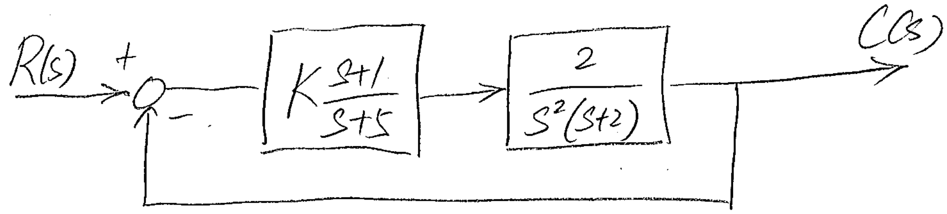
AAE 364 | HW#7 | Fall 2019

## Problem 1

B-6-7

Given:




Required: Root loci for the system and determine the range of K for stability

Solution:

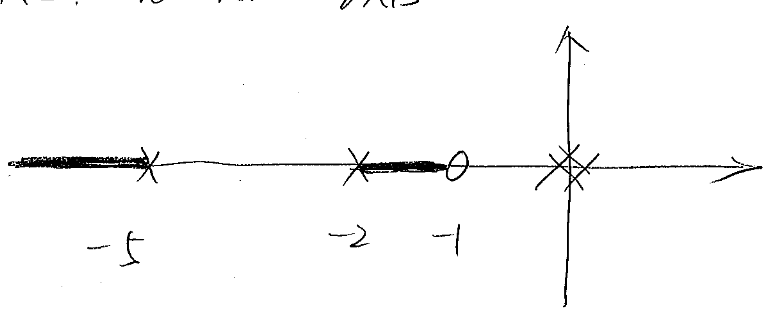
$$CE: \quad 1 + \underbrace{K \frac{s+1}{s+5} \frac{2}{s^2(s+2)}}_{L(s)} = 0$$

1) poles: $-5, -2, 0, 0.$ $\quad n=4$

zeros: $-1$ $\quad m=1$

2) Symmetry

3) R.L. on real axis



4) Asymptotes : $n-m=3$ asymptotes

$$\theta_a = \frac{180° + 360° l}{n-m}, \quad l = 0, 1, 2$$

$$= 60°, 180°, 300°$$

$$\sigma_a = \frac{\Sigma_i \text{ poles} - \Sigma_i \text{ zeros}}{n-m} = -2$$

5) Break-in/away points :

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$$\Downarrow$$

$$\frac{d}{ds} \frac{(s+5)s^2(s+2)}{s+1} = 0$$

$$LHS = \frac{d}{ds} \frac{s^2(s^2+7s+10)}{s+1} = \frac{d}{ds} \frac{s^4+7s^3+10s^2}{s+1}$$

$$= \frac{(s+1)(4s^3+21s^2+20s) - (s^4+7s^3+10s^2)(1)}{(s+1)^2} = 0$$

$$\Downarrow$$

$$(s+1)(4s^3+21s^2+20s) - (s^4+7s^3+10s^2) = 0$$

or

$$4s^4 + 21s^3 + 20s^2 + 4s^3 + 21s^2 + 20s - s^4 - 7s^3 - 10s^2 = 0$$

$$3s^4 + 18s^3 + 31s^2 + 20s = 0$$

roots: $\underbrace{0}_{\text{break-away point}}$, $\underbrace{-3.6885,\ -1.1557 \pm j0.6868}_{\text{NOT on RL} \cap \text{real-axis}}$

6) N/A

7) Intersection with imaginary-axis:

$$1 + kL(j\omega) = 0$$

$$\Downarrow$$

$$-\omega^2(j\omega+2)(j\omega+5) + k(j\omega+1)\cdot 2 = 0$$

$$-\omega^2(-\omega^2 + 7j\omega + 10) + 2jk\omega + 2k = 0$$

$$\omega^4 - 7j\omega^3 - 10\omega^2 + 2jk\omega + 2k = 0$$

$$\begin{cases} \omega^4 - 10\omega^2 + 2k = 0 & ① \\ -7\omega^3 + 2k\omega = 0 & ② \end{cases}$$

② $\Rightarrow$ $-\omega(7\omega^2 - 2k) = 0 \Rightarrow$ either $\omega = 0$ (i)
or $2k = 7\omega^2$ (ii)

If (i) is true, $k = 0 \Rightarrow$ NOT appliable.

If (ii) is true, $\omega^4 - 10\omega^2 + 7\omega^2 = 0$

$$\omega^2(\omega^2 - 3) = 0$$

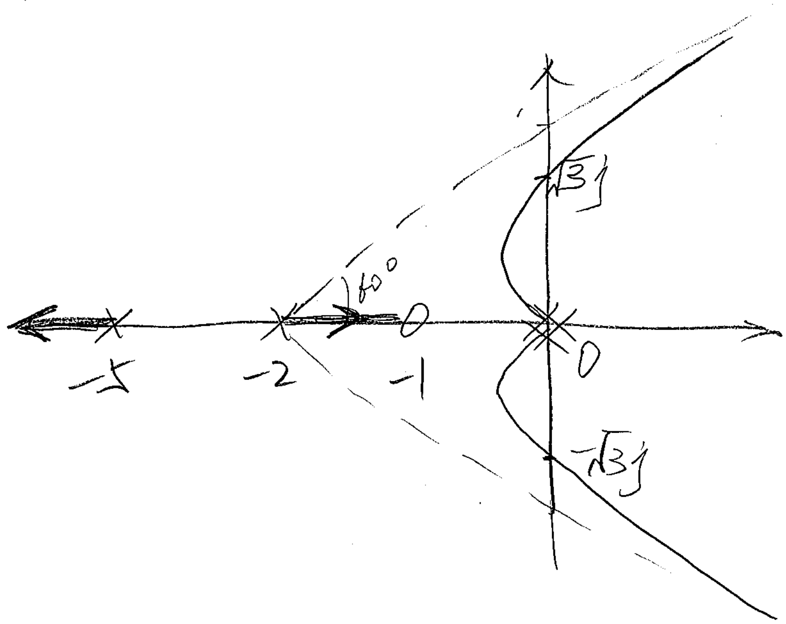
$$\omega = 0 \quad \text{or} \quad \omega = \pm\sqrt{3}$$

The intersections are $\boxed{\pm j\sqrt{3}}$.

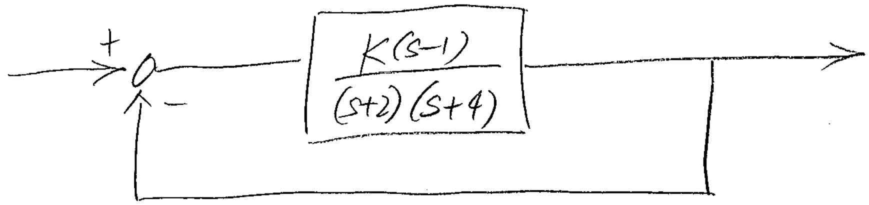
If $w^2 = 3$

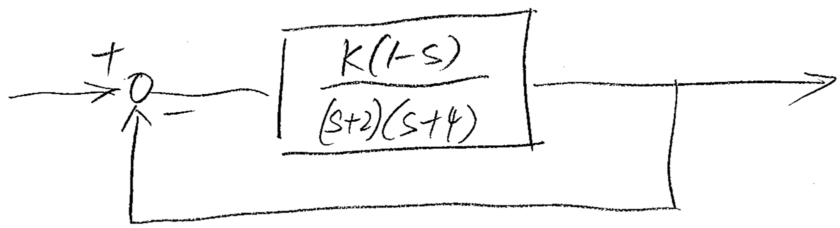
$k = \frac{7}{2} w^2 = \frac{21}{2}$

$$\boxed{K \in \left(0, \frac{21}{2}\right) \text{ for stability}}$$



B-6-12

(a) 


(b) 


Required: Root loci.

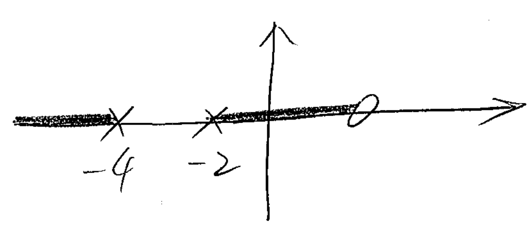
Solution:

(a) CE: $1 + K L(s) = 0$

where $L(s) = \dfrac{s-1}{(s+2)(s+4)}$

(1) $\begin{cases} \text{poles: } -2, -4 & n=2 \\ \text{zeros } 1 & m=1 \end{cases}$

(2) Symmetry

(3) R.L. on Re-axis 

(4) Asymptote :

$$\theta_a = 180^\circ$$

$$\sigma_a = -7$$

(5) Break-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$$\Downarrow$$

$$\frac{d}{ds}\frac{s^2+6s+8}{s-1} = \frac{(s-1)(2s+6)-(s^2+6s+8)1}{(s-1)^2} = 0$$

$$2s^2+4s-6-s^2-6s-8=0$$

$$s^2-2s-14=0$$

$s = 4.8730, -2.8730$ both are not on R.L.

(6) N/A

(7) Intersection with Im-axis

$$1+kL(j\omega)=0$$

$$(j\omega+2)(j\omega+4)+k(j\omega-1)=0$$

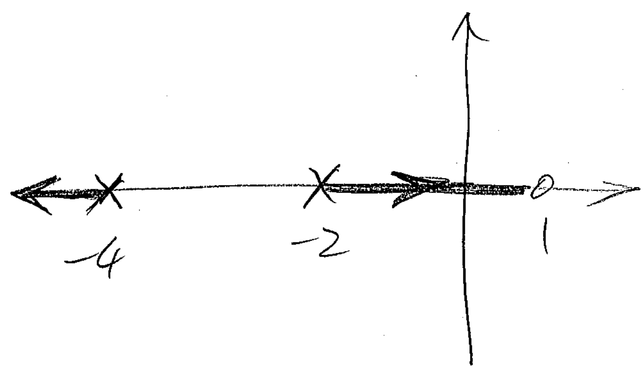
$$-\omega^2+6j\omega+8+kj\omega-k=0$$

$$\begin{cases} -\omega^2+8-k=0 & \text{①} \\ 6\omega+k\omega=0 & \text{②} \end{cases}$$

② $\Rightarrow$ either (i) $w = 0$ or (ii) $k = -6$

NOT applicable.

The only intersection is at the origin.




(b). CE: $1 + \frac{K(1-s)}{(s+2)(s+4)} = 0$

$$\boxed{\begin{array}{l} \text{or } 1 - KL(s) = 0 \\ \text{where } L(s) = \frac{s-1}{(s+2)(s+4)} \end{array}}$$

※ In the "standard form" of $L(s)$, all coefficients of $s$ is $+1$.

※ Once the "standard form" of $L(s)$ is obtained, the sign before $KL(s)$ decides the rule: "+" $\Rightarrow$ $-180°$-rule

"−" $\Rightarrow$ $0°$-rule

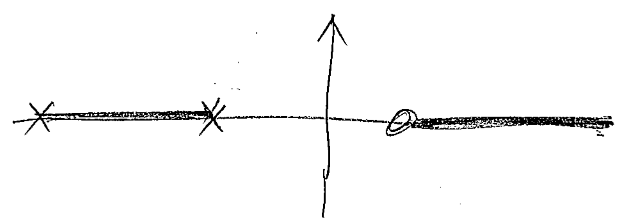
Since we are considering the roots of

$$1 - KL(s) = 0,$$

we should use the $0^\circ$-rule.

Rule (1) & (2) are same with part (a).

(3) R.L. on Re-axis:



(4) Asymptotes:

$$\theta_a = 0^\circ$$

$$\sigma_a = -7$$

(5) Break-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0 \Rightarrow s = 4.8730,\ -2.8730$$

break-in point      break-away point

(6) N/A

(7) Intersection with Im-axis:

$$1-kL(j\omega)=0$$

$$\Downarrow$$

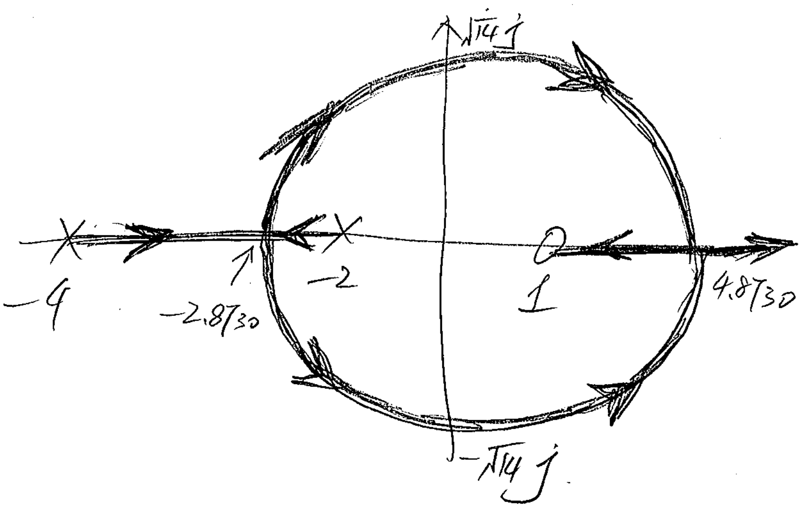
$$(j\omega+2)(j\omega+4)-k(j\omega-1)=0$$

$$-\omega^2+6j\omega+8-kj\omega+k=0$$

$$\begin{cases} -\omega^2+8+k=0 & ① \\ 6\omega-k\omega=0 & ② \end{cases}$$

② $\Rightarrow$ $k=6$ since $\omega\neq 0$

$k=6 \Rightarrow$ ①: $\omega^2=14 \Rightarrow \omega \doteq \sqrt{14}$



Problem 2

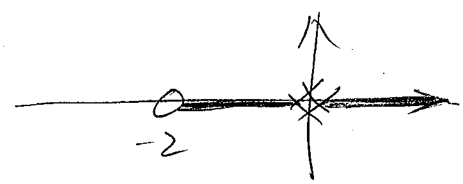
1.

CE: $1 - k \underbrace{\frac{s+2}{s^2}}_{L(s)} = 0$

$0^\circ$ — rule.

(1) $\begin{cases} \text{poles} = 0,\ 0 & n = 2 \\ \text{zeros}: -2 & m = 1. \end{cases}$

(2) Symmetry

(3) R.L on Re-axis: 


(4) Asymptotes:

$\theta_a = 0^\circ$

$\sigma_a = 2$

(5) Break-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = -\frac{(s+2)(2s) - s^2}{(s+1)^2} = 0$$

$\Downarrow$

$s = 0, -4$ — NOT on Re-axis $\cap$ R.L.

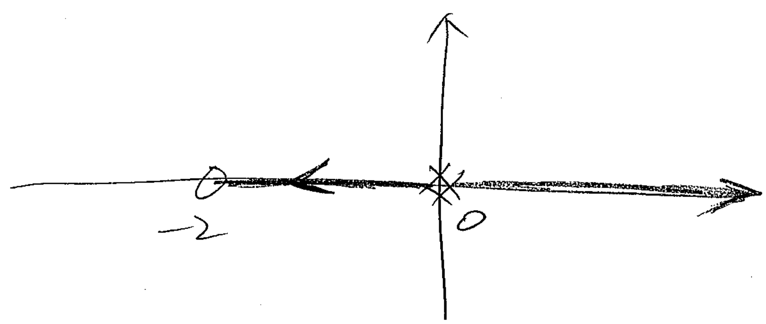
(6) N/A

(7) Intersection with Im-axis:

$$1 - kL(j\omega) = 0$$

$$-\omega^2 - k(j\omega + 2) = 0$$

$$\begin{cases} -\omega^2 - 2k = 0 \\ -k\omega = 0 \end{cases} \Rightarrow \begin{cases} \omega = 0 \\ k = 0 \end{cases} \Rightarrow \text{NOT applicable.}$$



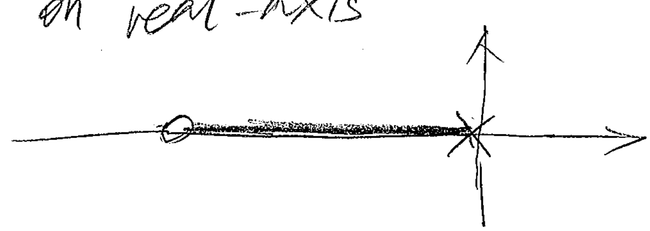

2. CE: $1 + K \underbrace{\dfrac{s+9}{s(s^2+4s+11)}}_{L(s)} = 0$

$\uparrow$ $-180°$-rule

(1) $\begin{cases} \text{poles: } 0, -2 \pm j\sqrt{7} & n = 3 \\ \text{zeros: } -9 & m = 1 \end{cases}$

(2) Symmetry

(3) R.L on real-axis



(4) Asymptotes:

$$\theta_a = \frac{180° + 360° l}{n-m}, \quad l = 0, 1$$

$$= 90°, 270°$$

$$\sigma_a = \frac{5}{2}$$

(5) Break-in/away points

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$$\Downarrow$$

$$2s^3 + 31s^2 + 72s + 99 = 0$$

$$s = -13.0284, \ -1.2358 \pm j1.5074$$

All are not applicable.

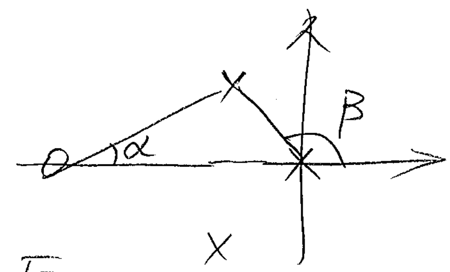
(6). Angle of departure

$\angle L(s^d) = -180°$ where $s^d$ is a point on R.L. and close to $-2+j\sqrt{7}$

$$\angle L(s^d) = \overbrace{\angle(s^d+9)}^{\alpha} - \overbrace{\angle s^d}^{\beta}$$

$$- \theta^d - 90° = -180°$$

$$\alpha = \tan^{-1}\left(\frac{\sqrt{7}}{7}\right)$$




$$\beta = 180^\circ - \tan^{-1}\left(\frac{\sqrt{7}}{2}\right)$$

$\alpha \approx 20.7048^\circ$ $\quad$ $\beta \approx 127.0867$.

$$\theta^d = 180^\circ + \alpha - \beta - 90^\circ \approx -16.3819^\circ$$

(7) Intersection with Im-axis

$$1 + kL(j\omega) = 0$$

$$\Downarrow$$

$$j\omega(-\omega^2 + 4j\omega + 11) + k(j\omega + 9) = 0$$

$$-j\omega^3 - 4\omega^2 + 11j\omega + kj\omega + 9k = 0$$

$$\begin{cases} -\omega^3 + 11\omega + k\omega = 0 & ① \\ -4\omega^2 + 9k = 0 & ② \end{cases}$$

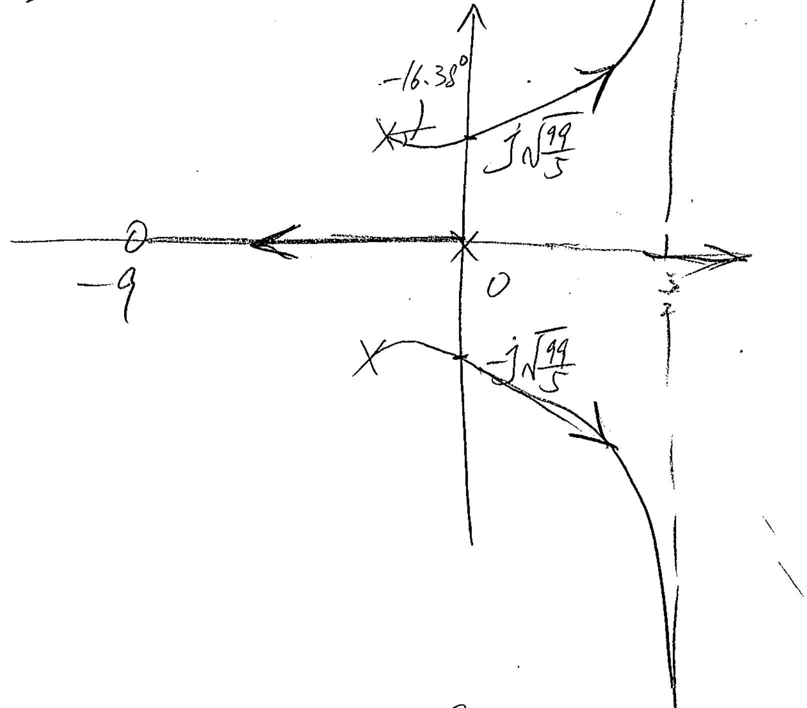
From ② and $\omega \neq 0$, we know $k = \frac{4}{9}\omega^2$

plugging into ①, we have

$$-\omega^3 + 11\omega + \frac{4}{9}\omega^3 = 0$$

$$\omega\left(-\frac{5}{9}\omega^2 + 11\right) = 0$$

$$\boxed{\omega = \pm\sqrt{\frac{99}{5}}}$$




3.

$$\text{C.E.}: \quad 1 + K\frac{(1-s)}{s^3+s^2+1} = 0$$

$$\Downarrow$$

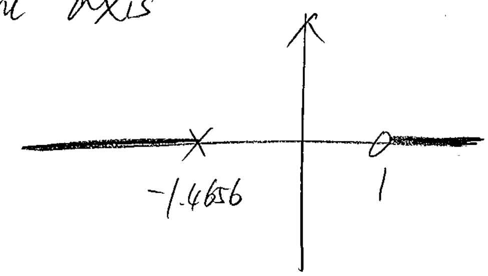
$$1 - KL(s) = 0 \quad 0^\circ\text{-rule}$$

$$\text{where} \quad L(s) = \frac{s-1}{s^3+s^2+1}$$

(1) $\begin{cases} \text{poles}: -1.4656, \quad 0.2328 \pm 0.7926j & n=3 \\ \text{zeros}: \quad 1 & m=1 \end{cases}$

(2) Symmetry

(3) R.L. on real axis




(4) Asyptotes :

$$\theta_a = \frac{360° l}{n-m} \qquad l = 0, 1.$$

$$= 0°, 180°$$

$$\sigma_a = -1$$

5) Break-in/away points.

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$$\Downarrow$$

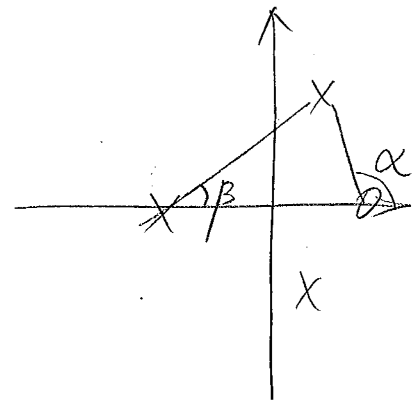
$$2s^3 - 2s^2 - 2s - 1 = 0$$

$$\Downarrow$$

$$s = 1.7399, \; -0.3700 \pm j\,0.3880.$$

↑
break-in point

6) Angle of departure




$$\alpha - \beta - 90^\circ - \theta_d = 0^\circ$$

$$\alpha = 180^\circ - \tan^{-1}\left(\frac{0.7926}{1-0.2328}\right) \approx 134.0671^\circ$$

$$\beta = \tan^{-1}\left(\frac{0.7926}{1.4656+0.2328}\right) \approx 25.0173^\circ$$

$$\underline{\theta_d = \alpha - \beta - 90^\circ \approx 19.0498^\circ}$$

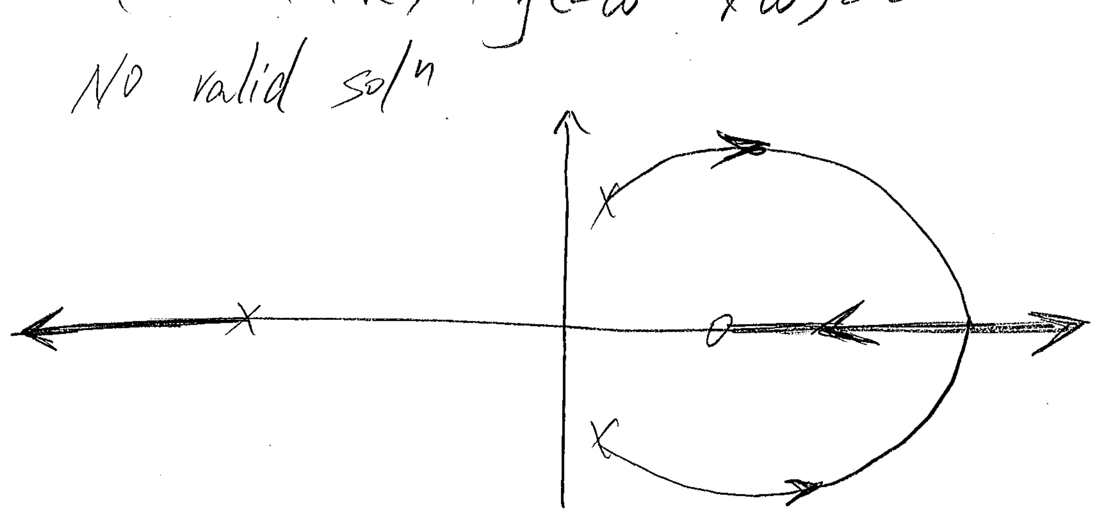
7) Intersection with Im-axis

$$1 - kL(j\omega) = 0$$

$$\Downarrow$$

$$(-\omega^2 + 1 + k) + j(-\omega^3 - k\omega) = 0$$

No valid sol$^n$



Problem 3.

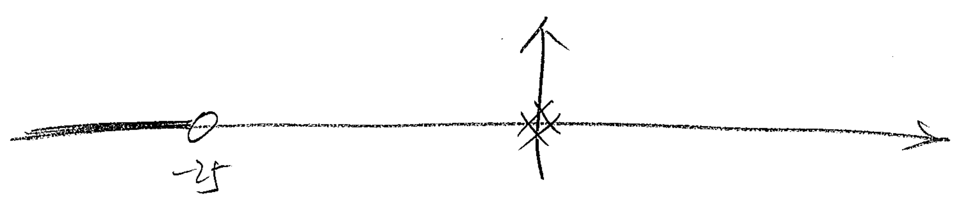
CE: $1 + KL(s) = 0$

where $L(s) = \dfrac{0.036\,(s+25)}{s^2\,(s^2 + 0.04s + 1)}$

(1) $\begin{cases} \text{poles: } 0,\ 0,\ -0.02 \pm j0.9998 & n = 4 \\ \text{zeros: } -25 & m = 1 \end{cases}$

(2) Symmetry

(3) R.L on the real-axis.




(4). Asymptotes: 3 asymptotes.

$\theta_a = \dfrac{180° + 360° l}{3}, \quad l = 0, 1, 2.$

$= 60°, 180°, 360°$

$\sigma_a \approx \frac{25}{3}$

(5) Break-in/away points.

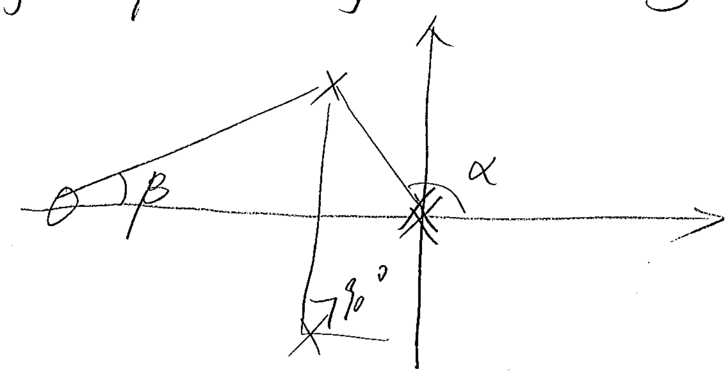
$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$$\frac{10s\,(75s^3 + 2502s^2 + 100s + 1250)}{9\,(s+25)^2} = 0$$

$s = 0, \quad -33.350, \quad -0.0125 \pm j0.7070$

break-away point — break-in point — NOT applicable.

(6). Angle of departure from $-0.02 \pm j0.9998$.




$$\beta - 2\alpha - \theta_d - 90^\circ = -180^\circ$$

$$\alpha = 180^\circ - \tan^{-1}\left(\frac{0.9998}{0.02}\right) \approx 91.146^\circ$$

$$\beta = \tan^{-1}\left(\frac{0.9998}{25 - 0.02}\right) \approx 2.292^\circ$$

$$\theta_d = 180^\circ + \beta - 2\alpha - 90^\circ \approx -90^\circ$$

(7) Intersection with Im-axis

$$1 + kL(j\omega) = 0$$

$$-\omega^2(-\omega^2 + 0.04j\omega + 1) + 0.036k(j\omega + 25) = 0$$

$$\begin{cases} \omega^4 - \omega^2 + (25 \times 0.036)k = 0 & ① \\ -0.04\omega^3 + 0.036k\omega = 0 & ② \end{cases}$$

For nonzero $\omega$,

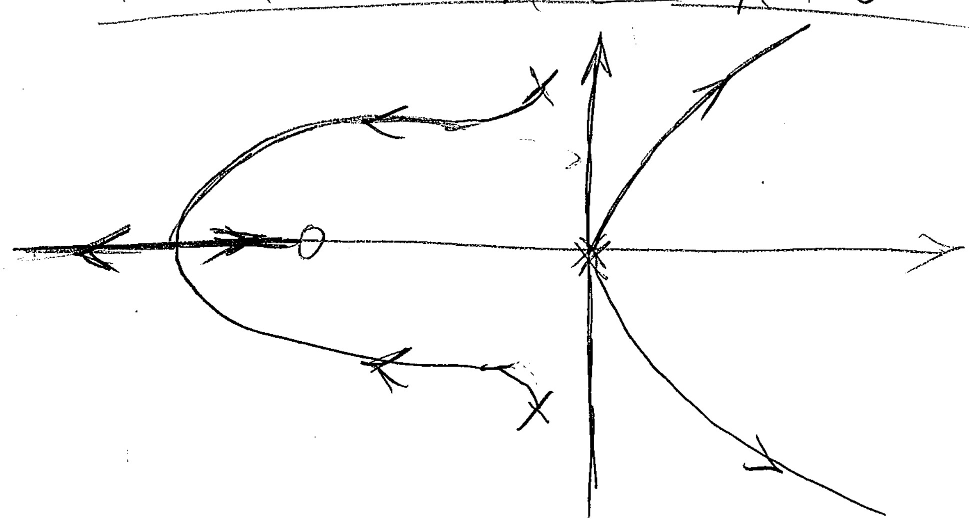
② $\Rightarrow 0.036k = 0.04\omega^2$

① then becomes $\omega^4 - \omega^2 + 25 \times 0.04\omega^2 = 0$

$$\Downarrow$$

$$\omega^4 = 0$$

No intersection as $k \neq 0$.



Problem 4.

1. $$G = \frac{1.1057s + 0.1900}{s^3 + 0.7385s^2 + 0.8008s}$$

2. $$G = \frac{1.1057s - 0.1900}{s^5 + 17.95s^4 + 123.3s^3 + 366.35s^2 + 112.2s}$$

Solution:

1. CE: $1 + kL(s) = 0$

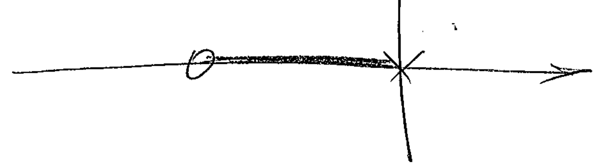
$$L(s) = \frac{1.1057s + 0.1900}{s^3 + 0.7385s^2 + 0.8008s}$$

1) poles: $0,\ -0.3693 \pm j0.8151$ $\quad n = 3$

zeros: $-0.1718$ $\quad m = 1$

2) Symmetry

3)



4) $\theta_a = \frac{180° + 360° l}{2}$, $l = 0, 1$

$= 90°, 270°$

$\sigma_a = -0.2834$

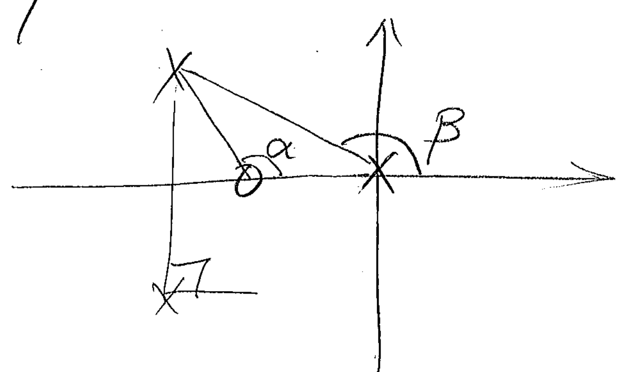
5) Break-in/away points.

$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$

$\Downarrow$ by MATLAB symbolic toolbox

No feasible real solution.

6). Angle of departure.




$\alpha - \beta - 90° - \theta_d = -180°$

$\alpha = 180° - \tan^{-1}\left(\frac{0.8151}{0.3693 - 0.1718}\right) \approx 103.6204°$

$\beta = 180° - \tan^{-1}\left(\frac{0.8151}{0.3693}\right) \approx 114.3740°$

$\boxed{\theta_d = 79.2464°}$

7) Intersection with Im-axis.

$$1 + kL(j\omega) = 0$$

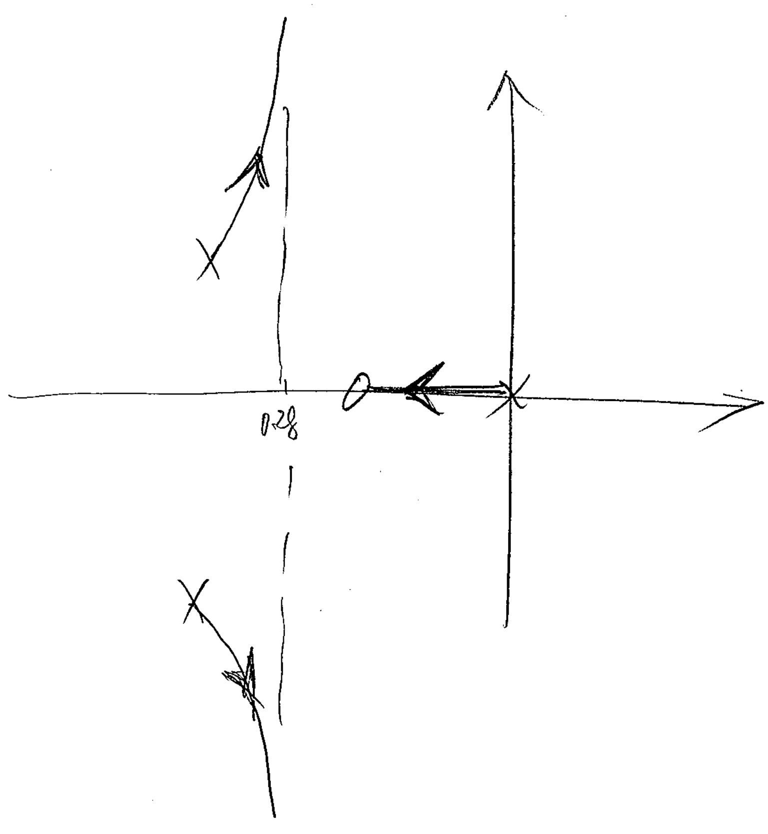
$$-j\omega^3 - 0.7385\omega^2 + 0.8008j\omega + k(1.1057j\omega + 0.19) = 0.$$

$$\begin{cases} -\omega^3 + 0.8008\omega + 1.1057k\omega = 0 \\ -0.7385\omega^2 + 0.19k = 0 \end{cases}$$

$$\Downarrow$$

No feasible solution

(Note that $k > 0$)



2. CE: $1 + KL(s) = 0$ ← 180°-rule

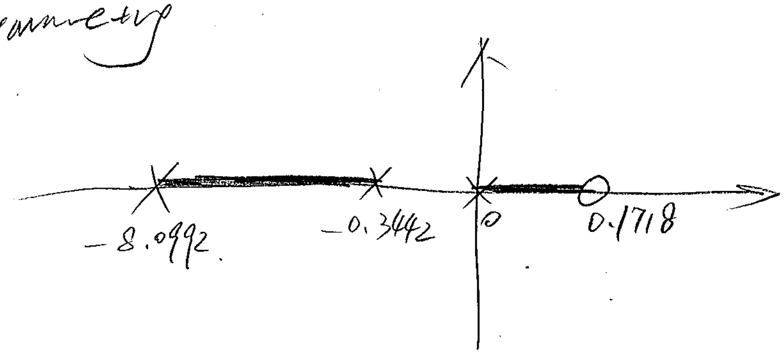
$$L(s) = \frac{1.1057s - 0.1900}{s^5 + 17.95s^4 + 123.3s^3 + 386.3s^2 + 112.2s}$$

1) poles : $0, -8.0992, -0.3442, -4.7533 \pm j4.2012$ $\quad n = 5$

zeros : $0.1718$ $\quad m = 1$

2) Symmetry

3)




4) $\theta_a = \dfrac{180° + 360° l}{4}$ $\quad l = 0, 1, 2, 3$

$= 45°, 135°, 225°, 315°$

$\sigma_a = -4.5305$

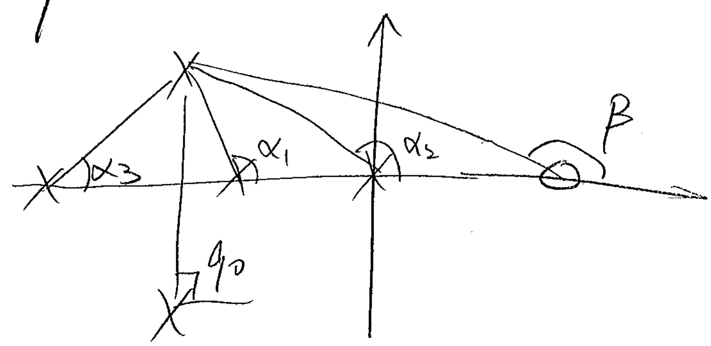
5) Break-in/away points.

$$\frac{d}{ds}\left(-\frac{1}{L(s)}\right) = 0$$

$\Downarrow$

The only break-away point is $-3.2759$.

6) Angle of departure




$$\beta - \alpha_1 - \alpha_2 - \alpha_3 - \theta_d - 90^\circ = -180^\circ$$

$$\beta = 180^\circ - \tan^{-1}\left(\frac{4.2012}{4.7533+0.1718}\right) \approx 139.3352^\circ$$

$$\alpha_1 = 180^\circ - \tan^{-1}\left(\frac{4.2012}{4.7533-0.3442}\right) \approx 136.3832^\circ$$

$$\alpha_2 = 180^\circ - \tan^{-1}\left(\frac{4.2012}{4.7533}\right) \approx 138.5282^\circ$$

$$\alpha_3 = \tan^{-1}\left(\frac{4.2012}{8.0992-4.7533}\right) \approx 51.4656^\circ$$

$$\theta_d = 180^\circ + \beta - \alpha_1 - \alpha_2 - \alpha_3 - 90^\circ$$

$$= -96.8418^\circ$$

7) Intersection with Im-axis

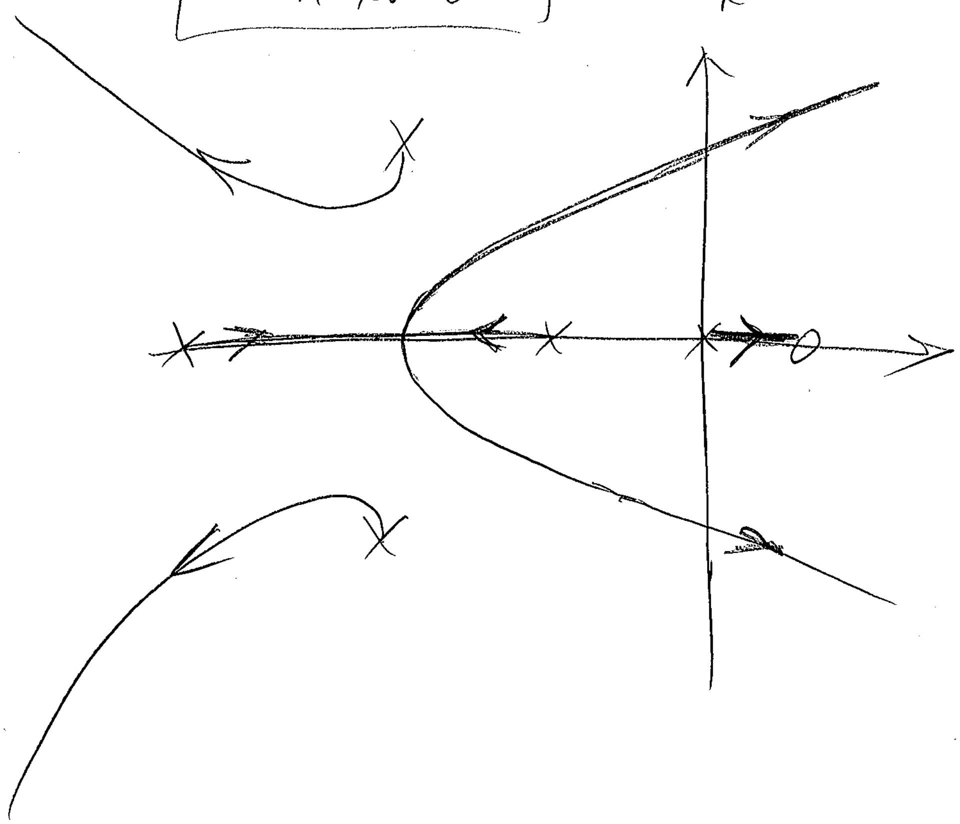
$$1 + kL(j\omega) = 0$$

$$\begin{cases} 17.95w^4 - 366.3w^2 - 0.19kw = 0 \\ w\left(w^4 - 123.3w^2 + 11.2 + 1.1059k\right) = 0 \end{cases}$$

for $w \neq 0$, , $1.1059k = 123.3w^2 - w^4 - 11.2$

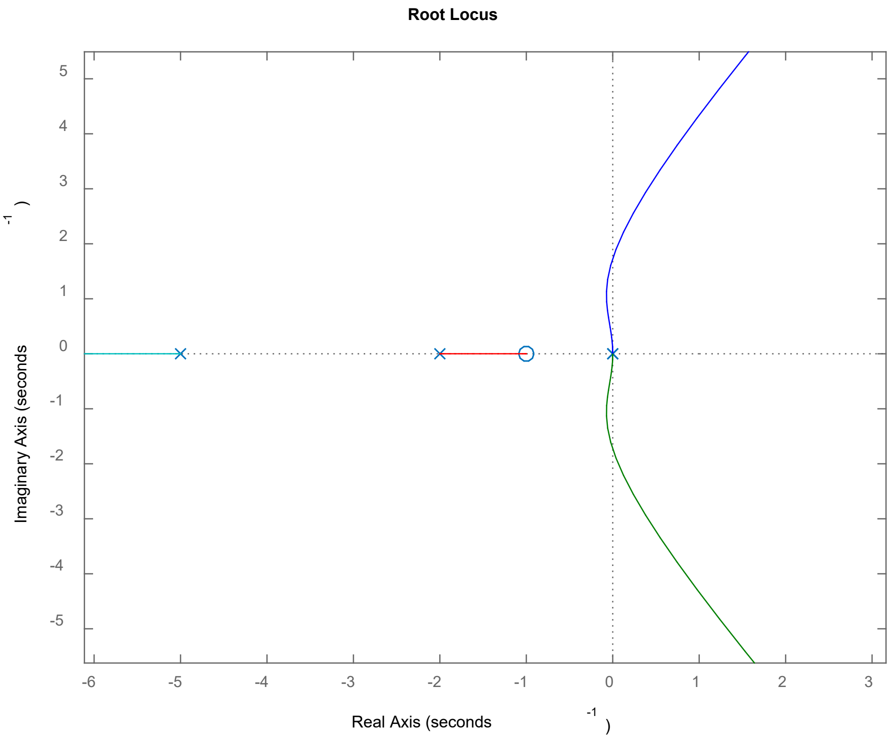
plugging into the first equation.

$\boxed{\begin{matrix} w = \pm 4.6187 \\ \Downarrow \\ k = 1865.8 \end{matrix}}$ or $\begin{matrix} \pm 0.2233 \\ \Downarrow \\ k < 0 \end{matrix}$ → NOT applicable

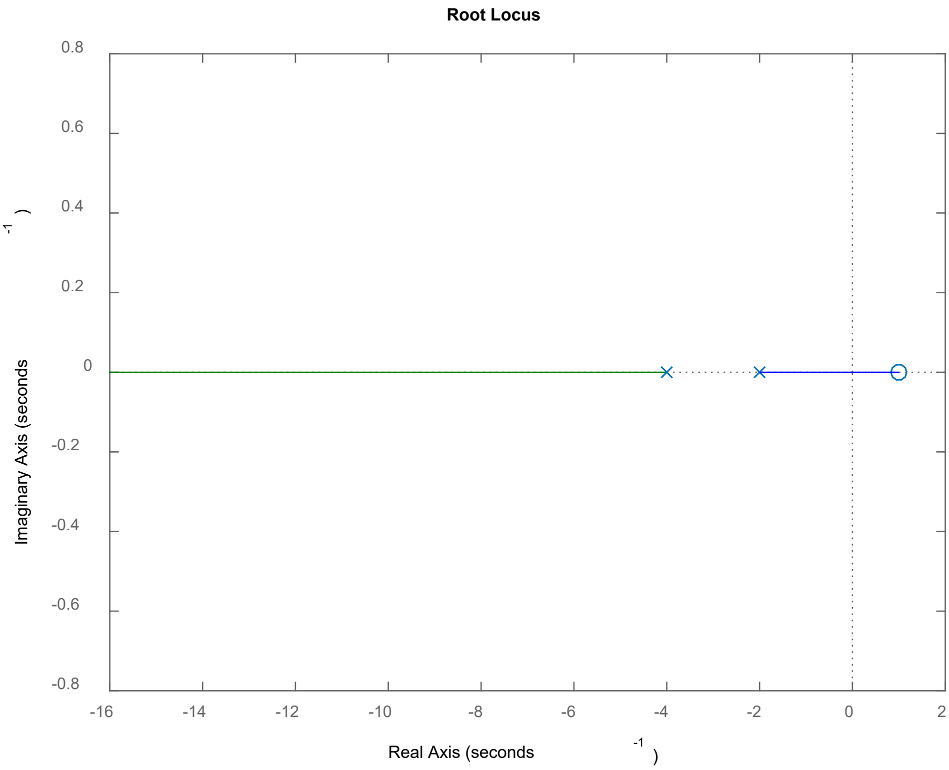


Problem 1

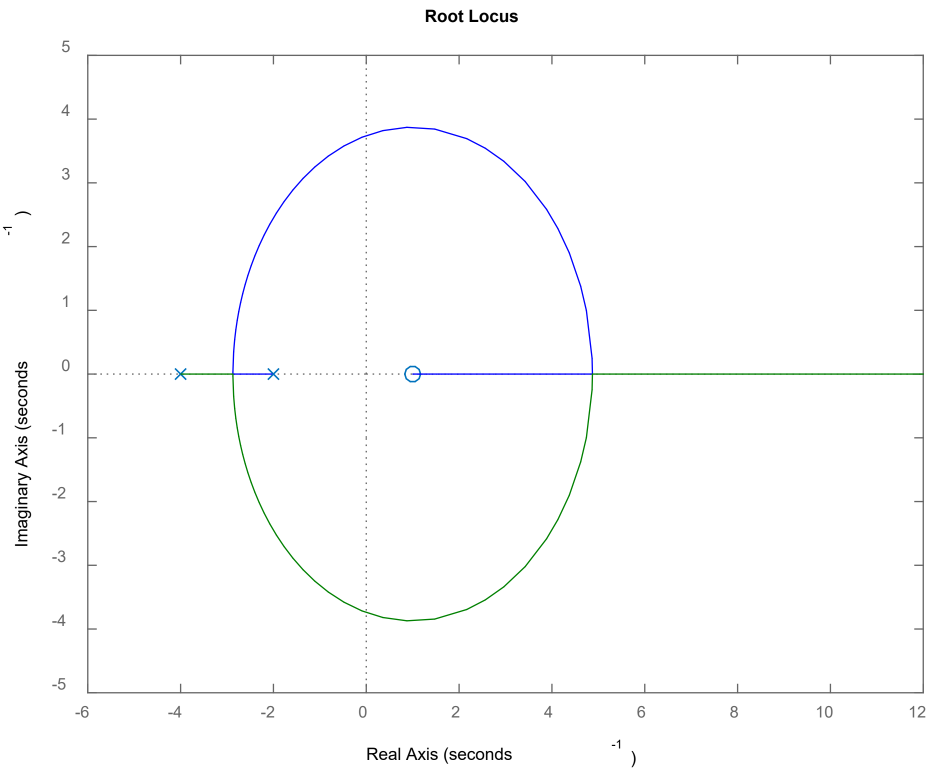
B-6-7



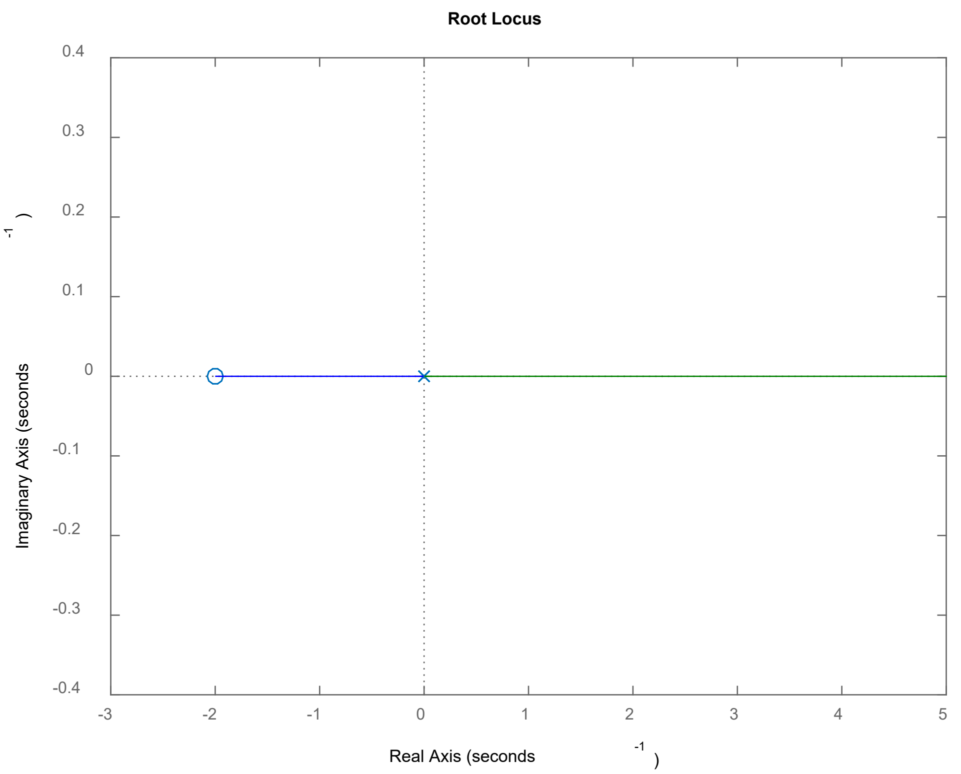

B-6-12 (a)



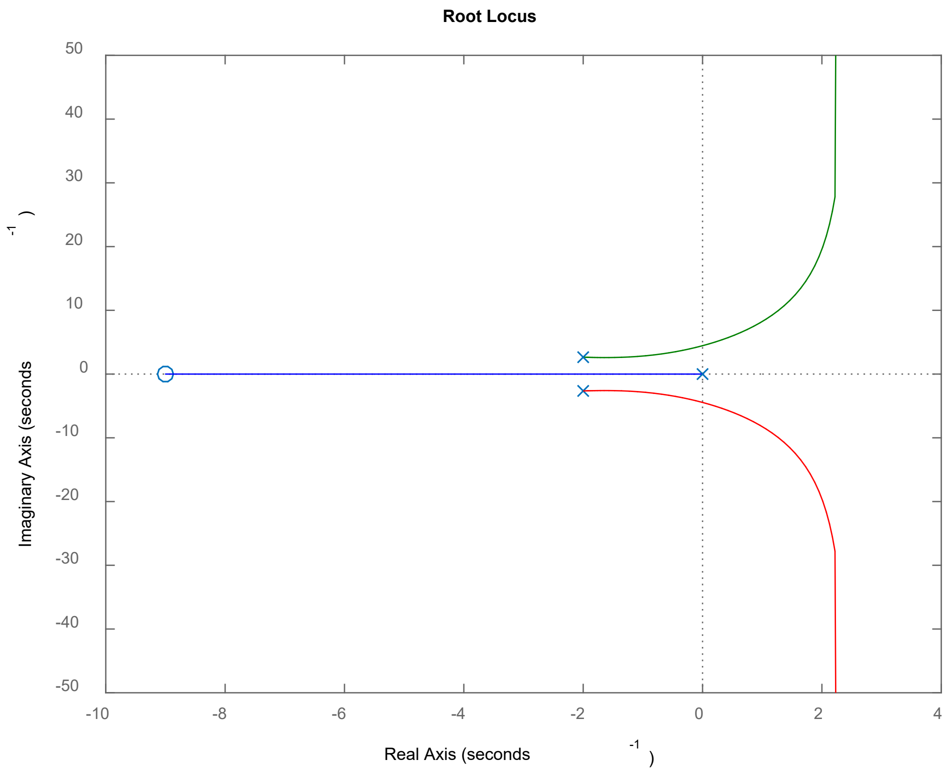
B-6-12 (b)



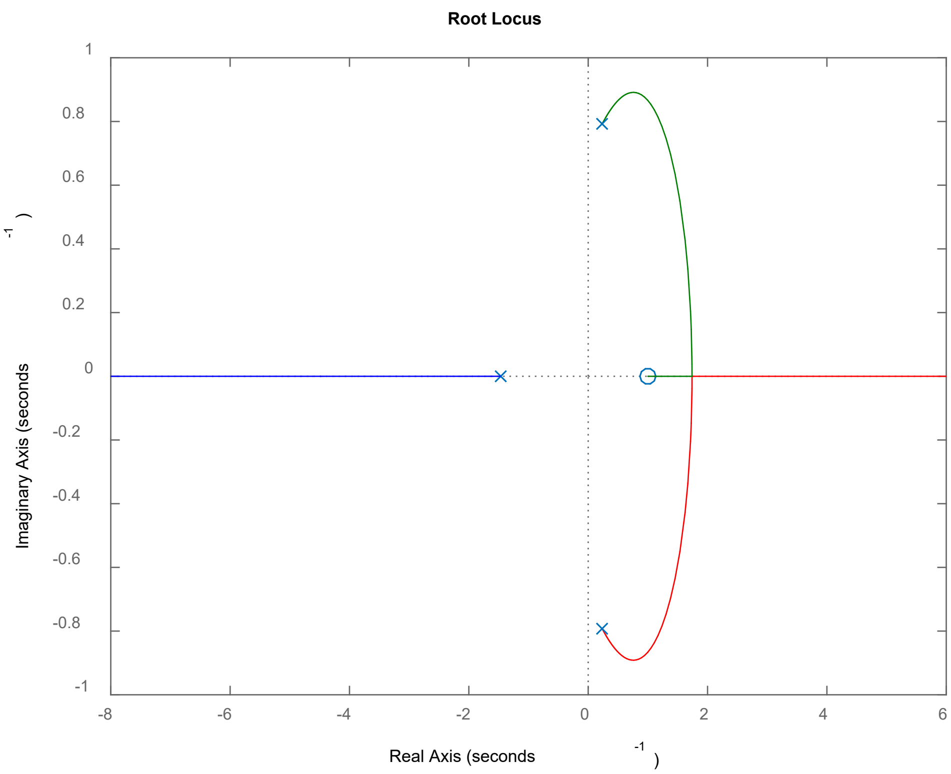

Problem 2-1



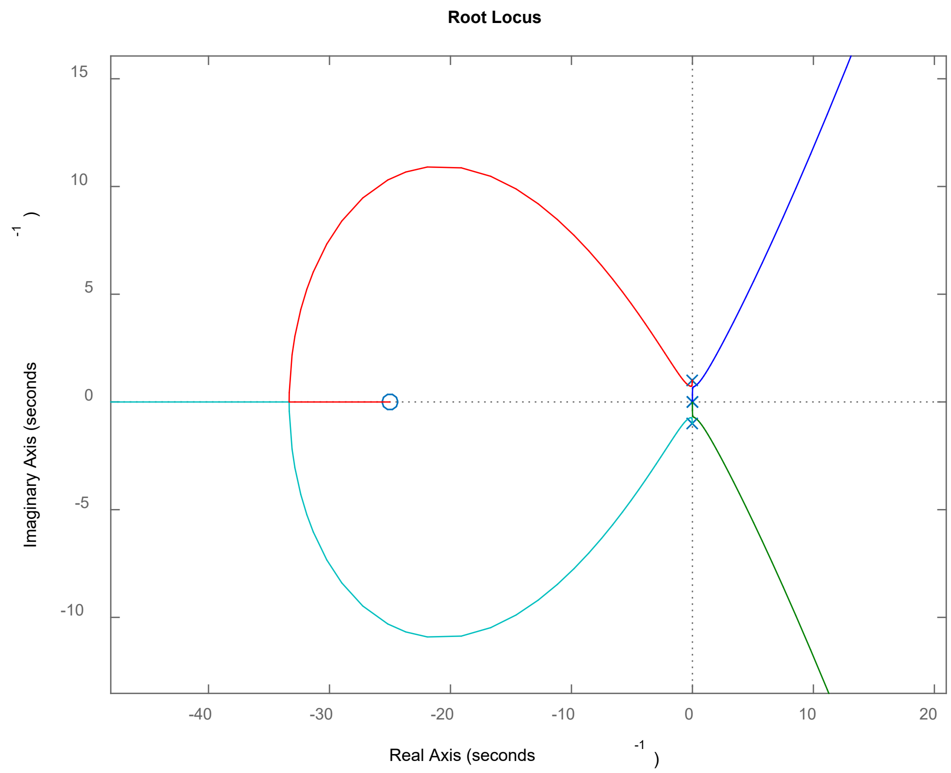
## Problem 2-2



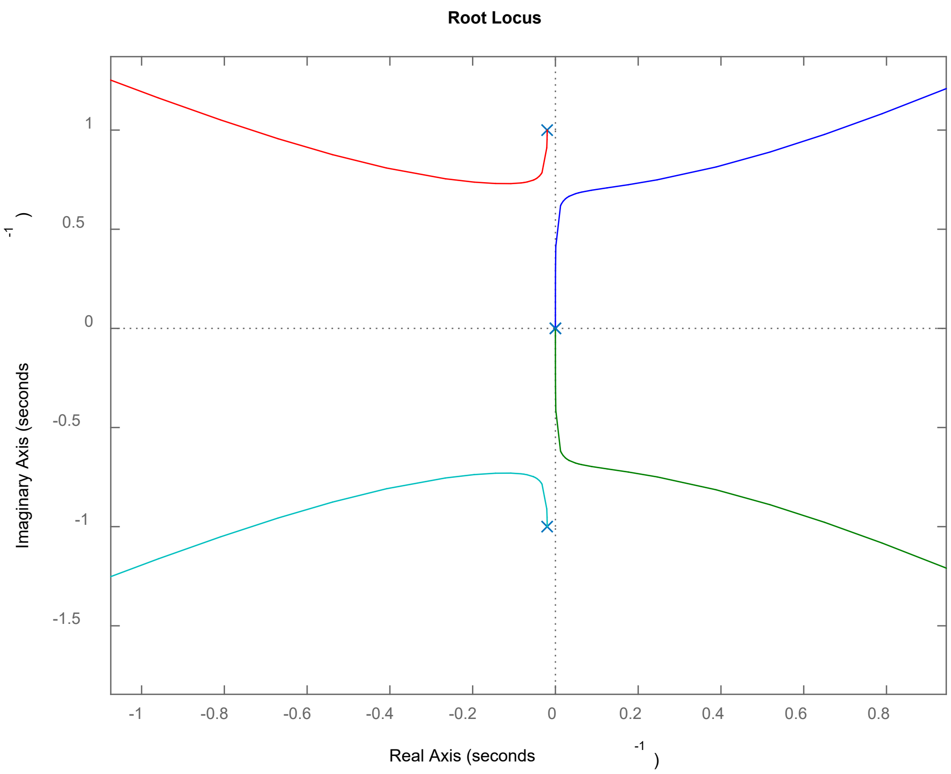

## Problem 2-3



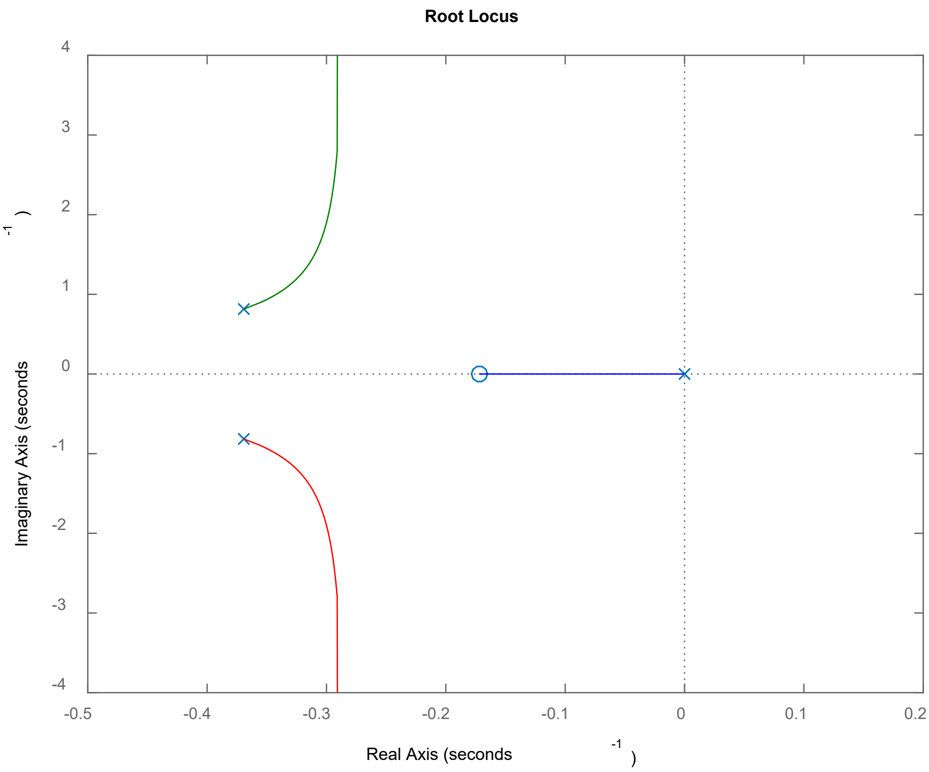
Problem 3



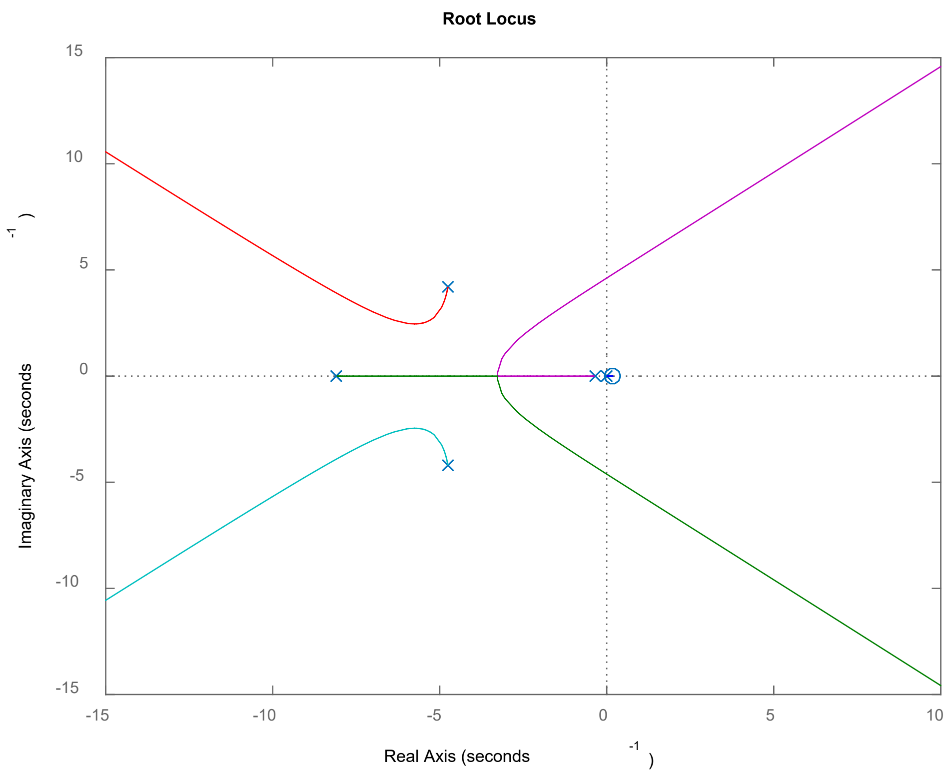

Zoom-in:



Problem 4-1




Problem 4-2



```
%P1
%B-6-7
L = zpk([-1], [-5 -2 0 0],2);
figure(1)
rlocus(L)
%B-6-12 (a)
L = zpk([1], [-2 -4], 1);
figure(2)
rlocus(L)
%B-6-12 (b)
L = zpk([1], [-2 -4], -1);
figure(3)
rlocus(L)
%P2-1
L = zpk([-2],[0 0],-1);
figure(4)
rlocus(L)
%P2-2
L = tf([1 9], [1 4 11 0]);
figure(5)
rlocus(L)
%P2-3
L = tf([-1 1], [1 1 0 1]);
figure(6)
rlocus(L)
%P3
L = tf([0.036 0.036*25], [1 0.04 1 0 0])
figure(7)
rlocus(L)
%P7-1
L = tf([1.1057 0.1900], [1 0.7385 0.8008 0])
figure(8)
rlocus(L)
%P7-2
L = tf([1.1057 -0.1900], [1 17.95 123.3 366.3 112.2 0])
figure(9)
rlocus(L)
```