College of Engineering
School of Aerospace Engineering

ME 6444: Nonlinear Systems

HW2

Professor:
Michael J. Leamy
Gtech ME Professor

Student:
Tomoki Koike
Gtech MS Student

September 20, 2021

Table of Contents

Problem 1

(10 PTS) 2DOF Sprung Pendulum

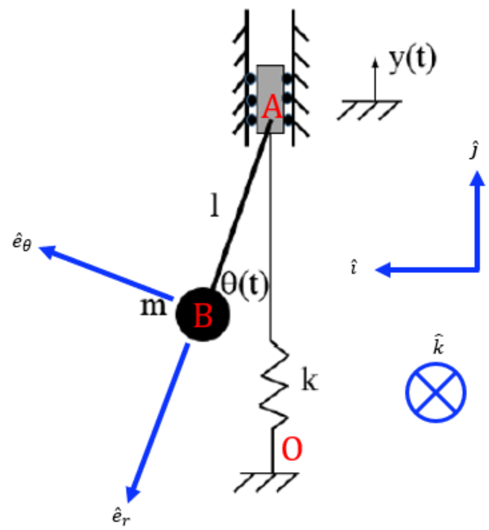

Figure 1: 2DOF Sprung Pendulum

In the sprung pendulum shown, the only appreciable mass comes from that shown at the end of the pendulum. The cart shown is free to move in the vertical direction. There is no friction or other source of dissipation in the problem. There is no forcing - i.e., non-trivial initial conditions are required for there to be motion.

a. Using Lagrange's equations, derive the two equations of motion governing $y(t)$ and $\theta(t)$.

b. Identify extra terms in the equation governing $\theta(t)$ absent from a simple pendulum.

c. Identify extra terms in the equation governing $y(t)$ absent from a simple linear oscillator.

Solution:
a) Define the following

- d: length of spring OA at equilibrium.
- r: rotational-frame (frame of the mass of pendulum) $\hat{e}_r, \hat{e}_\theta, \hat{k}$.
- i: inertial-frame $\hat{\imath}, \hat{\jmath}, \hat{k}$.

- $y(t)$: This variable is defined from the ground at the same level of point O.

The relation between r-frame and i-frame is

$$
\begin{aligned}
{}^{r}R_i &= \begin{pmatrix} \hat{\imath}\cdot\hat{e}_r & \hat{\jmath}\cdot\hat{e}_r & \hat{k}\cdot\hat{e}_r \\ \hat{\imath}\cdot\hat{e}_\theta & \hat{\jmath}\cdot\hat{e}_\theta & \hat{k}\cdot\hat{e}_\theta \\ \hat{\imath}\cdot\hat{k} & \hat{\jmath}\cdot\hat{k} & \hat{k}\cdot\hat{k} \end{pmatrix} \\
&= \begin{pmatrix} cos(\frac{\pi}{2}-\theta) & cos(\pi-\theta) & 0 \\ cos(\theta) & cos(\frac{\pi}{2}-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} sin(\theta) & -cos(\theta) & 0 \\ cos(\theta) & sin(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

where the rotation matrix transforms the i-frame to r-frame. The position of the mass is

$$
\begin{aligned}
\vec{r}_{OB} &= \vec{r}_{OA} + \vec{r}_{AB} \\
&= y(t)\hat{\jmath} + l\hat{e}_r
\end{aligned}
$$

To compute the velocity we take the derivative of this position based on the i-frame.

$$
\begin{aligned}
\frac{{}^{i}d}{dt}\vec{r}_{OB} &= \dot{y}\hat{\jmath} + \frac{{}^{i}d}{dt}(l\hat{e}_r) \\
{}^{i}\vec{v}_{OB} &= \dot{y}\hat{\jmath} + \frac{{}^{r}d}{dt}(l\hat{e}_r) +{}^{i}\vec{\omega}^{r} \times l\hat{e}_r \\
{}^{i}\vec{v}_{OB} &= \dot{y}\hat{\jmath} + \dot{\theta}\hat{k} \times l\hat{e}_r \\
{}^{i}\vec{v}_{OB} &= \dot{y}\hat{\jmath} + l\dot{\theta}\hat{e}_\theta
\end{aligned}
$$

From the rotation matrix ${}^{r}R_i$, we know that

$$\hat{e}_\theta = cos(\theta)\hat{\imath} + sin(\theta)\hat{\jmath}.$$

Thus,

$${}^{i}\vec{v}_{OB} = l\dot{\theta}cos(\theta)\hat{\imath} + (\dot{y} + l\dot{\theta}sin(\theta))\hat{\jmath}$$

Now the kinetic energy, T becomes

$$
\begin{aligned}
T &= \frac{1}{2}m({}^{i}\vec{v}_{OB}) \cdot ({}^{i}\vec{v}_{OB}) \\
&= \frac{1}{2}m(\dot{y} + l\dot{\theta}cos(\theta))^2 + \frac{1}{2}(l\dot{\theta}sin(\theta))^2 \\
&= \frac{1}{2}m(\dot{y}^2 + 2l\dot{y}\dot{\theta}cos(\theta) + l^2\dot{\theta}^2cos^2(\theta)) + \frac{1}{2}ml^2\dot{\theta}^2sin^2(\theta) \\
&= \frac{1}{2}m\dot{y}^2 + ml\dot{y}\dot{\theta}cos(\theta) + \frac{1}{2}ml^2\dot{\theta}^2
\end{aligned}
$$

The potential energy, V for this system becomes

$$V = mgl(1 - cos(\theta)) + \frac{1}{2}k(d-y)^2$$

Using Lagrange's equation $L = T - V$ and the Lagrangian with the conditions of

$$n = 2, \quad q_1 = y, \quad q_2 = \theta, \quad Q_1 = Q_2 = 0.$$

$$L = \frac{1}{2}m\dot{y}^2 + ml\dot{y}\dot{\theta}cos(\theta) + \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - cos(\theta)) - \frac{1}{2}k(d-y)^2$$

we can compute the following.

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + ml\dot{\theta}cos(\theta).$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y} + ml\ddot{\theta}cos(\theta) - ml\dot{\theta}^2 sin(\theta)$$

$$\frac{\partial L}{\partial q} = k(d-y)$$

Then becomes,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial q} = m\ddot{y} + ml\ddot{\theta}cos(\theta) - ml\dot{\theta}^2 sin(\theta) - k(d-y) = Q_1.$$

Let the constant term $d = 0$. Thus, the first EOM ($EOM(y)$) is

$$\boxed{\ddot{y} + l\ddot{\theta}cos(\theta) - l\dot{\theta}^2 sin(\theta) + \frac{k}{m}y = 0.}$$

Next,

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{y}cos(\theta) + ml^2\dot{\theta}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml\ddot{y}cos(\theta) - ml\dot{y}\dot{\theta}sin(\theta) + ml\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{y}\dot{\theta}sin(\theta) - mglsin(\theta)$$

which gives

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = ml^2\ddot{\theta} + ml\ddot{y}cos(\theta) - ml\dot{y}\dot{\theta}sin(\theta) + ml\dot{y}\dot{\theta}sin(\theta) + mglsin(\theta) = Q_2$$

Hence, the second EOM ($EOM(\theta)$) is

$$\boxed{\ddot{\theta} + \frac{\ddot{y}}{l}cos(\theta) + \frac{g}{l}sin(\theta) = 0}$$

b) From the equation $EOM(\theta)$ that we derived and by comparing this to the simple pendulum EOM which is

$$\ddot{\theta} + \frac{g}{l}sin(\theta) = 0$$

we can tell that the extra term in $EOM(\theta)$ is

$$\boxed{\frac{\ddot{y}}{l}cos(\theta).}$$

c) The simple linear oscillator (without damping) has a EOM of

$$\ddot{y} + \frac{k}{m}y = 0.$$

Comparing this equation to $EOM(y)$ that we have derived, we can tell that the extra terms in $EOM(y)$ are

$$\boxed{l\ddot{\theta}cos(\theta) \quad \text{and} \quad -l\dot{\theta}^2 sin(\theta).}$$

Problem 2

(10 PTS) Forced Inverted Pendulum

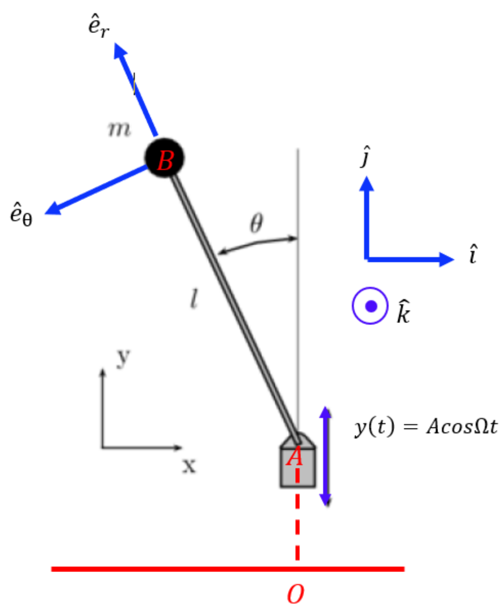

Figure 2: Forced inverted pendulum

a. Derive the equations of motion for a forced inverted pedulum of end-mass m and length l. Neglect the mass of the rod. Assume base-excitation in the form of $y = Acos(\Omega t)$. Keep the full nonlinear equation - i.e., do not expand sine.

b. Explore the behavior of the system, for a parameter set of your choice, by plotting trajectories in the phase plane (use Maple or another mathematics package). Demonstrate that low frequency forcing leads to unstable response, while high frequency forcing can stabilize the pendulum about the $\theta = 0$ position.

Solution:
a) Define the following

- r: rotational-frame (frame of the mass of pendulum) $\hat{e}_r, \hat{e}_\theta, \hat{k}$.
- i: inertial-frame $\hat{\imath}, \hat{\jmath}, \hat{k}$.

The relation between r-frame and i-frame is

$$
{}^{r}R_i = \begin{pmatrix} \hat{\imath}\cdot\hat{e}_r & \hat{\jmath}\cdot\hat{e}_r & \hat{k}\cdot\hat{e}_r \\ \hat{\imath}\cdot\hat{e}_\theta & \hat{\jmath}\cdot\hat{e}_\theta & \hat{k}\cdot\hat{e}_\theta \\ \hat{\imath}\cdot\hat{k} & \hat{\jmath}\cdot\hat{k} & \hat{k}\cdot\hat{k} \end{pmatrix}
$$

$$
= \begin{pmatrix} -sin(\theta) & cos(\theta) & 0 \\ -cos(\theta) & -sin(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}
$$

where the rotation matrix transforms the i-frame to r-frame. The position of the mass is

$$
\begin{aligned}
\vec{r}_{OB} &= \vec{r}_{OA} + \vec{r}_{AB} \\
&= y(t)\hat{\jmath} + l\hat{e}_r
\end{aligned}
$$

To compute the velocity we take the derivative of this position based on the i-frame.

$$
\begin{aligned}
\frac{{}^{i}d}{dt}\vec{r}_{OB} &= \dot{y}\hat{\jmath} + \frac{{}^{i}d}{dt}(l\hat{e}_r) \\
{}^{i}\vec{v}_{OB} &= \dot{y}\hat{\jmath} + \frac{{}^{r}d}{dt}(l\hat{e}_r) +{}^{i}\vec{\omega}^{r} \times l\hat{e}_r \\
{}^{i}\vec{v}_{OB} &= \dot{y}\hat{\jmath} + \dot{\theta}\hat{k} \times l\hat{e}_r \\
{}^{i}\vec{v}_{OB} &= \dot{y}\hat{\jmath} + l\dot{\theta}\hat{e}_\theta
\end{aligned}
$$

From the rotation matrix ${}^{r}R_i$, we know that

$$
\hat{e}_\theta = -cos(\theta)\hat{\imath} - sin(\theta)\hat{\jmath}.
$$

Thus,

$$
{}^{i}\vec{v}_{OB} = -l\dot{\theta}cos(\theta)\hat{\imath} + (\dot{y} - l\dot{\theta}sin(\theta))\hat{\jmath}
$$

Now the kinetic energy, T becomes

$$
\begin{aligned}
T &= \frac{1}{2}m({}^{i}\vec{v}_{OB}) \cdot ({}^{i}\vec{v}_{OB}) \\
&= \frac{1}{2}m(\dot{y} - l\dot{\theta}sin(\theta))^2 + \frac{1}{2}(l\dot{\theta}cos(\theta))^2 \\
&= \frac{1}{2}m(\dot{y}^2 - 2l\dot{y}\dot{\theta}sin(\theta) + l^2\dot{\theta}^2 sin^2(\theta)) + \frac{1}{2}ml^2\dot{\theta}^2 cos^2(\theta) \\
&= \frac{1}{2}m\dot{y}^2 - ml\dot{y}\dot{\theta}sin(\theta) + \frac{1}{2}ml^2\dot{\theta}^2
\end{aligned}
$$

Since $y = Acos(\Omega t)$, we can substitute this in to compute the kinetic energy

$$
T = \frac{1}{2}mA^2\Omega^2 sin^2(\Omega t) + mlA\Omega\dot{\theta}sin(\Omega t)sin(\theta) + \frac{1}{2}ml^2\dot{\theta}^2
$$

The potential energy, V for this system becomes

$$V = mglcos(\theta)$$

Using Lagrange's equation $L = T - V$ and the Lagrangian with the conditions of

$$n = 1, \quad q_1 = \theta, \quad Q_1 = 0.$$

$$L = \frac{1}{2}mA^2\Omega^2 sin^2(\Omega t) + mlA\Omega\dot{\theta}sin(\Omega t)sin(\theta) + \frac{1}{2}ml^2\dot{\theta}^2 - mglcos(\theta)$$

we can compute the following.

$$\frac{\partial L}{\partial \dot{y}} = mlA\Omega sin(\Omega t)sin(\theta) + ml^2\dot{\theta}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) = mlA\Omega^2 cos(\Omega t)sin(\theta) + mlA\Omega\dot{\theta}sin(\Omega t)cos(\theta) + ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial q} = mlA\Omega\dot{\theta}sin(\Omega t)cos(\theta) + mglsin(\theta)$$

Then becomes,

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial q} &= mlA\Omega^2 cos(\Omega t)sin(\theta) + mlA\Omega\dot{\theta}sin(\Omega t)cos(\theta) + ml^2\ddot{\theta} \\ &\qquad - mlA\Omega\dot{\theta}sin(\Omega t)cos(\theta) - mglsin(\theta) \\ &= ml^2\ddot{\theta} + mlA\Omega^2 cos(\Omega t)sin(\theta) - mglsin(\theta)\end{aligned}$$

Hence, the EOM becomes

$$\boxed{\ddot{\theta} - \frac{g}{l}sin(\theta) + \frac{A\Omega^2}{l}cos(\Omega t)sin(\Omega) = 0}$$

b) For the simulation, we will use the following parameters

$$g = 9.8m/s^2, \;\; l = 1.0m, \;\; A = 0.2m$$

with initial conditions of

$$\theta(0) = 0 \quad rad, \qquad \dot{\theta}(0) = 0.01 \quad rad/s$$

The simulations are done using the Python code in the Appendix.

The values of Ω used to represent different frequencies are [0, 10, 20, 30, 40, 50] rad/s. This will produce the following simulation results

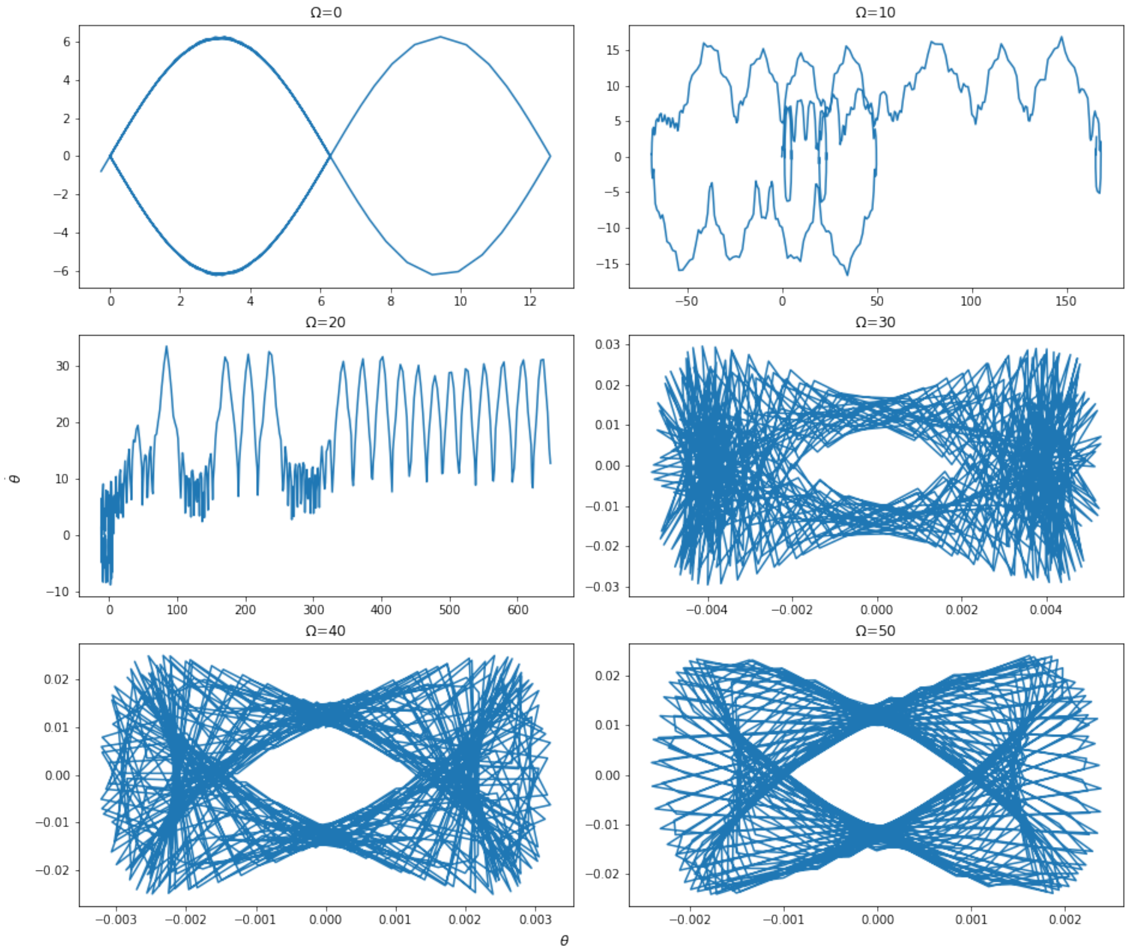

Figure 3: Phase Plane of inverted pendulum with different frequencies

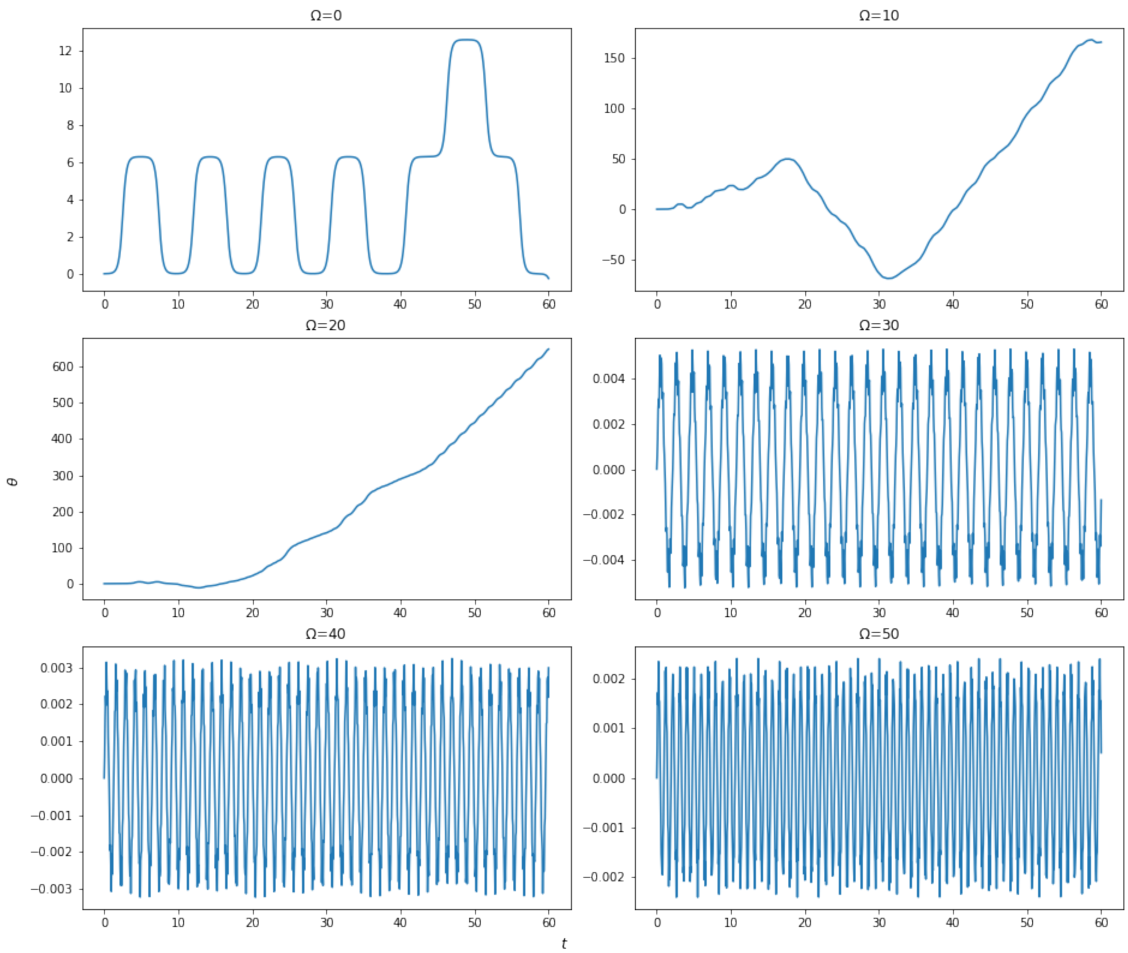

Figure 4: Angle over time for the inverted pendulum with different frequencies

The plots show the clear results of when the frequency of the vertical excitation is more than 30 rad/s (a high frequency) the inverted pendulum will stay in the initial position of $\theta = 0$. However, when the frequency is smaller or none, the inverted pendulum will not be able to stabilize and will eventually rotate and swing around.

Problem 3

(10 PTS) Variational Operator

a. Show that for small virtual displacements, the variation of the Lagrangian $L = L(u, u', u'', \dot{u})$ as define by

$$L(u + \delta u, u' + \delta u', u'' + \delta u'', \dot{u} + \delta \dot{u}) - L(u, u', u'', \dot{u})$$

is given to first order by

$$\delta L = \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \frac{\partial L}{\partial u''}\delta u'' + \frac{\partial L}{\partial \dot{u}}\delta \dot{u}.$$

b. For larger virtual displacements a second-order term can be defined such that

$$L(u + \delta u, u' + \delta u', u'' + \delta u'', \dot{u} + \delta \dot{u}) - L(u, u', u'', \dot{u}) = \delta L + \frac{1}{2}\delta^2 L + OH.$$

Find an expression for $\delta^2 L$ and show that $\delta^2 L = \delta(\delta L)$ - i.e., is equivalent to two operations of the variational operator.

Hint: for both a. and b. it is helpful to introduce a small parameter by letting $u + \delta u = u + \epsilon\phi(x,t)$ and to then use a Taylor expansion on ϵ. Alternatively, you can use a multivariable Taylor expansion as done in class.

Solution:

a) With small virtual displacements the variation of the Lagrangian becomes

$$\begin{aligned}
\Delta L &= L(u + \delta u, u' + \delta u', u'' + \delta u'', \dot{u} + \delta \dot{u}) - L(u, u', u'', \dot{u}) \\
&\qquad \text{with Taylor expansion becomes} \\
&\approx L(u, u', u'', \dot{u}) + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \frac{\partial L}{\partial u''}\delta u'' + \frac{\partial L}{\partial \dot{u}}\delta \dot{u} - L(u, u', u'', \dot{u}) \\
&= \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \frac{\partial L}{\partial u''}\delta u'' + \frac{\partial L}{\partial \dot{u}}\delta \dot{u}.
\end{aligned}$$

For a small δu, $\Delta L \approx \delta L$. Thus,

$$\boxed{\therefore \delta L = \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \frac{\partial L}{\partial u''}\delta u'' + \frac{\partial L}{\partial \dot{u}}\delta \dot{u}.}$$

b) For a larger virtual displacement we use the 2nd order Taylor expansion.

$$\Delta L = L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u}) - L(u,u',u'',\dot{u})$$
$$= \sum_{n=0}^{2} \frac{1}{n!}\left(\delta u \frac{\partial}{\partial u}\delta u' \frac{\partial}{\partial u'}\delta u'' \frac{\partial}{\partial u''}\delta\dot{u}\frac{\partial}{\partial \dot{u}}\right)^n L(u,u',u'',\dot{u}) - L(u,u',u'',\dot{u})$$

or using the Hessian matrix, H_L

$$= \begin{bmatrix}\delta u & \delta u' & \delta u'' & \delta\dot{u}\end{bmatrix} H_L \begin{bmatrix}\delta u \\ \delta u' \\ \delta u'' \\ \delta\dot{u}\end{bmatrix} - L(u,u',u'',\dot{u})$$

$$= \begin{bmatrix}\delta u & \delta u' & \delta u'' & \delta\dot{u}\end{bmatrix} \begin{bmatrix} \frac{\partial^2}{\partial u^2} & \frac{\partial^2}{\partial u \partial u'} & \frac{\partial^2}{\partial u \partial u''} & \frac{\partial^2}{\partial u \partial \dot{u}} \\ \frac{\partial^2}{\partial u' \partial u} & \frac{\partial^2}{\partial u'^2} & \frac{\partial^2}{\partial u' \partial u''} & \frac{\partial^2}{\partial u' \partial \dot{u}} \\ \frac{\partial^2}{\partial u'' \partial u} & \frac{\partial^2}{\partial u'' \partial u'} & \frac{\partial^2}{\partial u''^2} & \frac{\partial^2}{\partial u'' \partial \dot{u}} \\ \frac{\partial^2}{\partial \dot{u} \partial u} & \frac{\partial^2}{\partial \dot{u} \partial u'} & \frac{\partial^2}{\partial \dot{u} \partial u''} & \frac{\partial^2}{\partial \dot{u}^2} \end{bmatrix} L \begin{bmatrix}\delta u \\ \delta u' \\ \delta u'' \\ \delta\dot{u}\end{bmatrix} - L(u,u',u'',\dot{u})$$

$$= L(u,u',u'',\dot{u}) + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial u'}\delta u' + \frac{\partial L}{\partial u''}\delta u'' + \frac{\partial L}{\partial \dot{u}}\delta\dot{u}$$
$$+ \frac{1}{2}\left(\frac{\partial^2 L}{\partial u^2}\delta u^2 + \frac{\partial^2 L}{\partial u'^2}\delta u'^2 + \frac{\partial^2 L}{\partial u''^2}\delta u''^2 + \frac{\partial^2 L}{\partial \dot{u}^2}\delta\dot{u}^2\right.$$
$$+ 2\frac{\partial^2 L}{\partial u \partial u'}\delta u \delta u' + 2\frac{\partial^2 L}{\partial u \partial u''}\delta u \delta u'' + 2\frac{\partial^2 L}{\partial u \partial \dot{u}}\delta u \delta\dot{u}$$
$$\left.+2\frac{\partial^2 L}{\partial u' \partial u''}\delta u' \delta u'' + 2\frac{\partial^2 L}{\partial u' \partial \dot{u}}\delta u' \delta\dot{u} + 2\frac{\partial^2 L}{\partial u'' \partial \dot{u}}\delta u'' \delta\dot{u}\right) - L(u,u',u'',\dot{u})$$

Thus, comparing this result with $\delta L + \frac{1}{2}\delta^2 L + OH$, we know that

$$\boxed{\begin{aligned}\delta^2 L = {}& \frac{\partial^2 L}{\partial u^2}\delta u^2 + \frac{\partial^2 L}{\partial u'^2}\delta u'^2 + \frac{\partial^2 L}{\partial u''^2}\delta u''^2 + \frac{\partial^2 L}{\partial \dot{u}^2}\delta\dot{u}^2 \\ &+ 2\frac{\partial^2 L}{\partial u \partial u'}\delta u \delta u' + 2\frac{\partial^2 L}{\partial u \partial u''}\delta u \delta u'' + 2\frac{\partial^2 L}{\partial u \partial \dot{u}}\delta u \delta\dot{u} \\ &+ 2\frac{\partial^2 L}{\partial u' \partial u''}\delta u' \delta u'' + 2\frac{\partial^2 L}{\partial u' \partial \dot{u}}\delta u' \delta\dot{u} + 2\frac{\partial^2 L}{\partial u'' \partial \dot{u}}\delta u'' \delta\dot{u}\end{aligned}}$$

Now if we compute two operations of variational operators

$$\begin{aligned}
\delta\left(\delta L\right) = &\delta u \frac{\partial}{\partial u}\left(L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u})\right) \\
&+ \delta u' \frac{\partial}{\partial u'}\left(L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u})\right) \\
&+ \delta u'' \frac{\partial}{\partial u''}\left(L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u})\right) \\
&+ \delta \dot{u} \frac{\partial}{\partial \dot{u}}\left(L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u})\right) \\
= &\frac{\partial^2 L}{\partial u^2}\delta u^2 + \frac{\partial^2 L}{\partial u \partial u'}\delta u \delta u' + \frac{\partial^2 L}{\partial u \partial u''}\delta u \delta u'' + \frac{\partial^2 L}{\partial u \partial \dot{u}}\delta u \delta \dot{u} \\
&+ \frac{\partial^2 L}{\partial u' \partial u}\delta u' \delta u + \frac{\partial^2 L}{\partial u'^2}\delta u'^2 + \frac{\partial^2 L}{\partial u' \partial u''}\delta u' \delta u'' + \frac{\partial^2 L}{\partial u' \partial \dot{u}}\delta u' \delta \dot{u} \\
&+ \frac{\partial^2 L}{\partial u'' \partial u}\delta u'' \delta u + \frac{\partial^2 L}{\partial u'' \partial u'}\delta u'' \delta u' + \frac{\partial^2 L}{\partial u''^2}\delta u''^2 + \frac{\partial^2 L}{\partial u'' \partial \dot{u}}\delta u'' \delta \dot{u} \\
&+ \frac{\partial^2 L}{\partial \dot{u} \partial u}\delta \dot{u} \delta u + \frac{\partial^2 L}{\partial \dot{u} \partial u'}\delta \dot{u} \delta u' + \frac{\partial^2 L}{\partial \dot{u} \partial u''}\delta \dot{u} \delta u'' + \frac{\partial^2 L}{\partial \dot{u}^2}\delta \dot{u}^2 \\
= &\frac{\partial^2 L}{\partial u^2}\delta u^2 + \frac{\partial^2 L}{\partial u'^2}\delta u'^2 + \frac{\partial^2 L}{\partial u''^2}\delta u''^2 + \frac{\partial^2 L}{\partial \dot{u}^2}\delta \dot{u}^2 \\
&+ 2\frac{\partial^2 L}{\partial u \partial u'}\delta u \delta u' + 2\frac{\partial^2 L}{\partial u \partial u''}\delta u \delta u'' + 2\frac{\partial^2 L}{\partial u \partial \dot{u}}\delta u \delta \dot{u} \\
&+ 2\frac{\partial^2 L}{\partial u' \partial u''}\delta u' \delta u'' + 2\frac{\partial^2 L}{\partial u' \partial \dot{u}}\delta u' \delta \dot{u} + 2\frac{\partial^2 L}{\partial u'' \partial \dot{u}}\delta u'' \delta \dot{u}.
\end{aligned}$$

Hence,

$$\boxed{\delta^2 L = \delta(\delta L).}$$

Appendix

Problem 2: Python Code

```python
from scipy.integrate import solve_ivp
import numpy as np
import matplotlib.pyplot as plt

def invPendulum(t, u, l, A, omega):
    g = 9.8
    return [
        u[1],
        g/l * np.sin(u[0]) - A*omega**2/l * np.cos(omega * t) * np.sin(u[0])
    ]

def solve_invPendulum(func, t, tspan, ic, parameters, algorithm='DOP853'):
    return solve_ivp(fun=func, t_span=tspan, t_eval=t, y0=ic, method=algorithm,
                     args=tuple(parameters.values()), atol=1e-8, rtol=1e-5)

# Define parameters
params = {
    'l': 1.0,
    'A': 0.2,
    'omega': 0
}

# Phase Plane
fig, ax = plt.subplots(3, 2, figsize=(13, 11), constrained_layout=True)
T = np.linspace(0, 60, 500)
t_span = (np.min(T), np.max(T))
IC = [0, 0.01]
i = 0
for w in range(0, 51, 10):
    params['omega'] = w
    res = solve_invPendulum(invPendulum, T, t_span, IC, params)
    ax[i//2][i%2].plot(res.y[0, :], res.y[1, :])
    ax[i//2][i%2].set_title(r'$\Omega$='+str(w))
    i += 1
fig.supxlabel(r'$\theta$')
fig.supylabel(r'$\dot{\theta}$')
plt.savefig('p2_phasePlane.png')
plt.show()

# Angle over time
```

```
fig, ax = plt.subplots(3, 2, figsize=(13, 11), constrained_layout=True)
T = np.linspace(0, 60, 1000)
t_span = (np.min(T), np.max(T))
IC = [0, 0.01]
i = 0
for w in range(0, 51, 10):
    params['omega'] = w
    res = solve_invPendulum(invPendulum, T, t_span, IC, params)
    ax[i//2][i%2].plot(T, res.y[0, :])
    ax[i//2][i%2].set_title(r'$\Omega$='+str(w))
    i += 1
fig.supylabel(r'$\theta$')
fig.supxlabel(r'$t$')
plt.savefig('p2_theta_over_time')
plt.show()
```