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Exercise 1

Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy.

$$\begin{bmatrix} PA + A^T P + C^T C + 2\alpha P & PB \\ B^T P & -\gamma^{-2} I \end{bmatrix} \leq 0.$$

Then the system (11.18)-(11.19) is globally asymptotically stable about the origin with rate of convergence α.

Where (11.18)

$$\dot{x} = Ax + B\phi(Cx)$$

and (11.19)

$$\|\phi(z)\| \leq \gamma \|z\|.$$

Solution:
From the Schur complement result we can rewrite the given matrix as

$$\begin{aligned} -\gamma^{-2} I &< 0 \\ PA + A^T P + 2\alpha P - PB(-\gamma^{-2} I)^{-1} B^T P &< 0 \end{aligned}$$

and the second equation can be organized to be

$$\begin{aligned} PA + A^T P + 2\alpha P + \gamma^2 PBB^T P &< 0 \\ PA + A^T P + \gamma^2 PBB^T P &< -2\alpha P < 0 \end{aligned}$$

and from **Theorem 18** we can say that if

$$PA + A^T P + \gamma^2 PBB^T P < 0$$

is satisfied the system (11.18)-(11.19) is globally asymptotically stable about the origin with Lyapunov matrix P. And now if we let $Q = \alpha P > 0$, we can see that

$$\begin{aligned} \lambda_{min}(P^{-1} Q) &= \lambda_{min}(P^{-1} \alpha P) \\ &= \alpha. \end{aligned}$$

Hence, the rate of convergence is α.

q.e.d

Exercise 2

Recall the double inverted pendulum of Exercise 34. Using the results of this section, obtain a value of the spring constant k which guarantees that this system is globally exponentially stable about the zero solution.

The double inverted pendulum is described as

$$\begin{aligned} \ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2k\theta_1 - k\theta_2 - \sin\theta_1 &= 0 \\ \ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - k\theta_1 + k\theta_2 - \sin\theta_2 &= 0 \end{aligned}$$

Solution:
The given system equations can be modified as

$$\begin{aligned} \ddot{\theta}_1 &= -2\dot{\theta}_1 + \dot{\theta}_2 - 2k\theta_1 + k\theta_2 + \sin\theta_1 \\ \ddot{\theta}_2 &= \dot{\theta}_1 - \dot{\theta}_2 + k\theta_1 - k\theta_2 + \sin\theta_2 \end{aligned}$$

In space-state representation it becomes

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin\theta_1 \\ \sin\theta_2 \end{bmatrix}$$

Now if we define $x_1 := \theta_1$, $x_2 := \theta_2$, $x_3 := \dot{\theta}_1$, and $x_4 := \dot{\theta}_2$, we can rewrite this as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin x_1 \\ \sin x_2 \end{bmatrix}.$$

We structure the nonlinearity to be

$$\begin{aligned} \psi_1(x) &= \sin x_1 \\ \psi_2(x) &= \sin x_2 \end{aligned}$$

and since

$$\begin{aligned} -1 \leq \sin x_1 \leq 1 \\ -1 \leq \sin x_2 \leq 1 \end{aligned}$$

The system matrices become

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Now if $z_1 := x_1$, $z_2 := x_2$, $\lambda_1 = 1$, and λ_2, we can say that

$$\tilde{\phi}(z) = \begin{bmatrix} \lambda_1 \phi_1(\lambda_1^{-1} z_1) \\ \lambda_2 \phi_2(\lambda_2^{-1} z_2) \end{bmatrix} \qquad \text{where} \qquad z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \textit{with} \quad z_i \in \mathbb{R}^{pi}.$$

Then the system can also be expressed as

$$\dot{x} = Ax + \tilde{B}\tilde{\phi}(\tilde{C}x)$$

with

$$\tilde{B} := \begin{bmatrix} \lambda_1^{-1} B_1 & \lambda_2^{-1} B_2 \end{bmatrix}, \qquad \tilde{C} := \begin{bmatrix} \lambda_1 C_1 \\ \lambda_2 C_2 \end{bmatrix}.$$

Provided what we have so far we can setup the LMI to be

$$\begin{bmatrix} PA + A^T P + \lambda_1^2 C_1^T C_1 + \lambda_2^2 C_2^T C_2 & \gamma P B_1 & \gamma P B_2 \\ \gamma B_1^T P & -\lambda_1^2 I & 0 \\ \gamma B_2^T P & 0 & -\lambda_2^2 I \end{bmatrix} < 0.$$

Since $\gamma = 1$, $\lambda_1 = 1$, and $\lambda_2 = 1$

$$\begin{bmatrix} PA + A^T P + C_1^T C_1 + C_2^T C_2 & PB_1 & PB_2 \\ B_1^T P & -I & 0 \\ B_2^T P & 0 & -I \end{bmatrix} < 0$$

$$0 < P$$

Now we solve this using MATLAB's LMI Toolbox, and the code is as follows.

As a result, we obtain the minimal spring constant k that guarantees that this system is GES about the zero solution to be

$$k = 18.0398$$

with a corresponding P matrix of

$$P = \begin{bmatrix} 119.9267 & -72.4921 & 1.1087 & -0.6401 \\ -72.4921 & 47.4346 & -0.6401 & 0.4686 \\ 1.1087 & -0.6401 & 2.6295 & -1.3891 \\ -0.6401 & 0.4686 & -1.3891 & 1.2405 \end{bmatrix}.$$

Exercise 3

Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy

$$\begin{aligned} PA + A'P + 2\alpha P &\leq 0 \\ B'P &= C \end{aligned}$$

Then the system (11.37)-(11.38) is globally exponentially stable about the origin with rate α and with Lyapunov matrix P.

Where (11.37) is

$$\dot{x} = Ax - B\phi(Cx)$$

and (11.38) is

$$z'\phi(z) \leq 0$$

for all z.

Solution:
If $V = x'Px$

$$\begin{aligned} \dot{V} &= \dot{x}'Px + x'P\dot{x} \\ &= 2x'P\dot{x} \\ &= 2x'P\Big(Ax - B\phi(Cx)\Big) \\ &= 2x'PAx - 2x'PB\phi(Cx) \\ &= x'(PA + A'P)x - 2x'C'\phi(Cx) \\ &= x'(PA + A'P)x - 2(Cx)'\phi(Cx) \\ &\leq x'(PA + A'P)x. \end{aligned}$$

Since from the given conditions we know that

$$PA + A'P < -2\alpha P$$

we can posit that

$$\dot{V} < -2\alpha P.$$

Hence, the system (11.37)-(11.38) is globally exponentially stable about the origin with rate α and with a Lyapunov matrix P.

q.e.d

Exercise 4

Consider the transfer function

$$\hat{g}(s) = \frac{\beta s + 1}{s^2 + s + 2}$$

Using Lemma 12, determine the range of β for which this transfer function is SPR. Verify your results with the KYSPR lemma.

Solution:
This transfer function can be expressed as the following state space model

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & \beta \end{bmatrix}, \quad D = 0$$

From Lemma 12 we first check the stability of this transfer function, so

$$det(\lambda I - A) = \lambda^2 + \lambda + 2 = 0$$

which gives us eigenvalues of

$$eig(A) = \frac{-1 \pm \sqrt{7}i}{2}.$$

Since the eigenvalues have a negative real part this system is stable.
Next, we check the dissipativity of the transfer function.

$$\begin{aligned}
\hat{g}(j\omega) &= \frac{\beta\omega j + 1}{-\omega^2 + j\omega + 2} \\
&= \frac{1 + \beta\omega j}{(2-\omega^2) + j\omega} \\
&= \frac{(1+\beta\omega j)\Big((2-\omega^2) - j\omega\Big)}{\Big((2-\omega^2)+j\omega\Big)\Big((2-\omega^2)-j\omega\Big)} \\
&= \frac{2 + (\beta - 1)\omega^2 - \Big((2-\omega^2)\beta\omega - \omega\Big)j}{(2-\omega^2)^2 + \omega^2}
\end{aligned}$$

and therefore,

$$\hat{g}(j\omega) + \hat{g}(j\omega)' = \frac{2 + (\beta - 1)\omega^2}{(2-\omega^2)^2 + \omega^2}$$

which is greater than 0 when $\beta \geq 1$, thus

$$\hat{g}(j\omega) + \hat{g}(j\omega)' > 0 \qquad \text{if} \quad \beta \geq 1$$

Finally, we check the asymptotic side condition

$$\lim_{\omega\to\infty} \omega^2 \frac{2+(\beta-1)\omega^2}{(2-\omega^2)^2+\omega^2} = \lim_{\omega\to\infty} \frac{\frac{2}{\omega^2}+(\beta-1)}{(\frac{2}{\omega^2}-1)^2+\frac{1}{\omega^2}}$$
$$= \beta - 1.$$

This becomes positive when only $\beta > 1$. Hence,

$$\lim_{|\omega|\to\infty} \omega^2 \hat{g}(j\omega) + \hat{g}(j\omega)' \neq 0.$$

Thus, from Lemma 12 we have proven this transfer function to be strictly positive real (SPR).

Let us verify this using the KYSPR lemma. First we check the observability and controllability of the system when $\beta = 1.2$.

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$
$$rank(Q_c) = 2.$$

Hence the system is controllable.

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1.2 \\ -2.4 & -0.2 \end{bmatrix}$$
$$rank(Q_o) = 2.$$

Hence the system is observable.