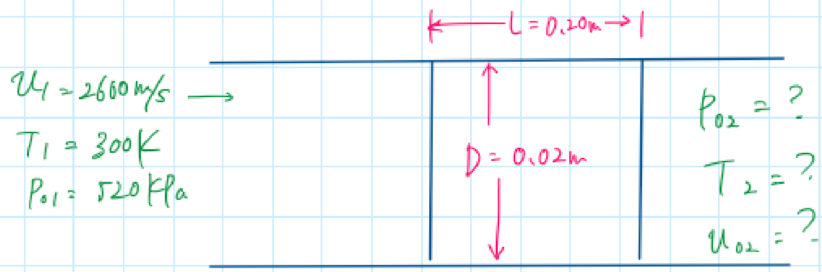
$$P_0^* = \frac{P_{02}}{1.00} = \frac{782445\,Pa}{1.000}$$

$$\boxed{P_0^* = 782445\ Pa}$$

Problem 4

Finally let's consider Fanno flow. Losses due to friction are unavoidable and common in internal flows of conventional chemical propulsion systems. Sources include momentum boundary layers in ducts; at entry and exit points where the flow can separate and recirculate; and through the rotors and stators of compressors and turbines. For both subsonic and supersonic flows, one of the effects of friction is a shift in flow Mach number towards 1. Depending on the Mach number and L/d of the duct, a Fanno flow analysis of the compressible flow should be considered and compared to an incompressible flow calculation.

Hydrogen enters a constant-area insulated duct with a velocity of 2600 m/s, a static temperature $T = 300$ K, and a stagnation pressure $p_o = 520$ kPa. The duct is 0.02 m in diameter, and 0.20 m long. For a friction coefficient, f ($f = 4C_f$), of 0.02, determine the static pressure p and temperature T at the end of the duct, and the velocity of the hydrogen at the exit. Calculate the loss in stagnation pressure. What length of duct would result in choked flow at its exit?



$$M_1 = \frac{u_1}{\sqrt{\gamma R T_{01}}} = \frac{2600 \text{ m/s}}{\sqrt{(1.4)(287.05 \frac{J}{kg\,K})(300K)}} = 7.4881$$

using VaTech calculator to calculate $\frac{P_0}{P_0^*}$ & $\frac{T}{T^*}$ & $\frac{fL^*}{D}$

@ state 1

$$\frac{P_{01}}{P_0^*} = 140.829, \quad \frac{T_1}{T^*} = 0.098245, \quad \frac{fL^*}{D} = 0.76103$$

$$\frac{V}{V^*} = 2.34708$$

from
$$\frac{fL}{D} = \frac{(0.02)(0.20)}{0.02} = 0.20$$

and
$$L^* = \frac{(0.76103)(0.02m)}{0.02} = 0.76103 \text{ m}$$

then

$$L_2^* = L_1^* - L$$

$$= 0.76103\,m - 0.20\,m = 0.56103\,m$$

at this length from the Fanno flow table

$$\frac{f L_2^*}{D} = \frac{(0.02)(0.56103\,m)}{(0.02\,m)} = 0.56103$$

from interpolation of the same table

$$\frac{p_{02}}{p_0^*} = \frac{1.2003 - 1.2130}{0.53174 - 0.57568}\left(0.56103 - 0.57568\right) + 1.2130$$

$$\frac{p_{02}}{p_0^*} = 1.2088$$

thus

$$\frac{p_{02}}{p_0^*} \cdot \frac{p_0^*}{p_{01}} = \frac{p_{02}}{p_{01}} = (1.2088)\left(\frac{1}{140.829}\right)$$

$$p_{02} = 8.5835 \times 10^{-3}\, p_{01}$$

$$p_{02} = 8.5835 \times 10^{-3} \cdot 520\ kPa$$

$$\boxed{p_{02} = 4.4634\ kPa}$$

similarly

$$\frac{T_2}{T^*} = \frac{1.1219 - 1.1244}{0.53174 - 0.57568}(0.51103 - 0.57568) + 1.1244$$

$$\frac{T_2}{T^*} = 1.12357$$

$$\frac{T_2}{\cancel{T^*}} \cdot \frac{\cancel{T^*}}{T_1} = (1.12357)\left(\frac{1}{0.098245}\right)$$

$$T_2 = 11.4364 \ T_1$$

$$T_2 = (11.4364)(300 \text{ K})$$

$$\boxed{T_2 = 3430.92 \text{ K}}$$

$$\frac{V_2}{V^*} = \frac{0.62492 - 0.61500}{0.53174 - 0.57568}(0.51103 - 0.57568) + 0.61500$$

$$\frac{V_2}{V^*} = 0.61831$$

finally

$$\frac{V_2}{\cancel{V^*}} \cdot \frac{\cancel{V^*}}{V_1} = (0.61831)\left(\frac{1}{2.34708}\right)$$

$$V_2 = (0.61831)\left(\frac{1}{2.34708}\right)(2600 \text{ m/s})$$

$$\boxed{V_2 = 684.94 \text{ m/s}}$$

the loss in stagnation pressure is therefore

$$|p_{02} - p_{01}| = |4.4784 \text{ kPa} - 520 \text{ kPa}|$$

$$= \boxed{515.52 \text{ kPa}}$$

for the ducts exit to be a throat

the exit should be $M = 1$

where from the table $\frac{fL^*}{D} = 0$

here $L_2^* = 0$ so the duct should equal

$L = L_1^*$ from

$$L_2^* = L_1^* - L$$

$$\therefore \boxed{L = L_1^* = 0.76103 \text{ m}}$$