Assignment #4 Due: October 24, 2021

1. Solve the following optimal control problem

$$\min \mathcal{J} = \frac{1}{2}\int_0^1 u^2 \, \mathrm{d}t$$

subject to

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = u$$

and boundary conditions $x_1(0) = 0$, $x_2(0) = 0$ and $x_1(1) = 1$.

2. Solve the following optimal control problem: Minimize

$$\frac{1}{2}\int_0^{t_f} u^2 \, \mathrm{d}t$$

subject to

$$\dot{x} = ax + bu$$

where $x, u \in \mathbb{R}$, $x(0) = x_0, x(t_f) = 0$ and t_0 and t_f are fixed.

3. A rocket is launched from the origin (0,0) with velocity $u(0)$ parallel to the x-axis and $v(0)$ parallel to the y-axis. Assuming a constant thrust, we wish to find the thrust direction $\theta(t)$ for minimum time to the point $(x_\mathrm{f}, 0)$. Using appropriately non-dimensionalized variables, the equations of motion can be written as

$$\begin{aligned}
\dot{u}(t) &= \cos\theta(t) \\
\dot{v}(t) &= \sin\theta(t) \\
\dot{x}(t) &= u(t) \\
\dot{y}(t) &= v(t)
\end{aligned}$$

   (a) Write down the Hamiltonian for this problem.
   (b) Derive the system of adjoint equations.
   (c) Write down the transversality condition(s) for this problem.
   (d) Write down the expression for the optimal control in terms of the state and adjoint (co-state) equations.

4. Minimize

$$\mathcal{J} = \frac{1}{2}(x(2)-1)^2 + \frac{1}{2}\int_0^2 (x^2+u^2)\, \mathrm{d}t$$

subject to

$$\dot{x} = u, \qquad x(0) = x(2)$$

where $x(2)$ is free.

5. Consider the following optimal control problem in the Bolza form with both a terminal and running cost

$$J(u,t_\mathrm{f},x(t_\mathrm{f})) = \phi(t_\mathrm{f},x(t_\mathrm{f})) + \int_{t_0}^{t_\mathrm{f}} L(x(t),u(t),t)\,\mathrm{d}t$$

with dynamics

$$\dot{x} = f(x,u,t)$$

Write down the Hamiltonian function $H(x,u,t,p)$ for this problem.

(a) Re-formulate the problem as a problem of Lagrange with only a running cost, that is, re-write the performance index as

$$J_a(u) = \int_{t_0}^{t_\mathrm{f}} L_a(x(t),u(t),t)\,\mathrm{d}t$$

Provide an explicit expression for $L_a(x(t),u(t),t)$.

(b) Write down the Hamiltonian function $H_a(x,u,t,\lambda)$ of the new problem formulation and compare with the Hamiltonian function $H(x,u,t,p)$ of the original problem.

(c) Using the new problem formulation in terms of a Lagrange cost, provide the transversality condition, and show that it is the same as the one given in the notes, computed directly by taking the directional differential of J, that is,

$$\big(H(x(t_\mathrm{f}),u(t_\mathrm{f}),t_\mathrm{f},p(t_\mathrm{f}))\delta t_\mathrm{f} - p^\intercal(t_\mathrm{f})\delta(t_\mathrm{f})\big) + \big(\phi_{t_\mathrm{f}}(t_\mathrm{f},x(t_\mathrm{f}))\delta t_\mathrm{f} + \phi_{x_\mathrm{f}}(t_\mathrm{f},x(t_\mathrm{f}))\delta x_\mathrm{f}\big) = 0$$