College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 2
State Space Representation, Linearization, and Transfer Functions

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Exercise 1 Obtain the A, B, C, D matrices for a state space representation of the following systems:

(a)

$$
\begin{aligned}
u &= a_0 q + a_1 \dot{q} + \ldots + a_{n-1} q^{(n-1)} + q^{(n)} \\
y &= \beta_0 q + \beta_1 \dot{q} + \ldots + \beta_{n-1} q^{(n-1)} + \gamma u
\end{aligned}
$$

where $u(t), y(t) \in \mathbb{R}$.

(b)

$$
\begin{aligned}
u(k) &= a_0 q(k) + a_1 q(k+1) + \ldots + a_{n-1} q(k+n-1) + q(k+n) \\
y(k) &= \beta_0 q(k) + \beta_1 q(k+1) + \ldots + \beta_{n-1} q(k+n-1) + \gamma u(k)
\end{aligned}
$$

where $y(k), u(k) \in \mathbb{R}$

(c) The 'simple structure' with outputs $y_1 = q_1$ and $y_2 = q_2$.

(a) From the given equation

$$u = a_0 q + a_1 \dot{q} + \cdots + a_{n-1} q^{(n-1)} + q^{(n)}$$

Say, $q = x_0$, $\dot{q} = x_1$, $\ddot{q} = x_2$, $q^{(3)} = x_3$,

$\cdots$ $q^{(n-1)} = x_{n-1}$, $q^{n} = x_n$

Then, generally $x_j = \dot{x}_{i-1}$ $(0 \leqq j \leqq n)$

Thus,

$$\dot{x}_{n-1} = x_n = -a_0 x_0 - a_1 x_1 - \cdots - a_{n-1} q^{(n-1)} + u$$

$$
\underbrace{\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-3} \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \end{bmatrix}}_{n\times 1}
=
\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & & \cdots & & -a_{n-1} \end{bmatrix}}_{\substack{n\times n \\ \text{matrix } A}}
\underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-3} \\ x_{n-2} \\ x_{n-1} \end{bmatrix}}_{n\times 1}
+
\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\substack{n\times 1 \\ \text{col vector } B}} u
$$

Also,

$$
y = \beta_0 q + \beta_1 \dot{q} + \cdots + \beta_{n-1} q^{(n-1)} + \gamma u
$$

$$
y = \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} \begin{bmatrix} q & \dot{q} & \cdots & q^{(n-1)} \end{bmatrix} + \gamma u
$$

$$
y = \underbrace{\begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix}}_{\text{row vector } C} \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} + \underbrace{\gamma}_{\text{scalar } D} u
$$

Thus,

$$\begin{cases} \dot{X} = AX + Bu \\ y = CX + Du \end{cases}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} \dot{X} &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & & \cdots & & -a_{n-1} \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} X + \gamma u \end{aligned}}$$

(b) Similar to what we did in (a), the answer is exactly the same by defining

$q(k) = x_0$, $q(k+1) = x_1$, $\ldots$, $q(k+n-1) = x_{n-1}$,

$q(k+n) = x_n$

Then, $x_0(k+1) = x_1$

$\vdots$

generally $x_i(k+1) = x_{i+1} \quad (0 \leq i \leq n-1)$

Thus,

$$x_{n-1}(k+n) = -a_0 x_0(k) - a_1 x_1(k) - \cdots - a_{n-1} x_{n-1}(k+n-1) + u(k)$$

This discreet time system ends up having the same state-space representation as (a)

$$\begin{cases} X(k+1) = AX(k) + Bu(k) \\ y(k) = CX(k) + Du(k) \end{cases}$$

$$\Downarrow$$

$$\boxed{\begin{aligned} X(k+1) &= \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & & \cdots & & -a_{n-1} \end{bmatrix} X(k) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} \beta_0 & \beta_1 & \cdots & \beta_{n-1} \end{bmatrix} X(k) + \gamma u(k) \end{aligned}}$$

(c) The simple structure is defined as (from the notes)

$$\begin{cases} m_1\ddot{q}_1 + (c_1+c_2)\dot{q}_1 + (k_1+k_2)q_1 - c_2\dot{q}_2 - k_2 q_2 = u_1 \\ m_2\ddot{q}_2 - c_2\dot{q}_1 - k_2 q_1 + c_2\dot{q}_2 + k_2 q_2 = u_2 \\ y_1 = q_1 \\ y_2 = q_2 \end{cases}$$

Say $x_0 = q_1$, $x_1 = \dot{q}_1$, $x_2 = q_2$, $x_3 = \dot{q}_2$

Then, $\dot{x}_1 = \ddot{q}_1$, $\dot{x}_3 = \ddot{q}_2$

$$\begin{cases} m_1\dot{x}_1 + (c_1+c_2)x_1 + (k_1+k_2)x_0 - c_2 x_3 - k_2 x_2 = u_1 \\ m_2\dot{x}_3 - c_2 x_1 - k_2 x_0 + c_2 x_3 + k_2 x_2 = u_2 \\ y_1 = x_0 \\ y_2 = x_2 \end{cases}$$

$$\begin{cases} \dot{x}_1 = -\frac{k_1+k_2}{m}x_0 - \frac{c_1+c_2}{m}x_1 + \frac{k_2}{m}x_2 - \frac{c_2}{m}x_3 + u_1 \\ \dot{x}_3 = \frac{k_2}{m}x_0 + \frac{c_2}{m}x_1 - \frac{k_2}{m}x_2 - \frac{c_2}{m}x_3 + u_2 \\ y_1 = x_0 \\ y_2 = x_2 \end{cases}$$

$$\begin{bmatrix} \dot{x}_0 \\ \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m} & -\frac{c_1+c_2}{m} & \frac{k_2}{m} & -\frac{c_2}{m} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m} & \frac{c_2}{m} & -\frac{k_2}{m} & -\frac{c_2}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad D = 0$$

Exercise 2 Linearize (if possible) the following systems about *each* of their equilibrium states (if not possible, state why) and obtain the system A matrix for these linearized systems.

(a)

$$\dot{x} = x^3 - x$$

(b)

$$\dot{x} = \sqrt{|x|}$$

(c) The 'attitude dynamics' system. Consider non-symmetric case ($I_1 \neq I_2 \neq I_3$).

(a)

$$\dot{x} = x^3 - x \longrightarrow \ddot{x} = 3x^2\dot{x} - \dot{x}$$

linearize

$$\Longrightarrow \delta\ddot{x} = 3(x^e)^2\delta\dot{x} + 6x^e\dot{x}^e\,\delta x - \delta\dot{x}$$

$$\delta\ddot{x} = [3(x^e)^2 - 1]\,\delta\dot{x} + 6x^e\dot{x}^e\,\delta x$$

Say, $x_1 := \delta x$, $x_2 := \delta\dot{x}$

$$A = \begin{bmatrix} 0 & 1 \\ 6x^e\dot{x}^e & 3(x^e)^2 - 1 \end{bmatrix}$$

$$\dot{x} = 0 \longrightarrow x(x-1)(x+1) = 0 \longrightarrow x = 0, 1, -1$$

$$\boxed{\begin{array}{ll} x^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \\ x^e = \begin{bmatrix} 1 \\ 0 \end{bmatrix} & A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \\ x^e = \begin{bmatrix} -1 \\ 0 \end{bmatrix} & A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \end{array}}$$

(b) $$\dot{x} = \sqrt{|x|}$$

the definition of linearization is characterized as setting the state derivative as zero.

Thus, $\dot{x} = \sqrt{|x|} = 0$

This is only when $x^e = 0$

however, an absolute function is not differentiable at $x = 0$

(c) The attitude dynamics system is defined as

$$\begin{cases} I_1 \dot{\omega}_1 = (I_2 - I_3)\omega_2 \omega_3 \\ I_2 \dot{\omega}_2 = (I_3 - I_1)\omega_3 \omega_1 \\ I_3 \dot{\omega}_3 = (I_1 - I_2)\omega_1 \omega_2 \end{cases}$$

$$\begin{array}{ccc} \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & a \\ 0 & a & 0 \\ 0 & 0 & a \\ a & 0 & 0 \end{array}$$

linearize $\Downarrow$

$$\boxed{\begin{cases} I_1 \delta \dot{\omega}_1 = (I_2 - I_3)\omega_3 \delta\omega_2 + (I_2 - I_3)\omega_2 \delta\omega_3 \\ I_2 \delta \dot{\omega}_2 = (I_3 - I_1)\omega_1 \delta\omega_3 + (I_3 - I_1)\omega_3 \delta\omega_1 \\ I_3 \delta \dot{\omega}_3 = (I_1 - I_2)\omega_2 \delta\omega_1 + (I_1 - I_2)\omega_1 \delta\omega_2 \end{cases}}$$

$$x_1 := \delta\omega_1, \quad x_2 := \delta\omega_2, \quad x_3 := \delta\omega_3$$

$$A = \begin{bmatrix} 0 & (I_2 - I_3)\omega_3 & (I_2 - I_3)\omega_2 \\ (I_3 - I_1)\omega_3 & 0 & (I_3 - I_1)\omega_1 \\ (I_1 - I_2)\omega_2 & (I_1 - I_2)\omega_1 & 0 \end{bmatrix}$$

$$\boxed{x^e = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix} \quad a \ \forall \text{ real number} \qquad A = \begin{bmatrix} 0 & (I_2 - I_3)a & 0 \\ (I_3 - I_1)a & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$\boxed{x^e = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \quad a \ \forall \text{ real number} \qquad A = \begin{bmatrix} 0 & 0 & (I_2 - I_3)a \\ 0 & 0 & 0 \\ (I_1 - I_2)a & 0 & 0 \end{bmatrix}}$$

$$x^e = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \quad a \ \forall \text{ real number}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & (I_3 - I_1)\,a \\ 0 & (I_1 - I_2)\,a & 0 \end{bmatrix}$$

Exercise 3 (a) Obtain all equilibrium states of the following system:

$$\begin{aligned}\dot{x}_1 &= 2x_2(1-x_1) - x_1 \\ \dot{x}_2 &= 3x_1(1-x_2) - x_2\end{aligned}$$

(b) Linearize the above system about the zero equilibrium state.

(a)
$$\begin{cases}\dot{x}_1 = (2x_2 - 1)x_1 + 2x_2 \\ \dot{x}_2 = 3x_1 - (3x_1+1)x_2\end{cases}$$

When $\dot{x}_1 = \dot{x}_2 = 0$

$$x_1 = \frac{2x_2}{1-2x_2}$$

$$\frac{6x_2}{1-2x_2} - \left(\frac{6x_2}{1-2x_2} + 1\right)x_2 = 0$$

$$\frac{6x_2}{1-2x_2} - \left(\frac{4x_2+1}{1-2x_2}\right)x_2 = 0$$

$$\frac{6x_2 - 4x_2^2 - x_2}{1-2x_2} = 0$$

$$\frac{4x_2^2 - 5x_2}{2x_2 - 1} = 0$$

$$x_2 \neq \frac{1}{2} \Rightarrow x_2 = 0, \frac{5}{4}$$

Then, equilibrium points are

$$\boxed{\begin{aligned} &x_1 = 0 \quad \& \quad x_2 = 0 \\ &x_1 = -\frac{5}{3} \quad \& \quad x_2 = \frac{5}{4}\end{aligned}}$$

(b) linearize it

$$\begin{cases} \delta\dot{x}_1 = (2x_2^e - 1)\delta x_1 + 2x_1^e \delta x_2 + 2\delta x_2 \\ \delta\dot{x}_2 = 3\delta x_1 - (3x_1^e + 1)\delta x_2 - 3x_2^e \delta x_1 \end{cases}$$

$$\Downarrow$$

$$\delta\dot{x}_1 = (2x_2^e - 1)\delta x_1 + (2x_1^e + 2)\delta x_2$$

$$\delta\dot{x}_2 = (3 - 3x_2^e)\delta x_1 - (3x_1^e + 1)\delta x_2$$

$$A = \begin{bmatrix} 2x_2^e - 1 & 2x_1^e + 2 \\ 3 - 3x_2^e & -3x_1^e - 1 \end{bmatrix}$$

$$\boxed{\begin{aligned} x^e &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} \\ x^e &= \begin{bmatrix} -\frac{5}{3} \\ \frac{5}{4} \end{bmatrix} \quad A = \begin{bmatrix} \frac{3}{2} & -\frac{4}{3} \\ -\frac{3}{4} & 4 \end{bmatrix} \end{aligned}}$$

Exercise 4 For each of the following systems, linearize about each equilibrium solution and obtain the system A-matrix for a state space representation of these linearized systems.

(a)

$$\ddot{y} + (y^2 - 1)\dot{y} + y = 0\,.$$

where $y(t)$ is a scalar.

(b)

$$\ddot{y} + \dot{y} + y - y^3 = 0$$

where $y(t)$ is a scalar.

(c)

$$\begin{aligned}(M+m)\ddot{y} + ml\,\ddot{\theta}\cos\theta - ml\,\dot{\theta}^2\sin\theta + ky &= 0\\ ml\ddot{y}\cos\theta + ml^2\,\ddot{\theta} + mgl\,\sin\theta &= 0\end{aligned}$$

where $y(t)$ and $\theta(t)$ are scalars.

(d)

$$\ddot{y} + 0.5\dot{y}|\dot{y}| + y = 0\,.$$

where $y(t)$ is a scalar.

(a)
$$\begin{cases}\dot{x}_1 = x_2\\ \dot{x}_2 = -(x_1^2-1)x_2 - x_1\end{cases}$$

$$\Big\downarrow \text{linearize}$$

$$\delta\dot{x}_1 = \delta x_2$$
$$\delta\dot{x}_2 = -[(x_1^e)^2 - 1]\,\delta x_2 - 2x_1^e x_2^e\,\delta x_1 - \delta x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ -2x_1^e x_2^e - 1 & 1-(x_1^e)^2 \end{bmatrix}$$

$$\boxed{x^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}}$$

(b) $y = x_1$, $\dot{y} = x_2$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^3 - x_1 - x_2 \end{cases}$$

$\downarrow$ linearize

$$\begin{cases} \delta\dot{x}_1 = \delta x_2 \\ \delta\dot{x}_2 = 3(x_1^e)^2\delta x_1 - \delta x_1 - \delta x_2 \end{cases}$$

if $\dot{x}_1 = x_2 = 0$, $\dot{x}_2 = x_1(x_1^2 - 1) = 0$

$\Rightarrow x_1 = 0, 1, -1$

$$A = \begin{bmatrix} 0 & 1 \\ 3(x_1^e)^2 - 1 & -1 \end{bmatrix}$$

$$\boxed{\begin{aligned} x^e &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \\ x^e &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \\ x^e &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \end{aligned}}$$

(c) linearized

$$
\begin{cases}
(M+m)\delta\ddot{y} + m\ell\cos\theta^e\,\delta\ddot{\theta} - m\ell\ddot{\theta}^e\sin\theta^e\,\delta\theta \\
\qquad -2m\ell\dot{\theta}^e\sin\theta^e\,\delta\dot{\theta} - m\ell(\dot{\theta}^e)^2\cos\theta^e\,\delta\theta + k\,\delta y = 0 \\
m\ell\cos\theta^e\,\delta\ddot{y} - m\ell\ddot{y}^e\sin\theta^e\,\delta\theta + m\ell^2\,\delta\ddot{\theta} + mg\ell\cos\theta^e\,\delta\theta = 0
\end{cases}
$$

let $x_1 := \delta y$, $x_2 := \delta\dot{y}$, $x_3 := \delta\theta$, $x_4 := \delta\dot{\theta}$

$$
\begin{cases}
(M+m)\dot{x}_2 + m\ell\cos\theta^e\,\dot{x}_4 - m\ell\ddot{\theta}^e\sin\theta^e\,x_3 \\
\qquad -2m\ell\dot{\theta}^e\sin\theta^e\,x_4 - m\ell(\dot{\theta}^e)^2\cos\theta^e\,x_3 + k x_1 = 0 \\
m\ell\cos\theta^e\,\dot{x}_2 - m\ell\ddot{y}^e\sin\theta^e\,x_3 + m\ell^2\,\dot{x}_4 + mg\ell\cos\theta^e\,x_3 = 0
\end{cases}
$$

$$\Downarrow$$

$$
\begin{cases}
a\dot{x}_2 + b\dot{x}_4 + c = 0 \\
d\dot{x}_2 + e\dot{x}_4 + f = 0
\end{cases}
$$

Solve this using MATLAB, we get

$$\dot{x}_2 = \frac{l\,m\,x_3\,\dot{\theta}^2\cos(\theta) + 2\,l\,m\,x_4\sin(\theta)\,\dot{\theta} + g\,m\,x_3\cos(\theta)^2 - m\,x_3\,\ddot{y}\sin(\theta)\cos(\theta) - k\,x_1 + l\,m\,\ddot{\theta}\,x_3\sin(\theta)}{-m\cos(\theta)^2 + M + m}$$

$$
\begin{aligned}
= & -\frac{k}{M+m-m\cos^2\theta^e}\,x_1 \\
& + \frac{m\ell(\dot{\theta}^e)^2\cos\theta^e + mg\cos^2\theta^e - m\ddot{y}^e\sin\theta^e\cos\theta^e + m\ell\ddot{\theta}^e\sin\theta^e}{M+m-m\cos^2\theta^e}\,x_3 \\
& + \frac{2m\ell\dot{\theta}^e\sin\theta^e}{M+m-m\cos^2\theta^e}\,x_4
\end{aligned}
$$

$$\dot{x}_4 =$$

$$-\frac{M\,g\,x_3\cos(\theta) - k\,x_1\cos(\theta) + g\,m\,x_3\cos(\theta) - M\,x_3\,\ddot{y}\sin(\theta) - m\,x_3\,\ddot{y}\sin(\theta) + l\,m\,\dot{\theta}^2 x_3\cos(\theta)^2 + 2\,l\,m\,\dot{\theta}\,x_4\cos(\theta)\sin(\theta) + l\,m\,\ddot{\theta}\,x_3\cos(\theta)\sin(\theta)}{l\,(-m\cos(\theta)^2 + M + m)}$$

$$= \frac{k}{l\,(M+m-m\cos^2\theta^e)}x_1$$

$$+ \frac{(M+m)\ddot{y}^e\sin\theta^e - (M+m)g\cos\theta^e - ml(\dot{\theta}^e)^2\cos^2\theta^e - ml\ddot{\theta}^e\cos\theta^e\sin\theta^e}{l\,(M+m-m\cos^2\theta^e)}x_3$$

$$- \frac{2m\dot{\theta}^e\cos\theta^e\sin\theta^e}{M+m-m\cos^2\theta^e}x_4$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_3 = x_4$$

now we have the state-space representation

$$x^e = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \longrightarrow y^e = 0 \\ \longrightarrow \dot{y}^e = 0 \\ \longrightarrow \theta^e = 0 \\ \longrightarrow \dot{\theta}^e = 0 \end{matrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M} & 0 & \frac{M}{m}g & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{Ml} & 0 & -\frac{(M+m)g}{Ml} & 0 \end{bmatrix}$$

(d) $$\ddot{y} = -0.5\dot{y}|\dot{y}| - y$$

When $\dot{y} \geqq 0$

$$\ddot{y} = -0.5\dot{y}^2 - y$$

$$\delta\ddot{y} = -\dot{y}^e\delta\dot{y} - \delta y$$

let $x_1 := \delta y$, $x_2 := \delta\dot{y}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\dot{y}^e x_2 - x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -\dot{y}^e \end{bmatrix}$$

$$\boxed{x^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

When $\dot{y} < 0$

$$\ddot{y} = 0.5\dot{y}^2 - y$$

$$\delta\ddot{y} = \dot{y}^e\delta\dot{y} - \delta y$$

let $x_1 := \delta y$, $x_2 := \delta\dot{y}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \dot{y}^e x_2 - x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & \dot{y}^e \end{bmatrix}$$

$\dot{y}^e \neq 0$ since $\dot{y} < 0$

There is no equilibrium state for $\dot{y} < 0$

Exercise 5 Obtain the transfer function (matrix) of the following system

$$\begin{aligned} \ddot{y}_1 + \ddot{y}_2 + y_1 + y_2 &= u_1 + \dot{u}_2 \\ 2\ddot{y}_1 + 3\ddot{y}_2 + y_1 - y_2 &= 0 \end{aligned}$$

MIMO and zero ICs

$$\ddot{y}_1 + \ddot{y}_2 + y_1 + y_2 = u_1 + \dot{u}_2$$
$$2\ddot{y}_1 + 3\ddot{y}_2 + y_1 - y_2 = 0$$

Laplace transform

$$s^2\hat{y}_1(s) - s\cancelto{0}{y_1(0)} - \cancelto{0}{\dot{y}_1(0)} + s^2\hat{y}_2(s) - s\cancelto{0}{y_2(0)} - \cancelto{0}{\dot{y}_2(0)}$$
$$+ \hat{y}_1(s) + \hat{y}_2(s) = \hat{u}_1(s) + s\hat{u}_2(s) - \cancelto{0}{u_2(0)}$$

$$2s^2\hat{y}_1(s) - 2s\cancelto{0}{y_1(0)} - 2\cancelto{0}{\dot{y}_1(0)} + 3s^2\hat{y}_2(s) - 3s\cancelto{0}{y_2(0)} - 3\cancelto{0}{\dot{y}_2(0)}$$
$$+ \hat{y}_1(s) - \hat{y}_2(s) = 0$$

$$\Downarrow$$

$$\begin{cases} s^2\hat{y}_1(s) + s^2\hat{y}_2(s) + \hat{y}_1(s) + \hat{y}_2(s) = \hat{u}_1(s) + s\hat{u}_2(s) \\ 2s^2\hat{y}_1(s) + 3s^2\hat{y}_2(s) + \hat{y}_1(s) - \hat{y}_2(s) = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} (s^2+1)\hat{y}_1(s) + (s^2+1)\hat{y}_2(s) = \hat{u}_1(s) + s\hat{u}_2(s) & \cdots ① \\ (2s^2+1)\hat{y}_1(s) + (3s^2-1)\hat{y}_2(s) = 0 & \cdots ② \end{cases}$$

$$① \times (3s^2-1) - ② \times (s^2+1)$$

$$\Rightarrow \quad (3s^2-1)(s^2+1)\hat{y}_1(s) - (2s^2+1)(s^2+1)\hat{y}_1(s) = (3s^2-1)\hat{u}_1(s) + s(3s^2-1)\hat{u}_2(s)$$

$$(s^2-2)(s^2+1)\hat{y}_1(s) = (3s^2-1)\hat{u}_1(s) + s(3s^2-1)\hat{u}_2(s)$$

$$\hat{y}_1(s) = \frac{3s^2-1}{(s^2-2)(s^2+1)}\hat{u}_1(s) \quad \frac{s(3s^2-1)}{(s^2-2)(s^2+1)}\hat{u}_2(s)$$

Then,

$$\hat{y}_2(s) = -\frac{2s^2+1}{3s^2-1}\left(\frac{3s^2-1}{(s^2-2)(s^2+1)}\hat{u}_1(s) + \frac{s(3s^2-1)}{(s^2-2)(s^2+1)}\hat{u}_2(s)\right)$$

$$\hat{y}_2(s) = -\frac{2s^2+1}{(s^2-2)(s^2+1)}\hat{u}_1(s) - \frac{s(2s^2+1)}{(s^2-2)(s^2+1)}\hat{u}_2(s)$$

Thus,

$$\boxed{G = \begin{bmatrix} \dfrac{3s^2-1}{(s^2-2)(s^2+1)} & \dfrac{s(3s^2-1)}{(s^2-2)(s^2+1)} \\[2ex] -\dfrac{2s^2+1}{(s^2-2)(s^2+1)} & -\dfrac{s(2s^2+1)}{(s^2-2)(s^2+1)} \end{bmatrix}}$$

Exercise 6 Obtain the transfer function of the system with input u and output y described by

$$\begin{aligned} \ddot{q}_1 + 3\dot{q}_2 + \dot{q}_1 + 2q_2 &= \dot{u} + 4u \\ \ddot{q}_1 + 4\dot{q}_2 + 3q_2 &= u \\ y &= q_1 + q_2 \end{aligned}$$

This is a SISO and zero IC system.

Take the Laplace transformation of the system

$$\rightarrow \quad s^2\hat{q}_1(s) + 3s\hat{q}_2(s) + s\hat{q}_1(s) + 2\hat{q}_2(s) = s\hat{u}(s) + 4\hat{u}(s)$$

$$(s^2+s)\hat{q}_1(s) + (3s+2)\hat{q}_2(s) = (s+4)\hat{u}(s) \quad \cdots ①$$

$$\rightarrow \quad s^2\hat{q}_1(s) + 4s\hat{q}_2(s) + 3\hat{q}_2(s) = \hat{u}(s)$$

$$s^2\hat{q}_1(s) + (4s+3)\hat{q}_2(s) = \hat{u}(s) \quad \cdots ②$$

$$\rightarrow \quad \hat{y}(s) = \hat{q}_1(s) + \hat{q}_2(s) \quad \cdots ③$$

$① \times (4s+3) - ② \times (3s+2)$

$$\Rightarrow \quad s(s+1)(4s+3)\hat{q}_1(s) - s^2(3s+2)\hat{q}_1(s)$$
$$= (s+4)(4s+3)\hat{u}(s) - (3s+2)\hat{u}(s)$$

$$s\left[(s+1)(4s+3) - s(3s+2)\right]\hat{q}_1(s)$$
$$= \left[(s+4)(4s+3) - 3s - 2\right]\hat{u}(s)$$

$$s\left(4s^2 + 7s + 3 - 3s^2 - 2s\right)\hat{q}_1(s)$$
$$= \left(4s^2 + 19s + 12 - 3s - 2\right)\hat{u}(s)$$

$$s(s^2+5s+3)\hat{q}_1(s) = (4s^2+16s+10)\hat{u}(s)$$

$$\hat{q}_1(s) = \frac{4s^2+16s+10}{s(s^2+5s+3)}\hat{u}(s)$$

Then,

$$\hat{q}_2(s) = -\frac{s^2}{4s+3}\hat{q}_1(s) + \frac{\hat{u}(s)}{4s+3}$$

Thus,

$$\begin{aligned}\hat{y}(s) &= \hat{q}_1(s) + \hat{q}_2(s) \\ &= \hat{q}_1(s) - \frac{s^2}{4s+3}\hat{q}_1(s) + \frac{\hat{u}(s)}{4s+3} \\ &= \frac{-s^2+4s+3}{4s+3}\hat{q}_1(s) + \frac{\hat{u}(s)}{4s+3} \\ &= \frac{(-s^2+4s+3)(4s^2+16s+10)}{s(4s+3)(s^2+5s+3)}\hat{u}(s) + \frac{\hat{u}(s)}{4s+3} \\ &= \frac{-s^3+s^2+17s+10}{s(s^2+5s+3)}\hat{u}(s)\end{aligned}$$

Answer confirmed in MATLAB (code in Appendix)

$$\boxed{G(s) = \frac{y(s)}{u(s)} = \frac{-s^3+s^2+17s+10}{s(s^2+5s+3)}}$$

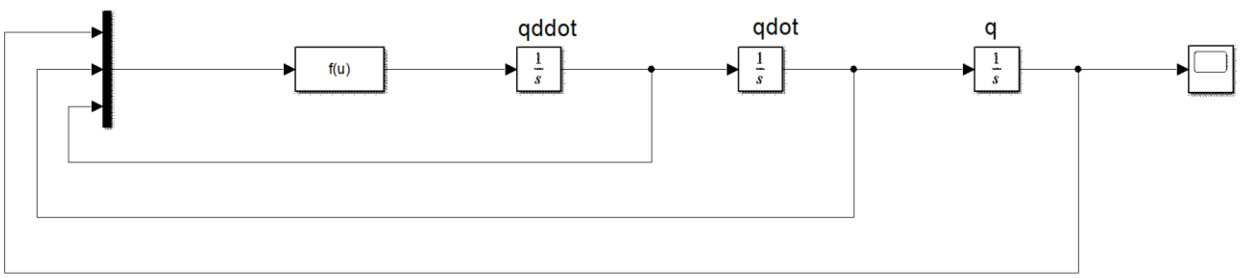
Exercise 7 Obtain a SIMULINK model of the following system.

$$\frac{d^3q}{dt^3} + 3\ddot{q} + \dot{q}\sin q + q^3 = 0\,.$$

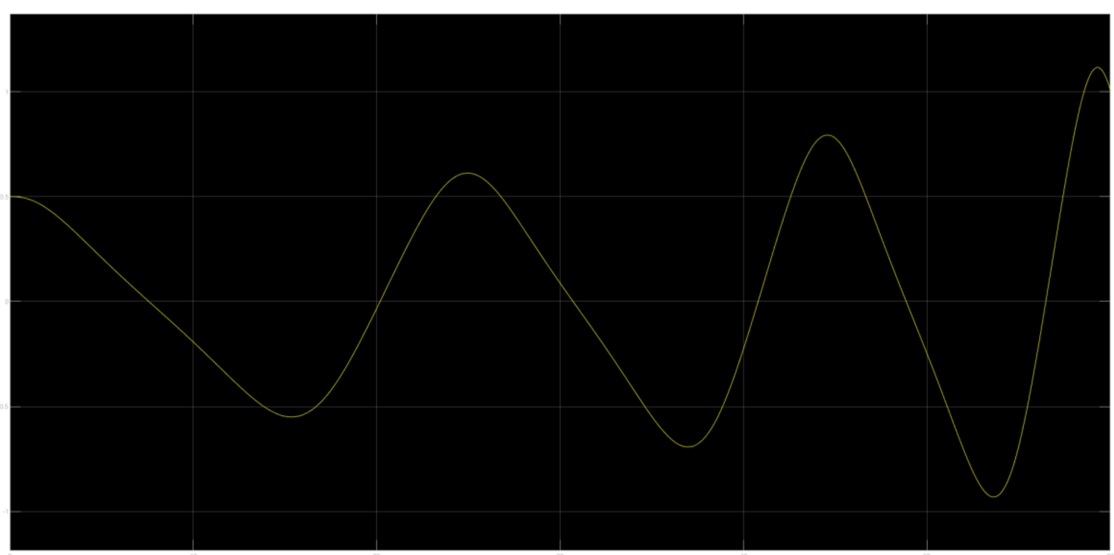
$$\frac{d^3q}{dt^3} = -3\ddot{q} - \dot{q}\sin q - q^3$$

fcn ⇒ -3*u(3) - u(2)* sin(u(1)) - u(1)* u(1)*u(1)

IC: $q(0) = 0.5$



Plot with the IC of q(0) = 0.5



Exercise 8 Obtain SIMULINK models of the following systems.

(a)

$$\begin{aligned} \ddot{q}_1 + 2\dot{q}_2 + q_1 &= 0 \\ \ddot{q}_2 + \dot{q}_1 + 6q_2 &= 0 \end{aligned}$$

Rewrite this as

$$\begin{cases} \ddot{q}_1 = -q_1 - 2\dot{q}_2 \\ \ddot{q}_2 = -\dot{q}_1 - 6q_2 \end{cases}$$

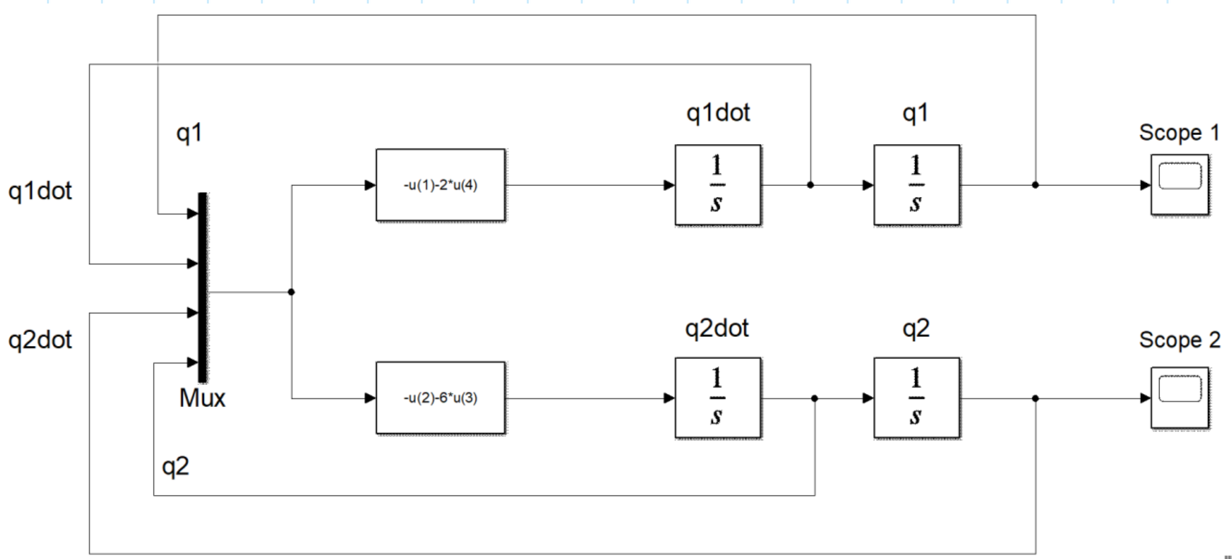
if $u(1) = q_1$, $u(2) = \dot{q}_1$, $u(3) = q_2$, $u(4) = \dot{q}_2$

fcn above will be

$-u(1) - 2*u(4)$

fcn below will be

$-u(2) - 6*u(3)$



plot for when

$$q_1(0) = 2, \quad q_2(0) = 0$$
$$\dot{q}_1(0) = 0, \quad \dot{q}_2(0) = 0$$
$$t = 60\ s$$

Output from Scope 1

Output from Scope 2

(b)

$$\begin{aligned} \ddot{q}_1 + \ddot{q}_2 + 6q_2 &= 0 \\ \ddot{q}_1 - \ddot{q}_2 + 4q_1 &= 0 \end{aligned}$$

Solve this as

$$\ddot{q}_1 + \ddot{q}_2 + 6q_2 = 0 \quad \cdots ①$$
$$\ddot{q}_1 - \ddot{q}_2 + 4q_1 = 0 \quad \cdots ②$$

① - ②

$$\Rightarrow \quad 2\ddot{q}_2 - 4q_1 + 6q_2 = 0$$
$$\ddot{q}_2 = 2q_1 - 3q_2$$

$$\therefore \quad \ddot{q}_1 = \ddot{q}_2 - 4q_1$$
$$\ddot{q}_1 = 2q_1 - 3q_2 - 4q_1$$
$$\ddot{q}_1 = -2q_1 - 3q_2$$

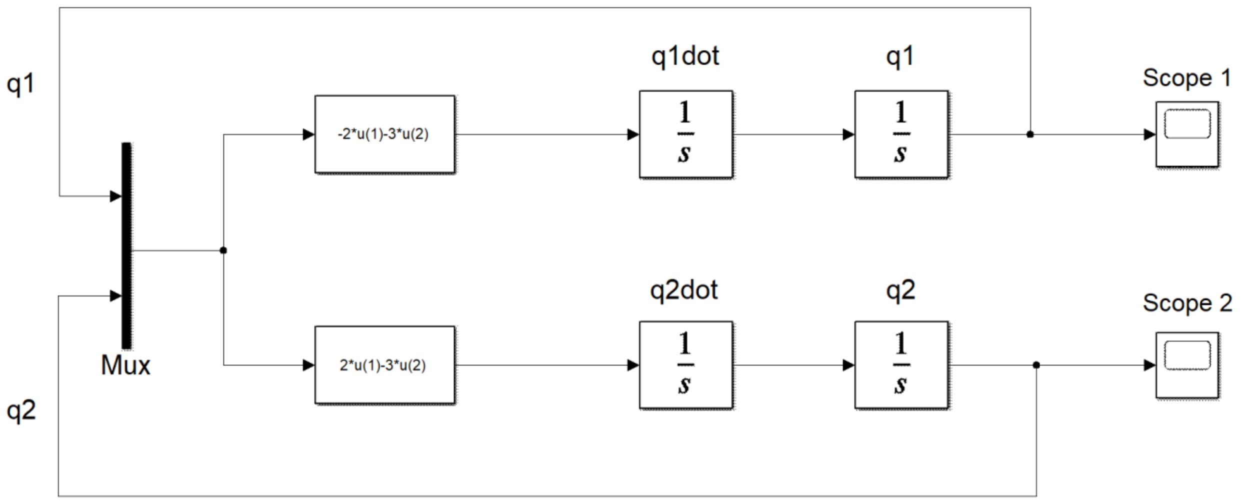
if $u(1) = q_1$, $u(2) = q_2$

fcn above will be

-2*u(1) -3*u(2)

fcn below will be

2*u(1) -3*u(2)



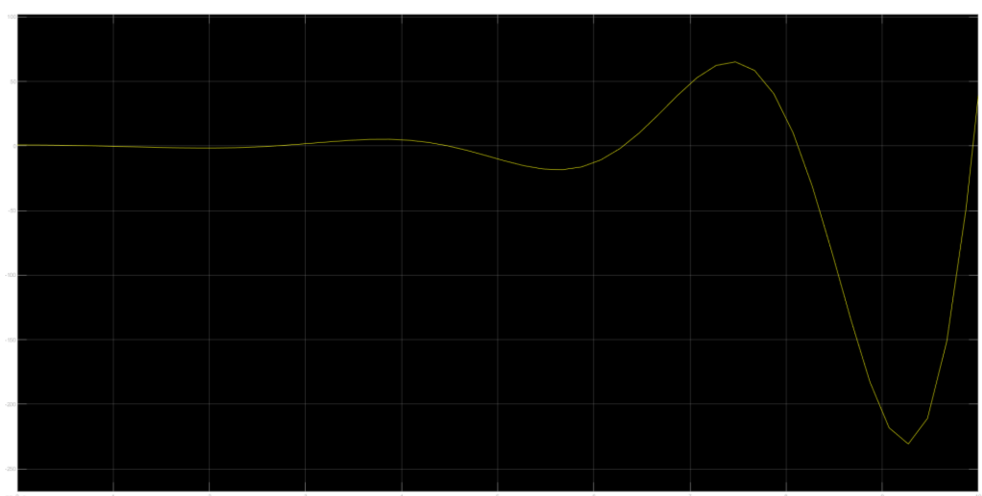
plot for when

$q_1(0) = 0.8$, $q_2(0) = 0.2$

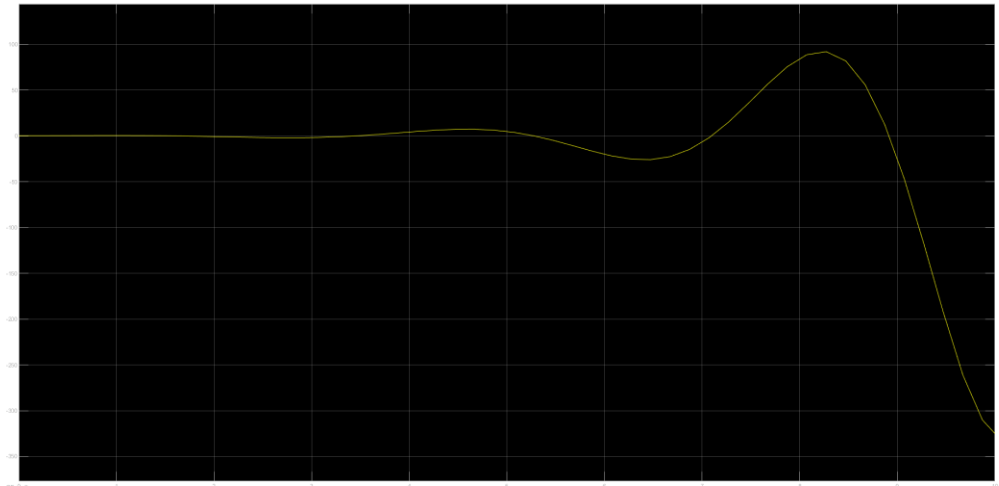
$\dot{q}_1(0) = 0$, $\dot{q}_2(0) = 0$

$t = 10\,s$

Output from Scope 1



Output from Scope 2



(c)

$$\begin{aligned} \ddot{q}_1 + \dot{q}_1 + q_1 q_2 &= 0 \\ (1+q_2^2)\dot{q}_2 + 2\dot{q}_1 q_2 &= 0 \end{aligned}$$

Take this system equation

2*

$$\ddot{q}_1 = -\dot{q}_1 - q_1 q_2 \quad \cdots ①$$

$$(1+q_2^2)\dot{q}_2 + 2\dot{q}_1 q_2 = 0 \quad \cdots ②$$

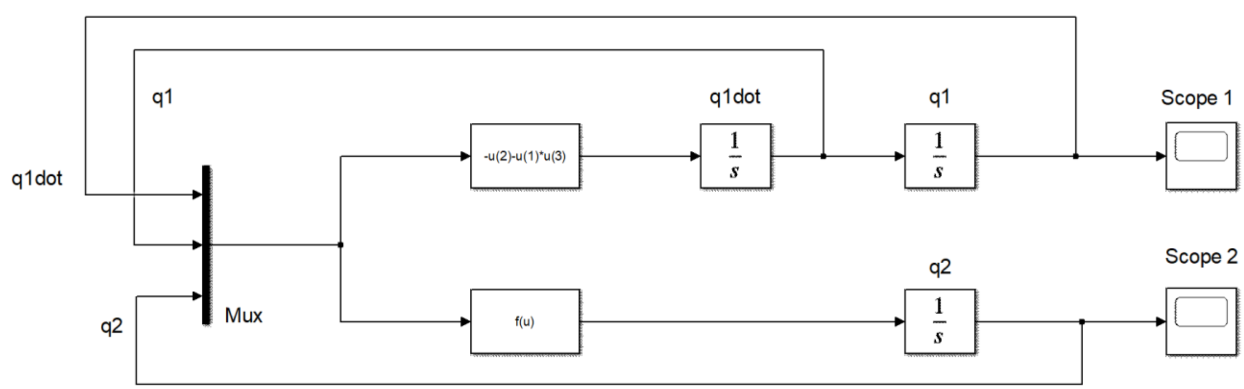
if $u(1) = q_1$, $u(2) = \dot{q}_1$, $u(3) = q_2$

fcn above will be

-u(2) - u(1)*u(3)

fcn below will be

(-2*u(2)*u(3))/(1+u(3)*u(3))

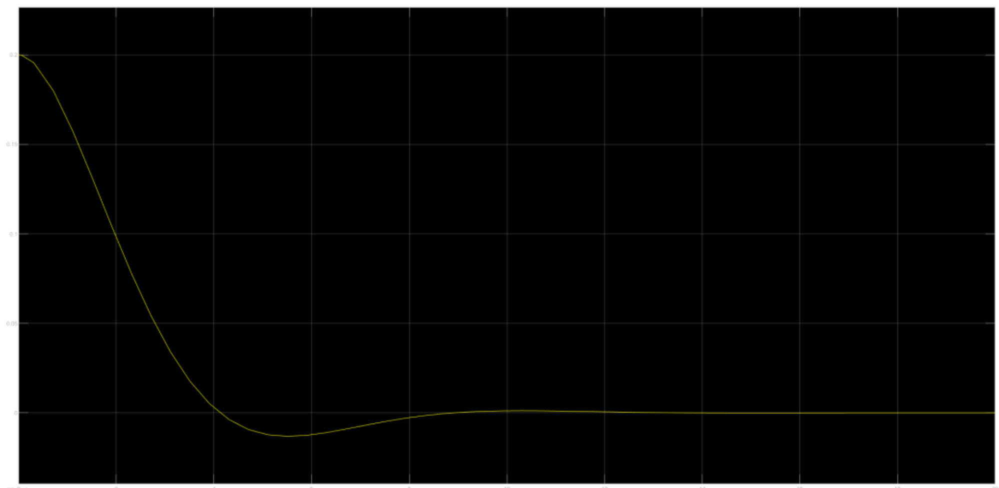


Plot with the ICs

$q_1(0) = 0.2, \quad q_2(0) = 0.5$

$\dot{q}_1(0) = 0 \qquad t = 20$

Output from Scope 1



Output from Scope 2