College of Engineering
School of Aerospace Engineering

AE6210: Advanced Dynamics I

Homework 3

Professor:
Mayuresh Patil
Gtech AE Professor

Student:
Tomoki Koike
AE MS Student

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I Instructions

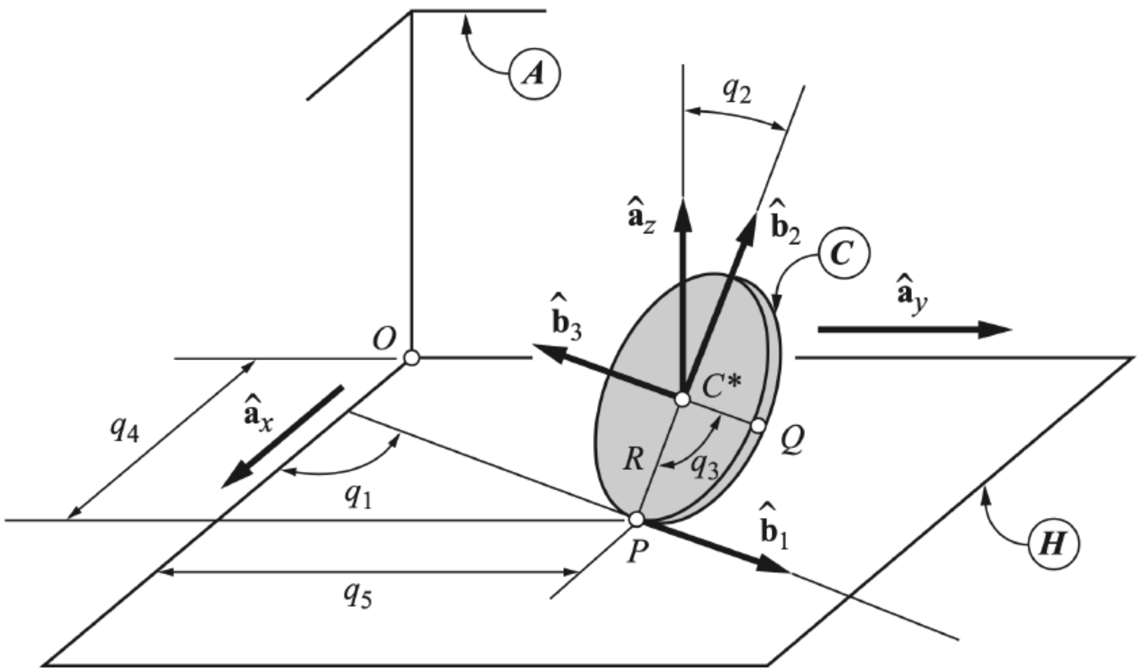

Figure 1: Kane's Problem 2.7 diagram

The figure above shows a circular disk C of radius R in contact with a horizontal plane H that is fixed in a reference frame A rigidly attached to the Earth. Mutually perpendicular unit vectors $\hat{a}_x$, $\hat{a}_y$, and $\hat{a}_z = \hat{a}_x \times \hat{a}_y$ are fixed in A, and $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$ form a dextral set of orthogonal unit vectors, with $\hat{b}_1$ parallel to the tangent to the periphery of C at the point of contact between H and C (denoted by $\hat{C}$), $\hat{b}_2$ parallel to the line connecting this contact point ($\hat{C}$) to $C^{\star}$, the center of C, and $\hat{b}_3$ normal to the plane of C.

The orientation of C in A can be described in terms of three angles q_1, q_2, q_3, where Q is a point fixed on the periphery of C. The two quantities q_4 and q_5 characterize the position in A of the point P (which represents a point on the path made by the disk on the H).

II Problem One

The angular velocity of C in A can be expressed as:

$$ {}^{A}\vec{\omega}^{C} = u_1\hat{b}_1 + u_2\hat{b}_2 + u_3\hat{b}_3. \tag{II.1} $$

Determine u_1, u_2, and u_3, which we can also choose as the generalized velocities for the problem. The expression for u_i's should be written in terms of the generalized coordinates and their time derivatives.

Solution:
Let a unit vector with no specific frame be expressed as

$$ \hat{X} = (1,0,0)^T, \quad \hat{Y} = (0,1,0)^T, \quad \hat{Z} = (0,0,1)^T, $$

and for clarity and my preference, let $(\hat{a}_x,\ \hat{a}_y,\ \hat{a}_z) = (\hat{a}_1,\ \hat{a}_2,\ \hat{a}_3)$ and define new coordinate frames: c-frame and h-frame. The new frames are observed in Figure 2.

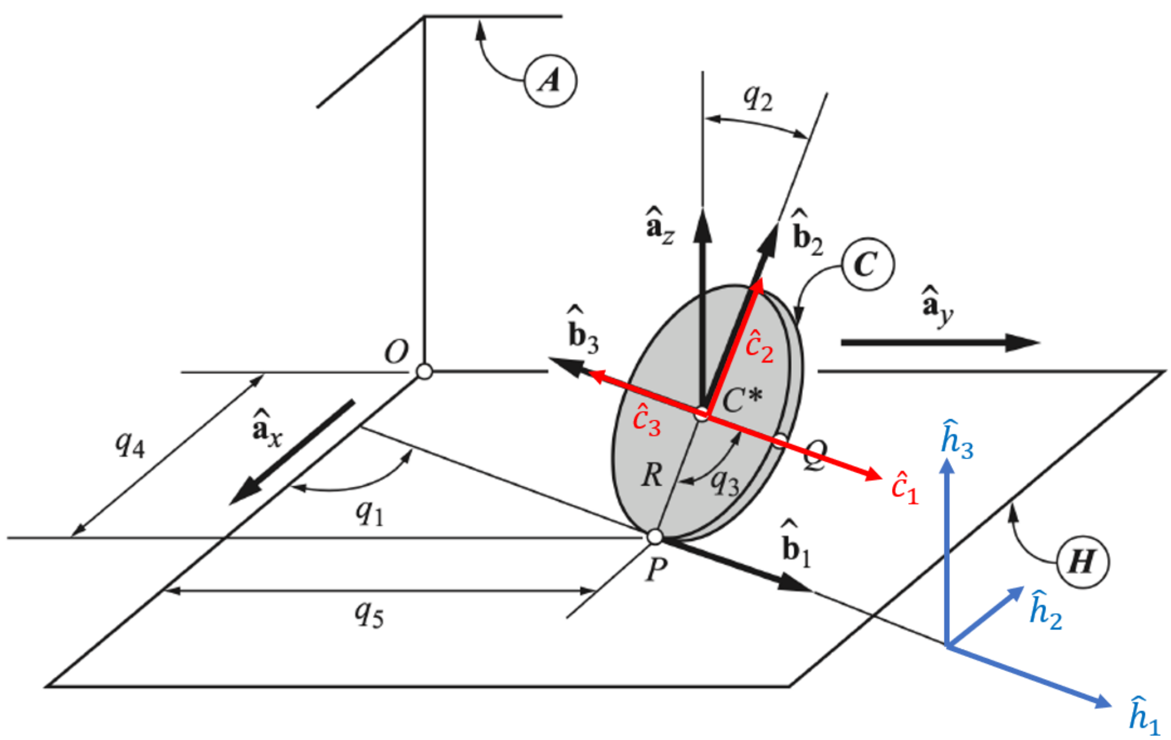

Figure 2: Problem diagram with extra coordinate frames.

To make the transitions between different frames easy, we will define the relations between each frames with rotations. If $S(\cdot)$ indicates the skew-symmetric matrix operation and $\otimes$ is the outer product (tensor product) then we have

$$\hat{h} = L\left(q_1, \hat{Z}\right)\hat{a} = R_a^h \hat{a}$$
$$\hat{h} = \left[(I - \hat{Z}\otimes\hat{Z})\cos(q_1) - S(\hat{Z})\sin(q_1) + \hat{Z}\otimes\hat{Z}\right]\hat{a}$$
$$\begin{bmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{bmatrix} = \begin{bmatrix}\cos(q_1) & \sin(q_1) & 0\\ -\sin(q_1) & \cos(q_1) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\\ \hat{a}_3\end{bmatrix}. \tag{II.2}$$

Similarly,

$$\hat{b} = L\left(\frac{\pi}{2} - q_2, \hat{X}\right)\hat{h} = R_h^b \hat{h}$$
$$\hat{b} = \left[(I - \hat{X}\otimes\hat{X})\cos\left(\frac{\pi}{2} - q_1\right) - S(\hat{X})\sin\left(\frac{\pi}{2} - q_2\right) + \hat{X}\otimes\hat{X}\right]\hat{h}$$
$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \sin(q_2) & \cos(q_2)\\ 0 & -\cos(q_2) & \sin(q_2)\end{bmatrix}\begin{bmatrix}\hat{h}_1\\ \hat{h}_2\\ \hat{h}_3\end{bmatrix}. \tag{II.3}$$

And finally,

$$\hat{c} = L\left(q_3 - \frac{\pi}{2}, \hat{Z}\right)\hat{b} = R_b^c \hat{b}$$
$$\hat{c} = \left[(I - \hat{Z}\otimes\hat{Z})\cos\left(q_3 - \frac{\pi}{2}\right) - S(\hat{Z})\sin\left(q_3 - \frac{\pi}{2}\right) + \hat{Z}\otimes\hat{Z}\right]\hat{b}$$
$$\begin{bmatrix}\hat{c}_1\\ \hat{c}_2\\ \hat{c}_3\end{bmatrix} = \begin{bmatrix}\sin(q_3) & -\cos(q_3) & 0\\ \cos(q_3) & \sin(q_3) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\\ \hat{b}_3\end{bmatrix}. \tag{II.4}$$

Now that we have that defined we can find ${}^A\vec{\omega}^C$ using the additive property of angular velocities. It becomes

$${}^A\omega^C = {}^a\omega^h + {}^h\omega^b + {}^b\omega^c$$

From the definition of angular velocity we have

$$-S({}^a\omega^h) = \dot{R}_a^h\left(R_a^h\right)^T = \dot{R}_a^h R_h^a \tag{II.5}$$
$$-\begin{bmatrix}0 & -{}^a\omega_3^h & {}^a\omega_2^h\\ {}^a\omega_3^h & 0 & -{}^a\omega_1^h\\ -{}^a\omega_2^h & {}^a\omega_1^h & 0\end{bmatrix} = \begin{bmatrix}-\dot{q}_1\sin(q_1) & \dot{q}_1\cos(q_1) & 0\\ -\dot{q}_1\cos(q_1) & -\dot{q}_1\sin(q_1) & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\cos(q_1) & -\sin(q_1) & 0\\ \sin(q_1) & \cos(q_1) & 0\\ 0 & 0 & 1\end{bmatrix} \tag{II.6}$$
$$\therefore {}^a\omega^h = \dot{q}_1\hat{a}_3. \tag{II.7}$$

Similarly we can compute,

$${}^h\omega^b = -\dot{q}_2\hat{h}_1 = -\dot{q}_2\hat{b}_1 \tag{II.8}$$
$${}^b\omega^c = \dot{q}_3\hat{b_3}. \tag{II.9}$$

Thus, from (II.7)-(II.9) we have

$${}^A\omega^C = \dot{q}_1\hat{a}_3 - \dot{q}_2\hat{b}_1 + \dot{q}_3\hat{b}_3. \tag{II.10}$$

From the transformations (II.2) and (II.3), we can convert $\hat{a}_3$ as follows

$$\hat{a}_3 = \left(\left(R_h^b R_a^h\right)^{-1} \hat{b}\right) \cdot \hat{Z} = \left(R_h^a R_b^h \hat{b}\right) \cdot \hat{Z}$$
$$\hat{a}_3 = \cos(q_2)\hat{b}_2 + \sin(q_2)\hat{b}_3.$$

The generalized velocities are

$$\begin{aligned} u_1 &= -\dot{q}_2 \\ u_2 &= \dot{q}_1 \cos(q_2) \\ u_3 &= \dot{q}_1 \sin(q_2) + \dot{q}_3. \end{aligned}$$

Hence,

$$\boxed{{}^A\vec{\omega}^C = -\dot{q}_2\hat{b}_1 + \dot{q}_1 \cos(q_2)\hat{b}_2 + \left(\dot{q}_1 \sin(q_2) + \dot{q}_3\right)\hat{b}_3.} \tag{II.11}$$

III Problem Two

Similarly determine ${}^{A}\vec{\omega}^{B}$ as represented in the B reference frame. You will need this to determine many of the quantities asked below. Try to write the angular velocity in terms of the generalized coordinates and generalized velocities only.

Solution:
Similar to Problem One, we compute angular velocity as

$$\begin{aligned} {}^{A}\omega^{B} &= {}^{a}\omega^{h} + {}^{h}\omega^{b} \\ &= \dot{q}_1\hat{a}_3 - \dot{q}_2\hat{b}_1 \end{aligned}$$

Hence,

$$\begin{aligned} {}^{A}\vec{\omega}^{B} &= -\dot{q}_2\hat{b}_1 + \dot{q}_1\cos(q_2)\hat{b}_2 + \dot{q}_1\sin(q_2)\hat{b}_3 \\ &= -\dot{q}_2\hat{b}_1 + \dot{q}_1\cos(q_2)\hat{b}_2 + \dot{q}_1\cos(q_2)\tan(q_2)\hat{b}_3 \end{aligned}$$

and

$$\boxed{{}^{A}\vec{\omega}^{B} = u_1\hat{b}_1 + u_2\hat{b}_2 + u_2\tan(q_2)\hat{b}_3.} \tag{III.1}$$

IV Problem Three

The angular acceleration of C in terms of A can be expressed as

$$^{A}\vec{\alpha}^{C} = \alpha_1 \hat{b}_1 + \alpha_2 \hat{b}_2 + \alpha_3 \hat{b}_3. \tag{IV.1}$$

Determine α_1, α_2, and α_3 in terms of the generalized coordinates, generalized velocities, and the time derivatives of generalized velocities.

Solution:
The acceleration is the derivative of the angular velocity, and therefore, from (II.11) and (III.1) we can compute this as follows.

$$^{A}\vec{\alpha}^{C} = \frac{^{b}d}{dt} {}^{A}\vec{\omega}^{C} + {}^{A}\vec{\omega}^{B} \times {}^{A}\vec{\omega}^{C}.$$

Hence,

$$\boxed{^{A}\vec{\alpha}^{C} = \left(\dot{u}_1 + u_2\left(u_3 - u_2\tan(q_2)\right)\right)\hat{b}_1 + \left(\dot{u}_2 - u_1\left(u_3 - u_2\tan(q_2)\right)\right)\hat{b}_2 + \dot{u}_3\hat{b}_3,} \tag{IV.2}$$

V Problem Four

Define $u_4 = \dot{q}_4$ and $u_5 = \dot{q}_5$, two more generalized velocities. The velocity of $C^\star$ in A can be expressed as:

$$ {}^A\vec{v}^{C^\star} = v_1\hat{b}_1 + v_2\hat{b}_2 + v_3\hat{b}_3. \tag{V.1}$$

Determine v_1, v_2, and v_3 in terms of the generalized coordinates (q_i's) and generalized velocities (u_i's) only.

Solution:
The position of point $C^\star$ with respect to the origin is

$$\begin{aligned} \vec{r}_{OC^\star} &= \vec{r}_{OP} + \vec{r}_{PC^\star} \\ &= q_4\hat{a}_1 + q_5\hat{a}_2 + R\hat{b}_2. \end{aligned} \tag{V.2}$$

Then the velocity is

$$\begin{aligned} {}^a\vec{v}_{OC^\star} &= \frac{{}^a d\vec{r}_{OC^\star}}{dt} \\ &= \frac{{}^a d}{dt}\left(q_4\hat{a}_1 + q_5\hat{a}_2\right) + {}^A\vec{\omega}^B \times (R\hat{b}_2) \\ &= \dot{q}_4\hat{a}_1 + \dot{q}_5\hat{a}_2 + \left(u_1\hat{b}_1 + u_2\hat{b}_2 + u_2\tan(q_2)\hat{b}_3\right) \times (R\hat{b}_2) \\ &= u_4\hat{a}_1 + u_5\hat{a}_2 + \left(u_1\hat{b}_1 + u_2\hat{b}_2 + u_2\tan(q_2)\hat{b}_3\right) \times (R\hat{b}_2) \\ &= u_4\hat{a}_1 + u_5\hat{a}_2 - Ru_2\tan(q_2)\hat{b}_1 + Ru_1\hat{b}_3. \end{aligned} \tag{V.3}$$

Since,

$$\begin{aligned} \hat{a}_1 &= \left(\left(R_h^b R_a^h\right)^{-1}\hat{b}\right)\cdot\hat{X} = \left(R_h^a R_b^h \hat{b}\right)\cdot\hat{X} \\ &= \cos(q_1)\hat{b}_1 - \sin(q_1)\sin(q_2)\hat{b}_2 + \sin(q_1)\cos(q_2)\hat{b}_3, \end{aligned}$$

and

$$\begin{aligned} \hat{a}_2 &= \left(\left(R_h^b R_a^h\right)^{-1}\hat{b}\right)\cdot\hat{Y} = \left(R_h^a R_b^h \hat{b}\right)\cdot\hat{Y} \\ &= \sin(q_1)\hat{b}_1 + \cos(q_1)\sin(q_2)\hat{b}_2 - \cos(q_1)\cos(q_2)\hat{b}_3. \end{aligned}$$

Plug in these results to (V.3) and we can compute the results

$$\boxed{\begin{aligned} v_1 &= -Ru_2\tan(q_2) + u_4\cos(q_1) + u_5\sin(q_1) \\ v_2 &= -u_4\cos(q_1)\sin(q_1) + u_5\cos(q_1)\sin(q_2) \\ v_3 &= Ru_1 + u_4\sin(q_1)\cos(q_2) - u_5\cos(q_1)\cos(q_2) \end{aligned}}$$

VI Problem Five

The acceleration of $C^\star$ in A can be expressed as:

$$ {}^A\vec{a}^{C^\star} = a_1\hat{b}_1 + a_2\hat{b}_2 + a_3\hat{b}_3. \tag{VI.1}$$

Determine a_1, a_2, and a_3 in terms of the generalized coordinates (q_i's), generalized velocities (u_i's), and the time derivatives of generalized velocities ($\dot{u}_i$'s).

Solution:
This is simply the derivative of the answer in Problem V in the b-frame which is

$$ {}^A\vec{a}^{C^\star} = \frac{{}^b d}{dt} {}^A\vec{v}^{C^\star} + {}^A\vec{\omega}^B \times {}^A\vec{v}^{C^\star}, $$

which reduces to

$$\boxed{\begin{aligned} a_1 &= -R\dot{u}_2\tan(q_2) + \dot{u}_4\cos(q_1) + \dot{u}_5\sin(q_1) + Ru_1u_2\left(1+\sec^2(q_2)\right) \\ a_2 &= -\dot{u}_4\sin(q_1)\sin(q_2) + \dot{u}_5\cos(q_1)\sin(q_2) - Ru_1^2 - Ru_2^2\tan^2(q_2) \\ a_3 &= R\dot{u}_1 + \dot{u}_4\sin(q_1)\cos(q_2) - \dot{u}_5\cos(q_1)\cos(q_2) + Ru_2^2\tan(q_2) \end{aligned}}$$

VII Problem Six

At any time, there exists precisely one point on the disk C that is in contact with the plane H. Calling this point $\hat{C}$, determine the velocity of this point ${}^{A}\vec{v}^{\hat{C}}$ as represented in the A reference frame.

Solution:
For this problem, we use the two-point formula since points $C^{\star}$ and $\hat{C}$ are on the same rigid body. Therefore, in the b-frame the velocity is expressed as

$$\begin{aligned} {}^{A}\vec{v}^{\hat{C}} &= {}^{A}\vec{v}^{C^{\star}} + {}^{A}\vec{\omega}^{C} \times \vec{r}_{C^{\star}\hat{C}} \\ &= {}^{A}\vec{v}^{C^{\star}} + {}^{A}\vec{\omega}^{C} \times R(-\hat{b}_2) \\ &= \begin{bmatrix} u_4\cos(q_1) + Ru_3 + u_5\sin(q_1) - Ru_2\tan(q_2) \\ \sin(q_2)\left(u_5\cos(q_1) - u_4\sin(q_1)\right) \\ -\cos(q_2)\left(u_5\cos(q_1) - u_4\sin(q_1)\right) \end{bmatrix}. \end{aligned}$$

To express this in the a-frame we plug the above result into (II.2) and (II.3) as a sequence of rotations

$$\begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix} = \begin{bmatrix} \cos(q_1) & \sin(q_1) & 0 \\ -\sin(q_1) & \cos(q_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin(q_2) & \cos(q_2) \\ 0 & -\cos(q_2) & \sin(q_2) \end{bmatrix}^T \begin{bmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \hat{b}_3 \end{bmatrix}.$$

Thus, we obtain

$$\boxed{\begin{aligned} {}^{A}\vec{v}^{\hat{C}} = &\left(u_4 + u_3R\cos(q_1) - u_2R\cos(q_1)\tan(q_2)\right)\hat{a}_1 \\ &+ \left(u_5 + u_3R\sin(q_1) - u_2R\sin(q_1)\tan(q_2)\right)\hat{a}_2. \end{aligned}}$$