# L BP

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# Transfer Orbits: Lambert Arcs

Two approaches to mission planning:

(a) Given the transfer orbit → initial and final positions are specified; relate to the time of flight } straight forward

(b) Given the initial (departure) and final (target) points → determine the orbit that passes through the points } challenging

create opportunities

## Transfer Orbit Design
## (special class of boundary value problem)

1. **Geometrical relationships**

   Conic paths connecting 2 points that are fixed in space (with focus at attracting center)

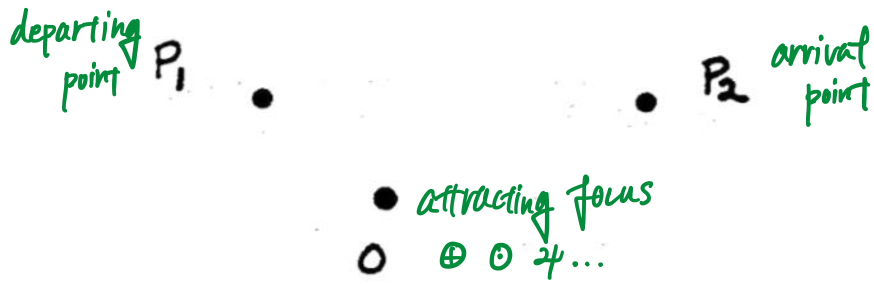
2. **Analytical Relationships**

3. **Lambert's Theorem**

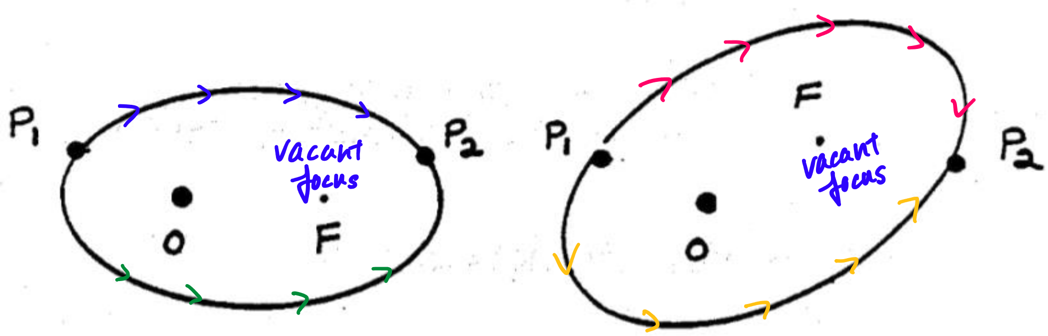
## Geometrical Relationships: Ellipse

Given two fixed points $P_1, P_2$; center of force at point $O$

Find: ellipse with focus at point $O$ that connects $P_1, P_2$ ← at some instance




what ellipse connect these two locations?



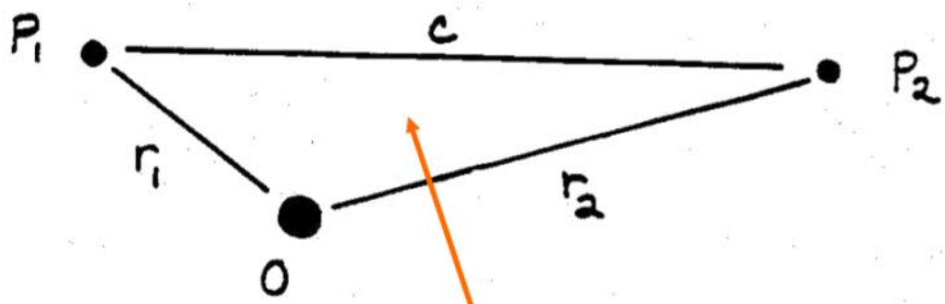

Example

2 ellipses ⇒ 4 options

If $F$ is not specified ⟹ $\infty$ number of solutions exist

Thus, find the locus of all possible $F$ locations ← the real problem
Pick one of the $F$ sites and the ellipse is determined

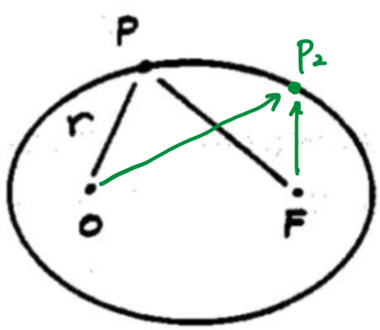
why?
How to get F?




"space triangle for transfer"

Let $\begin{cases} OP_1 = r_1 \\ OP_2 = r_2 \\ P_1P_2 = c \end{cases}$

Since $P_1$ and $P_2$ must both lie on the same ellipse, $F$ must be selected such that




$$\overbrace{OP_1}^{r_1} + P_1F = 2a = \overbrace{OP_2}^{r_2} + P_2F$$

(always true for an ellipse)

OR

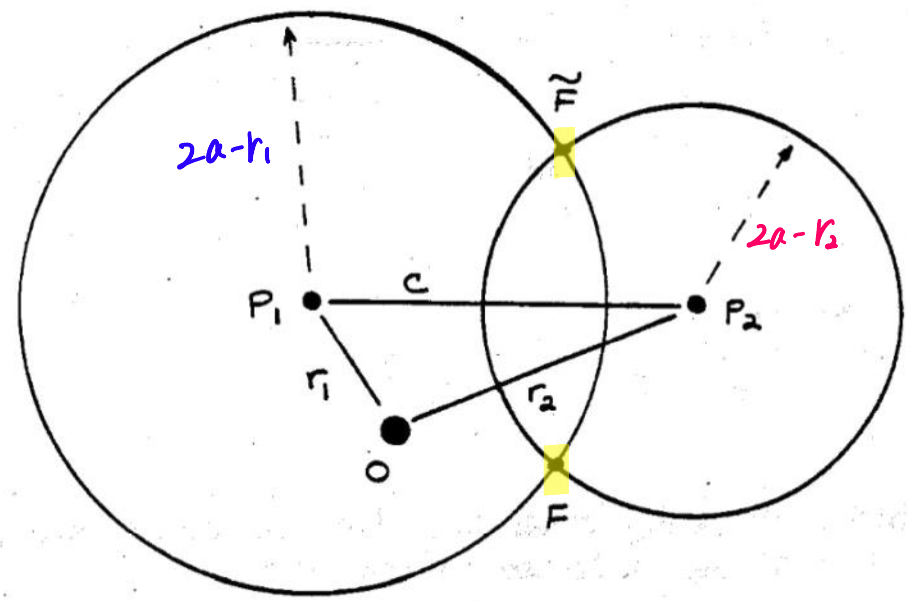
$$\left.\begin{aligned} P_1F &= 2a - r_1 \\ P_2F &= 2a - r_2 \end{aligned}\right\} \text{given } r_1, r_2$$

if we knew "a" can solve for F

For ellipse with major axis $2a$, point $F$ determined as the intersection of two circles centered at $P_1$ and $P_2$ with radii $2a - r_1$ and $2a - r_2$

$$\left.\begin{array}{l} P_1F = 2a - r_1 \\ P_2F = 2a - r_2 \end{array}\right\}$$
F lies on circle about $P_1$
F lies on circle about $P_2$

$r_1 < r_2$

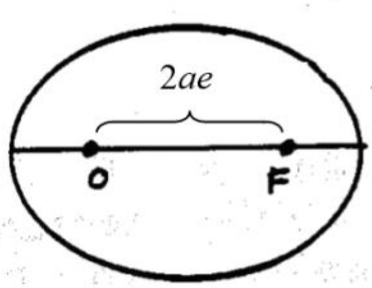



For a given "$a$" two possible intersection points

→ 2 possible elliptic paths between $P_1$ & $P_2$

Closest to $O$ ⟶ $F$ , $\tilde{F}$ ← furthest from O

Given "$a$" ⟹ distance between foci $O$ and $F$ = $2ae$




$\therefore \tilde{F}$ associated with
larger distance to O
larger e
smaller $p = a(1-e^2)$

Choose 3 different values of "$a$"

$\Rightarrow$ as "$a$" gets smaller, circles shrink

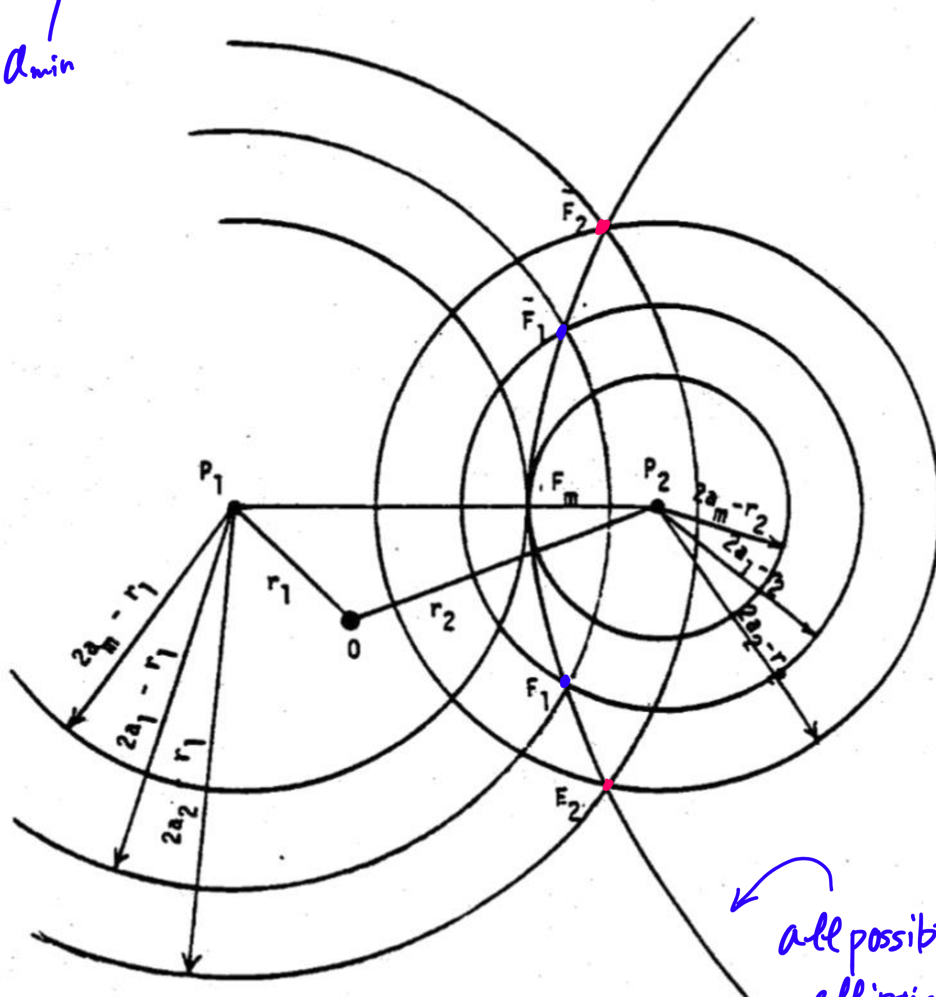
$a_2 \to a_1 \to a_m$

Note: there is a smallest value of "$a$" ($a_m$) below which there is <u>no ellipse</u> that connects $P_1$ and $P_2$ because the circles do not intersect

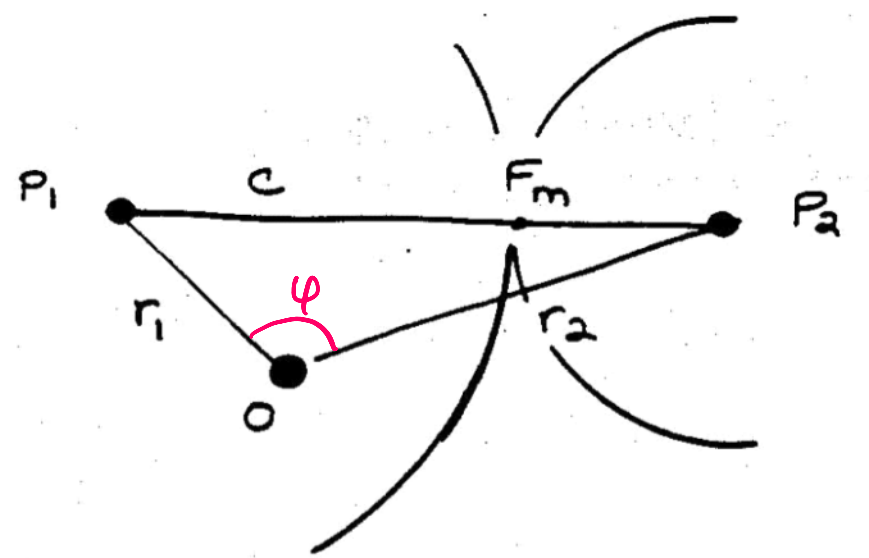
$a = a_m$ ➡ circles are tangent

➡ $F_m$ lies on chord

$a_{min}$




all possible elliptical arcs




$$(2a_m - r_1) + (2a_m - r_2) = c$$

$$4a_m = r_1 + r_2 + c \qquad \text{OR}$$

$2a_{min} = \frac{1}{2}\underbrace{(r_1 + r_2 + c)}_{s}$

s
semi-perimeter
given space triangle

$a_{min} = \frac{s}{2}$

smallest possible semi-major axis

→ $F_m$ defines minimum energy elliptic path from $P_1$ to $P_2$

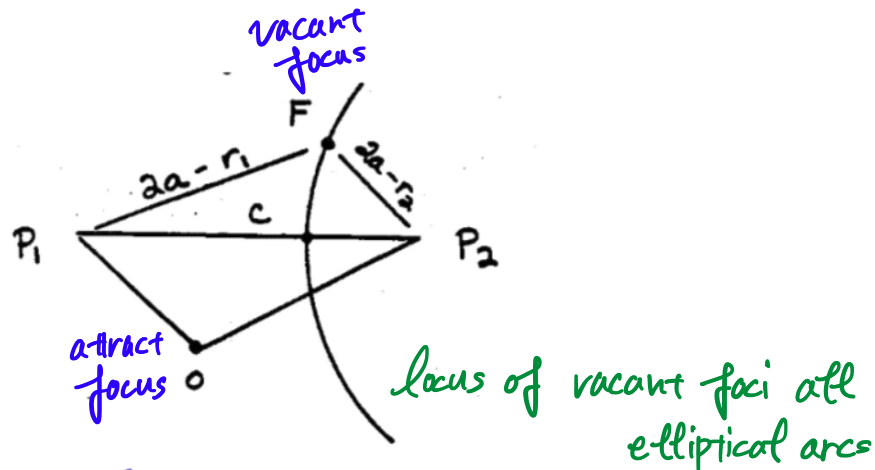
$$\left( \mathcal{E} = -\frac{\mu}{\underbrace{2a_m}_{a_{min}}} \quad \text{when } a_m \text{ small as possible, } \mathcal{E} \text{ is min} \right)$$

$a_{min}$ smallest value
$\left|-\frac{\mu}{2a_m}\right|$ largest

$\mathcal{E}$ is neg
smallest largest
$\mathcal{E}$

Note: choosing different values of "$a$", produces pairs of vacant foci ( $F, \tilde{F}$)

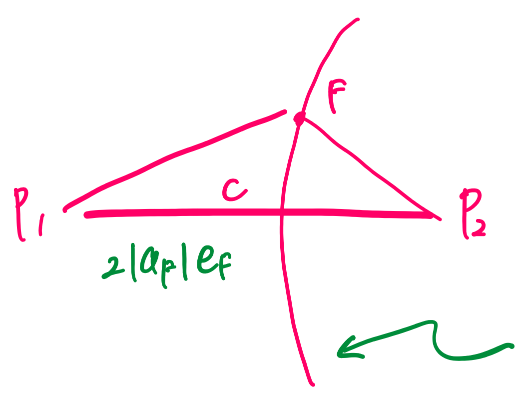
Sketch curve through all vacant foci $F$'s
What does curve look like?




Equations for circles $\begin{cases} P_1F = 2a - r_1 \\ P_2F = 2a - r_2 \end{cases}$

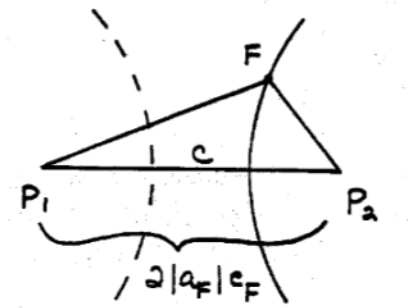
Subtract equations $\quad P_1F \underset{\uparrow \text{ minus}}{-} P_2F = \underbrace{r_2 - r_1}_{\text{const.}}$ $\Big\}$ only unknown is $F$!

Equation of a hyperbola: $F$ is point on hyperbola
$P_1, P_2$ are foci
constant on right side: $2|a_F|$




$|a_F| = \dfrac{r_2 - r_1}{2}$

$e_F = \dfrac{c}{2|a_F|} = \dfrac{c}{r_2 - r_1}$




## Geometrical Relationships: Hyperbola

Given two fixed points $P_1, P_2$; center of force at point $O$

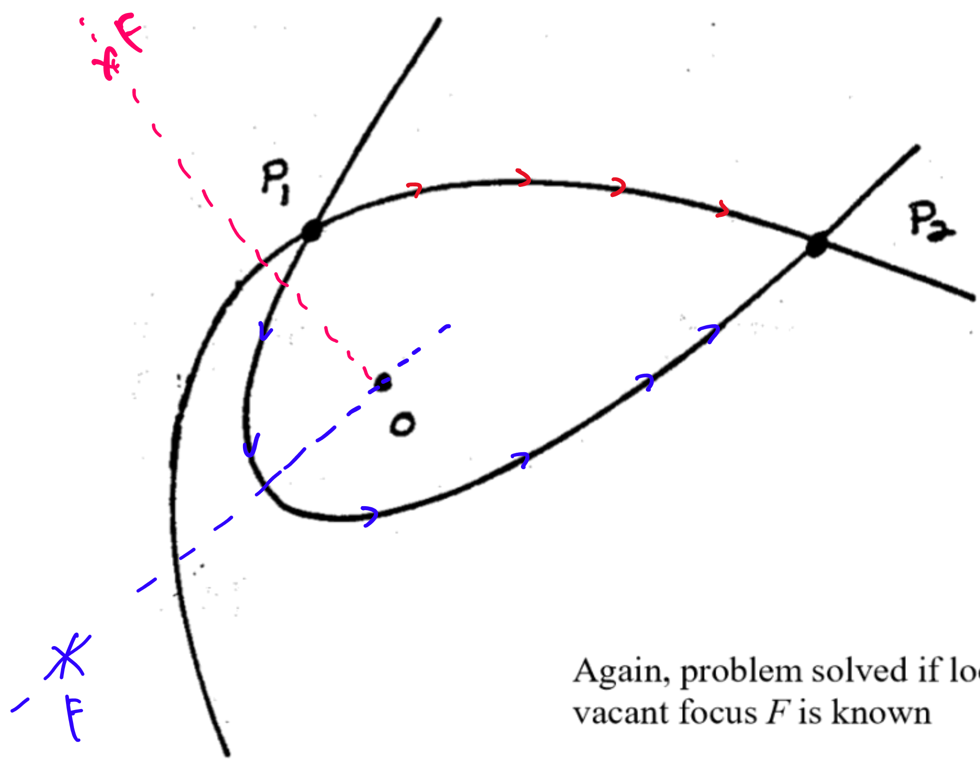
Find: hyperbola with focus at point $O$ that connects $P_1, P_2$

dep point

arr point


attracting focus

what hyp connects these 2 pts?




Again, problem solved if location of vacant focus $F$ is known

Since $P_1$ and $P_2$ must both lie on the same hyperbola, $F$ must be selected such that

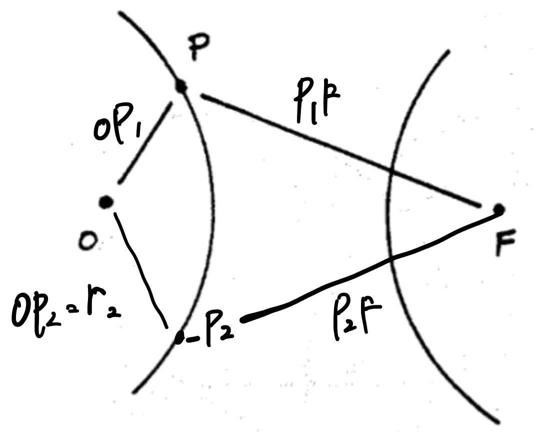
use this hyp for transfer

$$P_1F - \overbrace{OP_1}^{r_1} = 2|a| = P_2F - \overbrace{OP_2}^{r_2}$$

always true for hyperbola

OR

$$P_1F = 2|a| + r_1$$
$$P_2F = 2|a| + r_2$$




Given $r_1, r_2$

$\rightarrow$ if we know $|a|$

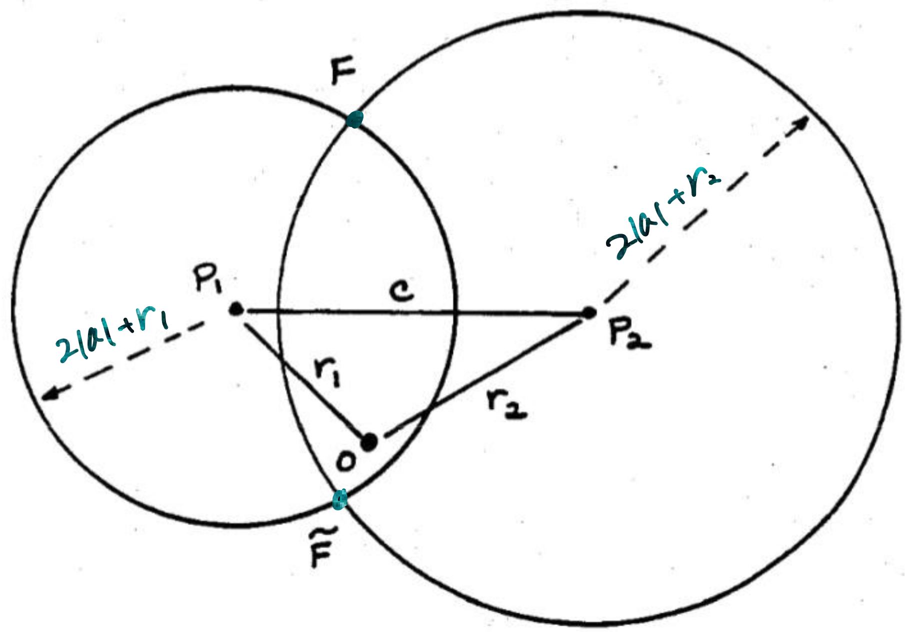
$\rightarrow$ solve for F

For hyperbola, with major axis $2|a|$, point $F$ determined as the intersection of two circles centered at $P_1$ and $P_2$ with radii $2|a| + r_1$ and $2|a| + r_2$

$$\left.\begin{aligned} P_1F &= 2|a| + r_1 \\ P_2F &= 2|a| + r_2 \end{aligned}\right\}$$

F must lie on circle about $P_1$

F " $P_2$

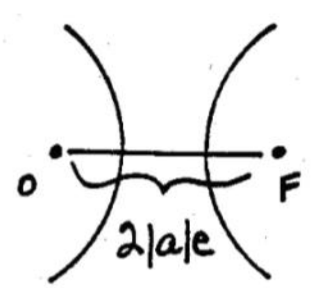



For a given $|a|$, two possible intersection points

→ 2 possible hyperbolic paths between $P_1$ and $P_2$

$F, \quad \tilde{F}$

Given $|a|$ ➡ distance between foci $O$ and $F = 2|a|e$




$\therefore F$ associated with $\begin{cases} \text{larger } e \\ \text{larger } p \end{cases}$

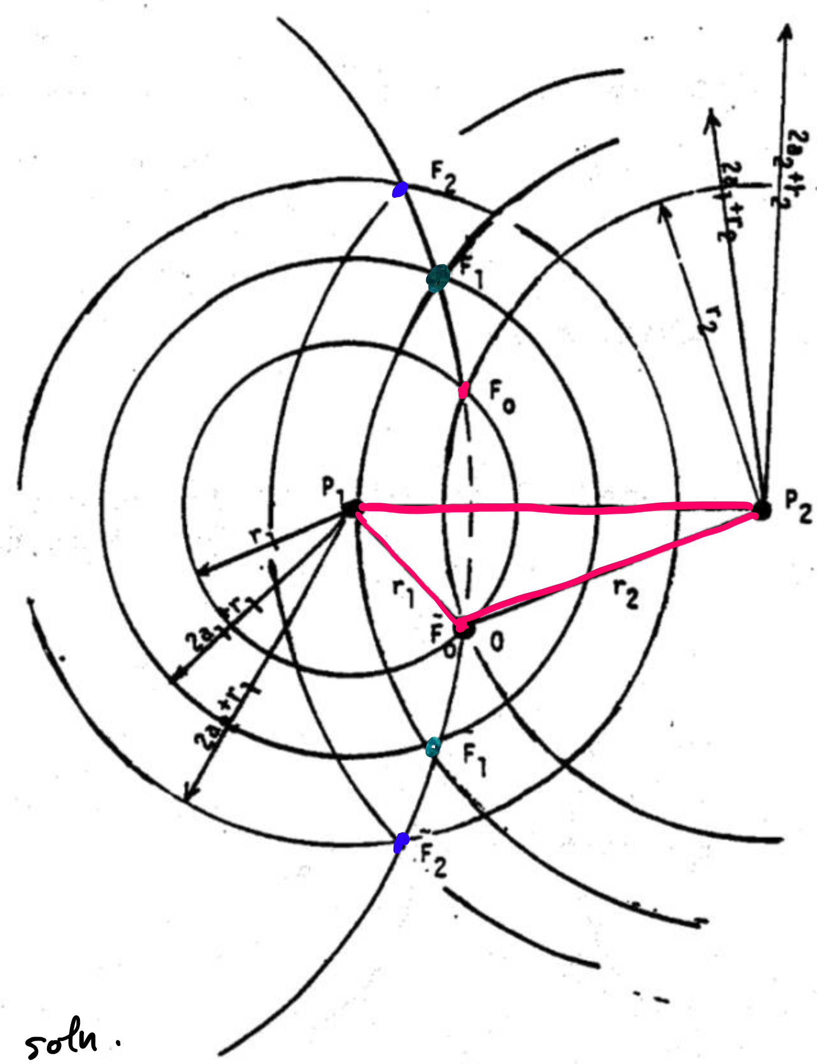
$p = |a|(e-1)$

Choose 3 different values of $|a|$

$\Rightarrow$ as $|a|$ gets smaller, circles shrink

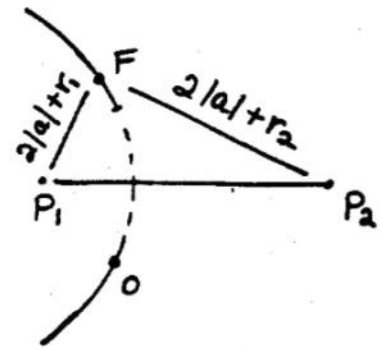
Note: smallest value of $|a|$ that is possible is

(then circles have radii $r_1$ and $r_2$) $\Rightarrow$




soln. curve $\rightarrow$ locus for vacant foci

Note: Now sketch a curve through all vacant $F$'s
What does the curve look like?



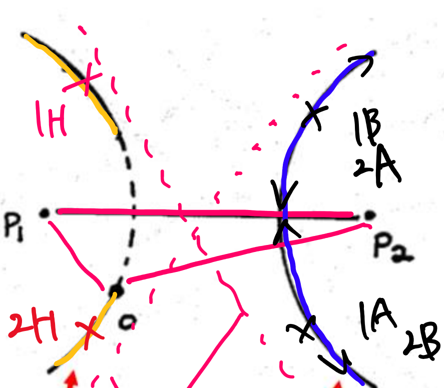

Locus of vacant foci **is** branch of a hyperbola

Equations for circles $\begin{cases} P_1F = 2a + r_1 \\ P_2F = 2a + r_2 \end{cases}$

Subtract equations $P_2F - P_1F = \underbrace{\overset{2|a_F|}{r_2 - r_1}}_{\text{same const.}}$ unknown is $F$ again!

⇨ Equation of a hyperbola: other branch of **same** hyperbola
$P_1, P_2$ are foci
constant on right side: $2|a_F|$

asymptotes represent the 2 parabolas that link $P_1$ & $P_2$




branches of a hyperbola with foci at $P_1, P_2$

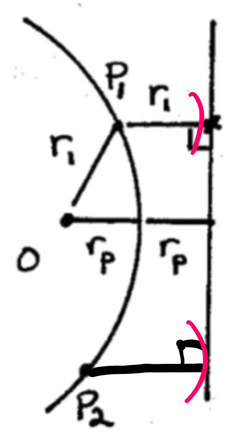
All possible F for hyp arcs

parabola

All possible F for elliptical transfer

## Geometrical Relationships: Parabola

Only two possible parabolas ← $a = \infty$ ; $F$ at $\infty$

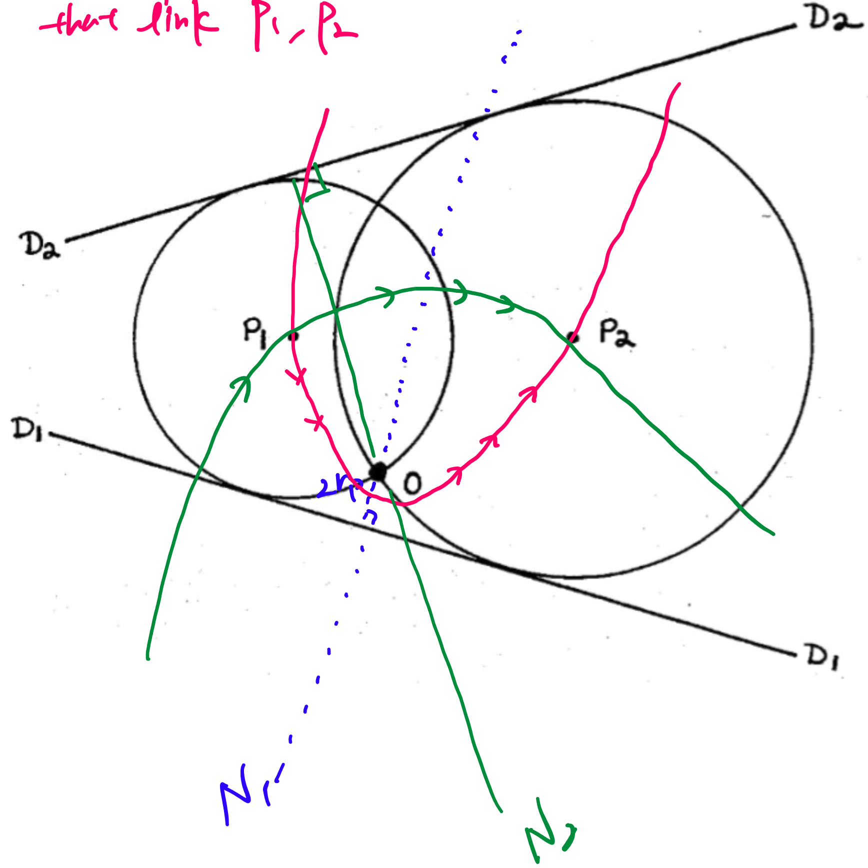



Definition of parabola:
$OP$ = distance to perpendicular intersection with directrix

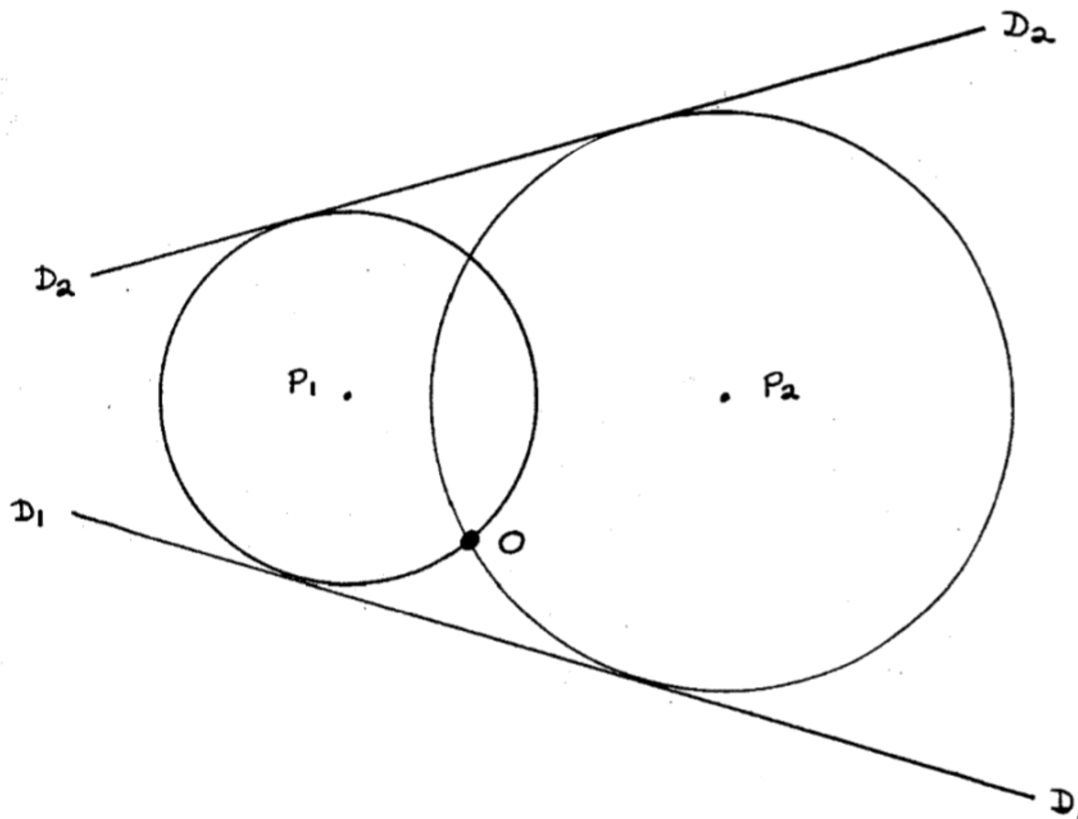
→ point on directrix must be on circle about $P_1$ of radius $r_1$

→ $P_1/P_2$ on same parabola so point on directrix on circle about $P_2$ of radius $r_2$

→ only 2 parabolas that link $P_1, P_2$



To construct parabolas: requires normals $N$ and vertices $V$

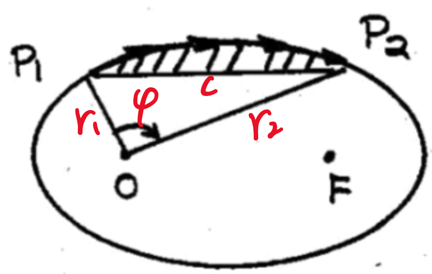


## Geometrical Relationships: Summary

Once $F$ is selected or otherwise identified, particular conic section is known

Necessary to define a method to categorize or classify transfers

| | | |
|---|---|---|
| Legend: | A – | Ellipse ($F$ NOT between chord and arc) |
| | B – | Ellipse ($F$ between chord and arc) |
| | H – | Hyperbola |
| | 1 – | Transfer Angle $< 180^\circ$ |
| | 2 – | Transfer Angle $> 180^\circ$ |

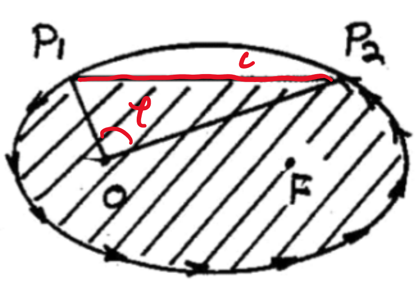



F is below c

TrA = $\varphi$

$\varphi < 180^\circ \rightarrow$ 1

1A - type

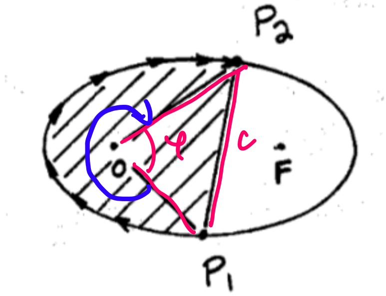



TrA = $360^\circ - \varphi > 180^\circ$

type $\rightarrow$ 2

between chord and arch

2B - type

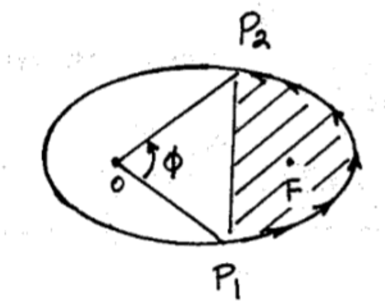



F "above" chord

opposite of O

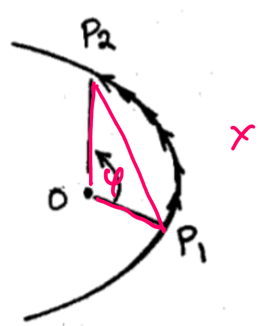
$TrA = 360° - \varphi > 180°$

2A




$TrA = \varphi < 180°$

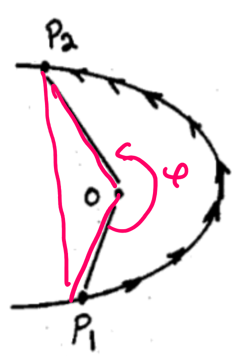
1B




$TrA = \varphi < 180°$

1H

F opposite side of chord from O



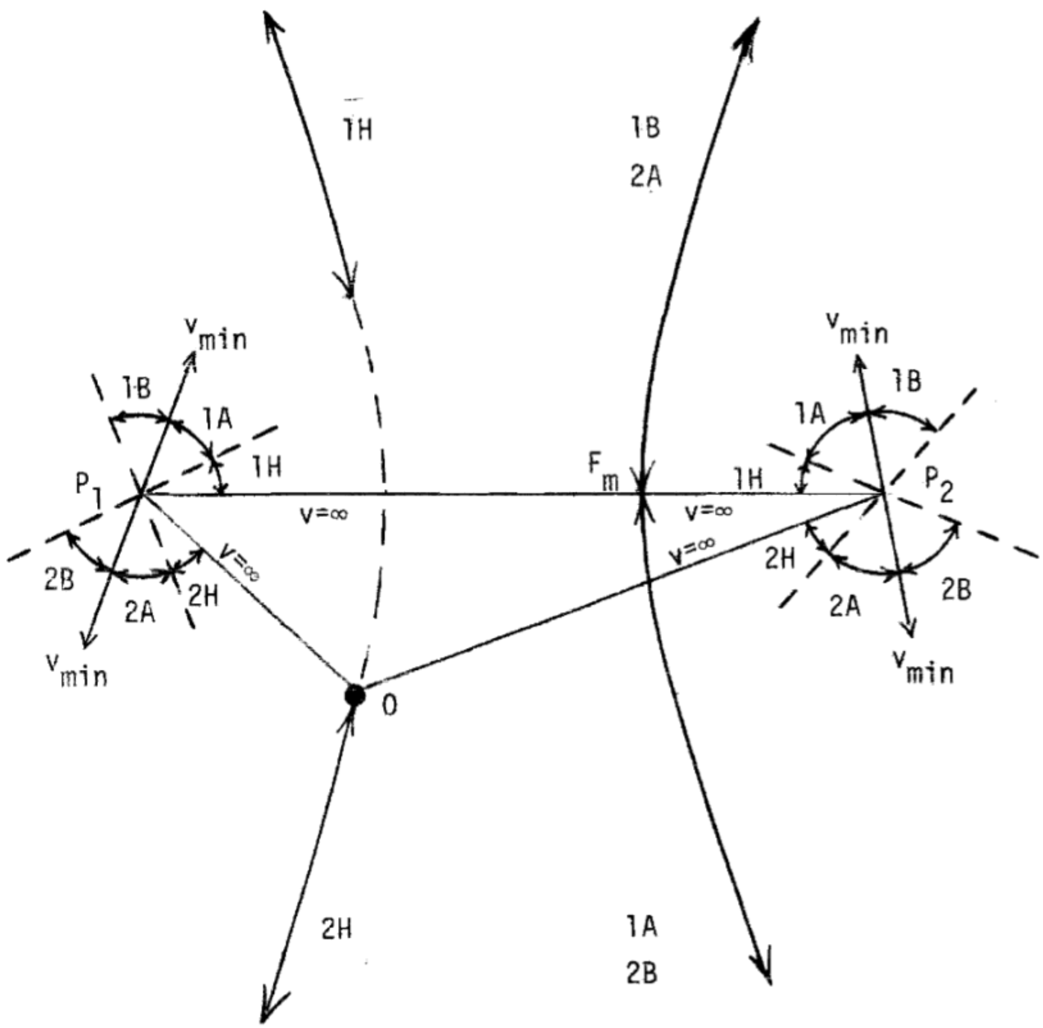

$TrA = \varphi > 180°$

F is on same side of c as O

2H

Various Orbits Between Two Points $P_1$, $P_2$

Locus of Vacant Focus F




Legend: A - Ellipse ( F not between chord and focus)

B - Ellipse ( F between chord and focus)

H - Hyperbola

1 - Transfer Angle < 180°

2 - Transfer Angle > 180°

We may suppose $r_2 \geq r_1$.