Homework #3 Due: Wednesday, October 13, 2021

1. Consider the problem of minimizing

$$\mathcal{J}(x) = \int_0^1 \left(-a^2 x^2(t) + \dot{x}^2(t) \right) \mathrm{d}t$$

subject to $x(0) = x(1) = 0$. Show that if these boundary conditions are satisfied, then all solutions of Euler's equation are of the form

$$\begin{aligned} &x(t) \equiv 0, && \text{if} \quad a \neq n\pi \\ &x(t) = A\sin(n\pi t) \text{ or } x(t) = 0, && \text{if} \quad a = n\pi \end{aligned}$$

   (a) Show that $\mathcal{J}(x(t)) = 0$ for all these solutions.

   (b) Do all the solutions actually minimize $\mathcal{J}$? What does the Legendre condition gives ?

   (c) Are there some values of a^2 such that $\mathcal{J}$ can be negative ? To answer this, evaluate $\mathcal{J}(x)$ for a few choices of $x(t)$

$$\begin{aligned} &x(t) = t(1-t) \\ &x(t) = t^m(1-t), \qquad m > 0 \\ &x(t) = \sin(\pi t) \end{aligned}$$

2. Consider a particle sliding along a ramp from point $(0,0)$ to the point (a,b) under the force of gravity with zero initial velocity.

   (a) Show that the trip takes t_f seconds, where

$$t_f = \sqrt{\frac{2(a^2+b^2)}{gb}}$$

   (b) Show that the brachistochrone solution is a cycloid given by

$$\begin{aligned} x &= \alpha + \beta(\psi + \sin\psi) \\ y &= \beta(1 + \cos\psi) \end{aligned}$$

   The curve is parameterized by ψ, with constants α and β. If ψ_1 and ψ_2 are the values of the parameter ψ at the initial and final points, respectively, show that the time to transverse the cycloid is

$$t_f = \sqrt{\frac{\beta}{g}}(\psi_2 - \psi_1)$$

   (c) Show that $\psi_2 = \theta + \pi$, where θ satisfies

$$(1 - \cos\theta) - \frac{b}{a}(\theta - \sin\theta) = 0$$

   and solve for ψ_1, α and β.

(d) How much faster than the ramp is the cycloid? Let $a = 4$ ft and $b = 2$ ft and compare the time difference. Where is the particle on the ramp when the particle on the cycloid finishes? Show that this distance is more pronounced for $a \gg b$. (Assume that the gravitational acceleration is $g = 32\ \text{ft}/\sec^2$.)

3. Analyze the following problem

$$\min \int_0^1 \left(\dot{y}^2(t) + 12ty(t)\right) \mathrm{d}t$$

subject to $y(0) = y(1) = 0$.

4. Find the extremals for the problem

$$\min \mathcal{J} = \int_{t_0}^{t_1} (3t^2x^2 + 2t^3 x\dot{x})\,\mathrm{d}t$$

with boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$. Calculate the optimal value of the cost $\mathcal{J}$.

5. Recall that the conjugate points for the problem

$$\min_{y(x)} \int_a^b F(x, y, y')\,\mathrm{d}x$$

are given by the solution $\phi(x)$ of the Euler-Lagrange equations of the accessory minimization problem

$$\min_{\phi(x)} \int_a^b (F_{yy}\phi^2 + 2F_{yy'}\phi\phi' + F_{y'y'}(\phi')^2)\,\mathrm{d}x$$

also known as the Jacobi equation.

(a) Show that the Jacobi equation can be written as follows

$$\left(F_{yy} - \frac{\mathrm{d}}{\mathrm{d}x}F_{yy'}\right)\phi - \frac{\mathrm{d}}{\mathrm{d}x}\left(F_{y'y'}\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = 0$$

where $F_{yy}, F_{yy'}, F_{y'y'}$ are evaluated at the candidate weak local minimizer, say $y^*(x)$.

(b) Show that the ratio $\phi_1(x)/\phi_2(x)$ is constant for all conjugate points where $\phi_1(x)$ and $\phi_2(x)$ are two independent solutions of the Jacobi equation. (Hint: Since the Jacobi equation is a second-order ordinary differential equation, its solutions are given by $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x)$ where c_1 and c_2 are some constants. For $\phi(x) = 0$ for $x = a$, then we have $\phi_1(a)/\phi_2(a) = -c_2/c_1$

6. Consider the problem of minimizing

$$J(y) = \int_{t_0}^{t_1} (\dot{y}^2(t) - 1)^2\,\mathrm{d}t$$

(a) Write down the Euler-Lagrange equations for this problem and show that the extremals for this problem are curves of constant slope (e.g., line segments).

(b) Using the Erdmann corner conditions, show that the only extremals with corners are those such that the slope is ± 1.

(c) Let $t_0 = 0$ and $t_1 = 3$, and assume that $y(0) = 1$ and $y(3) = 2$. Find the *global* minimizer for this case.

(d) What about the case when $t_0 = 0$ and $t_1 = 1$ and $y(0) = 0$ and $y(1) = 2$?