College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 13
Output Feedback and Lyapunov Theory

Author:
Tomoki Koike

Supervisor:
Martin Corless

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Purdue University
West Lafayette, Indiana

Exercise 1

Determine whether or not each of the following systems are observable, detectable, or not detectable.

(a)

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + u \\ y &= x_1 + x_2 \end{aligned}$$

(b)

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + u \\ y &= x_1 \end{aligned}$$

(c)

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 + u \\ y &= x_1 + x_2 \end{aligned}$$

(a)

The state matrices are

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 1], \qquad D = 0 \ .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow rank(Q_o) = 2 \ .$$

This system is observable, and therefore, this system is detectable.

(b)

The state matrices are

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0], \qquad D = 0 \ .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(Q_o) = 1 \, < \, 2 \ .$$

This system is unobservable. Next, we have to find the unobservable eigenvalues with the PBH test. The eigenvalues are

$$det(A - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \ .$$

If $\lambda = -1$,

$$\mathrm{Z} = \begin{bmatrix} A - \lambda\, I \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathrm{rank}(Z) = 2 \ .$$

This eigenvalue is observable.

If $\lambda = 1$,

$$\mathrm{Z} = \begin{bmatrix} A - \lambda\, I \\ C \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathrm{rank}(Z) = 1 \, < \, 2 \ .$$

This eigenvalue is unobservable.

The unobservable eigenvalue has a positive real part, and therefore, this system is not detectable.

(c)

The state matrices are

$$\mathrm{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 1], \qquad D = 0 \ .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(Q_o) = 1 \, < \, 2 \ .$$

This system is unobservable. Next, we have to find the unobservable eigenvalues with the PBH test. The eigenvalues are

$$det(A - \lambda I) = \lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = 1 \ .$$

If $\lambda = 1$,

$$Z = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(Z) = 1 < 2 \ .$$

This eigenvalue is unobservable.

The unobservable eigenvalue has a positive real part, and therefore, this system is not detectable.

Exercise 2

Consider the system described by

$$\begin{aligned}\dot{x}_1 &= -x_2 + u\\ \dot{x}_2 &= -x_1 - u\\ y &= x_1 - x_2\end{aligned}$$

Where all quantities are scalars.

(a) Is this system observable?
(b) Is this system detectable?
(c) Does there exist an asymptotic state estimator for this system? If an estimator does not exist, explain why; if one does exist, give an example of one.
(d) If the answer to part (c) is yes, illustrate the effectiveness of your observer with a simulation.

(a)

The state matrices are

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & -1 \end{bmatrix}, \qquad D = 0\ .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(Q_o) = 1 < 2\ .$$

This system is unobservable.

(b)

The eigenvalues are

$$det(A - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1\ .$$

If $\lambda = -1$,

$$Z = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathrm{rank}(Z) = 1\ .$$

This eigenvalue is unobservable.

If $\lambda = 1$,

$$Z = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow rank(Z) = 2 \ .$$

This eigenvalue is unobservable.

The unobservable eigenvalue has a negative real part, and therefore, this system is detectable.

(c)

Let

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} .$$

we have

$$A + LC = \begin{bmatrix} l_1 & -l_1 - 1 \\ l_2 - 1 & -l_2 \end{bmatrix} .$$

Hence,

$$det(sI - A - LC) = \begin{vmatrix} s - l_1 & l_1 + 1 \\ -l_2 + 1 & s + l_2 \end{vmatrix} = (s - l_1)(s + l_2) - (l_1 + 1)(-l_2 + 1)$$

$$\Rightarrow s^2 + (l_2 - l_1)s - l_1 + l_2 + 1 = 0 \ .$$

And $A + LC$ is asymptotically stable if

$$l_2 - l_1 > 0 \quad and \quad -l_1 + l_2 + 1 > 0 \quad \Rightarrow \quad l_2 > l_1 \ .$$

Thus, there exists an asymptotic state estimator.

An example of an asymptotic observer is then given by

$$\begin{aligned} \dot{\hat{x}}_1 &= 2l_1\hat{x}_1 - (2l_1 + 1)\hat{x}_2 + u - l_1 y \\ \dot{\hat{x}}_2 &= (2l_2 - 1)\hat{x}_1 - 2l_2\hat{x}_2 - u - l_2 y \end{aligned}$$

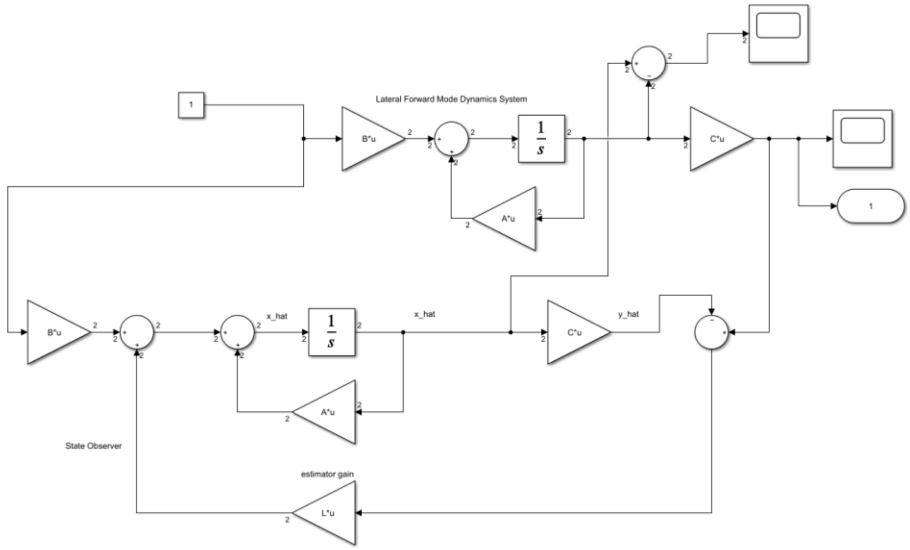
If $l_1 = 1, l_2 = 2$

$$\begin{aligned} \dot{\hat{x}}_1 &= 2\hat{x}_1 - 3\hat{x}_2 + u - y \\ \dot{\hat{x}}_2 &= 3\hat{x}_1 - 4\hat{x}_2 - u - 2y \end{aligned}$$

This is an example of an asymptotic observer.

(d)

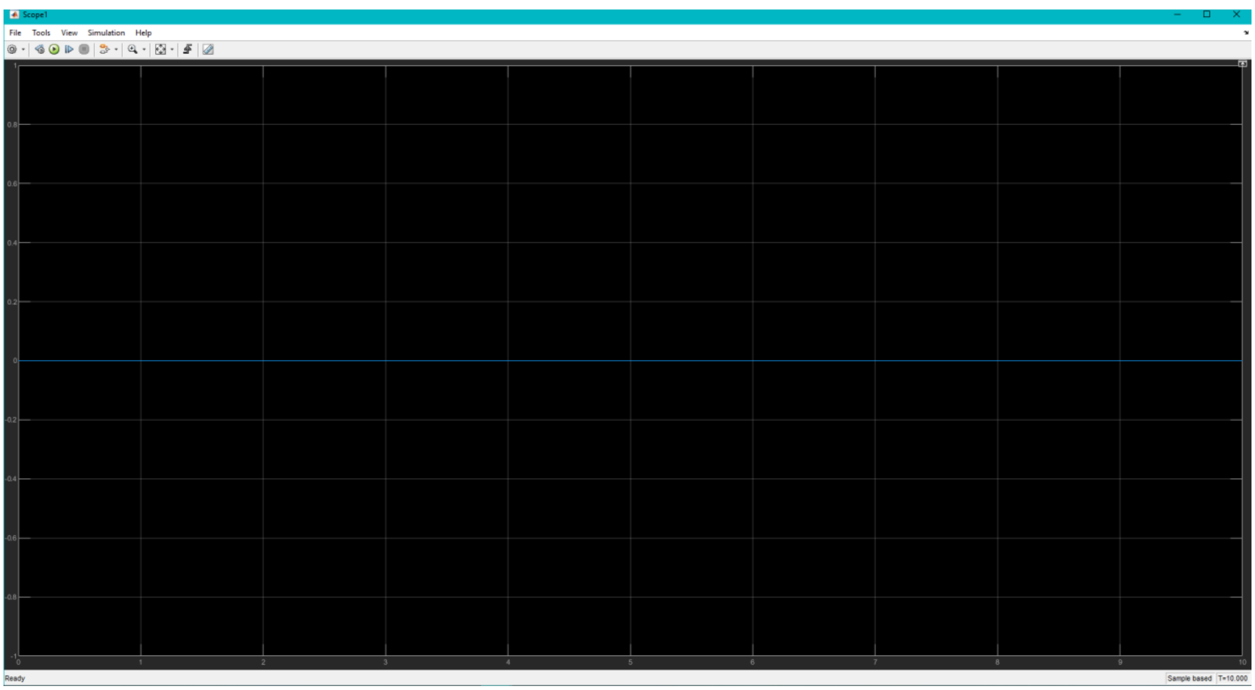
Using the following Simulink model



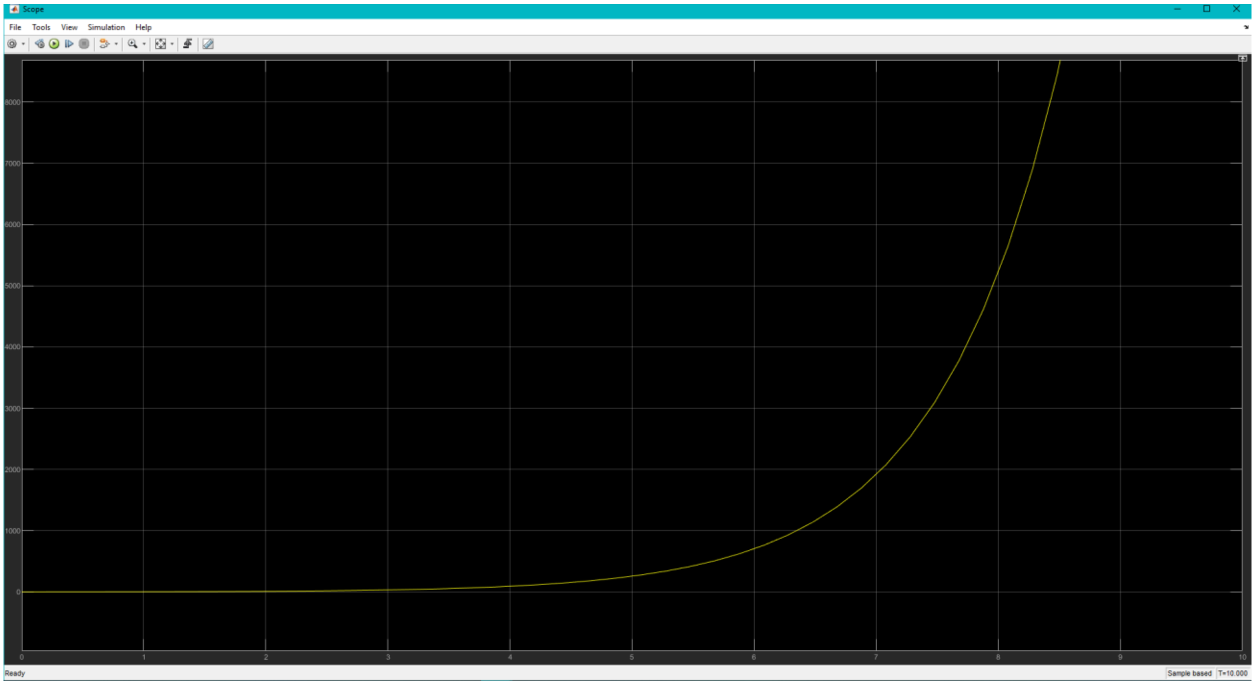
We define

$$L = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

And simulate the model. Then $\tilde{x} = \hat{x} - x$ becomes



And the output, y being



Since the difference between the estimated states and actual states are zero we can see that the observer is effective.

Exercise 3

Consider the system,

$$\begin{aligned} \dot{x}_1 &= x_2 + u_1 \\ \dot{x}_2 &= u_2 \\ y &= x_1 \end{aligned}$$

where all quantities are scalar. Obtain (by hand) an output feedback controller which results in an asymptotically stable closed loop system.

The state matrices are

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D = 0 \ .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow rank(Q_o) = 2 \ .$$

This system is observable, and thus detectable.

The controllability matrix

$$Q_c = \begin{bmatrix} A & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow rank(Q_c) = 2.$$

This system is controllable, and thus stabilizable.

Let the state feedback and observer matrix be

$$K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

The closed loop system can be characterized as

$$\begin{aligned} \dot{\hat{x}} &= [A + BK + L(C + DK)]\hat{x} - Ly \\ u &= K\hat{x} \end{aligned}$$

Now,

$$A_c = A + BK + L(C + DK) = \begin{bmatrix} k_1 + l_1 & k_2 + 1 \\ k_3 + l_2 & k_4 \end{bmatrix}$$

$$B_c = -L = -\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

$$C_c = K = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}.$$

Set our desired poles as $p = -2, -5$.

We first find K for these desired poles using the Brogan's algorithm

Find

$$\phi = (xI_{n\times n} - A)^{-1} = \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} \\ 0 & \frac{1}{x} \end{bmatrix}.$$

Compute

$$\psi = \phi B = \begin{bmatrix} \frac{1}{x} & \frac{1}{x^2} \\ 0 & \frac{1}{x} \end{bmatrix}.$$

Calculate

$$\bar{\psi} = [\psi_1(\lambda_1) \quad \psi_2(\lambda_2)] = \begin{bmatrix} -0.5 & 0.04 \\ 0 & -0.2 \end{bmatrix}$$

Where $\psi_1(x), \psi_2(x)$ correspond to the columns of ψ.

Find the gains with

$$K = E\bar{\psi}^{-1}$$

Where

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\mathrm{K} = \begin{bmatrix} -2 & -0.04 \\ 0 & -5 \end{bmatrix}.$$

Then,

$$A_{cl} = \mathrm{A} + \mathrm{BK} = \begin{bmatrix} -2 & 0.6 \\ 0 & -5 \end{bmatrix} \Rightarrow eig(A_{cl}) = -2, -5 .$$

Next, for the same desired poles we will find the observer gains L.

The characteristic equation for the desired poles is

$$CE \coloneqq (s+2)(s+5) = s^2 + 7s + 10 .$$

Now,

$$det\big(sI - (A' + C'L')\big) = s^2 - l_1 s - l_2 .$$

Thus,

$$\begin{aligned} -l_1 &= 7 \\ -l_2 &= 10 \end{aligned} \Rightarrow \begin{aligned} l_1 &= -7 \\ l_2 &= -10 \end{aligned}$$

Hence,

$$\mathrm{K} = \begin{bmatrix} -2 & -0.04 \\ 0 & -5 \end{bmatrix}, \qquad \mathrm{L} = \begin{bmatrix} -7 \\ -10 \end{bmatrix}$$

And

$$A_c = \mathrm{A} + \mathrm{BK} + \mathrm{L}(C + DK) = \begin{bmatrix} -9 & 0.6 \\ -10 & -5 \end{bmatrix}$$

$$B_c = -L = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

$$C_c = K = \begin{bmatrix} -2 & -0.04 \\ 0 & -5 \end{bmatrix}.$$

Now we check that it is asymptotically stable

$$eig(A_c) = -7 \pm 1.4142j \ .$$

It is asymptotically stable.

Thus, the output feedback controller is

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} -9 & 0.6 \\ -10 & -5 \end{bmatrix} \hat{x} + \begin{bmatrix} 7 \\ 10 \end{bmatrix} y \\ u &= \begin{bmatrix} -2 & -0.04 \\ 0 & -5 \end{bmatrix} \hat{x} \end{aligned}$$

Exercise 4

Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_3 \\ \dot{x}_2 &= u \\ \dot{x}_3 &= x_2 \\ y &= x_3 \end{aligned}$$

with scalar control input u and scalar measured output y.

(a) Obtain (by hand) an observer-based output feedback controller which results in an asymptotically stable closed loop system.
(b) Can all the eigenvalues of the closed loop system be arbitrarily placed?

(a)

The state matrices are

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C = [0 \quad 0 \quad 1], \qquad D = 0 \; .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(Q_o) = 2 \; < \; 3 \; .$$

This system is unobservable.

$$det(sI - A) = s^3 + s^2 \quad \Rightarrow \quad eig(A) = 0, -1$$

If $\lambda = -1$,

$$Z_o = \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(Z_o) = 2 < 3$$

This eigenvalue is unobservable.

If $\lambda = 0$,

$$Z_o = \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{rank}(Z_o) = 3$$

This eigenvalue is observable. Now, since the unobservable eigenvalue has a negative real part, this system is detectable.

The controllability matrix

$$Q_c = [A \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow rank(Q_c) = 3.$$

This system is controllable, and thus stabilizable.

Let the state feedback and observer matrix be

$$K = [k_1 \quad k_2 \quad k_3]$$

$$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}.$$

Now,

$$det(sI - A - BK) = s^3 + (1 - k_2)\, s^2 + (-k_2 - k_3)\, s - k_1 - k_3$$

The eigenvalues become asymptotically stable if all the coefficients are positive.

$$\begin{array}{rcl} 1 - k_2 & > & 0 \\ -k_2 - k_3 & > & 0 \\ -k_1 - k_3 & > & 0 \end{array}.$$

$$det(sI - A - LC) = s^3 + (1 - l_3)\, s^2 + (-l_2 - l_3)\, s - l_2$$

The eigenvalues become asymptotically stable if all the coefficients are positive.

$$\begin{array}{rcl} 1 - l_3 & > & 0 \\ -l_2 - l_3 & > & 0 \\ -l_2 & > & 0 \end{array}.$$

For the L matrix we find the positive combinations of a, b, c that satisfy the following

$$\begin{array}{rcl} 1 - l_3 & = & a \\ -l_2 - l_3 & = & b \\ -l_2 & = & c \end{array}$$

Which means that a, c can be arbitrary but b must satisfy $b = a + c - 1$. Thus, we choose

$$a = 5, \qquad b = 10, \qquad c = 6$$

Where

$$l_2 = -6, \qquad l_3 = -4$$

And l_1 can be any value so we select

$$l_1 = -1 \ .$$

Corresponding to the coefficients a, b, c, the characteristic equation becomes

$$CE \coloneqq s^3 + 5s^2 + 10s + 6$$

Then,

$$\begin{array}{rcl} 1 - k_2 & = & 5 \\ -k_2 - k_3 & = & 10 \\ -k_1 - k_3 & = & 6 \end{array} \Rightarrow \begin{array}{rcl} k_1 & = & 0 \\ k_2 & = & -4 \\ k_3 & = & -6 \end{array} .$$

Hence,

$$\mathrm{K} = [0 \quad -4 \quad -6], \ \ \mathrm{L} = \begin{bmatrix} -1 \\ -6 \\ -4 \end{bmatrix}$$

The closed loop system can be characterized as

$$\begin{array}{rcl} \dot{\hat{x}} & = & [A + BK + L(C + DK)]\hat{x} - Ly \\ u & = & K\hat{x} \end{array}$$

And

$$A_c = \mathrm{A} + \mathrm{BK} + \mathrm{L}(C + DK) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & -12 \\ 0 & 1 & -4 \end{bmatrix}$$

$$B_c = -L = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

$$C_c = K = [0 \quad -4 \quad -6].$$

Now we check that it is asymptotically stable

$$eig(A_c) = -4 \pm 3.464j, -1 \ .$$

It is asymptotically stable.

Thus, the output feedback controller is

$$\begin{array}{rcl} \dot{\hat{x}} & = & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -4 & -12 \\ 0 & 1 & -4 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} y \\ u & = & [0 \quad -4 \quad -6]\hat{x} \end{array} .$$

(b)

From the relation in part (a),

$$det(sI - A - LC) = s^3 + (1 - l_3)\, s^2 + (-l_2 - l_3)\, s - l_2$$

$$\begin{array}{rcl} 1 - l_3 & > & 0 \\ -l_2 - l_3 & > & 0 \\ -l_2 & > & 0 \end{array}.$$

We can see that the matrix representing these three inequality equations can be represented by the matrix

$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This shows that the eigenvalues have to be selected so that the following relationship is satisfied

$$\begin{array}{rcl} 1 - l_3 & = & a \\ -l_2 - l_3 & = & b \\ -l_2 & = & c \end{array}$$

Thus, the desired poles CANNOT be selected arbitrarily.

Exercise 5

Consider the system,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 + u\\ y &= x_1\end{aligned}$$

where all quantities are scalar. Obtain (by hand) an output feedback controller which results in an asymptotically stable closed loop system.

The state matrices are

$$\mathrm{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathrm{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0], \qquad D = 0 \ .$$

The observability matrix

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow rank(Q_o) = 2 \ .$$

This system is observable, and therefore, detectable.

The controllability matrix

$$Q_c = [A \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow rank(Q_c) = 2.$$

This system is controllable, and thus stabilizable.

Let the state feedback and observer matrix be

$$K = [k_1 \quad k_2]$$

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} .$$

Let the desired poles be $p = -2, -3$, the corresponding characteristic equation becomes

$$CE \coloneqq (s+2)(s+3) = s^2 + 5s + 6$$

Now,

$$det(sI - A - BK) = s^2 - k_2\, s - k_1 - 1$$

$$det(sI - A - LC) = s^2 - l_1\, s - l_2 - 1$$

Hence,

$$K = [-7 \quad -5], \ L = \begin{bmatrix} -5 \\ -7 \end{bmatrix} .$$

The closed loop system can be characterized as

$$\begin{aligned} \dot{\hat{x}} &= [A + BK + L(C + DK)]\hat{x} - Ly \\ u &= K\hat{x} \end{aligned}$$

And

$$A_c = \mathrm{A} + \mathrm{BK} + \mathrm{L}(C + DK) = \begin{bmatrix} -5 & 1 \\ -13 & -5 \end{bmatrix}$$

$$B_c = -L = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$C_c = K = [-7 \quad -5].$$

Now we check that it is asymptotically stable

$$eig(A_c) = -5 \pm 3.6056j \ .$$

It is asymptotically stable.

Thus, the output feedback controller is

$$\begin{aligned} \dot{\hat{x}} &= \begin{bmatrix} -5 & 1 \\ -13 & -5 \end{bmatrix}\hat{x} + \begin{bmatrix} 5 \\ 7 \end{bmatrix} y \\ u &= [-7 \quad -5]\hat{x} \end{aligned} \ .$$

Exercise 6

(Stabilization of cart pendulum system via output feedback.) consider the cart pendulum system with the displacement y as the measured output. Carry out the following for parameter P4 and equilibriums *E1* and *E2*. Illustrate the effectiveness of your controllers with numerical simulations.
using eigenvalue placement techniques, obtain a output feedback controller which stabilizes the nonlinear system about the equilibrium.
What is the largest value of δ (in degrees) for which your controller guarantees convergence of the closed loop system to the equilibrium for initial condition

$$\left(y, \theta_1, \theta_2, \dot{y}, \dot{\theta}_1, \dot{\theta}_2\right)(0) = (0, \theta_1^e - \delta, \theta_2^e + \delta, 0,0,0)$$

Where θ_1^e and θ_2^e are the equilibrium values of θ_1 and θ_2.

The system equation for the double pendulum cart system is

$$\begin{aligned}
(m_0 + m_1 + m_2)\ddot{y} - m_1 l_1 cos\theta_1 \ddot{\theta}_1 - m_2 l_2 cos\theta_2 \ddot{\theta}_2 + m_1 l_1 sin\theta_1 \dot{\theta}_1^2 + m_2 l_2 sin\theta_2 \dot{\theta}_2^2 &= u \\
-m_1 l_1 cos\theta_1 \ddot{y} + m_1 l_1^2 \ddot{\theta}_1 \quad + \quad m_1 l_1 g sin\theta_1 &= 0 \\
-m_2 l_2 cos\theta_2 \ddot{y} + m_2 l_2^2 \ddot{\theta}_2 \quad + \quad m_2 l_2 g sin\theta_2 &= 0
\end{aligned}$$

Have the system be a single output of the displacement y.

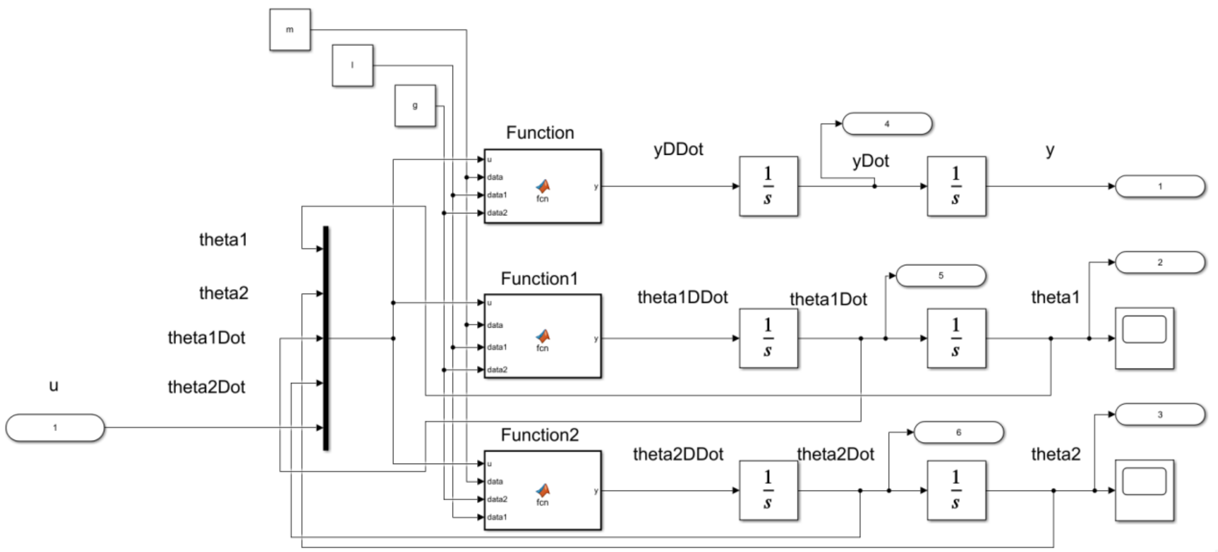
$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

$$E2: \quad (y^e, \theta_1^e, \theta_2^e) = (0, \pi, \pi)$$

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

L1	P1	E1
L2	P1	E2
L3	P2	E1
L4	P2	E2
L5	P3	E1
L6	P3	E2
L7	P4	E1
L8	P4	E2

The nonlinear Simulink model used for this is shown below,



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

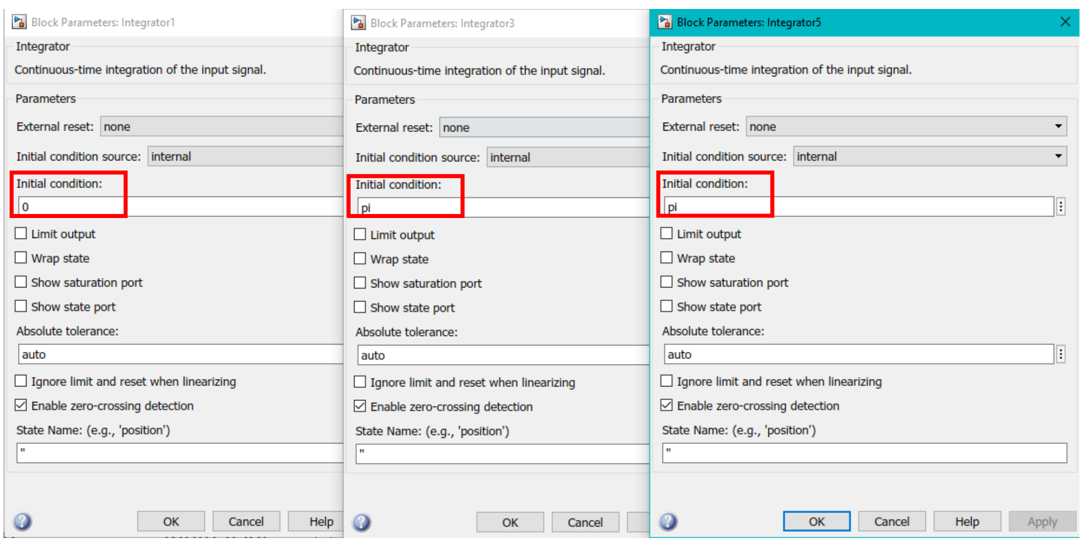
Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



The system state space matrices and poles computed from the linearization for L7 and L8 are the following

L7 (P4 & E1):

```
A = 6×6
      0         0         0    1.0000         0         0
      0         0         0         0    1.0000         0
      0         0         0         0         0    1.0000
      0   -0.5000   -0.5000         0         0         0
      0   -1.5000   -0.5000         0         0         0
      0   -1.0000   -3.0000         0         0         0

B = 6×1
      0
      0
      0
 0.5000
 0.5000
 1.0000

C = 1×6
   1   0   0   0   0   0

D = 0
```

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.8113i
   0.0000 - 1.8113i
  -0.0000 + 1.1042i
  -0.0000 - 1.1042i
```

L8 (P4 & E2):

```
A = 6×6
      0         0         0    1.0000         0         0
      0         0         0         0    1.0000         0
      0         0         0         0         0    1.0000
      0   -0.5000   -0.5000         0         0         0
      0    1.5000    0.5000         0         0         0
      0    1.0000    3.0000         0         0         0

B = 6×1
       0
       0
       0
  0.5000
 -0.5000
 -1.0000

C = 1×6
   1   0   0   0   0   0

D = 0
```

```
eigVal = 6×1
         0
         0
   -1.8113
   -1.1042
    1.8113
    1.1042
```

Using the following MATLAB code we can simulate the nonlinear system controlled by the output feedback controller. The K and L gains are computed using the pole placement with arbitrarily selected poles.

```
close all; clear all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
addpath(genpath("C:\Users\Tomo\Desktop\studies\2020-Fall\AAE564\matlab_simulink"))

% System requirements
p = [-1, -1.22, -1.5, -2, -2.3, -2.7];

global m l g ye theta1e theta2e
warning('off');
param_combo = ["L7","L8"]; %["L7","L8"]
for i = 1:numel(param_combo)
    define_params(param_combo(i));
    xe = trim('db_pend_cart_lin');
    [A, B, C, D] = linmod('db_pend_cart_lin',xe);
    lin_sys(i).Amat = A;
    lin_sys(i).Bmat = B;
    lin_sys(i).Cmat = C;
    lin_sys(i).Dmat = D;

    % Compute the gains
    K    = -place(A, B, p);
    lin_sys(i).K = K;
    L    = -place(A', C', p)';
    lin_sys(i).L = L;

    % Output feedback matrices
    Ac = A + B*K + L*C;
    Bc = -L;
    [B_rows, B_cols] = size(Bc);
    Cc = K;
    [C_rows, C_cols] = size(Cc);
    Dc = zeros(C_rows, B_cols);
    ICc = zeros(B_rows, 1);

    % Plotting
    % Initialize figure
    fig = figure('Renderer',"painters", 'Position', [10 10 900 1000]);
    delta_max = "0";
    inc_history = [];
    while true
        % delta = linspace(0,deg2rad(str2double(delta_max(i))),50)
        delta = str2double(delta_max);
        % Initial conditions
        dyi = 0;
        u = 0;
        yi = ye + dyi;
        theta1i = theta1e - deg2rad(delta);
        theta2i = theta2e + deg2rad(delta);
```

```
        IC_ss = [yi, theta1i, theta2i, 0, 0, 0];

        % Simulate
        simout = sim('db_pend_cart_lin_outputFeedback');
        lin_sys(i).simout = simout;

        % Plot
        time = simout.tout;
        data = simout.res.signals.values;
        y = data(:,1);
        theta1 = data(:,2);
        theta2 = data(:,3);

        if i == 1
            if abs(theta1(end)-theta1e) > 0.1 || abs(theta2(end)-theta2e) > 0.1
                break;
            elseif abs(theta1(end)-theta1e) > 0.08 || abs(theta2(end)-theta2e) > 0.08
                inc = 0.01;
            elseif abs(theta1(end)-theta1e) > 0.05 || abs(theta2(end)-theta2e) > 0.05
                inc = 0.1;
            elseif abs(theta1(end)-theta1e) > 0.01 || abs(theta2(end)-theta2e) > 0.01
                inc = 0.5;
            elseif abs(theta1(end)-theta1e) > 0.001 || abs(theta2(end)-theta2e) >
0.001
                inc = 0.8;
            else
                inc = 1;
            end
        else
            if abs(theta1(end)-theta1e) > 0.1 || abs(theta2(end)-theta2e) > 0.1
                break;
            else
                inc = 0.01;
            end
        end

        subplot(3,1,1)
        grid on; grid minor; box on;
        plot(time,y)
        hold on; grid on; grid minor; box on;
        ylabel('y [m]')
        subplot(3,1,2)
        grid on; grid minor; box on;
        plot(time,theta1)
        hold on; grid on; grid minor; box on;
        ylabel('$\theta_1$ [rad]')
        subplot(3,1,3)
        grid on; grid minor; box on;
        plot(time,theta2)
        hold on; grid on; grid minor; box on;
        ylabel('$\theta_2$ [rad]')

        delta_max = num2str(str2double(delta_max) + inc);
        inc_history = [inc_history, inc];
    end
```

```
    hold off;
    xlabel('time, [sec]')
    line1 = param_combo(i)+' Time Histories for Output Feedback Controlled';
    delta_char = compose("%d", str2double(delta_max));
    line2 = 'Cart Pendulum System for $\delta\in[0,'+delta_char+'^{\circ}]$ - T. 
Koike';
    title_string = {line1,line2};
    sgtitle(title_string)
    saveas(fig, 'p6_'+param_combo(i)+'.png');
end

function define_params(L)
    % Function to define parameters
    global m l g ye theta1e theta2e
    if L == "L1"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L2"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L3"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L4"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L5"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L6"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L7"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L8"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; theta1e = pi; theta2e = pi; % E2
    else
        print('error: did not match any')
    end
end
```

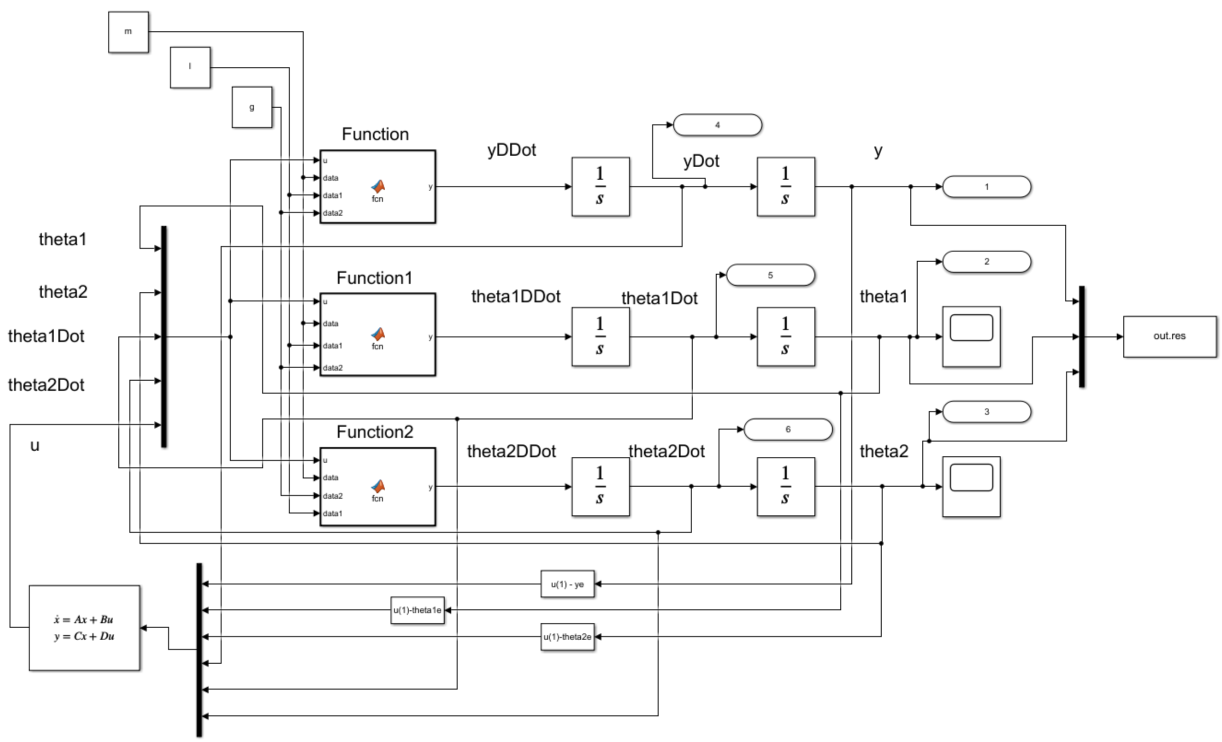
We implement an output feedback controller by finding the K and L gains using eigenvalue placement. The controller is defined as

$$\begin{aligned}\dot{\hat{x}} &= [A + BK + L(C + DK)]\hat{x} - Ly \\ u &= K\hat{x}\end{aligned}$$

and the poles are

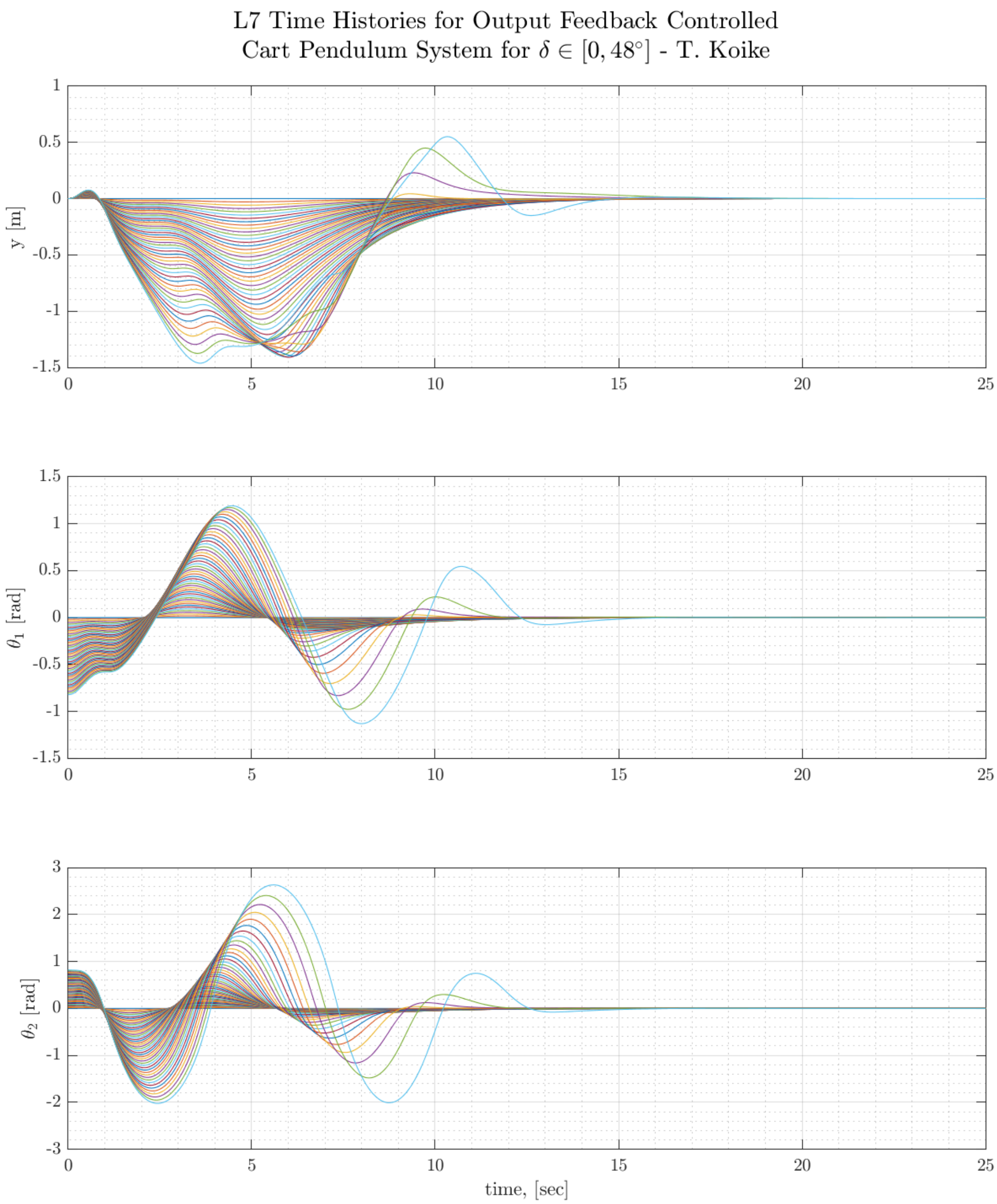
$$p = -1 \quad -1.22 \quad -1.5 \quad -2 \quad -2.3 \quad -2.7\ .$$

The Simulink model used for the output feedback controlled system is



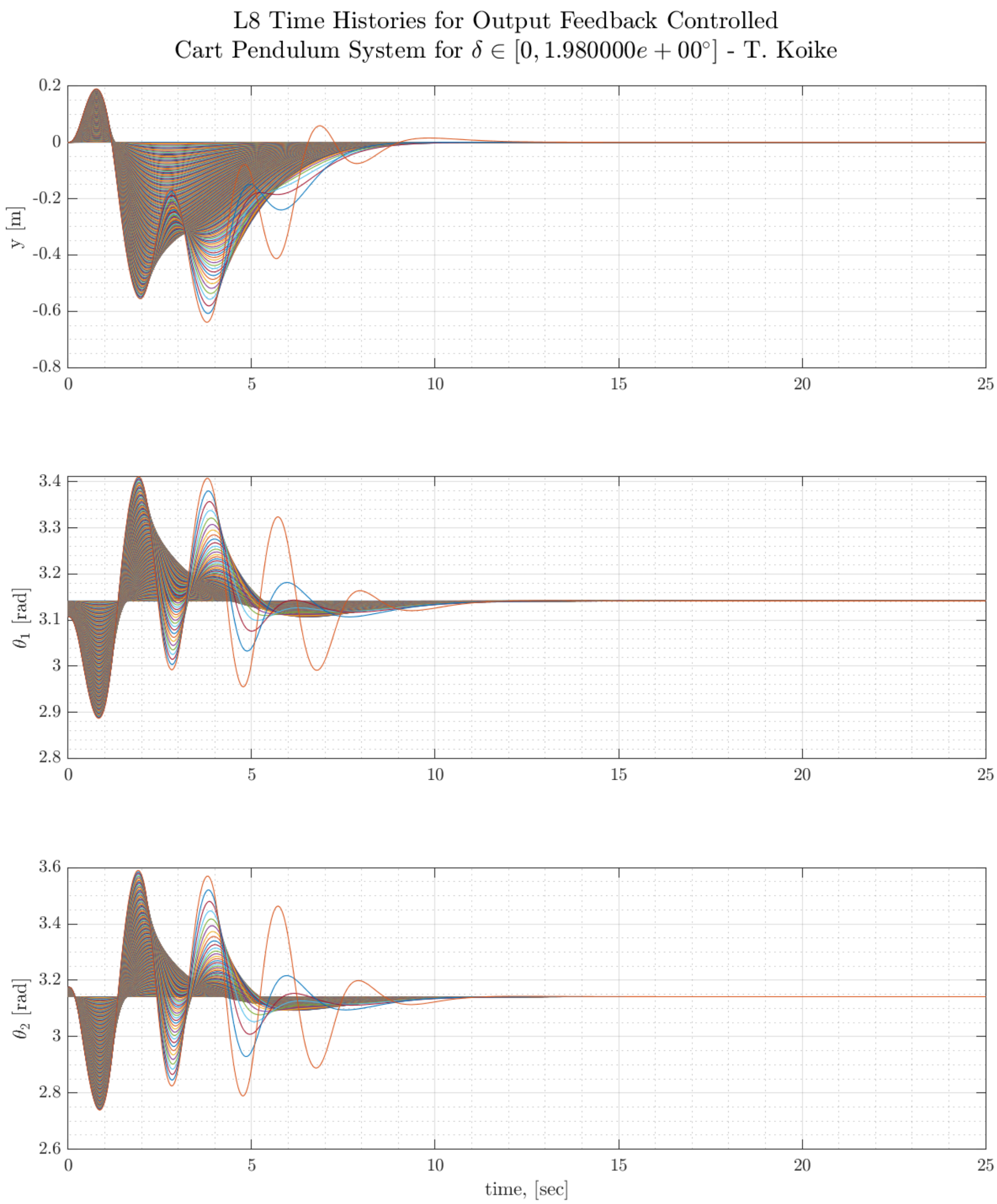
The simulation results are the following

L7 (P4 & E1):



The results show that $max(\delta) = 48°$.

L8 (P4 & E2):



The results show that $max(\delta) = 1.98°$.

Exercise 7

Using the Lyapunov equation determine (by hand) whether or not the system $\dot{x} = Ax$ is asymptotically stable for each one of the following A matrices.

(a)

$$\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Check your answers using the MATLAB command lyap.

(a)

Let a Hermitian matrix P be

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} .$$

Say

$$Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} .$$

Since,

$$eig(Q) = 0.3820, \;\; 2.6180 \Rightarrow Q > 0 \;\; .$$

Then from the Lyapunov equation

$$PA + A'P + Q = 0$$

$$\Rightarrow \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 - 2\,p_{11} = 0 & 2\,p_{11} - 2\,p_{12} - 1 = 0 \\ 2\,p_{11} - 2\,p_{12} - 1 = 0 & 4\,p_{12} - 2\,p_{22} + 2 = 0 \end{pmatrix}$$

Solving these equations, we get

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} .$$

$$eig(P) = 0.5, \;\; 1 \Rightarrow P > 0 \;\; .$$

Thus, the system $\dot{x} = Ax$ is asymptotically stable.

MATLAB Verification:

```
% verify
A = [-1, 2; 0, -1];
Q = [1, -1; -1, 2];
P = lyap(A', Q)
eigVal = eig(P)
```

```
P = 2×2
    0.5000         0
         0    1.0000
```

```
eigVal = 2×1
    0.5000
    1.0000
```

(b)

Let a Hermitian matrix P be

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} .$$

Say

$$Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} .$$

Since,

$$eig(Q) = 0.3820,\ \ 2.6180 \Rightarrow Q > 0 \ .$$

Then from the Lyapunov equation

$$PA + A'P + Q = 0$$

$$\Rightarrow \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 - 2\,p_{11} = 0 & 2\,p_{11} - 1 = 0 \\ 2\,p_{11} - 1 = 0 & 4\,p_{12} + 2\,p_{22} + 2 = 0 \end{pmatrix}$$

Solving these equations, we get

$$\mathrm{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & -0.5 \\ -0.5 & 0 \end{pmatrix} .$$

$$eig(P) = -0.3090,\ \ 0.8090 \Rightarrow P\ is\ not\ positive\ definite\ .$$

Thus, the system $\dot{x} = Ax$ is NOT asymptotically stable.

MATLAB Verification:

```
% verify
A = [-1, 2; 0, 1];
Q = [1, -1; -1, 2];
P = lyap(A', Q)
```

Error using **lyap** (line 73)
The solution of this Lyapunov equation does not exist or is not unique.

(c)

Let a Hermitian matrix P be

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} .$$

Say

$$Q = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} .$$

Since,

$$eig(Q) = 0.3820, \quad 2.6180 \Rightarrow Q > 0 \; .$$

Then from the Lyapunov equation

$$PA + A'P + Q = 0$$

$$\Rightarrow \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2\,p_{11} + 1 = 0 & 2\,p_{11} + 2\,p_{12} - 1 = 0 \\ 2\,p_{11} + 2\,p_{12} - 1 = 0 & 4\,p_{12} + 2\,p_{22} + 2 = 0 \end{pmatrix}$$

Solving these equations, we get

$$\mathrm{P} = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} = \begin{pmatrix} -0.5 & 1 \\ 1 & -3 \end{pmatrix} .$$

$$eig(P) = -0.3090, \quad -0.1492 \Rightarrow P \; is \; not \; positive \; definite \; .$$

Thus, the system $\dot{x} = Ax$ is NOT asymptotically stable.

MATLAB Verification:

```
% verify
A = [1, 2; 0, 1];
Q = [1, -1; -1, 2];
P = lyap(A', Q)
eigVal = eig(P)
```

P = 2×2 -0.5000 1.0000 1.0000 -3.0000	eigVal = 2×1 -3.3508 -0.1492

Exercise 8

Consider the system with disturbance input w and performance output z described by

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 + w \\ \dot{x}_2 &= -x_1 - 4x_2 + 2w \\ z &= x_1 \end{aligned} .$$

Using an appropriate Lyapunov equation, determine (by hand)

$$\int_0^\infty ||z(t)||^2\, dt$$

for each of the following situations.

(a)

$$w = 0 \quad and \quad x(0) = (1, 0) .$$

(b)

$$w(t) = \delta(t) \quad and \quad x(0) = 0 .$$

From the given system we know that

$$\mathrm{A} = \begin{pmatrix} -1 & 1 \\ -1 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \qquad D = 0$$

Then

$$Q = C'C = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} .$$

Let

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{pmatrix} .$$

Then solving the Lyapunov equation we get

$$PA + A'P + Q = 0$$

$$\Rightarrow \begin{pmatrix} 1 - 2\,p_{12} - 2\,p_{11} = 0 & p_{11} - 5\,p_{12} - p_{22} = 0 \\ p_{11} - 5\,p_{12} - p_{22} = 0 & 2\,p_{12} - 8\,p_{22} = 0 \end{pmatrix}$$

$$\Rightarrow p_{11} = 0.4200, \qquad p_{12} = 0.0800, \qquad p_{22} = 0.0200$$

$$P = \begin{pmatrix} 0.4200 & 0.0800 \\ 0.0800 & 0.0200 \end{pmatrix} .$$

$$eig(P) = 0.0046,\ \ 0.4354 \Rightarrow P\ is\ positive\ definite\ .$$

(a)

With $w = 0$ and $x(0) = (1, 0)'$

$$\int_0^\infty \left||z(t)|\right|^2 dt = x(0)'Px(0) = (1, 0)\begin{pmatrix} 0.4200 & 0.0800 \\ 0.0800 & 0.0200 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0.4200 \ .$$

(b)

With $x(0) = 0$ and $w(t) = \delta(t)$

$$\int_0^\infty \left||z(t)|\right|^2 dt = B'PB = (1 \quad 2)\begin{pmatrix} 0.4200 & 0.0800 \\ 0.0800 & 0.0200 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0.8200 \ .$$