Homework solution Exercise 1.4.2 page 22

1 Exercise 🚲🚲🚲🚲🚲🚲🚲🚲🚲🚲🚲🚲

1.1 Problem 1.

Consider the system $\dot{x} = Ax$ and $y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

Is the pair $\{C, A\}$ observable? Find the optimal initial condition $\widehat{x}_0$ and the error d in the observability optimization problem

$$d^2 = \inf\{\int_0^\infty |f(t) - Ce^{At}x_0|^2 \, dt : x_0 \in \mathbb{C}^2\}$$

where $f = e^{-t}$. Is your choice of $\widehat{x}_0$ unique? Finally, compute the error d.

Solution. The eigenvalues for A are $\{-1, -2\}$. Hence A is stable and the optimization problem makes sense. Notice that

$$CA = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -4 & -5 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix} \quad \text{and} \quad \text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = 2$$

The pair $\{C, A\}$ is observable.

Because the pair $\{C, A\}$ is observable, the optimal solution $\widehat{x}_0$ is unique and given by

$$\widehat{x}_0 = P^{-1} \int_0^\infty e^{A^* t} C^* e^{-t} dt$$
$$P = \int_0^\infty e^{A^* t} C^* C e^{At} dt$$

Moreover, P is the unique solution to the Lyapunov equation

$$A^*P + PA + C^*C = 0$$

By using the lyap command in Matlab

$$P = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad P^{-1} = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}$$

Clearly, P is strictly positive. By employing Matlab

$$\widehat{x}_0 = P^{-1}\int_0^\infty e^{A^*t}C^*e^{-t}dt = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Finally,

$$d^2 = \|e^{-t}\|^2_{L^2(0,\infty)} - (P\widehat{x}_0, \widehat{x}_0) = \int_0^\infty |e^{-t}|^2 dt - \widehat{x}_0^* P \widehat{x}_0 = \frac{1}{2} - \frac{1}{2} = 0$$

In other words,

$$d = 0$$

Notice $d = 0$ because e^{-t} is in the subspace $\mathcal{H} = \{Ce^{At}x_0 : x_0 \in \mathbb{C}^2\}$ of $L^2(0,\infty)$. To be more precise,

$$Ce^{At} = \begin{bmatrix} 3e^{-2t} - 2e^{-t} & \quad 3e^{-2t} - e^{-t} \end{bmatrix}$$

This shows that $\mathcal{H} = \text{span}\{e^{-t}, e^{-2t}\}$. Moreover,

$$Ce^{At}\widehat{x}_0 = Ce^{At}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-t}$$

Therefore $e^{-t} = Ce^{At}\widehat{x}_0$, and thus, $d = 0$.

1.2 Problem 2.

Repeat Problem 1 with $f = e^{-3t}$.

Solution. In this case,

$$\widehat{x}_0 = P^{-1} \int_0^\infty e^{A^* t} C^* e^{-3t} dt = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix}$$

Finally,

$$d^2 = \|e^{-3t}\|^2_{L^2(0,\infty)} - (P\widehat{x}_0, \widehat{x}_0) = \int_0^\infty |e^{-3t}|^2 dt - \widehat{x}_0^* P \widehat{x}_0 = \frac{1}{600}$$

In other words,

$$d^2 = \frac{1}{600} \quad \text{and} \quad d = \frac{1}{10\sqrt{6}}$$

1.3 Problem 3.

Consider the system $\dot{x} = Ax$ and $y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Is the pair $\{C, A\}$ observable? Find all optimal initial condition $\widehat{x}_0$ and the error d in the observability optimization problem

$$d^2 = \inf\{\int_0^\infty |f(t) - Ce^{At}x_0|^2\, dt : x_0 \in \mathbb{C}^2\}$$

where $f = e^{-t}$. Is your choice of $\widehat{x}_0$ unique? Finally, compute the error d.

Solution. The eigenvalues for A are $\{-1, -2\}$. Hence A is stable and the optimization problem makes sense. Notice that

$$CA = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \quad \text{and} \quad \text{rank} \begin{bmatrix} C \\ CA \end{bmatrix} = 1$$

The pair $\{C, A\}$ is not observable.

Because the pair $\{C, A\}$ is not observable, an optimal solution $\widehat{x}_0$ is not unique. However, all solutions $\widehat{x}_0$ are given by solving

$$P\widehat{x}_0 = \int_0^\infty e^{A^*t}C^*e^{-t}dt$$
$$P = \int_0^\infty e^{A^*t}C^*Ce^{At}dt$$

Moreover, P is the unique solution to the Lyapunov equation

$$A^*P + PA + C^*C = 0$$

By using the lyap command in Matlab

$$P = \begin{bmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

Notice that P is positive and not strictly positive. The eigenvalues for P are given by $\{0, \frac{5}{2}\}$. By employing pinv in Matlab the unique optimal solution $\widehat{x}_{opt}$ in $\mathcal{N}(P)^{\perp}$ is given by

$$\widehat{x}_{opt} = \text{pinv}(P) \int_0^{\infty} e^{A^* t} C^* e^{-t} dt = \frac{1}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Here $\mathcal{N}(M)$ denotes the null space of matrix M and $\mathcal{N}(M)^{\perp}$ its orthogonal complement. Notice that

$$\mathcal{N}(P) = \text{span} \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

Therefore all optimal solutions are given by

$$\widehat{x}_0 = \widehat{x}_{opt} + \mathcal{N}(P) = \frac{1}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad (\alpha \in \mathbb{C})$$

Finally,

$$d^2 = \|e^{-t}\|_{L^2}^2 - (P\widehat{x}_0, \widehat{x}_0) = \int_0^{\infty} |e^{-t}|^2 dt - \widehat{x}_{opt}^* P \widehat{x}_{opt} = \frac{1}{2} - \frac{1}{2} = 0$$

In other words,

$$d = 0$$

Notice $d = 0$ because e^{-t} is in the subspace $\mathcal{H} = \{Ce^{At}x_0 : x_0 \in \mathbb{C}^2\}$ of $L^2(0, \infty)$. To be more precise,

$$Ce^{At} = \begin{bmatrix} 2e^{-t} & e^{-t} \end{bmatrix}$$

This shows that $\mathcal{H}$ is the one dimensional space spanned by e^{-t}, that is, $\mathcal{H} = \text{span}\{e^{-t}\}$ in $L^2(0, \infty)$. Finally,

$$Ce^{At} \left(\widehat{x}_{opt} + \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right) = Ce^{At}\widehat{x}_{opt} = \frac{1}{5} Ce^{At} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = e^{-t},$$

as expected.

1.4 Problem 4.

Repeat Problem 3 with $f = e^{-3t}$.

Solution. Recall that $P = \int_0^\infty e^{A^*t}C^*Ce^{At}dt$ is given by

$$P = \begin{bmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}$$

Notice that P is positive and not strictly positive. The eigenvalues for P are given by $\{0, \frac{5}{2}\}$. By employing pinv in Matlab the unique optimal solution $\widehat{x}_{opt}$ in $\mathcal{N}(P)^\perp$ is given by

$$\widehat{x}_{opt} = \text{pinv}(P) \int_0^\infty e^{A^*t}C^*e^{-3t}dt = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$$

Here $\mathcal{N}(M)$ denotes the null space of matrix M and $\mathcal{N}(M)^\perp$ its orthogonal complement. Notice that

$$\mathcal{N}(P) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

Therefore all optimal solutions are given by

$$\widehat{x}_0 = \widehat{x}_{opt} + \mathcal{N}(P) = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad (\alpha \in \mathbb{C})$$

Finally,

$$\begin{aligned} d^2 &= \|e^{-3t}\|^2_{L^2(0,\infty)} - (P\widehat{x}_0, \widehat{x}_0) \\ &= \int_0^\infty e^{-6t}dt - \widehat{x}^*_{opt}P\widehat{x}_{opt} = \frac{1}{6} - \frac{1}{8} = \frac{1}{24} \end{aligned}$$

In other words,

$$d^2 = \frac{1}{24} \quad \text{and} \quad d = \frac{1}{2\sqrt{6}}$$