AAE 440: Spacecraft Attitude Dynamics

PS3: Successive Rotations and Orientational Angles

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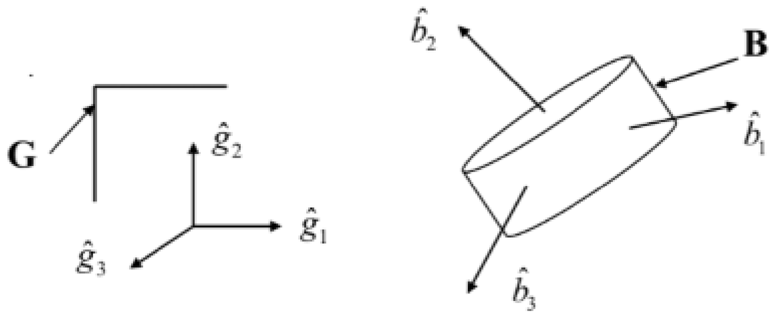
Problem 1: Assume that a rigid body B (e.g., a spacecraft) can move with respect to a frame G. Let a dextral set of orthogonal unit vectors $\hat{b}_i$ be fixed in the body B; unit vectors $\hat{g}_i$ are fixed in G such that $\hat{b}_i = \hat{g}_i$ initially.



Initially $\hat{b}_i = \hat{g}_i$. B then undergoes two rotations in succession as follows:

Rot #1	A rotation described by the following set of Euler parameters $\bar{\varepsilon} = .4\hat{g}_1 - .1\hat{g}_2 + .2\hat{g}_3$, $\varepsilon_4 > 0$
Rot #2	A $+60^\circ$ rotation about a line parallel to $\hat{g}_3$

(a) For each rotation, write the corresponding direction cosine matrix and the set of Euler parameters. Find the final orientation of B in G and represent it in terms of both ${}^G C^B$ and ${}^G\bar{\varepsilon}^B$, ${}^G\varepsilon_4^B$. But demonstrate that you generate the same result either of two ways: (i) use the direction cosine rule for successive rotations; (ii) use only the Euler parameter rule for successive rotations.

Define the rotation as a sequence

$G \rightarrow B' \rightarrow B$ where B' is the intermediate

$\langle G \rightarrow B' \rangle$

from the provided $\bar{\varepsilon} = 0.4\hat{g}_1 - 0.1\hat{g}_2 + 0.2\hat{g}_3$ $\varepsilon_4 > 0$

since this represents rotation # 1 $(B \rightarrow G')$

$${}^G\varepsilon_1^{B'} = 0.4, \quad {}^G\varepsilon_2^{B'} = -0.1, \quad {}^G\varepsilon_3^{B'} = 0.2$$

using the relation

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1$$

$${}^G\varepsilon_4^{B'} = \sqrt{1 - 0.4^2 - (-0.1)^2 - 0.2^2} \simeq 0.88882$$

the DCM ${}^{G}C^{B'}$ becomes

$$
\begin{aligned}
C_{11} &= 1 - \varepsilon_2^2 - 2\varepsilon_3^2 &&= 0.9 \\
C_{12} &= 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) &&= -0.4355 \\
C_{13} &= 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4) &&= -0.0178 \\
C_{21} &= 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) &&= 0.2755 \\
C_{22} &= 1 - 2\varepsilon_3^2 - 2\varepsilon_1^2 &&= 0.6 \\
C_{23} &= 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) &&= -0.7511 \\
C_{31} &= 2(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4) &&= 0.3378 \\
C_{32} &= 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) &&= 0.6711 \\
C_{33} &= 1 - 2\varepsilon_1^2 - 2\varepsilon_2^2 &&= 0.6600
\end{aligned}
$$

calculated by MATLAB

$$\Downarrow$$

$$\therefore \quad {}^{G}C^{B'} = \begin{bmatrix} 0.9 & -0.4355 & -0.0178 \\ 0.2755 & 0.6 & -0.07511 \\ 0.3378 & 0.6711 & 0.6600 \end{bmatrix}$$

<B' → B>

${}^{B'}\hat{\lambda}^{B} = \hat{g}_3$ is given, this is in the g-vector basis

convert this into the b'-vector basis

$$\rightarrow {}^{B'}\hat{\lambda}^{B} = {}^{B'}\hat{\lambda}^{B}\,{}^{G}C^{B'}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.9 & -0.4355 & -0.0178 \\ 0.2755 & 0.6 & -0.07511 \\ 0.3378 & 0.6711 & 0.6600 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3378 & 0.6711 & 0.6600 \end{bmatrix} = \begin{bmatrix} {}^{B'}\lambda^{B}_1 & {}^{B'}\lambda^{B}_2 & {}^{B'}\lambda^{B}_3 \end{bmatrix}$$

rotation angle $\theta = 60^\circ = \frac{\pi}{3}$

then the DCM ${}^{B'}C^{B}$ becomes

denote ${}^{B'}\lambda_1^{B}$, ${}^{B'}\lambda_2^{B}$, ${}^{B'}\lambda_3^{B}$ as λ_1 , λ_2 , λ_3

for convenience

$$
\begin{aligned}
C_{11} &= \cos\theta + \lambda_1^2(1-\cos\theta) &&= 0.5570 \\
C_{12} &= -\lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta) &&= -0.4582 \\
C_{13} &= \lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta) &&= 0.6926 \\
C_{21} &= \lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta) &&= 0.6849 \\
C_{22} &= \cos\theta + \lambda_2^2(1-\cos\theta) &&= 0.7252 \\
C_{23} &= -\lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta) &&= -0.0711 \\
C_{31} &= -\lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta) &&= -0.04697 \\
C_{32} &= \lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta) &&= 0.5140 \\
C_{33} &= \cos\theta + \lambda_3^2(1-\cos\theta) &&= 0.7178
\end{aligned}
$$

calculated by MATLAB

$$\Downarrow$$

$$\therefore \; {}^{B'}C^{B} = \begin{bmatrix} 0.5570 & -0.4582 & 0.6926 \\ 0.6849 & 0.7252 & -0.0711 \\ -0.4697 & 0.5140 & 0.7178 \end{bmatrix}$$

then the Euler parameters are computed

denote ${}^{B'}C_{11}^{B} = C_{11}$, ${}^{B'}C_{12}^{B} = C_{12}$ $\cdots$ and so on

$${}^{B'}\varepsilon_4^{B} = \frac{1}{2}\sqrt{1 + C_{11} + C_{22} + C_{33}} = 0.8660$$

$${}^{B'}\varepsilon_1^{B} = \frac{C_{32} - C_{23}}{4\varepsilon_4} = 0.1689$$

$${}^{B'}\varepsilon_2^{B} = \frac{C_{13} - C_{31}}{4\varepsilon_4} = 0.3355$$

$${}^{B'}\varepsilon_3^{B} = \frac{C_{21} - C_{12}}{4\varepsilon_4} = 0.3300$$

calculate with MATLAB

then

$$\boxed{({}^{G}C^{B})_1 = {}^{G}C^{B'}\,{}^{B'}C^{B} = \begin{bmatrix} 0.2114 & -0.7374 & 0.6416 \\ 0.9172 & -0.0772 & -0.3909 \\ 0.3378 & 0.6711 & 0.6600 \end{bmatrix}}$$

Euler parameters becomes ${}^{G}\varepsilon^{B}$, ${}^{G}\varepsilon_4^{B}$

$${}^{G}\bar{\varepsilon}^{B} = {}^{G}\bar{\varepsilon}^{B'}\,{}^{B'}\varepsilon_4^{B} + {}^{B'}\varepsilon^{B}\,{}^{G}\varepsilon_4^{B'} + {}^{B'}\varepsilon^{B} \times {}^{G}\varepsilon^{B'}$$

$$= \boxed{0.5966\,\hat{b}_1' + 0.3098\,\hat{b}_2' + 0.3154\,\hat{b}_3'}$$

$${}^{G}\varepsilon_4^{B} = {}^{G}\varepsilon_4^{B'}\,{}^{B'}\varepsilon_4^{B} - {}^{G}\varepsilon^{B'} \cdot {}^{B'}\varepsilon^{B}$$

$${}^{G}\varepsilon_4^{B} = \boxed{0.6697}$$

now convert this ${}^{G}\bar{\varepsilon}^{B}$ to b-vector basis

$${}^{G}\bar{\varepsilon}^{B} = {}^{G}\bar{\varepsilon}^{B}\,{}^{B'}C^{B}$$

$${}^{G}\bar{\varepsilon}^{B} = \begin{array}{ccc} [\,0.5966 & 0.3098 & 0.3154\,] \\ \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \end{array} \begin{bmatrix} 0.5570 & -0.4582 & 0.6926 \\ 0.6049 & 0.7252 & -0.0711 \\ -0.4697 & 0.5140 & 0.7178 \end{bmatrix}$$

$$\boxed{{}^{G}\bar{\varepsilon}^{B} = [\,0.3964 \quad 0.1134 \quad 0.6176\,]}$$

compute ${}^{G}C^{B}$ from ${}^{G}\vec{\varepsilon}^{B}$ & ${}^{G}\varepsilon_4^{B}$ denote as $\sum_{i=1}^{4}\varepsilon_i$

$$
\begin{aligned}
C_{11} &= 1 - \varepsilon_2^2 - 2\varepsilon_3^2 = 0.2114 \\
C_{12} &= 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4) = -0.7374 \\
C_{13} &= 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4) = 0.6416 \\
C_{21} &= 2(\varepsilon_1\varepsilon_2 + \varepsilon_3\varepsilon_4) = 0.9172 \\
C_{22} &= 1 - 2\varepsilon_3^2 - 2\varepsilon_1^2 = -0.0772 \\
C_{23} &= 2(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4) = -0.3909 \\
C_{31} &= 2(\varepsilon_3\varepsilon_1 - \varepsilon_2\varepsilon_4) = 0.3378 \\
C_{32} &= 2(\varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_4) = 0.6711 \\
C_{33} &= 1 - 2\varepsilon_1^2 - 2\varepsilon_2^2 = 0.6600
\end{aligned}
$$

$$\Downarrow$$

$$
\boxed{({}^{G}C^{B})_2 = \begin{bmatrix} 0.2114 & -0.7374 & 0.6416 \\ 0.9172 & -0.0772 & -0.3909 \\ 0.3378 & 0.6711 & 0.6600 \end{bmatrix}}
$$

Calculate the error of $({}^{G}C^{B})_1$ & $({}^{G}C^{B})_2$

$$\text{error} = ({}^{G}C^{B})_1 - ({}^{G}C^{B})_2$$

$$
\boxed{\text{error} = \begin{bmatrix} -4.1633 & 0 & 2.2204 \\ 1.1102 & -1.8041 & 1.6653 \\ -2.7756 & 0 & -3.3307 \end{bmatrix} \times 10^{-16}}
$$

(b) Express the final result for ${}^G\bar{\varepsilon}^B$, ${}^G\varepsilon_4^B$ in terms of $\hat{g}, \hat{b}, \hat{b}'$. (Note that $\hat{b}'$ is the intermediate vector basis frozen after the first rotation.)

from results in (a)

→ b′-vector frame

$$\boxed{\begin{aligned} {}^G\bar{\varepsilon}^B &= [\,0.5966 \quad 0.3098 \quad 0.3154\,] \\ {}^G\varepsilon_4^B &= 0.6697 \end{aligned}}$$

→ b-vector frame

$$\boxed{\begin{aligned} {}^G\bar{\varepsilon}^B &= [\,0.3964 \quad 0.1134 \quad 0.6176\,] \\ {}^G\varepsilon_4^B &= 0.6697 \end{aligned}}$$

for g-vector frame

$$({}^G\bar{\varepsilon}^B)_g = ({}^G\bar{\varepsilon}^B)_{b'} \, ({}^G C^{b'})^T$$

$$= [\,0.5966 \quad 0.3098 \quad 0.3154\,] \begin{bmatrix} 0.9 & 0.2755 & 0.3378 \\ -0.4355 & 0.6 & 0.6711 \\ -0.0178 & -0.07511 & 0.6600 \end{bmatrix}$$

$$= \boxed{[\,0.3964 \quad 0.1134 \quad 0.6176\,]}$$

$$\boxed{({}^G\bar{\varepsilon}_4^B)_g = 0.6697}$$

(c) Determine the <u>equivalent single rotation</u> ${}^G\hat{\lambda}^B$, ${}^G\theta^B$ that orients B with respect to G att he final time.

$$ {}^G\hat{\lambda}^B = \frac{{}^G\varepsilon_1^B \hat{g}_1 + {}^G\varepsilon_2^B \hat{g}_2 + {}^G\varepsilon_3^B \hat{g}_3}{\sqrt{({}^G\varepsilon_1^B)^2 + ({}^G\varepsilon_2^B)^2 + ({}^G\varepsilon_3^B)^3}} $$

$$ \boxed{{}^G\hat{\lambda}^B = 0.5338\,\hat{g}_1 + 0.1527\,\hat{g}_2 + 0.8317\,\hat{g}_3} $$

$$ \theta = 2\arccos({}^G\varepsilon_4^B) \cdot \frac{180}{\pi} $$

$$ \boxed{\theta = 95.91^{0}} $$

Problem 2: Under Blackboard→Supplementary Materials is a file with the final form of the direction cosine matrices corresponding to various sets of angle sequences.

(a) Derive the final form for the direction cosine matrix for the following types of successive rotation sequences:

Body-three $1-2-3$
Body-two $1-2-1$

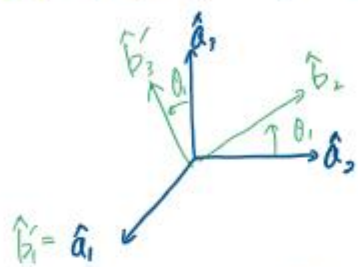
for convenience $\cos\theta = C_\theta$ $\sin\theta = S_\theta$

Body-three $1 - 2 - 3$

$\quad\;\uparrow\;\;\uparrow\;\;\uparrow$

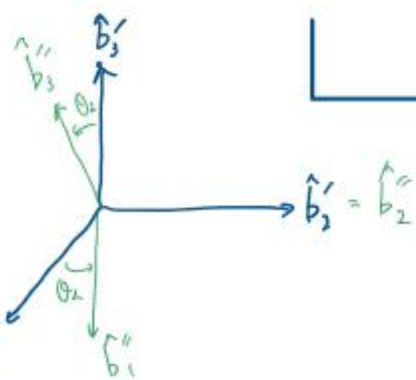
$\;\;$(i) (ii) (iii)

(i) rot #1 ${}^{A}\hat{\lambda}^{B'} = \hat{a}_1 = \hat{b}_1$



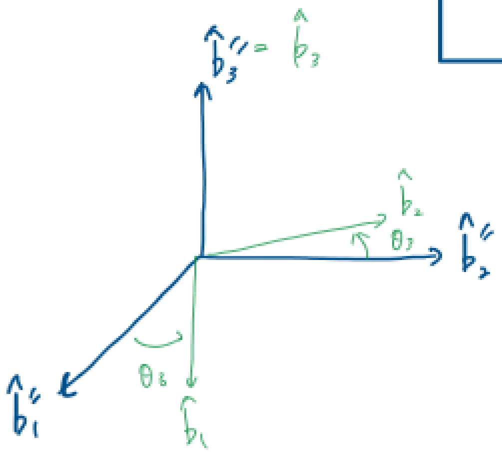
${}^{A}C^{B'}$	$\hat{b}_1'$	$\hat{b}_2'$	$\hat{b}_3'$
$\hat{a}_1$	1	0	0
$\hat{a}_2$	0	C_1	$-S_1$
$\hat{a}_3$	0	S_1	C_1

(ii) rot #2 ${}^{B'}\lambda^{B''} = \hat{b}_2' = \hat{b}_2''$



${}^{B'}C^{B''}$	$\hat{b}_1''$	$\hat{b}_2''$	$\hat{b}_3''$
$\hat{b}_1'$	C_2	0	S_2
$\hat{b}_2'$	0	1	0
$\hat{b}_3'$	$-S_2$	0	C_2

(iii) rot #3 $\quad {}^{B''}\lambda^{B} = \hat{b}_3'' = \hat{b}_3$



$$
\begin{array}{c|ccc}
{}^{B''}C^{B} & \hat{b}_1 & \hat{b}_2 & \hat{b}_3 \\
\hline
\hat{b}_1'' & C_3 & -S_3 & 0 \\
\hat{b}_2'' & S_3 & C_3 & 0 \\
\hat{b}_3'' & 0 & 0 & 1
\end{array}
$$

$$
{}^{A}C^{B''} = {}^{A}C^{B'}\,{}^{B'}C^{B''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & -S_1 \\ 0 & S_1 & C_1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & S_2 \\ 0 & 1 & 0 \\ -S_2 & 0 & C_2 \end{bmatrix}
$$

$$
= \begin{bmatrix} C_2 & 0 & S_2 \\ S_1 S_2 & C_1 & -S_1 C_2 \\ -C_1 S_2 & S_1 & C_1 C_2 \end{bmatrix}
$$

$$
{}^{A}C^{B} = {}^{A}C^{B''}\,{}^{B''}C^{B} = \begin{bmatrix} C_2 & 0 & S_2 \\ S_1 S_2 & C_1 & -S_1 C_2 \\ -C_1 S_2 & S_1 & C_1 C_2 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

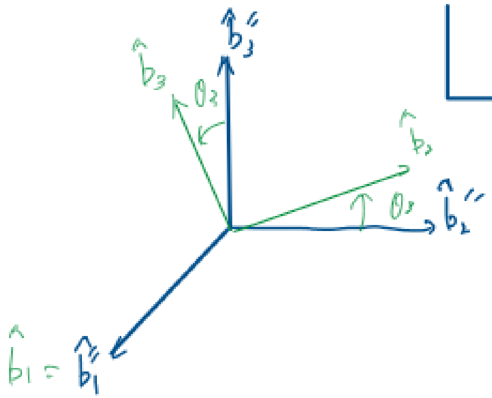
$$
DCM = \boxed{\begin{bmatrix} C_2 C_3 & -C_2 S_3 & S_2 \\ S_1 S_2 C_3 + S_3 C_1 & -S_1 S_2 S_3 + C_3 C_1 & -S_1 C_2 \\ -C_1 S_2 C_3 + S_2 S_1 & C_1 S_2 S_3 + C_3 S_1 & C_1 C_2 \end{bmatrix}}
$$

Body-two 1-2-1

(i) rot #1 ${}^{A}\lambda^{B'} = \hat{a}_1 = \hat{b}'_1$ $\Rightarrow$ same as previous (i)

(ii) rot #2 ${}^{B'}\lambda^{B''} = \hat{b}'_2 = \hat{b}''_2$ $\Rightarrow$ same as previous (ii)

(iii) rot #3 ${}^{B''}\lambda^{B} = \hat{b}''_1 = \hat{b}_1$



${}^{B''}C^{B}$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_3$
$\hat{b}''_1$	1	0	0
$\hat{b}''_2$	0	C_3	$-S_3$
$\hat{b}''_3$	0	S_3	C_3

$${}^{A}C^{B''} = {}^{A}C^{B'}\,{}^{B'}C^{B''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_1 & -S_1 \\ 0 & S_1 & C_1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & S_2 \\ 0 & 1 & 0 \\ -S_2 & 0 & C_2 \end{bmatrix}$$

$$= \begin{bmatrix} C_2 & 0 & S_2 \\ S_1 S_2 & C_1 & -S_1 C_2 \\ -C_1 S_2 & S_1 & C_1 C_2 \end{bmatrix}$$

$${}^{A}C^{B} = {}^{A}C^{B''}\,{}^{B''}C^{B} = \begin{bmatrix} C_2 & 0 & S_2 \\ S_1 S_2 & C_1 & -S_1 C_2 \\ -C_1 S_2 & S_1 & C_1 C_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_3 & -S_3 \\ 0 & S_3 & C_3 \end{bmatrix}$$

$$\boxed{DCM = \begin{bmatrix} C_2 & S_2 S_3 & S_2 C_3 \\ S_1 S_2 & C_1 C_3 - S_1 C_2 S_3 & -C_1 S_3 - S_1 C_2 C_3 \\ -C_1 S_2 & S_1 C_3 + C_1 C_2 S_3 & -S_1 S_3 + C_1 C_2 C_3 \end{bmatrix}}$$

(b) Assume that the final orientation of body B in frame N is given by

$$^N\bar{\varepsilon}^B = -.5\hat{n}_1 + .2\hat{n}_2 - .25\hat{n}_3 , \qquad ^N\varepsilon_4^B > 0$$

Determine the equivalent single rotation $^N\hat{\lambda}^B$ and $^N\theta^B$.

$$\boxed{^N\hat{\lambda}^B = \frac{^N\bar{\varepsilon}^B}{|^N\bar{\varepsilon}^B|} = -0.8422\,\hat{n}_1 + 0.3369\,\hat{n}_2 - 0.4211\,\hat{n}_3}$$

$$\boxed{^N\theta^B = 72.84^\circ \quad (1.2713 \text{ rad})}$$

(c) Determine the following set of angles that also describes the orientation in (b):

Body-three 1 – 2 – 3
Body-two 1– 2 – 1

$$\hat{\lambda} = -0.8422\,\hat{n}_1 + 0.3369\,\hat{n}_2 - 0.4211\,\hat{n}_3$$

$$\theta = 72.84^\circ = 1.2713 \text{ rad}$$

DCM is $^NC^B$ (calculated with MATLAB)

$$C_{11} = \cos\theta + \lambda_1^2(1-\cos\theta) = 0.7950$$
$$C_{12} = -\lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta) = 0.2023$$
$$C_{13} = \lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta) = 0.5719$$
$$C_{21} = \lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta) = -0.6023$$
$$C_{22} = \cos\theta + \lambda_2^2(1-\cos\theta) = 0.3750$$
$$C_{23} = -\lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta) = 0.7047$$
$$C_{31} = -\lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta) = -0.0719$$
$$C_{32} = \lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta) = -0.9047$$
$$C_{33} = \cos\theta + \lambda_3^2(1-\cos\theta) = 0.4200$$

$$\Downarrow$$

$$ {}^N C^B = \begin{bmatrix} 0.795 & 0.2023 & 0.5719 \\ -0.6023 & 0.375 & -0.7047 \\ -0.0719 & 0.9047 & 0.42 \end{bmatrix} $$

⟨I⟩ Body-three 1-2-3

$$ \begin{bmatrix} C_2C_3 & -C_2S_3 & S_2 \\ S_1S_2C_3+S_3C_1 & -S_1S_2S_3+C_3C_1 & -S_1C_2 \\ -C_1S_2C_3+S_3S_1 & C_1S_2S_3+C_3S_1 & C_1C_2 \end{bmatrix} = \begin{bmatrix} 0.795 & 0.2023 & 0.5719 \\ -0.6023 & 0.375 & -0.7047 \\ -0.0719 & 0.9047 & 0.42 \end{bmatrix} $$

$$ S_2 = 0.5719 \iff \theta_2 = 0.6088 \text{ rad or } 2.5328 \text{ rad} $$
$$ = 34.8828^\circ \text{ or } 145.1172 $$

when $\theta_2 = 34.88^\circ$

$C_2 = 0.8203$

$$ \Rightarrow \quad -S_1C_2 = -0.7047 \iff S_1 = \frac{0.7047}{C_2} $$
$$ \theta_1 = -59.2042^\circ, -120.7958^\circ $$

$$ C_1C_2 = 0.42 \iff C_1 = \frac{0.42}{C_2} $$
$$ \theta_1 = 59.2042^\circ, -59.2042^\circ $$

$$ C_2C_3 = 0.795 \iff C_3 = \frac{0.795}{C_2} $$

$$\theta_3 = 14.2793^\circ, \ \underline{-14.2793^\circ}$$

$$-C_2 S_3 = 0.2023 \iff S_3 = \frac{-0.2023}{C_2}$$

$$\theta_3 = \underline{-14.2793^\circ}, \ -165.7207^\circ$$

when $\theta_2 = 145.1193^\circ$

$$C_2 = -0.8203$$

$$\Rightarrow -S_1 C_2 = -0.7047$$

$$S_1 = \frac{0.7047}{C_2} \Rightarrow \theta_1 = 59.2042^\circ, \ \underline{120.7958^\circ}$$

$$C_1 C_2 = 0.42$$

$$C_1 = \frac{0.42}{C_2} \Rightarrow \theta_1 = \underline{120.7958^\circ}, \ -120.7958^\circ$$

$$C_2 C_3 = 0.795$$

$$C_3 = \frac{0.795}{C_2} \Rightarrow \theta_3 = \underline{165.7207^\circ}, \ -165.7207^\circ$$

$$-C_2 S_3 = 0.2023$$

$$S_3 = -\frac{0.2033}{C_2} \Rightarrow \theta_3 = 14.2793^\circ, \ \underline{165.7207^\circ}$$

$$\boxed{\begin{array}{lll|l} \text{Possible combos} & & & \\ \hat{a}_1 & \theta_1 = -59.2042^\circ & & \theta_1 = 120.7958^\circ \\ \hat{b}_2' & \theta_2 = 34.8807^\circ & & \theta_2 = 145.1193^\circ \\ \hat{b}_3 & \theta_3 = -14.2793^\circ & & \theta_3 = 165.7217^\circ \end{array}}$$

<II> Body-Two 1-2-1

$$\begin{bmatrix} c_2 & s_2 s_3 & s_2 c_3 \\ s_1 s_2 & c_1 c_3 - s_1 c_2 s_3 & -c_1 s_3 - s_1 c_2 c_3 \\ -c_1 s_2 & s_1 c_3 + c_1 c_2 s_3 & -s_1 s_3 + c_1 c_2 c_3 \end{bmatrix} = \begin{bmatrix} 0.795 & 0.2023 & 0.5719 \\ -0.6023 & 0.375 & -0.7047 \\ -0.0719 & 0.9047 & 0.42 \end{bmatrix}$$

$$c_2 = 0.795 \implies \theta_2 = 37.3447^\circ, -37.3447^\circ$$

when $\theta_2 = 37.3447^\circ$

- $s_2 = 0.6066$

$$s_1 s_2 = -0.6023$$

$$s_1 = \frac{-0.6023}{s_2} \implies \theta_1 = -83.1958^\circ, -96.8042$$

$\cdot \quad -c_1 s_2 = -0.0719$

$$c_1 = \frac{0.0719}{s_2} \Rightarrow \theta_1 = 83.1958,\ -83.1958$$

$\cdot \quad s_2 s_3 = 0.2023$

$$s_3 = \frac{0.2023}{s_2} \Rightarrow \theta_3 = 19.4846^\circ,\ 160.5154^\circ$$

$\cdot \quad s_2 c_3 = 0.5719$

$$c_3 = \frac{0.5719}{s_2} \Rightarrow \theta_3 = 19.4846^\circ,\ -19.4846^\circ$$

when $\theta_2 = -37.3447^\circ$

$\therefore \sin\theta_2 = -0.6066$

$s_1 s_2 = -0.6023$

$$s_1 = \frac{-0.6023}{s_2} \Rightarrow \theta_1 = 83.1958^\circ,\ 96.8042^\circ$$

$c_1 s_2 = -0.0719$

$$c_1 = \frac{-0.0719}{s_2} \Rightarrow \theta_1 = 96.8042,\ -96.8042$$

$s_2 s_3 = 0.2023$

$$s_3 = \frac{0.2023}{s_2} \Rightarrow \theta_3 = -19.4846^\circ,\ -160.5154^\circ$$

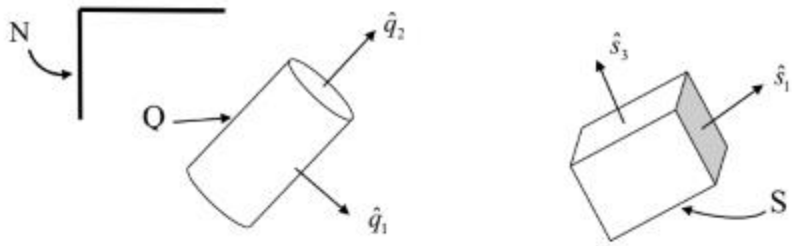
$s_2 c_3 = 0.5719$

$$c_3 = \frac{0.5719}{s_2} \Rightarrow \theta_3 = 160.5154^\circ,\ -160.5154^\circ$$

Possible combos

$\hat{a}_1$	$\theta_1 = -83.1458^\circ$	$\theta_1 = 96.8042^\circ$
$\hat{b}_2'$	$\theta_2 = 37.3447^\circ$	$\theta_2 = -37.3447^\circ$
$\hat{b}_1$	$\theta_3 = 19.4846^\circ$	$\theta_3 = -160.5154^\circ$

Problem 3: Two satellites Q and S can move in the Earth-fixed frame N.
Let $\hat{n}_i, \hat{q}_i, \hat{s}_i$ be unit vectors fixed in N, Q, and S respectively; each set is a dextral orthonormal triad.



At a given instant, $\hat{n}_i = \hat{q}_i$ and the orientation of S with respect to N is known and defined in terms of the following direction cosine matrix:

$$^{N}C^{S} = \begin{bmatrix} 0.5357 & 0.6229 & 0.5701 \\ -0.7658 & 0.6429 & 0.0172 \\ -0.3558 & -0.4457 & 0.8214 \end{bmatrix}$$

(a) Define the orientation of S with respect to Q at this instant in terms of the body three angles 3-2-1.

from the supplementary notes
for Body-three 3-2-1

$$^{N}C^{S} = \begin{bmatrix} C_1C_2 & C_1S_2S_3 - C_3S_1 & C_1S_2C_3 - S_3S_1 \\ S_1C_2 & S_1S_2S_3 + C_3C_1 & S_1S_2C_3 - S_3C_1 \\ -S_2 & C_2S_3 & C_2C_3 \end{bmatrix}$$

So

$$\begin{bmatrix} C_1C_2 & C_1S_2S_3 - C_3S_1 & C_1S_2C_3 - S_3S_1 \\ S_1C_2 & S_1S_2S_3 + C_3C_1 & S_1S_2C_3 - S_3C_1 \\ -S_2 & C_2S_3 & C_2C_3 \end{bmatrix} = \begin{bmatrix} 0.5357 & 0.6229 & 0.5701 \\ -0.7658 & 0.6429 & 0.0172 \\ -0.3558 & -0.4457 & 0.8214 \end{bmatrix}$$

$$S_2 = 0.3558$$

$$\Rightarrow \theta_2 = 20.8425^\circ,\ 159.1575^\circ$$

when $\theta_2 = 20.8425°$

$c_1 c_2 = 0.5357$

$c_1 = \frac{0.5357}{c_2} \Rightarrow \theta_1 = 55.0257°, -55.0257°$

$s_1 c_2 = -0.7658$

$s_1 = -\frac{0.7658}{c_2} \Rightarrow \theta_1 = -55.0257°, -124.9731°$

$c_2 c_3 = 0.8214$

$c_3 = \frac{0.8214}{c_2} \Rightarrow \theta_3 = 28.4883°, -28.4883°$

$c_2 s_3 = -0.4457$

$s_3 = -\frac{0.4457}{c_2} \Rightarrow \theta_3 = -28.4886°, -151.5164°$

when $\theta_2 = 159.1575$

$c_2 = -0.9346$

- $c_1 c_2 = 0.5357$

$c_1 = \frac{0.5357}{c_2} \rightarrow \theta_1 = 124.9743°, -124.9743°$

- $s_1 c_2 = -0.7658$

$s_1 = -\frac{0.7658}{c_2} \Rightarrow \theta_1 = 55.0269, 124.9731$

- $c_2 c_3 = 0.8214$

$c_3 = \frac{0.8214}{c_2} \Rightarrow \theta_3 = 151.5117°, -151.5117°$

- $c_2 s_3 = -0.4457$

$s_3 = -\frac{0.4457}{c_2} \rightarrow \theta_3 = 28.4836°, 151.5164°$

Possible combo

$\hat{n}_3$	$\theta_1 = -55.03°$	$\theta_1 = 124.97°$
$\hat{s}_2'$	$\theta_2 = 20.84°$	$\theta_2 = 159.16°$
$\hat{s}_1$	$\theta_3 = -28.49°$	$\theta_3 = 151.51°$

(b) The satellite Q then undergoes a rotation with respect to N that is described as ${}^N\bar{L}^Q = -0.25\hat{n}_1 + \hat{n}_3$, ${}^N\theta^Q = 30^\circ$. Determine the new orientation of S with respect to Q. Describe the new orientation in terms of Euler parameters ${}^Q\bar{\varepsilon}^S$, ${}^Q\varepsilon_4^S$.

from ${}^N\bar{L}^Q$

$$\begin{cases} \lambda_1 = \dfrac{-0.25}{\sqrt{(-0.25)^2 + 1^2}} = -0.2425 \\ \lambda_2 = 0 \\ \lambda_3 = \dfrac{1}{\sqrt{(-0.25)^2 + 1^2}} = 0.9701 \end{cases}$$

$$\theta = 30^\circ = \frac{\pi}{6}$$

dcm becomes from ${}^N C^Q$

$$\begin{aligned}
C_{11} &= \cos\theta + \lambda_1^2(1-\cos\theta) &&= 0.8739 \\
C_{12} &= -\lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta) &&= -0.4850 \\
C_{13} &= \lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta) &&= -0.0315 \\
C_{21} &= \lambda_3\sin\theta + \lambda_1\lambda_2(1-\cos\theta) &&= 0.4850 \\
C_{22} &= \cos\theta + \lambda_2^2(1-\cos\theta) &&= 0.8660 \\
C_{23} &= -\lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta) &&= 0.1212 \\
C_{31} &= -\lambda_2\sin\theta + \lambda_3\lambda_1(1-\cos\theta) &&= -0.0315 \\
C_{32} &= \lambda_1\sin\theta + \lambda_2\lambda_3(1-\cos\theta) &&= -0.1212 \\
C_{33} &= \cos\theta + \lambda_3^2(1-\cos\theta) &&= 0.9921
\end{aligned}$$

Find ${}^{a}C^{s}$

$$ {}^{a}C^{s} = {}^{a}C^{N}\,{}^{N}C^{s} $$

$$ = \left({}^{N}C^{a}\right)^{T}\,{}^{N}C^{s} $$

$$ = \begin{bmatrix} 0.8739 & 0.4851 & -0.0315 \\ -0.4851 & 0.8660 & -0.1213 \\ -0.0315 & 0.1213 & 0.9921 \end{bmatrix} \begin{bmatrix} 0.5357 & 0.6229 & 0.5701 \\ -0.7658 & 0.6429 & 0.0172 \\ -0.3558 & -0.4457 & 0.8214 \end{bmatrix} $$

$$ = \begin{bmatrix} 0.1079 & 0.8703 & 0.4807 \\ -0.8799 & 0.3087 & -0.3613 \\ -0.4627 & -0.3839 & 0.7990 \end{bmatrix} $$

Find ${}^{a}\bar{\epsilon}^{s}$ & ${}^{a}\varepsilon_4^{s}$

$$ {}^{a}\varepsilon_4^{s} = \frac{1}{2}\sqrt{1 + C_{11} + C_{22} + C_{33}} = 0.7442 $$

$$ {}^{a}\varepsilon_1^{s} = \frac{C_{32} - C_{23}}{4\varepsilon_4} = -0.0026 $$

$$ {}^{a}\varepsilon_2^{s} = \frac{C_{13} - C_{31}}{4\varepsilon_4} = 0.3169 $$

$$ {}^{a}\varepsilon_3^{s} = \frac{C_{21} - C_{12}}{4\varepsilon_4} = -0.5879 $$

$$\boxed{\begin{aligned} {}^{a}\bar{\varepsilon}^{s} &= \begin{bmatrix} -0.0076 & 0.3169 & -0.5879 \end{bmatrix} \\ {}^{a}\varepsilon_4^{s} &= 0.7442 \end{aligned}}$$

AAE440 PS3 MATLAB CODE

problem 1

```
clear all; close all; clc;
```

<a>

```
% Rot#1 (G->B')
% Define the Euler Parameters/epsilons
e1_1 = 0.4;
e1_2 = -0.1;
e1_3 = 0.2;
e1_4 = sqrt(1-e1_1^2-e1_2^2-e1_3^2);
% Set as vector
e1 = [e1_1,e1_2,e1_3,e1_4];


% Compute the DCM
C_1 = DCM_euler_para(e1);


% Rot#2 (B'->B)
lambda_g = [0 0 1];  % in g vector basis
lambda_b_prime = lambda_g*C_1;  % in b' vector basis
theta = pi/3;  % rotation angle in radians


% Update DCM with next rotation
C_2 = DCM_lambda_theta(lambda_b_prime,theta);


% Compute the Euler parameters
e2 = epsilons_with_DCM(C_2);


% Compute the overall DCM
C = C_1*C_2;


% Compute the successive Euler parameter
e_new_Bprime = eulerPara_successive_rot(e1, e2);


% Convert this to b-vector basis
e_new_vec = e_new_Bprime(1:3)*C_2;
```

```
e_new_B = [e_new_vec e_new_Bprime(4)];


C_final = DCM_euler_para(e_new_B);


% Calculating the error
C_error = C - C_final;
```

<b>

```
% Epsilon in g frame
e_g = e_new_Bprime(1:3)*C_1.';
```

<c>

```
% Compute the lambda and theta for one single rotation
e_g = [e_g e_new_Bprime(4)];
[lambda_SRT, theta_SRT] = lambda_and_theta_fromEpsilon(e_g);
% Covnert radians to degrees
theta_SRT = theta_SRT/pi*180;
```

problem 2

```
clear all; close all; clc
```

<b>

```
e_NB = [-0.5 0.2 -0.25];
e_NB_4 = sqrt(1 - sum(e_NB.^2));
e_NB = [e_NB e_NB_4];
% Compute the lambda and theta for SRT
[lambda_NB theta_NB] = lambda_and_theta_fromEpsilon(e_NB);
% Convert lambda to degrees from radians
theta_NB_deg = theta_NB*180/pi;
```

<b>

```
% Calculating the DCM
C_NB = DCM_lambda_theta(lambda_NB, theta_NB);
```

problem 3

<b>

```
clear all; close all; clc;
L = [-0.25 0 1];
theta = pi/6;
lambdas = zeros([1 3]);
lambdas(1) = L(1)/sqrt(L(1)^2+L(2)^2+L(3)^2);
lambdas(2) = L(2)/sqrt(L(1)^2+L(2)^2+L(3)^2);
lambdas(3) = L(3)/sqrt(L(1)^2+L(2)^2+L(3)^2);
% DCM from N basis to Q basis
C_NQ = DCM_lambda_theta(lambdas, theta);


% DCM from N basis to S basis
C11 = 0.5357;
C12 = 0.6229;
C13 = 0.5701;
C21 = -0.7658;
C22 = 0.6429;
C23 = 0.0172;
C31 = -0.3558;
C32 = -0.4457;
C33 = 0.8214;
C_NS = [C11 C12 C13; C21 C22 C23; C31 C32 C33];


% Compute the DCM from Q to S
C_QS = C_NQ.'*C_NS


% Compute epsilons for Q to S basis
epsilon = epsilons_with_DCM(C_QS);
```

```matlab
function C_mat = DCM_euler_para(epsilons)
    % Epsilon vector
    epsilon1 = epsilons(1);
    epsilon2 = epsilons(2);
    epsilon3 = epsilons(3);
    epsilon4 = epsilons(4);


    % Calculating DCM with Euler parameters
    C11 = 1 - 2*epsilon2^2 - 2*epsilon3^2;
    C12 = 2*(epsilon1*epsilon2 - epsilon3*epsilon4);
    C13 = 2*(epsilon3*epsilon1 + epsilon2*epsilon4);
    C21 = 2*(epsilon1*epsilon2 + epsilon3*epsilon4);
    C22 = 1 - 2*epsilon3^2 - 2*epsilon1^2;
    C23 = 2*(epsilon2*epsilon3 - epsilon1*epsilon4);
    C31 = 2*(epsilon3*epsilon1 - epsilon2*epsilon4);
    C32 = 2*(epsilon2*epsilon3 + epsilon1*epsilon4);
    C33 = 1 - 2*epsilon1^2 - 2*epsilon2^2;

    C_mat = [C11 C12 C13; C21 C22 C23; C31 C32 C33];
end
```

```matlab
function C_mat = DCM_lambda_theta(lambdas, theta)
    % Lambda vector
    lambda1 = lambdas(1);
    lambda2 = lambdas(2);
    lambda3 = lambdas(3);

    % Calculating DCM with lambdas and theta
    C11 = cos(theta) + lambda1^2*(1-cos(theta));
    C12 = -lambda3*sin(theta) + lambda1*lambda2*(1-cos(theta));
    C13 = lambda2*sin(theta) + lambda3*lambda1*(1-cos(theta));
    C21 = lambda3*sin(theta) + lambda1*lambda2*(1-cos(theta));
    C22 = cos(theta) + lambda2^2*(1-cos(theta));
    C23 = -lambda1*sin(theta) + lambda2*lambda3*(1-cos(theta));
    C31 = -lambda2*sin(theta) + lambda3*lambda1*(1-cos(theta));
    C32 = lambda1*sin(theta) + lambda2*lambda3*(1-cos(theta));
    C33 = cos(theta) + lambda3^2*(1-cos(theta));

    C_mat = [C11 C12 C13; C21 C22 C23; C31 C32 C33];
end
```

```matlab
function epsilons = epsilons_with_DCM(C_mat)
    epsilon4 = 0.5*sqrt(1+C_mat(1,1)+C_mat(2,2)+C_mat(3,3));
    epsilon1 = (C_mat(3,2)-C_mat(2,3))/4/epsilon4;
    epsilon2 = (C_mat(1,3)-C_mat(3,1))/4/epsilon4;
    epsilon3 = (C_mat(2,1)-C_mat(1,2))/4/epsilon4;
    epsilons = [epsilon1 epsilon2 epsilon3 epsilon4];
end

function e_new = eulerPara_successive_rot(e1, e2)
    e1_v = e1(1:3);
    e1_4 = e1(4);
    e2_v = e2(1:3);
    e2_4 = e2(4);

    % Calculate the successive epsilon
    e_v_new = e1_v*e2_4 + e2_v*e1_4 + cross(e2_v, e1_v);
    e4_new = e1_4 * e2_4 - dot(e1_v,e2_v);
    e_new = [e_v_new e4_new];
end
```

```matlab
function [lambda, theta] = lambda_and_theta_fromEpsilon(epsilons)
    % Calculating the lambda unit vector and the angle theta for a simple
    % rotation using the epsilon values
    e_vec = epsilons(1:3);
    e4 = epsilons(4);

    % compute lambda
    lambda = e_vec / sqrt(sum(e_vec.^2));
    theta = 2*acos(e4);
end
```