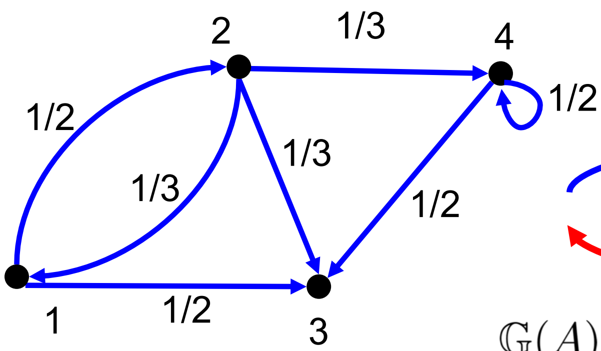
Lecture: Laplacian Matrices

Shaoshuai Mou

Assistant Professor
School of Aeronautics and Astronautics
PURDUE UNIVERSITY

Review

Graph and Matrix



an n-node **graph** $\mathbb{G}$

Adjacency Matrix $A_{\mathbb{G}}$

$\mathbb{G}(A)$ Graph of an non-negative matrix.

$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

an n×n **matrix A**

There exists a directed edge $i \to j$ ⇔ $A_{ij} > 0$

- **Adjacency Matrix** of an n-node graph is an n×n **matrix A**

$$A_{ij} = \begin{cases} w_{ij}, & i \to j; \\ 0, & \text{otherwise} \end{cases} \qquad j \to i$$

- Given an non-negative matrix $A \in \mathbb{R}^{n \times n}$,

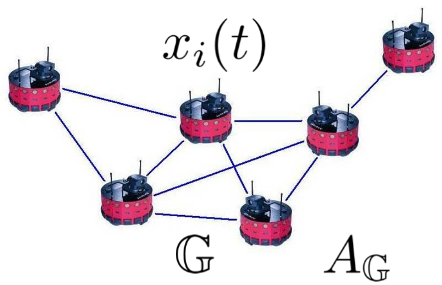
  the ***graph of a matrix*** A *is* a directed graph of ***n*** nodes such that there exists a directed **edge** $i \to j$ with the weight A_{ij} if and only if $A_{ij} > 0$.

Review

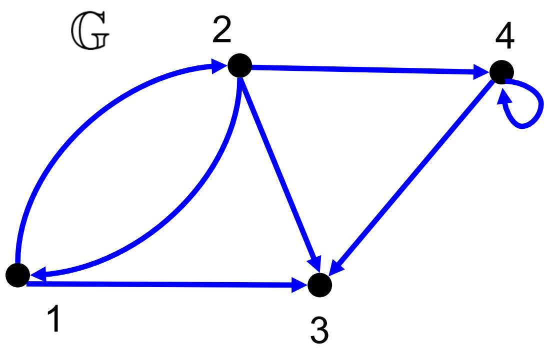
- **Connections between Graphs and its Adjacency Matrices**

$\mathbb{G}$ is **strongly connected**	$\Longleftrightarrow$	A is **irreducible**.	$\sum_{i=0}^{n-1} A^i > 0$
$\mathbb{G}$ is **strongly connected** and aperiodic	$\Longleftrightarrow$	A is **primitive**	$A^k > 0$

- **Compact Form.**



$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad A_{\mathbb{G}} x = \begin{bmatrix} \sum_{j \in \mathbb{N}_1} w_{1j} x_j \\ \vdots \\ \sum_{j \in \mathbb{N}_i} w_{ij} x_j \\ \vdots \\ \sum_{j \in \mathbb{N}_m} w_{mj} x_j \end{bmatrix}$$



$$\mathbb{G} = \{\mathcal{V}, \mathcal{E}\} \qquad \mathcal{V} = \{1,2,3,4\}$$

$$\mathcal{E} = \{(1,2),(1,3),(2,1),(2,3),(2,4),(4,3),(4,4)\}$$

$$A_{\mathbb{G}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad L_{\mathbb{G}} = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

- Adjacency Matrix: $A_{\mathbb{G}} = [a_{ij}]_{n \times n}$

$$a_{ij} = \begin{cases} 1, & i \to j; \\ 0, & \text{otherwise} \end{cases}$$

$$\boxed{L_{\mathbb{G}} = \mathrm{diag}(A_{\mathbb{G}}\mathbf{1}) - A_{\mathbb{G}}}$$

- **Laplacian Matrix:** $L_{\mathbb{G}} = [l_{ij}]_{n \times n}$

$$l_{ij} = \begin{cases} -1, & i \to j,\, i \neq j; \\ d_i, & i = j; \\ 0, & \text{otherwise} \end{cases}$$

Self-arcs does not count.

d_i : number of edges from *i* to other different nodes.

$$\boxed{(L_{\mathbb{G}} x)_i = \sum_{j=1}^{n} l_{ij}(x_i - x_j)}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

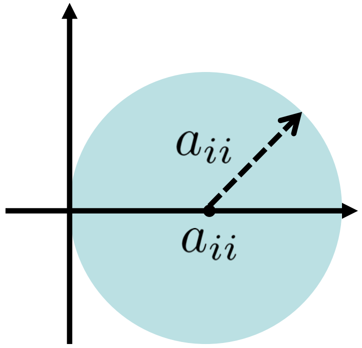
Properties of the Laplacian Matrix

$$L_{\mathbb{G}} = \mathrm{diag}(A_{\mathbb{G}}\mathbf{1}) - A_{\mathbb{G}}$$

- By definition, one has $L_{\mathbb{G}}\mathbf{1} = 0$. Then 0 is an eigenvalue.

- All eigenvalues other than 0 are with **strictly positive real part**.

Gershgorin Circle Theorem: $|\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| = a_{ii}$



- The graph has a **globally reachable** node if and only if

$$\mathrm{rank}\ L_{\mathbb{G}} = n - 1$$

  - For a graph with a globally reachable node, one has

$$\ker\ L_{\mathbb{G}} = \mathrm{span}\ \mathbf{1}$$

  - Consensus in a graph with a globally reachable node

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x_1 = x_2 = \cdots = x_n \quad \Longleftrightarrow \quad L_{\mathbb{G}} x = 0$$

  - For a graph with a globally reachable node, 0 is a **simple** eigenvalue.

Further Properties of Laplacian Matrix for Undirected Graphs

- $L_{\mathbb{G}}$ is symmetric.

$$L_{\mathbb{G}} = \mathrm{diag}(A_{\mathbb{G}}\mathbf{1}) - A_{\mathbb{G}}$$

- Each column sum is 0, namely $\mathbf{1}' L_{\mathbb{G}} = 0$

$$L_{\mathbb{G}}\mathbf{1} = 0$$

- All eigenvalues are real, non-negative, with 0 the smallest eigenvalue.

$$0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

- $\lambda_2 \neq 0$ if and only if $\mathbb{G}$ is connected.

$$\mathrm{rank}\ L_{\mathbb{G}} = n - 1 \qquad \ker\ L_{\mathbb{G}} = \mathrm{span}\ \mathbf{1}$$

- $L_{\mathbb{G}}$ is positive semi-definite.

$$x' L_{\mathbb{G}} x = \sum_{(i,j)\in\mathcal{E}} (x_i - x_j)^2$$

$$= \sum_{i=1}^{n} x_i \Big(\sum_{j=1}^{n} l_{ij}(x_i - x_j)\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}(x_i^2 - x_i x_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}\Big(\frac{1}{2}x_i^2 - x_i x_j + \frac{1}{2}x_j^2\Big) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}(x_i - x_j)^2$$

Continuous-Time Distributed Consensus

Given a network $\mathbb{G}$, which is **undirected and connected**.

$$x_1 = x_2 = \cdots = x_m \quad \Longleftrightarrow \quad L_{\mathbb{G}} x = 0$$

$L_{\mathbb{G}}$ is positive semi-definite,

Then one introduce a Lyapunov function $V = \frac{1}{2} x' L_{\mathbb{G}} x$

$V \geq 0$ with equality holding if and only if consensus is reached

One could use the following **gradient method** to achieve one equilibrium of V

$$\dot{x} = -\frac{\partial V}{\partial x} = -L_{\mathbb{G}} x \qquad \dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j)$$

Distributed

$$x(t) \to x^* \text{ such that } L_{\mathbb{G}} x^* = 0$$

The convergence is **exponentially fast** for LTI

Does this also work for directed networks?

Given a directed network $\mathbb{G}$ with **a globally reachable node**.

$$x_1 = x_2 = \cdots = x_m \iff L_{\mathbb{G}} x = 0 \qquad V = \frac{1}{2} x' L_{\mathbb{G}} x \cancel{\geq} 0$$

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j) \qquad \dot{x} = -L_{\mathbb{G}} x$$

$L_{\mathbb{G}}$ is not symmetric, not positive semidefinite.

Prove: $x(t) \to x^* \text{ such that } L_{\mathbb{G}} x^* = 0$

Hint:

- All eigenvalues other than 0 are with **strictly positive real part**.
- For a graph with a globally reachable node, 0 is a **simple** eigenvalue.

Then the Jordan form of $L_{\mathbb{G}}$ is $L_{\mathbb{G}} = T \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} T^{-1}$

Consensus is reached!

$$\lim_{t \to \infty} e^{-L_{\mathbb{G}} t} = T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} T^{-1} = v w' \qquad x(t) \to \mathbf{1} w' x(0)$$

Write $T^{-1} = \begin{bmatrix} w' \\ w_2' \\ \vdots \\ w_n' \end{bmatrix}$ $\quad T = \begin{bmatrix} v & v_2 & \cdots v_n \end{bmatrix}$ $\quad L_{\mathbb{G}} T = T \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}$ $\quad L_{\mathbb{G}} v = 0 \quad v = \mathbf{1}$

$$T^{-1} T = I \qquad w' v = 1 \qquad T^{-1} L_{\mathbb{G}} = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix} T^{-1} \quad w' L_{\mathbb{G}} = 0$$

w is one left eigenvector of L corresponding to 0, and $w' v = 1$

Summary

- What is the Laplacian Matrix for a graph?

- Properties of the Laplacian. (eigenvalues, eigenvectors, rank, kernel)

- Properties of the Laplacian for undirected graphs.
(eigenvalues, eigenvectors, rank, kernel, positive semi-definite.)

- Continuous-Time Distributed Consensus.

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j) \qquad \dot{x} = -L_{\mathbb{G}} x$$

B. Gharesifard and J. Cortes. Distributed continuous-time convex optimization on weight-balanced diagraphs. IEEE Trans. Automatic Control. 2014.