AAE 567 Spring 2021 Midterm 1 Tomoki Koike

Problem 1

(i) $A = \begin{pmatrix} 1 & 1 \\ a & b \\ a^2 & b^2 \\ a^3 & b^3 \\ \vdots & \vdots \end{pmatrix}$ where $a = \frac{1}{2}$, $b = \frac{2}{3}$

$$\begin{cases} a^2 = \frac{1}{4} & b^2 = \frac{4}{9} \\ ab = \frac{1}{3} & ac = \frac{3}{8} \\ bc = \frac{1}{2} \end{cases}$$

$y = \begin{pmatrix} 1 \\ c \\ c^2 \\ c^3 \\ \vdots \end{pmatrix}$ where $c = \frac{3}{4}$

here we can say that from projection theorem

$$\hat{x} = (A^*A)^{-1}A^*y$$

$$A^*A = \begin{pmatrix} 1 & a & a^2 & a^3 & \cdots \\ 1 & b & b^2 & b^3 & \cdots \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a & b \\ a^2 & b^2 \\ a^3 & b^3 \\ \vdots & \vdots \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\infty} a^{2j} & \sum_{j=0}^{\infty} (ab)^j \\ \sum_{j=0}^{\infty} (ab)^j & \sum_{j=0}^{\infty} b^{2j} \end{pmatrix}$$

using : if $|r| < 1$ $\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$

$$A^*A = \begin{pmatrix} \frac{1}{1-\frac{1}{4}} & \frac{1}{1-\frac{1}{3}} \\ \frac{1}{1-\frac{1}{3}} & \frac{1}{1-\frac{4}{9}} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & \frac{3}{2} \\ \frac{3}{2} & \frac{4}{5} \end{pmatrix}$$

$$(A^*A)^{-1} = \begin{pmatrix} 12 & -10 \\ -10 & \frac{80}{9} \end{pmatrix}$$

$$A^*y = \begin{pmatrix} 1 & a & a^2 & a^3 \cdots \\ 1 & b & b^2 & b^3 \cdots \end{pmatrix} \begin{pmatrix} 1 \\ c \\ c^2 \\ c^3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \sum_{j=0}^{\infty} (ac)^j \\ \sum_{j=0}^{\infty} (bc)^j \end{pmatrix}$$

similar to previous calculations

$$A^*y = \begin{pmatrix} \frac{1}{1-3/8} \\ \frac{1}{1-1/2} \end{pmatrix} = \begin{pmatrix} \frac{8}{5} \\ 2 \end{pmatrix}$$

$$\therefore \quad \hat{x} = (A^*A)^{-1}A^*y = \begin{pmatrix} 12 & -10 \\ -10 & 80/9 \end{pmatrix} \begin{pmatrix} \frac{8}{5} \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} -\frac{4}{5} \\ \frac{16}{9} \end{pmatrix}}$$

(ii) This is unique $\boxed{\text{TRUE}}$

(iii)

$$d^2 = \|y\|^2 - \langle \hat{x}, A^*y \rangle$$

$$\|y\|^2 = \sum_{j=0}^{\infty} \left(\frac{3}{4}\right)^j = \frac{1}{1-\frac{3}{4}} = 4$$

$$\langle \hat{x}, A^*y \rangle = \begin{pmatrix} -4/5 \\ 16/9 \end{pmatrix} \cdot \begin{pmatrix} 8/5 \\ 2 \end{pmatrix} = -\frac{32}{25} + \frac{32}{9} = \frac{512}{225}$$

$$d^2 = 4 - \frac{512}{225} = \frac{388}{225}$$

$$\boxed{d^2 = 1.7244}$$

Problem 2

$$A = \begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix} \quad c = \begin{pmatrix} 2 & 3 \end{pmatrix} \quad f(t) = 26e^{-3t}$$

(i) $Q_0 = \begin{pmatrix} C \\ CA \end{pmatrix}$ $\quad CA = \begin{pmatrix} 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 6 \\ -2 & -5 \end{pmatrix} = \begin{pmatrix} -2 & -3 \end{pmatrix}$

$Q_0 = \begin{pmatrix} 2 & 3 \\ -2 & -3 \end{pmatrix}$ $\quad Q_0 = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}$ $\quad \mathrm{rank}(Q_0) = 1$ $\quad \boxed{\text{FALSE}}$

(ii) Not an unique optimal solution $\boxed{\text{FALSE}}$

(iii)

$$e^{At} = \begin{pmatrix} 4e^{-t} - 3e^{-2t} & 6e^{-t} - 6e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 4e^{-2t} - 3e^{-t} \end{pmatrix}$$

$$P = \int_0^\infty e^{A^*t} C^* C e^{At} \, dt = \begin{pmatrix} 2 & 3 \\ 3 & 4.5 \end{pmatrix}$$

$$\hat{x} = (P^*P)^{-1} P \int_0^\infty e^{A^*t} c \, f(t) \, dt \quad \because P^\dagger = \begin{pmatrix} 0.0473 & 0.0710 \\ 0.0710 & 0.1065 \end{pmatrix}$$

$$\boxed{\hat{x} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \mathrm{Null}(P)} \quad \mathrm{Null}(P) = q\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$\boxed{\hat{x}_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + q\begin{pmatrix} 3 \\ -2 \end{pmatrix}} \quad \hat{x}_{opt} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

(iv) $d^2 = \int_0^\infty \| 26e^{-3t} - Ce^{At}\hat{x}_{opt} \| \, dt$

$= 28.1667$

Problem 3

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & o/w \end{cases} \qquad f_U(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & o/w \end{cases}$$

$$f_V(v) = \begin{cases} 1 & 0 \le v \le 1 \\ 0 & o/w \end{cases} \qquad EX = EU = EV = 0.5$$

$$\mathcal{H} = \text{span}\{1, y\}$$

(i)

$$g = \begin{pmatrix} 1 \\ y \end{pmatrix}$$

$$R_{xg} = EXEg = EXE(1 \;\; y) = (EX \;\; EXY)$$

$$EXY = E\,x(x+u+v) = Ex^2 + Exu + Exv$$

$$Ex^2 = \int_0^1 x^2 dx = \frac{1}{3}$$

$$EXY = \frac{1}{3} + \frac{1}{4} + \frac{1}{4} = \frac{1}{3} + \frac{2}{4} = \frac{4}{12} + \frac{6}{12} = \frac{5}{6}$$

$$R_{xg} = \left(\frac{1}{2} \;\; \frac{5}{6}\right)$$

$$R_g = Egg^* = E\begin{pmatrix} 1 \\ y \end{pmatrix}(1 \;\; y) = \begin{pmatrix} E1 & Ey \\ Ey & Ey^2 \end{pmatrix}$$

$$Ey = EX + EU + EV = 1.5$$

$$Ey^2 = E(x^2 + u^2 + v^2 + 2xu + 2uv + 2xv)$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$

$$= 1 + 1.5 = 2.5$$

$$R_g = \begin{pmatrix} 1 & 1.5 \\ 1.5 & 2.5 \end{pmatrix}$$

$$(\alpha\ \beta) = R_{xy} R_y^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{5}{6} \end{pmatrix} \begin{pmatrix} 1 & 1.5 \\ 1.5 & 2.5 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \end{pmatrix}$$

$$\boxed{\begin{aligned} \alpha &= 0 \\ \beta &= 1/3 \end{aligned}}$$

$$\boxed{\hat{x} = \frac{1}{3} y}$$

(ii) $E(x - \hat{x})^2$

$= R_x - R_{xy} R_y^{-1} R_{yx}$

$= EX^2 - \begin{pmatrix} 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1/2 \\ 5/6 \end{pmatrix}$

$= 1/3 - \frac{5}{18}$

$= 0.1556$

$$\boxed{E(x - \hat{x})^2 = 0.0556}$$

Problem 4

$X, V \quad f_X(x) = \begin{cases} xe^{-x} & \text{if } x \geq 0 \\ 0 & \text{o/w} \end{cases}$

$f_V(v) = \begin{cases} e^{-v} & \text{if } v \geq 0 \\ 0 & \text{o/w} \end{cases}$

$y = x + v$

$$EX = \int_0^\infty x^2 e^{-x} dx = -x^2 e^{-x} \Big|_0^\infty + 2\int x e^{-x} dx$$

$$= -x^2 e^{-x} \Big|_0^\infty - 2xe^{-x} \Big|_0^\infty + 2\int e^{-x} dx$$

$$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} \Big|_0^\infty = 0 - (-2) = 2$$

$$EV = \int_0^\infty ve^{-v} dv = -ve^{-v} \Big|_0^\infty + \int e^{-v} dv = -ve^{-v} - e^{-v} \Big|_0^\infty$$

$$= 1$$

$$f_{X,Y}(x,y) = f_{X,V}(x,v) \left| \det\left(\nabla(xe^{-x}, xe^{-x} + e^{-v})\right) \right|^{-1}$$

$$\sim\sim = \begin{vmatrix} e^{-x} - xe^{-x} & 0 \\ e^{-x} - xe^{-x} & -e^{-v} \end{vmatrix}^{-1} = \left[-e^{-v}(e^{-x}(1-x))\right]^{-1}$$

$$f_{X,Y}(x,y) = \frac{(xe^{-x})(e^{-v})}{(-e^{-v})(e^{-x})(1-x)} = \frac{x}{x+1}$$

$$f_Y(y) = \int_0^y \frac{x}{x+1}\,dx$$

$$= \int_0^y \left(1 - \frac{1}{x+1}\right) dx = \Big[\, x - \ln|x+1| \,\Big]_0^y$$

$$= y - \ln|y+1|$$

$$\hat{g}(y) = \int_0^y x \, \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx$$

$$= \frac{1}{y - \ln|y+1|} \int_0^y \frac{x^2}{x+1}\, dx$$

$$= \frac{1}{y - \ln|y+1|} \int_0^y \left(x + \frac{1}{x+1} - 1 \right) dx$$

$$= \frac{1}{y - \ln|y+1|} \left[\frac{x^2}{2} + \ln|x+1| - x \right]_0^y$$

$$= \frac{\frac{y^2}{2} + \ln|y+1| - y}{y - \ln|y+1|}$$

$$\hat{g}(y) = \frac{\frac{y^2}{2} + \ln|y+1| - y}{y - \ln|y+1|}$$

$$\frac{(x+1)^2 - 2x - 1}{x+1}$$

$$x + 1 - \frac{2x}{x+1} - \frac{1}{x+1}$$

$$= x + 1 - \left[\frac{2(x+1)}{x+1} - \frac{2}{x+1} \right] - \frac{1}{x+1}$$

$$= x + 1 - 2 + \frac{2}{x+1} - \frac{1}{x+1}$$

$$= x + \frac{1}{x+1} - 1$$

(ii)

$$R_{fg} = EXE(1\ y) = (EX\ \ EXY)$$

$$EXY = EX(X+V) = EX^2 + EXV$$

$$EX^2 = (2+1)! = 6$$

$$EXV = EXEV = 2 \cdot 1 = 2$$

$$EXY = 6 + 2 = 8$$

$$R_{fg} = (2\ \ 8)$$

$$R_g = \begin{pmatrix} E1 & EY \\ EY & EY^2 \end{pmatrix}$$

$$EY = EX + EV = 2 + 1 = 3$$

$$EY^2 = E(X+V)^2 = EX^2 + 2EXEV + EV^2$$

$$EX^2 = 6 \qquad EV^2 = \int_0^\infty v^2 e^{-v}\, dv = 2$$

$$EY^2 = 6 + 2 \cdot 2 \cdot 1 + 2 = 12$$

$$R_g = \begin{pmatrix} 1 & 3 \\ 3 & 12 \end{pmatrix}$$

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = (2\ \ 8) \begin{pmatrix} 1 & 3 \\ 3 & 12 \end{pmatrix}^{-1} = \boxed{(0 \quad 0.6667)}$$