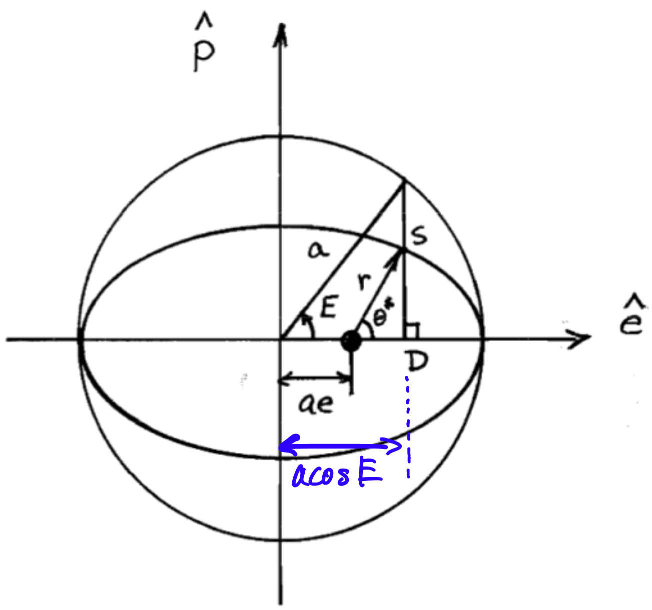
f and g Functions

E and H are most definitely useful for "time" relationships
But they are also useful in other ways.

⟶ new expressions for $\bar{r}, \bar{v}$

Begin with the elliptic case



$$\bar{r} = a(\cos E - e)\,\hat{e} + \overbrace{b\sin E}^{SD}\,\hat{p}$$

$$\bar{v} = \dot{\bar{r}} = -a\dot{E}\sin E\,\hat{e} + b\dot{E}\cos E\,\hat{p}$$

$$\frac{d}{dt}(M = nt = E - e\sin E) \rightarrow \quad n = \dot{E}\underbrace{(1-e\cos E)}_{r/a}$$

$$M = n(t - t_p)$$

$$\longrightarrow \quad n = \frac{\dot{E}r}{a} \quad \text{sub}$$

$\bar{r} = a(\cos E - e)\,\hat{e} + b\sin E\,\hat{p}$

$$\bar{v} = -\frac{a^2 n}{r}\sin E\,\hat{e} + \frac{abn}{r}\cos E\,\hat{p}$$

Evaluate $\hat{e}, \hat{p}$ at $t = t_0$ (i.e., $\bar{r}_0, \bar{v}_0$) $\bar{r}(t_0) = \bar{r}_0 \quad \bar{v}(t_0) = \bar{v}_0$

$$\hat{e} = \frac{1}{a(\cos E_0 - e)}\bar{r}_0 - \frac{b\sin E_0}{a(\cos E_0 - e)}\hat{p}$$

Substitute into $\bar{v}$ equation

$$\begin{cases} \hat{p} = \dfrac{(\cos E_0 - e)}{r\sqrt{ap}}\bar{v}_0 + \sqrt{\dfrac{a}{p}}\dfrac{\sin E_0}{r_0}\bar{r}_0 \\ \hat{e} = \dfrac{\cos E_0}{r_0}\bar{r}_0 - \dfrac{\sin E_0}{an}\bar{v}_0 \end{cases}$$

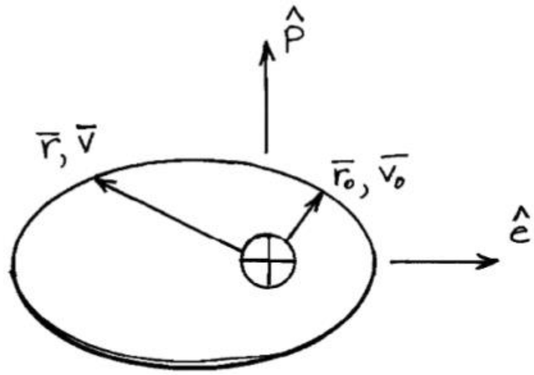
Substitute $\hat{e}, \hat{p}$ into original expressions

t_0 might be t_p but doesn't have to be!

$$\boxed{\begin{aligned} \bar{r} &= \underbrace{\left\{1 - \frac{a}{r_0}\left[1 - \cos(E - E_0)\right]\right\}}_{f}\bar{r}_0 + \underbrace{\left\{(t - t_0) + \left[\frac{\sin(E - E_0) - (E - E_0)}{n}\right]\right\}}_{g}\bar{v}_0 \\ \bar{v} &= \underbrace{-\frac{na^2}{rr_0}\sin(E - E_0)}_{\dot{f}}\bar{r}_0 + \underbrace{\left\{1 - \frac{a}{r}\left[1 - \cos(E - E_0)\right]\right\}}_{\dot{g}}\bar{v}_0 \end{aligned}}$$

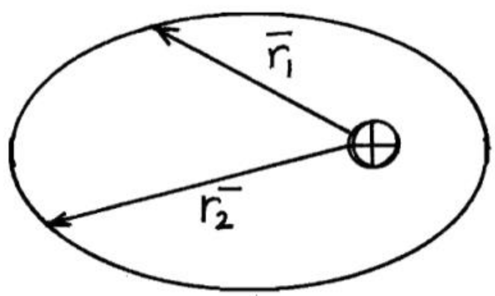
$\bar{r} = f\,\bar{r}_0 + g\,\bar{v}_0$

$\bar{v} = \dot{f}\,\bar{r}_0 + \dot{g}\,\bar{v}_0$



$$\boxed{\begin{aligned}\bar{r} &= f_0 \bar{r}_0 + g_0 \bar{v}_0 \\ \bar{v} &= \dot{f}_0 \bar{r}_0 + \dot{g}_0 \bar{v}_0\end{aligned}}$$

Example:



$$\begin{cases} t = t_1 \\ t = t_2 \end{cases}$$

$\bar{r}_2 = f\,\bar{r}_1 + g\,\bar{v}_1$ ⇒ $\bar{v}_1 = \dfrac{\bar{r}_2 - f\bar{r}_1}{g}$

⇒ Do same in terms of θ^*;
Do same in hyperbolic orbits

f and g Relationships

Any conic

$$\bar{r} = \underbrace{\left\{1-\frac{r}{p}\left[1-\cos\left(\theta^{*}-\theta_0^{*}\right)\right]\right\}}_{f}\bar{r}_0 + \underbrace{\frac{r\,r_0}{\sqrt{\mu p}}\sin\left(\theta^{*}-\theta_0^{*}\right)}_{g}\bar{v}_0$$

f, g should agree

$$\bar{v} = \left\{\frac{\bar{r}_0 \square \bar{v}_0}{p\,r_0}\left[1-\cos\left(\theta^{*}-\theta_0^{*}\right)\right]-\frac{1}{r_0}\sqrt{\frac{\mu}{p}}\sin\left(\theta^{*}-\theta_0^{*}\right)\right\}\bar{r}_0 + \left\{1-\frac{r_0}{p}\left[1-\cos(\theta^{*}-\theta_0^{*})\right]\right\}\bar{v}_0$$

Elliptic Orbits ΔE

$$\bar{r} = \left\{1-\frac{a}{r_0}\left[1-\cos\left(E-E_0\right)\right]\right\}\bar{r}_0 + \left\{\left(t-t_0\right)-\sqrt{\frac{a^3}{\mu}}\left[\left(E-E_0\right)-\sin\left(E-E_0\right)\right]\right\}\bar{v}_0$$

$$\bar{v} = -\frac{\sqrt{\mu a}}{r r_0}\sin\left(E-E_0\right)\bar{r}_0 + \left\{1-\frac{a}{r}\left[1-\cos\left(E-E_0\right)\right]\right\}\bar{v}_0$$

f, g should agree!

Hyperbolic Orbits ΔH

$$\bar{r} = \left\{1-\frac{|a|}{r_0}\left[\cosh\left(H-H_0\right)-1\right]\right\}\bar{r}_0 + \left\{\left(t-t_0\right)-\sqrt{\frac{|a|^3}{\mu}}\left[\sinh\left(H-H_0\right)-\left(H-H_0\right)\right]\right\}\bar{v}_0$$

$$\bar{v} = -\frac{\sqrt{\mu|a|}}{r r_0}\sinh\left(H-H_0\right)\bar{r}_0 + \left\{1-\frac{|a|}{r}\left[\cosh\left(H-H_0\right)-1\right]\right\}\bar{v}_0$$