Homework Problems

1 Exercise

1.1 Problem 1.

Let $\mathbf{x}$ be a uniform random variable over the interval $[0, 4]$. Moreover, $\mathbf{v}$ is a uniform random variable over the interval $[-1, 1]$. Assume that $\mathbf{x}$ and $\mathbf{v}$ are independent. Let $\mathbf{y}$ be the random variable given by $\mathbf{y} = \mathbf{x} + \mathbf{v}$.

- Let $\mathcal{H}$ be the space spanned by $\{1, \mathbf{y}, \mathbf{y}^2, \mathbf{y}^3\}$. Then compute

$$P_{\mathcal{H}}\mathbf{x} = a + b\mathbf{y} + c\mathbf{y}^2 + d\mathbf{y}^3 \quad \text{and} \quad d_4^2 = E|\mathbf{x} - P_{\mathcal{H}}\mathbf{x}|^2$$

- Compute the conditional expectation

$$\widehat{g}(y) = E(\mathbf{x}|\mathbf{y} = y)$$

- Plot $\widehat{g}(y)$ and its approximation $a + by + cy^2 + dy^3$ on the same graph over the interval $[-1, 5]$.

1.2 Problem 2.

Let $\mathbf{x}$ and $\mathbf{y}$ be two independent uniform random variables both over the interval $[0, 1]$. Let $\mathbf{a}$ be the random variable defined by the area $\mathbf{a} = \mathbf{x}\mathbf{y}$. Clearly, the area $0 \leq \mathbf{a} \leq 1$. Our problem is given the area $\mathbf{a}$ find the best estimate $\widehat{\mathbf{x}}$ of $\mathbf{x}$.

- Let $\mathcal{H}$ be the space spanned by $\{1, \mathbf{a}, \mathbf{a}^2, \mathbf{a}^3\}$. Then compute

$$P_{\mathcal{H}}\mathbf{x} = \alpha + \beta\mathbf{a} + \gamma\mathbf{a}^2 + \delta\mathbf{a}^3 \quad \text{and} \quad d_4^2 = E|\mathbf{x} - P_{\mathcal{H}}\mathbf{x}|^2$$

- Compute the conditional expectation

$$\widehat{g}(a) = E(\mathbf{x}|\mathbf{a} = a) \quad \text{and} \quad d_\infty^2 = E|\mathbf{x} - \widehat{g}(\mathbf{a})|^2$$

- Plot $\widehat{g}(a)$ and its approximation $\alpha + \beta a + \gamma a^2 + \delta a^3$ on the same graph over the interval $[0, 1]$. Is $d_\infty < d_4$? Explain why or why not.