AAE 666 Homework 5 Solution

February 27, 2021

Exercise 1

Let $x_1 = q$, $x_2 = \dot{q}$, $x_3 = \ddot{q}$, $x_4 = q^{(3)}$, therefore we have the state space system:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= sinx_1
\end{aligned}$$

Linearizing about the origin, we get:

$$\begin{aligned}
\delta\dot{x}_1 &= \delta x_2 \\
\delta\dot{x}_2 &= \delta x_3 \\
\delta\dot{x}_3 &= \delta x_4 \\
\delta\dot{x}_4 &= cosx_1^3 \delta x_1 \\
A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}
\end{aligned}$$

Eigenvalues for this system are $\lambda_{1,2} = \pm 1$ and $\lambda_{1,2} = \pm i$, which means the system will be **unstable**.

Exercise 2

Let $x_1 = q$, $x_2 = \dot{q}$, therefore we have the state space system:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1^3 - x_2
\end{aligned}$$

Linearization about the origin gives:

$$\begin{aligned}
\delta\dot{x}_1 &= \delta x_2 \\
\delta\dot{x}_2 &= 3(x_1^e)^2 \delta x_1 - \delta x_2
\end{aligned}$$

Substitute the value for the equilibrium at the origin gives;

$$\delta\dot{x}_1 = \delta x_2$$
$$\delta\dot{x}_2 = -\delta x_2$$
$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

Eigenvalues for this system are given by $\lambda_1 = 0$,$\lambda_2 = -1$, which means the system stability is **undetermined**.

Exercise 3

(i)

$$\dot{x}_1 = (1 + x_1^2)x_2$$
$$\dot{x}_2 = -x_1^3$$

Linearize about origin gives:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_1^e x_2^e & 1 + (x_1^e)^2 \\ -3(x_1^e)^2 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Eigenvalues for this system are given by $\lambda_{1,2} = 0$, which means the system stability is **undetermined**.

(ii)

$$\dot{x}_1 = sinx_2$$
$$\dot{x}_2 = (cosx_1)x_3$$
$$\dot{x}_3 = e^{x_1}x_2$$

Linearize about origin gives:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & cosx_2^e & 0 \\ (-sinx_1^e)x_3^e & 0 & cosx_1^e \\ e^{x_1^e}x_2^e & e^{x_1^e} & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix}$$
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Eigenvalues for this system are given by $\lambda_1 = 0$,$\lambda_{2,3} = \pm 1$ which means the system stability is **unstable**.

Exercise 4

$$x_1(k+1) = x_1(k)^2 + sin(x_2(k))$$
$$x_2(k+1) = 0.4cos(x_2(k))x_1(k)$$

Linearize about origin gives:

$$\begin{bmatrix}\delta x_1(k+1)\\ \delta x_2(k+1)\end{bmatrix} = \begin{bmatrix}2x_1^e(k) & cos x_2^e(k)\\ 0.4cos(x_2^e(k)) & -0.4sin(x_2^e(k))x_1^e(k)\end{bmatrix}\begin{bmatrix}\delta x_1(k)\\ \delta x_2(k)\end{bmatrix}$$
$$A = \begin{bmatrix}0 & 1\\ 0.4 & 0\end{bmatrix}$$

Eigenvalues for this system are given by $\lambda_{1,2} = \pm\sqrt{0.4} = \pm 0.6325$. Since $|\lambda_{1,2}| < 1$, it means the system stability is **exponentially stable**.

Exercise 5

$$x_1(k+1) = (1 + x_1(k)^3)x_2(k)$$
$$x_2(k+1) = x_1(k)^3 + x_2(k)^5$$

Linearize about origin gives:

$$\begin{bmatrix}\delta x_1(k+1)\\ \delta x_2(k+1)\end{bmatrix} = \begin{bmatrix}3x_1^e(k)^2x_2^e(k) & 1 + x_1^e(k)^3\\ 3x_1^e(k)^2 & 5x_2^e(k)^4\end{bmatrix}\begin{bmatrix}\delta x_1(k)\\ \delta x_2(k)\end{bmatrix}$$
$$A = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

Eigenvalues for this system are given by $\lambda_{1,2} = 0$. Since $|\lambda_{1,2}| < 1$, it means the system stability is **exponentially stable**.

Exercise 6

$$x_1(k+1) = x_2(k)$$
$$x_2(k+1) = sin(x_1(k)) + x_2(k)^5$$

Linearize about origin gives:

$$\begin{bmatrix}\delta x_1(k+1)\\ \delta x_2(k+1)\end{bmatrix} = \begin{bmatrix}0 & 1\\ cos x_1^e(k) & 5x_2^e(k)^4\end{bmatrix}\begin{bmatrix}\delta x_1(k)\\ \delta x_2(k)\end{bmatrix}$$
$$A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Eigenvalues for this system are given by $\lambda_{1,2} = \pm 1$. Since $|\lambda_{1,2}| = 1$, it means the system stability is **undetermined**.

Exercise 7

$$\dot{x}_1 = \sigma(x_2 - x_1)$$
$$\dot{x}_2 = rx_1 - x_2 - x_1 x_3$$
$$\dot{x}_3 = -bx_3 + x_1 x_2$$

Let $V(x) = rx_1^2 + \sigma x_2^2 + \sigma(x_3 - 2r)^2$. Since $\sigma, r > 0$, $V(x)$ is **radially unbounded**.

$$DV(x) = \begin{bmatrix} 2rx_1 & 2\sigma x_2 & 2\sigma(x_3 - 2r) \end{bmatrix}$$
$$DV(x)f(x) = -2\sigma(rx_1^2 + x_2^2) - 2\sigma b x_3(x_3 - 2r)$$

Using the ineuqality relation $ab \leq \frac{1}{2}\epsilon a^2 + \epsilon^{-1} b^2$, where $\epsilon > 0$, we have

$$x_3 \leq \frac{1}{2}\epsilon x_3^2 + \epsilon^{-1} 1^2$$
$$4b\sigma r x_3 \leq 2b\sigma r \epsilon x_3^2 + \frac{2b\sigma r}{\epsilon}$$

Therefore $DV(x)f(x)$ becomes:

$$DV(x)f(x) \leq -2\sigma(rx_1^2 + x_2^2) - 2\sigma b x_3^2(\epsilon r - 1) + \frac{2b\sigma r}{\epsilon}$$

$DV(x)f(x) \leq 0$ when $|x| \geq R$ where R is big enough. Therefore, all solutions of the system are bounded.