College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 12
State Feedback and Stabilization

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Exercise 1

(By hand) Determine whether or not each of the following systems are controllable, stabilizable, or not stabilizable.

(a)

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

(b)

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

(c)

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

(a)

The A matrix and B matrix are

$$\mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore rank(Q_c) = 2 \ .$$

Thus, this system is controllable.

For any gain matrix K

$$A + BK = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} (k_1 \quad k_2) = \begin{pmatrix} k_1 - 1 & k_2 \\ k_1 & k_2 + 1 \end{pmatrix}.$$

Then

$$\det\big(\lambda I - (A + BK)\big) = 0$$

$$\Rightarrow \lambda^2 + (-k_1 - k_2)\,\lambda + k_1 - k_2 - 1 = 0$$

If $k_1 = 1$ and $k_2 = -4$

$$\Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

The characteristics polynomial has all positive coefficients, so the corresponding eigenvalues have a negative real part and is asymptotically stable. Thus, this system is stabilizable.

(b)

The A matrix and B matrix are

$$\mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} .$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore rank(Q_c) = 1 \ .$$

Thus, this system is uncontrollable.

The eigenvalues of this system are

$$det(A - \lambda I) = 0 \Rightarrow (-1 - \lambda)(1 - \lambda) = 0 \ .$$

The uncontrollable eigenvalue can be found by the PBH test

For $\lambda = 1$

$$\mathrm{Z} = \mathrm{rank}\big((A - \lambda I \quad B)\big) = \mathrm{rank}\left(\begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = 2$$

For $\lambda = -1$

$$Z = rank\big((A - \lambda I \quad B)\big) = rank\left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}\right) = 1 \neq 2$$

Thus, the uncontrollable eigenvalue has a negative real part so this system is stabilizable.

(c)

The A matrix and B matrix are

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore rank(Q_c) = 1 \ .$$

Thus, this system is uncontrollable.

The eigenvalues of this system are

$$det(A - \lambda I) = 0 \Rightarrow (1 - \lambda)(-\lambda) = 0 \ .$$

The uncontrollable eigenvalue can be found by the PBH test

For $\lambda = 0$

$$Z = \mathrm{rank}\big((A - \lambda I \quad B)\big) = \mathrm{rank}\left(\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}\right) = 1 \neq 2$$

For $\lambda = 1$

$$Z = \mathrm{rank}\big((A - \lambda I \quad B)\big) = \mathrm{rank}\left(\begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \end{pmatrix}\right) = 2$$

Thus, the uncontrollable eigenvalue does not have a negative real part so this system is not stabilizable.

Exercise 2

Obtain an open loop control which drives the following system from $x(0) = -1$ to $x(1) = 1$.

$$\dot{x} = x + u$$

From the problem we know that

$$x_0 = x(0) = -1$$

$$T = 1: \quad x_f = x(T) = 1$$

$$A = 1, \qquad B = 1$$

The controllability grammian is going to be

$$Wc(T) \coloneqq \int_0^T e^{At} BB' e^{A't} dt$$

$$W_c(1) = \int_0^1 e^t e^t dt = \int_0^1 e^{2t} dt = \frac{1}{2}(e^2 - 1) = 3.1945 .$$

Then,

$$\tilde{x} = W_c(T)^{-1}\left(x_f - e^{AT}x_0\right) = \frac{1}{3.1945}\left(1 - e(-1)\right) = 1.1640 .$$

Then the control input will become

$$u(t) = B' e^{A'(T-t)} \tilde{x} = e^{1-t}(1.1640)$$

Thus, the open loop control that drives x_0 to x_f for a given time frame of $[0, T]$ is

$$u(t) = 1.1640 e^{1-t}$$

Exercise 3

(By hand) Consider the system described by

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 + u \\ \dot{x}_2 &= u \end{aligned}$$

Obtain a state feedback controller which results in a closed loop system which is asymptotically stable about the zero state.

The A, B matrix of this system are

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The controllability matrix is

$$Q_c = (B \quad AB) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore rank(Q_c) = 2 \ .$$

This system is controllable.

Choose two asymptotically stable poles

$$p = -1, -2 \ .$$

Use the eigenvalue placement method to stabilize the system.

Say the feedback gains are

$$K = (k_1 \quad k_2) \ .$$

Then

$$A + BK = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 \quad k_2) = \begin{pmatrix} 1 & 1 \\ k_1 & k_2 \end{pmatrix} \ .$$

Then

$$det\big(\lambda I - (A + BK)\big) = 0 \quad \Rightarrow \quad \lambda^2 + (-k_2 - 1)\lambda - k_1 + k_2 = 0$$

The characteristic equation with the desired poles is

$$(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2 = 0 \ .$$

Then, equate

$$\begin{aligned} -k_2 - 1 &= 3 \\ -k_1 + k_2 &= 2 \end{aligned} \ \Rightarrow \ k_1 = -6, k_2 = -4 \ \Rightarrow \ K = (-6 \quad -4)$$

Exercise 4

(By hand) Consider the system described by

$$\dot{x}_1 = -x_2 + u$$

$$\dot{x}_2 = -x_1 - u$$

Where all quantities are scalars.

(a) Is this system stabilizable via state feedback?
(b) Does there exist a linear state feedback controller which results in closed loop eigenvalues -1, -4?
(c) Does there exist a linear state feedback controller which results in closed loop eigenvalues -2, -4?

In parts (b) and (c): If no controller exists, explain why; if one does exist, give an example of one.

(a)

The A matrix and B matrix are

$$\mathrm{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} .$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow rank(Q_c) = 1 \neq 2.$$

Thus, this system is uncontrollable.

The eigenvalues of this system are

$$det(A - \lambda I) = 0 \Rightarrow (-1 - \lambda)(1 - \lambda) = 0 \ .$$

The uncontrollable eigenvalue can be found by the PBH test

For $\lambda = 1$

$$\mathrm{Z} = \mathrm{rank}\big((A - \lambda \mathrm{I} \quad B)\big) = \mathrm{rank}\left(\begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\right) = 2$$

For $\lambda = -1$

$$\mathrm{Z} = \mathrm{rank}\big((A - \lambda I \quad B)\big) = \mathrm{rank}\left(\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}\right) = 1 \neq 2$$

From the PBH test we know that the uncontrollable eigenvalue has a negative real part so this is stabilizable.

For a feedback gain matrix K

$$A + BK = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} (k_1 \quad k_2) = \begin{pmatrix} k_1 & k_2 - 1 \\ -k_1 - 1 & -k_2 \end{pmatrix} .$$

Then

$$det\big(\lambda I - (A + BK)\big) = 0$$

$$\Rightarrow \lambda^2 + (k_2 - k_1)\,\lambda - k_1 + k_2 - 1 = 0$$

(b)

If the desired poles are -1 and -4, the corresponding characteristic equation will be

$$\lambda^2 + 5\lambda + 4 = 0 \ .$$

Then from part (a) we can find the gains of the state feedback controller.

$$\begin{aligned} k_2 - k_1 \quad &= \quad 5 \\ -k_1 + k_2 - 1 \quad &= \quad 4 \end{aligned}$$

The augmented matrix is

$$\begin{pmatrix} -1 & 1 & 5 \\ -1 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

So, there are infinite amount of gains that satisfy

$$k_1 - k_2 = -5 \ .$$

And one example is

$$k_1 = -4, \qquad k_2 = 1$$

Thus, there exists a linear state feedback controller of $K = (-4 \quad 1)$.

(c)

If the desired poles are -2 and -4, the corresponding characteristic equation will be

$$\lambda^2 + 6\lambda + 8 = 0 \ .$$

Then from part (a) we can find the gains of the state feedback controller.

$$\begin{aligned} k_2 - k_1 \quad &= \quad 6 \\ -k_1 + k_2 - 1 \quad &= \quad 8 \end{aligned}$$

The augmented matrix is

$$\begin{pmatrix} -1 & 1 & 6 \\ -1 & 1 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -6 \\ 0 & 0 & 2 \end{pmatrix}$$

In the second row, 2 cannot be equal to 0. Thus, there does NOT exist a linear state controller for the eigenvalues of -2, -4.

Exercise 5

(Stabilization of cart pendulum system via state feedback.) (MATLAB)

Carry out the following parameter sets P2 and P4 and equilibriums E1 and E2. Illustrate the effectiveness of your controllers with numerical simulations.

Using eigenvalue placement techniques, obtain a state feedback controller which stabilizes the nonlinear system about the equilibrium.

What is the largest value of δ (in degrees) for which your controller guarantees convergence of the closed loop system to the equilibrium for initial conditions

$$\left(y, \theta_1, \theta_2, \dot{y}, \dot{\theta}_1, \dot{\theta}_2\right)(0) = (0, \theta_1^e - \delta, \theta_2^e + \delta, 0,0,0)$$

Where θ_1^e and θ_2^e are the equilibrium values of θ_1 and θ_2.

The system equation for the double pendulum cart system is

$$\begin{aligned}
(m_0 + m_1 + m_2)\ddot{y} - m_1 l_1 cos\theta_1 \ddot{\theta}_1 - m_2 l_2 cos\theta_2 \ddot{\theta}_2 + m_1 l_1 sin\theta_1 \dot{\theta}_1^2 + m_2 l_2 sin\theta_2 \dot{\theta}_2^2 &= u \\
-m_1 l_1 cos\theta_1 \ddot{y} + m_1 l_1^2 \ddot{\theta}_1 \quad + \quad m_1 l_1 g sin\theta_1 &= 0 \\
-m_2 l_2 cos\theta_2 \ddot{y} + m_2 l_2^2 \ddot{\theta}_2 \quad + \quad m_2 l_2 g sin\theta_2 &= 0
\end{aligned}$$

Have the system be a single output of the displacement y.

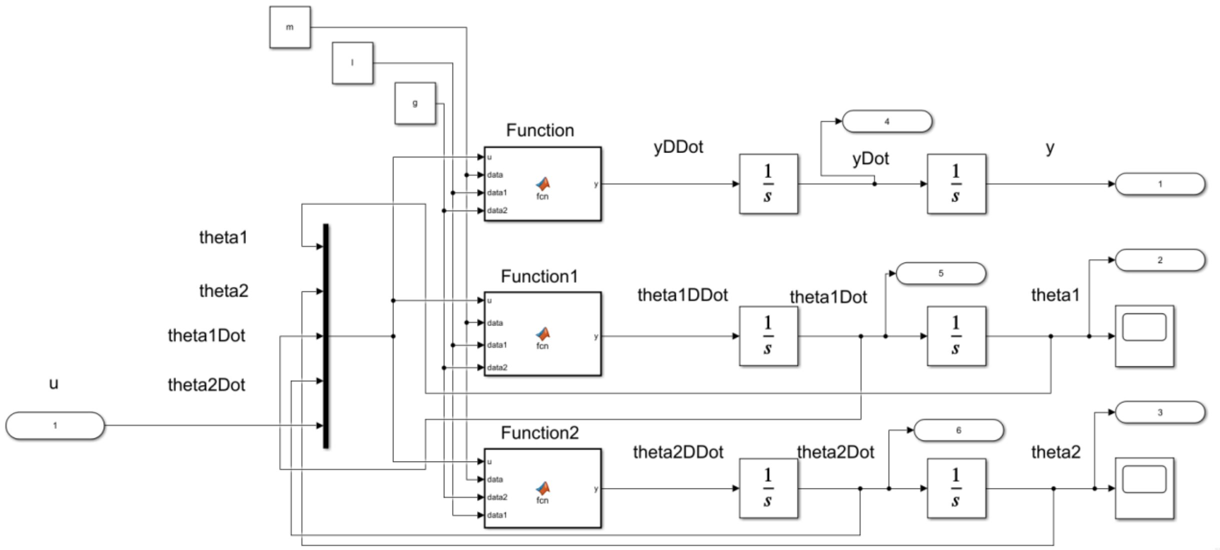
$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

$$E2: \quad (y^e, \theta_1^e, \theta_2^e) = (0, \pi, \pi)$$

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

L1	P1	E1
L2	P1	E2
L3	P2	E1
L4	P2	E2
L5	P3	E1
L6	P3	E2
L7	P4	E1
L8	P4	E2

The Simulink model used for this is shown below,



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

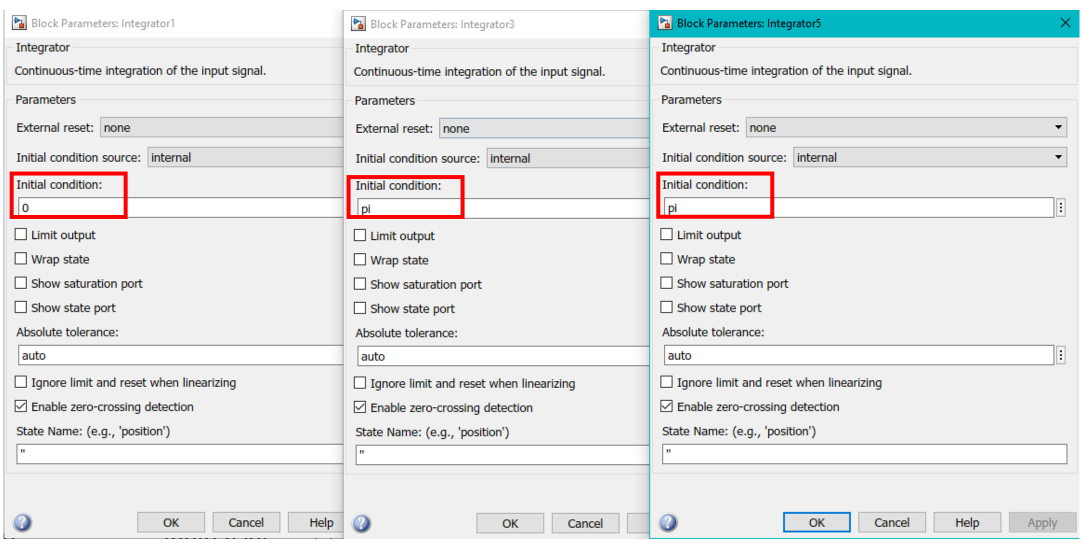
Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4)...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

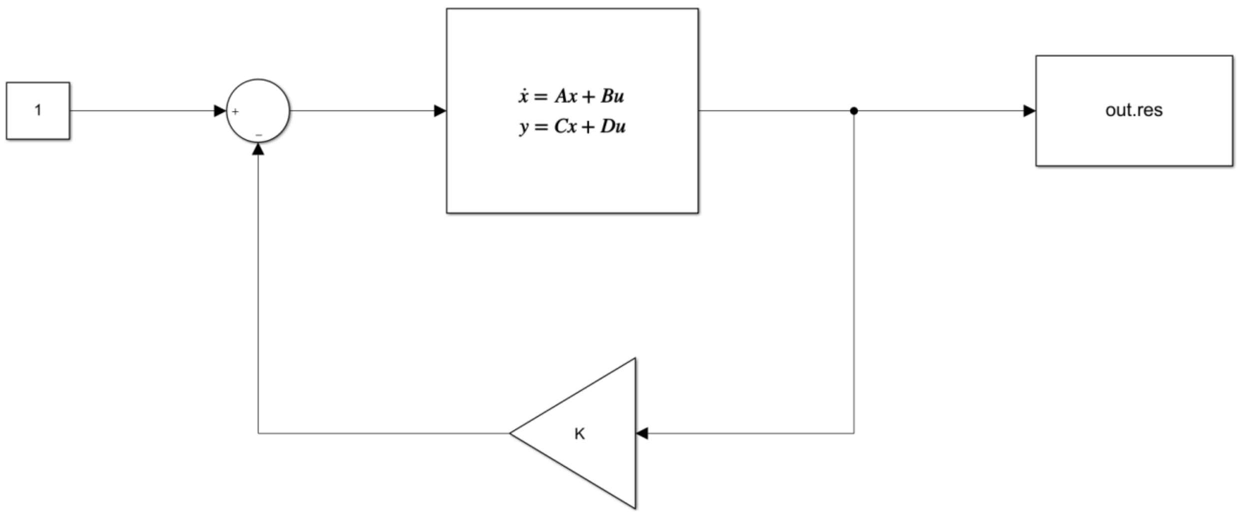
```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) + 
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



The feedback control system is the following Simulink model

“hw12_feedbackControl”



Procedure:

1. Linearize the system for P# and E# and obtain the A, B, C, D matrices for corresponding equilibrium conditions
2. Define the system requirements – eigenvalues to be placed
3. Compute the feedback gain K with eigenvalue placement
4. Plot the response of the controlled system with the system above

L3 (P2 & E1):

1.

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -0.5051   -1.5152         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5051

C = 1×6
     1     0     0     0     0     0

D = 0
```

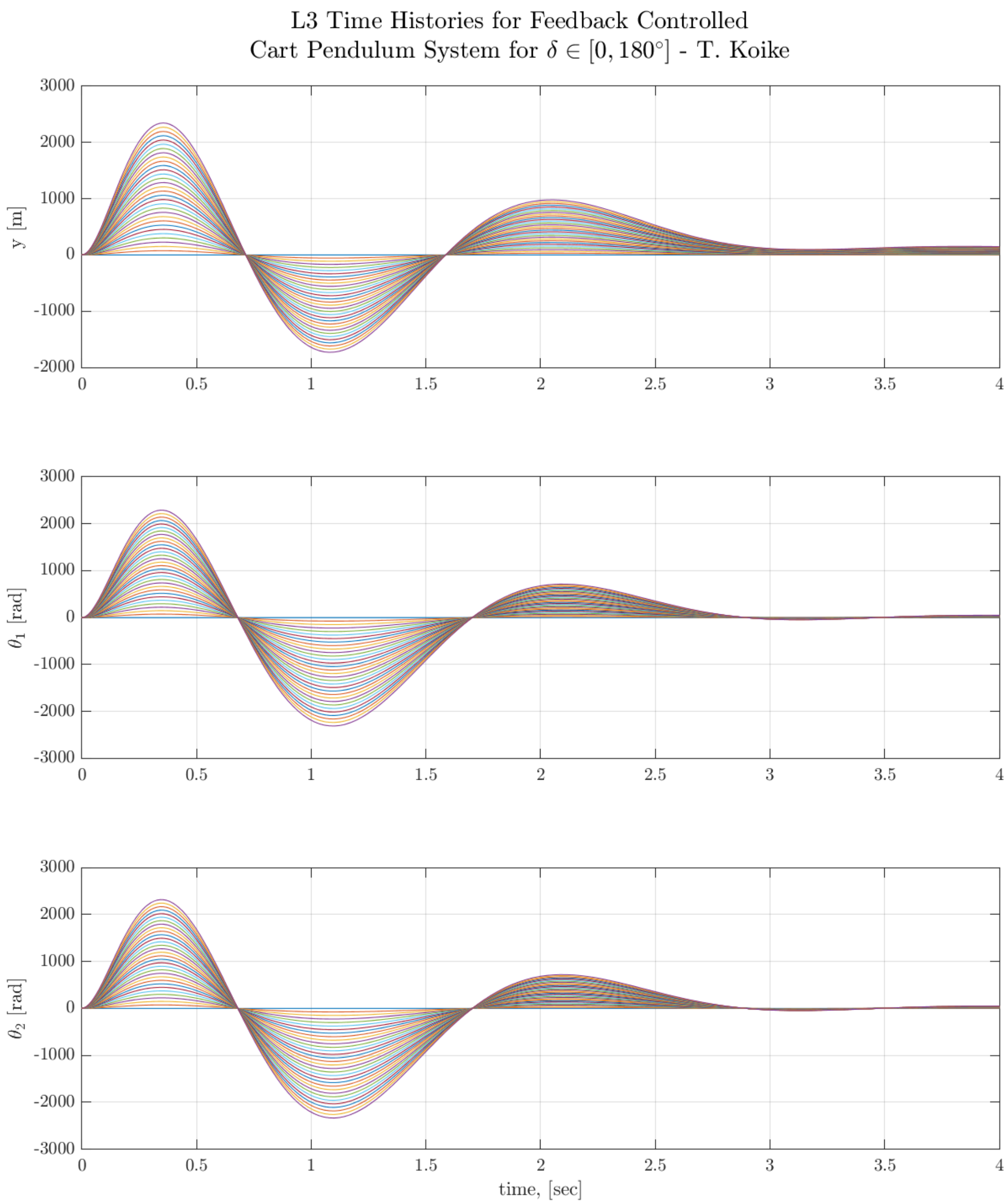
2.

```
p =
     -23/15     + 1207/363i       -11/50     +    0i            -3        +    0i
     -23/15     - 1207/363i        -3/2      +    0i            -9/2      +    0i
```

3.

```
K = 1×6
10^4 ×
    0.0118    6.2267   -6.1632    0.0709   -2.7555    2.6601
```

4.


L3 Time Histories for Feedback Controlled
Cart Pendulum System for $\delta \in [0, 180°]$ - T. Koike
y [m]
θ_1 [rad]
θ_2 [rad]
time, [sec]

$\delta = 180°$

L4 (P2 & E2):

1.

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0    1.5000    0.5000         0         0         0
         0    0.5051    1.5152         0         0         0

B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -0.5051

C = 1×6
     1     0     0     0     0     0

D = 0
```

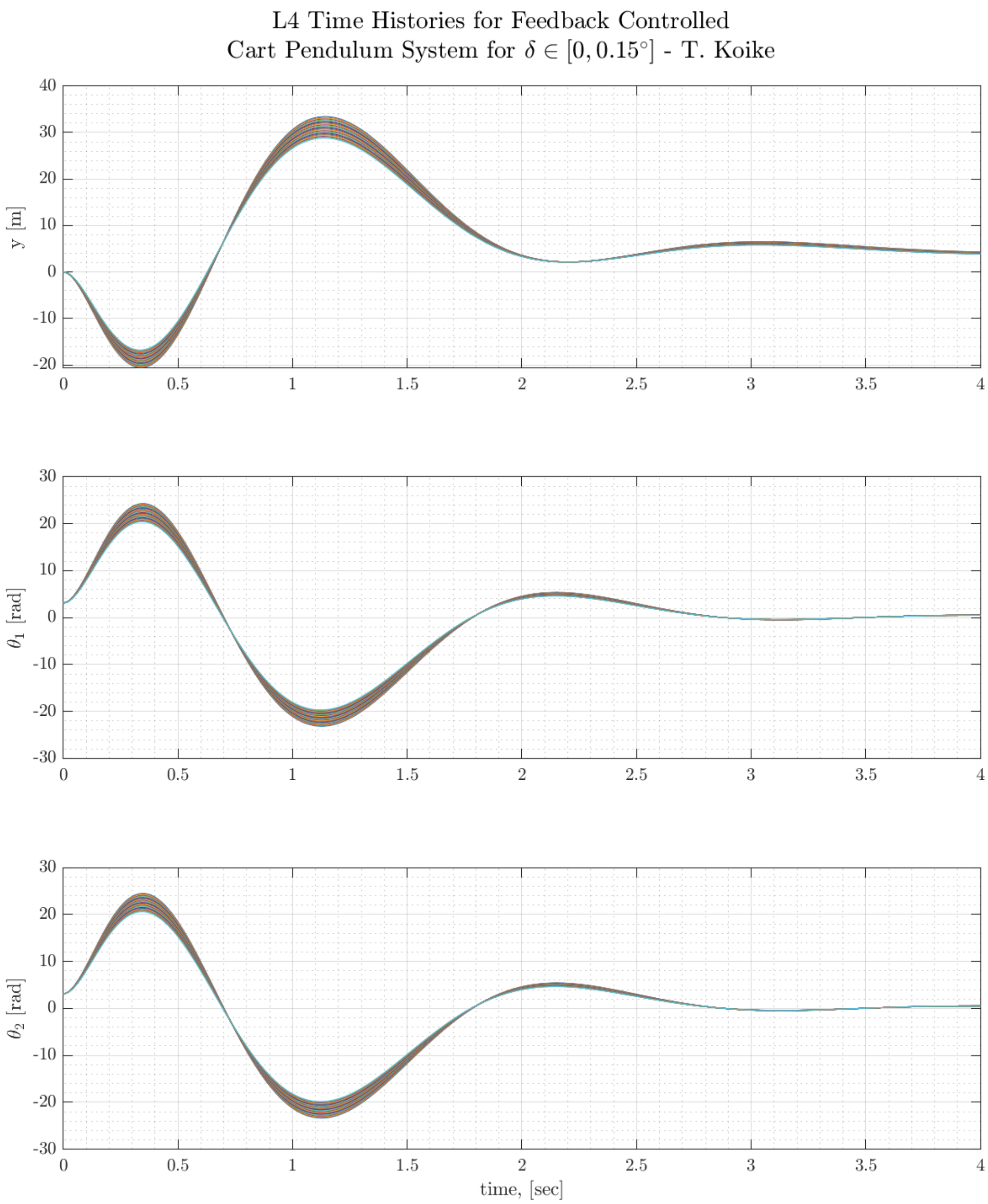
2.

```
p =
    -23/15     + 1207/363i       -11/50     +    0i             -3        +    0i
    -23/15     - 1207/363i        -3/2      +    0i             -9/2      +    0i
```

3.

```
K = 1×6
10^5 ×
    0.0012    1.1301   -1.1191    0.0071    1.1914   -1.1727
```

4.



L4 Time Histories for Feedback Controlled
Cart Pendulum System for $\delta \in [0, 0.15°]$ - T. Koike
y [m]
θ_1 [rad]
θ_2 [rad]
time, [sec]

$\delta = 0.15°$

L7 (P4 & E1):

1.

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -1.0000   -3.0000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    1.0000

C = 1×6
     1     0     0     0     0     0

D = 0
```

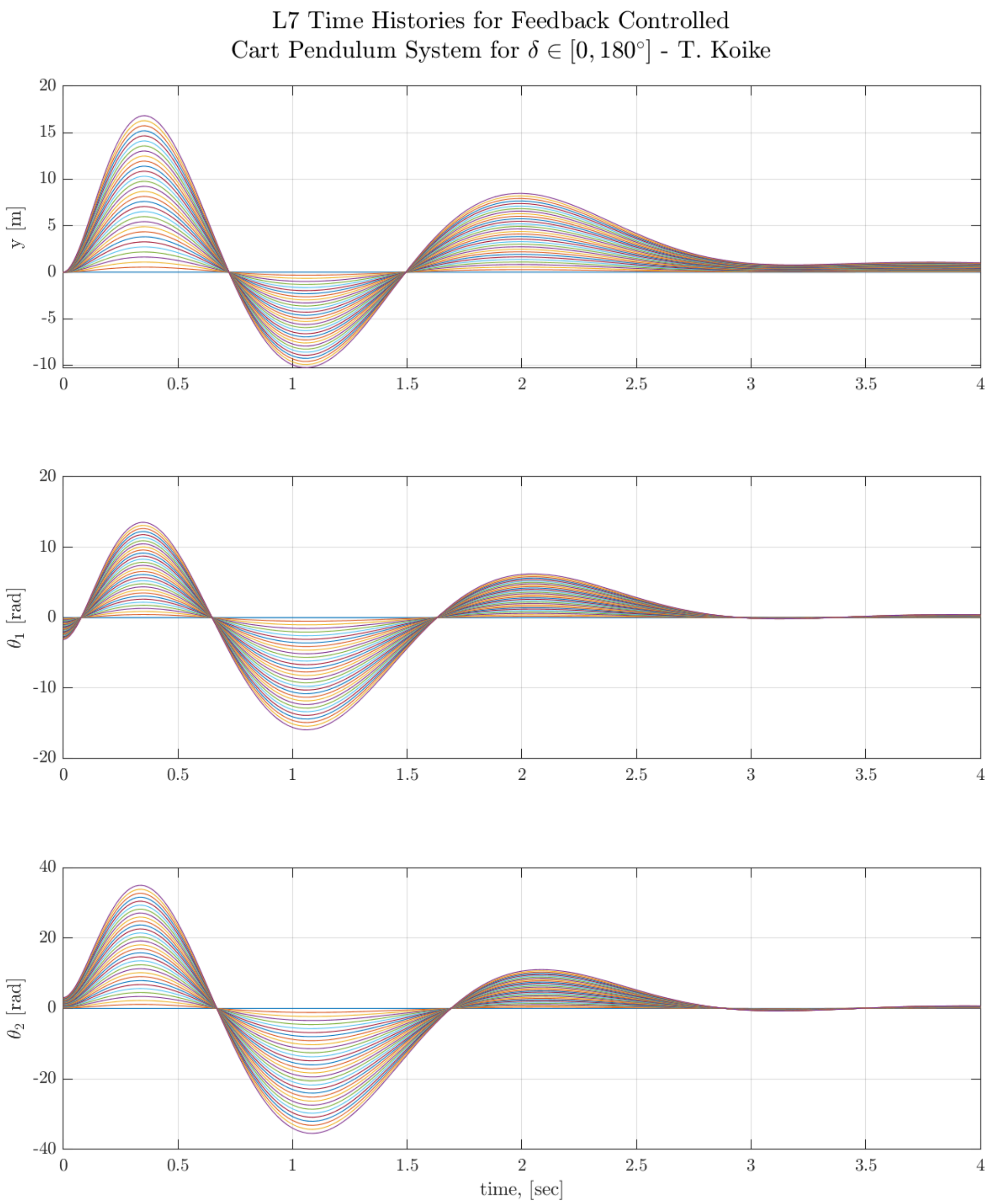
2.

```
p =
    -23/15     + 1207/363i      -11/50     +    0i           -3        +    0i
    -23/15     - 1207/363i       -3/2      +    0i           -9/2      +    0i
```

3.

```
K = 1×6
   59.7291  627.9728 -279.9391  358.1599 -278.3304  -27.6281
```

4.


L7 Time Histories for Feedback Controlled
Cart Pendulum System for $\delta \in [0, 180^\circ]$ - T. Koike
y [m]
θ_1 [rad]
θ_2 [rad]
time, [sec]

$\delta = 180°$

L8 (P4 & E2):

1.

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0    1.5000    0.5000         0         0         0
         0    1.0000    3.0000         0         0         0
```

```
B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -1.0000
```

```
C = 1×6
     1     0     0     0     0     0
```

```
D = 0
```

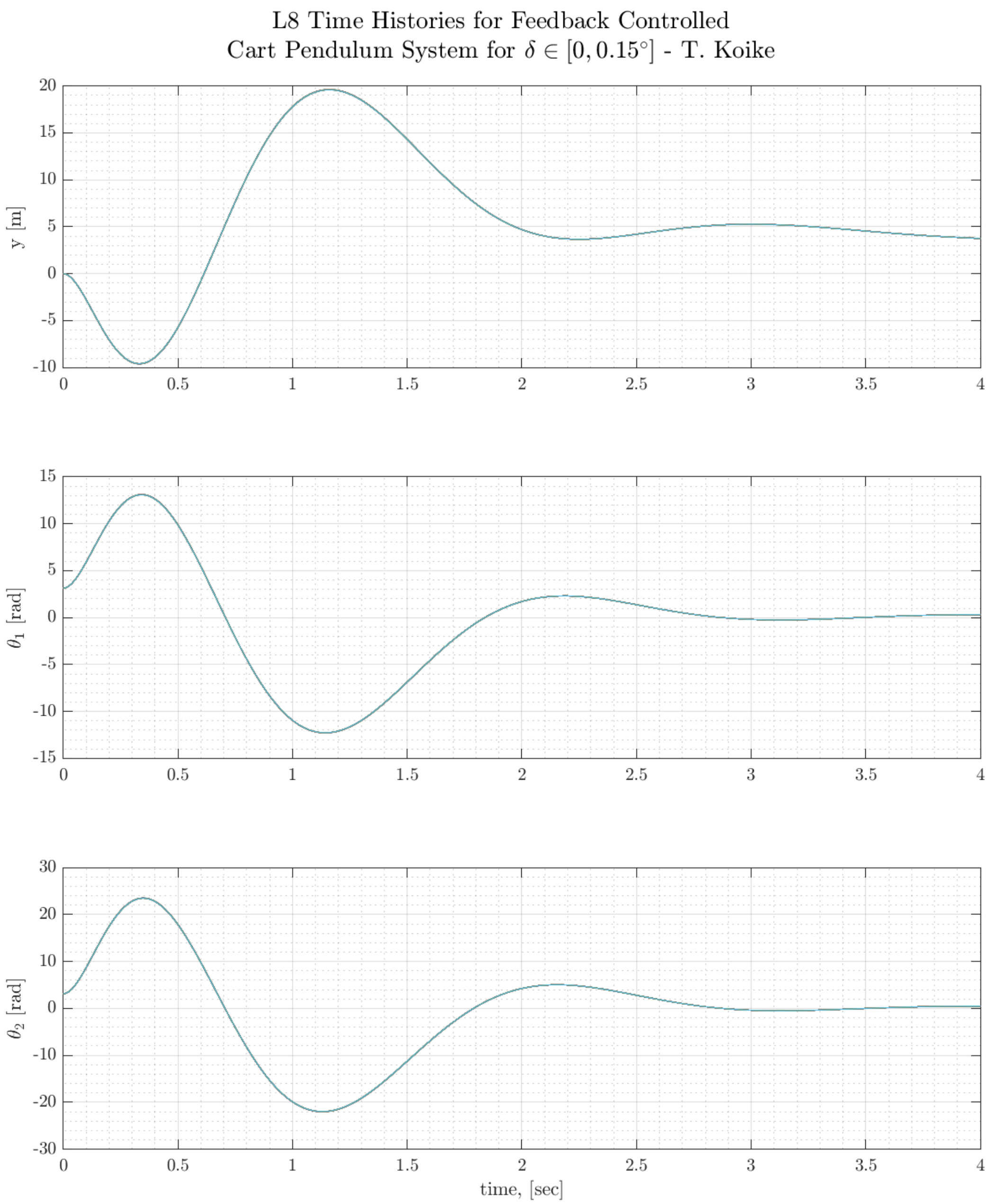
2.

```
p =
    -23/15     + 1207/363i      -11/50     +    0i          -3        +    0i
    -23/15     - 1207/363i       -3/2      +    0i          -9/2      +    0i
```

3.

```
K = 1×6
10³ ×
    0.0597    1.1405   -0.6133    0.3582    1.2035   -0.4349
```

4.


L8 Time Histories for Feedback Controlled
Cart Pendulum System for $\delta \in [0, 0.15°]$ - T. Koike
y [m]
θ_1 [rad]
θ_2 [rad]
time, [sec]

$\delta = 0.15°$

Exercise 6

(By hand) Consider the discrete time system

$$x(k+1) = x_1(k) + x_2(k)$$

$$x_2(k+1) = x_2(k) + u(k)$$

Obtain a state feedback controller which always drives the state of this system to zero in at most two steps.

The state matrices become

$$A_d = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The controllability matrix is

$$Q_c = (B \quad AB) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore rank(Q_c) = 2 \ .$$

This system is controllable.

Define a gain K so that $u = -Kx$.

$$A_d + B_d K = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (k_1 \quad k_2) = \begin{pmatrix} 1 & 1 \\ k_1 & k_2 + 1 \end{pmatrix}$$

Take the determinant, and it becomes

$$\lambda^2 + (-k_2 - 2)\,\lambda - k_1 + k_2 + 1 = 0$$

If the system goes to zero in two steps (or two seconds with a sampling time of one second), the Cayley-Hamilton Theorem implies that the following should be satisfied

$$A_d^2 = 0$$

Thus, the characteristic polynomial of the desired poles become

$$s^2 = 0$$

Thus, we solve

$$-k_2 - 2 = 0$$

$$-k_1 + k_2 + 1 = 0$$

Which gives us

$$K = (-1 \quad -2) \ .$$

Exercise 7

For the system described by

$$2\ddot{q}_1 + \ddot{q}_2 - q_2 = u_1$$

$$\ddot{q}_1 + 2\ddot{q}_2 - q_1 = u_2$$

Obtain a feedback controller generating u_1 and u_2 which stabilizes this system. Assume that $q_1, q_2, \dot{q}_1, \dot{q}_2$ can be measured. You can use MATLAB for some of this.

Manipulating the given equations, we get the following relations

$$\ddot{q}_1 = -\frac{1}{3}q_1 + \frac{2}{3}q_2 + \frac{2}{3}u_1 - \frac{1}{3}u_2$$

$$\ddot{q}_2 = -\frac{1}{3}q_1 + \frac{2}{3}q_2 - \frac{1}{3}u_1 + \frac{2}{3}u_2$$

Let, $x_1 \coloneqq q_1, x_2 \coloneqq q_2, x_3 \coloneqq \dot{q}_1, x_4 \coloneqq \dot{q}_2$. Then A, B matrices of this system are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

The controllability matrix becomes

$$Q_c = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 & -\frac{4}{9} & \frac{5}{9} \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 & -\frac{4}{9} & \frac{5}{9} \\ \frac{2}{3} & -\frac{1}{3} & 0 & 0 & -\frac{4}{9} & \frac{5}{9} & 0 & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 & -\frac{4}{9} & \frac{5}{9} & 0 & 0 \end{bmatrix}$$

The row reduced echelon form of this matrix is

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{4}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & -\frac{4}{3} & \frac{5}{3} \end{bmatrix}$$

Thus,

$$rank(Q_c) = 4$$

And this system is controllable, so we can choose any poles to control this system. The arbitrary poles that we choose are

$$\lambda = [-1.25 + 2.5j \quad -1.25 - 2.5j \quad -0.3536 \quad -0.7071]\,.$$

Now we find the controller gains using the Brogan's Algorithm

Step 1:

Find

$$\phi = (xI_{n\times n} - A)^{-1}$$

$$\Rightarrow \begin{bmatrix} -\frac{3\,x^2-2}{x-3\,x^3} & -\frac{2}{x-3\,x^3} & -\frac{3\,x^2-2}{x^2-3\,x^4} & -\frac{2}{x^2-3\,x^4} \\ \frac{1}{x-3\,x^3} & -\frac{3\,x^2+1}{x-3\,x^3} & \frac{1}{x^2-3\,x^4} & -\frac{3\,x^2+1}{x^2-3\,x^4} \\ -\frac{1}{3\,x^2-1} & \frac{2}{3\,x^2-1} & -\frac{3\,x^2-2}{x-3\,x^3} & -\frac{2}{x-3\,x^3} \\ -\frac{1}{3\,x^2-1} & \frac{2}{3\,x^2-1} & \frac{1}{x-3\,x^3} & -\frac{3\,x^2+1}{x-3\,x^3} \end{bmatrix}$$

Step 2:

Compute

$$\psi = \phi B$$

$$\Rightarrow \begin{bmatrix} \frac{0.6667}{x^2-3\,x^4} - \frac{0.6667\,(3\,x^2-2)}{x^2-3\,x^4} & \frac{0.3333\,(3\,x^2-2)}{x^2-3\,x^4} - \frac{1.3333}{x^2-3\,x^4} \\ \frac{0.3333\,(3\,x^2+1)}{x^2-3\,x^4} + \frac{0.6667}{x^2-3\,x^4} & -\frac{0.6667\,(3\,x^2+1)}{x^2-3\,x^4} - \frac{0.3333}{x^2-3\,x^4} \\ \frac{0.6667}{x-3\,x^3} - \frac{0.6667\,(3\,x^2-2)}{x-3\,x^3} & \frac{0.3333\,(3\,x^2-2)}{x-3\,x^3} - \frac{1.3333}{x-3\,x^3} \\ \frac{0.3333\,(3\,x^2+1)}{x-3\,x^3} + \frac{0.6667}{x-3\,x^3} & -\frac{0.6667\,(3\,x^2+1)}{x-3\,x^3} - \frac{0.3333}{x-3\,x^3} \end{bmatrix}$$

Step 3:

Calculate

$$\bar{\psi} = [\psi_1(\lambda_1) \quad \psi_1(\lambda_2) \quad \psi_2(\lambda_3) \quad \psi_2(\lambda_4)]$$

Where $\psi_1(x), \psi_2(x)$ correspond to the columns of ψ.

$$\bar{\psi} = \begin{bmatrix} -0.0494 + 0.0751\,j & -0.0494 - 0.0751\,j & 0.5000 & 3.8362\text{e-}06 \\ 0.0274 - 0.0273\,j & 0.0274 + 0.0273\,j & 1.5000 & 0.5000 \\ -0.1261 - 0.2175\,j & -0.1261 + 0.2175\,j & -0.5000 & -5.4251\text{e-}06 \\ 0.0339 + 0.1025\,j & 0.0339 - 0.1025\,j & -1.5000 & -0.7071 \end{bmatrix}$$

Step 4:

Find the gains with

$$K = -E\bar{\psi}^{-1}$$

Where

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus,

$$K = -\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -0.0494 + 0.0751\,j & -0.0494 - 0.0751\,j & 0.5000 & 3.8362\text{e-}06 \\ 0.0274 - 0.0273\,j & 0.0274 + 0.0273\,j & 1.5000 & 0.5000 \\ -0.1261 - 0.2175\,j & -0.1261 + 0.2175\,j & -0.5000 & -5.4251\text{e-}06 \\ 0.0339 + 0.1025\,j & 0.0339 - 0.1025\,j & -1.5000 & -0.7071 \end{bmatrix}$$

$$K = \begin{bmatrix} 8.5075 & -8.3006 & 1.2140 & -5.8694 \\ 0.6225 & 3.7283 & 1.6559 & 4.0505 \end{bmatrix}$$

And

$$A_{cl} = A - BK = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5.7975 & 7.4432 & -0.2574 & 5.2631 \\ 2.0875 & -4.5858 & -0.6993 & -4.6568 \end{bmatrix}.$$

The eigenvalues become

$$eig(A_{cl}) = \begin{bmatrix} -1.4142 \\ -1 \\ -1.2500 - 2.5000\,j \\ -1.2500 + 2.5000\,j \end{bmatrix}$$

Exercise 8

Consider the system described by

$$\dot{x}_1 = x_1 + u$$

$$\dot{x}_2 = -x_2 + 2u$$

Obtain (by hand) a state-feedback controller (it will not be a static controller) which always results in the state of the closed loop going to zero in at most 2 secs. Illustrate your results with a simulation.

The A and B matrices of this continuous time system is

$$A_c = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The discrete time state space becomes

$$A_d = e^{A_c T},\ \mathrm{B_d} = \left(\int_0^{\mathrm{T}} \mathrm{e}^{\mathrm{A_c}\,\tau}\, \mathrm{d}\tau \right) \mathrm{B_c}$$

Let the sampling time, T be 1 second. Then

$$A_d = e^{A_c},\ B_d = A_d B_c$$

$$\Rightarrow A_d = \begin{bmatrix} 2.7183 & 0 \\ 0 & 0.3679 \end{bmatrix}, \qquad B_d = \begin{bmatrix} 1.7183 \\ 1.2642 \end{bmatrix}$$

The controllability matrix is

$$Q_c = (B_d \quad A_d B_d) = \begin{bmatrix} 1.7183 & 4.6708 \\ 1.2642 & 0.4651 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore rank(Q_c) = 2 \ .$$

This system is controllable.

Define a gain K so that $u = -Kx$.

$$\begin{aligned} A_d + B_d K &= \begin{bmatrix} 2.7183 & 0 \\ 0 & 0.3679 \end{bmatrix} + \begin{bmatrix} 1.7183 \\ 1.2642 \end{bmatrix} [k_1 \quad k_2] \\ &= \begin{bmatrix} 1.7183\, k_1 + 2.7183 & 1.7183\, k_2 \\ 1.2642\, k_1 & 1.2642\, k_2 + 0.3679 \end{bmatrix} \end{aligned}$$

Take the determinant, and it becomes

$$\lambda^2 + (-1.7183\, k_1 - 1.2642\, k_2 - 3.0862)\,\lambda + 0.6321\, k_1 + 3.4366\, k_2 + 1.0000 = 0$$

If the system goes to zero in two steps (or two seconds with a sampling time of one second), the Cayley-Hamilton Theorem implies that the following should be satisfied

$$A_d^2 = 0$$

Thus, the characteristic polynomial of the desired poles become

$$s^2 = 0$$

Thus, we solve

$$-1.7183\,k_1 - 1.2642\,k_2 - 3.0862 = 0$$

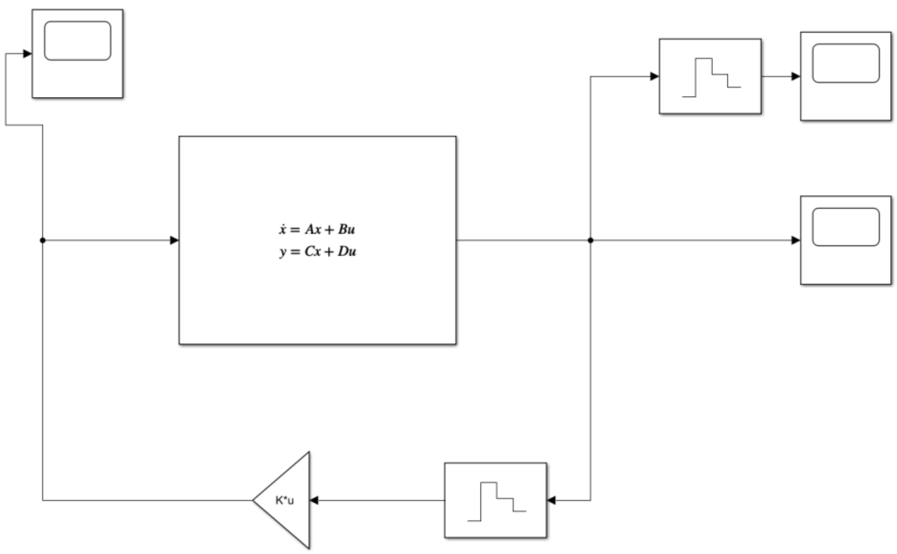
$$0.6321\,k_1 + 3.4366\,k_2 + 1.0000 = 0$$

Solving these linear equations, we obtain

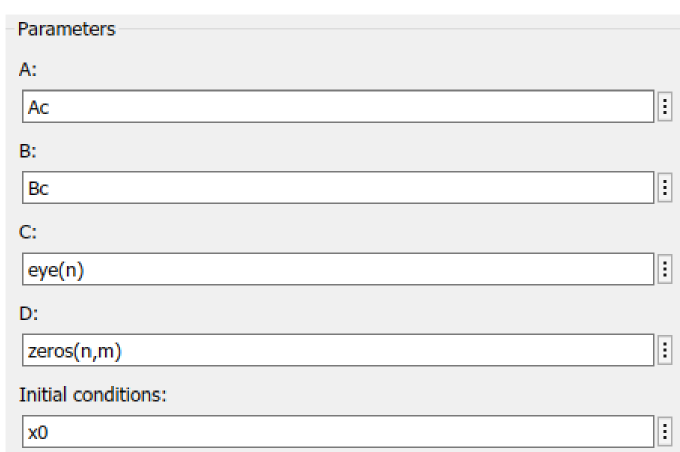
$$k_1 = -1.8296, \qquad k_2 = 0.0455$$

$$K = [-1.8296 \quad 0.0455]$$

Now we create the following Simulink to simulate the system.



Where the state space block has


Parameters

A:

Ac

B:

Bc

C:

eye(n)

D:

zeros(n,m)

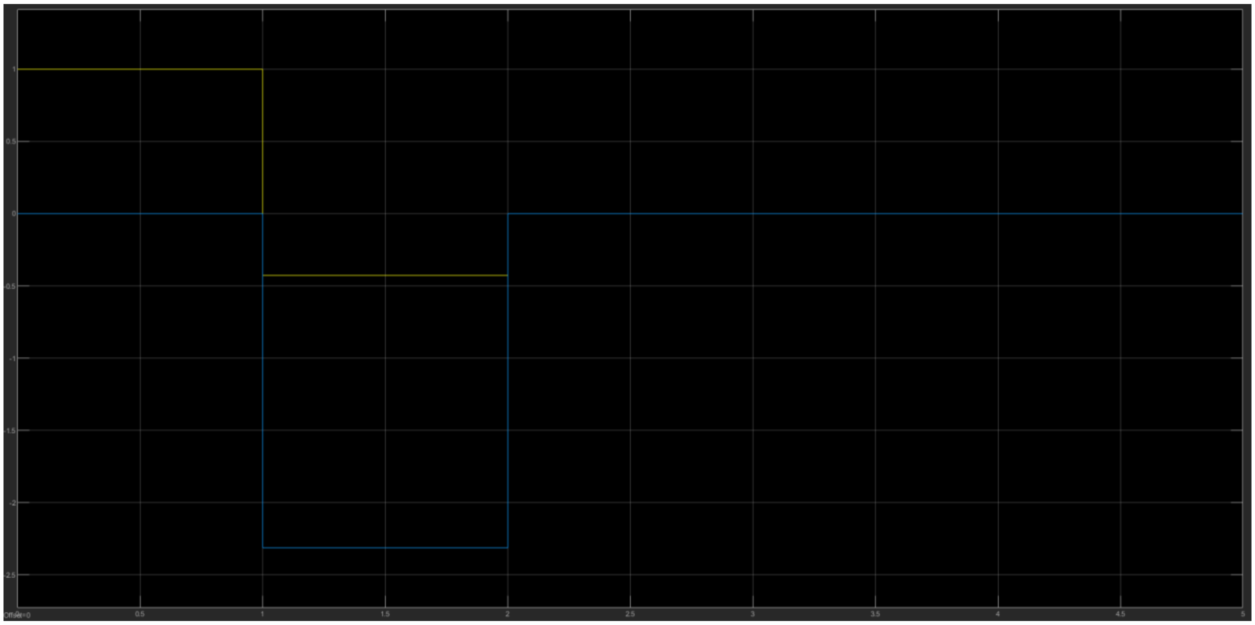
Initial conditions:

x0

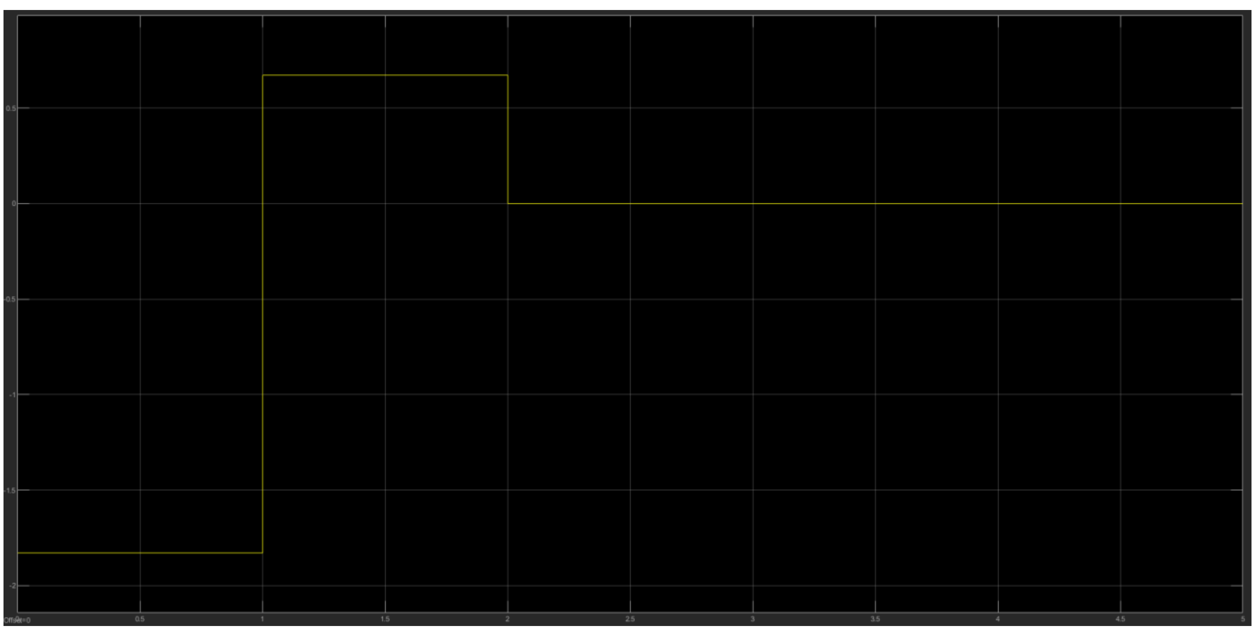
And

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The input is



The discrete output is



The continuous output is