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Stability Analysis

Now that a particular solution is available, is there any information that can be gained analytically that might be useful to help assess the numerical investigation?

Stability information – not only is it useful but it can also help focus the numerical simulations

How to actually accomplish a stability analysis?

Note: word "stable" has many shades of meaning so it must be precisely defined to be useful; there are various types of definitions

Two (at least) important concepts:

1. "infinitesimal" or Lagrange stability

"boundedness" concept: if a small deviation from some equilibrium point remains bounded

→ then motion is Lagrange or infinitesimally stable

2. Liapunov stability

A solution to DE being sufficiently close to equilibrium point must remain arbitrarily close to itself after perturbation

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Lagrange stability requires a solution to remain within a finite distance from equilibrium; Liapunov stability requires arbitrarily small deviations

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Liapunov stability implies Lagrange stability

Not vice versa

We will be strict here and utilize the Liapunov concept; it is complicated by the fact that we have a nonlinear system

Liapunov criteria is most easily applied to a linear system so we must build an approximate linear model

→ linearize wrt nominal / ref motion

Analytical evaluation of stability:

ALWAYS test a given particular solution to the diff equations (the motion of interest; nominal motion; reference soln)

The nominal motion is this example

For all time t, $\hat{c}_i = \hat{a}_i$

$${}^N\bar{\omega}^B = \omega_{3_0}\hat{a}_3$$

$${}^A\bar{\omega}^C = \bar{0} \quad \rightarrow \quad s = \omega_{3_0} - \Omega$$

Corresponds to:

$$\boxed{\begin{array}{l} \omega_1 = {}^N\omega_2^B = 0 \\ \omega_3 = \omega_{30} \\ \varepsilon_1 = \varepsilon_2 = {}^A\varepsilon_3^C = 0 \\ \varepsilon_4 = 1 \end{array}}$$

remove

This is the solution that we are testing for stability

Is it stable?

Introduce a comparison by introducing perturbations $\tilde{\varepsilon}_i$, $\tilde{\omega}_i$

State variables = state vars that re present particular solutions + perturbations

$$\Downarrow$$

$$i = 1,2,3 \quad \varepsilon_i = 0 + \tilde{\varepsilon}_i$$

$$\varepsilon_4 = 1 + \tilde{\varepsilon}_4$$

$$j = 1,2 \quad \omega_j = 0 + \tilde{\omega}_j$$

const.

Substitute these expressions into the nonlinear equations and linearize
About the particular solution

$$2\dot{\varepsilon}_1 = \varepsilon_2\left(\omega_3 - s + \Omega\right) - \varepsilon_3\omega_2 + \varepsilon_4\omega_1$$
$$2\dot{\varepsilon}_2 = \varepsilon_3\omega_1 + \varepsilon_4\omega_2 - \varepsilon_1\left(\omega_3 - s + \Omega\right)$$
$$2\dot{\varepsilon}_3 = \varepsilon_4\left(\omega_3 - s - \Omega\right) + \varepsilon_1\omega_2 - \varepsilon_2\omega_1$$
$$2\dot{\varepsilon}_4 = -\varepsilon_1\omega_1 - \varepsilon_2\omega_2 - \varepsilon_3\left(\omega_3 - s - \Omega\right)$$
$$\dot{\omega}_1 = -s\omega_2 + \left(1 - \frac{J}{I}\right)\left[\omega_2\omega_3 - 12\Omega^2\left(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4\right)\left(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4\right)\right]$$
$$\dot{\omega}_2 = s\omega_1 - \left(1 - \frac{J}{I}\right)\left[\omega_1\omega_3 - 6\Omega^2\left(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4\right)\left(1 - 2\varepsilon_2^2 - 2\varepsilon_3^2\right)\right]$$

$$2\dot{\tilde{\varepsilon}}_1 = \tilde{\varepsilon}_2\overbrace{\left(\omega_3 - (\omega_{30} - \Omega) + \Omega\right)}^{2\Omega} - \tilde{\varepsilon}_2\tilde{\omega}_2 + (1 + \tilde{\varepsilon}_4)\tilde{\omega}_1$$

neglect HOTs

$$2\dot{\tilde{\varepsilon}}_1 = 2\Omega\tilde{\varepsilon}_2 + \tilde{\omega}_1$$

Use $s = \omega_{30} + \Omega, \quad x = \frac{I}{J} - 1 \quad y = \frac{\omega_{30}}{\Omega} - 1$

$$\dot{\tilde{\varepsilon}}_1 = \Omega\tilde{\varepsilon}_2 + \frac{\tilde{\omega}_1}{2}$$
$$\dot{\tilde{\varepsilon}}_2 = \frac{\tilde{\omega}_2}{2} - \Omega\tilde{\varepsilon}_1$$
$$\dot{\tilde{\varepsilon}}_3 = 0$$
$$\dot{\tilde{\varepsilon}}_4 = 0$$
$$\dot{\tilde{\omega}}_1 = -\left[y + x(1+y)\right]\Omega\tilde{\omega}_2 = -Q\Omega\tilde{\omega}_2$$
$$\dot{\tilde{\omega}}_2 = \left[y + x(1+y)\right]\Omega\tilde{\omega}_1 - 6x\Omega^2\tilde{\varepsilon}_2 = Q\Omega\tilde{\omega}_1 - 6x\Omega^2\tilde{\varepsilon}_2$$

1^{st} – order system; constant coefficients

Let $z = \begin{bmatrix} \tilde{\varepsilon}_1 & \tilde{\varepsilon}_2 & \tilde{\varepsilon}_3 & \tilde{\varepsilon}_4 & \tilde{\omega}_1 & \tilde{\omega}_2 \end{bmatrix}$

Then $\boxed{\dot{z} = z\,A}$

$$A = \begin{bmatrix} 0 & -\Omega & 0 & 0 & 0 & 0 \\ \Omega & 0 & 0 & 0 & 0 & -6x\Omega^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & Q\Omega \\ 0 & \frac{1}{2} & 0 & 0 & -Q\Omega & 0 \end{bmatrix}$$

How to get stability information about this linear system?

Eigenvalues of A

But a 6×6 matrix is inconvenient to work with

Back up, re-examine the equations, decouple some of the linear equations so do not result in a 6^{th} – order system Characteristic equation?

Yes 6th order

$$\dot{\tilde{\varepsilon}}_1 = \Omega\tilde{\varepsilon}_2 + \frac{\tilde{\omega}_1}{2}$$

$$\dot{\tilde{\varepsilon}}_2 = \frac{\tilde{\omega}_2}{2} - \Omega\tilde{\varepsilon}_1$$

$$\left.\begin{aligned} \dot{\tilde{\varepsilon}}_3 &= 0 \\ \dot{\tilde{\varepsilon}}_4 &= 0 \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow \lambda_1 = 0 \\ &\Rightarrow \end{aligned}$$

$$\left.\begin{aligned}\dot{\tilde{\varepsilon}}_3 &= 0\\ \dot{\tilde{\varepsilon}}_4 &= 0\end{aligned}\right\} \quad \begin{aligned}&\Rightarrow \lambda_1 = 0\\ &\Rightarrow \lambda_2 = 0\end{aligned}$$

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$$\dot{\tilde{\omega}}_2 = \left[y + x(1+y)\right]\Omega\tilde{\omega}_1 - 6x\Omega^2\tilde{\varepsilon}_2 = Q\Omega\tilde{\omega}_1 - 6x\Omega^2\tilde{\varepsilon}_2$$

$$\left[\dot{\tilde{\varepsilon}}_1 \quad \dot{\tilde{\varepsilon}}_2 \quad \dot{\tilde{\omega}}_1 \quad \dot{\tilde{\omega}}_2\right] = \left[\tilde{\varepsilon}_1 \quad \tilde{\varepsilon}_2 \quad \tilde{\omega}_1 \quad \tilde{\omega}_2\right]\begin{bmatrix} 0 & -\Omega & 0 & 0 \\ \Omega & 0 & 0 & -6x\Omega^2 \\ \frac{1}{2} & 0 & 0 & Q\Omega \\ 0 & \frac{1}{2} & -Q\Omega & 0 \end{bmatrix}$$

$$\left|A - \lambda U\right| = \begin{bmatrix} -\lambda & -\Omega & 0 & 0 \\ \Omega & -\lambda & 0 & -6x\Omega^2 \\ \frac{1}{2} & 0 & -\lambda & Q\Omega \\ 0 & \frac{1}{2} & -Q\Omega & -\lambda \end{bmatrix}$$

CE: $= \lambda^4 + (3x + Q^2 + 1)\Omega^2\lambda^2 + (3xQ + Q^2)\Omega^4 = 0$

Characteristic Equation –

Eigenvalues are a function of BOTH shape, spin

What do the eigenvalues mean for the actual nonlinear system?

Analytical determination of stability:

1. Nonlinear DE
2. Exact (particular) solution ⟵
3. Linearized DE about nominal solution
4. Eigenvalues of A ⟵ λ's for linear system

constant A constant soln.

observe

(i) Any eigenvalues have POSITIVE real parts

particular solution modeled as linear system

$\Longrightarrow$ unstable

$\hookrightarrow$ $\Rightarrow$ also there for NL

(ii) All eigenvalues have NEGATIVE real parts

$\rightarrow$ asymptotically stable

$\llcorner$ also NL

(iii) No eigenvalues have POSITIVE real parts

$\Longrightarrow$ marginally stable (L but ~~NL~~)

under NL, could be marginally stable or maybe not.