AAE 440: Spacecraft Attitude Dynamics

PS8*

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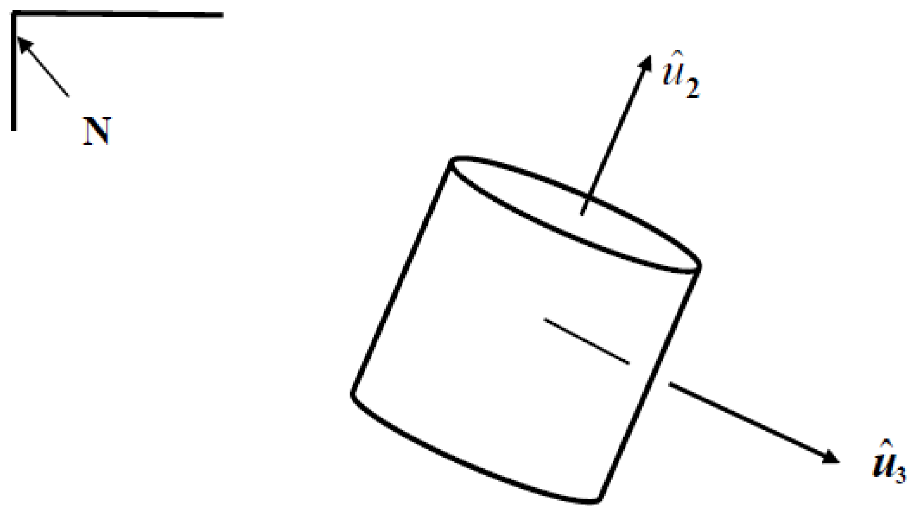
Problem 1: In PS7, the rigid body U was examined in a torque-free environment; this body has the same inertia characteristics as the body from PS5.

$$\bar{\bar{I}}^{U/U^*} = 400\hat{u}_1\hat{u}_1 + 100\hat{u}_2\hat{u}_2 + 400\hat{u}_3\hat{u}_3 \text{ kg-met}^2$$

Let $\hat{n}_i$ be fixed in the inertial frame N and $\hat{u}_i$ define body-fixed unit vectors parallel to central principal axes of inertia. At the initial time ($t = 0$), $\left|{}^N\bar{\omega}^U\right| = 4$ rad/s and ${}^N\bar{\omega}^U$ is directed 60° relative to the axis of symmetry in the $\hat{u}_2 - \hat{u}_3$ plane.



(a) You have already computed the inertia ellipsoid for this body. Now add the following vectors and quantities to the plot:

${}^N\bar{H}^{U/U^*}$
${}^N\bar{\omega}^U$
invariable plane π
nutation angle

What plane contains the body axis of symmetry, the angular velocity and the angular momentum vectors?

Using the same MATLAB code from PS7 problem 2 part (a); we have the inertia ellipsoid, from that we can compute the energy ellipsoid.

$$\|{}^N\bar{\omega}^U\| = 4 \text{ rad/s} \quad \text{and} \quad {}^N\hat{\omega}^U = \cos 60^\circ \hat{u}_2 + \sin 60^\circ \hat{u}_3$$

$$\Rightarrow {}^N\bar{\omega}^U = \|{}^N\bar{\omega}^U\| \, {}^N\hat{\omega}^U = (2\hat{u}_2 + 2\sqrt{3}\,\hat{u}_3) \text{ rad/s}$$

$$T_{rot} = \frac{1}{2}\, {}^N\bar{\omega}^U \cdot \bar{\bar{I}}^{U/U*} \cdot {}^N\bar{\omega}^U$$

$$= \frac{1}{2}(2\hat{u}_2 + 2\sqrt{3}\,\hat{u}_3)\cdot(400\hat{u}_1\hat{u}_1 + 100\hat{u}_2\hat{u}_2 + 400\hat{u}_3\hat{u}_3)\cdot(2\hat{u}_2 + 2\sqrt{3}\,\hat{u}_3)$$

$$= \frac{1}{2}(400 + 4800) = 2600 \ \frac{kg-m^2}{s^2}$$

then

$$T_{rot} = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2$$

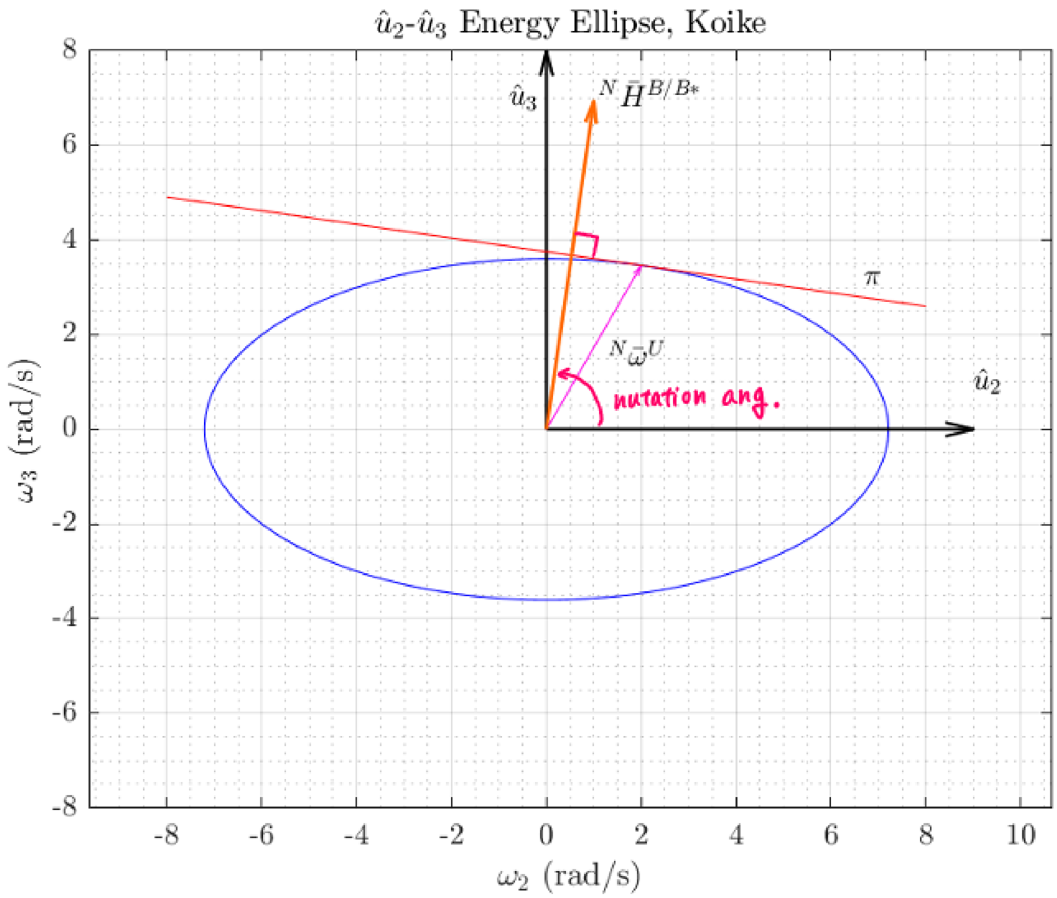
$$1 = \frac{\omega_1^2}{2T_{rot}I_1^{-1}} + \frac{\omega_2^2}{2T_{rot}I_2^{-1}} + \frac{\omega_3^2}{2T_{rot}I_3^{-1}}$$

semi-diameters become

$$d_1 = (2T_{rot}I^{-1})^{0.5} \qquad d_2 = (2T_{rot}J^{-1})^{0.5} \qquad d_3 = d_1$$

$$= 3.6056 \qquad\qquad = 7.2111$$

thus, the energy ellipse on $\hat{u}_2 - \hat{u}_3$ frame becomes



Discussion

Plane $\hat{u}_2$-$\hat{u}_3$ includes all 3 $\hat{u}_2$, ${}^N\bar{\omega}^U$, & ${}^N\bar{H}^{U/U*}$

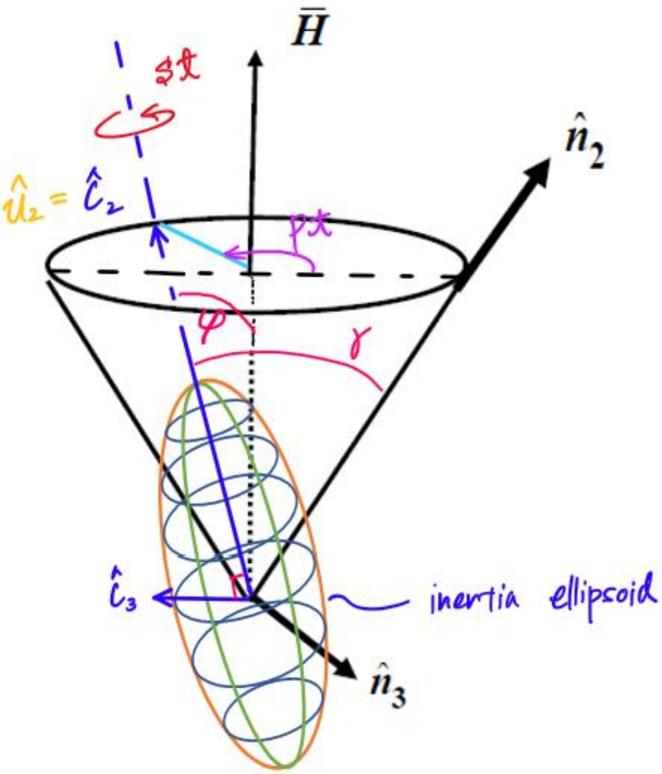
(b) Given a figure similar to the one on the next page, sketch the orientation of the unit vectors $\hat{c}_2$ and $\hat{c}_3$ with respect to $\hat{n}_2$ and $\hat{n}_3$ at an arbitrary time. (Recall that $\hat{c}_j = \hat{n}_j$ at the initial time.) Define γ as the angle the angle between $\hat{n}_2$ and $\hat{u}_2$. Where is γ in the sketch?

Determine the following quantities at t = 0.25 sec; 3.5 sec:

precession, nutation, spin angles

γ -- angle between $\hat{n}_2$ and $\hat{u}_2$

Add the inertia ellipsoid to this sketch!



the angular momentum

$$ {}^{N}\bar{H}^{B/B^*} = \bar{\bar{I}}^{U/U^*} \cdot {}^{N}\bar{\omega}^{B} $$

$$ = (400\,\hat{u}_1\hat{u}_1 + 100\,\hat{u}_2\hat{u}_2 + 400\,\hat{u}_3\hat{u}_3)\cdot(2\hat{u}_2 + 2\sqrt{3}\,\hat{u}_3) $$

$$ = (0.2000\,\hat{u}_2 + 1.3856\,\hat{u}_3)\times 10^3 $$

then precession rate

$$ p = \frac{\| {}^{N}\bar{H}^{B/B^*} \|}{I} = \frac{1400\ \frac{kg\text{-}m^2}{s}}{400\ kg\text{-}m^2} = 3.5\ rad/s $$

spin rate

$$ s = \frac{I-J}{I}\omega_2 = \frac{400-100}{400}\times 2\ rad/s = 1.5\ rad/s $$

and nutation angle

$$ \varphi = \arccos\left(\frac{{}^{N}\bar{H}^{B/B^*}\cdot \hat{u}_2}{\| {}^{N}\bar{H}^{B/B^*} \|}\right) = \arccos\left(\frac{200}{1400}\right) $$

$$ \varphi = 81.7868^\circ $$

from $p\pi$ we know that

$$ {}^{N}\bar{\varepsilon}^{c} = \hat{h}\sin\frac{p\pi}{2} \Rightarrow \varepsilon_1, \varepsilon_2, \varepsilon_3 $$

$$ {}^{N}\varepsilon_4^{c} = \cos\frac{p\pi}{2} \Rightarrow \varepsilon_4 $$

then using

$$\boxed{\begin{aligned}
{}^{N}C^{C}_{11} &= 1 - \varepsilon_2^2 - 2\varepsilon_3^2 = 0.6410 \\
C_{12} &= 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) = 0.7597 \\
C_{13} &= 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4) = 0.1096 \\
C_{21} &= 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) = -0.7597 \\
C_{22} &= 1 - 2\varepsilon_3^2 - 2\varepsilon_1^2 = 0.6483 \\
C_{23} &= 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) = -0.0503 \\
C_{31} &= 2(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4) = -0.1096 \\
C_{32} &= 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) = -0.0508 \\
C_{33} &= 1 - 2\varepsilon_1^2 - 2\varepsilon_2^2 = 0.9927
\end{aligned}}$$

@ $t = 0.25\,s$

$$\Rightarrow {}^{N}C^{C}\Big|_{t=0.25} = \begin{bmatrix} 0.6410 & 0.7597 & 0.1096 \\ -0.7597 & 0.6483 & -0.0503 \\ -0.1096 & -0.0508 & 0.9927 \end{bmatrix}$$

samely,

$${}^{N}C^{C}\Big|_{t=3.5} = \begin{bmatrix} 0.9504 & -0.3079 & -0.044 \\ 0.3079 & 0.9514 & -0.0070 \\ 0.0444 & -0.0070 & 0.9990 \end{bmatrix}$$

Next, from the relation

$$\hat{c}_2 \cdot \hat{n}_2 = \cos\gamma \quad (@\ t = t_i)$$

$$\gamma = \arccos(\hat{c}_2 \cdot \hat{n}_2)$$

from ${}^{N}C^{C}{}_{s}$ above

@ $t = 0.25$ $\quad \hat{c}_2\big|_{t=0.25} = -0.7597\,\hat{n}_1 + 0.6483\,\hat{n}_2 - 0.0503\,\hat{n}_3$

@ $t = 3.5$ $\quad \hat{c}_2\big|_{t=3.5} = 0.3079\,\hat{n}_1 + 0.9514\,\hat{n}_2 - 0.0070\,\hat{n}_3$

then @ $t = 0.25$

$$\gamma = \arccos(0.6483) = 49.5847^\circ$$

@ $t = 3.5$

$$\gamma = \arccos(0.9514) = 17.9392^\circ$$

@ $t = 0.25$ sec

$$\boxed{\begin{array}{l} \text{precession angle} \triangleq \sigma = pt = 50.1338^\circ \\ \text{nutation angle} \triangleq \varphi = 81.7868^\circ \\ \text{spin angle} \triangleq \eta = st = 21.4859^\circ \\ \gamma = 49.5847^\circ \end{array}}$$

@ $t = 3.5$ sec

$$\boxed{\begin{array}{l} \text{precession angle} \triangleq \sigma = pt = 341.8733^\circ \\ \text{nutation angle} \triangleq \varphi = 81.7868^\circ \\ \text{spin angle} \triangleq \eta = st = 300.8028^\circ \\ \gamma = 17.9392^\circ \end{array}}$$

(c) What are the Euler parameters ${}^{N}\bar{\varepsilon}^{U}$, ${}^{N}\varepsilon_4^{U}$ that correspond to these orientations at the specified times? Write the Euler vector in terms of unit vectors $\hat{c}$ as well as body-fixed unit vectors $\hat{u}$.

using the formula, and divide N to S rotation into 2 rotations
N to C & C to U

$$\begin{cases} {}^{N}\bar{\varepsilon}^{U} = {}^{N}\bar{\varepsilon}^{C}\,{}^{C}\varepsilon_4^{U} + {}^{C}\bar{\varepsilon}^{U}\,{}^{N}\varepsilon_4^{C} + {}^{C}\bar{\varepsilon}^{U} \times {}^{N}\bar{\varepsilon}^{C} \\ {}^{N}\varepsilon_4^{U} = {}^{N}\varepsilon_4^{C}\,{}^{C}\varepsilon_4^{U} - {}^{N}\bar{\varepsilon}^{C} \cdot {}^{C}\bar{\varepsilon}^{U} \end{cases}$$

where

$$\begin{cases} {}^{N}\bar{\varepsilon}^{C} = \hat{h} \sin \frac{p t}{2} \\ {}^{N}\varepsilon_4^{C} = \cos \frac{p t}{2} \\ {}^{C}\bar{\varepsilon}^{U} = \hat{c}_2 \sin \frac{s t}{2} \\ {}^{C}\varepsilon_4^{U} = \cos \frac{s t}{2} \end{cases}$$

where

$$\hat{h} = \cos\varphi\, \hat{c}_2 - \sin\varphi\, \hat{c}_3$$

$${}^{N}\bar{\varepsilon}^{U} = \hat{h} \sin \frac{pt}{2} \cos \frac{st}{2} + \hat{c}_2 \sin \frac{st}{2} \cos \frac{pt}{2} + \hat{c}_2 \sin \frac{st}{2} \times \hat{h} \sin \frac{pt}{2}$$

$${}^{N}\varepsilon_4^{U} = \cos \frac{pt}{2} \cos \frac{st}{2} - \hat{h} \sin \frac{pt}{2} \cdot \hat{c}_2 \sin \frac{st}{2}$$

@ $t = 0.25$ sec

$${}^{N}\bar{\varepsilon}^{U}\Big|_{t=0.25} = -0.0782\,\hat{c}_1 + 0.2283\,\hat{c}_2 - 0.4120\,\hat{c}_3$$

$${}^{N}\varepsilon_4^{U}\Big|_{t=0.25} = 0.8787$$

@ $t = 3.5$ sec

$${}^{N}\bar{\varepsilon}^{U}\Big|_{t=3.5} = 0.0770\,\hat{c}_1 + 0.5073\,\hat{c}_2 - 0.1356\,\hat{c}_3$$

$${}^{N}\varepsilon_4^{U}\Big|_{t=3.5} = -0.8475$$

next for $\hat{c}$-frame to be $\hat{u}$-frame we must use DCM

$$ {}^{c}C^{u} = \begin{bmatrix} \cos(5t) & 0 & -\sin(5t) \\ 0 & 1 & 0 \\ \sin(5t) & 0 & \cos(5t) \end{bmatrix} $$

then $$\underset{\hat{u}\text{-frame}}{{}^{N}\bar{\varepsilon}^{U}} = \underset{c\text{-frame}}{{}^{N}\bar{\varepsilon}^{U}} \; {}^{c}C^{u}$$

@ $t = 0.25$ sec

$$\left. {}^{N}\bar{\varepsilon}^{U} \right|_{t=0.25} = -0.2236\,\hat{u}_1 + 0.2283\,\hat{u}_2 - 0.3547\,\hat{u}_3$$

$$\left. {}^{N}\varepsilon_4^{U} \right|_{t=0.25} = 0.8787$$

@ $t = 3.5$ sec

$$\left. {}^{N}\bar{\varepsilon}^{U} \right|_{t=3.5} = 0.1559\,\hat{u}_1 + 0.5073\,\hat{u}_2 - 0.0033\,\hat{u}_3$$

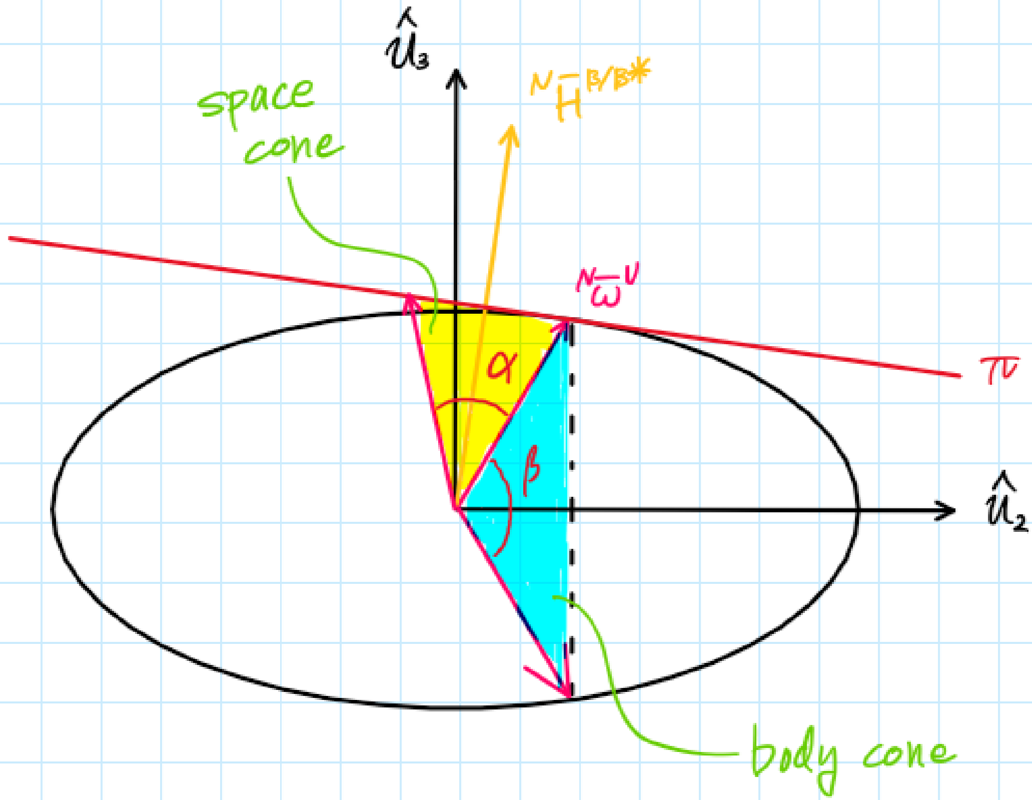
$$\left. {}^{N}\varepsilon_4^{U} \right|_{t=3.5} = -0.8475$$

(d) What is the maximum value of γ?

when $pt = 180°$ $\gamma = \gamma_{max}$

$$\boxed{\gamma_{max} = 2\varphi = 163.5736°}$$

(e) Use the ellipsoid in (a) to help define the space and body cones in this problem. What are the cone angles?
Sketch the space and body cones. How are they related to the motion described in part (b)? Is this body undergoing direct or retrograde precession? How do you know? What does that mean for this motion?

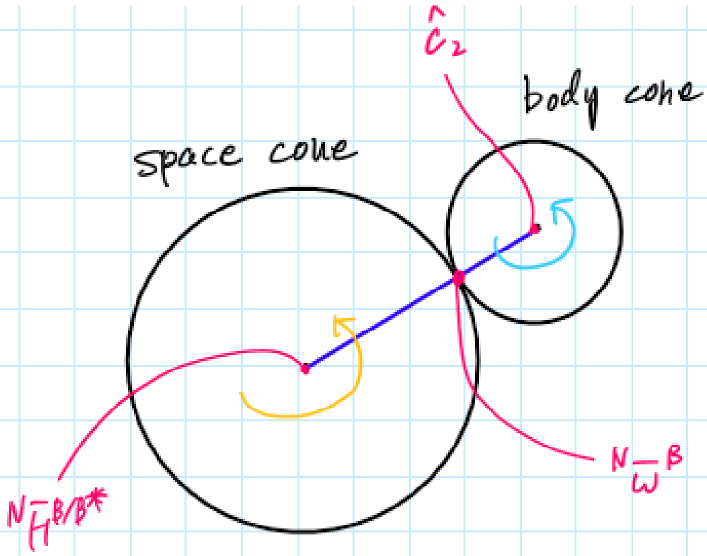


space cone angle α

$$\alpha = (\varphi - 60°) \times 2$$
$$= (81.7868 - 60°) \times 2$$
$$= \boxed{43.5736°}$$

body cone angle β

$$\beta = 60° \times 2 = \boxed{120°}$$



Discussion

We can observe that $\hat{c}_2$, $^N\bar{H}^{B/B*}$, & $^N\bar{\omega}^B$ all reside on the precession plane, and the body motion occurs as if the body cone rolls on the space cone. The cone depicted in (b) is the surface where the space cone and body cone touch each other.

The two cones rotate in the same direction, which implies that this is a direct precession.

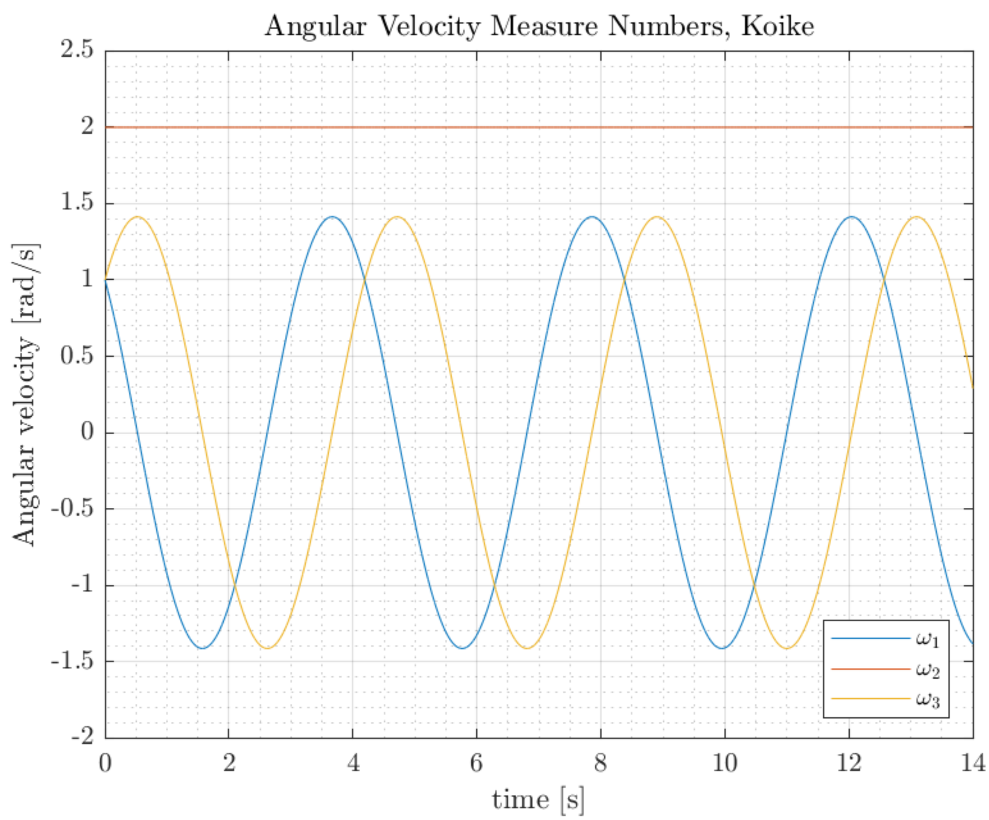
For a rod-like body, in which it rotates in the axis with maximum rotational energy, due to energy dissipation the body tends to end up spinning around the inertial long axis. This means that with direct precession the motion of the body is unstable.

Problem 2: Again, recall Problem Set 5. The axisymmetric rigid body U (spacecraft) moves in an inertial reference frame N. But the environment is now torque-free. Let $\hat{n}_i$ and $\hat{u}_i$ be unit vectors fixed in N and U, respectively. Assume that the body is axisymmetric such that the inertia dyadic is

$$\bar{\bar{I}}^{U/U^*} = 400\hat{u}_1\hat{u}_1 + 100\hat{u}_2\hat{u}_2 + 400\hat{u}_3\hat{u}_3 \text{ kg-met}^2$$

Return to your script for PS5. Now, it is torque-free. Assume that $\hat{u}_k = \hat{n}_k$ at $t = 0$ and modify the initial conditions $\omega_1(0) = +1.0$ rad/s, $\omega_2(0) = +2.0$ rad/s, $\omega_3(0) = +1.0$ rad/s, and $T = 0$ N-met. Again, plot all three angular velocity measure numbers on the same plot. [It may be most straightforward to use a 2-1-2 sequence. But be specific and clear about the sequence you are choosing to employ.] In what vector basis are these angular velocity components? Should they be constant? Why or why not? Are they oscillatory? Are they periodic?

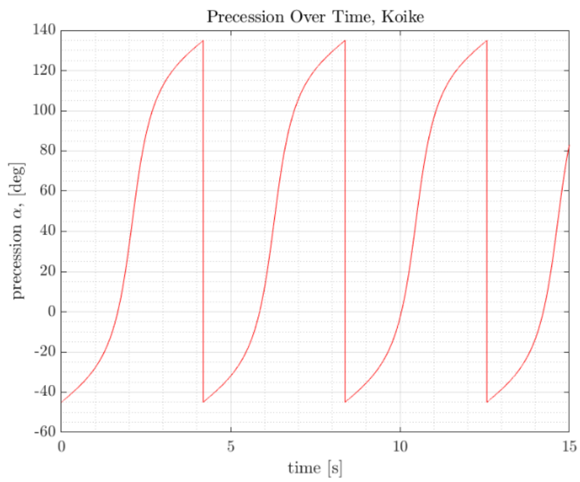
Angular velocity plot for Body-two 2-1-2 sequence

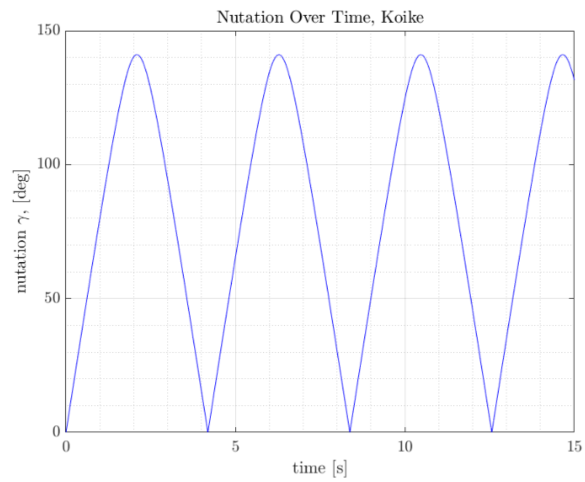


Discussion

- the angular velocities are in $\hat{u}$-basis
- ω_2 is zero because the rate of change is zero.
- ω_1 & ω_3 are oscillatory and periodic because in the differential eqn $\dot{\omega}_1$ is a function of ω_1 and $\dot{\omega}_3$ is a function of ω_3.

(a) Numerically integrate and plot the angle time histories from PS 5 for the angles that are defined as α, γ in PS5 in Prob 1(d). (Update your quadrant checks if necessary!) Are they now oscillatory? Periodic? Initially, $\gamma = 0^{\circ}$. How does it change over time? Does it return to zero? Do you know why?





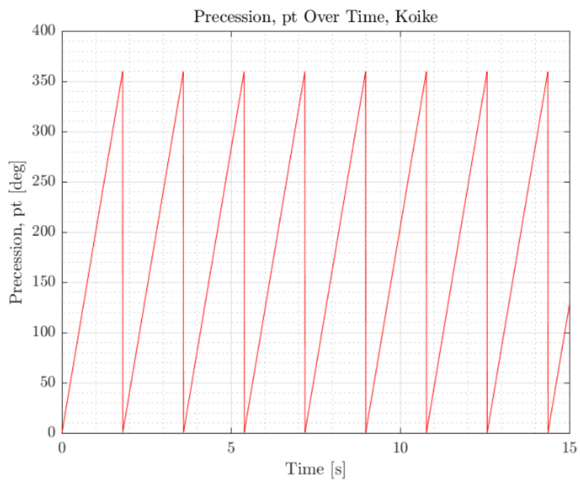
Discussion

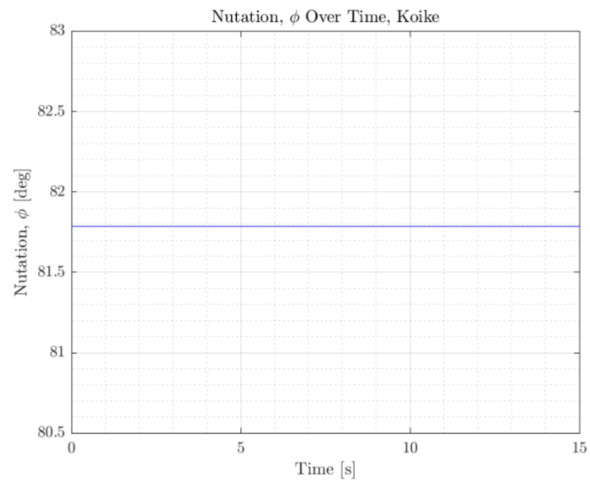
→ The precession, α is in a periodic motion but the repeated pattern is not oscillating.

→ The nutation, γ is in a oscillatory & periodic motion. This is purely periodic since this a torque free system allowing γ to go back to zero for each cycle.

(b) Why do these 'precession, nutation' angles (α, γ) behave differently than the 'precession, nutation' angles (pt, ϕ) that have been recently discussed for torque-free motion? How are they related? Given the simulation results for α, γ, can you determine pt, ϕ? If so, give examples.

Are the Euler parameters different from the torque-free ε_i discussed in class? Why?





Discussion

→ For this problem (α, γ) are numerically integrated from DCM which end up in the $\hat{n}$-basis or inertial frame. However, in our previous analysis involving (pt, ϕ), the parameters were derived by using a ficticious $\hat{c}$-frame and the properties I and H which are unique to the physical characteristics of the body. Thus, the values ended up being in the $\hat{u}$-basis or body frame. The matter is the difference in which frame they are defined in, and in actuality they are interchangeable by transforming the vector basis and rotation axis based on, and their orientations. the behavior (pt, ϕ) is evidently from (α, γ); however, with the manipulation of vector basis we can transform from one to the other

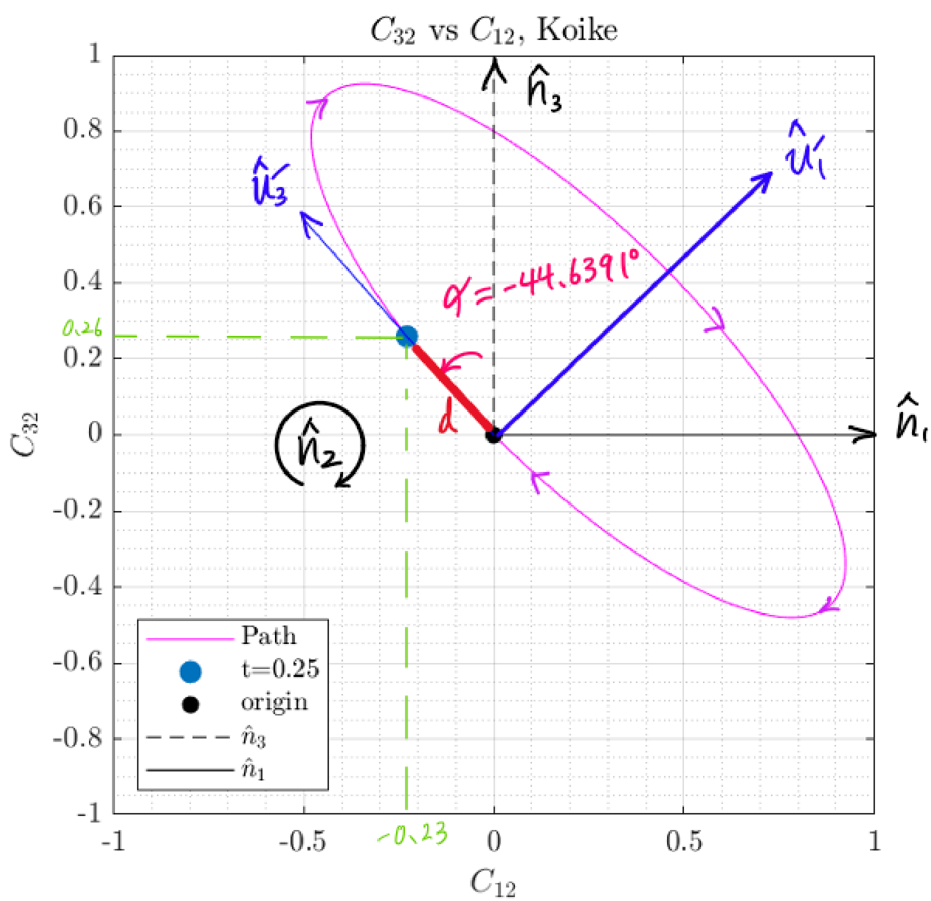
→ The Euler parameters are also defined in different vector basis, hence the signs are different for the values and can be transformed with the appropriate DCM; however, the magnitudes of their values are equivalent because the Euler parameters are independent properties of vector bases.

The relation between (q, δ) & (p_T, φ) and example

From the simulated values of q we cannot deduce the values of p_T.

However,
$$\varphi = \frac{\max(\delta)}{2}$$

(c) Again plot C_{32} as a function of C_{12}; be sure that you scale the plot so both axes cover C_{ij} values -1 to +1. This plot results in a view down the $\hat{n}_2$ axis. Is the positive $\hat{n}_2$ direction into or out of the page?
Is the curve now periodic? Be sure to mark the direction of motion!
On the plot, mark the time $t = 0.25$ sec. At this time, sketch the precession angle. If you measure the angle, does it match the value for α that you computed in the simulation? Sketch the value d (the distance d from the origin to the projection of the tip of $\hat{u}_2$ on the plane) and compute the angle γ. Does it match the value for γ that you computed in the simulation? You should also be able to add $\hat{u}_1'$ and $\hat{u}_3'$ to the $C_{12} - C_{32}$ plot at $t =$ 0.25 sec. How are these unit vectors related to the α, γ angles? What does that mean for this motion?



for precession, α

Discussion

→ $\hat{n}_2$ is directed into the page.

→ The curve seems to be purely periodic and moving clockwise

→ The actual measured angle $= -42.7°$
The computed angle $= -44.6391°$
They are almost the same.

→ The precession can be deduced as the angles between $\hat{n}_3$, $\hat{u}_3'$ & $\hat{n}_1$, $\hat{u}_1'$ from the plot. For each cycle the angles become equal which mean that the motion is cyclic and stable.

nutation, γ (instruction highlighted in red)

from the plot (by eyeballing) can be calculated as

$$\boxed{d = \sqrt{(0.23)^2 + (0.26)^2} = 0.34713}$$

since @ $t = 0.25\,s$

$C_{22} = 0.9382 > 0$ (* C_{22} is obtained from MATLAB simulation)

the nutation calculated from the plot becomes

$$\gamma_{ex} = \arcsin(d) = \arcsin(0.34713)$$

$$\boxed{\gamma_{ex} = 20.312°}$$

and the nutation angle from the simulation is

$$\gamma = \arccos(C_{22})$$

$$\gamma = \arccos(0.9382)$$

$$\boxed{\gamma = 20.2438^\circ}$$

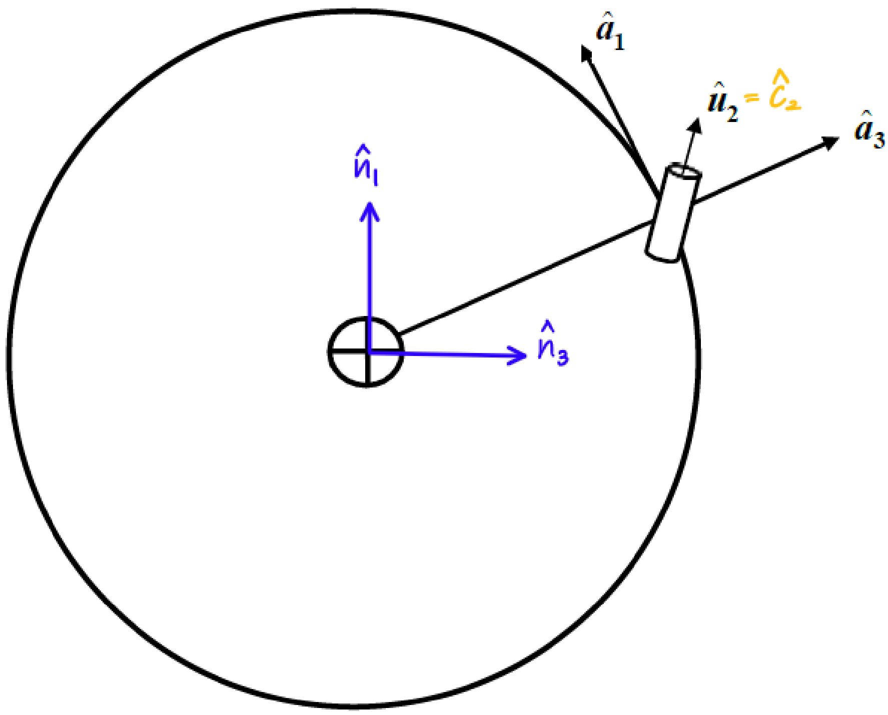
Discussion

The γ_{ex} (nutation calculated by computing α from plot) and γ (nutation angle computed from the simulation) are very close to each other and are accurate.

Problem 3: Again, the axisymmetric rigid body U (spacecraft) can move in an inertial reference frame N but we are now going to place the vehicle in a circular orbit. Let $\hat{n}_i$ and $\hat{u}_i$ be unit vectors fixed in N and U, respectively. Assume again

$$\bar{\bar{I}}^{U/U^*} = 400\hat{u}_1\hat{u}_1 + 100\hat{u}_2\hat{u}_2 + 400\hat{u}_3\hat{u}_3 \quad \text{kg-met}^2$$

The mass of the vehicle is 200 kg. Assume that the spacecraft moves in a circular Earth orbit at a constant rate Ω with respect to N. Define an orbit-fixed frame A such that $\hat{a}_3$ is directed radially outward from the Earth toward U*, $\hat{a}_1$ is 90° from $\hat{a}_3$ and in the direction of motion. Then, $\hat{a}_2$ is parallel to orbital angular momentum and ${}^N\bar{\omega}^A = \Omega\,\hat{a}_2$.



Consistent with the class discussion, an intermediate frame C is introduced such that $\hat{c}_2 = \hat{u}_2$ at all times. Define the measure numbers such that

$$\left|{}^C\bar{\omega}^U\right| = q \qquad \text{and} \qquad {}^N\bar{\omega}^U = \omega_i\hat{c}_i$$

(a) Derive the kinematic and dynamic differential equations that govern the attitude over time. Include the gravity torque and consider the kinematic variables.

(i) Use directions cosines as the kinematic variables. Derive the form of the complete set of differential equations.

The frames are

$\hat{a}_i$: orbit frame

$\hat{n}_i$: inertial frame

$\hat{c}_i$: intermediate frame

$\hat{u}_i$: body fixed frame

We are given that $\|{}^{C}\bar{\omega}^{U}\| = q$, ${}^{N}\bar{\omega}^{U} = \omega_i \hat{c}_i$, ${}^{N}\bar{\omega}^{A} = \Omega \hat{a}_2$

$$\bar{\bar{I}}^{U/U^*} = I\hat{c}_1\hat{c}_1 + J\hat{c}_2\hat{c}_2 + I\hat{c}_3\hat{c}_3$$

Dynamic Differential Equations

$$\rightarrow \bar{M}^{U^*} = \frac{{}^{N}d\,{}^{N}\bar{H}^{U/U^*}}{dt}$$

since, ${}^{N}\bar{\omega}^{U} = \omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3 = {}^{N}\bar{\omega}^{C} + {}^{C}\bar{\omega}^{U} = {}^{N}\bar{\omega}^{C} + q\hat{c}_2$

then,

$${}^{N}\bar{H}^{U/U^*} = \bar{\bar{I}}^{U/U^*} \cdot {}^{N}\bar{\omega}^{U}$$

$$= I\omega_1\hat{c}_1 + J\omega_2\hat{c}_2 + I\omega_3\hat{c}_3$$

$$\frac{{}^{N}d\,{}^{N}\bar{H}^{U/U^*}}{dt} \underset{=}{\overset{BKE}{=}} \frac{{}^{C}d\,{}^{N}\bar{H}^{U/U^*}}{dt} + {}^{N}\bar{\omega}^{C} \times {}^{N}\bar{H}^{U/U^*} \quad \left(\because {}^{N}\bar{\omega}^{C} = {}^{N}\bar{\omega}^{V} - {}^{C}\bar{\omega}^{V}\right)$$

$$\bar{M}^{U^*} = I\dot{\omega}_1\hat{c}_1 + J\dot{\omega}_2\hat{c}_2 + I\dot{\omega}_3\hat{c}_3$$
$$+ \left[\omega_1\hat{c}_1 + (\omega_2 - q)\hat{c}_2 + \omega_3\hat{c}_3\right] \times \left(I\omega_1\hat{c}_1 + J\omega_2\hat{c}_2 + I\omega_3\hat{c}_3\right)$$

$$\bar{M}^{U^*} = I\dot{\omega}_1\hat{c}_1 + J\dot{\omega}_2\hat{c}_2 + I\dot{\omega}_3\hat{c}_3$$
$$+ J\omega_1\omega_2\hat{c}_3 - \cancel{I\omega_1\omega_3\hat{c}_2}$$
$$- I\omega_1(\omega_2 - q)\hat{c}_3 + I(\omega_2 - q)\omega_3\hat{c}_1$$
$$+ \cancel{I\omega_1\omega_3\hat{c}_2} - J\omega_2\omega_3\hat{c}_1$$

$$\bar{M}^{U^*} = \left[I(\dot{\omega}_1 - q\omega_3) + (I - J)\omega_2\omega_3\right]\hat{c}_1$$
$$+ J\dot{\omega}_2\hat{c}_2$$

$$+ \left[I(\dot{\omega}_3 + q\omega_1) - (I-J)\omega_1\omega_2 \right]\hat{c}_3$$

$$\cdots \; (1)$$

$$\rightarrow \quad \bar{M}^{U*} = \frac{3\mu}{R^3}\hat{a}_3 \times \bar{\bar{I}}^{U/U*} \cdot \hat{a}_3 = 3\Omega^2 \hat{a}_3 \times \bar{\bar{I}}^{U/U*} \cdot \hat{a}_3$$

transform $\bar{\bar{I}}^{U/U*}$: $\hat{c}_i \rightarrow \hat{a}_i$ using DCM

${}^{A}C^{C}$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	×	×	×
$\hat{a}_2$	×	×	×
$\hat{a}_3$	C_{31}	C_{32}	C_{33}

$I_1 = I_3$

$$\bar{M}^{U*} = 3\Omega^2 \left[(I_3 - I_2)C_{32}C_{33}\hat{c}_1 + \cancel{(I_1 - I_3)}C_{33}C_{31}\hat{c}_2 + (I_2 - I_1)C_{31}C_{32}\hat{c}_3 \right]$$

$$\bar{M}^{U*} = 3\Omega^2 \left[(I-J)C_{32}C_{33}\hat{c}_1 - (I-J)C_{31}C_{32}\hat{c}_3 \right] \quad \cdots \; (2)$$

since (1) = (2)

$\hat{c}_1$: $\quad I(\dot{\omega}_1 - q\omega_3) + (I-J)\omega_2\omega_3 = 3\Omega^2(I-J)C_{32}C_{33}$

$\hat{c}_2$: $\quad J\dot{\omega}_2 = 0$

$\hat{c}_3$: $\quad I(\dot{\omega}_3 + q\omega_1) - (I-J)\omega_1\omega_2 = -3\Omega^2(I-J)C_{31}C_{32}$

this becomes

$$\boxed{\begin{aligned} \dot{\omega}_1 &= q\,\omega_3 + \frac{I-J}{I}\left(3\Omega^2 C_{32} C_{33} - \omega_2 \omega_3\right) \\ \dot{\omega}_2 &= 0 \\ \dot{\omega}_3 &= -q\,\omega_1 - \frac{I-J}{I}\left(3\Omega^2 C_{31} C_{32} - \omega_1 \omega_2\right) \end{aligned}}$$

Kinematic Differential Equations

from

$$^N\bar{\omega}^C = {}^N\bar{\omega}^A + {}^A\bar{\omega}^C$$

$$^A\bar{\omega}^C = {}^N\bar{\omega}^C - {}^N\bar{\omega}^A$$

$$\underline{^A\bar{\omega}^C = \omega_i \hat{c}_i - q\,\hat{c}_2 - \Omega\,\hat{a}_2}$$

since

$^A C^C$	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$
$\hat{a}_1$	×	×	×
$\rightarrow \hat{a}_2$	C_{21}	C_{22}	C_{23}
$\hat{a}_3$	×	×	×

$\rightarrow \hat{a}_2 = C_{21}\hat{c}_1 + C_{22}\hat{c}_2 + C_{23}\hat{c}_3$

then

$$
\begin{aligned}
&= \omega_1 \hat{c}_1 + \omega_2 \hat{c}_2 + \omega_3 \hat{c}_3 - q\, \hat{c}_2 \\
&\qquad - c_{21}\Omega\, \hat{c}_1 - c_{22}\Omega\, \hat{c}_2 - c_{23}\Omega\, \hat{c}_3 \\
&= (\omega_1 - c_{21}\Omega)\, \hat{c}_1 \\
&\qquad + (\omega_2 - q - c_{22}\Omega)\, \hat{c}_2 \\
&\qquad + (\omega_3 - c_{23}\Omega)\, \hat{c}_3
\end{aligned}
$$

therefore

$$
{}^{A}\tilde{\omega}^{C} = \begin{bmatrix} 0 & -(\omega_3 - c_{23}\Omega) & (\omega_2 - q - c_{22}\Omega) \\ (\omega_3 - c_{23}\Omega) & 0 & -(\omega_1 - c_{21}\Omega) \\ -(\omega_2 - q - c_{22}\Omega) & (\omega_1 - c_{21}\Omega) & 0 \end{bmatrix}
$$

finally,

$$
\dot{C} = {}^{A}C^{C}\ {}^{A}\tilde{\omega}^{C}
$$

$$
= \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} 0 & -(\omega_3 - c_{23}\Omega) & (\omega_2 - q - c_{22}\Omega) \\ (\omega_3 - c_{23}\Omega) & 0 & -(\omega_1 - c_{21}\Omega) \\ -(\omega_2 - q - c_{22}\Omega) & (\omega_1 - c_{21}\Omega) & 0 \end{bmatrix}
$$

$$
\begin{aligned}
\dot{c}_{11} &= c_{12}(\omega_3 - c_{23}\Omega) - c_{13}(\omega_2 - q - c_{22}\Omega) \\
\dot{c}_{12} &= -c_{11}(\omega_3 - c_{23}\Omega) + c_{13}(\omega_1 - c_{21}\Omega) \\
\dot{c}_{13} &= c_{11}(\omega_2 - q - c_{22}\Omega) - c_{12}(\omega_1 - c_{21}\Omega) \\
\dot{c}_{21} &= c_{22}(\omega_3 - \cancel{c_{23}\Omega}) - c_{23}(\omega_2 - q - \cancel{c_{22}\Omega}) \\
\dot{c}_{22} &= -c_{21}(\omega_3 - \cancel{c_{23}\Omega}) + c_{23}(\omega_1 - \cancel{c_{21}\Omega}) \\
\dot{c}_{23} &= c_{21}(\omega_2 - q - \cancel{c_{22}\Omega}) - c_{22}(\omega_1 - \cancel{c_{21}\Omega}) \\
\dot{c}_{31} &= c_{32}(\omega_3 - c_{23}\Omega) - c_{33}(\omega_2 - q - c_{22}\Omega) \\
\dot{c}_{32} &= -c_{31}(\omega_3 - c_{23}\Omega) + c_{33}(\omega_1 - c_{21}\Omega) \\
\dot{c}_{33} &= c_{31}(\omega_2 - q - c_{22}\Omega) - c_{12}(\omega_1 - c_{21}\Omega)
\end{aligned}
$$

$$\boxed{\begin{aligned}
\dot{c}_{11} &= c_{12}(\omega_3 - c_{23}\Omega) - c_{13}(\omega_2 - q - c_{22}\Omega) \\
\dot{c}_{12} &= -c_{11}(\omega_3 - c_{23}\Omega) + c_{13}(\omega_1 - c_{21}\Omega) \\
\dot{c}_{13} &= c_{11}(\omega_2 - q - c_{22}\Omega) - c_{12}(\omega_1 - c_{21}\Omega) \\
\dot{c}_{21} &= c_{22}\,\omega_3 - c_{23}(\omega_2 - q) \\
\dot{c}_{22} &= -c_{21}\,\omega_3 + c_{23}\,\omega_1 \\
\dot{c}_{23} &= c_{21}(\omega_2 - q) - c_{22}\,\omega_1 \\
\dot{c}_{31} &= c_{32}(\omega_3 - c_{23}\Omega) - c_{33}(\omega_2 - q - c_{22}\Omega) \\
\dot{c}_{32} &= -c_{31}(\omega_3 - c_{23}\Omega) + c_{33}(\omega_1 - c_{21}\Omega) \\
\dot{c}_{33} &= c_{31}(\omega_2 - q - c_{22}\Omega) - c_{32}(\omega_1 - c_{21}\Omega)
\end{aligned}}$$

(ii) Use Euler parameters as the kinematic variables and derive the complete set of differential equations.

Dynamic Differential Equations

Substitute

$$C_{31} = 2(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4)$$
$$C_{32} = 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4)$$
$$C_{33} = 1 - 2\varepsilon_1^2 - 2\varepsilon_2^2$$

into the Dynamic Differential Equation

$$\begin{aligned}\dot{\omega}_1 &= q\omega_3 + \frac{I-J}{I}\left(3\Omega^2 C_{32}C_{33} - \omega_2\omega_3\right)\\ &= q\omega_3 + \frac{I-J}{I}\left\{3\Omega^2\left[2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)\right](1-2\varepsilon_1^2+2\varepsilon_2^2) - \omega_2\omega_3\right\}\\ &= q\omega_3 + \frac{I-J}{I}\left[6\Omega^2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2+2\varepsilon_2^2) - \omega_2\omega_3\right]\end{aligned}$$

$$\dot{\omega}_2 = 0$$

$$\begin{aligned}\dot{\omega}_3 &= -q\omega_1 - \frac{I-J}{I}\left(3\Omega^2 C_{31}C_{32} - \omega_1\omega_2\right)\\ &= -q\omega_1 - \frac{I-J}{I}\left[12\Omega^2(\varepsilon_3\varepsilon_1-\varepsilon_2\varepsilon_4)(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) - \omega_1\omega_2\right]\end{aligned}$$

$$\boxed{\begin{aligned}\dot{\omega}_1 &= q\omega_3 + \frac{I-J}{I}\left[6\Omega^2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2+2\varepsilon_2^2) - \omega_2\omega_3\right]\\ \dot{\omega}_2 &= 0\\ \dot{\omega}_3 &= -q\omega_1 - \frac{I-J}{I}\left[12\Omega^2(\varepsilon_3\varepsilon_1-\varepsilon_2\varepsilon_4)(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) - \omega_1\omega_2\right]\end{aligned}}$$

Kinematic Differential Equations

$$
\begin{aligned}
{}^N\bar{\omega}^A &= \Omega \hat{a}_2 \\
&= \Omega (C_{21} \hat{c}_1 + C_{22} \hat{c}_2 + C_{23} \hat{c}_3) \\
&= \Omega \left[2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4)\hat{c}_1 + (1 - 2\varepsilon_3^2 - 2\varepsilon_1^2)\hat{c}_2 + 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)\hat{c}_3 \right]
\end{aligned}
$$

then, from $\quad {}^N\bar{\omega}^A + {}^A\bar{\omega}^C = {}^N\bar{\omega}^C = \omega_i \hat{c}_i$

$$
\begin{aligned}
{}^A\bar{\omega}^C &= ({}^N\bar{\omega}^C - {}^N\bar{\omega}^A) \\
&= \left[\omega_1 - 2\Omega(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) \right] \hat{c}_1 \\
&\quad + \left[\omega_2 - q - \Omega(1 - 2\varepsilon_3^2 - 2\varepsilon_1^2) \right] \hat{c}_2 \\
&\quad + \left[\omega_3 - 2\Omega(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \right] \hat{c}_3
\end{aligned}
$$

plug this into

$$
{}^A\dot{\varepsilon}^C = \frac{1}{2} {}^A\bar{\omega}^C E^T
$$

where

$$
E^T = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}
$$

$\rightarrow \quad 2\, {}^A\dot{\varepsilon}^C = 2 \left[\dot{\varepsilon}_1 \quad \dot{\varepsilon}_2 \quad \dot{\varepsilon}_3 \quad \dot{\varepsilon}_4 \right]$

$\rightarrow \quad {}^A\bar{\omega}^C E^T$

$$= \begin{bmatrix} \omega_1 - 2\Omega(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) \\ \omega_2 - q - \Omega(1 - 2\varepsilon_3^2 - 2\varepsilon_1^2) \\ \omega_3 - 2\Omega(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) \\ 0 \end{bmatrix}^T \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

Col #1

$$\omega_1\varepsilon_4 - 2\Omega(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4)\varepsilon_4 - \omega_2\varepsilon_3 + \Omega(1 - 2\varepsilon_3^2 - 2\varepsilon_1^2) + q\varepsilon_3 + \omega_3\varepsilon_2 - 2\Omega(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)\varepsilon_2$$

$$= \omega_1\varepsilon_4 - \omega_2\varepsilon_3 + q\varepsilon_3 + \omega_3\varepsilon_2$$

$$\quad - \Omega(\cancel{2\varepsilon_1\varepsilon_2\varepsilon_4} + 2\varepsilon_3\varepsilon_4^2 - \cancel{2\varepsilon_1\varepsilon_2\varepsilon_4} + 2\varepsilon_2^2\varepsilon_3 - \varepsilon_3 + 2\varepsilon_3^3 + 2\varepsilon_1^2\varepsilon_3)$$

$$= \omega_1\varepsilon_4 - \omega_2\varepsilon_3 + q\varepsilon_3 + \omega_3\varepsilon_2 - \Omega[2\varepsilon_3(\underbrace{\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2}_{1}) - \varepsilon_3]$$

$$= \omega_1\varepsilon_4 - \omega_2\varepsilon_3 + q\varepsilon_3 + \omega_3\varepsilon_2 - \Omega\varepsilon_3$$

$$= \omega_3\varepsilon_2 - (\omega_2 - q + \Omega)\varepsilon_3 + \omega_1\varepsilon_4$$

Col #2

$$\omega_1\varepsilon_3 - 2\Omega(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4)\varepsilon_3 + \omega_2\varepsilon_4 - \Omega(1 - 2\varepsilon_3^2 - 2\varepsilon_1^2)\varepsilon_4$$

$$\quad - q\varepsilon_4 - \omega_3\varepsilon_1 + 2\Omega(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)\varepsilon_1$$

$$= \omega_1\varepsilon_3 + \omega_2\varepsilon_4 - q\varepsilon_4 - \omega_3\varepsilon_1 - \Omega(\cancel{2\varepsilon_1\varepsilon_2\varepsilon_3} + \cancel{2\varepsilon_3^2\varepsilon_4} + \varepsilon_4 - \cancel{2\varepsilon_3^2\varepsilon_4}$$

$$\quad - \cancel{2\varepsilon_1^2\varepsilon_4} - \cancel{2\varepsilon_1\varepsilon_2\varepsilon_3} + \cancel{2\varepsilon_1^2\varepsilon_4})$$

$$= -\omega_3\varepsilon_1 + \omega_1\varepsilon_3 + (\omega_2 - q - \Omega)\varepsilon_4$$

Col #3

$$-\omega_1\varepsilon_2 + 2\Omega(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4)\varepsilon_2 + \omega_2\varepsilon_1 - \Omega(1-2\varepsilon_3^2-2\varepsilon_1^2)\varepsilon_1 - q\varepsilon_1$$
$$+\omega_3\varepsilon_4 - 2\Omega(\varepsilon_2\varepsilon_3-\varepsilon_1\varepsilon_4)\varepsilon_4$$
$$= -\omega_1\varepsilon_2 + \omega_2\varepsilon_1 - q\varepsilon_1 + \omega_3\varepsilon_4 + \Omega(2\varepsilon_1\varepsilon_2^2 + \cancel{2\varepsilon_2\varepsilon_3\varepsilon_4} - \varepsilon_1 + 2\varepsilon_1\varepsilon_3^2 + 2\varepsilon_1^3$$
$$-\cancel{2\varepsilon_2\varepsilon_3}\varepsilon_4 + 2\varepsilon_1\varepsilon_4^2)$$
$$= -\omega_1\varepsilon_2 + \omega_2\varepsilon_1 - q\varepsilon_1 + \omega_3\varepsilon_4 + \Omega\left[2\varepsilon_1\cancelto{1}{(\varepsilon_1^2+\varepsilon_2^2+\varepsilon_3^2+\varepsilon_4^2)} - \varepsilon_1\right]$$
$$= \underline{(\omega_2 - q + \Omega)\varepsilon_1 - \omega_1\varepsilon_2 + \omega_3\varepsilon_4}$$

Col #4

$$-\omega_1\varepsilon_1 + 2\Omega(\varepsilon_1\varepsilon_2+\varepsilon_3\varepsilon_4)\varepsilon_1 - \omega_2\varepsilon_2 + \Omega(1-2\varepsilon_3^2-2\varepsilon_1^2)\varepsilon_2 + q\varepsilon_2$$
$$-\omega_3\varepsilon_3 + 2\Omega(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)\varepsilon_3$$
$$= -\omega_1\varepsilon_1 - \omega_2\varepsilon_2 + q\varepsilon_2 - \omega_3\varepsilon_3 + \Omega(\cancel{2\varepsilon_1^2\varepsilon_2} + \cancel{2\varepsilon_1\varepsilon_3\varepsilon_4} + \varepsilon_2$$
$$-\cancel{2\varepsilon_2\varepsilon_3^2} - \cancel{2\varepsilon_1^2\varepsilon_2} + \cancel{2\varepsilon_2\varepsilon_3^2} - \cancel{2\varepsilon_1\varepsilon_3\varepsilon_4})$$
$$= \underline{-\omega_1\varepsilon_1 - (\omega_2 - q - \Omega)\varepsilon_2 - \omega_3\varepsilon_3}$$

Thus,

$$\boxed{\begin{aligned}
2\dot{\varepsilon}_1 &= \omega_3\varepsilon_2 - (\omega_2 - q + \Omega)\varepsilon_3 + \omega_1\varepsilon_4\\
2\dot{\varepsilon}_2 &= -\omega_3\varepsilon_1 + \omega_1\varepsilon_3 + (\omega_2 - q - \Omega)\varepsilon_4\\
2\dot{\varepsilon}_3 &= (\omega_2 - q + \Omega)\varepsilon_1 - \omega_1\varepsilon_2 + \omega_3\varepsilon_4\\
2\dot{\varepsilon}_4 &= -\omega_1\varepsilon_1 - (\omega_2 - q - \Omega)\varepsilon_2 - \omega_3\varepsilon_3
\end{aligned}}$$

(iii) What sets of unit vectors do the kinematic variables relate? How do you know?
What angular velocity components appear in the equations of motion? In what vector basis are they expressed?

Discussion

→ $\hat{a}_i$ is related to $\hat{c}_j$ by the kinematic variables because they yield values that show body moment w.r.t the orbit frame.

→ The body's angular velocity is expressed in the equations of motion, from the inertial frame to the body frame, which are in the $\hat{c}_i$-frame.

(iv) Assume Euler parameters as the kinematic variables, check the differential equations in Notes R. Since the orbit normal in class is $\hat{a}_3$ and the orbit normal in this problem $\hat{a}_2$, should your equations be exactly the same as those in the notes?

$$2\dot{\varepsilon}_1 = \varepsilon_2(\omega_3 - s + \Omega) - \varepsilon_3\omega_2 + \varepsilon_4\omega_1$$

$$2\dot{\varepsilon}_2 = \varepsilon_3\omega_1 + \varepsilon_4\omega_2 - \varepsilon_1(\omega_3 - s + \Omega)$$

$$2\dot{\varepsilon}_3 = \varepsilon_4(\omega_3 - s - \Omega) + \varepsilon_1\omega_2 - \varepsilon_2\omega_1$$

$$2\dot{\varepsilon}_4 = -\varepsilon_1\omega_1 - \varepsilon_2\omega_2 - \varepsilon_3(\omega_3 - s - \Omega)$$

$$\dot{\omega}_1 = -s\omega_2 + \left(1 - \frac{J}{I}\right)\left[\omega_2\omega_3 - 12\Omega^2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4)(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)\right]$$

$$\dot{\omega}_2 = s\omega_1 - \left(1 - \frac{J}{I}\right)\left[\omega_1\omega_3 - 6\Omega^2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)(1 - 2\varepsilon_2^2 - 2\varepsilon_3^2)\right]$$

$$\dot{\omega}_3 = 0$$

from notes R

Discussion

→ the equations are similar but not exactly the same due to how the way the vector bases are defined or oriented differently.

(b) Continue with Euler parameters as the kinematic variables. In the differential equations, change the independent variable from time (t) to number of revolutions (ν). This change generalizes the results and makes it easier to interpret any numerical data. Nondimensionalize the differential equations such that the independent variable is ν and the dependent variables are ε_i and w_i, where w_i are the nondimensional angular velocities, i.e., $w_i = \omega_i / \Omega$. Do you need to nondimensionalize ε_i? Why not?

nondimensionalize the Dynamic Differential Equations

$$\dot{\omega}_1 = q\omega_3 + \frac{I-J}{I}\left[6\Omega^2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2+2\varepsilon_2^2) - \omega_2\omega_3\right] \quad \cdots (1)$$

$$\dot{\omega}_2 = 0 \quad \cdots (2)$$

$$\dot{\omega}_3 = -q\omega_1 - \frac{I-J}{I}\left[12\Omega^2(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4)(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) - \omega_1\omega_2\right] \quad \cdots (3)$$

(1)

$$\frac{1}{sec} \rightarrow \frac{1}{rev} \Rightarrow \frac{df}{dt}\cdot\frac{2\pi}{\Omega} = \frac{df}{d\nu}$$

then

$$\frac{2\pi}{\Omega^2}\dot{\omega}_1 = \omega_3\left(\frac{2\pi}{\Omega}\right)\cdot q\left(\frac{2\pi}{\Omega}\right) + \frac{I-J}{I}\left[6\Omega^2(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2-2\varepsilon_2^2) - \omega_2\omega_3\right]\left(\frac{2\pi}{\Omega}\right)^2$$

since $q = \omega_{20} - \Omega$ & say $\frac{I-J}{I} = -x$ (shape factor)

$$\frac{2\pi\dot{\omega}_1}{\Omega^2} = 4\pi^2\cdot\frac{\omega_3}{\Omega}\cdot\frac{\omega_{20}-\Omega}{\Omega} - x\left[6(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2-2\varepsilon_2^2) - \frac{\omega_2}{\Omega}\frac{\omega_3}{\Omega}\right](4\pi^2)$$

provided that $w_i = \frac{\omega_i}{\Omega}$ & $\dot{w}_i = \frac{\dot{\omega}_i}{\Omega^2}$

& $y = \frac{\omega_{20}-\Omega}{\Omega}$ (spin factor)

$$\dot{w}_1 = 2\pi w_3 \cdot y - x\left[6(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2-2\varepsilon_2^2) - w_2 w_3\right](2\pi)$$

$$\dot{w}_1 = 2\pi\left\{w_3 y - x\left[6(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2-2\varepsilon_2^2) - w_2 w_3\right]\right\}$$

3 similar to 1

$$\dot{w}_3 = 2\pi\left\{-w_1 y + x\left[12(\varepsilon_3\varepsilon_1-\varepsilon_2\varepsilon_4)(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) - w_1 w_2\right]\right\}$$

thus, all 3 nondimensionalized equations become

$$\boxed{\begin{aligned}
\dot{w}_1 &= 2\pi\left\{w_3 y - x\left[6(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4)(1-2\varepsilon_1^2-2\varepsilon_2^2) - w_2 w_3\right]\right\}\\
\dot{w}_2 &= 0\\
\dot{w}_3 &= 2\pi\left\{-w_1 y + x\left[12(\varepsilon_3\varepsilon_1-\varepsilon_2\varepsilon_4)(\varepsilon_2\varepsilon_3+\varepsilon_1\varepsilon_4) - w_1 w_2\right]\right\}
\end{aligned}}$$

nondimensionalize the Kinematic Differential Equations

$$2\dot{\varepsilon}_1 = \omega_3\varepsilon_2 - (\omega_2 - q + \Omega)\varepsilon_3 + \omega_1\varepsilon_4 \quad \cdots \quad (4)$$

$$2\dot{\varepsilon}_2 = -\omega_3\varepsilon_1 + \omega_1\varepsilon_3 + (\omega_2 - q - \Omega)\varepsilon_4 \quad \cdots \quad (5)$$

$$2\dot{\varepsilon}_3 = (\omega_2 - q + \Omega)\varepsilon_1 - \omega_1\varepsilon_2 + \omega_3\varepsilon_4 \quad \cdots \quad (6)$$

$$2\dot{\varepsilon}_4 = -\omega_1\varepsilon_1 - (\omega_2 - q - \Omega)\varepsilon_2 - \omega_3\varepsilon_3 \quad \cdots \quad (7)$$

4

plug $q = \omega_{20} - \Omega$ & multiply by $\frac{2\pi}{\Omega}$

$$2\dot{\varepsilon}_1 = \left\{\omega_3\varepsilon_2 - \left[\omega_2 - (\omega_{20} - \Omega) + \Omega\right]\varepsilon_3 + \omega_1\varepsilon_4\right\}\frac{2\pi}{\Omega}$$

since $y = \frac{w_{20}}{\Omega} - 1 \Rightarrow -\frac{w_{20}}{\Omega} = -y - 1$

$$\mathring{\varepsilon}_1 = \left[w_3\varepsilon_2 - (w_2 - y + 1)\varepsilon_3 + w_1\varepsilon_4\right]\pi \quad \because w_i = \frac{\omega_i}{\Omega}$$

⑤ ~ ⑦ similar to ④

⑤ : $\dot{\varepsilon}_2 = \left[-w_3\varepsilon_1 + w_1\varepsilon_3 + (w_2 - y - 1)\varepsilon_4 \right]\pi v$

⑥ : $\dot{\varepsilon}_3 = \left[(w_2 - y + 1)\varepsilon_1 - w_1\varepsilon_2 + w_3\varepsilon_4 \right]\pi v$

⑦ : $\dot{\varepsilon}_4 = \left[-w_1\varepsilon_1 - (w_2 - y - 1)\varepsilon_2 - w_3\varepsilon_3 \right]\pi v$

Hence,

$$\boxed{\begin{aligned}
\dot{\varepsilon}_1 &= \left[w_3\varepsilon_2 - (w_2 - y + 1)\varepsilon_3 + w_1\varepsilon_4 \right]\pi v \\
\dot{\varepsilon}_2 &= \left[-w_3\varepsilon_1 + w_1\varepsilon_3 + (w_2 - y - 1)\varepsilon_4 \right]\pi v \\
\dot{\varepsilon}_3 &= \left[(w_2 - y + 1)\varepsilon_1 - w_1\varepsilon_2 + w_3\varepsilon_4 \right]\pi v \\
\dot{\varepsilon}_4 &= \left[-w_1\varepsilon_1 - (w_2 - y - 1)\varepsilon_2 - w_3\varepsilon_3 \right]\pi v
\end{aligned}}$$

Appendix

AAE440 HW8 PROBLEM 1 MATLAB

```
clear all; close all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Spring\AAE440\MATLAB\outputs\HW8';
set(groot, 'defaulttextinterpreter',"latex");
set(groot, 'defaultAxesTickLabelInterpreter',"latex");
set(groot, 'defaultLegendInterpreter',"latex");
% Arrow drawing function
drawArrow = @(x,y,varargin) quiver( x(1),y(1),x(2)-x(1),y(2)-y(1),0,
varargin{:} );
```

(a)

```
% Given properties
I_body = [400 0 0; 0 100 0; 0 0 400];  % [kg-m2]
I = I_body(1,1); J = I_body(2,2);
w_NU_mag = 4;  % magnitude of angular velocity [rad/s]
w_NU_hat = [0 cosd(60) sind(60)];  % angle of w_NU relative to u_2 [deg]
w_NU = w_NU_mag*w_NU_hat;

% Kinetic rotational energy
Trot = 0.5*w_NU*I_body*w_NU.'
% Semi-diameters of energy ellipsoid
d1 = sqrt(2*Trot*I^(-1))
d2 = sqrt(2*Trot*J^(-1))
d3 = d1
% Plotting the inertial ellipsoid
theta = 0:0.01:2*pi;
u_str = ["$\hat{u}_1$","$\hat{u}_2$","$\hat{u}_3$"];

% u2-u3
fig1 = figure("Renderer","painters");
    plot(d2*cos(theta), d3*sin(theta), 'b')
    title('$\hat{u}_2$-$\hat{u}_3$ Energy Ellipse, Koike')
    xlabel('$\omega_2$ (rad/s)')
    ylabel('$\omega_3$ (rad/s)')
    hold on
    drawArrow([0 9], [0 0],'k','linewidth',1);
text(9,1,u_str(2),"Interpreter","Latex");
    drawArrow([0 0], [0 8],'k','linewidth',1); text(-
0.8,7,u_str(3),"Interpreter","Latex");

    % Angular velocity
    drawArrow([0 w_NU(2)], [0 w_NU(3)],'color','#FD07EA');
    text(1.3,1.5,'${}^N\bar{\omega}^U$','Interpreter','Latex');

    % Invariable plane PI
    [a, b] = line_tangent2ellipse(w_NU(2),w_NU(3),d2,d3);
```

```
    x = -8:0.1:8;
    y = a*x + b;
    plot(x,y,'-r'); text(6.7,3.2,'$\pi$','Interpreter',"latex");

    % H_body
    drawArrow([0 1.0],[0 1.0*(-1/a)],'color','#FF6800','linewidth',1.2)
    text(1.1,7,'${}^N\bar{H}^{B/B*}$')

    hold off
    xlim([-9 9]); ylim([-8 8]);
    grid on; grid minor; box on; axis equal;
saveas(fig1, fullfile(fdir,'P1-a-u2_u3_EN-ellipse.png'));
```

(b)

```
% Angular momentum
H_NU = I_body*w_NU.';
H_NU_mag = norm(H_NU);

% Computing p, s, and phi
p = H_NU_mag/I;
s = (I - J)/I*w_NU(2);
phi = acos(H_NU(2)/H_NU_mag)
phi_deg = rad2deg(phi)

% Computing the precession, nutation, and spin angles
% @ t = 0.25
t = 0.25;
sigma = p*t
sigma_deg = rad2deg(sigma)
eta = s*t
eta_deg = rad2deg(eta)

% @ t = 3.5
t = 3.5;
sigma = mod(p*t,2*pi)
sigma_deg = rad2deg(sigma)
eta = mod(s*t,2*pi)
eta_deg = rad2deg(eta)
h_hat_U = H_NU/H_NU_mag;
h_hat_C = [0 cos(phi) -sin(phi)];
% gamma
syms t1
e_NC = h_hat_C*sin(p*t1/2);
e4_NC = cos(p*t1/2);

% DCM
% @ t= 0.25
e_NC_025 = double(subs(e_NC,t1,0.25));
e4_NC_025 = double(subs(e4_NC,t1,0.25));
C_NC_025 = DCM_from_EulerPara([e_NC_025 e4_NC_025])
% @ t= 3.5
```

```
e_NC_35 = double(subs(e_NC,t1,3.5));
e4_NC_35 = double(subs(e4_NC,t1,3.5));
C_NC_35 = DCM_from_EulerPara([e_NC_35 e4_NC_35])

gamma_025 = acosd(C_NC_025(2,2))
gamma_35 = acosd(C_NC_35(2,2))
```

(c)

```
% Euler parameters
c2_hat = [0 1 0];
syms t
e_NC = h_hat_C*sin(p*t/2);
e4_NC = cos(p*t/2);
e_CU = c2_hat*sin(s*t/2);
e4_CU = cos(s*t/2);

e_NU = e_NC*e4_CU + e_CU*e4_NC + cross(e_CU,e_NC);
e4_NU = e4_NC*e4_CU - dot(e_NC,e_CU);
% C-frame
% @ t = 0.25
e_NU_025_C = double(subs(e_NU,t,0.25))
e4_NU_025 = double(subs(e4_NU,t,0.25))

% @ t = 3.5
e_NU_35_C = double(subs(e_NU,t,3.5))
e4_NU_35 = double(subs(e4_NU,t,3.5))
% U-frame
syms t2
C_CU = [cos(s*t2) 0 -sin(s*t2);
                0 1          0;
        sin(s*t2) 0  cos(s*t2)];
% @ t = 0.25
e_NU_025_U = double(e_NU_025_C*subs(C_CU,t2,0.25))
C_NU_025_1 = double(C_NC_025*subs(C_CU,t2,0.25))
C_NU_025_2C = DCM_from_EulerPara([e_NU_025_C e4_NU_025])
C_NU_025_2U = DCM_from_EulerPara([e_NU_025_U e4_NU_025])
% @ t = 3.5
e_NU_35_U = double(e_NU_35_C*subs(C_CU,t2,3.5))
```

(d)

```
gamma_max = 2*phi_deg;
```

(e)

```
alpha = 2*(phi_deg - 60)
```

ADDITIONAL (FOR PROBLEM 2)

```
tspan = 0:0.005:15;
fig2 = figure("Renderer","painters")
    plot(tspan,rad2deg(mod(p*tspan,2*pi)),'r')
    title("Precession, pt Over Time, Koike")
    ylabel('Precession, pt [deg]')
    xlabel('Time [s]')
    grid on; grid minor; box on;
```

```
saveas(fig2,fullfile(fdir,"pt_precession.png"))

fig3 = figure("Renderer","painters")
    plot(tspan,rad2deg(phi).*ones(size(tspan)),'b')
    title("Nutation, $\phi$ Over Time, Koike")
    ylabel('Nutation, $\phi$ [deg]')
    xlabel('Time [s]')
    grid on; grid minor; box on;
saveas(fig3,fullfile(fdir,"phi_nutation.png"))
```

FUNCTION

```
function [slope, y_intercept] = line_tangent2ellipse(x1,y1,a,b)
    slope = -x1/y1*b^2/a^2;
    y_intercept = y1 - x1*slope;
end
```

AAE440 HW8 PROBLEM 2 MATLAB

```
clear all; close all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Spring\AAE440\MATLAB\outputs\HW8';
set(groot, 'defaulttextinterpreter',"latex");
set(groot, 'defaultAxesTickLabelInterpreter',"latex");
set(groot, 'defaultLegendInterpreter',"latex");
```

```
% Defining System Properties
T       = 0;                            % Torque [N m]
I_cm    = [400   0   0;
             0 100   0;
             0   0 400];                % Inertia Dyadic [kg m2]
I       = 400;
J       = 100;

% Given Initial Conditions
w0      = [1 2 1];                      % Initial AngVel [rad s-1]
e0      = [0 0 0 1];                    % Initial Euler Parameters
C0      = [1 0 0 0 1 0 0 0 1];          % Initial DCM

% Numerically integrating dynamic and kinematic EOMs
tspan = [0 14]; % Integration time
y0 = [w0 e0 0 C0]; % Initial conditions
opt = odeset('RelTol', 1e-13, 'AbsTol', 1e-13); % Integration Tolerance
[t1, res1] = ode45(@(t,y) EOM(t,y,I,J,T), tspan, y0, opt);

% Plotting three angular velocity measure numbers over time
fig1 = figure("Renderer","painters");
    plot(t1, res1(:,1:3))
    ylabel('Angular velocity [rad/s]')
    xlabel('time [s]')
    title({'Angular Velocity Measure Numbers, Koike'})
    axis([tspan -2 2.5])
    legend('$\omega_1$', '$\omega_2$', '$\omega_3$','Location',"best")
    grid on; grid minor; box on;
```

```
saveas(fig1, fullfile(fdir, 'angVel_measure_nums.png'));
```

(a)

```
% Plotting precession and nutation angles
tspan_a = 0:0.001:15;
[t_a, res_a] = ode45(@(t,y) EOM(t,y,I,J,T), tspan_a, y0, opt);
% Assigning computed C12, C22, and C32 to a variable
C_a = res_a(:,9:end);
[alpha_a, gamma_a] = ang_calc_body212(C_a);

fig2 = figure(2);
    plot(t_a, alpha_a, 'r')
    xlabel('time [s]')
    ylabel('precession $\alpha$, [deg]')
    title('Precession Over Time, Koike')
    grid on; grid minor; box on;
saveas(fig2, fullfile(fdir,'alpha.png'));

fig3 = figure(3);
    plot(t_a, gamma_a, 'b')
    xlabel('time [s]')
    ylabel('nutation $\gamma$, [deg]')
    title('Nutation Over Time, Koike')
    grid on; grid minor; box on
saveas(fig3, fullfile(fdir,'gamma.png'));
```

(c)

```
% Integration with smaller time step
tspan_c = 0:0.05:15;
[t_c, res_c] = ode45(@(t,y) EOM(t,y,I,J,T), tspan_c, y0, opt);
% C_new = res_c(:,9:17);

% Assigning computed C12, C22, and C32 to a variable
C11s_c = res_c(:,9);
C12s_c = res_c(:,10);
C13s_c = res_c(:,11);
C21s_c = res_c(:,12);
C22s_c = res_c(:,13);
C23s_c = res_c(:,14);
C31s_c = res_c(:,15);
C32s_c = res_c(:,16);
C33s_c = res_c(:,17);

% Finding the index when t=0.2 and t=1.5 and corresponding C12 C22 C32
index_t0p25 = find(t_c==0.25);
C11_t025 = C11s_c(index_t0p25);
C12_t025 = C12s_c(index_t0p25);
C13_t025 = C13s_c(index_t0p25);
C21_t025 = C21s_c(index_t0p25);
C22_t025 = C22s_c(index_t0p25);
C23_t025 = C23s_c(index_t0p25);
```

```matlab
C31_t025 = C31s_c(index_t0p25);
C32_t025 = C32s_c(index_t0p25);
C33_t025 = C33s_c(index_t0p25);
C_025 = [C11_t025 C12_t025 C13_t025;
         C21_t025 C22_t025 C23_t025;
         C31_t025 C32_t025 C33_t025]

% Computing gamma
gamma = acosd(C22_t025)

% Plotting at t = 0.2 and 1.5
fig4 = figure(4);
    plot(C12s_c, C32s_c,'-m','MarkerSize',15)
    title('$C_{32}$ vs $C_{12}$, Koike')
    xlabel('$C_{12}$')
    ylabel('$C_{32}$')
    hold on
    plot(C12_t025, C32_t025, '.','MarkerSize',26)
    plot(0,0,'.k','MarkerSize',20)
    plot([0 0],[0 1],'--k')
    plot([0 1],[0 0],'-k')
    d = linspace(0,-0.5,100);
    plot(d,d.*(C32_t025/C12_t025),'-b')
    hold off

legend('Path','t=0.25','origin','$\hat{n}_3$','$\hat{n}_1$',"location",'southwest
')
    grid on; grid minor; axis equal; box on;
    xlim([-1 1]); ylim([-1 1]);
saveas(fig4, fullfile(fdir, 'C12_vs_C32.png'));
```

FUNCTION

```matlab
function [alphas, gammas] = ang_calc_body212(DCM)
    %{
       This function calculates the precession, nutation, and spin angle
       from the provided DCM
     %}

    % DCM is 1 by 9 matrix with each column being C_ij
    C12s = DCM(:,2);
    C21s = DCM(:,4);
    C22s = DCM(:,5);
    C23s = DCM(:,6);
    C32s = DCM(:,8);

    % Preallocating alpha and gamma arrays
    alphas = zeros([length(C12s),1]);
    gammas = zeros([length(C12s),1]);

    % For loop to construct alpha and gamma arrays interatively
    for i = 1:length(alphas)
```

```
        % calculating gamma
        gammas(i) = acos(C22s(i));

        % calculating and verfying alpha
        alpha1 = round([acos(C32s(i)/sin(gammas(i))), ...
                        -acos(C32s(i)/sin(gammas(i))), ...
                        -acos(C32s(i)/sin(gammas(i)))+2*pi],4);
        alpha2 = round([asin(C12s(i)/sin(gammas(i))), ...
                        pi-asin(C12s(i)/sin(gammas(i))), ...
                        pi-asin(C12s(i)/sin(gammas(i)))],4);
        if i == 1
            alphas(i) = deg2rad(-44.9829164957209);
        else
            alphas(i) = intersect(alpha1, alpha2);
        end
        gammas(i) = rad2deg(gammas(i));
        alphas(i) = rad2deg(alphas(i));
    end
end
```

```
function dwdt = EOM(t,y,I,J,T)
    %{
      inputs:  1) t: time lapse
               2) y: angular velocities, euler parameters, initial
                     euler constraint constant, DCM
               3) I: moment of inertia about the non-rotating axis
               4) J: moment of inertia about the rotating axis
               5) T: torque
      outputs: 1) dwdt: differential y
    %}
    dwdt = zeros(17,1);
    % Dynamics EOMs
    dwdt(1) = T/I - (I-J)/I*y(3)*y(2);
    dwdt(2) = 0;
    dwdt(3) = (I-J)/I*y(1)*y(2);
    % Kinematic EOM of angular velocities and Euler parameters
    dedt1 =  0.5*( y(1)*y(7)-y(2)*y(6)+y(3)*y(5));
    dedt2 =  0.5*( y(1)*y(6)+y(2)*y(7)-y(3)*y(4));
    dedt3 =  0.5*(-y(1)*y(5)+y(2)*y(4)+y(3)*y(7));
    dedt4 = -0.5*( y(1)*y(4)+y(2)*y(5)+y(3)*y(6));

    dwdt(4) = dedt1;
    dwdt(5) = dedt2;
    dwdt(6) = dedt3;
    dwdt(7) = dedt4;

    dwdt(8) = y(4)^2 + y(5)^2 + y(6)^2 + y(7)^2 - 1; % Euler Constraint

    e = [y(4) y(5) y(6) y(7)];
```

```
    C = DCM_from_EulerPara(e); % DCM

    % Kinematic EOM of angular velocities and direction cosines
    dwdt(9) = C(1,2)*y(3)-C(1,3)*y(2);
    dwdt(10) = C(1,3)*y(1)-C(1,1)*y(3);
    dwdt(11) = C(1,1)*y(2)-C(1,2)*y(1);
    dwdt(12) = C(2,2)*y(3)-C(2,3)*y(2);
    dwdt(13) = C(2,3)*y(1)-C(2,1)*y(3);
    dwdt(14) = C(2,1)*y(2)-C(2,2)*y(1);
    dwdt(15) = C(3,2)*y(3)-C(3,3)*y(2);
    dwdt(16) = C(3,3)*y(1)-C(3,1)*y(3);
    dwdt(17) = C(3,1)*y(2)-C(3,2)*y(1);
end
```

```
function C_mat = DCM_from_EulerPara(epsilons)

    % Euler Parameters
    epsilon1 = epsilons(1);
    epsilon2 = epsilons(2);
    epsilon3 = epsilons(3);
    epsilon4 = epsilons(4);

    % Calculating DCM from Euler Parameters
    C11 = 1 - 2*epsilon2^2 - 2*epsilon3^2;
    C12 = 2*(epsilon1*epsilon2 - epsilon3*epsilon4);
    C13 = 2*(epsilon3*epsilon1 + epsilon2*epsilon4);
    C21 = 2*(epsilon1*epsilon2 + epsilon3*epsilon4);
    C22 = 1 - 2*epsilon3^2 - 2*epsilon1^2;
    C23 = 2*(epsilon2*epsilon3 - epsilon1*epsilon4);
    C31 = 2*(epsilon3*epsilon1 - epsilon2*epsilon4);
    C32 = 2*(epsilon2*epsilon3 + epsilon1*epsilon4);
    C33 = 1 - 2*epsilon1^2 - 2*epsilon2^2;

    C_mat = [C11 C12 C13; C21 C22 C23; C31 C32 C33];

end
```