

# AAE 666 Homework 2 Solution

January 29, 2021

## Exercise 1

$$\ddot{y} + 0.1\dot{y} - y + y^3 = 0$$

Define the state variable as  $x_1 = y$  and  $x_2 = \dot{y}$ , and the system can be described as:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -0.1x_2 + x_1 - x_1^3$$

At equilibrium, we have  $\dot{x}_1 = 0$ , and  $\dot{x}_2 = 0$ . Thus:

$$x_2^e = 0$$

$$-0.1x_2^e + x_1^e - (x_1^e)^3 = 0$$

Therefore, we have the second equation as:

$$0 + x_1^e - (x_1^e)^3 = 0$$

$$x_1^e(1 - (x_1^e)^2) = 0$$

and we have three equilibrium solutions,  $x_1^e = 0, -1, 1$  and  $x_2^e = 0$ .

By linearizing about  $(x_1^e, x_2^e)$ :

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & -0.1 \end{bmatrix} \bigg|_{(x_1^e, x_2^e)} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

1. At  $(x_1^e, x_2^e) = (0, 0)$ :

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -0.1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

The eigenvalues can be computed:  $\lambda_1 = -1.0512$ , and  $\lambda_2 = 0.9512$ , which means a **saddle point**.

2. At  $(x_1^e, x_2^e) = (\pm 1, 0)$ :

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -0.1 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$

The eigenvalues can be computed:  $\lambda_{1,2} = -0.05 \pm 1.4133i$ , which means a **stable focus**. We can first numerically integrate the nonlinear system with multiple initial states so that we see the behavior of the system and plot the phase portrait

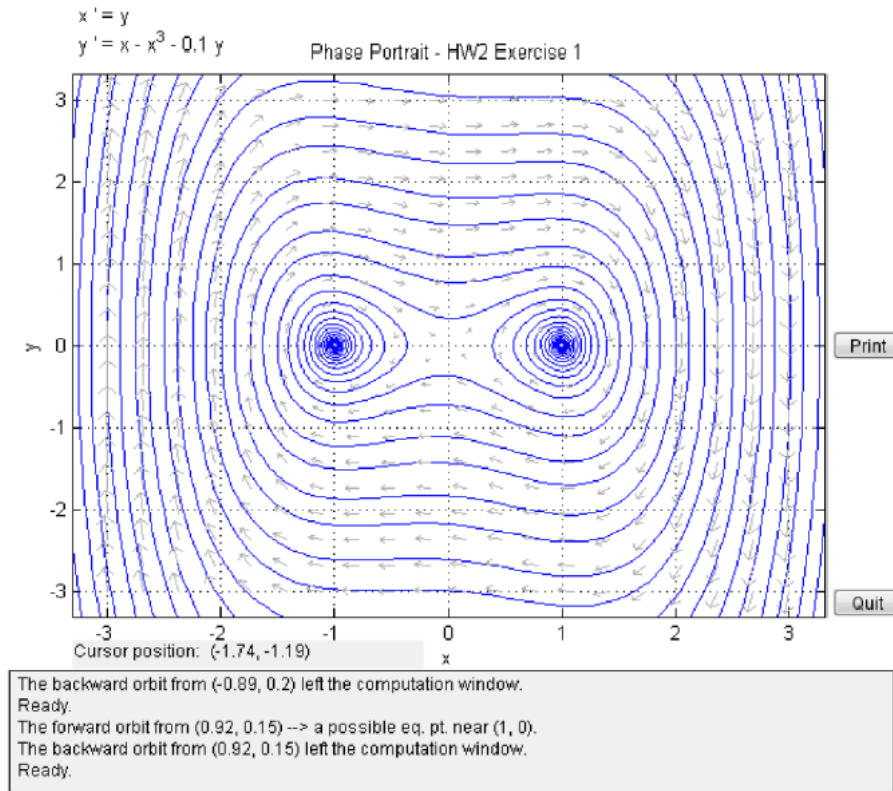


Figure 1: Exercise 1 Plot

## Exercise 2

First define the state variables,  $x_1 = y$ ,  $x_2 = \dot{y}$

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1\end{aligned}$$

We identify two kinds of equilibrium solutions from the above equations. 1.  
When  $x_2 = 0$ ,  $x_1^e = 2n\pi$

$$\begin{aligned}\delta \dot{x}_1 &= \delta x_2 \\ \delta \dot{x}_2 &= -\cos x_1^\delta x_1 = -\delta x_1 \\ A &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\end{aligned}$$

The eigenvalues can be computed:  $\lambda_{1,2} = \pm i$ , which means a **center**.

2. When  $x_2 = 0$ ,  $x_1^e = 2n + 1\pi$

$$\delta \dot{x}_1 = \delta x_2$$

$$\delta \dot{x}_2 = -\cos x_1^e \delta x_1 = \delta x_1$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The eigenvalues can be computed:  $\lambda_{1,2} = \pm 1$ , which means a **saddle point**.

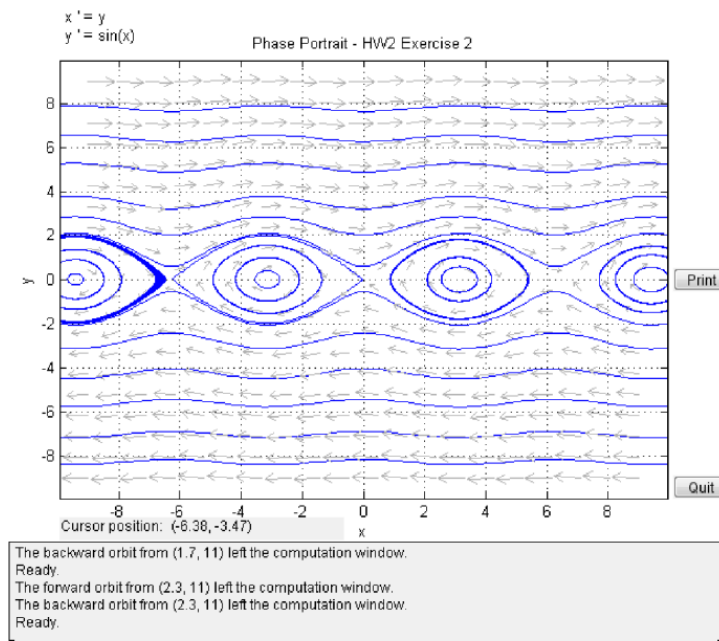


Figure 2: Exercise 2 Plot

### Exercise 3

a.  $\dot{x} = -x - x^3$

$$-x(x^2 + 1) = 0$$

$$x^e = 0$$

The only real equilibrium is  $\dot{x} = 0$ , where the region of attraction is  $(-\infty, +\infty)$ .

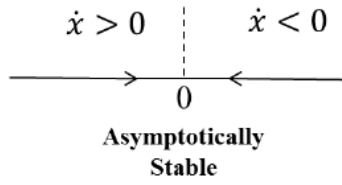


Figure 3: Exercise 3a Plot

b.  $\dot{x} = -x + x^3$

$$-x(-x^2 + 1) = 0$$

$$x^e = 0, -1, +1$$

b.  $\dot{x} = x - 2x^2 + x^3$

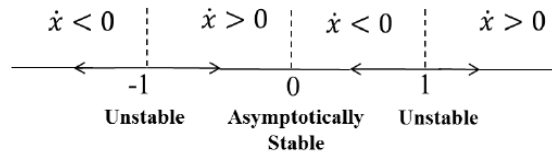


Figure 4: Exercise 3b Plot

$$x(1 - 2x + x^2) = 0$$

$$x^e = 0, +1$$

There is no region of attraction for these equilibrium solutions.

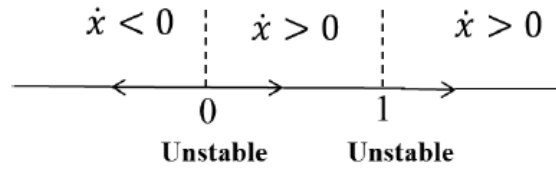


Figure 5: Exercise 3c Plot

## Exercise 4

$$\begin{aligned}\dot{x} &= x^3 \\ \int_{x_0}^x \frac{1}{x^3} dx &= \int_0^t dt \\ \left. \frac{-1}{2x^2} \right|_{x_0}^x &= t \\ \frac{-1}{2x^2} + \frac{1}{2x_0^2} &= t \\ \frac{1 - 2x_0^2 t}{2x_0^2} &= \frac{1}{2x^2} \\ x(t) &= \frac{x_0}{\sqrt{1 - 2x_0^2 t}} \\ t_{x \text{ inf}} &= \frac{1}{2x_0^2}\end{aligned}$$

## Exercise 5

$$\begin{aligned}\dot{x} &= \frac{x}{1+x^2} + \sin(x) \\ \left| \frac{x}{1+x^2} \right| &= \left| \frac{1}{1+x^2} \right| |x| \leq |x| \\ |\sin(x)| &\leq 1 \\ \left| \frac{x}{1+x^2} + \sin(x) \right| &\leq |x| + 1\end{aligned}$$

## Exercise 6

$$\dot{x} = -\sqrt{(1-x)^2}$$

If  $x < 1$ , then  $\dot{x} = x - 1$

If  $x > 1$ , then  $\dot{x} = -x + 1$

Differentiability of function  $f(x)$  guarantees uniqueness,  $f(x)$  is not differentiable at  $x=1$ . Therefore,  $x_0 < 1$  or  $x_0 > 1$  guarantees unique solutions.

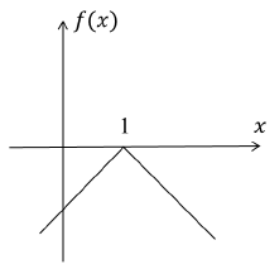


Figure 6: Exercise 6 Plot