AAE 334: Aerodynamics

HW8: Compressible Isentropic Relation & Pitot Tube

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1. [25 pts] Consider several wind tunnels with fixed Mach number $M = 2.0$, but different working gases. Neglect the possibility of phase change and chemical reaction for this problem. Assume that the stagnation conditions are fixed at 300 kPa and 500 K. What are the pressure and temperature in the test section if the working gas is: argon ($\gamma = 1.667$), nitrogen ($\gamma = 1.400$), carbon dioxide ($\gamma = 1.289$), octane ($\gamma = 1.044$), or per-fluoro-*n*-butane (C_4F_{10}, $\gamma = 1.024$). (Note that, although this does not directly affect this particular problem, the Mach number is determined by the area ratio of the nozzle and γ. So, if the gas is changed, the nozzle would have to have a different area ratio to maintain a test section Mach number of M = 2.)

<i> Ar : $\gamma_{Ar} = 1.667$

since we are given that

$$P_0 = 300 \times 10^3\,Pa,\quad T_0 = 500\,K,\quad M = 2.0$$

$$P_{Ar} = P_0\left[1 + \frac{\gamma_{Ar}-1}{2}M^2\right]^{-\gamma_{Ar}/\gamma_{Ar}-1}$$

$$= (300 \times 10^3\,Pa)\left[1 + \frac{1.667-1}{2}(2.0)^2\right]^{-1.667/1.667-1}$$

$$= 3.6070 \times 10^4\,Pa$$

$$T_{Ar} = T_0\left[1 + \frac{\gamma_{Ar}-1}{2}M^2\right]^{-1}$$

$$= (500\,K)\left[1 + \frac{1.667-1}{2}(2.0)^2\right]^{-1}$$

$$= 214.2245\,K$$

$$\boxed{\begin{aligned} P_{Ar} &= 36.070\,kPa \\ T_{Ar} &= 214.22\,K \end{aligned}}$$

exact same procedure as <i> will continue for the following <ii> ~ <V>

<ii> N_2 : $\gamma_{N_2} = 1.400$

$$P_{N_2} = P_0\left[1 + \frac{\gamma_{N_2}-1}{2}M^2\right]^{-\gamma_{N_2}/\gamma_{N_2}-1} = 3.8341 \times 10^4\,Pa$$

$$T_{N_2} = T_0\left[1 + \frac{\gamma_{N_2}-1}{2}M^2\right]^{-1} = 277.7778\,K$$

$$\boxed{\begin{aligned} P_{N_2} &= 38.341\,kPa \\ T_{N_2} &= 277.78\,K \end{aligned}}$$

<iii> CO_2 : $\gamma_{CO_2} = 1.289$

$$P_{CO_2} = P_0 \left[1 + \frac{\gamma_{CO_2} - 1}{2} M^2 \right]^{-\gamma_{CO_2}/\gamma_{CO_2}-1} = 3.9221 \times 10^4 \, Pa$$

$$T_{CO_2} = T_0 \left[1 + \frac{\gamma_{CO_2} - 1}{2} M^2 \right]^{-1} = 316.8568 \, K$$

$$\boxed{\begin{aligned} P_{CO_2} &= 39.221 \text{ kPa} \\ T_{CO_2} &= 316.86 \text{ K} \end{aligned}}$$

<iv> C_8H_{18} : $\gamma_{C_8H_{18}} = 1.044$

$$P_{C_8H_{18}} = P_0 \left[1 + \frac{\gamma_{C_8H_{18}} - 1}{2} M^2 \right]^{-\gamma_{C_8H_{18}}/\gamma_{C_8H_{18}}-1} = 4.0552 \times 10^4 \, Pa$$

$$T_{C_8H_{18}} = T_0 \left[1 + \frac{\gamma_{C_8H_{18}} - 1}{2} M^2 \right]^{-1} = 459.5588 \, K$$

$$\boxed{\begin{aligned} P_{C_8H_{18}} &= 40.552 \text{ kPa} \\ T_{C_8H_{18}} &= 459.56 \text{ K} \end{aligned}}$$

<v> C_4F_{10} : $\gamma_{C_4F_{10}} = 1.024$

$$P_{C_4F_{10}} = P_0 \left[1 + \frac{\gamma_{C_4F_{10}} - 1}{2} M^2 \right]^{-\gamma_{C_4F_{10}}/\gamma_{C_4F_{10}}-1} = 4.0586 \times 10^4 \, Pa$$

$$T_{C_4F_{10}} = T_0 \left[1 + \frac{\gamma_{C_4F_{10}} - 1}{2} M^2 \right]^{-1} = 477.0992 \, K$$

$$\boxed{\begin{aligned} P &= 40.586 \text{ kPa} \\ T &= 477.10 \text{ K} \end{aligned}}$$

2. [25 pts] In the movie *The Martian*, the crew of a spaceship use a bomb to blow a hole in an airlock in order to produce thrust from the resulting jet of air flowing out into space. Assume that the spacecraft atmosphere is air at 100 kPa and 300 K, that space is a perfect vacuum, and that the Mach number is $M = 1$ at the hole in the ship (choked flow). Also assume that the hole is a circle of radius of 1.0 m. Estimate the initial mass flow $\dot{m} = \rho u A = \rho^* a^* A$ and thrust $T = \dot{m}u + pA = (\rho^* a^{*2} + p^*)A$ resulting from the hole. (Initial means before the ship's pressure drops; the star * means conditions for $M = 1$, and A is the area of the hole.) How does the thrust compare to that of a RD-180 rocket engine?

since inside a spacecraft we can think of the air flow to be almost stagnant, so

$$P_0 = 100 \text{ kPa}$$
$$T_0 = 300 \text{ kPa}$$

Thus, at the hole, from isentropic relations

$$p^* = P_0 \left[1 + \frac{\gamma - 1}{2} M^2\right]^{-\gamma/\gamma-1}$$

where $M = 1.0$, $\gamma = 1.4$

$$\Rightarrow p^* = (100 \text{ kPa}) \left[1 + \frac{1.4-1}{2} \times 1.0^2\right]^{1.4/1.4-1}$$

$$= 52.828 \text{ kPa}$$

$$T^* = T_0 \left[1 + \frac{\gamma-1}{2} M^2\right]^{-1}$$

$$= (300 \text{K}) \left[1 + \frac{1.4-1}{2} \times 1.0^2\right]^{-1}$$

$$= 250 \text{K}$$

now since $p^* = \rho^* R T^*$

$$\rho^* = p^*/RT^* = (52.828 \text{ kPa}) \Big/ \left(287.05 \tfrac{\text{J}}{\text{kg·K}}\right)(250 \text{K})$$

$$\rho^* = 0.7362 \text{ kg/m}^3$$

also, $a^* = M\sqrt{\gamma R T^*}$

$$= (1.0)\sqrt{(1.4)(287 \tfrac{J}{kg \cdot K})(250K)}$$

$$= 316.9661 \ m/s$$

thus,

$$\dot{m} = \rho^* a^* A$$

$$\dot{m} = (0.7362 \ kg/m^3)(316.9661 \ m/s)\ \pi(1.0m)^2$$

$$\boxed{\dot{m} = 733.0452 \ kg/s}$$

then, the thrust becomes

$$F_T = \dot{m}a^* + p^* A$$

$$F_T = (733.0452 \ kg/s)(316.9661 \ m/s) + (52.828 kPa)\ \pi(1.0m)^2$$

$$F_T = 3.9832 \times 10^5 \ N$$

$$\boxed{F_T = 398.32 \ kN}$$

Discussion

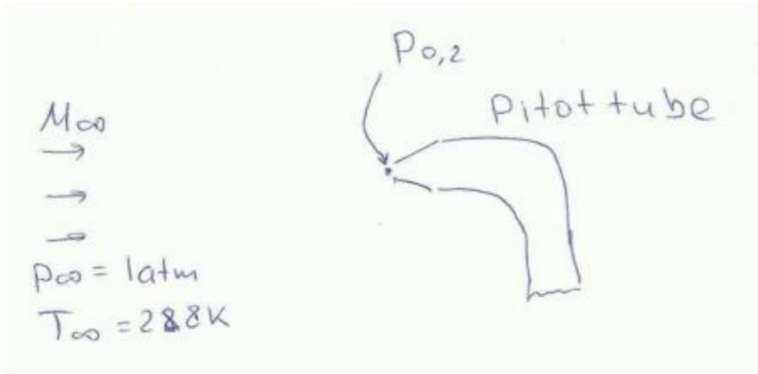
the RD-180 rocket engine has a thrust of approximately

4.15 MN @ vacuum

Compared to this the hypothetical thrust provided by blowing a 1m radius hole into a spacecraft is unrealistic since

$$\frac{F_T}{4.15MN} = \frac{398.32\,kN}{4.15\,MN} = 0.0960$$

the thrust is less than 10% of the RD-180.



3. [25 pts] Consider a flow with a pressure of p_∞=1 atm and a temperature of T_∞=288 K.

(a) A Pitot tube inserted into the flow measures stagnation pressure of $p_{0,2}$ =1.524 atm.

(i) Using the flow tables given in the Appendices of Anderson's textbook (uploaded online on the Blackboard) analyze whether the flow is supersonic or subsonic;

(ii) Using the flow tables, find the Mach number of the flow;

(iii) Find the velocity of the flow.

(b) In the same settings as in part (a), the Pitot tube now measures stagnation pressure of $p_{0,2}$ =5.900 atm.

(iv) Using the flow tables given in the Appendices of Anderson's textbook (uploaded online on the Blackboard) analyze whether the flow is supersonic or subsonic;

(v) Using the flow tables, find the Mach number of the flow;

(vi) Find the velocity of the flow.

(a)

<i> the pressure ratio for the given condition is

$$\frac{P_{02}}{P_\infty} = \frac{1.524\,\text{atm}}{1.0\,\text{atm}} = 1.524$$

from the isentropic flow table of Anderson

@ $M = 1.0$ $\quad \frac{P_0}{P} = 1.893$

since $\quad \frac{P_{02}}{P_\infty} = 1.524 < 1.893 \Longrightarrow \boxed{\text{subsonic}}$

<ii> from the table @ $\frac{P_0}{P} = 1.524$

$$\Longrightarrow \boxed{M = 0.8}$$

<iii> $\quad u_\infty = M\sqrt{\gamma R T_\infty}$

since

$M = 0.8$, $\gamma = 1.4$, $R = 287.05 \frac{J}{\cancel{kg}\,K}$

$$\boxed{u_\infty = 272.1630 \text{ m/s}}$$

(b)

<iv> with the same method

$$\frac{P_{02}}{P_\infty} = \frac{5.900\,\text{atm}}{1.0\,\text{atm}} = 5.900 > 1.893 \Rightarrow \boxed{\text{supersonic}}$$

<v> from table, using interpolation

$$M = \frac{1.820 - 1.800}{5.924 - 5.746}(5.900 - 5.746) + 1.800$$

$$\boxed{M_2 = 1.8173}$$

<vi> the velocity becomes

$$u_2 = M_2\sqrt{\gamma R T_\infty}$$

$$u_2 = (1.8173)\sqrt{(1.4)\left(287.05\ \frac{J}{kg\ K}\right)(288\ K)}$$

$$\boxed{u_2 = 618.2533\ m/s}$$

4. [25 pts] The pressure coefficient is defined as

$$C_p = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}$$

Assuming incompressible flow, use the incompressible form of Bernoulli's equation to show that the value of C_p at a stagnation point is given by

$$C_{p,sp}^{\text{inc}} = 1.$$

Now assume isentropic compressible flow of a gas with constant specific heats and derive the following relation for the value of the pressure coefficient at a stagnation point:

$$C_{p,sp} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_0}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^2}\left[\left(1 + \frac{\gamma - 1}{2} M_\infty^2\right)^{\gamma/(\gamma-1)} - 1\right]$$

(Note that this relation is not valid for supersonic Mach numbers, since there would be a shock wave in front of a stagnation point, and the flow would then no longer be isentropic.)

Determine the freestream Mach number for which the value of Cp at the stagnation point deviates from the incompressible value by (i) 1%, (ii) 5% and (iii) 10%.

Bernoulli's equation is

$$p_0 = p_\infty + \frac{1}{2}\rho_\infty u_\infty^2$$

$$\Longrightarrow \quad \frac{p_0 - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = 1$$

$$\Longrightarrow \quad \boxed{C_{p,sp}^{inc} = \frac{p_0 - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = 1}$$

from

$$C_p = \frac{p_0 - p_\infty}{\frac{1}{2}\rho_\infty u_\infty^2} = \frac{2}{\rho_\infty u_\infty^2} p_\infty \left(\frac{p_0}{p_\infty} - 1\right)$$

$$\because \rho_\infty = \frac{p_\infty}{RT_\infty}$$

$$C_p = \frac{2 p_\infty}{\frac{p_\infty}{RT_\infty} u_\infty^2}\left(\frac{p_0}{p_\infty} - 1\right) = \frac{2RT_\infty}{u_\infty^2}\left(\frac{p_0}{p_\infty} - 1\right)$$

$$\because u_\infty^2 = M_\infty^2 (\gamma R T_\infty)$$

$$C_p = \frac{2RT_\infty}{M_\infty^2 \gamma R T_\infty}\left(\frac{p_0}{p_\infty} - 1\right)$$

$$\boxed{C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p_0}{p_\infty} - 1\right)}$$

$$\because \frac{p_0}{p_\infty} = \left[1 + \frac{\gamma - 1}{2} M_\infty^2\right]^{\gamma/\gamma-1}$$

$$\therefore \quad \boxed{C_{p,sp} = \frac{2}{\gamma M_\infty^2}\left[\left(1 + \frac{\gamma-1}{2}M_\infty^2\right)^{\gamma/\gamma-1} - 1\right]}$$

<i> $C_{p,sp} = 1.01$

using the formula above

plug in $C_{p,sp} = 1.01$ and solve for M_∞

computationally using MATLAB (code in Appendix)

$$\boxed{M_1 = 0.1996}$$

<ii> $C_{p,sp} = 1.05$

with the same method as <i>

$$\boxed{M_2 = 0.4429}$$

<iii> $C_{p,sp} = 1.10$

$$\boxed{M_3 = 0.6205}$$

Appendix

AAE334 HW8 MATLAB

```
clear all; close all; clc;
```

P1

```
P0 = 300e3;  % stagnation pressure [Pa]
T0 = 500;  % stagnation temperature [K]
M = 2.0;  % Mach number


% Argon
gamma_Ar = 1.667;
P_Ar = p_from_M_and_gamma(P0,M,gamma_Ar,'static')
T_Ar = T_from_M_and_gamma(T0,M,gamma_Ar,'static')


% Nitrogen
gamma_N2 = 1.400;
P_Ar = p_from_M_and_gamma(P0,M,gamma_N2,'static')
T_Ar = T_from_M_and_gamma(T0,M,gamma_N2,'static')


% Carbon Dioxide
gamma_CO2 = 1.289;
P_Ar = p_from_M_and_gamma(P0,M,gamma_CO2,'static')
T_Ar = T_from_M_and_gamma(T0,M,gamma_CO2,'static')


% Octane
gamma_C8H18 = 1.044;
P_Ar = p_from_M_and_gamma(P0,M,gamma_C8H18,'static')
T_Ar = T_from_M_and_gamma(T0,M,gamma_C8H18,'static')


% Per-fluoro-n-butane
gamma_C4F10 = 1.024;
P_Ar = p_from_M_and_gamma(P0,M,gamma_C4F10,'static')
T_Ar = T_from_M_and_gamma(T0,M,gamma_C4F10,'static')
```

P2

```
P0 = 100e3;  % stagnation pressure [Pa]
T0 = 300;  % stagnation temperature [K]
```

```
M = 1.0;  % Mach number
gamma = 1.4;  % heat capacity ratio
R = 287.05;  % gas constant [J/kg/K]
A = pi*1.0^2;  % hole area [m2]


P = p_from_M_and_gamma(P0,M,gamma,'static')  % static pressure at hole
T = T_from_M_and_gamma(T0,M,gamma,'static')  % static temperature at hole
rho = P/R/T  % static density at hole
u = M*sqrt(gamma*R*T)  % velocity at hole
m_dot = rho*u*A  % mass flow at hole
Ft = m_dot*u + P*A  % thrust at hole [N]


% Compare to RD-180 rocket engine
rat = Ft/4.15e6
```

P3

```
P02 = 1.524;  % [atm]
P = 1;  % [atm]
T = 288;  % [K]
M = 0.8;


% <iii>
u = M*sqrt(gamma*R*T)


% <v>
M2 = two_point_interpolate(5.9,5.746,5.924,1.8,1.82)


% <vi>
u2 = M2*sqrt(gamma*R*T)
```

P4

```
% Cp = 1.01
M1 = calc_M_from_Cp(1.01)


% Cp = 1.05
M2 = calc_M_from_Cp(1.05)


% Cp = 1.10
M3 = calc_M_from_Cp(1.10)
```

FUNCTION

```
function M = calc_M_from_Cp(Cp)
    gamma = 1.4;
    syms M
    a1 = 2/gamma/M^2;
    a2 = (1 + (gamma - 1)/2*M^2)^(gamma/(gamma - 1));
    eqn = Cp == a1*(a2 - 1);
    M = double(solve(eqn,M));
    M = M(M==real(M) & real(M)>0);
end
```

```
function y = two_point_interpolate(x,x_low,x_high,y_low,y_high)
    slope = (y_high - y_low) / (x_high - x_low);
    y = slope * (x - x_low) + y_low;
end
```

```
function  T2 = T_from_M_and_gamma(T1, M, gamma, type)
    if type == "stagnation"
        T2 = T1 * (1 + (gamma - 1) / 2 * M^2);
    elseif type == "static"
        T2 = T1 / (1 + (gamma - 1) / 2 * M^2);
    else
        disp("Error. Incorrect type. Type can only be 'stagnation' or 'static'.")
    end
end
```

```
function  p2 = p_from_M_and_gamma(p1, M, gamma, type)
    if type == "stagnation"
        p2 = p1 * (1 + (gamma - 1) / 2 * M^2)^(gamma/(gamma - 1));
    elseif type == "static"
        p2 = p1 / (1 + (gamma - 1) / 2 * M^2)^(gamma/(gamma - 1));
    else
        disp("Error. Incorrect type. Type can only be 'stagnation' or 'static'.")
    end
end
```