College of Engineering
School of Aerospace Engineering

AE6210 Advanced Dynamics I

Assignment

Space Robot Simulation

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I Problem Statement

In this assignment a simulation of a 3D rigid body motion of a space robot (show in blue below). The robot has two arms – one rotating about the z-axis by an angle θ_z (shown in green) and another rotating about the y-axis by an angle θ_y (shown in yellow). Assume that $\theta_z(t)$ and $\theta_y(t)$ are prescribed functions of time. Assume the satellite to have a dimension of $(L \times L/5 \times L/5)$ and mass M, and assume the arms to be of length L and mass $M/4$ each. The Kane's approach is used for the derivation of the equation of motion (EOM).

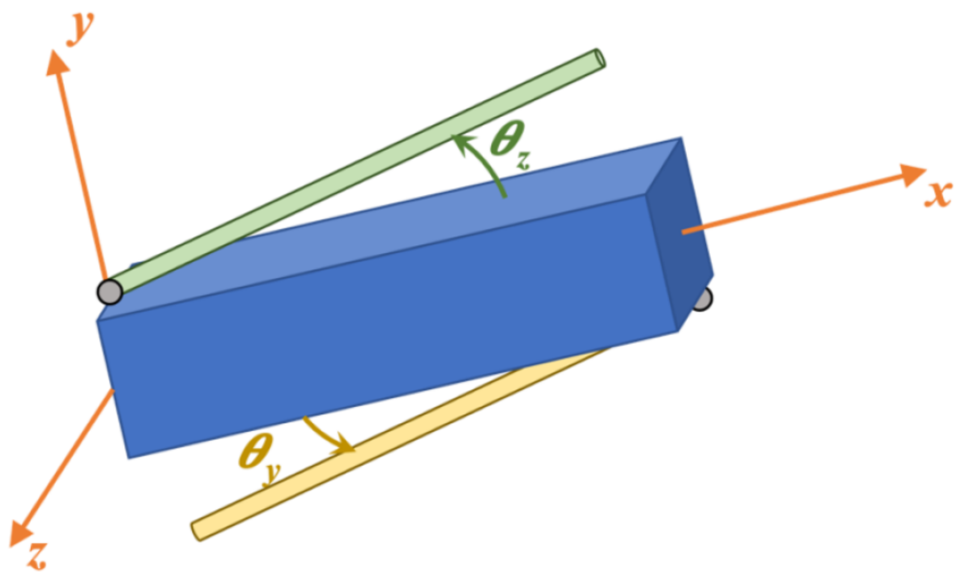

Figure 1: Space robot diagram.

The following sections are organized in the following manner.

1. Formulation of the equations of motion.
2. Simulation results with its verification.
3. Further discussions.

II Problem Formulation

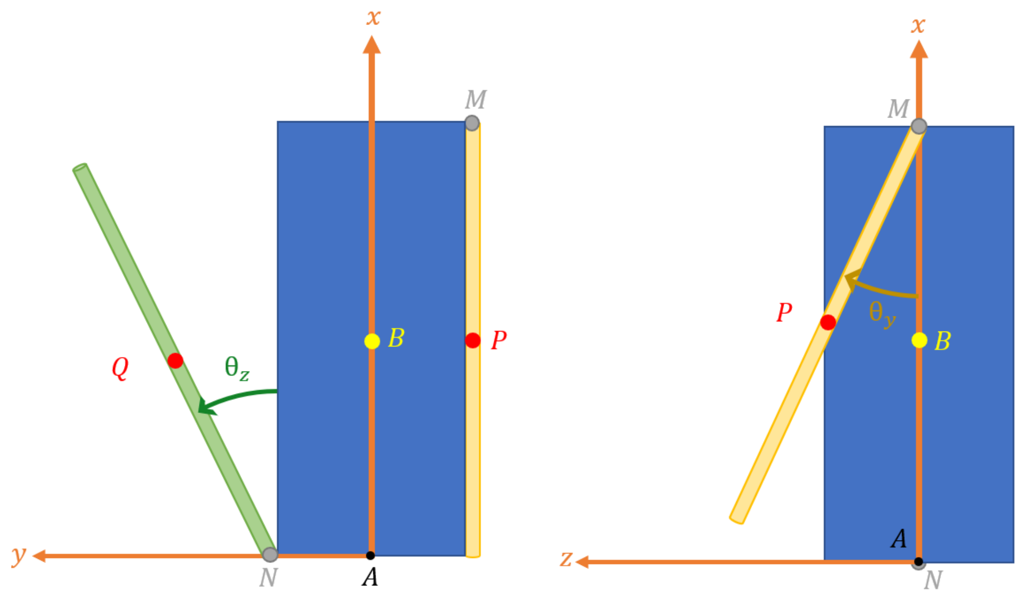

Figure 2: Space robot in side view (left) and bottom view (right).

The diagram above shows the robot from the side and bottom. Let the following points be defined

- A: origin of xyz coordinate frame.
- B: space robot center of mass (COM).
- M: hinge of yellow arm.
- N: hinge of green arm.
- P: yellow arm COM which is $L/2$ away from the hinge M.
- Q: green arm COM which is $L/2$ away from the hinge N.

For the prescribed angles of the arms we will assume the following equations depending on time

$$\theta_y(t) = \theta_{y0}\sin\Omega_y t \qquad \text{and} \qquad \theta_{z0} = \theta_{z0}(1-\cos\Omega_z t) \tag{II.1}$$

and let the unit vectors representing the body coordinate frame (xyz) be

$$B : \hat{\mathbf{b}}_1, \quad \hat{\mathbf{b}}_2, \quad \hat{\mathbf{b}}_3. \tag{II.2}$$

Further, let the inertial reference frame be

$$E : \hat{\mathbf{e}}_1, \quad \hat{\mathbf{e}}_2, \quad \hat{\mathbf{e}}_3. \tag{II.3}$$

Suppose that the COM of the blue body is expressed as the following from the origin O in the inertial frame

$$\mathbf{r}_{OB} = x_B\hat{\mathbf{e}}_1 + y_B\hat{\mathbf{e}}_2 + z_B\hat{\mathbf{e}}_3. \tag{II.4}$$

The velocity of this is simply the derivative of (II.4) which is

$$\mathbf{v}^1 = \mathbf{v}_{OB} = \dot{x}_B\hat{\mathbf{e}}_1 + \dot{y}_B\hat{\mathbf{e}}_2 + \dot{z}_B\hat{\mathbf{e}}_3, \tag{II.5}$$

Now if we assume that we have an orientation angle sequence of Body-three 1-2-3 then we have the following equations

$$^E\omega_1^B = \dot{\phi}c_\theta c_\psi + \dot{\theta}s_\psi \tag{II.6}$$
$$^E\omega_2^B = -\dot{\phi}c_\theta s_\psi + \dot{\theta}c_\psi \tag{II.7}$$
$$^E\omega_3^B = \dot{\phi}s_\theta + \dot{\psi}, \tag{II.8}$$

where

$$\boldsymbol{\omega}^1 = {}^E\boldsymbol{\omega}^B = {}^E\omega_1^B\hat{\mathbf{b}}_1 + {}^E\omega_2^B\hat{\mathbf{b}}_2 + {}^E\omega_3^B\hat{\mathbf{b}}_3. \tag{II.9}$$

Let the generalized coordinates for the COM position of the blue body in the inertial frame be

$$q_1 = x_B, \quad q_2 = y_B, \quad q_3 = z_B, \tag{II.10}$$

then

$$\mathbf{v}^1 = \dot{q}_1\hat{\mathbf{e}}_1 + \dot{q}_2\hat{\mathbf{e}}_2 + \dot{q}_3\hat{\mathbf{e}}_3. \tag{II.11}$$

Also, note that the rotation matrix to convert the inertial frame coordinate to the body frame coordinate is

$$R_E^B = \begin{bmatrix} c_\theta c_\psi & s_\phi s_\theta c_\psi + s_\psi c_\phi & -c_\phi s_\theta c_\psi + s_\psi s_\phi \\ -c_\theta s_\psi & -s_\phi s_\theta s_\psi + c_\psi c_\phi & c_\phi s_\theta s_\psi + c_\psi s_\phi \\ s_\theta & -s_\phi c_\theta & c_\phi c_\theta. \end{bmatrix}. \tag{II.12}$$

Let the orientation angles be additional generalized coordinates

$$q_4 = \phi, \quad q_5 = \theta, \quad q_6 = \psi. \tag{II.13}$$

Now we let the generalized velocities to be

$$\begin{aligned} u_1 &= \dot{q}_1 \\ u_2 &= \dot{q}_2 \\ u_3 &= \dot{q}_3 \\ u_4 &= \dot{q}_4 c_5 c_6 + \dot{q}_5 s_6 \\ u_5 &= -\dot{q}_4 c_5 s_6 + \dot{q}_5 c_6 \\ u_6 &= \dot{q}_4 s_5 + \dot{q}_6. \end{aligned} \tag{II.14}$$

Now, since the angles of the arms are prescribed we can compute the angular velocity of the two arms

$$\boldsymbol{\omega}^2 = \boldsymbol{\omega}^{yellow} = \boldsymbol{\omega}^1 + \dot{\theta}_y\hat{\mathbf{b}}_2 = u_4\hat{\mathbf{b}}_1 + (u_5 + \dot{\theta}_y)\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3 \tag{II.15}$$

$$\boldsymbol{\omega}^3 = \boldsymbol{\omega}^{green} = \boldsymbol{\omega}^1 + \dot{\theta}_z\hat{\mathbf{b}}_3 = u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + (u_6 + \dot{\theta}_z)\hat{\mathbf{b}}_3. \tag{II.16}$$

Using the two point formula we can compute the velocity of the hinge point M as

$$\begin{aligned} \mathbf{v}_{OM} &= \mathbf{v}^1 + \boldsymbol{\omega}^1 \times \mathbf{r}_{BM} = \mathbf{v}^1 + \boldsymbol{\omega}^1 \times \frac{L}{2}\hat{\mathbf{b}}_1 \\ &= \dot{q}_1\hat{\mathbf{e}}_1 + \dot{q}_2\hat{\mathbf{e}}_2 + \dot{q}_3\hat{\mathbf{e}}_3 + (u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3) \times \frac{L}{2}\hat{\mathbf{b}}_1 \\ &= u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3 + \frac{Lu_6}{2}\hat{\mathbf{b}}_2 - \frac{Lu_5}{2}\hat{\mathbf{b}}_3. \end{aligned} \tag{II.17}$$

Subsequently, we can compute the velocity for the COM of the yellow bar using the two point formula again as follows.

$$\begin{aligned} \mathbf{v}^2 &= \mathbf{v}_{OM} + \boldsymbol{\omega}^2 \times \mathbf{r}_{MP} = \mathbf{v}_{OM} + \boldsymbol{\omega}^2 \times \frac{L}{2}\left(-c_{\theta_y}\hat{\mathbf{b}}_1 + s_{\theta_y}\hat{\mathbf{b}}_3\right) \\ &= (\dot{q}_1\hat{\mathbf{e}}_1 + \dot{q}_2\hat{\mathbf{e}}_2 + \dot{q}_3\hat{\mathbf{e}}_3 + \frac{Lu_6}{2}\hat{\mathbf{b}}_2 - \frac{Lu_5}{2}\hat{\mathbf{b}}_3) + (u_4\hat{\mathbf{b}}_1 + (u_5 + \dot{\theta}_y)\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3) \times \frac{L}{2}\left(-c_{\theta_y}\hat{\mathbf{b}}_1 + s_{\theta_y}\hat{\mathbf{b}}_3\right) \\ &= u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3 \\ &\qquad + \frac{L}{2}\left\{(u_5 + \dot{\theta}_y)s_{\theta_y}\hat{\mathbf{b}}_1 + \left[(1 - c_{\theta_y})u_6 - u_4 s_{\theta_y}\right]\hat{\mathbf{b}}_2 + \left[(c_{\theta_y} - 1)u_5 + \dot{\theta}_y c_{\theta_y}\right]\hat{\mathbf{b}}_3\right\} \end{aligned} \tag{II.18}$$

Similarly, for the green arm we have

$$\begin{aligned}
\mathbf{v}_{ON} &= \mathbf{v}^1 + \boldsymbol{\omega}^1 \times \mathbf{r}_{BN} = \mathbf{v}^1 + \boldsymbol{\omega}^1 \times \left(-\frac{L}{2}\hat{\mathbf{b}}_1 + \frac{L}{10}\hat{\mathbf{b}}_2\right) \\
&= \dot{q}_1\hat{\mathbf{e}}_1 + \dot{q}_2\hat{\mathbf{e}}_2 + \dot{q}_3\hat{\mathbf{e}}_3 + (u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + u_6\hat{\mathbf{b}}_3) \times \left(-\frac{L}{2}\hat{\mathbf{b}}_1 + \frac{L}{10}\hat{\mathbf{b}}_2\right) \\
&= u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3 - \frac{L}{10}u_6\hat{\mathbf{b}}_1 - \frac{L}{2}u_6\hat{\mathbf{b}}_2 + \left(\frac{L}{10}u_4 + \frac{L}{2}u_5\right)\hat{\mathbf{b}}_3.
\end{aligned} \tag{II.19}$$

and

$$\begin{aligned}
\mathbf{v}^3 &= \mathbf{v}_{ON} + \boldsymbol{\omega}^3 \times \mathbf{r}_{NQ} = \mathbf{v}_{ON} + \boldsymbol{\omega}^3 \times \frac{L}{2}\left(c_{\theta_z}\hat{\mathbf{b}}_1 + s_{\theta_z}\hat{\mathbf{b}}_2\right) \\
&= \left(\dot{q}_1\hat{\mathbf{e}}_1 + \dot{q}_2\hat{\mathbf{e}}_2 + \dot{q}_3\hat{\mathbf{e}}_3 - \frac{L}{10}u_6\hat{\mathbf{b}}_1 - \frac{L}{2}u_6\hat{\mathbf{b}}_2 + \left(\frac{L}{10}u_4 + \frac{L}{2}u_5\right)\hat{\mathbf{b}}_3\right) \\
&\qquad + (u_4\hat{\mathbf{b}}_1 + u_5\hat{\mathbf{b}}_2 + (u_6 + \dot{\theta}_z)\hat{\mathbf{b}}_3) \times \frac{L}{2}\left(c_{\theta_z}\hat{\mathbf{b}}_1 + s_{\theta_z}\hat{\mathbf{b}}_2\right) \\
&= u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3 + \frac{L}{10}\left\{\left[-(1 + 5s_{\theta_z})u_6 - 5\dot{\theta}_z s_{\theta_z}\right]\hat{\mathbf{b}}_1\right. \\
&\qquad \left. +5\left[(c_{\theta_z} - 1)u_6 + \dot{\theta}_z c_{\theta_z}\right]\hat{\mathbf{b}}_2 + \left[(1 + 5s_{\theta_z})u_4 + 5(1 - c_{\theta_z})u_5\right]\hat{\mathbf{b}}_3\right\}.
\end{aligned} \tag{II.20}$$

Next we compute the partial velocities

$$\begin{aligned}
\mathbf{v}_1^1 &= \frac{\partial \mathbf{v}^1}{\partial u_1} = \hat{\mathbf{e}}_1 \\
\mathbf{v}_2^1 &= \frac{\partial \mathbf{v}^1}{\partial u_2} = \hat{\mathbf{e}}_2 \\
\mathbf{v}_3^1 &= \frac{\partial \mathbf{v}^1}{\partial u_3} = \hat{\mathbf{e}}_3 \\
\mathbf{v}_4^1 &= \frac{\partial \mathbf{v}^1}{\partial u_4} = 0 \\
\mathbf{v}_5^1 &= \frac{\partial \mathbf{v}^1}{\partial u_5} = 0 \\
\mathbf{v}_6^1 &= \frac{\partial \mathbf{v}^1}{\partial u_6} = 0
\end{aligned} \tag{II.21}$$

and

$$\begin{aligned}
\mathbf{v}_1^2 &= \frac{\partial \mathbf{v}^2}{\partial u_1} = \hat{\mathbf{e}}_1 \\
\mathbf{v}_2^2 &= \frac{\partial \mathbf{v}^2}{\partial u_2} = \hat{\mathbf{e}}_2 \\
\mathbf{v}_3^2 &= \frac{\partial \mathbf{v}^2}{\partial u_3} = \hat{\mathbf{e}}_3 \\
\mathbf{v}_4^2 &= \frac{\partial \mathbf{v}^2}{\partial u_4} = -\frac{L}{2}s_{\theta_y}\hat{\mathbf{b}}_2 \\
\mathbf{v}_5^2 &= \frac{\partial \mathbf{v}^2}{\partial u_5} = \frac{L}{2}\left[s_{\theta_y}\hat{\mathbf{b}}_1 + (c_{\theta_y} - 1)\hat{\mathbf{b}}_3\right] \\
\mathbf{v}_6^2 &= \frac{\partial \mathbf{v}^2}{\partial u_6} = \frac{L}{2}(1 - c_{\theta_y})\hat{\mathbf{b}}_2
\end{aligned} \tag{II.22}$$

and

$$
\begin{aligned}
\mathbf{v}_1^3 &= \frac{\partial \mathbf{v}^3}{\partial u_1} = \hat{\mathbf{e}}_1 \\
\mathbf{v}_2^3 &= \frac{\partial \mathbf{v}^3}{\partial u_2} = \hat{\mathbf{e}}_2 \\
\mathbf{v}_3^3 &= \frac{\partial \mathbf{v}^3}{\partial u_3} = \hat{\mathbf{e}}_3 \\
\mathbf{v}_4^3 &= \frac{\partial \mathbf{v}^3}{\partial u_4} = \frac{L}{10}(1 + 5s_{\theta_z})\hat{\mathbf{b}}_3 \\
\mathbf{v}_5^3 &= \frac{\partial \mathbf{v}^3}{\partial u_5} = \frac{L}{2}(1 - c_{\theta_z})\hat{\mathbf{b}}_3 \\
\mathbf{v}_6^3 &= \frac{\partial \mathbf{v}^3}{\partial u_6} = \frac{L}{10}\left[(-1 - 5s_{\theta_z})\hat{\mathbf{b}}_1 + 5(c_{\theta_z} - 1)\hat{\mathbf{b}}_2\right]
\end{aligned}
\tag{II.23}
$$

Similarly the partial angular velocities are

$$
\begin{aligned}
\boldsymbol{\omega}_1^1 &= \frac{\partial \boldsymbol{\omega}^1}{\partial u_1} = 0 \\
\boldsymbol{\omega}_2^1 &= \frac{\partial \boldsymbol{\omega}^1}{\partial u_2} = 0 \\
\boldsymbol{\omega}_3^1 &= \frac{\partial \boldsymbol{\omega}^1}{\partial u_3} = 0 \\
\boldsymbol{\omega}_4^1 &= \frac{\partial \boldsymbol{\omega}^1}{\partial u_4} = \hat{\mathbf{b}}_1 \\
\boldsymbol{\omega}_5^1 &= \frac{\partial \boldsymbol{\omega}^1}{\partial u_5} = \hat{\mathbf{b}}_2 \\
\boldsymbol{\omega}_6^1 &= \frac{\partial \boldsymbol{\omega}^1}{\partial u_6} = \hat{\mathbf{b}}_3
\end{aligned}
\tag{II.24}
$$

and the rest are the exact same

$$
\begin{aligned}
\boldsymbol{\omega}_1^2 &= \boldsymbol{\omega}_1^3 = 0 \\
\boldsymbol{\omega}_2^2 &= \boldsymbol{\omega}_2^3 = 0 \\
\boldsymbol{\omega}_3^2 &= \boldsymbol{\omega}_3^3 = 0 \\
\boldsymbol{\omega}_4^2 &= \boldsymbol{\omega}_4^3 = \hat{\mathbf{b}}_1 \\
\boldsymbol{\omega}_5^2 &= \boldsymbol{\omega}_5^3 = \hat{\mathbf{b}}_2 \\
\boldsymbol{\omega}_6^2 &= \boldsymbol{\omega}_6^3 = \hat{\mathbf{b}}_3.
\end{aligned}
\tag{II.25}
$$

The MOI of each body are

$$
\mathbf{I}_1 = \begin{bmatrix} \frac{ML^2}{125} & 0 & 0 \\ 0 & \frac{13ML^2}{125} & 0 \\ 0 & 0 & \frac{13ML^2}{125} \end{bmatrix}, \quad \mathbf{I}_2 = \begin{bmatrix} \frac{1}{12}ML^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}ML^2 \end{bmatrix}, \quad \mathbf{I}_3 = \begin{bmatrix} \frac{1}{12}ML^2 & 0 & 0 \\ 0 & \frac{1}{12}ML^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{II.26}
$$

Next we compute the angular accelerations

$$
\boldsymbol{\alpha}_1 = \dot{\boldsymbol{\omega}}_1 = \dot{u}_4\hat{\mathbf{b}}_1 + \dot{u}_5\hat{\mathbf{b}}_2 + \dot{u}_6\hat{\mathbf{b}}_3. \tag{II.27}
$$

$$
\boldsymbol{\alpha}_2 = \dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2 = (\dot{u}_4 - u_6\dot{\theta}_y)\hat{\mathbf{b}}_1 + (\ddot{\theta}_y + \dot{u}_5)\hat{\mathbf{b}}_2 + (u_4\dot{\theta}_y + \dot{u}_6)\hat{\mathbf{b}}_3 \tag{II.28}
$$

$$
\boldsymbol{\alpha}_3 = \dot{\boldsymbol{\omega}}_3 + \boldsymbol{\omega}_1 \times \boldsymbol{\omega}_3 = (u_5\dot{\theta}_z + \dot{u}_4)\hat{\mathbf{b}}_1 + (\dot{u}_5 - u_4\dot{\theta}_z)\hat{\mathbf{b}}_2 + (\ddot{\theta}_z + \dot{u}_6)\hat{\mathbf{b}}_3 \tag{II.29}
$$

Further, we compute the acceleration by taking the derivatives of the velocities

$$\mathbf{a}_1 = \dot{u}_1\hat{\mathbf{e}}_1 + \dot{u}_2\hat{\mathbf{e}}_2 + \dot{u}_3\hat{\mathbf{e}}_3 \tag{II.30}$$

$$\begin{aligned}
\mathbf{a}_2 = \dot{u}_1\hat{\mathbf{e}}_1 &+ \dot{u}_2\hat{\mathbf{e}}_2 + \dot{u}_3\hat{\mathbf{e}}_3 \\
&+ \Big\{ Lc_{\theta_y}\dot{\theta}_y^2 - \frac{L}{2}(1-c_{\theta_y})u_5^2 - \frac{L}{2}(1-c_{\theta_y})u_6^2 \\
&\quad -\frac{L}{2}\left[(1-c_{\theta_y})\dot{\theta}_y - 2c_{\theta_y}\dot{\theta}_y\right]u_5 + \frac{L}{2}s_{\theta_y}(\ddot{\theta}_y + \dot{u}_5) + \frac{L}{2}s_{\theta_y}u_4u_6 \Big\}\hat{\mathbf{b}}_1 \\
&+ \Big\{ \frac{L}{2}\dot{u}_6 + \frac{L}{2}(1-c_{\theta_y})u_4u_5 - \frac{L}{2}c_{\theta_y}u_6 \\
&\quad -\frac{L}{2}s_{\theta_y}\dot{u}_4 - Lc_{\theta_y}\dot{\theta}_y u_4 + Ls_{\theta_y}\dot{\theta}_y u_6 + \frac{L}{2}s_{\theta_y}u_5u_6 \Big\}\hat{\mathbf{b}}_2 \\
&+ \Big\{ \frac{L}{2}c_{\theta_y}(\ddot{\theta}_y + \dot{u}_5) - Ls_{\theta_y}\dot{\theta}_y^2 - \frac{L}{2}\dot{u}_5 - \frac{L}{2}s_{\theta_y}u_4^2 \\
&\quad -\frac{L}{2}s_{\theta_y}u_5^2 + \frac{L}{2}(1-c_{\theta_y})u_4u_6 - \frac{3L}{2}s_{\theta_y}\dot{\theta}_y u_5 \Big\}\hat{\mathbf{b}}_3
\end{aligned} \tag{II.31}$$

$$\begin{aligned}
\mathbf{a}_3 = \dot{u}_1\hat{\mathbf{e}}_1 &+ \dot{u}_2\hat{\mathbf{e}}_2 + \dot{u}_3\hat{\mathbf{e}}_3 \\
&+ \Big\{ \frac{L}{2}\left[(1-c_{\theta_z})\dot{\theta}_z - 2c_{\theta_z}\dot{\theta}_z\right]u_6 + \frac{L}{2}(1-c_{\theta_z})u_5^2 + \frac{L}{2}(1-c_{\theta_z})u_6^2 \\
&\quad -\frac{L}{10}\dot{u}_6 - Lc_{\theta_z}\dot{\theta}_z^2 - \frac{L}{2}s_{\theta_z}(\ddot{\theta}_z + \dot{u}_6) + \frac{L}{10}(1+5s_{\theta_z})u_4u_5 \Big\}\hat{\mathbf{b}}_1 \\
&+ \Big\{ \frac{L}{2}c_{\theta_z}(\ddot{\theta}_z + \dot{u}_6) - \frac{L}{10}(1+5s_{\theta_z})u_4^2 - \frac{L}{10}(1+5s_{\theta_z})u_6^2 - \frac{L}{2}\dot{u}_6 - Ls_{\theta_z}\dot{\theta}_z^2 \\
&\quad -\frac{L}{10}\left[(1+5s_{\theta_z})\dot{\theta}_z + 10s_{\theta_z}\dot{\theta}_z\right]u_6 - \frac{L}{2}(1-c_{\theta_z})u_4u_5 \Big\}\hat{\mathbf{b}}_2 \\
&+ \Big\{ \frac{L}{10}\dot{u}_4 + \frac{L}{2}\dot{u}_5 - \frac{L}{2}(1-c_{\theta_z})u_4u_6 - \frac{L}{2}c_{\theta_z}\dot{u}_5 \\
&\quad +\frac{L}{2}s_{\theta_z}\dot{u}_4 + \frac{L}{10}(1+5s_{\theta_z})u_5u_6 + Lc_{\theta_z}\dot{\theta}_z u_4 + Ls_{\theta_z}\dot{\theta}_z u_5 \Big\}\hat{\mathbf{b}}_3
\end{aligned} \tag{II.32}$$

The masses of the three bodies are

$$m_1 = M, \qquad m_2 = M/4, \qquad m_3 = M/4. \tag{II.33}$$

Finally the generalized inertia forces can be found by the following formula

$$F_r^* = \sum_{k=1}^{3}\left(\boldsymbol{\omega}_r^k \cdot \mathbf{T}_k^*\right) + \sum_{k=1}^{3}\left[\mathbf{v}_r^k \cdot \mathbf{R}_k^*\right], \tag{II.34}$$

where

$$\mathbf{T}_k^* = -\mathbf{I}_k \cdot \boldsymbol{\alpha}_k - \boldsymbol{\omega}_k \times (\mathbf{I}_k \cdot \boldsymbol{\omega}_k) \qquad \text{and} \qquad \mathbf{R}_k^* = -m_k\mathbf{a}_k. \tag{II.35}$$

We can compute

$$\mathbf{T}_1^* = -\frac{ML^2}{125}\dot{u}_4\hat{\mathbf{b}}_1 + \left(\frac{12ML^2}{125}u_4u_6 - \frac{13ML^2}{125}\dot{u}_5\right)\hat{\mathbf{b}}_2 + \left(-\frac{13ML^2}{125}\dot{u}_6 - \frac{12ML^2}{125}u_4u_5\right)\hat{\mathbf{b}}_3 \tag{II.36}$$

$$\mathbf{T}_2^* = \left[\frac{ML^2}{12}\left(\dot{\theta}_y u_6 - \dot{u}_4\right) - \frac{ML^2}{12}u_6(\dot{\theta}_y + u_5)\right]\hat{\mathbf{b}}_1 + \left[\frac{ML^2}{12}u_4(\dot{\theta}_y + u_5) - \frac{ML^2}{12}(\dot{\theta}_y u_4 + \dot{u}_6)\right]\hat{\mathbf{b}}_3 \tag{II.37}$$

$$\mathbf{T}_3^* = \left[\frac{ML^2}{12}u_5(\dot{\theta}_z + u_6) - \frac{ML^2}{12}(\dot{\theta}_z u_5 + \dot{u}_4)\right]\hat{\mathbf{b}}_1 + \left[\frac{ML^2}{12}(\dot{\theta}_z u_4 - \dot{u}_5) - \frac{ML^2}{12}u_4(\dot{\theta}_z + u_6)\right]\hat{\mathbf{b}}_2 \tag{II.38}$$

Thus, we have

$$F_1^* = -M\dot{u}_1 - \frac{M}{4}\dot{u}_1 - \frac{M}{4}\dot{u}_1 = -\frac{3M}{2}\dot{u}_1 \tag{II.39}$$

$$F_2^* = -M\dot{u}_2 - \frac{M}{4}\dot{u}_2 - \frac{M}{4}\dot{u}_2 = -\frac{3M}{2}\dot{u}_2 \tag{II.40}$$

$$F_3^* = -M\dot{u}_3 - \frac{M}{4}\dot{u}_3 - \frac{M}{4}\dot{u}_3 = -\frac{3M}{2}\dot{u}_3 \tag{II.41}$$

and

$$\begin{aligned}\frac{F_4^*}{ML^2} = \frac{\beta^2}{400}(\dot{u}_4 + u_5 u_6) - \frac{131}{750}\dot{u}_4 + \frac{\beta\gamma}{80}(\dot{u}_5 - u_6 u_4) + \frac{s_{\theta_y}\gamma}{16}(\dot{u}_6 + u_4 u_5)\\ + \frac{s_{\theta_y}\beta}{80}(u_4^2 + u_6^2) + \frac{\beta\dot{\theta}_z}{80}(2c_{\theta_z}u_4 + 2s_{\theta_z}u_5 + s_{\theta_y}u_6)\\ + \frac{s_{\theta_y}s_{\theta_z}\dot{\theta}_z}{8}u_6 + \frac{s_{\theta_y}}{16}(2s_{\theta_z}\dot{\theta}_z^2 - c_{\theta_z}\ddot{\theta}_z)\end{aligned} \tag{II.42}$$

$$\begin{aligned}\frac{F_5^*}{ML^2} = \frac{1}{80}\beta(\gamma-\delta)(\dot{u}_4 + u_5 u_6) + \frac{1}{16}\gamma(\gamma-\delta)(\dot{u}_5 - u_4 u_6) + \frac{281}{1500}\delta\dot{u}_5 + \frac{19}{1500}u_4 u_6\\ + \frac{1}{80}s_{\theta_y}\beta(\dot{u}_6 + u_4 u_5) + \frac{1}{16}s_{\theta_y}\gamma(u_5^2 + u_6^2) + \frac{1}{8}(\gamma-\delta)\dot{\theta}_z(c_{\theta_z}u_4 + s_{\theta_z}u_5)\\ + \frac{1}{16}s_{\theta_y}(\gamma\dot{\theta}_z - 2c_{\theta_z}\dot{\theta}_z)u_6 + \frac{1}{16}s_{\theta_y}(c_{\theta_z}\dot{\theta}_z^2 + s_{\theta_z}\ddot{\theta}_z)\end{aligned} \tag{II.43}$$

$$\begin{aligned}\frac{F_6^*}{ML^2} = \left(\frac{\gamma^2}{16} - \frac{\beta^2}{400} - \frac{\gamma\delta}{16}\right)(\dot{u}_6 + u_4 u_5) - \frac{281}{1500}\dot{u}_6 - \frac{19}{1500}u_4 u_5 + \frac{1}{80}\beta\left[(\gamma-\delta)u_4^2 - \gamma u_5^2 - \delta u_6^2\right]\\ + \frac{1}{80}\left[\dot{\theta}_z\beta(\gamma-\delta) + 10s_{\theta_z}\dot{\theta}_z(\gamma-\delta) - 5\dot{\theta}_z\beta\gamma + 10c_{\theta_z}\dot{\theta}_z\beta\right]u_6\\ + \frac{1}{80}\beta(2c_{\theta_z}\dot{\theta}_z^2 + s_{\theta_z}\ddot{\theta}_z) + \frac{1}{16}(2s_{\theta_z}\dot{\theta}_z^2 - c_{\theta_z}\ddot{\theta}_z)(\gamma-\delta),\end{aligned} \tag{II.44}$$

where

$$\beta = 5s_{\theta_z} + 1, \qquad \gamma = 1 - c_{\theta_z}, \qquad \delta = 1 - c_{\theta_y}. \tag{II.45}$$

Since there is no generalized active forces on the system we have the dynamic differential equations become

$$F_r^* = 0. \tag{II.46}$$

Note, the kinematic differential equations are (II.14).

III Simulation

For the simulation we will use the following values for the prescribed angles for the arms and the initial conditions for the numerical integration using `ode45`.

$$\theta_y(t) = 0.3491 \sin(1.5t) \qquad \text{and} \qquad \theta_z(t) = -0.1745(1 - \cos(0.8t)), \tag{III.1}$$

and

$$\begin{bmatrix} q_{10} \\ q_{20} \\ q_{30} \\ q_{40} \\ q_{50} \\ q_{60} \\ u_{10} \\ u_{20} \\ u_{30} \\ u_{40} \\ u_{50} \\ u_{60} \end{bmatrix} = \begin{bmatrix} 5681 \\ 3161 \\ 4437 \\ 0.7854 \\ -1.4661 \\ 0.1047 \\ 2 \\ 4 \\ 1 \\ 0.045 \\ 0.03 \\ -0.02 \end{bmatrix}. \tag{III.2}$$

The position of the space robot is assumed to be in LEO with an altitude of 1500 km which is 7871 km away from the center of the Earth. Now running the simulation with the code in the appendix we obtain the plots below.

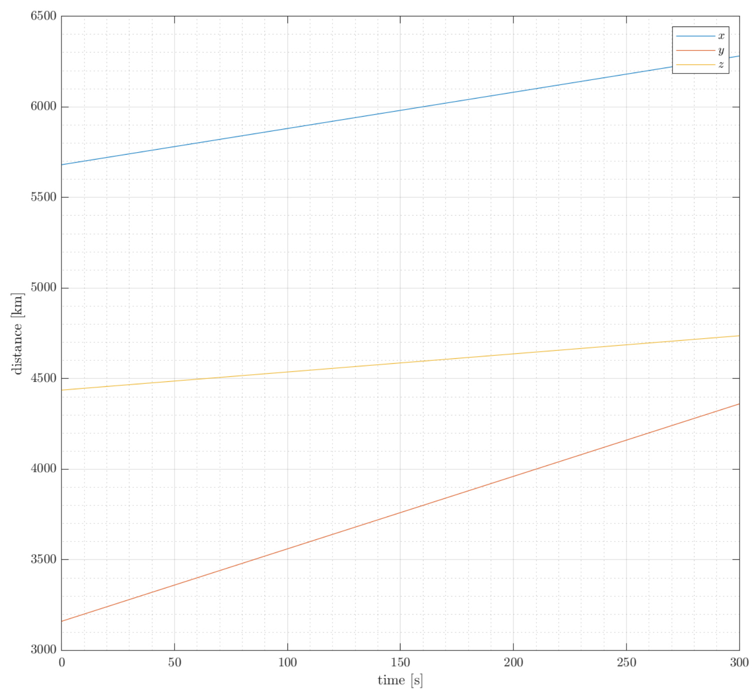

Figure 3: Space robot: position over time.

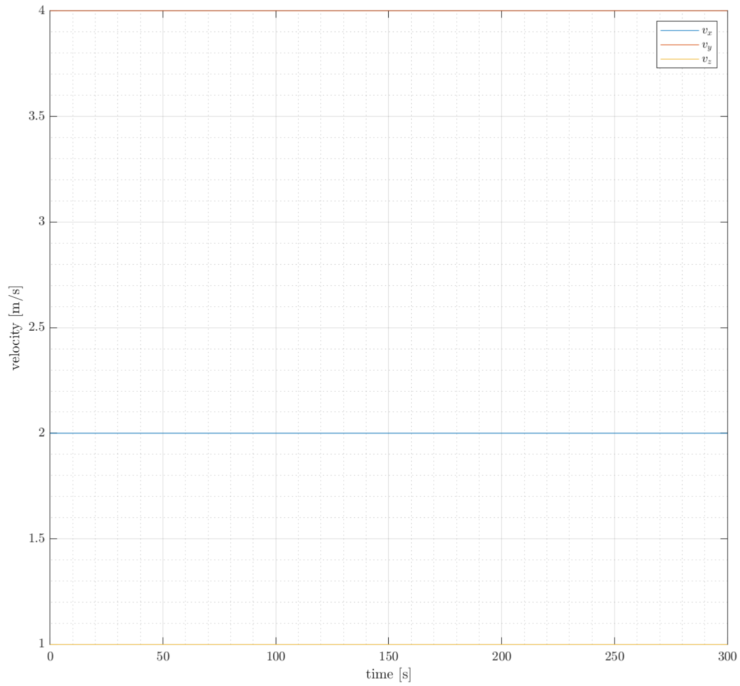

Figure 4: Space robot: velocity over time.

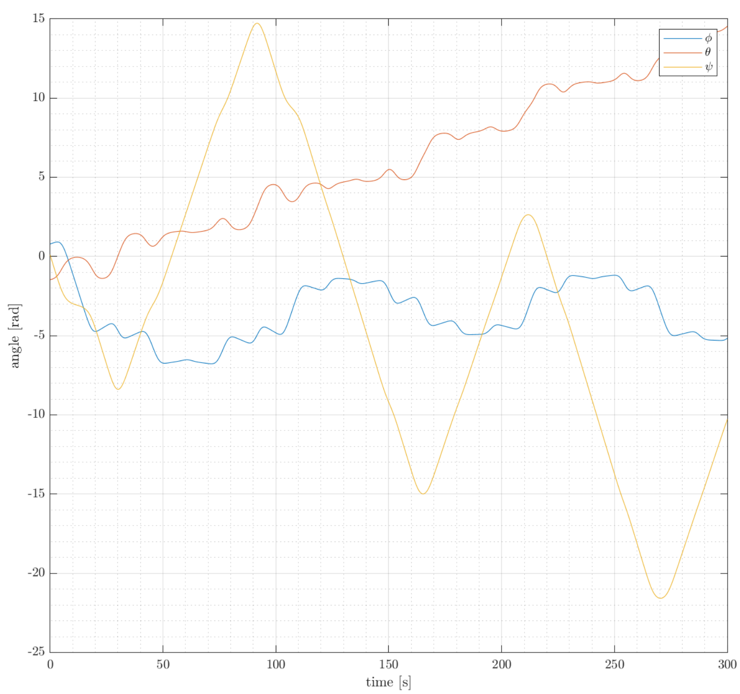

Figure 5: Space robot: orientation angles over time.

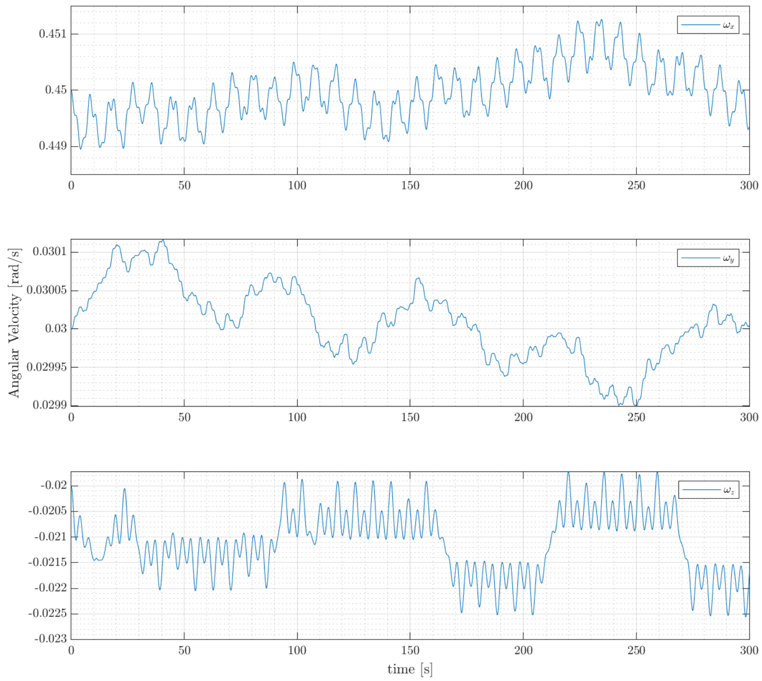

Figure 6: Space robot: angular velocity over time.

IV Verification & Discussion

For the verification we shall check whether the angular momentum is approximately constant.

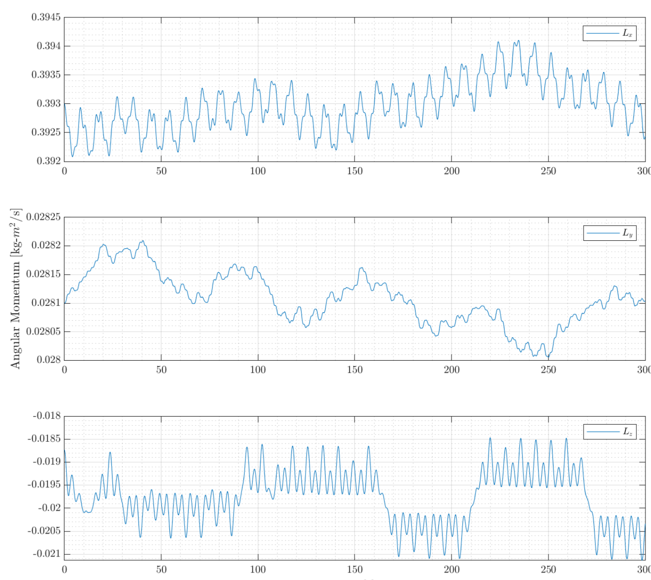

Figure 7: Space robot: angular momentum over time.

We say approximately since the numerical integration does add on some errors to the theoretical case in which the angular momentum should be constant with no external forces applied. To compute the angular momentum we can use the sum of the moment of inertias shown in (II.26) with the formula of

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}. \tag{IV.1}$$

The plot of the angular momentum can be observed in the Figure 7. Further, the variance of each $\hat{\mathbf{b}}_1 - \hat{\mathbf{b}}_2 - \hat{\mathbf{b}}_3$ component of the angular momentum are tabulated below.

Angular Momentum, $\mathbf{L}$	Variance
$\mathbf{L}_x$	1.8261e-07
$\mathbf{L}_y$	2.1511e-09
$\mathbf{L}_z$	3.4094e-07

Table 1: Variance of angular momentum components.

We can observe that the variance for each angular momentum component are very small indicating that the scattering is only occurring within a very small bound. Hence, this verifies that the angular momentum is approximately conserved and our simulation is valid.

V Appendix

Simulation Code

```
%% AE6210 HW8 Matlab Code
% Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% Setup

% Parameters
params.M = 5;
params.L = 1;
params.theta_y0 = deg2rad(20);
params.theta_z0 = deg2rad(-10);
params.Omega_y = 1.5;
params.Omega_z = 0.8;

% Tolerance
opts = odeset('RelTol',1e-8,'AbsTol',1e-10);

% Initial conditions
% LEO -> altitude 1500 km -> radius 7871 km from the center of the Earth
x0 = 5681; % km
y0 = 3161;
z0 = 4437;
% Orientation angles
phi0 = deg2rad(45);
theta0 = deg2rad(-84);
psi0 = deg2rad(6);
% Velocities
vx0 = 2;
vy0 = 4;
vz0 = 1;
% Angular Velocities
wx0 = 0.45;
wy0 = 0.03;
wz0 = -0.020;

IC = [x0;y0;z0;phi0;theta0;psi0;vx0;vy0;vz0;wx0;wy0;wz0];
tspan = [0,300];
[t,res] = ode45(@(t,z) spaceRobot(t,z,params),tspan,IC,opts);

%% Plot

fig = figure(Renderer="painters",Position=[60 60 900 800]);
    plot(t,res(:,1),DisplayName="$x$")
    hold on; grid on; grid minor; box on;
    plot(t,res(:,2),DisplayName="$y$")
```

```
    plot(t,res(:,3),DisplayName="$z$")
    hold off; legend;
    xlabel('time [s]')
    ylabel('distance [km]')
saveas(fig,'plots/position.png')

fig = figure(Renderer="painters",Position=[60 60 900 800]);
    plot(t,res(:,4),DisplayName="$\phi$")
    hold on; grid on; grid minor; box on;
    plot(t,res(:,5),DisplayName="$\theta$")
    plot(t,res(:,6),DisplayName="$\psi$")
    hold off; legend;
    xlabel('time [s]')
    ylabel('angle [rad]')
saveas(fig,'plots/orientation.png')

fig = figure(Renderer="painters",Position=[60 60 900 800]);
    plot(t,res(:,7),DisplayName="$v_x$")
    hold on; grid on; grid minor; box on;
    plot(t,res(:,8),DisplayName="$v_y$")
    plot(t,res(:,9),DisplayName="$v_z$")
    hold off; legend;
    xlabel('time [s]')
    ylabel('velocity [m/s]')
saveas(fig,'plots/velocity.png')

fig = figure(Renderer="painters",Position=[60 60 900 800]);
subplot(3,1,1)
    plot(t,res(:,10),DisplayName="$\omega_x$")
    grid on; grid minor; box on; legend;
subplot(3,1,2)
    plot(t,res(:,11),DisplayName="$\omega_y$")
    grid on; grid minor; box on; legend;
subplot(3,1,3)
    plot(t,res(:,12),DisplayName="$\omega_z$")
    grid on; grid minor; box on; legend;
    % Give common xlabel, ylabel and title to your figure
    han=axes(fig,'visible','off');
    han.Title.Visible='on';
    han.XLabel.Visible='on';
    han.YLabel.Visible='on';
    yl = ylabel(han,'Angular Velocity [rad/s]');
    yl.Position(1) = -0.07; % change horizontal position of ylabel
    xlabel(han,'time [s]');
saveas(fig,'plots/angVelocity.png')

%% Verification

% MoI
ml = params.M * params.L^2;
I1 = ml * diag([1/125, 13/125, 13/125]);
I2 = ml * diag([1/12, 0, 1/12]);
I3 = ml * diag([1/12, 1/12, 0]);
I = I1 + I2 + I3;
```

```

% Angular Momentum
L = zeros(length(t),3);
for i = 1:length(t)
    L(i,:) = I * res(i,10:12)';
end

fig = figure(Renderer="painters",Position=[60 60 900 800]);
subplot(3,1,1)
    plot(t,L(:,1),DisplayName="$L_x$")
    grid on; grid minor; box on; legend;
subplot(3,1,2)
    plot(t,L(:,2),DisplayName="$L_y$")
    grid on; grid minor; box on; legend;
subplot(3,1,3)
    plot(t,L(:,3),DisplayName="$L_z$")
    grid on; grid minor; box on; legend;
    % Give common xlabel, ylabel and title to your figure
    han=axes(fig,'visible','off');
    han.Title.Visible='on';
    han.XLabel.Visible='on';
    han.YLabel.Visible='on';
    yl = ylabel(han,'Angular Momentum [kg-$m^2$/s]');
    yl.Position(1) = -0.07; % change horizontal position of ylabel
    xlabel(han,'time [s]');
saveas(fig,'plots/angMomentum.png')

fprintf("Variance of Lx: %.4e\n",var(L(:,1)));
fprintf("Variance of Ly: %.4e\n",var(L(:,2)));
fprintf("Variance of Lz: %.4e\n",var(L(:,3)));


%% Functions

function dzdt = spaceRobot(t,z,params)
    % Unpack parameters
    ty0 = params.theta_y0; Wy = params.Omega_y;
    tz0 = params.theta_z0; Wz = params.Omega_z;

    % Prescribed angles of the arms
    ty = ty0*sin(Wy*t);
    tz = tz0*(1 - cos(Wz*t));
    % dty = ty0*Wy*cos(Wy*t);
    dtz = tz0*Wz*sin(Wz*t);
    ddtz = tz0*Wz^2*cos(Wz*t);

    % Generalized coordinates
    % xB, yB, zB
    % q1 = z(1); q2 = z(2); q3 = z(3);
    % roll, pitch, yaw
    % q4 = z(4);
    q5 = z(5); q6 = z(6);

    % Generalized velocities
```

```matlab
    % Velocity of CoM
    u1 = z(7); u2 = z(8); u3 = z(9);
    % Angular velocity
    q4dot = z(10); q5dot = z(11); q6dot = z(12);
    u4 = q4dot*cos(q5)*cos(q6) + q5dot*sin(q6);
    u5 = -q4dot*cos(q5)*sin(q6) + q5dot*cos(q6);
    u6 = q4dot*sin(q5) + q6dot;

    % Preallocate output
    dzdt = zeros(12,1);

    % Kinematic differential equations
    dzdt(1:6) = [u1; u2; u3; u4; u5; u6];

    % Some additional variables
    beta = 5*sin(tz)+1;
    gamma = 1-cos(tz);
    delta = 1-cos(ty);

    % Dynamical differential equations
    E = zeros(3,3);
    E(1,1) = beta^2/400 - 131/750;
    E(1,2) = beta*gamma/80;
    E(1,3) = sin(ty)*gamma/16;
    E(2,1) = beta*(gamma-delta)/80;
    E(2,2) = gamma*(gamma-delta)/16 + 281/1500*delta;
    E(2,3) = beta*sin(ty)/80;
    E(3,1) = 0;
    E(3,2) = 0;
    E(3,3) = gamma^2/16 - beta^2/300 - gamma*delta/16 - 281/1500;

    dzdt(7:9) = 0;
    dzdt(10) = beta*dtz/80*(2*cos(tz)*u4 + 2*sin(tz)*u5 + sin(ty)*u6) + sin(ty)*sin(tz)*dtz/8*u6 ...
        + sin(ty)*gamma/16*u4*u5 + beta^2/400*u5*u6 - beta*gamma/80*u6*u4 ...
        + sin(ty)*beta/80*(u4^2 + u6^2) + sin(ty)/16*(2*sin(tz)*dtz^2 - cos(tz)*ddtz);
    dzdt(11) = (gamma-delta)*dtz/8*(cos(tz)*u4 + sin(tz)*u5) + sin(tz)/16*(gamma*dtz - 2*cos(tz)*dtz)*u6 ...
        + sin(ty)*beta/80*u4*u5 + beta*(gamma-delta)/80*u5*u6 + (19/1500 - gamma*(gamma-delta)/16)*u6*u4 ...
        + sin(ty)*gamma/16*(u5^2 + u6^2) + sin(ty)*(cos(tz)*dtz^2 + sin(tz)*ddtz)/16;
    dzdt(12) = 1/80*(dtz*beta*(gamma-delta) + 10*sin(tz)*dtz*(gamma-delta) - 5*dtz*beta*gamma + 10*cos(tz)*dtz*beta)*u6 ...
        + (gamma^2/16 - beta^2/400 - gamma*delta/16 - 19/1500)*u4*u5 + beta/80*((gamma-delta)*u4^2 - gamma*u5^2 - delta*u6^2) ...
        + beta/80*(2*cos(tz)*dtz^2 + sin(tz)*ddtz) + (2*sin(tz)*dtz^2 - cos(tz)*ddtz)*(gamma-delta)/16;

    dzdt(10:12) = E * dzdt(10:12);

end
```