College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 9
Observability of Control Systems

Author:
Tomoki Koike

Supervisor:
Martin Corless

November 2nd, 2020 Friday
Purdue University
West Lafayette, Indiana

Exercise 1

Determine (by hand) whether or not each of the following systems are observable.

(a)

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= x_2 + u\\ y &= x_1 + x_2\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= -x_1\\ \dot{x}_2 &= x_2 + u\\ y &= x_2\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2 + u\\ y &= x_1 + x_2\end{aligned}$$

(d)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 4x_1 + u\\ y &= -2x_1 + x_2\end{aligned}$$

(a)

The A and C matrix for this system is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The observability matrix becomes

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$det(Q_o) = 2 \neq 0$$

Thus, this system is observable.

(b)

The A and C matrix for this system is

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The observability matrix becomes

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$
$$det(Q_o) = 0$$

Thus, this system is unobservable.

(c)

The A and C matrix for this system is

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = (1 \quad 1)$$

The observability matrix becomes

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
$$det(Q_o) = 0$$

Thus, this system is unobservable.

(d)

The A and C matrix for this system is

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \qquad C = (-2 \quad 1)$$

The observability matrix becomes

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$
$$det(Q_o) = 0$$

Thus, this system is unobservable.

MATLAB code for verification

```
function res = checkObservability(A, C)
    dim = size(A); n = dim(1);
    Qo = obsv(A, C);
    res.check = rank(Qo) == n;
    res.Qo = Qo;
end
```

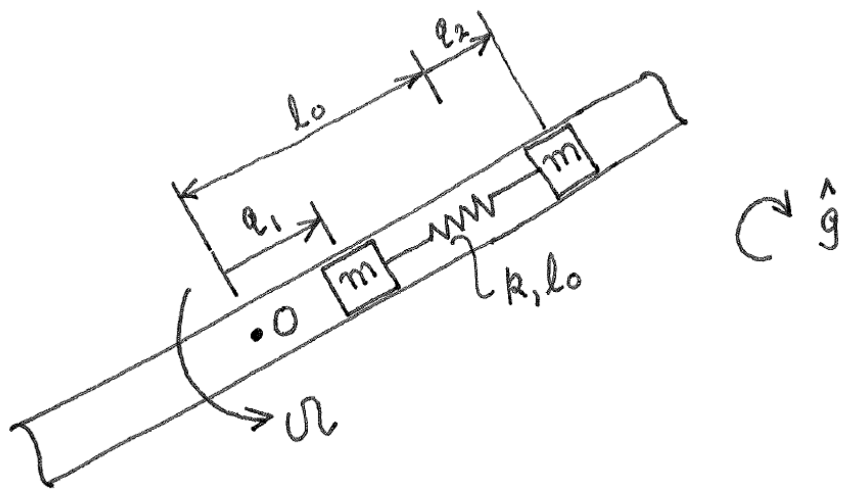
```
% Ex1
% (a)
A = [-1, 0; 0, 1];
C = [1, 1];
res = checkObservability(A, C);
res.check
res.Qo
% (b)
A = [-1, 0; 0, 1];
C = [0, 1];
res = checkObservability(A, C);
res.check
res.Qo
% (c)
A = [1, 0; 0, 1];
C = [1, 1];
res = checkObservability(A, C);
res.check
res.Qo
% (d)
A = [0, 1; 4, 0];
C = [-2, 1];
res = checkObservability(A, C);
res.check
res.Qo
```

Exercise 2

(BB in laundromat) Obtain a state space representation of the following system.

$$\begin{aligned} m\ddot{q}_1 - m\Omega^2 q_1 + k(q_1 - q_2) &= 0 \\ m\ddot{q}_2 - m\Omega^2 q_2 - k(q_1 - q_2) &= 0 \\ y &= q_1 \end{aligned}$$

Determine whether or not your state space representation is observable.



Manipulating the system, we obtain

$$\begin{aligned} \ddot{q}_1 &= \frac{m\Omega^2 - k}{m} q_1 + \frac{k}{m} q_2 \\ \ddot{q}_2 &= \frac{k}{m} q_1 + \frac{m\Omega^2 - k}{m} q_2 \\ y &= q_1 \end{aligned}$$

If $x_1 \coloneqq q_1,\ x_2 \coloneqq q_2,\ x_3 \coloneqq \dot{q}_1,\ x_4 \coloneqq \dot{q}_2$, the A and C matrices become

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & 0 & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Then the observability matrix becomes

$$Q_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & 0 & 0 \\ 0 & 0 & \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} \end{pmatrix}.$$

It is very easy to tell that the reduced echelon form of this observability matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And the rank of this is 4.

Thus, this system is observable.

Exercise 3

For each system in Exercise 1 which is not observable, obtain a basis for the unobservable subspace.

The unobservable systems were (b), (c), and (d).

(b)

Since we know that

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

The basis of the null space is

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow c\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ where } c \neq 0$$

Then the basis of the unobservable subspace is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

(c)

Since we know that

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

The basis of the null space is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow c\begin{pmatrix} -1 \\ 1 \end{pmatrix} \text{ where } c \neq 0$$

Then the basis of the unobservable subspace is

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(d)

Since we know that

$$Q_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$$

The basis of the null space is

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -0.5 \\ 0 & 0 \end{pmatrix} \Rightarrow c\begin{pmatrix} 0.5 \\ 1 \end{pmatrix} \text{ where } c \neq 0$$

Then the basis of the unobservable subspace is

$$\begin{pmatrix} 0.5 \\ 1 \end{pmatrix}.$$

Exercise 4

Determine the unobservable eigenvalues for each of the systems of Exercise 1.

(a)

Since the system is observable there are no unobservable eigenvalues.

(b)

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda = \pm 1$.

For $\lambda = 1$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$rank(Z) = 2$$

This eigenvalue is observable.

For $\lambda = -1$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$rank(Z) = 1$$

The unobservable eigenvalue is -1.

(c)

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

The eigenvalue is $\lambda = 1$.

For $\lambda = 1$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$rank(Z) = 1$$

The unobservable eigenvalue is 1.

(d)

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} -2 & 1 \end{pmatrix}$$

The eigenvalues are $\lambda = \pm 2$.

For $\lambda = 2$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \\ -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -0.5 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$rank(Z) = 1$$

The eigenvalue 2 is unobservable.

For $\lambda = -2$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \\ -2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$rank(Z) = 2$$

This eigenvalue is observable.

MATLAB Code for verification

```
function res = find_unobsv_eigVal(A, C)
    [v, d] = eig(A);
    sz = size(d);
    n = sz(1);
    for i = 1:n
        lambda = d(i,i);
        Z = [A-lambda*eye(n); C];
        res(i).observability = rank(Z) == n;
        res(i).Z = Z;
        res(i).rrefZ = rref(Z);
        res(i).lambda = lambda;
    end
end
```

```
% Ex4
% (b)
A = [-1, 0; 0, 1];
C = [0, 1];
res = find_unobsv_eigVal(A, C)
% (c)
A = [1, 0; 0, 1];
C = [1, 1];
```

```
res = find_unobsv_eigVal(A, C)
% (d)
A = [0, 1; 4, 0];
C = [-2, 1];
res = find_unobsv_eigVal(A, C)
```

Exercise 5

Determine (by hand) whether or not the following system is observable.

$$
\begin{aligned}
\dot{x}_1 &= 5x_1 - x_2 - 2x_3 \\
\dot{x}_2 &= x_1 + 3x_2 - 2x_3 \\
\dot{x}_3 &= -x_1 - x_2 + 4x_3 \\
y_1 &= x_1 + x_2 \\
y_2 &= x_2 + x_3
\end{aligned}
$$

If the system is unobservable, compute the unobservable eigenvalues.

The A and C matrix of this system is

$$A = \begin{pmatrix} 5 & -1 & -2 \\ 1 & 3 & -2 \\ -1 & -1 & 4 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The corresponding observability matrix

$$Q_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 6 & 2 & -4 \\ 0 & 2 & 2 \\ 36 & 4 & -32 \\ 0 & 4 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

The rank of this observability matrix is

$$rank(Q_o) = 2 \neq 3 \ .$$

The system is unobservable.

To find the unobservable eigenvalues we first find the eigenvalues of this system

$$A - \lambda I = \begin{pmatrix} 5-\lambda & -1 & -2 \\ 1 & 3-\lambda & -2 \\ -1 & -1 & 4-\lambda \end{pmatrix} .$$

$$
\begin{aligned}
det(A - \lambda \mathrm{I}) &= (5-\lambda)\begin{vmatrix} 3-\lambda & -2 \\ -1 & 4-\lambda \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ -1 & 4-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 3-\lambda \\ -1 & -1 \end{vmatrix} \\
&= (5-\lambda)[(3-\lambda)(4-\lambda) - 2] + (4 - \lambda - 2) - 2(-1 + 3 - \lambda) \\
&= (5-\lambda)(\lambda^2 - 7\lambda + 10) + (2-\lambda) - 2(2-\lambda) \\
&= (5-\lambda)(5-\lambda)(2-\lambda) + (2-\lambda) - 2(2-\lambda) \\
&= (2-\lambda)[(5-\lambda)^2 + 1 - 2] \\
&= (2-\lambda)(\lambda^2 - 10\lambda + 24)
\end{aligned}
$$

$$= (2-\lambda)(\lambda-4)(\lambda-6)$$

$$\therefore \lambda = 2, 4, 6 \ .$$

For $\lambda = 2$,

$$Z = \begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = \begin{pmatrix} 3 & -1 & -2 \\ 1 & 1 & -2 \\ -1 & -1 & 2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$rank(Z) = 3$$

This eigenvalue is observable.

For $\lambda = 4$,

$$Z = \begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = \begin{pmatrix} 1 & -1 & -2 \\ 1 & -1 & -2 \\ -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$rank(Z) = 2 \neq 3$$

This eigenvalue of 4 is unobservable.

For $\lambda = 6$,

$$Z = \begin{pmatrix} A-\lambda I \\ C \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 1 & -3 & -2 \\ -1 & -1 & -2 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$rank(Z) = 3$$

This eigenvalue is observable.

Exercise 6

Consider a system described by

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + b_1 u \\ \dot{x}_2 &= \lambda_2 x_2 + b_2 u \\ &\vdots \\ \dot{x}_n &= \lambda_n x_n + b_n u \\ y &= c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \end{aligned}$$

where all quantities are scalar. Obtain conditions on the numbers $\lambda_1, \cdots, \lambda_n$ and $c_1, \cdots, c_n$ which are necessary and sufficient for the observability of this system. (Hint: PBH time.)

The A matrix of this system is

$$A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

The C matrix is

$$C = \begin{pmatrix} c_1 & \cdots & c_n \end{pmatrix}$$

Since A is a diagonal matrix the diagonal values are the eigenvalues. Thus, for the observability of the system to hold true the PBH test for all eigenvalues must be true. This means that

$$rank \begin{pmatrix} A - \lambda_i I \\ C \end{pmatrix} = n \ .$$

For this to be true,

$$\begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

cannot have linearly dependent rows, which means that

$$\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$$

and

$$c_i \neq 0 \in [c \mid 1 \leq i \leq n]$$

Exercise 7

Using MATLAB, carry out the following for linearizations L1, L3, L7 of the two pendulum cart system.

(a) Determine which linearizations are observable.
(b) Determine the unobservable eigenvalues for the unobservable linearizations.

The system equation for the double pendulum cart system is

$$
\begin{aligned}
(m_0 + m_1 + m_2)\ddot{y} - m_1 l_1 cos\theta_1 \ddot{\theta}_1 - m_2 l_2 cos\theta_2 \ddot{\theta}_2 + m_1 l_1 sin\theta_1 \dot{\theta}_1^2 + m_2 l_2 sin\theta_2 \dot{\theta}_2^2 &= u \\
-m_1 l_1 cos\theta_1 \ddot{y} + m_1 l_1^2 \ddot{\theta}_1 \quad + \quad m_1 l_1 g sin\theta_1 &= 0 \\
-m_2 l_2 cos\theta_2 \ddot{y} + m_2 l_2^2 \ddot{\theta}_2 \quad + \quad m_2 l_2 g sin\theta_2 &= 0
\end{aligned}
$$

Have the system be a single output of the displacement y.

$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

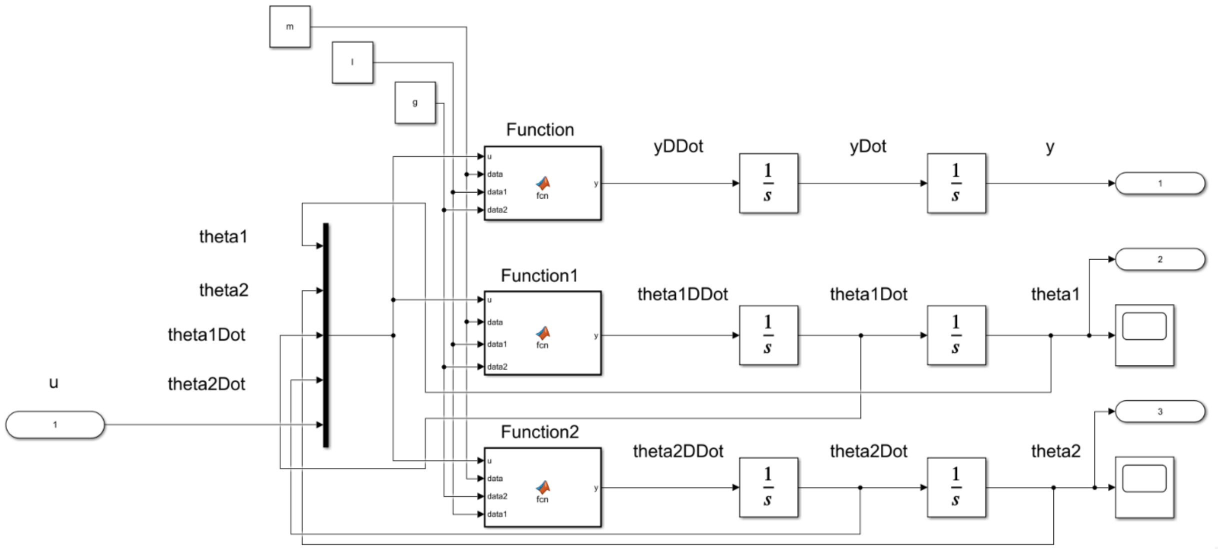
$$E2: \quad (y^e, \theta_1^e, \theta_2^e) = (0,\pi,\pi)$$

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

L1	P1	E1
L2	P1	E2
L3	P2	E1
L4	P2	E2
L5	P3	E1
L6	P3	E2
L7	P4	E1
L8	P4	E2

(a)

The Simulink model used for this is shown below,



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

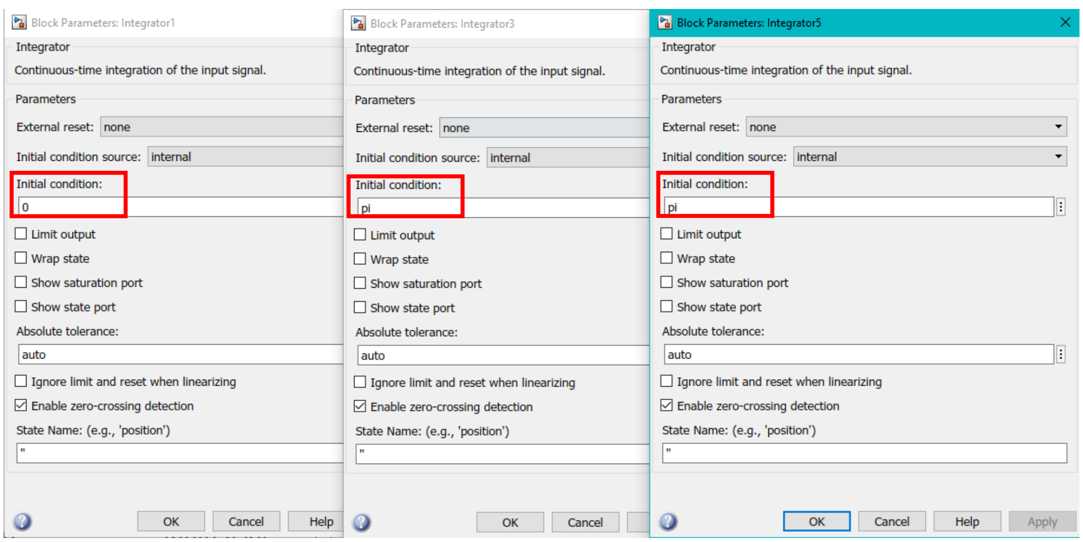
Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



L1:

```
A = 6×6
     0          0          0     1.0000          0          0
     0          0          0          0     1.0000          0
     0          0          0          0          0     1.0000
     0    -0.5000    -0.5000          0          0          0
     0    -1.5000    -0.5000          0          0          0
     0    -0.5000    -1.5000          0          0          0
```

```
B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5000
```

```
C = 1×6
     1     0     0     0     0     0
```

```
D = 0
```

The observability matrix for this system is

```
Qo_L1 = 6×6
    1.0000          0          0          0          0          0
         0          0          0     1.0000          0          0
         0    -0.5000    -0.5000          0          0          0
         0          0          0          0    -0.5000    -0.5000
         0     1.0000     1.0000          0          0          0
         0          0          0          0     1.0000     1.0000
```

The reduced echelon form of this matrix is

```
Qo_L1_rref = 6×6
     1     0     0     0     0     0
     0     1     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     1
     0     0     0     0     0     0
     0     0     0     0     0     0
```

Thus,

$$rank(Q_o) = 4 \; < \; 6$$

This system linearized by L1 is unobservable.

L3:

```
A = 6×6
     0         0         0    1.0000         0         0
     0   -0.5000... 
```

The observability matrix for this system is

```
Qo_L3 = 6×6
    1.0000         0         0         0         0         0
         0         0         0    1.0000         0         0
         0   -0.5000   -0.5000         0         0         0
         0         0         0         0   -0.5000   -0.5000
         0    1.0025    1.0076         0         0         0
         0         0         0         0    1.0025    1.0076
```

The reduced echelon form of this matrix is

```
Qo_L3_rref = 6×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

Thus,

$$rank(Q_o) = 6$$

This system linearized by L3 is observable.

L7:

```
A = 6×6
     0         0         0    1.0000         0         0
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
     0   -0.5000   -0.5000         0         0         0
     0   -1.5000   -0.5000         0         0         0
     0   -1.0000   -3.0000         0         0         0

B = 6×1
          0
          0
          0
     0.5000
     0.5000
     1.0000

C = 1×6
     1     0     0     0     0     0

D = 0
```

The observability matrix for this system is

```
Qo_L7 = 6×6
    1.0000         0         0         0         0         0
         0         0         0    1.0000         0         0
         0   -0.5000   -0.5000         0         0         0
         0         0         0         0   -0.5000   -0.5000
         0    1.2500    1.7500         0         0         0
         0         0         0         0    1.2500    1.7500
```

The reduced echelon form of this matrix is

```
Qo_L7_rref = 6×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

Thus,

$$rank(Q_o) = 6$$

This system linearized by L3 is observable.

(b)

The unobservable system is only L1.

The eigenvalues for L1 are

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

For $\lambda = 0$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

```
Z = 7×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -0.5000   -1.5000         0         0         0
    1.0000         0         0         0         0         0
```

The reduced echelon form of Z is

```
Z_rref = 7×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
     0     0     0     0     0     0
```

$$rank(Z) = 6$$

The eigenvalue 0 is observable.

For $\lambda = 1.4142j$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

```
Z = 7×6 complex
  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.4142i
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

The reduced echelon form of Z is

```
Z_rref = 7×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
     0     0     0     0     0     0
```

$$rank(Z) = 6$$

The eigenvalue 1.4142j is observable.

For $\lambda = -1.4142j$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

```
Z = 7×6 complex
  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.4142i
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

The reduced echelon form of Z is

```
Z_rref = 7×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
     0     0     0     0     0     0
```

$$rank(Z) = 6$$

The eigenvalue -1.4142j is observable.

For $\lambda = j$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

```
Z = 7×6 complex
   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

The reduced echelon form of Z is

```
Z_rref = 7×6 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

$$rank(Z) = 5$$

The eigenvalue j is unobservable.

For $\lambda = -j$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

```
Z = 7×6 complex
   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

The reduced echelon form of Z is

```
Z_rref = 7×6 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

$$rank(Z) = 5$$

The eigenvalue -j is unobservable.

MATLAB code

```
% AAE 564 HW9 Ex7
% Tomoki Koike
close all; clear all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

% (a)
global m l g ye theta1e theta2e
param_combo = ["L1","L3","L7"];
for i = 1:numel(param_combo)
    define_params(param_combo(i));
    [A, B, C, D] = linmod('db_pend_cart_lin');
    lin_sys(i).Amat = A;
    lin_sys(i).Bmat = B;
    lin_sys(i).Cmat = C;
    lin_sys(i).Dmat = D;
    sys_ss = ss(A, B, C, D);  % get the state space system
    OB(i) = checkObservability(A, C);  % check the observability of the system
    eigOB{i} = find_unobsv_eigVal(A, C);  % check the observability of the
eigenvalues
end

function define_params(L)
    % Function to define parameters
    global m l g ye theta1e theta2e
    if L == "L1"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L2"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L3"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L4"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L5"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L6"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L7"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L8"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; theta1e = pi; theta2e = pi; % E2
```

```
    else
        print('error: did not match any')
    end
end

function res = checkObservability(A, C)
    dim = size(A); n = dim(1);
    Qo = obsv(A, C);
    res.check = rank(Qo) == n;
    res.Qo = Qo;
end

function res = find_unobsv_eigVal(A, C)
    [v, d] = eig(A);
    sz = size(d);
    n = sz(1);
    for i = 1:n
        lambda = d(i,i);
        Z = [A-lambda*eye(n); C];
        res(i).observability = rank(Z) == n;
        res(i).Z = Z;
        res(i).rrefZ = rref(Z);
        res(i).lambda = lambda;
    end
end
```

Exercise 8

(BB in laundromat: mass center observations.) Obtain a state space representation of the following system.

$$\begin{aligned} m\ddot{q}_1 - m\Omega^2 q_1 + k(q_1 - q_2) &= 0 \\ m\ddot{q}_2 - m\Omega^2 q_2 - k(q_1 - q_2) &= 0 \\ y &= 0.5(q_1 + q_2) \end{aligned}$$

(a) Obtain a basis for its unobservable subspace.

(b) Determine the unobservable eigenvalues. Consider $\omega := \sqrt{\frac{k}{2m}} > \Omega$.

(a)

Manipulating the system, we obtain

$$\begin{aligned} \ddot{q}_1 &= \frac{m\Omega^2 - k}{m} q_1 + \frac{k}{m} q_2 \\ \ddot{q}_2 &= \frac{k}{m} q_1 + \frac{m\Omega^2 - k}{m} q_2 \\ y &= 0.5 q_1 + 0.5 q_2 \end{aligned}$$

If $x_1 := q_1,\ x_2 := q_2,\ x_3 := \dot{q}_1,\ x_4 := \dot{q}_2$, the A and C matrices become

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{m\Omega^2 - k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 \end{pmatrix}$$

Then the observability matrix becomes

$$Q_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{pmatrix} = 0.5 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ \frac{k}{m} + \frac{m\Omega^2 - k}{m} & \frac{k}{m} + \frac{m\Omega^2 - k}{m} & 0 & 0 \\ 0 & 0 & \frac{k}{m} + \frac{m\Omega^2 - k}{m} & \frac{k}{m} + \frac{m\Omega^2 - k}{m} \end{pmatrix}.$$

The reduced echelon form of this becomes

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus,

$$rank(Q_o) = 2 \neq 4$$

This system is unobservable.

From the reduced echelon form of the observability matrix we can get the null space bases

$$c_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \quad c_1, c_2 \neq 0$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(b)

The eigenvalues of this system is

$$\lambda = \pm\Omega, \quad \pm\frac{\sqrt{-m(2k - \Omega^2 m)}}{m}$$

When $\lambda = \Omega$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

$$Z = \begin{pmatrix} -\Omega & 0 & 1 & 0 \\ 0 & -\Omega & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & -\Omega & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & -\Omega \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$rank(Q_o) = 4$$

The eigenvalue of Ω is observable.

When $\lambda = -\Omega$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

$$Z = \begin{pmatrix} \Omega & 0 & 1 & 0 \\ 0 & \Omega & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & \Omega & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & \Omega \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$rank(Q_o) = 4$$

The eigenvalue of $-\Omega$ is observable.

When $\lambda = -\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

$$Z = \begin{pmatrix} -\dfrac{\sqrt{-m(2k - \Omega^2 m)}}{m} & 0 & 1 & 0 \\ 0 & -\dfrac{\sqrt{-m(2k - \Omega^2 m)}}{m} & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & -\dfrac{\sqrt{-m(2k - \Omega^2 m)}}{m} & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & -\dfrac{\sqrt{-m(2k - \Omega^2 m)}}{m} \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & -\dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} \\ 0 & 1 & 0 & \dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$rank(Q_o) = 3 \neq 4$$

The eigenvalue of $-\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$ is unobservable.

When $\lambda = \frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$,

$$Z = \begin{pmatrix} A - \lambda I \\ C \end{pmatrix}$$

$$Z = \begin{pmatrix} \dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} & 0 & 1 & 0 \\ 0 & \dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & \dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & \dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} \\ 0.5 & 0.5 & 0 & 0 \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & \dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} \\ 0 & 1 & 0 & -\dfrac{\sqrt{-m(2k-\Omega^2 m)}}{m} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$rank(Q_o) = 3 \neq 4$$

The eigenvalue of $\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$ is unobservable.