P1: C BP

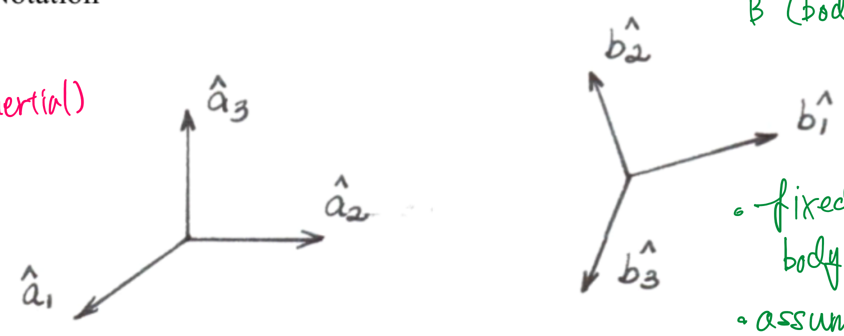
2020年1月27日 月曜日 午後0:30

$\boxed{\text{Direction Cosines}}$ → describe orientation

Notation

A (inertial)

B (body)



- fixed in the body
- assume B is rigid body

Direction cosine element $\quad C_{ij} \triangleq \hat{a}_i \cdot \hat{b}_j$

$$\underbrace{\left[\hat{b}_1\ \hat{b}_2\ \hat{b}_3\right]}=\left[\hat{a}_1\ \hat{a}_2\ \hat{a}_3\right]{}^AC^B$$ row vector format

transforms any vector from $\hat{a}_i$ to $\hat{b}_j$

do not have to be unit vector

$${}^AC^B=\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$ direction cosine matrix

${}^AC^B$ denotes transformation from A to B

${}^BC^A=\left({}^AC^B\right)^T$ from B to A

$$\left[\hat{a}_1, \hat{a}_2, \hat{a}_3\right]=\left[\hat{b}_1, \hat{b}_2, \hat{b}_3\right]{}^BC^A$$

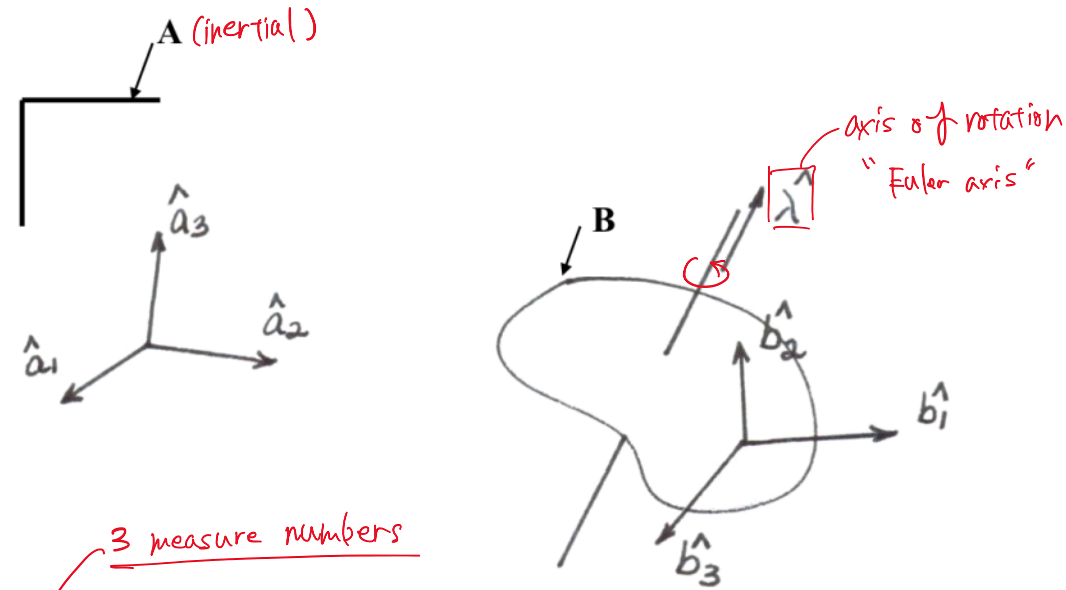
This set is unique since it actually serves two purposes for us:

1. Transformation matrix – transforms elements of any tensor of any rank with correct transformation equation (already noted)
2. Set of kinematic variables – solve directly for C_{ij} at any time and deduce the orientation of B in A

1) Direction cosines as descriptive (kinematic) variables

3 DOF / 9 variables → $\boxed{\text{6 constraint equations}}$ (Orthogonality conditions) } ★ check

2) Use the 9 C_{ij} variables to describe a simple rotation



3 measure numbers

$$\underbrace{\lambda_i \triangleq \hat{\lambda} \bullet \hat{a}_i = \hat{\lambda} \bullet \hat{b}_i}_{\hat{\lambda}\text{ remains fixed in both}}$$

$$\left[\hat{b}_1\ \hat{b}_2\ \hat{b}_3\right] = \left[\hat{a}_1\ \hat{a}_2\ \hat{a}_3\right]\ {}^{A}C^{B}$$

recall @ $t = t_0$ $\hat{a}_i = \hat{b}_i$

${}^{A}C^{B}$ @ $t = t_0$ is U

How can we write direction cosine elements as functions of $\hat{\lambda}(\lambda_i), \theta$?

$\star$ $\hat{a}_i = \hat{b}_i$ @ $t = t_0$

C3

Different ways of expressing the relationship

(a) Use original theorem (SRT)

$$\bar{b} = \bar{a}\cos\theta - \bar{a}\times\hat{\lambda}\sin\theta + \bar{a}\cdot\hat{\lambda}\hat{\lambda}(1-\cos\theta)$$

$\bar{a}$ = initial orientation

$\downarrow$

$\bar{b}$ = final orientation after rotation

Recall

$${}^{A}C^{B}_{lk} = \hat{a}_l \cdot \hat{b}_k$$

Of course, a unit vector IS a vector!! So…

$$\hat{b}_i = \hat{a}_i\cos\theta - \hat{a}_i\times\hat{\lambda}\sin\theta + \hat{a}_i\cdot\hat{\lambda}\,\hat{\lambda}(1-\cos\theta)$$

each unit vector in B

$$C_{lk} = \hat{a}_l \cdot \left[\hat{a}_k\cos\theta - \hat{a}_k\times\hat{\lambda}\sin\theta + \hat{a}_k\cdot\hat{\lambda}\,\hat{\lambda}(1-\cos\theta)\right]$$

$\hat{\lambda} = \lambda_i\hat{a}_i = \lambda_i\hat{b}_i$

$$\boxed{\begin{aligned}
C_{11} &= \cos\theta + \lambda_1^2(1-\cos\theta)\\
C_{12} &= -\lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta)\\
C_{13} &= \lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta)\\
C_{21} &= \lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta)\\
C_{22} &= \cos\theta + \lambda_2^2(1-\cos\theta)\\
C_{23} &= -\lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta)\\
C_{31} &= -\lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta)\\
C_{32} &= \lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta)\\
C_{33} &= \cos\theta + \lambda_3^2(1-\cos\theta)
\end{aligned}}$$

scalar relationships that relate DCM elements with euler axis $\hat{\lambda}$ and angle θ

$\hat{\lambda}_i, \theta = C_{ij}$

C4

$\hat{\lambda} = \lambda_1\hat{a}_1 + \lambda_2\hat{a}_2 + \lambda_3\hat{a}_3$

$= \lambda_1\hat{b}_1 + \lambda_2\hat{b}_2 + \lambda_3\hat{b}_3$

Caution: ${}^{A}C^{B}_{12} = -\lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta)$

must be vector components in

$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3$

must be vector components in vector basis A or B (always)

(b) Use permutation parameter

$$\varepsilon_{ijk} \triangleq \frac{1}{2}(i-j)(j-k)(k-i)$$

$$\boxed{C_{ij} = \delta_{ij}\cos\theta - \varepsilon_{ijk}\lambda_k \sin\theta + \lambda_i \lambda_j (1-\cos\theta)}$$

Simple rotation theorem (SRT) actually an expression for C as a function of λ_i, θ

(c) Write in terms of the simple rotation dyadic

$$C_{ij} = \hat{a}_i \bullet \hat{b}_j$$

but $\hat{b}_j = \hat{a}_j \bullet \bar{\bar{R}}$

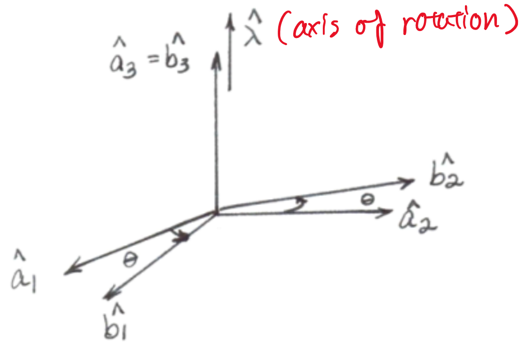
$$\boxed{C_{ij} = \hat{a}_i \cdot (\hat{a}_j \cdot \bar{\bar{R}})}$$

All methods should yield same result.

s/c

Example: Reorient vehicle about $\hat{\lambda}$ through θ
Direction cosine elements to describe orientation?

@ $t = t_0$: $\hat{a}_i = \hat{b}_i$



A frame : inertial frame

Assume $\hat{\lambda} = \hat{a}_3$

B frame : fixed in body frame

$$\begin{bmatrix} \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix} {}^A C^B$$

${}^A C^B$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{a}_1$	C_θ	$-S_\theta$	0
$\hat{a}_2$	S_θ	C_θ	0
$\hat{a}_3$	0	0	1

by inspection

Might not always be able to visualize C easily but if you know $\hat{\lambda}, \theta$ and can write $\hat{\lambda}$ in $\hat{a}$ or $\hat{b}$ components

$$\hat{\lambda} = \hat{a}_3 = \hat{b}_3 \quad \Longrightarrow \quad \lambda_1 = \lambda_2 = 0 \quad \lambda_3 = 1$$

$$
\begin{aligned}
{}^A C^B_{11} &= \cos\theta + \cancelto{0}{\lambda_1^2}(1 - \cos\theta) = \cos\theta \\
{}^A C^B_{12} &= -\cancelto{1}{\lambda_3}\sin\theta + \cancelto{0}{\lambda_1\lambda_2}(1 - \cos\theta) = -\sin\theta \\
{}^A C^B_{23} &= -\cancelto{0}{\lambda_1}\sin\theta + \cancelto{0}{\lambda_2\lambda_3}(1 - \cos\theta) = 0
\end{aligned}
$$