College of Engineering
Daniel Guggenheim School of Aerospace Engineering

AE6210: Advanced Dynamics I

Homework 5

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Table of Contents

I Problem One

Consider the double compound pendulum moving in the x-y plane as shown below. The rods are equal length, equal mass, and rigid. The hinges are frictionless. Derive the equations of motion using Lagrangian mechanics or Hamilton's principle or Lagrange's equations.

Solution:

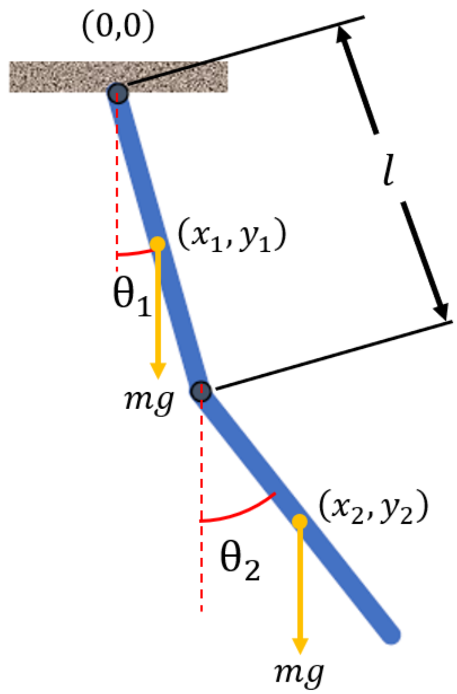

Figure 1: Double compound pendulum with generalized coordinates.

Let the mass and moment of inertia (about the CoM of the rod) of each rod be m and $I = ml^2/12$ respectively. The center of mass of the first rod is

$$x_1 = \frac{l}{2}\sin\theta_1 \tag{I.1}$$

$$y_1 = -\frac{l}{2}\cos\theta_1, \tag{I.2}$$

and the center of mass for the second rod is similarly

$$x_2 = l\left(\sin\theta_1 + \frac{1}{2}\sin\theta_2\right) \tag{I.3}$$

$$y_2 = -l\left(\cos\theta_1 + \frac{1}{2}\cos\theta_2\right) \tag{I.4}$$

If we define the coordinates to be unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ which they are in the direction of left to right and bottom to top respectively, then the velocity of the first rod (the velocity of its center of mass) becomes

$$\mathbf{v}_1 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{l}{2}\sin\theta_1\hat{\mathbf{i}} - \frac{l}{2}\cos\theta_1\hat{\mathbf{j}}\right) = \frac{l}{2}\dot{\theta}_1\cos\theta_1\hat{\mathbf{i}} + \frac{l}{2}\dot{\theta}_1\sin\theta_1\hat{\mathbf{j}} \tag{I.5}$$

Likewise the velocity of the second rod is

$$\mathbf{v}_2 = \left(l\dot{\theta}_1\cos\theta_1 + \frac{l}{2}\dot{\theta}_2\cos\theta_2\right)\hat{\mathbf{i}} + \left(l\dot{\theta}_1\sin\theta_1 + \frac{l}{2}\dot{\theta}_2\sin\theta_2\right)\hat{\mathbf{j}}. \tag{I.6}$$

Now the kinetic energy from translation becomes

$$\begin{aligned}
T_t &= \frac{1}{2}m(\mathbf{v}_1\cdot\mathbf{v}_1 + \mathbf{v}_2\cdot\mathbf{v}_2) \\
&= \frac{1}{2}m\left[\frac{l^2}{4}\dot{\theta}_1^2\cos^2\theta_1 + \frac{l^2}{4}\dot{\theta}_1^2\sin^2\theta_1 + \left(l\dot{\theta}_1\cos\theta_1 + \frac{l}{2}\dot{\theta}_2\cos\theta_2\right)^2 + \left(l\dot{\theta}_1\sin\theta_1 + \frac{l}{2}\dot{\theta}_2\sin\theta_2\right)^2\right] \\
&= \frac{1}{8}ml^2\dot{\theta}_1^2 + \frac{1}{2}m\left(l^2\dot{\theta}_1^2\cos^2\theta_1 + l^2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2 + \frac{1}{4}l^2\dot{\theta}_2^2\cos^2\theta_2\right. \\
&\qquad\left. + l^2\dot{\theta}_1^2\sin^2\theta_1 + l^2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2 + \frac{1}{4}l^2\dot{\theta}_2^2\sin^2\theta_2\right) \\
&= \frac{1}{8}ml^2\dot{\theta}_1^2 + \frac{1}{2}ml^2\dot{\theta}_1^2 + \frac{1}{8}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) \\
&= \frac{5}{8}ml^2\dot{\theta}_1^2 + \frac{1}{8}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2).
\end{aligned} \tag{I.7}$$

The kinetic energy due to rotation is

$$\begin{aligned}
T_r &= \frac{1}{2}I(\dot{\theta}_1^2 + \dot{\theta}_2^2) \\
&= \frac{1}{2}\left(\frac{1}{12}ml^2\right)\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) \\
&= \frac{1}{24}ml^2\dot{\theta}_1^2 + \frac{1}{24}ml^2\dot{\theta}_2^2.
\end{aligned} \tag{I.8}$$

Thus the total kinetic energy becomes

$$T = T_t + T_r = \frac{2}{3}ml^2\dot{\theta}_1^2 + \frac{1}{6}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2). \tag{I.9}$$

The potential energy is only due to gravity, which is

$$U = -\frac{1}{2}mgl\cos\theta_1 - mgl\left(\cos\theta_1 + \frac{1}{2}\cos\theta_2\right) = -\frac{3}{2}mgl\cos\theta_1 - \frac{1}{2}mgl\cos\theta_2. \tag{I.10}$$

The Lagrangian is then

$$L = T - U = \frac{2}{3}ml^2\dot{\theta}_1^2 + \frac{1}{6}ml^2\dot{\theta}_2^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + \frac{3}{2}mgl\cos\theta_1 + \frac{1}{2}mgl\cos\theta_2. \tag{I.11}$$

Now we can find the EOM for each θ_1 and θ_2.

$$\frac{\partial L}{\partial\dot{\theta}_1} = \frac{4}{3}ml^2\dot{\theta}_1 + \frac{1}{2}ml^2\dot{\theta}_2\cos(\theta_1-\theta_2). \tag{I.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial t}\right) = \frac{4}{3}ml^2\ddot{\theta}_1 + \frac{1}{2}ml^2\ddot{\theta}_2\cos(\theta_1-\theta_2) - \frac{1}{2}ml^2\dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)\sin(\theta_1-\theta_2). \tag{I.13}$$

$$\frac{\partial L}{\partial\theta_1} = -\frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - \frac{3}{2}mgl\sin\theta_1. \tag{I.14}$$

and

$$\frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{3}ml^2\dot{\theta}_2 + \frac{1}{2}ml^2\dot{\theta}_1\cos(\theta_1 - \theta_2). \tag{I.15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = \frac{1}{3}ml^2\ddot{\theta}_2 + \frac{1}{2}ml^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \frac{1}{2}ml^2\dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2). \tag{I.16}$$

$$\frac{\partial L}{\partial \theta_2} = \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) - \frac{1}{2}mgl\sin\theta_2. \tag{I.17}$$

Then we can obtain the EOM for θ_1 with (I.13) and (I.14) as follows.

$$\frac{4}{3}ml^2\ddot{\theta}_1 + \frac{1}{2}ml^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) - \frac{1}{2}ml^2\dot{\theta}_2(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2) + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{3}{2}mgl\sin\theta_1 = 0$$

$$\frac{4}{3}ml^2\ddot{\theta}_1 + \frac{1}{2}ml^2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + \frac{1}{2}ml^2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + \frac{3}{2}mgl\sin\theta_1 = 0. \tag{I.18}$$

Likewise the EOM for θ_2 can be found with (I.16) and (I.17)

$$\frac{1}{3}ml^2\ddot{\theta}_2 + \frac{1}{2}ml^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \frac{1}{2}ml^2\dot{\theta}_1(\dot{\theta}_1 - \dot{\theta}_2)\sin(\theta_1 - \theta_2) - \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1 - \theta_2) + \frac{1}{2}mgl\sin\theta_2 = 0$$

$$\frac{1}{3}ml^2\ddot{\theta}_2 + \frac{1}{2}ml^2\ddot{\theta}_1\cos(\theta_1 - \theta_2) - \frac{1}{2}ml^2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + \frac{1}{2}mgl\sin\theta_2 = 0 \tag{I.19}$$

If we organize the two equations (I.18) and (I.19) we end up with the following equations.

$$\boxed{\begin{aligned} &8l\ddot{\theta}_1 + 3l\ddot{\theta}_2\cos(\theta_1 - \theta_2) + 3l\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + 9g\sin\theta_1 = 0 \\ &2l\ddot{\theta}_2 + 3l\ddot{\theta}_1\cos(\theta_1 - \theta_2) - 3l\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + 3g\sin\theta_2 = 0 \end{aligned}} \tag{I.20}$$

II Problem Two

Next derive the equations of motion using Newton's law or Newton-Euler equations.

Solution:

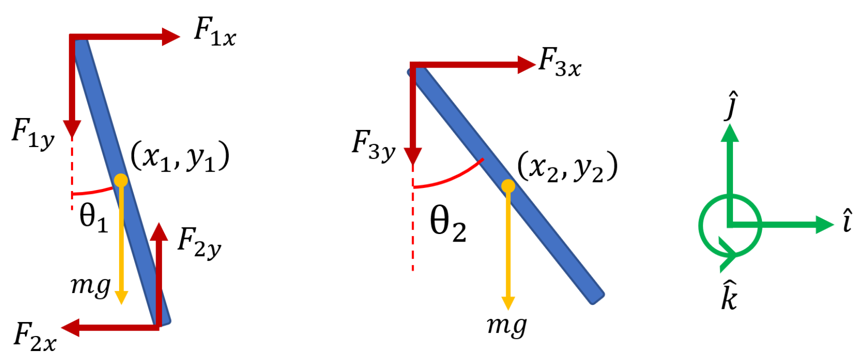

Figure 2: FBD of rod 1 and rod 2.

Let the CoM positions of the two rods be denoted as $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then, from Problem 1, we know that these positions are

$$\mathbf{r}_1 = \frac{l}{2}\sin\theta_1\hat{\mathbf{i}} - \frac{l}{2}\cos\theta_1\hat{\mathbf{j}} \tag{II.1}$$

$$\mathbf{r}_2 = \left(l\sin\theta_1 + \frac{l}{2}\sin\theta_2\right)\hat{\mathbf{i}} - \left(l\cos\theta_1 + \frac{l}{2}\cos\theta_2\right)\hat{\mathbf{j}} \tag{II.2}$$

Then we can find the velocities and the accelerations of the CoM for these two rods by taking the derivatives.

$$\dot{\mathbf{r}}_1 = \frac{l}{2}\dot{\theta}_1\cos\theta_1\hat{\mathbf{i}} + \frac{l}{2}\dot{\theta}_1\sin\theta_1\hat{\mathbf{j}} \tag{II.3}$$

$$\ddot{\mathbf{r}}_1 = \left(\frac{l}{2}\ddot{\theta}_1\cos\theta_1 - \frac{l}{2}\dot{\theta}_1^2\sin\theta_1\right)\hat{\mathbf{i}} + \left(\frac{l}{2}\ddot{\theta}_1\sin\theta_1 + \frac{l}{2}\dot{\theta}_1^2\cos\theta_1\right)\hat{\mathbf{j}} \tag{II.4}$$

and

$$\dot{\mathbf{r}}_2 = \left(l\dot{\theta}_1\cos\theta_1 + \frac{l}{2}\dot{\theta}_2\cos\theta_2\right)\hat{\mathbf{i}} + \left(l\dot{\theta}_1\sin\theta_1 + \frac{l}{2}\dot{\theta}_2\sin\theta_2\right)\hat{\mathbf{j}} \tag{II.5}$$

$$\ddot{\mathbf{r}}_2 = \left(l\ddot{\theta}_1\cos\theta_1 - l\dot{\theta}_1^2\sin\theta_1 + \frac{l}{2}\ddot{\theta}_2\cos\theta_2 - \frac{l}{2}\dot{\theta}_2^2\sin\theta_2\right)\hat{\mathbf{i}} \tag{II.6}$$

$$+\left(l\ddot{\theta}_1\sin\theta_1 + l\dot{\theta}_1^2\cos\theta_1 + \frac{l}{2}\ddot{\theta}_2\sin\theta_2 + \frac{l}{2}\dot{\theta}_2^2\cos\theta_2\right)\hat{\mathbf{j}} \tag{II.7}$$

Now from the FBD we can write out the Newton-Euler equations with the sum of all forces and moments about the center of mass of the two rods. The equations of moment are

$$I\ddot{\theta}_1 = \frac{1}{12}ml^2\ddot{\theta}_1 = -F_{1x}\frac{l}{2}\cos\theta_1 + F_{1y}\frac{l}{2}\sin\theta_1 - F_{2x}\frac{l}{2}\cos\theta_1 + F_{2y}\frac{l}{2}\sin\theta_1 \tag{II.8}$$

$$I\ddot{\theta}_2 = \frac{1}{12}ml^2\ddot{\theta}_2 = -F_{3x}\frac{l}{2}\cos\theta_2 + F_{3y}\frac{l}{2}\sin\theta_2 \tag{II.9}$$

The equations of forces are

$$F_{1x} - F_{2x} = m\left(\frac{l}{2}\ddot{\theta}_1\cos\theta_1 - \frac{l}{2}\dot{\theta}_1^2\sin\theta_1\right) \tag{II.10}$$

$$F_{2y} - F_{1y} - mg = m\left(\frac{l}{2}\ddot{\theta}_1\sin\theta_1 + \frac{l}{2}\dot{\theta}_1^2\cos\theta_1\right) \tag{II.11}$$

$$F_{3x} = m\left(l\ddot{\theta}_1\cos\theta_1 - l\dot{\theta}_1^2\sin\theta_1 + \frac{l}{2}\ddot{\theta}_2\cos\theta_2 - \frac{l}{2}\dot{\theta}_2^2\sin\theta_2\right) \tag{II.12}$$

$$-F_{3y} - mg = m\left(l\ddot{\theta}_1\sin\theta_1 + l\dot{\theta}_1^2\cos\theta_1 + \frac{l}{2}\ddot{\theta}_2\sin\theta_2 + \frac{l}{2}\dot{\theta}_2^2\cos\theta_2\right) \tag{II.13}$$

Furthermore, we have a constraint at the joint of the two rods where the tension between the two rods must be equal, i.e.

$$F_{2x} = F_{3x} \qquad F_{2y} = F_{3y}. \tag{II.14}$$

By plugging in the equations (II.12) and (II.13) into (II.9) we are able to get the EOM for θ_2. Then using the constraint (II.14) and plugging in (II.12) into (II.10) and (II.13) into (II.11) respectively we can find the expression for F_{1x} and F_{1y}. Thus, we found the expressions for forces F_{1x}, F_{1y}, F_{2x}, and F_{2y}. In order to find the EOM for θ_1 we can plug these expressions into (II.8). Solving the trivial algebra we are able to find the following expressions (the algebra was done using MATLAB and the code is given in the Appendix).

$$\boxed{\begin{aligned} &8l\ddot{\theta}_1 + 3l\ddot{\theta}_2\cos(\theta_1-\theta_2) + 3l\dot{\theta}_2^2\sin(\theta_1-\theta_2) + 9g\sin\theta_1 = 0 \\ &2l\ddot{\theta}_2 + 3l\ddot{\theta}_1\cos(\theta_1-\theta_2) - 3l\dot{\theta}_1^2\sin(\theta_1-\theta_2) + 3g\sin\theta_2 = 0 \end{aligned}} \tag{II.15}$$

III Appendix

MATLAB Code

```
clear; close all; clc;

syms F_1x F_2x F_1y F_2y
syms theta_1(t) theta_2(t) l m g

dtheta1 = diff(theta_1,t);
ddtheta1 = diff(dtheta1,t);
dtheta2 = diff(theta_2,t);
ddtheta2 = diff(dtheta2,t);
I = 1/12*m*l^2;

F_3x = m*(l*ddtheta1*cos(theta_1)-l*dtheta1^2*sin(theta_1) ...
    + l/2*ddtheta2*cos(theta_2)-l/2*dtheta2^2*sin(theta_2));
F_3y = -m*(l*ddtheta1*sin(theta_1)+l*dtheta1^2*cos(theta_1) ...
    + l/2*ddtheta2*sin(theta_2)+l/2*dtheta2^2*cos(theta_2)) - m*g;

% EOM of theta2
eom2 = I*ddtheta2 + F_3x*l/2*cos(theta_2) - F_3y*l/2*sin(theta_2) == 0;
eom2 = simplify(expand(eom2))
%%

eqn1 = F_1x - F_2x == m*(l/2*ddtheta1*cos(theta_1) - l/2*dtheta1^2*sin(theta_1));
eqn2 = F_2y - F_1y - m*g == m*(l/2*ddtheta1*sin(theta_1)+l/2*dtheta1^2*cos(theta_1));

eqn1 = subs(eqn1,F_2x,F_3x);
eqn2 = subs(eqn2,F_2y,F_3y);

F1x = solve(eqn1,F_1x);
F1y = solve(eqn2,F_1y);

eqn5 = I*ddtheta1 - (-F_1x*l/2*cos(theta_1)+F_1y*l/2*sin(theta_1) ...
    -F_2x*l/2*cos(theta_1)+F_2y*l/2*sin(theta_1)) == 0;

% EOM of theta1
eom1 = subs(eqn5,[F_1x F_1y F_2x F_2y],[F1x F1y F_3x F_3y]);
eom1 = simplify(expand(eom1))
```