College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW5

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Exercise 1

Using linearization, determine (if possible) the stability properties of the following system about the zero solution.

$$\frac{d^4q}{dt^4} - \sin(q) = 0.$$

If not possible, explain why.

Solution:
If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\begin{aligned} \delta q^{(4)} - \sin \delta q \cos q_e = 0 \\ \delta q^{(4)} - \delta q = 0. \end{aligned}$$

Let

$$\begin{aligned} x_1 &:= \delta q \\ x_2 &:= \delta \dot{q} \\ x_3 &:= \delta \ddot{q} \\ x_4 &:= \delta q^{(3)} \end{aligned}$$

then the system matrix A becomes

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0. \end{bmatrix}$$

For this linearized system

$$eig(A) = \pm 1, \pm j.$$

Since there is a positive real eigenvalue the linearized system is unstable. Hence, the nonlinear system is unstable.

Exercise 2

Using linearization, determine (if possible) the stability properties of the following system about the zero solution.

$$\ddot{q} + \dot{q} - q^3 = 0.$$

If not possible, explain why.

Solution:
If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\begin{aligned} \delta\ddot{q} + \delta\dot{q}3q_e^3\delta q = 0 \\ \delta\ddot{q} + \delta\dot{q} = 0. \end{aligned}$$

Let

$$\begin{aligned} x_1 &:= \delta q \\ x_2 &:= \delta\dot{q} \end{aligned}$$

then the system matrix A becomes

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

For this linearized system

$$eig(A) = -1, 0.$$

Since there is a negative real eigenvalue and a eigenvalue at the origin, this linearized system is stable. However, for the nonlinear system the eigenvalue on the origin makes the system stability undetermined.

Exercise 3

If possible, use linearization to determine the stability properties of each of the following systems about the zero equilibrium state.

(i)

$$\begin{aligned}\dot{x}_1 &= (1 + x_1^2)x_2 \\ \dot{x}_2 &= -x_1^3\end{aligned}$$

(ii)

$$\begin{aligned}\dot{x}_1 &= \sin x_2 \\ \dot{x}_2 &= (\cos x_1)x_3 \\ \dot{x}_3 &= e^{x_1}x_2\end{aligned}$$

Solution:
(i) If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\begin{aligned}\delta\dot{x}_1 &= \delta x_2 + 2x_{1e}x_{2e}\delta x_1 + x_{1e}^2\delta x_2 \\ \delta\dot{x}_2 &= -3x_{1e}^2\delta x_1.\end{aligned}$$

$$\begin{aligned}\delta\dot{x}_1 &= \delta x_2 \\ \delta\dot{x}_2 &= 0.\end{aligned}$$

Then the system matrix A becomes

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For this linearized system

$$eig(A) = 0.$$

Since there is only an eigenvalue at the origin, this linearized system is unstable. However, for the nonlinear system the eigenvalue on the origin makes the system stability undetermined.

(ii) If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\begin{aligned}\delta\dot{x}_1 &= \delta x_2 \cos x_{2e} \\ \delta\dot{x}_2 &= \delta x_3 \cos x_{1e} - \delta x_1 x_{3e} \sin x_{1e} \\ \delta\dot{x}_3 &= e^{x_{1e}}x_{2e}\delta x_1 + e^{x_{1e}}\delta x_2.\end{aligned}$$

$$
\begin{aligned}
\delta \dot{x}_1 &= \delta x_2 \\
\delta \dot{x}_2 &= \delta x_3 \\
\delta \dot{x}_3 &= \delta x_2.
\end{aligned}
$$

Then the system matrix A becomes

$$
A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
$$

For this linearized system

$$
eig(A) = -1, 0, 1.
$$

Since there is a positive real eigevalue this linearized system is unstable. So the nonlinear system is also unstable.

Exercise 4

If possible, use linearization to determine the stability properties of the following system about the zero equilibrium state.

$$\begin{aligned} x_1(k+1) &= x_1(k)^2 + \sin(x_2(k)) \\ x_2(k+1) &= 0.4\cos(x_2(k))x_1(k) \end{aligned}$$

Solution:
If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\begin{aligned} \delta x_1(k+1) &= 2x_{1e}(k)\delta x_1(k) + \cos(x_{2e}(k))\delta x_2(k) \\ \delta x_2(k+1) &= -0.4\sin(x_{2e}(k))x_{1e}(k)\delta x_2(k) + 0.4\cos(x_{2e}(k))\delta x_1(k) \end{aligned}$$

$$\begin{aligned} \delta x_1(k+1) &= \delta x_2(k) \\ \delta x_2(k+1) &= 0.4\delta x_1(k) \end{aligned}$$

then the system matrix A becomes

$$A = \begin{bmatrix} 0 & 1 \\ 0.4 & 0 \end{bmatrix}$$

For this linearized system

$$eig(A) = \pm 0.6325.$$

Since the eigenvalues for this linearized discrete time system have a magnitude contained in the unit circle it is asymptotically stable. Thus, the nonlinear system is stable.

Exercise 5

If possible, use linearization to determine the stability properties of the following system about the zero equilibrium state.

$$x_1(k+1) = (1x_1(k)^3)x_2(k)$$
$$x_2(k+1) = x_1(k)^3 + x_2(k)^5$$

Solution:
If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\delta x_1(k+1) = 3_x 1e(k)^2 x_{2e}(k)\delta x_1(k) + x_{1e}(k)^3 \delta x_2(k) + \delta x_2(k)$$
$$\delta x_2(k+1) = 3x_{1e}(k)^2 \delta x_1(k) + 5x_{2e}(k)^4 \delta x_2(k)$$

$$\delta x_1(k+1) = \delta x_2(k)$$
$$\delta x_2(k+1) = 0$$

then the system matrix A becomes

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

For this linearized system

$$eig(A) = 0, 0.$$

Since the eigenvalues have a magnitude of 0 which is smaller than 1 this linearized system is asymptotically stable. Thus, the nonlinear system is stable.

Exercise 6

If possible, use linearization to determine the stability properties of the following system about the zero equilibrium state.

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= \sin(x_1(k)) + x_2(k)^5 \end{aligned}$$

Solution:
If the equilibrium state is $q_e = 0$, we can linearize the system as

$$\begin{aligned} \delta x_1(k+1) &= \delta x_2(k) \\ \delta x_2(k+1) &= \cos(x_{1e}(k))\delta x_1(k) + 5x_{2e}(k)^4 \delta x_2(k) \end{aligned}$$

$$\begin{aligned} \delta x_1(k+1) &= \delta x_2(k) \\ \delta x_2(k+1) &= 0 \end{aligned}$$

then the system matrix A becomes

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For this linearized system

$$eig(A) = \pm 1.$$

Since the eigenvalues of the linearized system is 1 the linearized system is stable. However, if there is at least one eigenvalue with a magnitude of 1 for the linearized system, the nonlinear system becomes undetermined.

Exercise 7

Recall the Lorenz system

$$\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= rx_1 - x_2 - x_1x_3 \\
\dot{x}_3 &= -bx_3 + x_1x_2
\end{aligned}$$

with $\sigma, r, b > 0$. Prove that all solutions of this system are bounded. (Hint: Consider $V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$.)

Solution:
Considering the candidate Lyapunov function

$$V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$$

since this function is

$$\lim_{\|x\| \to \infty} V(x) = \infty$$

this Lyapunov function is radially unbounded. Then if we calculate

$$\begin{aligned}
DV(x)f(x) &= \begin{bmatrix} 2rx_1 & 2\sigma x_2 & 2\sigma(x_3 - 2r) \end{bmatrix} \begin{bmatrix} \dot{x}_1 & = \sigma(x_2 - x_1) \\ \dot{x}_2 & = rx_1 - x_2 - x_1x_3 \\ \dot{x}_3 & = -bx_3 + x_1x_2 \end{bmatrix} \\
&= 2rx_1\sigma(x_2 - x_1) + 2\sigma x_2(rx_1 - x_2 - x_1x_3) + 2\sigma(x_3 - 2r)(-bx_3 + x_1x_2) \\
&= 2\sigma rx_1x_2 - 2\sigma rx_1^2 + 2\sigma rx_1x_2 - 2\sigma x_2^2 - 2\sigma x_1x_2x_3 \\
&\qquad - \sigma bx_3^2 + 2\sigma x_1x_2x_3 + 4\sigma rbx_3 - 4\sigma rx_1x_2 \\
&= -2\sigma(rx_1^2 + x_2^2 + bx_3^2) + 4\sigma rbx_3
\end{aligned}$$

Now since if $\|x\|$ was a very large number $DV(x)f(x)$ would go to negative infinity due to the $-x_1^2, -x_2^2, -x_3^2$ terms. Thus,

$$DV(x)f(x) = -2\sigma(rx_1^2 + x_2^2 + bx_3^2) + 4\sigma rbx_3 \leq 0 \qquad \text{for} \qquad \|x\| \geq R$$

and we have proven that all solutions of $\dot{x} = f(x)$ are radially bounded.