Homework #1 <u>Due</u>: September 18, 2019

1. For each of the following functions $f\colon \mathcal{D} \mapsto \mathbb{R}$ determine whether a minimum and/or an infimum of $f(\mathcal{D})$ exists and explain why or why not Weierstrass's theorem applies:

   i) $\mathcal{D} = (-1,1)$, $f(x) = x^2$.

   ii) $\mathcal{D} = (1,2]$, $f(x) = \dfrac{1}{1-x}$.

   iii) $\mathcal{D} = [0,1]$, $f(0) = 0, f(x) = 1, x \in (0,1]$.

2. Determine $\mathrm{vcone}(\mathcal{D},(x_0,y_0))$ for the following sets $\mathcal{D} \subset \mathbb{R}^2$ and $(x_0,y_0) \in \mathcal{D}$:

   i) $\mathcal{D} = \{(x,y)\colon y \geq 0\}$ and $(x_0,y_0) = (4,0)$.

   ii) $\mathcal{D} = \{(x,y)\colon x^2+y^2 \leq 1\}$ and $(x_0,y_0) = (1,0)$.

3. Let $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$ be given by $f(x_1,x_2) = \sqrt{x_1^2+x_2^2}$. Evaluate $\mathrm{D}_+ f((0,0);(\xi_1,\xi_2))$.

4. Minimize the function $f : \mathcal{D} \mapsto \mathbb{R}$

   $$f(x_1,x_2) = x_1^3 + x_2^3,$$

   where $\mathcal{D} = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}$.

5. Assume a steady and level flight of an airplane and consider the propulsive thrust given by

   $$T = \tfrac{1}{2}\rho V^2 S C_{D_{\mathrm{par}}} + \frac{KW^2}{\frac{1}{2}\rho V^2 S},$$

   where ρ is air density, V is aircraft velocity, $C_{D_{\mathrm{par}}}$ is the zero-lift (parasitic) drag coefficient, K is the drag polar constant, and S is wing surface area. The drag coefficient C_D is given by the drag polar

   $$C_D = C_{D_{\mathrm{par}}} + KC_L^2,$$

   and the lift coefficient is

   $$C_L = \frac{W}{\frac{1}{2}\rho V^2 S}$$

   .

   Consider the problem of finding the aircraft velocity V that minimizes the thrust T. Determine whether this problem is convex, and find all local and global minimizers and the corresponding values of T, C_L, C_D, and C_L/C_D.

6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

   $$f(y,z) = (z - py^2)(z - qy^2)$$

   where $0 < p < q$.

(a) Show that $x_0 = (0,0)$ is a local minimizer of f along every line that passes through $(0,0)$, that is, for all $h \in \mathbb{R}^2$, the function $g(a) = f(x_0 + ah)$ is locally minimized by $a = 0$.

(b) Show that $f'(x_0) = 0$.

(c) Show that x_0 is <u>not</u> a local minimizer of f. (Hint: If $p < m < q$, then $f(y, my^2) < 0$ for $y \neq 0$ while $f(0,0) = 0$.)

iv) Plot the function f using $p = 1, q = 2$ to illustrate the fact that for this function x_0 is not a local minimizer even though x_0 is a local minimizer along every line through the origin.

This example demonstrates why working with the Gâteaux differential (which looks at the derivative of a function along one direction at a time) may lead to erroneous conclusions when solving optimization problems. This is more that a theoretical curiosity. A numerical algorithm based on screening potential minimizers by searching points where the Gâteaux differential zero will yield erroneous results. In this case, such an algorithm will return the origin as a strict local minimizer for this function whereas, as seen above, the origin is not a local minimizer.