Well done, take a look at the couple of minor comments.

College of Engineering
School of Aerospace Engineering

Fall2022 AE6230: Structural Dynamics

Homework 2

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Table of Contents

I Problem One

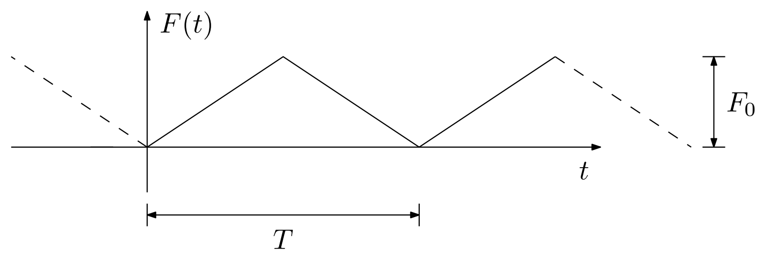

Figure 1: Periodic excitation applied to a single-degree-of-freedom system.

Consider a single-degree-of-freedom system subject to the periodic excitation in Fig. 1, with parameters given in Table 1. Answer the following questions:

1. Determine the expressions of the coefficients of the Fourier series representation of $F(t)$

$$\mathcal{F}(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos p\omega_0 t + \sum_{p=1}^{\infty} b_p \sin p\omega_0 t \tag{I.1}$$

2. Plot the discrete frequency spectrum associated with Eq. (I.1)

$$c_p = \sqrt{a_p^2 + b_p^2} \text{ vs. } p \tag{I.2}$$

   for $p = 0, \ldots, 12$;
3. Determine how many terms must be kept in Eq. (I.1) such that the highest-order harmonic has an amplitude below $0.05F_0$, $0.025F_0$, and $0.005F_0$;
4. Plot the truncated Fourier series representations of $F(t)$ identified via the convergence study in Question 3 against the true function in Fig. 1 for $t \in [0, T]$;
5. Determine the expression of the steady-state response of the system $x(t)$ subject to $F(t)$;
6. Plot the discrete frequency spectrum for $x(t)$ (that is, the amplitude of each harmonic) for $p = 0, \ldots, 12$;
7. Plot $x(t)$ for each truncated Fourier series representation of $F(t)$ identified in Question 3 for $t \in [0, T]$;
8. Motivate the trends observed in the plots for Questions 2, 4, 6, 7 for increasing p.

Guidelines:

- **Questions 1 and 5:** do not substitute the values of the parameters for these questions;
- **Question 3:** you can solve this analytically or numerically (or both).

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Excitation peak value	F_0	1 N
Excitation period	T	0.2 s
Natural frequency	ω_n	$5\omega_0$
Viscous damping factor	ζ	0.05
Stiffness constant	k	10 N/m

Solution

Question (1)

By observing Figure 1, we can tell that the given function is a piecewise linear function. This function can be represented as

$$F(t) = \begin{cases} \frac{2F_0}{T}t & 0 \le t \le \frac{T}{2} \\ -\frac{2F_0}{T}t + 2F_0 & \frac{T}{2} \le t \le T \end{cases} \tag{I.3}$$

The Fourier series representation of this function is defined as

$$\mathcal{F}(t) = \sum_{p=-\infty}^{\infty} a_p e^{-ip\omega_0 t} \quad \text{where} \quad \omega_0 = \frac{2\pi}{T}, \quad a_p = \frac{1}{T}\int_0^T e^{ip\omega_0 t} F(t)dt. \tag{I.4}$$

The coefficient a_p can be calculated for each $p = \{0, 1, 2, ...\}$. For example,

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T F(t)dt = \frac{1}{T}\int_0^{T/2} \frac{2F_0}{T} t dt + \frac{1}{T}\int_{T/2}^T \left(-\frac{2F_0}{T}t + 2F_0\right) dt \\ &= \frac{1}{T}\left[\frac{F_0}{T}t^2\right]_0^{T/2} + \frac{1}{T}\left[-\frac{F_0}{T}t^2 + 2F_0 t\right]_{T/2}^T = \frac{F_0}{2}. \end{aligned} \tag{I.5}$$

The remaining coefficients for $p > 0$ are calculated with the same procedure as a_0.

$$\begin{aligned} a_p &= \frac{1}{T}\int_0^{T/2} e^{ip\omega_0 t}\left(\frac{2F_0}{T}t\right) dt + \frac{1}{T}\int_{T/2}^T e^{ip\omega_0 t}\left(-\frac{2F_0}{T}t + 2F_0\right) dt \\ &= -\frac{F_0\left(2 - 2e^{\frac{ip\omega_0 T}{2}} + ipTe^{\frac{ip\omega_0 T}{2}}\right)}{T^2p^2\omega_0^2} + \frac{F_0 e^{\frac{ip\omega_0 T}{2}}\left(2 - 2e^{\frac{ip\omega_0 T}{2}} + ip\omega_0 T\right)}{T^2p^2\omega_0^2} \\ &= -\frac{2F_0}{T^2p^2\omega_0^2}\left(e^{ip\omega_0 T} - 2e^{\frac{ip\omega_0 T}{2}} + 1\right). \end{aligned}$$

Now if we expand the exponential terms with Euler's formula, we obtain

$$a_p = -\frac{2F_0}{T^2p^2\omega_0^2}\left(\cos(p\omega_0 T) + i\sin(p\omega_0 T) - 2\cos\left(\frac{p\omega_0 T}{2}\right) - 2i\sin\left(\frac{p\omega_0 T}{2}\right) + 1\right). \tag{I.6}$$

Let the coefficients for the Fourier Series be represented as α_k and β_p for the cosine and sine terms respectively. Then, it follows that

$$\alpha_p = 2\mathfrak{Re}(a_p) = -\frac{4F_0}{T^2p^2\omega_0^2}\left(\cos(p\omega_0 T)) - 2\cos\left(\frac{p\omega_0 T}{2}\right) + 1\right) \tag{I.7}$$

$$\beta_p = 2\mathfrak{Im}(a_p) = -\frac{4F_0}{T^2p^2\omega_0^2}\left(\sin(p\omega_0 T) - 2\sin\left(\frac{p\omega_0 T}{2}\right)\right) \tag{I.8}$$

The code used to compute the coefficients is in the MATLAB code section IV.i. Then, from the (I.5), (I.7), and (I.8) we can formulate the Fourier series representation as

$$\mathcal{F}(t) = \frac{F_0}{2} + \sum_{p=1}^{\infty} \alpha_p \cos\left(\frac{2\pi p}{T}t\right) + \sum_{p=1}^{\infty} \beta_p \sin\left(\frac{2\pi p}{T}t\right) \tag{I.9}$$

Question (2)

The discrete frequency spectrum is plotted using MATLAB in IV.i.

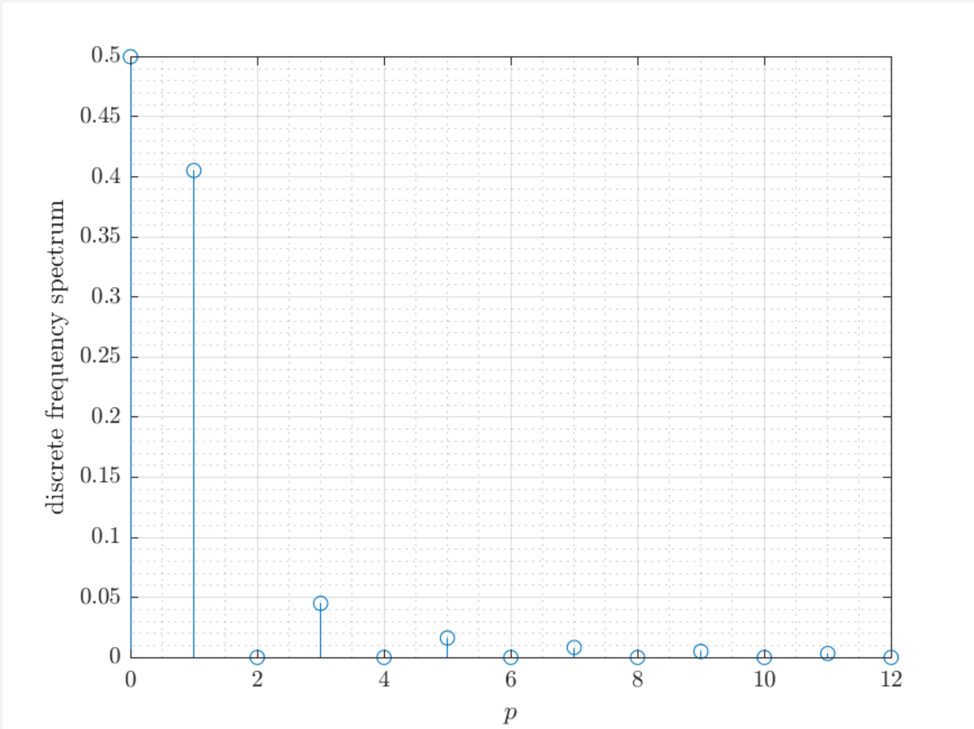

Figure 2: Discrete frequency spectrum over $p = 0, 1, \ldots, 12$.

Question (3)

If we plot the discrete frequency spectrum for more values of p, we have

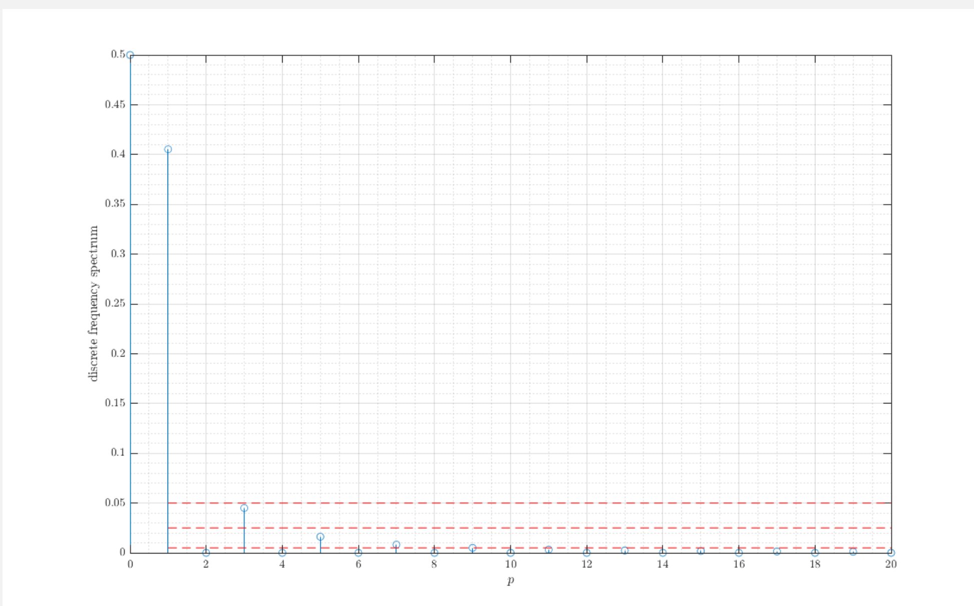

Figure 3: Discrete frequency spectrum with $0.05F_0$, $0.025F_0$, and $0.005F_0$ lines.

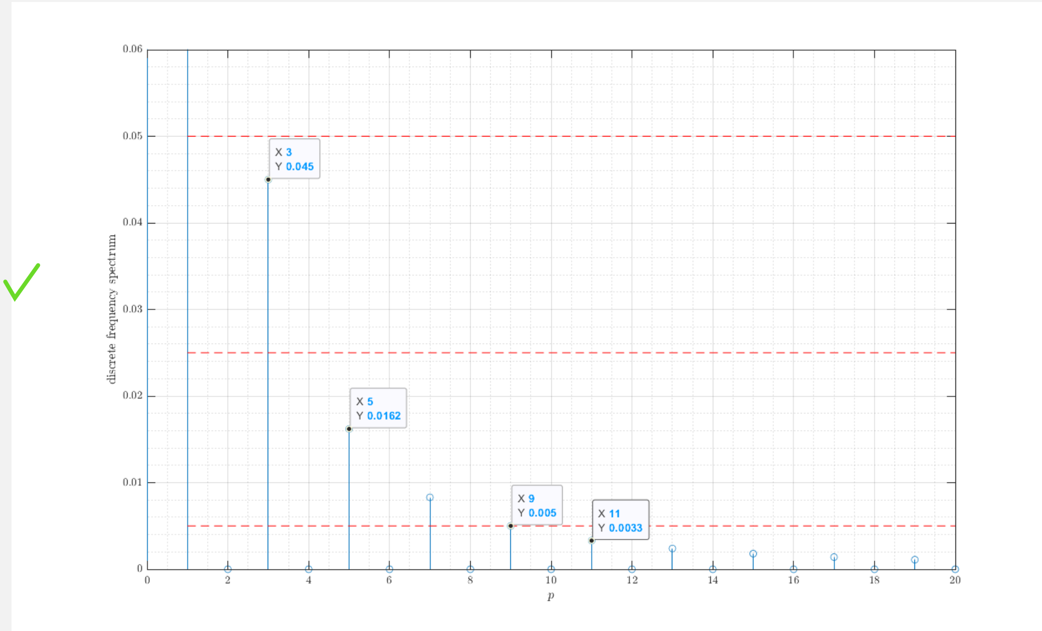

Figure 4: Close up view of Figure 3.

For this problem we want to find the highest order harmonic, p which has an amplitude below 0.05, 0.025, and 0.005 since $F_0 = 1$. From Figure 3 and 4, we can tell that those terms are $p = 3,\ 5,\ 11$ respectively.

Question (4)

The time step is a bit too coarse.

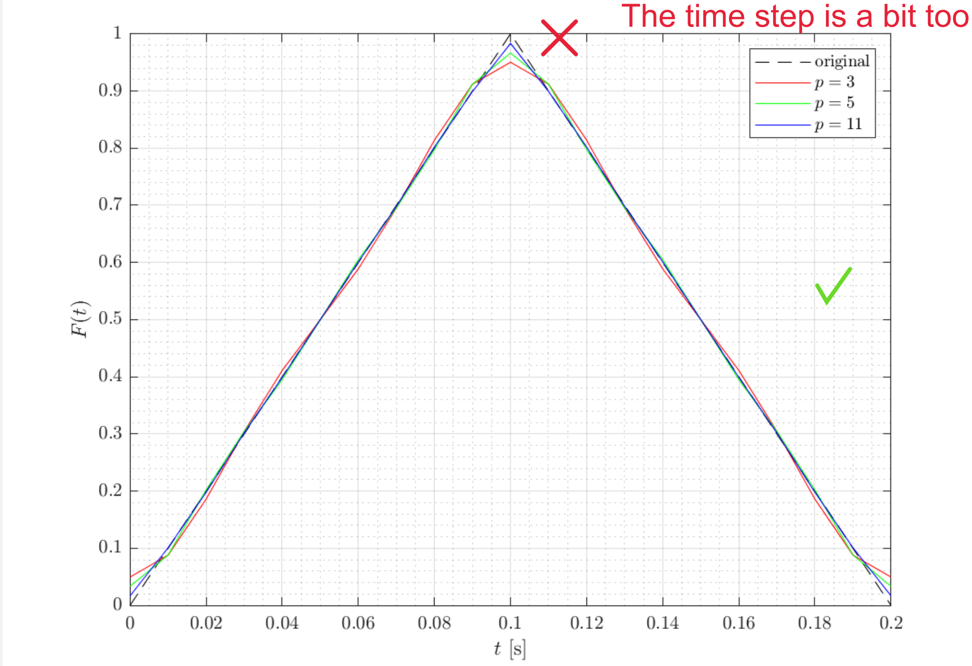

Figure 5: Plots of original $F(t)$ and truncated Fourier series for $p = 3, 5, 11$.

Question (5)

The equation of motion for a SDOF with periodic excitation is represented as

$$m\ddot{x}(t) + c\dot{x}(t) + kx = \frac{F_0}{2} + \sum_{p=1}^{\infty} \alpha_p \cos(p\omega_0 t) + \sum_{p=1}^{\infty} \beta_p \sin(p\omega_0 t). \tag{I.10}$$

The steady state response for this system takes the form of

$$x_p(t) = x_{a_0}(t) + \sum_{p=1}^{\infty} \left[x_{\alpha_p}(t) + x_{\beta_p}(t) \right], \tag{I.11}$$

where

$$x_{a_0}(t) = \frac{F_0}{2k}, \qquad x_{\alpha_p} = \alpha_p |H(i\omega)| \cos\left[\omega t - \theta(\omega)\right]\Big|_{\omega = p\omega_0}, \qquad x_{\beta_p} = \beta_p |H(i\omega)| \sin\left[\omega t - \theta(\omega)\right]\Big|_{\omega = p\omega_0}$$

Hence, knowing the expressions of α_p, β_p and the complex frequency response relations for viscously damped SDOT system, the steady state response becomes

$$x_p(t) = \frac{F_0}{2k} + \sum_{p=1}^{\infty} \left[\alpha_p |H(i\omega)| \cos\left[\omega t - \theta(\omega)\right]\Big|_{\omega = p\omega_0} + \beta_p |H(i\omega)| \sin\left[\omega t - \theta(\omega)\right]\Big|_{\omega = p\omega_0} \right], \tag{I.12}$$

where

$$\alpha_p = -\frac{4F_0}{T^2 p^2 \omega_0^2} \left(\cos(p\omega_0 T)) - 2\cos\left(\frac{p\omega_0 T}{2}\right) + 1 \right) \tag{I.13}$$

$$\beta_p = -\frac{4F_0}{T^2 p^2 \omega_0^2} \left(\sin(p\omega_0 T) - 2\sin\left(\frac{p\omega_0 T}{2}\right) \right) \tag{I.14}$$

$$|H(i\omega)| = \frac{1}{k} \left[\left(1 - \omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2 \right]^{-\frac{1}{2}} \tag{I.15}$$

$$\theta(\omega) = \arctan\left(\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2} \right) \tag{I.16}$$

Question (6)

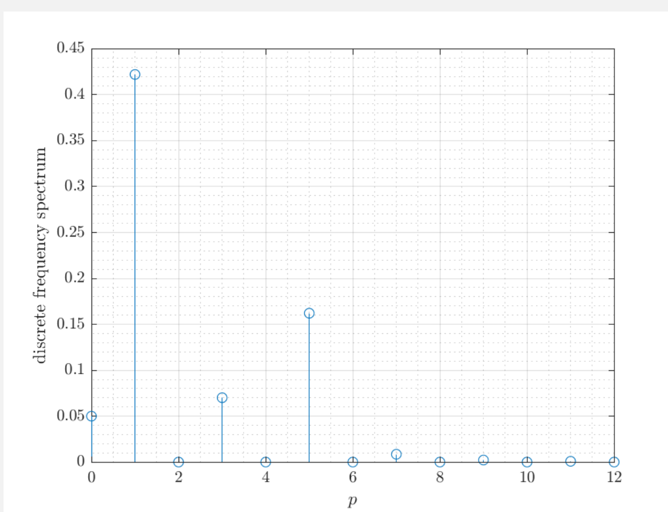

Figure 6: Discrete frequency spectrum of d_p with respect to the steady state response.

The discrete frequency spectrum for the steady state response is in terms of the coefficients of the Fourier series and the amplitude of the complex frequency response, i.e.

$$\gamma_p = \alpha_p |H(i\omega)| \qquad \delta_p = \beta_p |H(i\omega)|. \tag{I.17}$$

Then

$$d_p = \sqrt{\gamma_p^2 + \delta_p^2} = |H(i\omega)| c_p. \tag{I.18}$$

The plotted result for $p = 0, 1, ..., 12$ is shown in Figure 6.

Question (7)

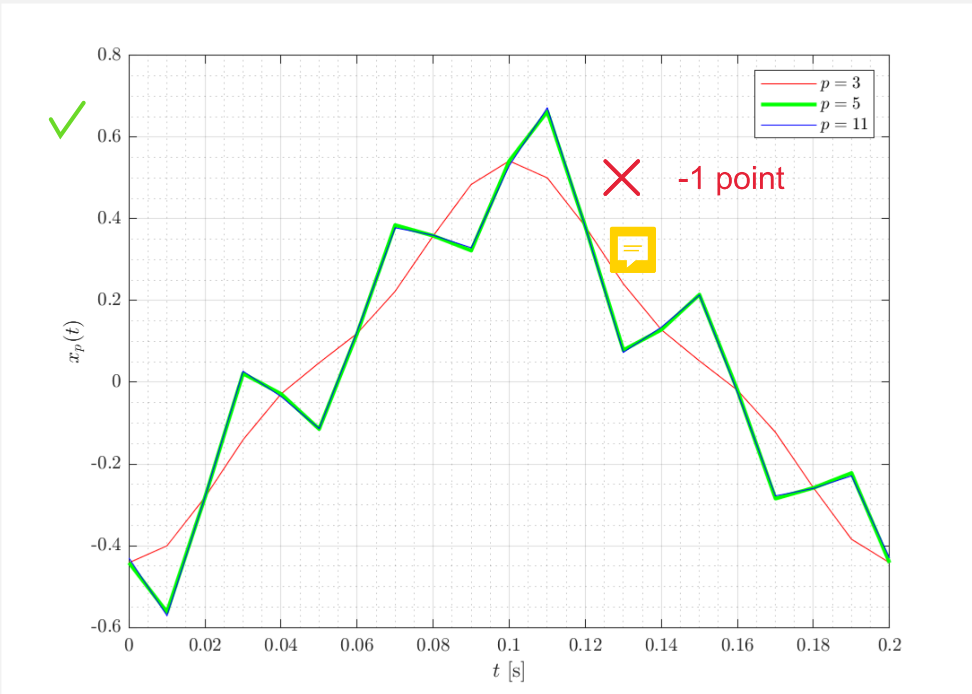

Figure 7: Steady state responses for periodic excitation of piecewise linear function for $p = 3, 5, 11$.

Question (8)

From the discrete frequency spectrum plots in Question 5 and 6 we observe that past $p = 11$ the amplitude of the higher order harmonics are very low that they barely contribute to the overall function. Thus, even if we increase the value of p our approximation of the periodic excitation nor the steady state response will improve. As we can observe in Figure 7, $p = 5$ and $p = 11$ are quite identical with small discrepancies.

II Problem Two

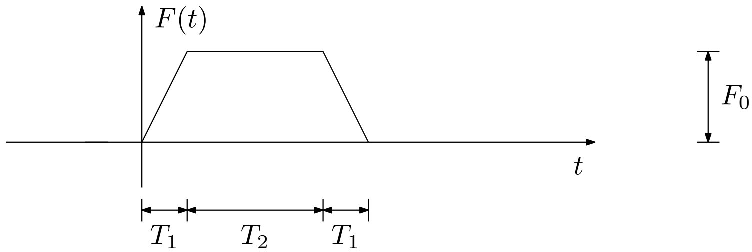

Figure 8: Trapezoidal input applied to a single-degree-of-freedom system.

Consider a single-degree-of-freedom system subject to the input in Fig. 8, with parameters in Table 2. Assuming the system at rest for $t \leq 0$ and neglecting damping, answer the following questions:

1. Using the convolution integral, show that the response for $0 \leq t \leq T_1$ is given by

$$x(t) = \frac{x_s}{T_1 \omega_n} \left(\omega_n t - \sin \omega_n t\right) \tag{II.1}$$

   where $\omega_n = 2\pi/T_n$ is the natural frequency of the system and $x_s = F_0/k$ is the response for a static input having the same amplitude as the trapezoidal input in Fig. 8;
2. Considering the other time intervals $T_1 \leq t \leq T_1 + T_2$, $T_1 + T_2 \leq t \leq 2T_1 + T_2$, and $t \geq 2T_1 + T_2$
   (a) Explain the approach you pursue to determine $x(t)$;
   (b) Derive the expression of $x(t)$ specialized to each time interval;
3. Plot $x(t)/x_s$ for $T_1 = 0.1T_n, 0.5T_n, T_n, 1.5T_n, 2T_n, 2.5T_n$ for $t \in [0, 1.5]$ s and $x(t)/x_s \in [-2, 2]$;
4. Determine the maximum value of $x(t)/x_s$ in the time interval $T_1 \leq t \leq T_1 + T_2$ as a function of T_1/T_n;
5. Plot the result from Question 4 for $T_1/T_n \in [0, 4]$;
6. Discuss the trends in the results for Question 3 and 5.

Guidelines:

- **Question 4:** you can use the plots from Question 3 to check the results for this question.

Table 2: Parameter values for Problem 2.

Parameter	Symbol	Value
Time length of constant input	T_2	0.5 s
Natural frequency of the system	ω_n	20π rad/s

Solution

Question (1)

This problem involves an arbitrary excitation. To solve this analytically, we first convert the equation to the Laplace domain and then revert it back to the time domain using the convolution integral for the inverse Laplace transformation. The equation of motion is simply

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t). \tag{II.2}$$

The trapezoidal input is represented as

$$F(t) = \begin{cases} \frac{F_0}{T_1}t & 0 \le t \le T_1 \\ F_0 & T_1 \le t \le T_1 + T_2 \\ -\frac{F_0}{T_1}t + \frac{2T_1+T_2}{T_1}F_0 & T_1 + T_2 \le t \le 2T_1 + T_2 \\ 0 & t \in (-\infty, 0) \cup (2T_1 + T_2, \infty) \end{cases} \tag{II.3}$$

Now the general solution of (II.2) in the Laplace domain is

$$X(s) = x_0 \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} + v_0 \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} + F(s)\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{II.4}$$

Taking the inverse Laplace transformation we have

$$x(t) = \mathcal{L}^{-1}[X(s)] = e^{-\zeta\omega_n t}\left(x_0 \cos\omega_d t + \frac{v_0 + x_0\zeta\omega_n}{\omega_d}\sin\omega_d t\right) + \mathcal{L}^{-1}[F(s)H(s)], \tag{II.5}$$

where $H(s)$ is the impulse response function $h(t)$ in the Laplace domain. Now since this problem neglects damping we set $\zeta = 0$ and $\omega_d = \omega_n$ which gives us

$$x(t) = x_0 \cos\omega_n t + \frac{v_0}{\omega_n}\sin\omega_n t + \mathcal{L}^{-1}[F(s)H(s)], \tag{II.6}$$

Here we use the convolution integral to solve

$$\mathcal{L}^{-1}[F(s)H(s)] = \int_0^t F(\tau)h(t-\tau)d\tau = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)d\tau. \tag{II.7}$$

Hence, the particular solution or the steady state response for $0 \le t \le T_1$ of this system can be found by computing

$$\begin{aligned} x_1(t) &= \frac{1}{m\omega_n}\int_0^t \frac{F_0}{T_1}\tau \sin\omega_n(t-\tau)d\tau = \frac{F_0}{T_1 m\omega_n^3}(\omega_n t - \sin\omega_n t) \\ &= \frac{x_s k}{T_1 m\omega_n \frac{k}{m}}(\omega_n t - \sin\omega_n t) = \frac{x_s}{T_1\omega_n}(\omega_n t - \sin\omega_n t) \end{aligned} \tag{II.8}$$

This was solved using MATLAB. The code can be found in IV.ii.

Question (2)

(a) For the other time intervals we superimpose functions to achieve the proper piecewise function in (II.3) for each time interval. Particularly, we consider the following four functions

$$F_1(t) = \frac{F_0}{T_1}t \qquad 0 < t \tag{II.9}$$

$$F_2(t) = \begin{cases} 0 & 0 < t < T_1 \\ -F_0\frac{t-T_1}{T_1} & T_1 \le t \end{cases} \tag{II.10}$$

$$F_3(t) = \begin{cases} 0 & 0 < t < T_1 + T_2 \\ -\frac{F_0}{T_1}t + \frac{T_1+T_2}{T_1}F_0 & T_1 + T_2 \le t \end{cases} \tag{II.11}$$

$$F_4(t) = \begin{cases} 0 & 0 < t < 2T_1 + T_2 \\ \frac{F_0}{T_1}t - \frac{2T_1+T_2}{T_1}F_0 & 2T_1 + T_2 \leq t \end{cases} \tag{II.12}$$

We can observe that the relationship $F(t) = F_1(t) + F_2(t) + F_3(t) + F_4(t)$ is satisfied. Now for each time intervals we can compute the response $x_i(t)$ by computing the convolution interval. Hence, we have

$$\begin{aligned} x_2(t) &= \frac{1}{\omega_n}\int_0^t F_2(\tau)\sin\omega_n(t-\tau)d\tau = \frac{1}{\omega_n}\int_{T_1}^t -F_0\frac{\tau - T_1}{T_1}\sin\omega_n(t-\tau)d\tau \\ &= \frac{F_0}{m\omega_n^3}\left[\sin\omega_n(t-T_1) - \omega_n(t-T_1)\right] = \frac{x_s}{T_1\omega_n}\left[\sin\omega_n(t-T_1) - \omega_n(t-T_1)\right] \end{aligned} \tag{II.13}$$

$$\begin{aligned} x_3(t) &= \frac{1}{\omega_n}\int_0^t F_3(\tau)\sin\omega_n(t-\tau)d\tau = \frac{1}{\omega_n}\int_{T_1+T_2}^t \left(-\frac{F_0}{T_1}\tau + \frac{T_1+T_2}{T_1}F_0\right)\sin\omega_n(t-\tau)d\tau \\ &= \frac{F_0}{m\omega_n^3}\left[\sin\omega_n(t-T_1-T_2) - \omega_n(t-T_1-T_2)\right] \\ &= \frac{x_s}{T_1\omega_n}\left[\sin\omega_n(t-T_1-T_2) - \omega_n(t-T_1-T_2)\right] \end{aligned} \tag{II.14}$$

$$\begin{aligned} x_4(t) &= \frac{1}{\omega_n}\int_0^t F_4(\tau)\sin\omega_n(t-\tau)d\tau = \frac{1}{\omega_n}\int_{2T_1+T_2}^t \left(\frac{F_0}{T_1}\tau - \frac{2T_1+T_2}{T_1}F_0\right)\sin\omega_n(t-\tau)d\tau \\ &= \frac{F_0}{m\omega_n^3}\left[\omega_n(t-2T_1-T_2) - \sin\omega_n(t-2T_1-T_2)\right] \\ &= \frac{x_s}{T_1\omega_n}\left[\omega_n(t-2T_1-T_2) - \sin\omega_n(t-2T_1-T_2)\right] \end{aligned} \tag{II.15}$$

(b) Now that we have computed the response for each $F_i(t)$, the total expression of $x(t)$ specialized for each time interval is given by the sum of $x_i(t)$ or

$$x(t) = \begin{cases} 0 & t < 0 \\ x_1(t) & 0 \leq t < T_1 \\ x_1(t) + x_2(t) & T_1 \leq t < T_1 + T_2 \\ x_1(t) + x_2(t) + x_3(t) & T_1 + T_2 \leq t < 2T_1 + T_2 \\ x_1(t) + x_2(t) + x_3(t) + x_4(t) & 2T_1 + T_2 \leq t \end{cases} \tag{II.16}$$

Which is with full expression

$$x(t) = \begin{cases} 0 & (-\infty, 0) \\ \frac{x_s}{T_1\omega_n}(\omega_n t - \sin\omega_n t) & [0, T_1) \\ \frac{x_s}{T_1\omega_n}(\omega_n T_1 + \sin\omega_n(t-T_1) - \sin\omega_n t) & [T_1, T_1+T_2) \\ \frac{x_s}{T_1\omega_n}(\sin\omega_n(t-T_1-T_2) + \sin\omega_n(t-T_1) - \sin\omega_n t - \omega_n(t-2T_1-T_2)) & [T_1+T_2, 2T_1+T_2) \\ \frac{x_s}{T_1\omega_n}(\sin\omega_n(t-T_1-T_2) - \sin\omega_n(t-2T_1-T_2) + \sin\omega_n(t-T_1) - \sin\omega_n t) & [2T_1+T_2, \infty) \end{cases} \tag{II.17}$$

Question (3)

The plot is as follows.

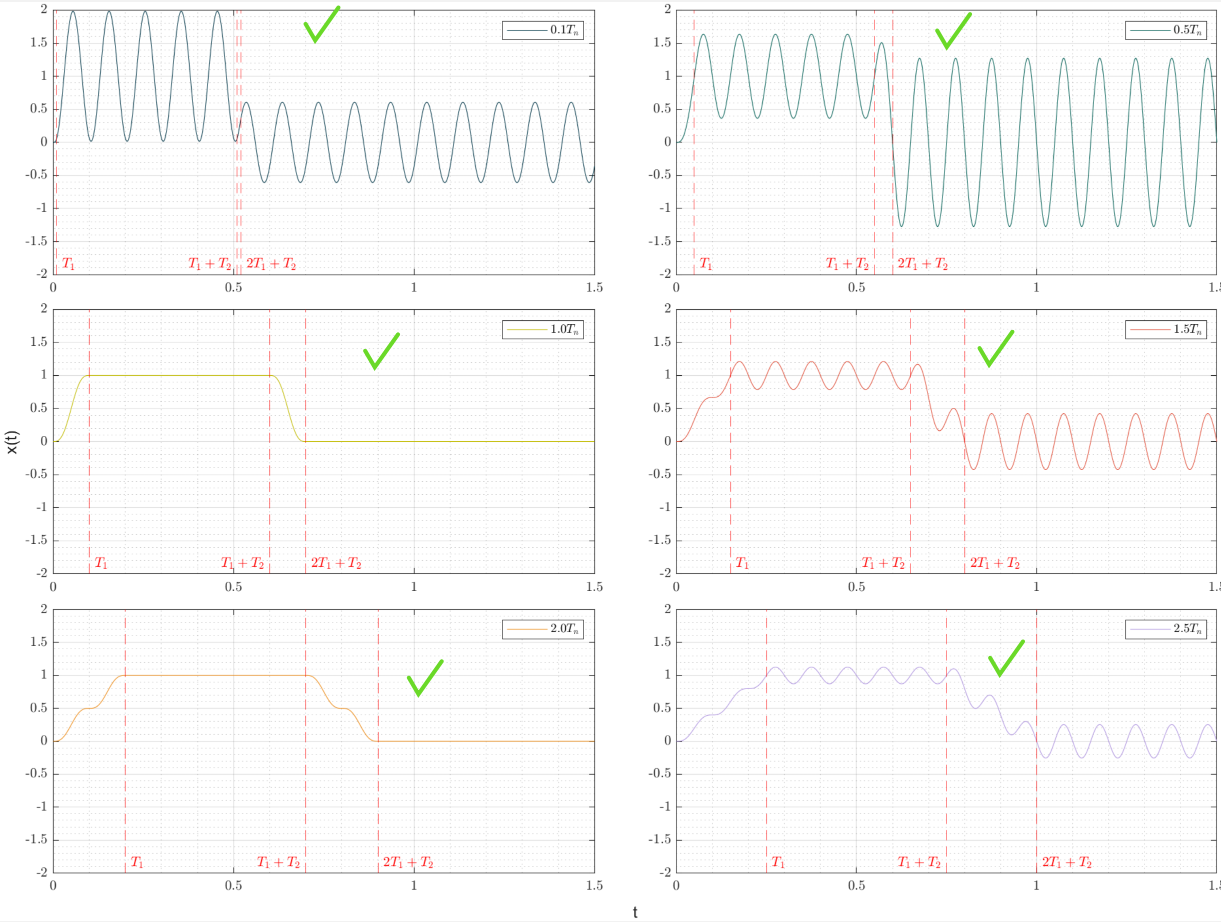

Figure 9: Plots for $x(t)/x_s$ for different T_1 values for the trapezoidal excitation.

Question (4)

From (II.17), we have the reponse for the interval of $t \in [T_1, T_1 + T_2)$. Then the maximum value of this functions is

$$\begin{aligned}
&\frac{1}{T_1\omega_n}(\omega_n T_1 + \sin\omega_n(t - T_1) - \sin\omega_n t) \\
&= 1 + \frac{1}{T_1\omega_n}\sin(\omega_n t - \omega_n T_1 - \omega_n t)\cos(\omega_n t - \omega_n T_1 + \omega_n t) \\
&= 1 - \frac{1}{T_1\omega_n}\sin\omega_n T_1 \cos\omega_n(2t - T_1) \\
&\leq 1 + \frac{1}{2\pi(T_1/T_n)}\sin 2\pi(T_1/T_n)
\end{aligned} \tag{II.18}$$

-1 point

Hence,

$$\max_{t\in[T_1,T_1+T_2]} x(t) = A(T_1/T_n) = 1 + \frac{\sin 2\pi(T_1/T_n)}{2\pi(T_1/T_n)}. \tag{II.19}$$

Question (5)

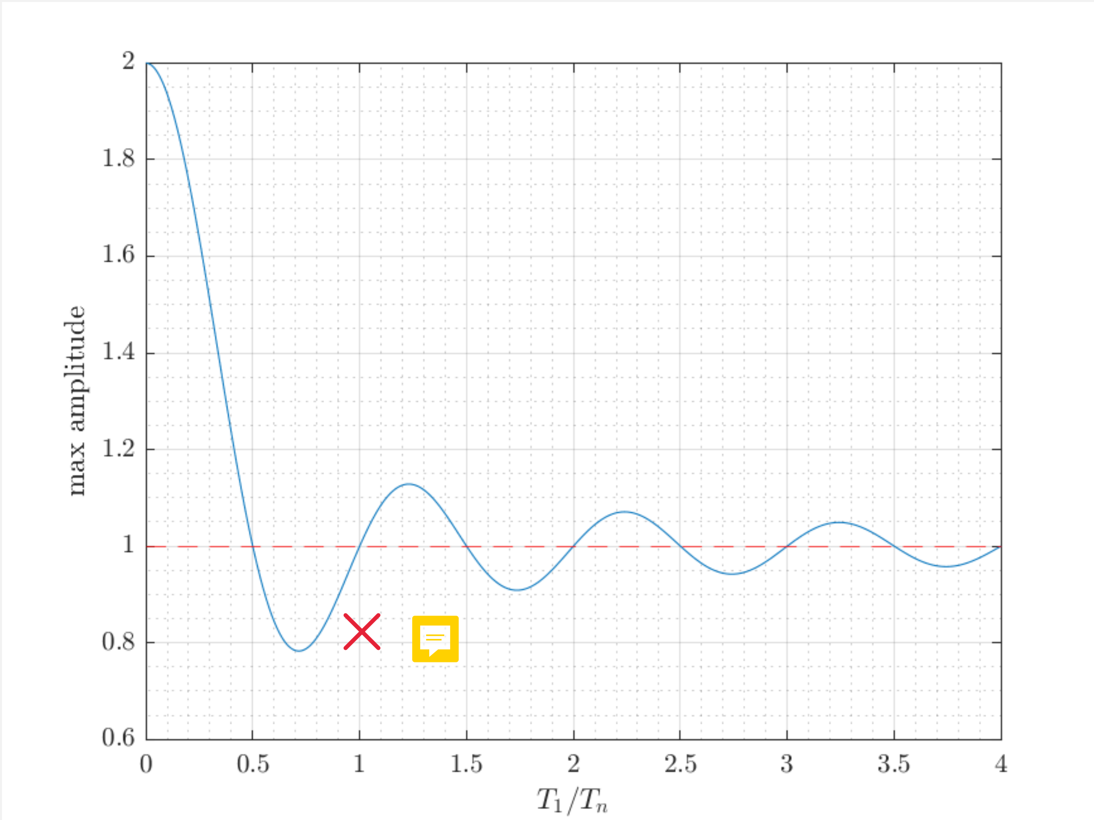

Figure 10: Maximum amplitude of the response function $x(t)$ for $t \in [T_1, T_1 + T_2]$ for $T_1/T_n \in [0, 4]$

Question (6)

- From Figures 9 and 10, we can see that the response in the time interval of $t \in [T_1, T_1 + T_2]$ becomes a constant value when $T_1 = jT_n$ where $j \in \mathbb{N} \setminus \emptyset = \mathbb{Z}^+$. When this does not apply the same interval becomes a sinusoidal wave and the peak amplitude decreases as T_1/T_n becomes larger.
- Another observation we can make is that, when $T_1/T_n < 1$ the amplitude of the waves become relatively large. Whereas when this ratio exceeds the value of 1 the amplitude begins to decrease in the interval of $t \in [T_1, \infty]$.
- When T_1 is an exact multiple (multiple of a positive integer value) of T_n another interesting fact is that the response completely dies out after $t = 2T_1 + T_2$. This is true because from (II.17) time interval $t \in [2T_1 + T_2, \infty)$, the sinusoidal waves cancel each other out evenly resulting in 0.

The derivations and visualizations were done using MATLAB in IV.ii.

III Problem Three

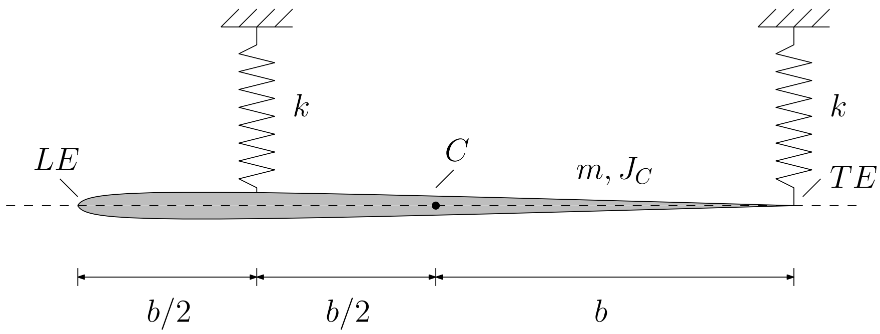

Figure 11: Schematic of wind-tunnel wing model undergoing plunge and pitch vibrations.

Consider a rigid wing mounted in a wind-tunnel test section (Fig. 11). The wing undergoes plunge (vertical translation) and pitch vibrations, which are restrained by two springs attached to the quarter-chord and trailing-edge points as shown in Fig. 11. The wing has mass m and pitch moment of inertia J_C about the center of mass C, located at the half-chord point. The chord has length $2b$ and the two springs both have spring constant k. The motion is described by choosing the vertical translations of the leading-edge and trailing-edge points, denoted by $h_{LE}(t)$ and $h_{TE}(t)$, as the generalized coordinates. The traslations are assumed to be positive in the upward direction and are measured from the horizontal configuration in Fig. 11. Neglecting gravity and assuming small-amplitude motions, answer the following questions:

1. Write the kinetic and potential energies of the system as functions of $h_{LE}(t)$ and $h_{TE}(t)$;
2. Derive the equations of motion in the matrix form

$$\mathbf{M}\begin{Bmatrix}\ddot{h}_{LE}(t)\\ \ddot{h}_{TE}(t)\end{Bmatrix} + \mathbf{K}\begin{Bmatrix}h_{LE}(t)\\ h_{TE}(t)\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix} \tag{III.1}$$

   using Lagrange's equations;
3. Define a coordinate transformation that results in inertial decoupling (but not necessarily elastic decoupling) and derive the corresponding transformation matrix $\mathbf{T}$;
4. Obtain the new mass and stiffness matrices based on the transformation matrix from Question 3;
5. Derive the equations of motion using the Netwonian approach based on the free-body diagram for the system (to be included in the solution) and compare the results with Question 4.

Guidelines:

- **Question 2:** show the steps in the process, not only the final form of the matrices;
- **Question 4:** you can verify the results by obtaining the new matrices directly from Lagrange's equations.

Solution

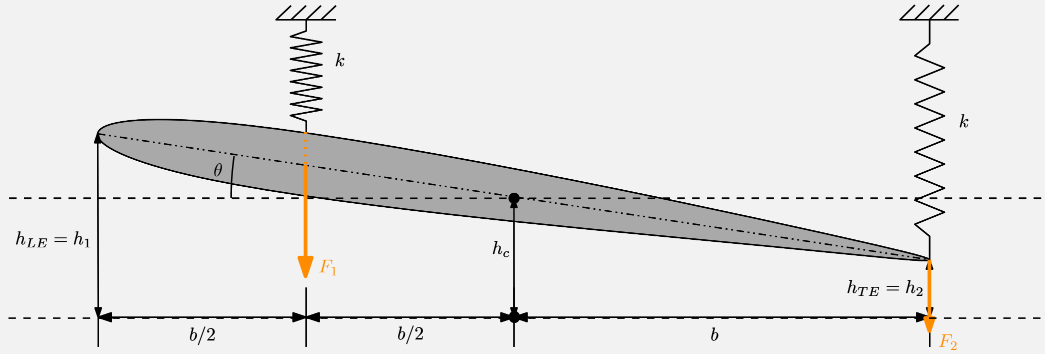

Figure 12: Free-body diagram of the system.

Question (1)

First, let us lay out the assumptions and the constraints for this system. The assumption we have is that the small angle approximation $\theta \ll 1$ holds, and there is one holonomic constraint of $2b\tan\theta = h_{LE} - h_{TE}$. Now let the vertical position of the center of gravity be h_c. Using basic geometry and the holonomic constraint we have

$$h_c = h_{LE} - b\tan\theta = h_{LE} - b \cdot \frac{h_{LE} - h_{TE}}{2b} = \frac{h_{LE} + h_{TE}}{2}. \tag{III.2}$$

Additionally, the constraint gives us

$$\theta = \frac{h_{LE} - h_{TE}}{2b}. \tag{III.3}$$

Then the kinetic energy of this system becomes

$$\begin{aligned} T &= \frac{1}{2}m\dot{h}_c^2 + \frac{1}{2}J_c\dot{\theta}^2 \\ &= \frac{1}{2}m\left(\frac{\dot{h}_{LE} + \dot{h}_{TE}}{2}\right)^2 + \frac{1}{2}J_c\left(\frac{\dot{h}_{LE} - \dot{h}_{TE}}{2b}\right)^2 \\ &= \frac{1}{8}m\left(\dot{h}_{LE} + \dot{h}_{TE}\right)^2 + \frac{1}{8b^2}J_c\left(\dot{h}_{LE} - \dot{h}_{TE}\right)^2. \end{aligned} \tag{III.4}$$

Next, neglecting gravity the potential energy of the system becomes

$$V = \cancelto{0}{-mgh_c} + \frac{1}{2}kh_1^2 + \frac{1}{2}kh_2^2$$

where h_1 and h_2 are the vertical positions measured from the horizontal configuration of the front and back springs respectively. We know that

$$h_1 = h_c + \frac{b}{2}\tan\theta = \frac{3h_{LE} + h_{TE}}{4}, \qquad h_2 = h_{TE}.$$

Thus, the potential energy is

$$\begin{aligned} V &= \frac{1}{2}k\left(\frac{3h_{LE} + h_{TE}}{4}\right)^2 + \frac{1}{2}kh_{TE}^2 \\ &= \frac{1}{32}k(3h_{LE} + h_{TE})^2 + \frac{1}{2}kh_{TE}^2 \end{aligned} \tag{III.5}$$

Question (2)

Since we have derived the expressions for the kinetic and potential energies in (III.4) and (III.5) the Lagrangian amounts to

$$L = T - V = \frac{1}{8}m\left(\dot{h}_{LE} + \dot{h}_{TE}\right)^2 + \frac{1}{8b^2}J_c\left(\dot{h}_{LE} - \dot{h}_{TE}\right)^2 - \frac{1}{32}k(3h_{LE} + h_{TE})^2 - \frac{1}{2}kh_{TE}^2. \tag{III.6}$$

Let $h_1 = h_{LE}$ and $h_2 = h_{TE}$ for simplicity. Then the equations of motion can be found by computing

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{h}_1}\right) - \frac{\partial L}{\partial h_1} = 0 \tag{III.7}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{h}_2}\right) - \frac{\partial L}{\partial h_2} = 0. \tag{III.8}$$

Let us solve (III.7) step-by-step. First we compute

$$\frac{\partial L}{\partial h_1} = -\frac{k}{32}(18h_1 + 6h_2) \tag{III.9}$$

Then,

$$\frac{\partial L}{\partial \dot{h}_1} = \left(\frac{m}{4} + \frac{J_c}{4b^2}\right)\dot{h}_1 + \left(\frac{m}{4} - \frac{J_c}{4b^2}\right)\dot{h}_2, \tag{III.10}$$

and it follows that

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{h}_1}\right) = \left(\frac{m}{4} + \frac{J_c}{4b^2}\right)\ddot{h}_1 + \left(\frac{m}{4} - \frac{J_c}{4b^2}\right)\ddot{h}_2. \tag{III.11}$$

Finally, the expression for (III.7) becomes

$$\left(\frac{m}{4} + \frac{J_c}{4b^2}\right)\ddot{h}_1 + \left(\frac{m}{4} - \frac{J_c}{4b^2}\right)\ddot{h}_2 + \frac{9k}{16}h_1 + \frac{3k}{16}h_2 = 0. \tag{III.12}$$

With the exact same procedure we can find the expression for (III.8)

$$\left(\frac{m}{4} - \frac{J_c}{4b^2}\right)\ddot{h}_1 + \left(\frac{m}{4} + \frac{J_c}{4b^2}\right)\ddot{h}_2 + \frac{3k}{16}h_1 + \frac{17k}{16}h_2 = 0. \tag{III.13}$$

Thus, we have

$$\begin{bmatrix} \frac{m}{4} + \frac{J_c}{4b^2} & \frac{m}{4} - \frac{J_c}{4b^2} \\ \frac{m}{4} - \frac{J_c}{4b^2} & \frac{m}{4} + \frac{J_c}{4b^2} \end{bmatrix}\begin{bmatrix} \ddot{h}_{LE} \\ \ddot{h}_{TE} \end{bmatrix} + \begin{bmatrix} \frac{9k}{16} & \frac{3k}{16} \\ \frac{3k}{16} & \frac{17k}{16} \end{bmatrix}\begin{bmatrix} h_{LE} \\ h_{TE} \end{bmatrix} = 0. \tag{III.14}$$

Question (3)

For our new coordinate system we employ new variables from the relations shown in (III.2) and (III.3). Let the new coordinates be $q_1 = h_c$ and $q_2 = \theta$, then we have

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2b} & -\frac{1}{2b} \end{bmatrix}\begin{bmatrix} h_{LE} \\ h_{TE} \end{bmatrix}. \tag{III.15}$$

Thus the transformation matrix $\mathbf{T}$ becomes

$$\mathbf{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2b} & -\frac{1}{2b} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}. \tag{III.16}$$

Question (4)

The new inertia and stiffness matrices can be computed as follows.

$$\widehat{\mathbf{M}} = \mathbf{T}^\top \mathbf{M} \mathbf{T} = \begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix} \begin{bmatrix} \frac{m}{4} + \frac{J_c}{4b^2} & \frac{m}{4} - \frac{J_c}{4b^2} \\ \frac{m}{4} - \frac{J_c}{4b^2} & \frac{m}{4} + \frac{J_c}{4b^2} \end{bmatrix} \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & J_c \end{bmatrix}. \tag{III.17}$$

Similarly

$$\widehat{\mathbf{K}} = \mathbf{T}^\top \mathbf{M} \mathbf{T} = \begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix} \begin{bmatrix} \frac{9k}{16} & \frac{3k}{16} \\ \frac{3k}{16} & \frac{17k}{16} \end{bmatrix} \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} = \begin{bmatrix} 2k & -\frac{bk}{2} \\ -\frac{bk}{2} & \frac{5b^2k}{4} \end{bmatrix}. \tag{III.18}$$

Question (5)

The free-body diagram is shown in Figure 12. According to the forces, we can find the Newton's equation as follows.

$$\begin{aligned} m\ddot{h}_c &= -F_1 - F_2 = -kh_2 - k(h_c + \frac{b}{2}\tan\theta) \approx -kh_2 - k(h_c + \frac{b}{2}\theta) \\ &= -k(h_c - b\theta) - k(h_c + \frac{b}{2}\theta) = -2kh_c + \frac{bk}{2}\theta. \end{aligned} \tag{III.19}$$

Similarly, the Euler's equation is

$$\begin{aligned} J_c\ddot{\theta} &= -F_1\frac{b}{2}\cos\theta + F_2 b\cos\theta \\ &= -k(h_c + \frac{b}{2}\theta)\frac{b}{2} + k(h_c - b\theta)b \\ &= \frac{bk}{2}h_c - \frac{5b^2k}{4}\theta \end{aligned} \tag{III.20}$$

Hence, we have

$$\begin{aligned} m\ddot{h}_c + 2kh_c - \frac{bk}{2}\theta = 0 \\ J_c\ddot{\theta} - \frac{bk}{2}h_c + \frac{5b^2k}{4}\theta = 0 \end{aligned} \tag{III.21}$$

This equation of motion agrees with the result (III.17) and (III.18) in Question (4). Thus, we have successfully derived the equation of motions. The derivations were done using MATLAB (refer to the code in IV.iii).

IV MATLAB Code

IV.i Problem 1

```
% AE6230 HW2 Problem 1
% Author: Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% Parameters
params.F0 = 1;  % [N]
params.T = 0.2;  % [s]
params.omega0 = 2*pi/params.T;  % [rad/s]
params.omega_n = 5*params.omega0;  % [rad/s]
params.zeta = 0.05;
params.k = 10;  % [N/m]

%% (a)
syms T F_0 k t p l zeta omega_0 omega_n real
assume(T > 0); assume(F_0 > 0); assume(p > 0);

% piecewise function
F1 = 2*F_0/T*t;
F2 = -2*F_0/T*t + 2*F_0;
F(t) = piecewise((0 <= t) & (t <= T/2), F1, (T/2 <= t) & (t <= T), F2);
F(t) = subs(F, [F_0,T], [params.F0,params.T]);

% Fourier Series
a0 = int(F1/T,t,0,T/2) + int(F2/T,t,T/2,T);
integrand1 = exp(1i*p*omega_0*t) * F1;
integrand2 = exp(1i*p*omega_0*t) * F2;
a1 = simplify(expand(int(integrand1 / T, t, 0, T/2)));
a2 = simplify(expand(int(integrand2 / T, t, T/2, T)));
a_k = expand(a1 + a2);
simplify(a_k);
a_k = rewrite(a_k, "sincos");
alpha(p) = 2*real(a_k);
beta(p) = 2*imag(a_k);

%% (b)
c(p) = sqrt(alpha^2 + beta^2);
c(p) = subs(c, [F_0, T, omega_0], [params.F0, params.T, params.omega0]);
pts = 1:12;
fig = figure(Renderer="painters");
    stem([0 pts],[subs(a0,F_0,params.F0) c(pts)])
    xlabel("$p$")
    ylabel("discrete frequency spectrum")
    grid on; grid minor; box on;
saveas(fig, "plots/p1_b.png");
```

```
%% (c)
pts = 1:20;
fig = figure(Renderer="painters");
    stem([0 pts],[subs(a0,F_0,params.F0) c(pts)])
    grid on; grid minor; box on; hold on;
    plot([pts(1) pts(end)], [0.05*params.F0, 0.05*params.F0], '--r')
    plot([pts(1) pts(end)], [0.025*params.F0, 0.025*params.F0], '--r')
    plot([pts(1) pts(end)], [0.005*params.F0, 0.005*params.F0], '--r')
    hold off
    xlabel("$p$")
    ylabel("discrete frequency spectrum")
% saveas(fig, "plots/p1_b.png");

fig = figure(Renderer="painters");
    stem([0 pts],[subs(a0,F_0,params.F0) c(pts)])
    grid on; grid minor; box on; hold on;
    plot([pts(1) pts(end)], [0.05*params.F0, 0.05*params.F0], '--r')
    plot([pts(1) pts(end)], [0.025*params.F0, 0.025*params.F0], '--r')
    plot([pts(1) pts(end)], [0.005*params.F0, 0.005*params.F0], '--r')
    hold off
    xlabel("$p$")
    ylabel("discrete frequency spectrum")
    ylim([0, 0.06])

%% (d)
F_f(t,p) = a0 + symsum(alpha(l)*cos(l*omega_0*t),l,1,p) + ...
  symsum(beta(l)*sin(l*omega_0*t),l,1,p);
F_f(t,p) = subs(F_f, [F_0,T,omega_0], [params.F0,params.T,params.omega0]);

tspan = 0:0.01:params.T;

fig = figure(Renderer="painters", Position=[60 60 700 500]);
    plot(tspan, F(tspan), '--k', DisplayName="original")
    grid on; grid minor; box on; hold on;
    plot(tspan, F_f(tspan,3), '-r', DisplayName="$p=3$")
    plot(tspan, F_f(tspan,5), '-g', DisplayName="$p=5$")
    plot(tspan, F_f(tspan,11), '-b', DisplayName="$p=11$")
    hold off; legend;
    xlabel("$t$ [s]")
    ylabel("$F(t)$")
saveas(fig, "plots/p1_d.png");

%% (e)
omega = p * omega_0;
Hi(p) = ( (1 - omega^2/omega_n^2)^2 + (2*zeta*omega/omega_n)^2 )^-0.5;
Hi(p) = subs(Hi, [omega_n, zeta], [params.omega_n, params.zeta]);

d(p) = Hi * c;
d(p) = subs(d, [F_0, T, omega_0], [params.F0, params.T, params.omega0]);

pts = 1:12;
fig = figure(Renderer="painters");
    stem([0 pts],[subs(a0,F_0,params.F0)/params.k d(pts)])
    xlabel("$p$")
```

```
    ylabel("discrete frequency spectrum")
    grid on; grid minor; box on;
saveas(fig, "plots/p1_e.png");

%% (f)
theta(p) = atan2( (2*zeta*omega/omega_n) , (1 - omega^2/omega_n^2) );

x_alpha(p) = alpha(p) * Hi(p) * cos(omega*t - theta(p));
x_beta(p) = beta(p) * Hi(p) * sin(omega*t - theta(p));
xp(t,p) = F_0/2/k + symsum(x_alpha(l) + x_beta(l), l, 1, p);
xp(t,p) = subs(xp, [F_0, T, omega_0, omega_n, zeta, k], ...
               [params.F0, params.T, params.omega0, params.omega_n, ...
                params.zeta, params.k]);

tspan = 0:0.01:params.T;

fig = figure(Renderer="painters", Position=[60 60 700 500]);
    plot(tspan, xp(tspan,3), '-r', DisplayName="$p=3$")
    grid on; grid minor; box on; hold on;
    plot(tspan, xp(tspan,5), '-g', DisplayName="$p=5$", LineWidth=2)
    plot(tspan, xp(tspan,11), '-b', DisplayName="$p=11$")
    hold off; legend;
    xlabel("$t$ [s]")
    ylabel("$x_p(t)$")
saveas(fig, "plots/p1_f.png");
```

IV.ii Problem 2

```
% AE6230 HW2 Problem 2
% Author: Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% Parameters
params.T2 = 0.5;  % [s]
params.omega_n = 20*pi;  % [rad/s]

%% (1)
syms m omega_d omega_n zeta F_0 T_1 T_2 tau t real positive
F1(tau) = F_0/T_1*tau;
h(tau) = sin(omega_n*(t-tau));
xp1(t) = int(F1*h/m/omega_n,tau,0,t);
xp1(t) = simplify(expand(xp1));

%% (2)
F2 = -F_0*(tau - T_1)/T_1;
xp2(t) = int(F2*h/m/omega_n,tau,T_1,t);
xp2(t) = simplify(expand(xp2));

```

```
F3 = -F_0/T_1*tau + (T_1+T_2)*F_0/T_1;
xp3(t) = int(F3*h/m/omega_n,tau,T_1+T_2,t);
xp3(t) = collect(simplify(expand(xp3)),[sin(omega_n*t) cos(omega_n*t)]);
xp3(t) = collect(xp3, F_0/T_1/m/omega_n^3);

F4 = F_0/T_1*tau - (2*T_1+T_2)*F_0/T_1;
xp4(t) = int(F4*h/m/omega_n,tau,2*T_1+T_2,t);
xp4(t) = collect(simplify(expand(xp4)),[sin(omega_n*t) cos(omega_n*t)]);
xp4(t) = collect(xp4, F_0/T_1/m/omega_n^3);

%% (3)

Tn = 2*pi/params.omega_n;
T1_inputs = [0.1 0.5 1.0 1.5 2.0 2.5] * Tn;

% Arrange steady state responses
xp1 = xp1 / F_0 * m * omega_n^2;
xp2 = xp2 / F_0 * m * omega_n^2;
xp3 = xp3 / F_0 * m * omega_n^2;
xp4 = xp4 / F_0 * m * omega_n^2;

% Create the summed response function
xp_seg1 = xp1;
xp_seg2 = xp1 + xp2;
xp_seg3 = xp1 + xp2 + xp3;
xp_seg4 = xp1 + xp2 + xp3 + xp4;

colors = ["#003547", "#005E54", "#C2BB00", "#E1523D", "#ED8B16", "#B096E0"];
labels = ["0.1", "0.5", "1.0", "1.5", "2.0", "2.5"] + "$T_n$";
ct = 1;

fig = figure(Renderer="painters", Position=[60 60 1200 900]);
t = tiledlayout(3,2,'TileSpacing','Compact', 'Padding','tight');
  xlabel(t,"t");
  ylabel(t,"x(t)");
  for T1 = T1_inputs
    xp = [];
    tint = [];

    % Substitute in the parameters
    xp_sub_seg1 = subs(xp_seg1, [T_1, omega_n], [T1, params.omega_n]);
    xp_sub_seg2 = subs(xp_seg2, [T_1, T_2, omega_n], [T1, params.T2, params.omega_n]);
    xp_sub_seg3 = subs(xp_seg3, [T_1, T_2, omega_n], [T1, params.T2, params.omega_n]);
    xp_sub_seg4 = subs(xp_seg4, [T_1, T_2, omega_n], [T1, params.T2, params.omega_n]);

    % Time intervals
    tint1 = 0:0.001:T1;
    tint2 = T1:0.001:T1+params.T2;
    tint3 = T1+params.T2:0.001:2*T1+params.T2;
    tint4 = 2*T1+params.T2:0.001:1.5;

    % Compute values
    xp1_data = xp_sub_seg1(tint1);
```

```
    xp2_data = xp_sub_seg2(tint2);
    xp3_data = xp_sub_seg3(tint3);
    xp4_data = xp_sub_seg4(tint4);


    % Combine all data
    tint = [tint tint1 tint2 tint3 tint4];
    xp = [xp xp1_data xp2_data xp3_data xp4_data];

    % Plot
    nexttile;
    plot(tint, xp, DisplayName=labels(ct), Color=colors(ct));
    xline(T1,'--r', "$T_1$", LabelVerticalAlignment='bottom', ...
          LabelOrientation='horizontal', LabelHorizontalAlignment='right', ...
          HandleVisibility='off', Interpreter='latex', FontSize=10)
    xline(T1+params.T2,'--r', "$T_1+T_2$", LabelVerticalAlignment='bottom', ...
          LabelOrientation='horizontal', LabelHorizontalAlignment='left', ...
          HandleVisibility='off', Interpreter='latex', FontSize=10)
    xline(2*T1+params.T2,'--r', "$2T_1+T_2$", LabelVerticalAlignment='bottom', ...
          LabelOrientation='horizontal', LabelHorizontalAlignment='right', ...
          HandleVisibility='off', Interpreter='latex', FontSize=10)
    grid on; grid minor; box on; legend;
    xlim([0,1.5])
    ylim([-2,2])
    ct = ct + 1;
  end
saveas(fig,"plots/p2/p2-x(t).png")

%% (5)

delta = 0:0.001:4;
A_T1Tn = 1 + 1./(2*pi*delta) .* sin(2*pi*delta);

fig = figure(Renderer="painters");
  plot(delta, A_T1Tn)
  grid on; grid minor; box on;
  xlabel("$T_1/T_n$")
  ylabel("max amplitude")
  yline(1,'--r')
saveas(fig,"plots/p2/p2-maxamp.png")
```

IV.iii Problem 3

```
% AE6230 HW2 Problem 3
% Author: Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
sympref('FloatingPointOutput', false);  % fractions in symbolic

```

```
%% (1)
syms t real
syms m b J_c k real positive
syms h_1(t) h_2(t) theta(t)

% let h1(t) = h_LE(t) and h2(t) = h_TE(t)
h1dot = diff(h_1,t);
h2dot = diff(h_2,t);
h1ddot = diff(h1dot,t);
h2ddot = diff(h2dot,t);

T = m*(h1dot + h2dot)^2/8 + J_c*(h1dot - h2dot)^2/8/b^2;  % kinetic energy
V = k*(3*h_1 + h_2)^2/32 + k*h_2^2/2;  % potential energy

% Lagrangian
L = T - V;

%% (2)
% EOM Derivation
Lq1 = diff(L,h_1)
Lq1dot = collect(simplify(expand(diff(L,h1dot))), [h1dot h2dot])
Lq1dot_dt = collect(simplify(expand(diff(Lq1dot,t))), [h1ddot h2ddot])
eqn1 = collect(simplify(expand(Lq1dot_dt - Lq1)), ...
               [h_1 h_2 h1ddot h2ddot])

Lq2 = simplify(expand(diff(L,h_2)))
Lq2dot = collect(simplify(expand(diff(L,h2dot))), [h1dot h2dot])
Lq2dot_dt = collect(simplify(expand(diff(Lq2dot,t))), [h1ddot h2ddot])
eqn2 = collect(simplify(expand(Lq2dot_dt - Lq2)), ...
               [h_1 h_2 h1ddot h2ddot])

%% (3)
eqn1_coeffs = coeffs(eqn1,[h1ddot h2ddot h_1 h_2])
eqn2_coeffs = coeffs(eqn2,[h1ddot h2ddot h_1 h_2])
eqn1_coeffs = formula(eqn1_coeffs);
eqn2_coeffs = formula(eqn2_coeffs);

M = [eqn1_coeffs(4) eqn1_coeffs(3); eqn2_coeffs(4) eqn2_coeffs(3)]
K = [eqn1_coeffs(2) eqn1_coeffs(1); eqn2_coeffs(2) eqn2_coeffs(1)]

T = [1/2 1/2; 1/2/b -1/2/b];
T = inv(T)
Mhat = expand(T.' * M * T)
Khat = expand(T.' * K * T)
```