College of Engineering

School of Aeronautical and Astronautical Engineering

AAE 567: Introduction To Applied Stochastic Processes

HW2

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Problem 1

[Problem 1 from p.22 of the notes] Consider the system $\dot{x} = Ax$ and $y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad and \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

Is the pair $\{C, A\}$ observable? Find the optimal initial condition $\hat{x}_0$ and the error d in the observability optimization problem

$$d^2 = inf\left\{ \int_0^\infty \left\| f(t) - Ce^{At}x_0 \right\|^2 dt \quad : x_0 \in \mathbb{C}^2 \right\}$$

where $f = e^{-t}$. Is your choice of $\hat{x}_0$ unique? Finally, compute the error d.

Solution:
The observability matrix becomes

$$Q_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}.$$

The observability matrix Q_0 has a rank of 2, and therefore, this system is observable.
Say P is the solution to the Lyapunov equation of

$$A^*P + PA + C^*C = 0.$$

This P is computed to be

$$P = \int_0^\infty e^{A^*t}C^*Ce^{At}dt$$

where

$$e^{At} = \begin{bmatrix} 2\,e^{-t} - e^{-2\,t} & e^{-t} - e^{-2\,t} \\ 2\,e^{-2\,t} - 2\,e^{-t} & 2\,e^{-2\,t} - e^{-t} \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}.$$

Next we can compute the initial state to be

$$\hat{x}_0 = P^{-1} \int_0^\infty e^{A^*t}C^*f(t)dt = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

From **THEOREM 1.4.3** we know that if the pair $\{C, A\}$ is observable $\hat{x}_0$ becomes a unique solution for the optimization problem. Hence, $\hat{x}_0$ is unique.
The error then can be found by the following process,

$$d^2 = \int_0^\infty \left\| f(t) - Ce^{At}x_0 \right\|^2 dt = \int_0^\infty \left\| e^{-t} - e^{-t} \right\|^2 dt = 0$$

$$\therefore d = 0.$$

MATLAB CODE:

```
% Housekeeping commands
clear all; close all; clc;

% Given system matrices
A = [0, 1; -2, -3];
C = [1, 2];

% Observability matrix
res = checkObservability(A,C);
disp(res.check);
disp(res.Qo);

% Exponential of A
syms t
eA = expm(A*t);
P = int(eA'*C'*C*eA, 0, inf);

% x_0
f = exp(-t);
x0 = inv(P) * int(eA'*C'*f, 0, inf);

% d
temp = C*eA*x0;
d2 = int((f - temp)^2, 0, inf);
```

Problem 2

[Problem 2 from p.22 of the notes] Consider the system $\dot{x} = Ax$ and $y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad and \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}.$$

Is the pair $\{C, A\}$ observable? Find the optimal initial condition $\hat{x}_0$ and the error d in the observability optimization problem

$$d^2 = inf\left\{ \int_0^\infty \left\| f(t) - Ce^{At}x_0 \right\|^2 dt \quad : x_0 \in \mathbb{C}^2 \right\}$$

where $f = e^{-3t}$. Is your choice of $\hat{x}_0$ unique? Finally, compute the error d.

Solution:
The observability matrix becomes

$$Q_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}.$$

The observability matrix Q_0 has a rank of 2, and therefore, this system is observable.
Say P is the solution to the Lyapunov equation of

$$A^*P + PA + C^*C = 0.$$

This P is computed to be

$$P = \int_0^\infty e^{A^*t}C^*Ce^{At}dt$$

where

$$e^{At} = \begin{bmatrix} 2\,e^{-t} - e^{-2\,t} & e^{-t} - e^{-2\,t} \\ 2\,e^{-2\,t} - 2\,e^{-t} & 2\,e^{-2\,t} - e^{-t} \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}.$$

Next we can compute the initial state to be

$$\hat{x}_0 = P^{-1} \int_0^\infty e^{A^*t}C^*f(t)dt = \begin{bmatrix} -0.1 \\ 0.5 \end{bmatrix}.$$

From **THEOREM 1.4.3** we know that if the pair $\{C, A\}$ is observable $\hat{x}_0$ becomes a unique solution for the optimization problem. Hence, $\hat{x}_0$ is unique.
The error then can be found by the following process,

$$d^2 = \int_0^\infty \left\| f(t) - Ce^{At}x_0 \right\|^2 dt = \int_0^\infty \left\| e^{-3t} - (1.2e^{-2t} - 0.3e^{-t}) \right\|^2 dt = 0.0017$$

$$\therefore d = \sqrt{d^2} = \sqrt{0.0017} = 0.0408.$$

MATLAB CODE:

```
% Housekeeping commands
clear all; close all; clc;

% Given system matrices
A = [0, 1; -2, -3];
C = [1, 2];

% Observability matrix
res = checkObservability(A,C);
disp(res.check);
disp(res.Qo);

% Exponential of A
syms t
eA = expm(A*t);
P = int(eA'*C'*C*eA, 0, inf);

% x_0
f = exp(-3*t);
x0 = inv(P) * int(eA'*C'*f, 0, inf);

% d
temp = C*eA*x0;
d2 = int((f - temp)^2, 0, inf);
d = sqrt(d2);
```

Problem 3

[Problem 3 from p.22 of the notes] Consider the system $\dot{x} = Ax$ and $y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad and \qquad C = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Is the pair $\{C, A\}$ observable? Find the optimal initial condition $\hat{x}_0$ and the error d in the observability optimization problem

$$d^2 = inf\left\{\int_0^\infty \left\|f(t) - Ce^{At}x_0\right\|^2 dt \quad : x_0 \in \mathbb{C}^2\right\}$$

where $f = e^{-t}$. Is your choice of $\hat{x}_0$ unique? Finally, compute the error d.

Solution:
The observability matrix becomes

$$Q_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

The observability matrix Q_0 has a rank of 1, and therefore, this system is NOT observable. Say P is the solution to the Lyapunov equation of

$$A^*P + PA + C^*C = 0.$$

This P is computed to be

$$P = \int_0^\infty e^{A^*t}C^*Ce^{At}dt$$

where

$$e^{At} = \begin{bmatrix} 2\,e^{-t} - e^{-2\,t} & e^{-t} - e^{-2\,t} \\ 2\,e^{-2\,t} - 2\,e^{-t} & 2\,e^{-2\,t} - e^{-t} \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Since, matrix P is non-invertible, we have to find the Moore-Penrose pseudo inverse of P and not the inverse which is denoted as $P^\dagger$. To accomplish this, we first take the singular value decomposition of P.

$$P = U\Sigma V^* \Longrightarrow U = \begin{bmatrix} 0.8944 & -0.4472 \\ 0.4472 & 0.8944 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0.8944 & 0.4472 \\ 0.4472 & -0.8944 \end{bmatrix}.$$

Then, the reciprocal of Σ becomes

$$\Sigma^{\dagger} = \begin{bmatrix} \frac{1}{2.5} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}$$

. Then,

$$P^{\dagger} = V\Sigma^{\dagger}U^{*} = \begin{bmatrix} 0.32 & 0.16 \\ 0.16 & 0.08 \end{bmatrix}.$$

Next we can compute the initial state to be

$$\hat{x}_0 = P^{\dagger}\int_0^{\infty} e^{A^* t}C^* f(t)dt = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}.$$

From **THEOREM 1.4.3** we know that if the pair $\{C, A\}$ is not observable $\hat{x}_0$ does not become a unique solution for the optimization problem. Hence, $\hat{x}_0$ is NOT unique.
The error then can be found by the following process,

$$d^2 = \int_0^{\infty} \left\| f(t) - Ce^{At}x_0 \right\|^2 dt = \int_0^{\infty} \left\| e^{-t} - e^{-t} \right\|^2 dt = 0$$

$$\therefore d = 0.$$

MATLAB CODE:

```
% Housekeeping commands
clear all; close all; clc;

% Given system matrices
A = [0, 1; -2, -3];
C = [2, 1];

% Observability matrix
res = checkObservability(A,C);
disp(res.check);
disp(res.Qo);

% Exponential of A
syms t
eA = expm(A*t);
P = int(eA'*C'*C*eA, 0, inf);

% x_0
f = exp(-t);
[U,S,V] = svd(P);
Sp = 1./S;
```

```
Sp(Sp == inf) = 0;
Pi =  V * Sp * U';
x0 = Pi * int(eA'*C'*f, 0, inf);

% d
temp = C*eA*x0;
d2 = int((f - temp)^2, 0, inf);
d = sqrt(d2);
```

Problem 4

[Problem 4 from p.22 of the notes] Consider the system $\dot{x} = Ax$ and $y = Cx$ where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \qquad and \qquad C = \begin{bmatrix} 2 & 1 \end{bmatrix}.$$

Is the pair $\{C, A\}$ observable? Find the optimal initial condition $\hat{x}_0$ and the error d in the observability optimization problem

$$d^2 = inf\left\{\int_0^\infty \left\|f(t) - Ce^{At}x_0\right\|^2 dt \quad : x_0 \in \mathbb{C}^2\right\}$$

where $f = e^{-3t}$. Is your choice of $\hat{x}_0$ unique? Finally, compute the error d.

Solution:
The observability matrix becomes

$$Q_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

The observability matrix Q_0 has a rank of 1, and therefore, this system is NOT observable. Say P is the solution to the Lyapunov equation of

$$A^*P + PA + C^*C = 0.$$

This P is computed to be

$$P = \int_0^\infty e^{A^*t}C^*Ce^{At}dt$$

where

$$e^{At} = \begin{bmatrix} 2\,e^{-t} - e^{-2\,t} & e^{-t} - e^{-2\,t} \\ 2\,e^{-2\,t} - 2\,e^{-t} & 2\,e^{-2\,t} - e^{-t} \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 0.5 \end{bmatrix}.$$

Since, matrix P is non-invertible, we have to find the Moore-Penrose pseudo inverse of P and not the inverse which is denoted as $P^\dagger$. To accomplish this, we first take the singular value decomposition of P.

$$P = U\Sigma V^* \Longrightarrow U = \begin{bmatrix} 0.8944 & -0.4472 \\ 0.4472 & 0.8944 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2.5 & 0 \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0.8944 & 0.4472 \\ 0.4472 & -0.8944 \end{bmatrix}.$$

Then, the reciprocal of Σ becomes

$$\Sigma^\dagger = \begin{bmatrix} \frac{1}{2.5} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}$$

. Then,

$$P^\dagger = V\Sigma^\dagger U^* = \begin{bmatrix} 0.32 & 0.16 \\ 0.16 & 0.08 \end{bmatrix}.$$

Next we can compute the initial state to be

$$\hat{x}_0 = P^\dagger \int_0^\infty e^{A^* t} C^* f(t) dt = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}.$$

From **THEOREM 1.4.3** we know that if the pair $\{C, A\}$ is not observable $\hat{x}_0$ does not become a unique solution for the optimization problem. Hence, $\hat{x}_0$ is NOT unique.
The error then can be found by the following process,

$$d^2 = \int_0^\infty \left\| f(t) - Ce^{At} x_0 \right\|^2 dt = \int_0^\infty \left\| e^{-3t} - 0.5e^{-t} \right\|^2 dt = 0.0417$$
$$\therefore d = \sqrt{d^2} = \sqrt{0.0417} = 0.2041.$$

MATLAB CODE:

```
% Housekeeping commands
clear all; close all; clc;

% Given system matrices
A = [0, 1; -2, -3];
C = [2, 1];

% Observability matrix
res = checkObservability(A,C);
disp(res.check);
disp(res.Qo);

% Exponential of A
syms t
eA = expm(A*t);
P = int(eA'*C'*C*eA, 0, inf);

% x_0
f = exp(-3*t);
[U,S,V] = svd(P);
Sp = 1./S;
```

```
Sp(Sp == inf) = 0;
Pi =  V * Sp * U';
x0 = Pi * int(eA'*C'*f, 0, inf);

% d
temp = C*eA*x0;
d2 = int((f - temp)^2, 0, inf);
d = sqrt(d2);
```

Appendix

Additional MATLAB Functions:

<checkObservability.m>

```
function res = checkObservability(A, C)
    dim = size(A); n = dim(1);
    Qo = obsv(A, C);
    res.check = rank(Qo) == n;
    res.Qo = Qo;
end
```