College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW4

Professor:
Martin Corless
Purdue AAE Professor

Student:
Tomoki Koike
Purdue AAE Senior

February 19, 2021

Table of Contents

Exercise 1

Determine whether or not the following functions is *positive definite*.

$$V(x) = x_1^4 - x_1^2 x_2 + x_2^2$$

Solution:
When $x = [0, 0]^T$,

$$V(0) = 0$$

and

$$\begin{aligned} V(x) &= x_1^4 - x_1^2 x_2 + \frac{1}{4}x_2^2 + \frac{3}{4}x_2^2 \\ &= (x_1^2 - \frac{1}{2}x_2)^2 + \frac{3}{4}x_2^2 \\ &> 0 \qquad \forall x \neq 0. \end{aligned}$$

Also, due to the dominance of x_1^4 and x_2^2

$$\lim_{\|x\|\to\infty} V(x) = \infty.$$

Thus, this function $V(x)$ is positive definite.

Exercise 2

Show that the following system is AS about zero

$$\dot{x} = -(1 + \sin x)x$$

Solution:
A candidate Lyapunov function $V(x)$ would be

$$V(x) = x^2.$$

Then V is *lpd* about zero and

$$DV(x)f(x) = -2x^2(1 + \sin x) < 0 \qquad for \quad |x| < \frac{\pi}{2}, x \neq 0.$$

Hence the origin is AS.

Exercise 3

By appropriate choice of Lyapunov function, show that the origin is an *asymptotically stable* equilibrium state for

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + x_1^3 - x_2\end{aligned}$$

Solution:
The candidate Lyapunov function is

$$V(x) = \frac{1}{2}x_1^2 - \frac{1}{2}x_1^4 - x_1x_2 + x_2^2.$$

Thus, we obtain

$$\begin{aligned}DV(x)f(x) &= \begin{bmatrix} x_1 - 2_1^3 - x_2 & -x_1 + 2x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 + x_1^3 - x_2 \end{bmatrix} \\ &= x_1x_2 - 2x_1^3x_2 - x_2^2 + x_1^2 - x_1^4 + x_1x_2 - 2x_1x_2 + 2x_1^3x_2 - 2x_2^2 \\ &= -3x_2^2 + x_1^2 - x_1^4 \\ &= -3x_2^2 - x_1^2(1 + x_1)(1 - x_1) \\ &< 0 \qquad for \quad \|x\| < 1, x \neq 0\end{aligned}$$

Hence the origin is an AS equilibrium state for the system.

Exercise 4

(Stabilization of the Duffing system) Consider the Duffing system with a scalar control input $u(t)$:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 - x_1^3 + u\end{aligned}$$

Obtain a linear controller of the form

$$u = -k_1 x_1 - k_2 x_2$$

which results in a closed loop system which is GAS about the origin. Numerically simulate the open loop system ($u = 0$) and the closed loop system for several initial conditions.

Solution:
A candidate Lyapunov function is

$$V(x) = \frac{1}{2}\lambda k_2^2 x_1 + \lambda k_2 x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{2}x_1^4 + \frac{1}{2}k_1 x_1^2.$$

where

$$0 < \lambda < 1$$

This Lyapunov function is positive definite because

$$V(0) = 0$$

$$DV(x) = \begin{bmatrix}\lambda k_2^2 x_1 + \lambda k_2 x_2 - x_1 + x_1^3 + k_1 x_1 & \lambda k_2 x_1 + x_2\end{bmatrix} \Longrightarrow DV(0) = \begin{bmatrix}0 & 0\end{bmatrix}$$

$$D^2V(x) = \begin{bmatrix}\lambda k_2^2 - 1 + 3x_1^2 + k_1 & \lambda k_2\\ \lambda k_2 & 1\end{bmatrix} > 0 \qquad \text{if } k_1 > 1 \text{ and } k_2 > 0$$

$$\therefore V(x) > 0 \qquad \forall x \neq 0$$

$$\lim_{\|x\|\to\infty} V(x) = \infty.$$

Now, if we calculate $DV(x)f(x)$ we obtain as follows.

$$\begin{aligned}DV(x)f(x) &= \begin{bmatrix}\lambda k_2^2 x_1 + \lambda k_2 x_2 - x_1 + x_1^3 + k_1 x_1 & \lambda k_2 x_1 + x_2\end{bmatrix}\begin{bmatrix}x_2\\ x_1 - x_1^3 - k_1 x_1 - k_2 x_2\end{bmatrix}\\
&= \lambda k_2^2 x_1 x_2 + \lambda k_2 x_2^2 - x_1 x_2 + x_1^3 x_2 + k_1 x_1 x_2\\
&\qquad + \lambda k_2 x_1^2 - \lambda k_2 x_1^4 - \lambda k_1 k_2 x_1^2 - \lambda k_2^2 x_1 x_2\\
&\qquad\qquad + x_1 x_2 - x_1^3 x_2 - k_1 x_1 x_2 - k_2 x_2^2\\
&= -(1-\lambda)k_2 x_2^2 - (k_1 - 1)\lambda k_2 x_1^2 - \lambda k_2 x_1^4\end{aligned}$$

For this system to be GAS about the zero state based on the Lyapunov function, the following must be satisfied.

$$\begin{cases} k_1 > 1 \\ k_2 > 0 \end{cases} .$$

Hence a probable combination for the constants k_1 and k_2 is

$$\begin{cases} k_1 = 1.25 \\ k_2 = 1.5 \\ \lambda = 0.5 \end{cases}$$

Where

$$DV(x)f(x) = -0.1875x_1^2 - 0.75x_2^2 - 0.75x_1^4 < 0 \qquad \forall x \neq 0 \quad .$$

Now using the following SIMULINK model, we test our results numerically with 6 randomly selected initial conditions in the range of $[-3, 3]$.

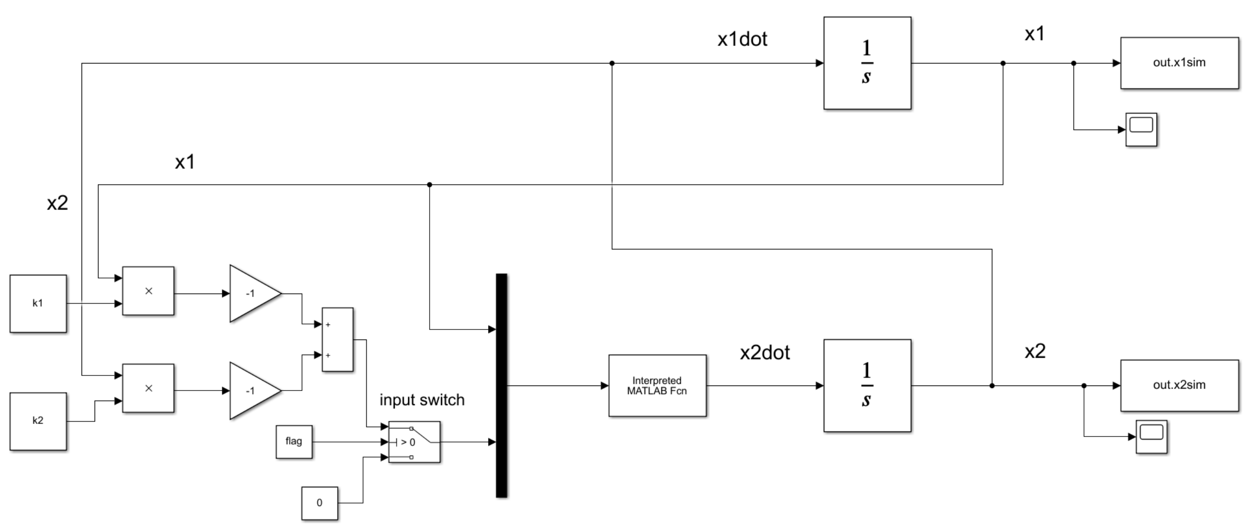
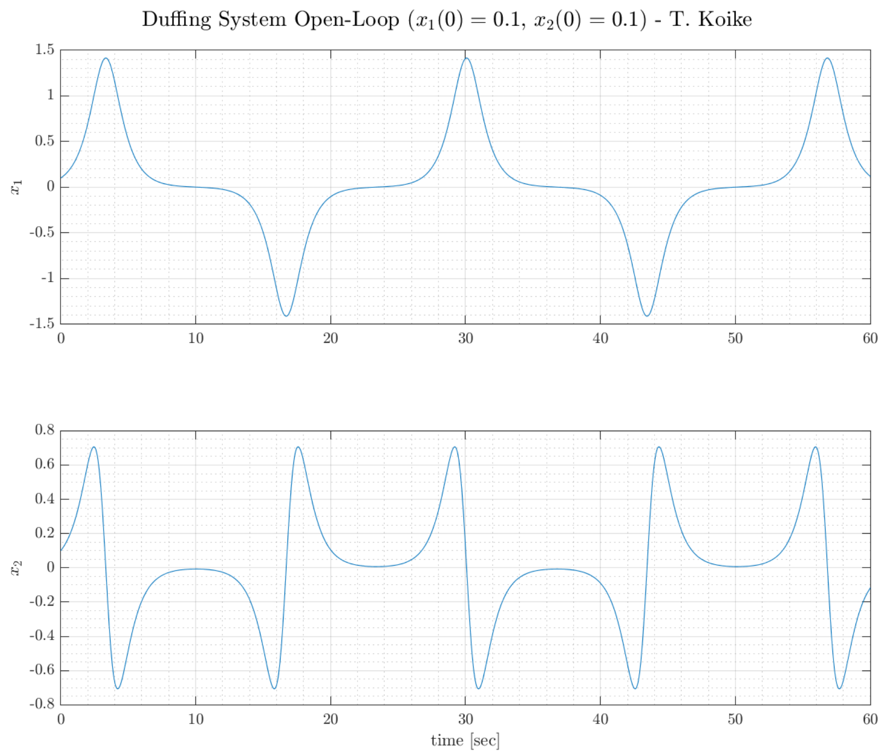
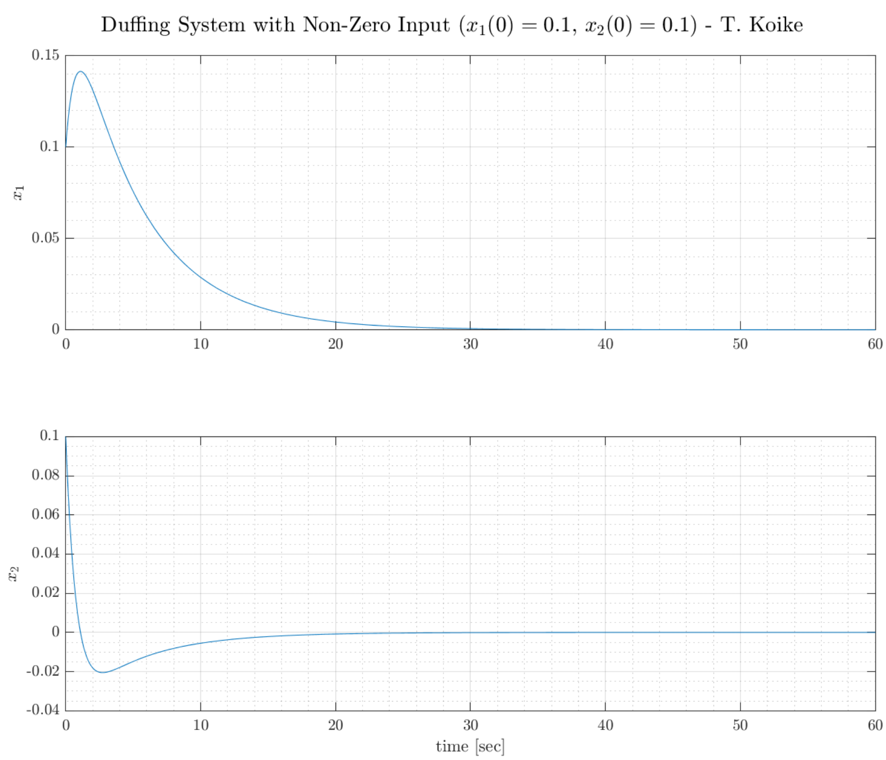


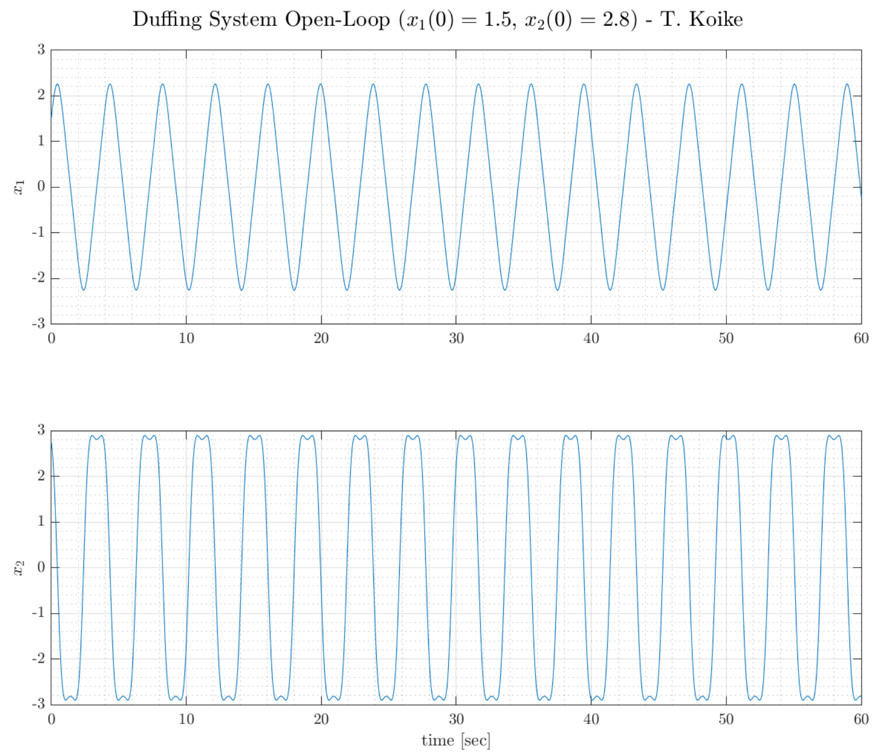
Figure 1: Simulink Model of Duffing System

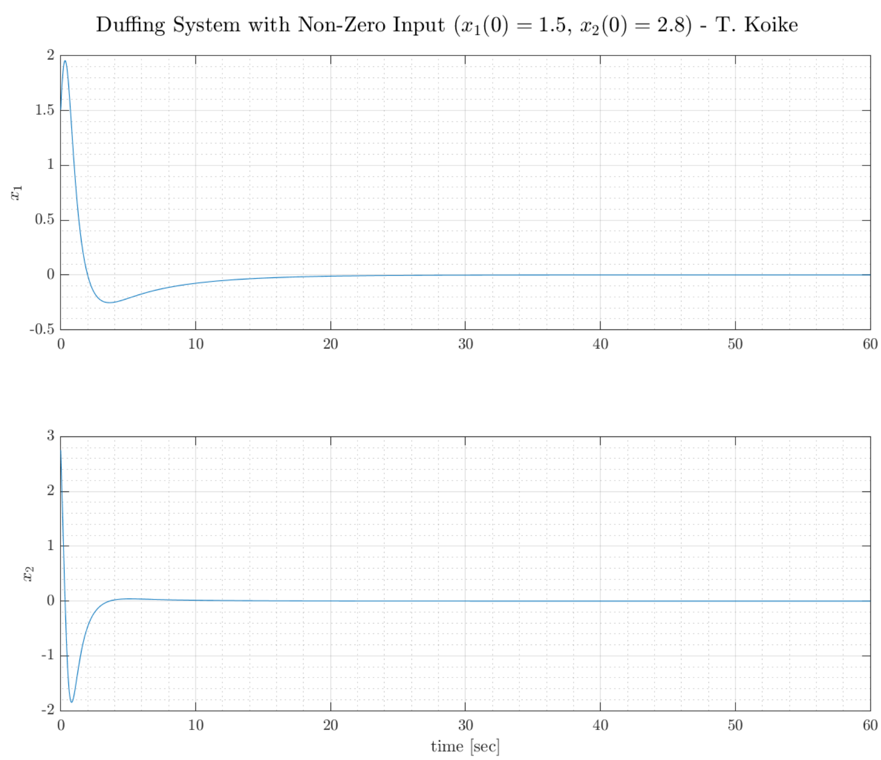
CASE 1: $x_0 = [0.1, 0.1]^T$



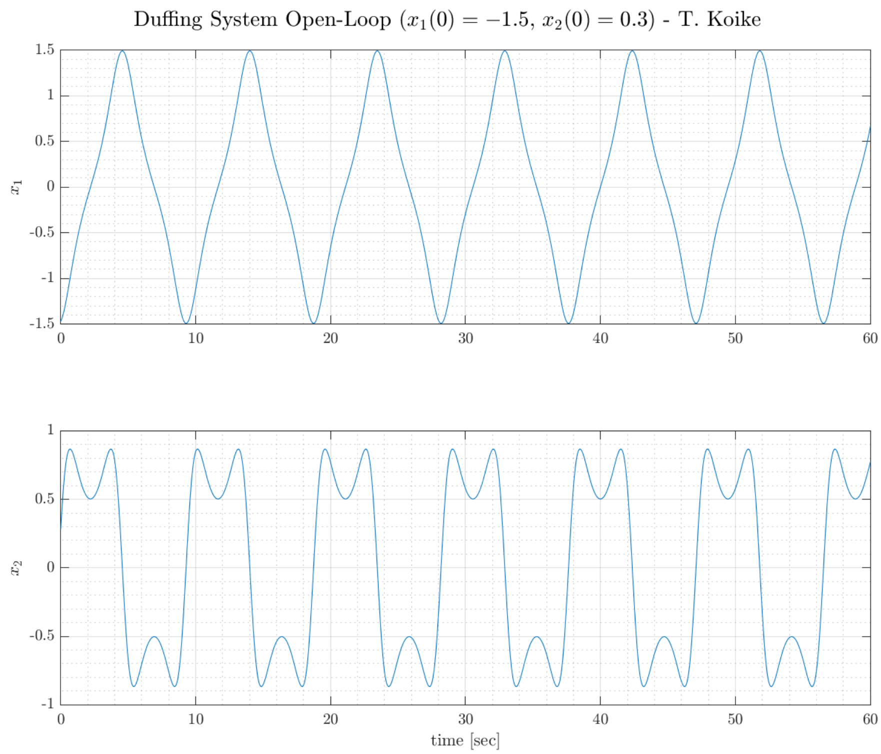


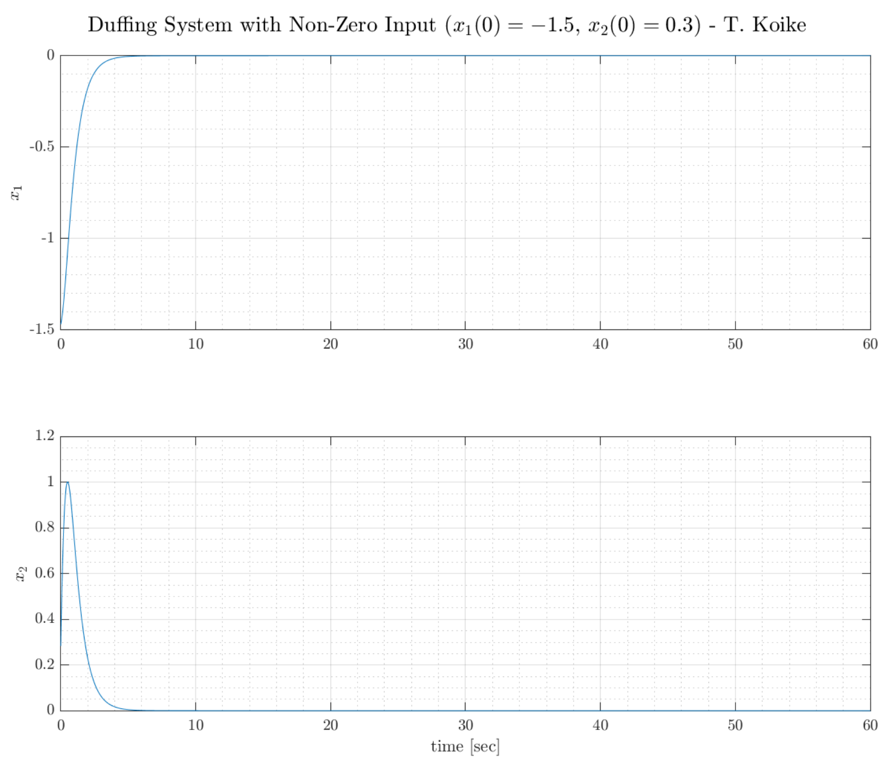
CASE 2: $x_0 = [1.5076, 2.7557]^T$



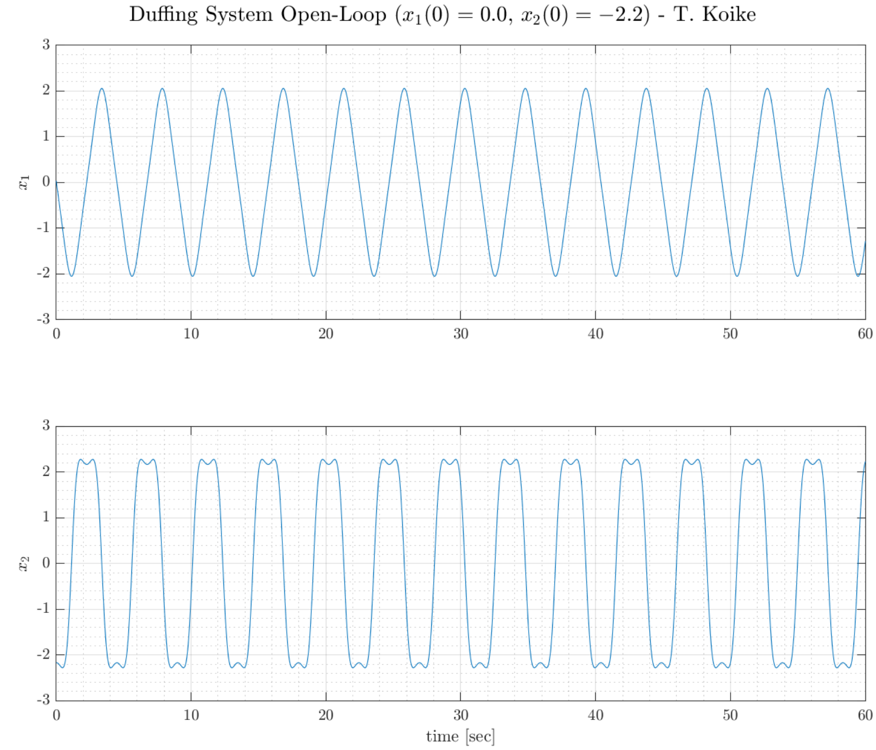


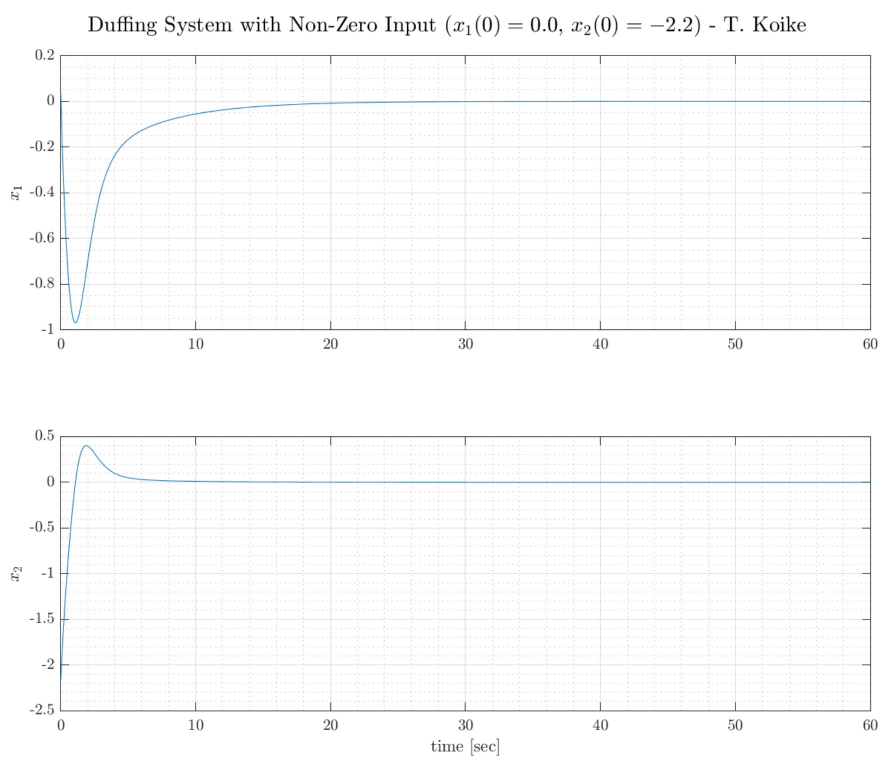
CASE 3: $x_0 = [-1.4694, 0.2833]^T$



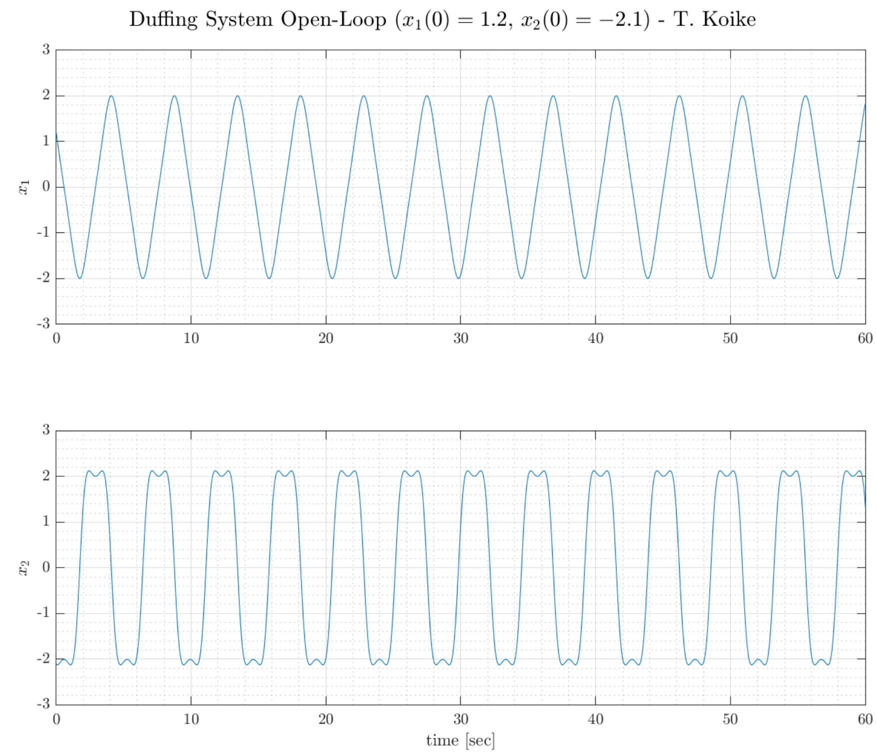


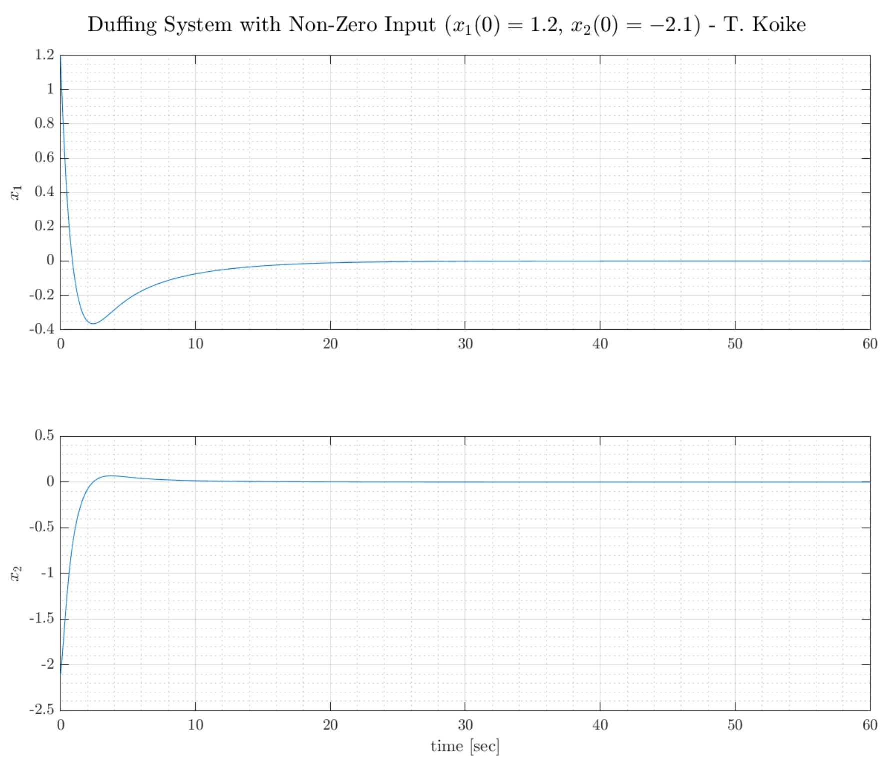
CASE 4: $x_0 = [0.0357, -2.1683]^T$





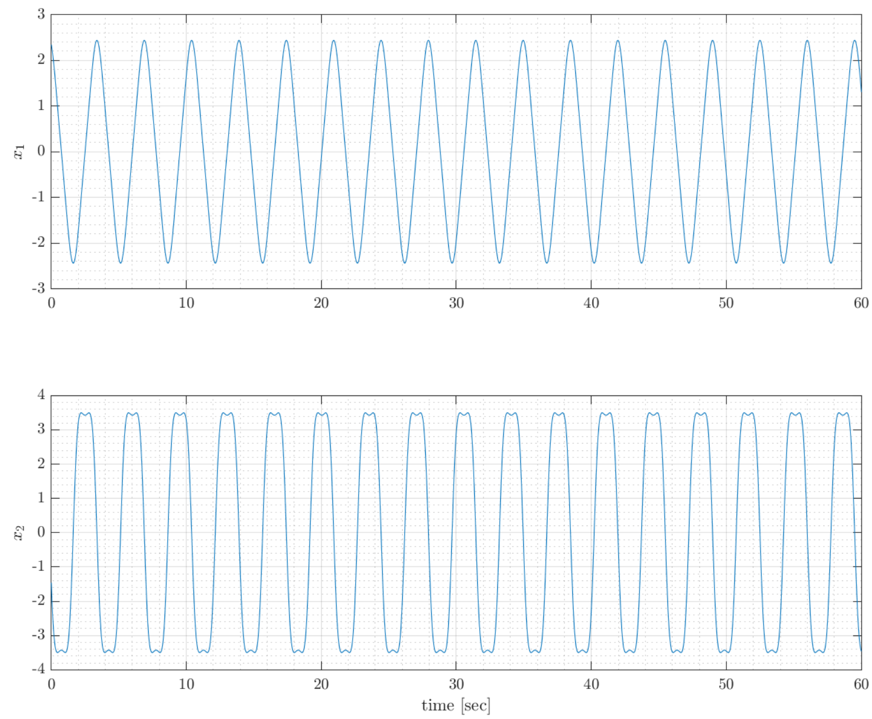
CASE 5: $x_0 = [1.1945, -2.1042]^T$



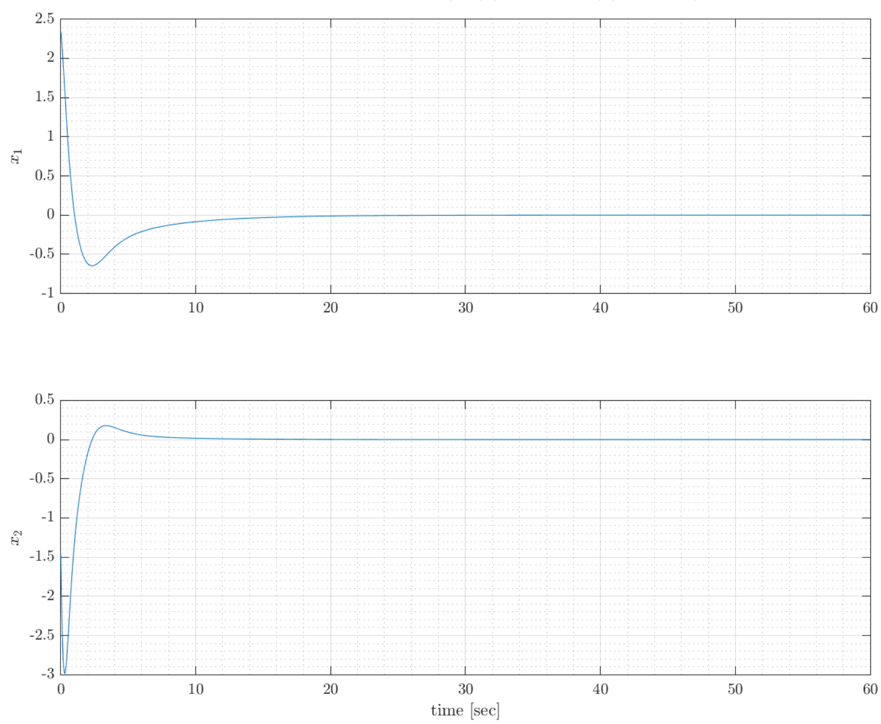


CASE 6: $x_0 = [2.3454, -1.4550]^T$

Duffing System Open-Loop ($x_1(0) = 2.3$, $x_2(0) = -1.5$) - T. Koike



Duffing System with Non-Zero Input ($x_1(0) = 2.3$, $x_2(0) = -1.5$) - T. Koike



Comment:
As we can see from the simulations, with $k_1 = 1.25$ and $k_2 = 1.5$ we can achieve GAS with the given system for any initial condition.

MATLAB CODE:

```
close all; clear all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2021-Spring\AAE666\matlab\hw4';
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
%%
% Constants
k1 = 1.25;
k2 = 1.5;

% Generate random ones
low  = -3;
high = 3;
rgen = @(a,b) (b-a).*rand(5,1) + a;

x1_0s = rgen(low, high);
x2_0s = rgen(low, high);
x1_0s = [0.1; x1_0s];
x2_0s = [0.1; x2_0s];
%%
% Open Loop Simulation
% - Set flag to switch SIMULINK to open loop
flag = -1;

for i = 1:length(x1_0s)
    x1_0 = x1_0s(i);
    x2_0 = x2_0s(i);

    % - Simulate
    simout = sim("duffingSystem.slx");

    % - Data rendering
    x1 = simout.x1sim.signals.values;
    x2 = simout.x2sim.signals.values;
    t  = simout.tout;

    % - Plot
    fig = figure("Renderer","painters","Position",[60 60 900 700]);
```

```
        subplot(2,1,1)
        plot(t, x1)
        grid on; grid minor; box on;
        ylabel('$x_1$')
        subplot(2,1,2)
        plot(t, x2)
        grid on; grid minor; box on;
        ylabel('$x_2$')
        xlabel('time [sec]')
        title_string = 'Duffing System Open—Loop ($x_1(0)=%0.1f$, $x_2(0)=%0.1f$) — T. Koike';
        title_S = sprintf(title_string, x1_0, x2_0);
        sgtitle(title_S)
    file_string = 'hw4_ex4_OL_case%d.png';
    file_S = sprintf(file_string, i);
    saveas(fig, file_S);
end
%%
% Closed Loop Simulation
% — Set flag to switch SIMULINK to open loop
flag = 1;

for i = 1:length(x1_0s)
    x1_0 = x1_0s(i);
    x2_0 = x2_0s(i);

    % — Simulate
    simout = sim("duffingSystem.slx");

    % — Data rendering
    x1 = simout.x1sim.signals.values;
    x2 = simout.x2sim.signals.values;
    t  = simout.tout;

    % — Plot
    fig = figure("Renderer","painters","Position",[60 60 900 700]);
        subplot(2,1,1)
        plot(t, x1)
        grid on; grid minor; box on;
        ylabel('$x_1$')
        subplot(2,1,2)
        plot(t, x2)
        grid on; grid minor; box on;
        ylabel('$x_2$')
        xlabel('time [sec]')
```

```
        title_string = 'Duffing System with Non-Zero Input ($x_1(0)=%0.1f$, $x_2(0)=%0.1f$) - T. Koike';
        title_S = sprintf(title_string, x1_0, x2_0);
        sgtitle(title_S)
    file_string = 'hw4_ex4_CL_case%d.png';
    file_S = sprintf(file_string, i);
    saveas(fig, file_S);
end
```

Exercise 5

Determine whether or not the following function is radially unbounded.

$$V(x) = x_1 - x_1^3 + x_1^4 - x_1^2 + x_2^4$$

Solution:
Since we have a $+x_1^4$ and $+x_2^4$,

$$\lim_{\|x\| \to \infty} V(x) = \infty$$

Thus, this Lyapunov function is radially unbounded.

Exercise 6

(Forced Duffing's equation with damping) Show that all solutions of the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - x_1^3 - cx_2 + 1 \qquad c > 0\end{aligned}$$

are bounded.
Hint: Consider

$$V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$

where $0 < \lambda < 1$. Letting

$$P = \frac{1}{2}\begin{bmatrix}\lambda c^2 & \lambda c \\ \lambda c & 1\end{bmatrix}$$

note that $P > 0$ and

$$\begin{aligned}V(x) &= x^T P x - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 \\ &\geq x^T P x - \frac{1}{4} \quad .\end{aligned}$$

Solution:
From what we are given we know that the Lyapunov function

$$V(x) = \frac{1}{2}\lambda c^2 x_1^2 + \lambda c x_1 x_2 + \frac{1}{2}x_2^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4$$

is radially unbounded and lpd about $\frac{1}{4}$.

Then,

$$\begin{aligned}DV(x)f(x) &= \begin{bmatrix}\lambda c^2 x_1 + \lambda c x_2 - x_1 + x_1^3 & \lambda c x_1 + x_2\end{bmatrix}\begin{bmatrix}x_2 \\ x_1 - x_1^3 - cx_2 + 1\end{bmatrix} \\ &= \lambda c^2 x_1 x_2 + \lambda c x_2^2 - x_1 x_2 + x_1^3 x_2 \\ &\qquad \lambda c x_1^2 - \lambda c x_1^4 - \lambda c^2 x_1 x_2 + \lambda c x_1 \\ &\qquad\qquad x_1 x_2 - x_1^3 x_2 - c x_2^2 + x_2 \\ &= -\lambda c x_1^4 - c x_2^2 + \lambda c x_1^2 + \lambda c x_2^2 + \lambda c x_1 + x_2\end{aligned}$$

For this equation when $\|x\| \geq R$ for a large $\|x\|$, $DV(x)f(x)$ is dominated by the terms $-\lambda c x_1^4$ and $-c x_2^2$. Hence

$$DV(x)f(x) \leq 0$$

and the solutions of the system for the given function are bounded.

Exercise 7

Show that all solutions of

$$\dot{x} = \cos x - x^3 + 100$$

are bounded.

Solution:
Using the Lyapunov function of

$$V(x) = x^2$$

we calculate

$$DV(x)f(x) = 2x(\cos x - x^3 + 100) = -x^3 + 2x \cos x + 100.$$

Observing this we can tell that in the range of $\|x\| \geq R$ for a large $\|x\|$ the function $DV(x)f(x)$ is dominated by the $-x^3$ term which results in

$$DV(x)f(x) \leq 0.$$

Hence, all solutions for the given equation is bounded.

Exercise 8

Show that all solutions of

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= \cos x - x^3 + 100\end{aligned}$$

are bounded.

Solution:
Using the Lyapunov function of

$$V(x) = 1 - \sin x_1 + \frac{1}{4}x_1^4 - 100x_1 + \frac{1}{2}x_2^2$$

knowing that this Lyapunov equation is radially unbounded, we calculate

$$\begin{aligned}DV(x)f(x) &= \begin{bmatrix}-\cos x_1 + x_1^3 - 100 & x_2\end{bmatrix}\begin{bmatrix}x_2 \\ \cos x_1 - x_1^3 + 100\end{bmatrix} \\ &= -x_2\cos x_1 + x_1^3x_2 - 100x_2 + x_2\cos x_1 - x_1^3x_2 + 100x_2 \\ &= 0 \\ &\leq 0\end{aligned}$$

Hence, all solutions for the given equation is bounded.

Exercise 9

Show that the following system is GES about zero.

$$\dot{x} = -(2 + \sin x)x$$

Give a rate of convergence.

Solution:
Considering

$$V(x) = x^2$$

we have

$$\begin{aligned} DV(x)f(x) &= -2(2 + \sin x)x^2 \\ &= -2(2 + \sin x)V \\ &= -4V - 2(\sin x)V \\ &\leq -4V \quad . \end{aligned}$$

Hence, we have GES about zero with rate of convergence 4.

Exercise 10

Show that the following system is GES about 1.

$$\dot{x} = -(2 + \sin x)(x - 1)$$

Give a rate of convergence.

Solution:
Considering

$$V(x) = (x - 1)^2$$

we have

$$\begin{aligned} DV(x)f(x) &= -2(2 + \sin x)(x - 1)^2 \\ &= -2(2 + \sin x)V \\ &= -4V - 2(\sin x)V \\ &\leq -4V \quad . \end{aligned}$$

Hence, we have GES about 1 with rate of convergence 4.

Exercise 11

Show that the following system is GES about the zero state.

$$\begin{aligned}
\dot{x}_1 &= -x_1 + (I_2 - I_3)x_2x_3 \\
\dot{x}_2 &= -2x_2 + (I_3 - I_1)x_1x_3 \\
\dot{x}_3 &= -3x_3 + (I_1 - I_2)x_1x_2
\end{aligned}$$

where I_1, I_2, I_3 are arbitrary constants. Give a rate of convergence.

Solution:
Considering

$$P = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$\begin{aligned}
x^T P f(x) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -x_1 + (I_2 - I_3)x_2x_3 \\ -2x_2 + (I_3 - I_1)x_1x_3 \\ -3x_3 + (I_1 - I_2)x_1x_2 \end{bmatrix} \\
&= -x_1^2 - 2x_2^2 - 3x_3^2 \\
&= - \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
&= -x^T Q x
\end{aligned}$$

Since the two matrices P and Q are positive definite symmetric matrices we can say that this system is GES about the zero state. The convergence rate is

$$\begin{aligned}
\alpha &= \lambda_{min}(P^{-1}Q) \\
&= \lambda_{min} \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right) \\
&= 1 \quad .
\end{aligned}$$