AE 6230 – HW3: Mode Shapes and Responses of MDOF Systems

Out: November 1, 2022; **Due:** November 8, 2022 by 11:59 PM ET in Canvas

Guidelines

- Read each question carefully before doing any work;
- If you find yourself doing pages of math, pause and consider if there is an easier approach;
- You can consult any relevant materials;
- You can discuss solution approaches with others, but your submission must be your own work;
- If you have doubts, please ask questions in class, during office hours, and/or Piazza (no questions via email);
- The solution to each question should concisely and clearly show the steps;
- Simplify your results as much as possible;
- Box the final answer for each question;
- Submit any code with the solution (but remember to also submit all relevant plots).

Problem 1 – 40 points

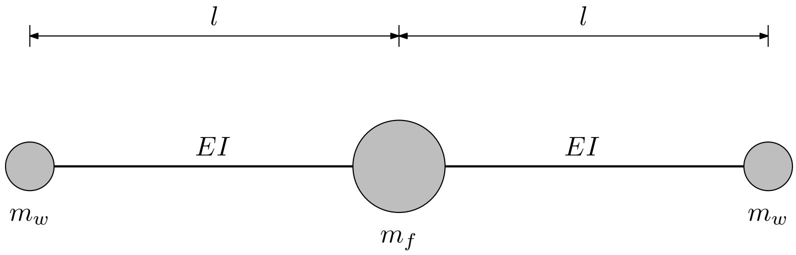


Figure 1: Schematic of an aircraft undergoing out-of-plane (vertical) bending vibrations in free flight.

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Half-wing mass	m_w	750 kg
Fuselage mass	m_f	$5m_w$
Wing semispan	l	10 m
Wing out-of-plane bending stiffness	EI	5×10^6 Nm^2

Figure 1 shows a simplified model for the out-of-plane (vertical) bending vibrations of a free-flying aircraft. The aircraft inertia is modeled by a concentrated mass m_f at the fuselage centerline and two concentrated masses m_w at the wing tips. The elasticity of each half wing is modeled by a beam of negligible mass with out-of-plane bending stiffness EI and length l, which behaves as a spring $k = 3EI/l^3$. The aircraft motion is described in terms of the vertical translations of the left, center, and right masses, denoted by $h_{wl}(t)$, $h_f(t)$, and $h_{wr}(t)$, respectively. These translations are positive upward and measured from the undeformed configuration of the aircraft in Fig. 1. Assuming small-amplitude vibrations and neglecting gravity, answer the following questions:

1. Considering the equations of motion

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_w & 0 \\ 0 & 0 & m_w \end{bmatrix} \begin{Bmatrix} \ddot{h}_f \\ \ddot{h}_{wl} \\ \ddot{h}_{wr} \end{Bmatrix} + \frac{3EI}{l^3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} h_f \\ h_{wl} \\ h_{wr} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{1}$$

   evaluate the natural frequencies for the parameters in Table 1 (in ascending order);

2. Evaluate the corresponding mode shapes normalized to have unit maximum displacement;

3. Plot the mode shapes from Question 2 and interpret their meaning;

4. Evaluate the inverse of the modal matrix $\mathbf{U}$ for the assumed mode shape normalization[1];

5. Assuming that a wind gust causes the initial conditions

$$\mathbf{q}(0) = \mathbf{q}_0 = \begin{Bmatrix} 0.5 \\ 0.0 \\ 0.0 \end{Bmatrix} \qquad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 = 0 \tag{2}$$

   determine the initial conditions for the modal equations;

6. Write the analytical expression of the damped free response in the form

$$\mathbf{q}(t) = \mathbf{U}\boldsymbol{\eta}(t) \tag{3}$$

   considering the modal viscous damping factors $\zeta_1 = 0, \zeta_2 = \zeta_3 = 0.04$;

7. Plot the components of $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ for $0 \leq t \leq 20$ s;

8. Explain the results from Question 7 (motivate the contribution from each mode).

[1]Note that the assumed mode shape normalization yields non-unit modal mass.

Problem 2 – 30 points

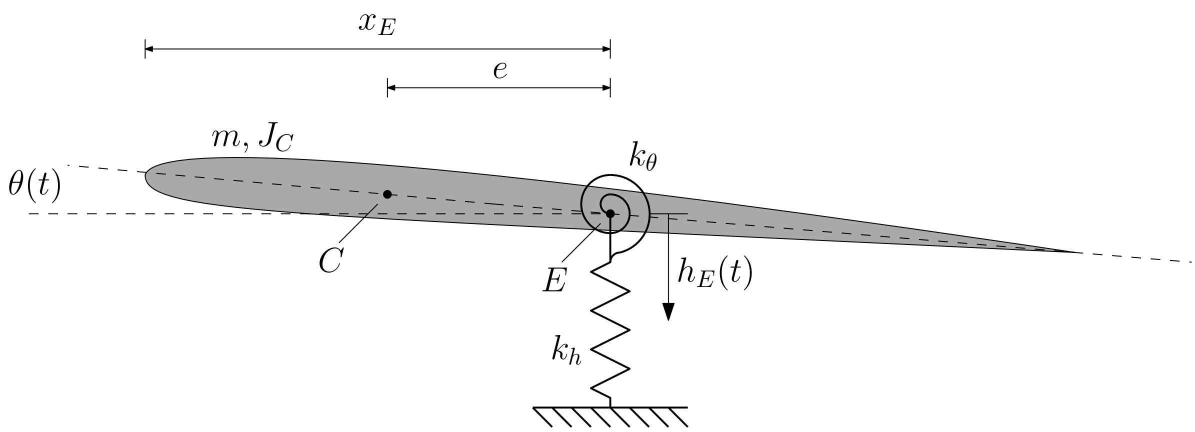


Figure 2: Schematic of typical section model.

Table 2: Parameter values for Problem 2.

Parameter	Symbol	Value
Mass	m	10 kg
Moment of inertia about E	J_E	0.08 kg·m^2
Chord	c	0.2 m
Offset of C from E (positive as in Fig. 2)	e	$-0.2c$
Position of E along the chord (positive as in Fig. 2)	x_E	$0.4c$
Translational spring stiffness	k_h	1000 N/m
Rotational spring stiffness	k_θ	200 Nm/rad

Consider the typical section model in Fig. 2, which is an abstraction for the cross section of a wing undergoing out-of-plane (vertical) bending and torsion. The typical section is subject to the excitation

$$\mathbf{Q}(t) = \mathbf{Q}_0 \sin \omega_0 t \tag{4}$$

with $Q_{0_1} = -10$ N, $Q_{0_2} = 1.5$ Nm, and $\omega_0 = 15$ rad/s. The modal mass and stiffness matrices along with the natural frequencies and mode shapes (normalized to have unit modal mass) can be computed using the script

`AE6230_Fall2022_L17_MDOF_Free_TypicalSection.m`

available in Canvas. Damping effects are captured by the modal viscous damping factors $\zeta_1 = \zeta_2 = 0.02$. Assuming small-amplitude vibrations and neglecting gravity, answer the following questions:

1. Determine the modal excitation $\mathbf{N}(t)$;
2. Considering the frequency response functions $H_1(\omega)$ and $H_2(\omega)$ associated with the modal coordinates $\boldsymbol{\eta}(t)$
   (a) Evaluate their magnitudes at the excitation frequency ω_0;
   (b) Evaluate their phase delays at that frequency;
3. Write the analytical expression of the damped steady-state response in the form of Eq. (3);
4. Plot the components of $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ for $0 \leq t \leq 2$ s;
5. Explain the results from Question 4 (motivate the contribution from each mode).

Problem 3 – 30 points

Consider the same typical section model as in Problem 2. The model experiences the step excitation

$$\mathbf{Q}(t) = \mathbf{Q}_0 \mathrm{u}(t) \tag{5}$$

with $Q_{0_1} = -10$, $Q_{0_2} = 1.5$ Nm, and zero initial conditions. Damping is captured by the proportional model

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{6}$$

where $\alpha = 1.0\ \mathrm{s}^{-1}$ and $\beta = 1 \times 10^{-5}$ s. Answer the following questions:

1. Evaluate the modal viscous damping factors ζ_1 and ζ_2;
2. Evaluate the damped frequencies ω_{d_1} and ω_{d_2};
3. Write the analytical expression of the damped response in the form of Eq. 3;
4. Plot the components of $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ for $0 \le t \le 10$ s;
5. Explain the results from Question 4 (motivate the contribution from each mode);
6. Obtain the results from Question 4 for $e = -0.05c$ and explain any qualitative changes.

Problem 1 Solution – 40 points

Question 1 – 5 points

To find the natural frequencies of the system, we solve the generalized eigenvalue problem

$$\left(\mathbf{K} - \omega^2\mathbf{M}\right)\mathbf{u} = \mathbf{0} \tag{7}$$

for non-trivial solutions $\mathbf{u} \neq \mathbf{0}$. This can be done using the matrix eigenvalue solver in MATLAB

```
[U, Omega2] = eig(K,M)
```

that returns the modal matrix `U` having the eigenvectors in its columns and the diagonal matrix `Omega2` of the eigenvalues (natural frequencies squared). Alternatively, we can set the determinant of the 3×3 coefficient matrix in Eq. (7) to vanish (required to have non-trivial solutions $\mathbf{u} \neq \mathbf{0}$) and solve the resulting characteristic equation. With the given problem parameters, the mass and stiffness matrices are

$$\mathbf{M} = \begin{bmatrix} 3750 & 0 & 0 \\ 0 & 750 & 0 \\ 0 & 0 & 750 \end{bmatrix} \text{kg} \qquad \mathbf{K} = \begin{bmatrix} 30000 & -15000 & -15000 \\ -15000 & 15000 & 0 \\ -15000 & 0 & 15000 \end{bmatrix} \text{N/m} \tag{8}$$

Using the MATLAB `eig` function and taking the square roots of the eigenvalues, we obtain the natural frequencies

$$\omega_1 = 0\,\text{rad/s} \qquad \omega_2 = 4.4721\,\text{rad/s} \qquad \omega_3 = 5.2915\,\text{rad/s} \tag{9}$$

The first natural frequency is zero, as expected for an aircraft free to translate vertically as a rigid body.

For completeness, the frequencies in Hz are

$$f_1 = 0\,\text{Hz} \qquad f_2 = 0.7118\,\text{Hz} \qquad f_3 = 0.8422\,\text{Hz} \tag{10}$$

Aircraft that have free-flight elastic frequencies below 1 Hz experience detrimental flight dynamics-aeroelastic coupling effects leading to issues such as reduced roll maneuverability and body-freedom flutter. These issues become more critical in next-generation transport aircraft, which tend to have more flexible wings due to the need for reducing fuel burn. Computing the natural frequencies of the unrestrained aircraft gives a first assessment of the potential for flight-dynamics aeroelastic coupling effects to arise, which is important for design.

Question 2 – 5 points

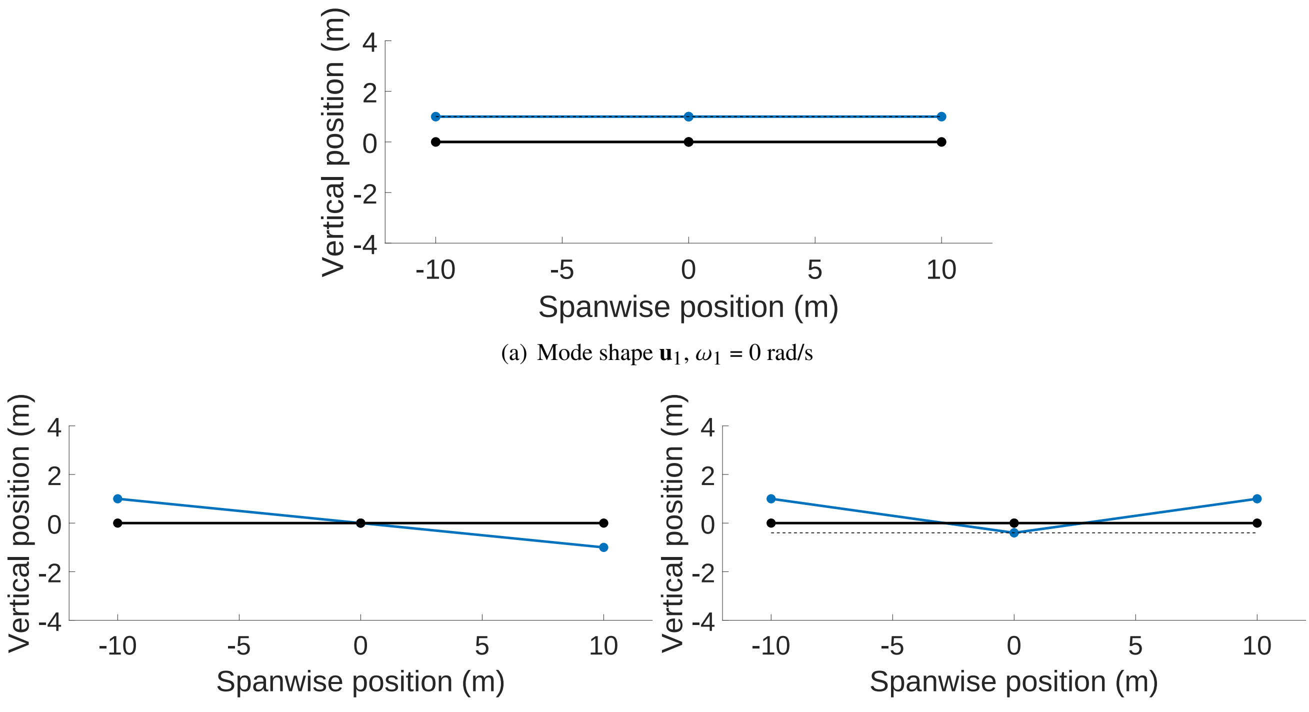
The mode shapes are given by the eigenvectors associated with the natural frequencies in Eq. (9). We can obtain these eigenvectors from the MATLAB matrix eigenvalue solver directly or by plugging each eigenvalue (natural frequency squared) into Eq. (7) and solving for the corresponding $\mathbf{u}$, whose components can only be determined within an arbitrary constant. In this problem, the eigenvectors are chosen to have unit maximum displacement:

$$\mathbf{u}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \text{m} \qquad \mathbf{u}_2 = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} \text{m} \qquad \mathbf{u}_3 = \begin{Bmatrix} -0.4 \\ 1 \\ 1 \end{Bmatrix} \text{m} \tag{11}$$

In the unit maximum displacement normalization, there are different conventions related on where to keep units. In this problem, we can keep the units of meters with the mode shape, which can be then interpreted as a dimensional displacement field, or with the modal coordinate, making the mode shapes dimensionless. In the end, it is important to verify that the solution vector $\mathbf{q}(t)$ has the correct units for each original coordinate.

Question 3 – 5 points

The mode shapes are plotted in Fig. 3. The first mode shape is a rigid-body mode representing a vertical translation of the aircraft as a whole. This is a non-zero displacement field that does not cause any elastic deformation, thus leaving the potential energy of the system unchanged. The second and third mode shapes are an anti-symmetric and a symmetric elastic mode, respectively. These modes approximate of the first anti-symmetric and symmetric out-of-plane bending modes of an aircraft in free flight. The displacements of the three masses in the elastic mode shapes are such that their weighted sum is zero, as required by the rigid-elastic orthogonality conditions. This means that the aircraft elastic motion does not change the position of the center of mass.



(a) Mode shape $\mathbf{u}_1$, $\omega_1 = 0$ rad/s

(b) Mode shape $\mathbf{u}_2$, $\omega_2 = 4.4721$ rad/s

(c) Mode shape $\mathbf{u}_3$, $\omega_3 = 5.2915$ rad/s

Figure 3: Mode shape plots for Problem 1 Question 3.

Question 4 – 5 points

The modal matrix inverse can be obtained considering that

$$\mathbf{U}^T\mathbf{M}\mathbf{U} = \overline{\mathbf{M}} \tag{12}$$

where $\overline{\mathbf{M}}$ is the modal mass matrix. In this problem, the modal mass matrix is a diagonal but not identity matrix because the mode shapes are chosen to have unit maximum displacement. Equation (12) gives

$$\mathbf{U}^{-1} = \left(\overline{\mathbf{M}}\right)^{-1}\mathbf{U}^T\mathbf{M} = \begin{bmatrix} 0.7143 & 0.1429 & 0.1429 \\ 0.0000 & 0.5000 & -0.5000 \\ -0.7143 & 0.3571 & 0.3571 \end{bmatrix} \text{m}^{-1} \tag{13}$$

Using Eq. (13), we can compute the modal matrix inverse even in cases where the number of modes retained in the solution is less than the number of original coordinates (degrees of freedom) and the modal matrix is rectangular. This situation is the standard, not the exception, when dealing with large-dimensional multi-degree-of-freedom systems such as those arising in finite element analysis. However, this problem involves a 3×3 modal matrix, whose inverse can also be computed using standard matrix inversion techniques.

Question 5 – 5 points

The initial conditions for the modal coordinates are given by

$$\boldsymbol{\eta}_0 = \mathbf{U}^{-1}\mathbf{q}_0 = \begin{Bmatrix} 0.3571 \\ 0 \\ -0.3571 \end{Bmatrix} \qquad \dot{\boldsymbol{\eta}}_0 = \mathbf{U}^{-1}\dot{\mathbf{q}}_0 = \mathbf{0} \tag{14}$$

The initial modal velocity vector $\dot{\boldsymbol{\eta}}_0$ is zero because the initial velocity vector $\dot{\mathbf{q}}_0$ is zero. The initial modal coordinate vector has a zero second component. This is expected because the initial displacements of the three masses are symmetric with respect to the aircraft centerline (only the fuselage mass is displaced) and do not excite the second mode that is anti-symmetric. In this problem, there would be no change in the free response results if we retained only the first and third mode in the solution.

Question 6 – 5 points

In the most general case, the free response is given by

$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \dot{\eta}_{1_0} t + \eta_{1_0} \\ e^{-\zeta_2 \omega_2 t} \left(\eta_{2_0} \cos \omega_{d_2} t + \dfrac{\dot{\eta}_{2_0} + \zeta_2\, \omega_2\, \eta_{2_0}}{\omega_{d_2}} \sin \omega_{d_2} t \right) \\ e^{-\zeta_3 \omega_3 t} \left(\eta_{3_0} \cos \omega_{d_3} t + \dfrac{\dot{\eta}_{3_0} + \zeta_3\, \omega_3\, \eta_{3_0}}{\omega_{d_3}} \sin \omega_{d_3} t \right) \end{Bmatrix} \tag{15}$$

The first modal coordinate is a linear function of time because the corresponding modal equation is

$$\ddot{\eta}_1(t) = 0 \tag{16}$$

In the provided script, the free response is implemented in the form of Eq. (15) for generality. However, several terms simplify due to the zero initial velocities and the second mode not contributing:

$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \eta_{1_0} \\ 0 \\ \eta_{3_0} e^{-\zeta_3 \omega_3 t} \left(\cos \omega_{d_3} t + \dfrac{\zeta_3}{\sqrt{1 - \zeta_3^2}} \sin \omega_{d_3} t \right) \end{Bmatrix} \tag{17}$$

or

$$\mathbf{q}(t) = \mathbf{u}_1\, \eta_{1_0} + \mathbf{u}_3\, \eta_{3_0} e^{-\zeta_3 \omega_3 t} \left(\cos \omega_{d_3} t + \frac{\zeta_3}{\sqrt{1 - \zeta_3^2}} \sin \omega_{d_3} t \right) \tag{18}$$

From Eq. (18), we expect the masses to undergo damped oscillations about the steady-state solution $\mathbf{u}_1 \eta_{1_0}$. Looking at the components of $\mathbf{u}_3$ in Eq. (11), we also expect the oscillation to be symmetric about the centerline and that the wing and fuselage masses always displace in opposite directions with respect to $\mathbf{u}_1 \eta_{1_0}$ at any given time.

Question 7 – 5 points

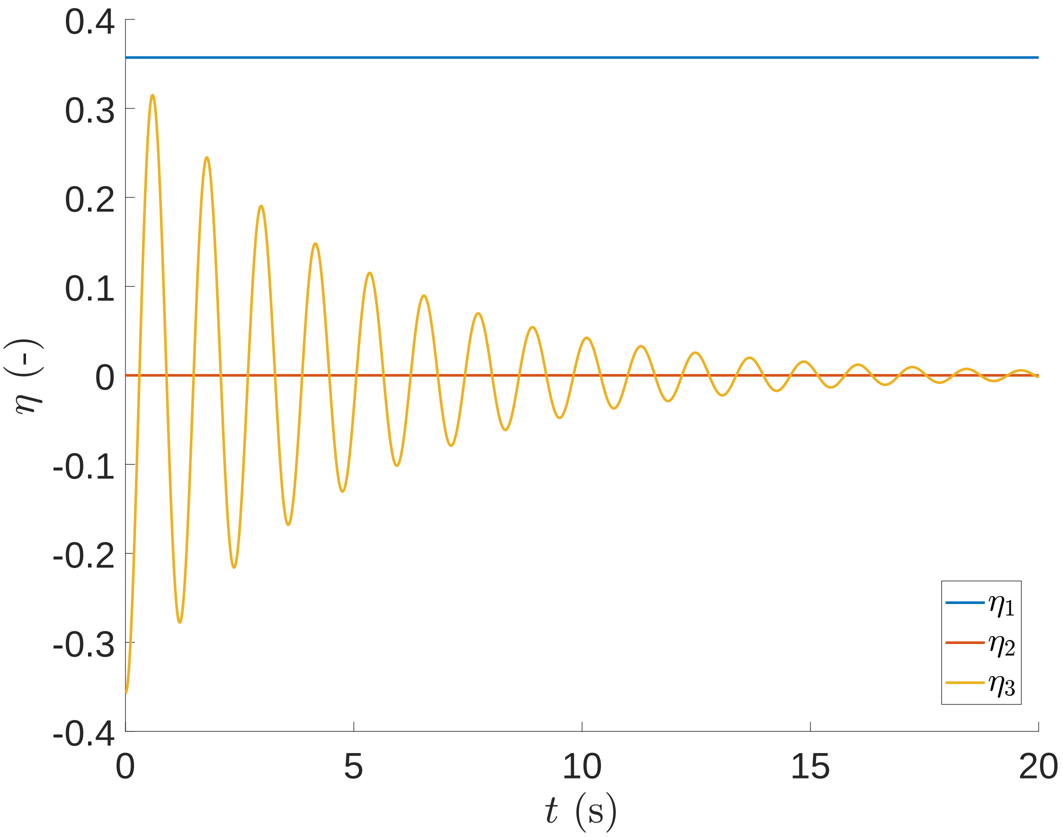
Figure (4) shows the time histories of the modal coordinates η_i and of the physical coordinates q_i $(i = 1, 2, 3)$.

Question 8 – 5 points

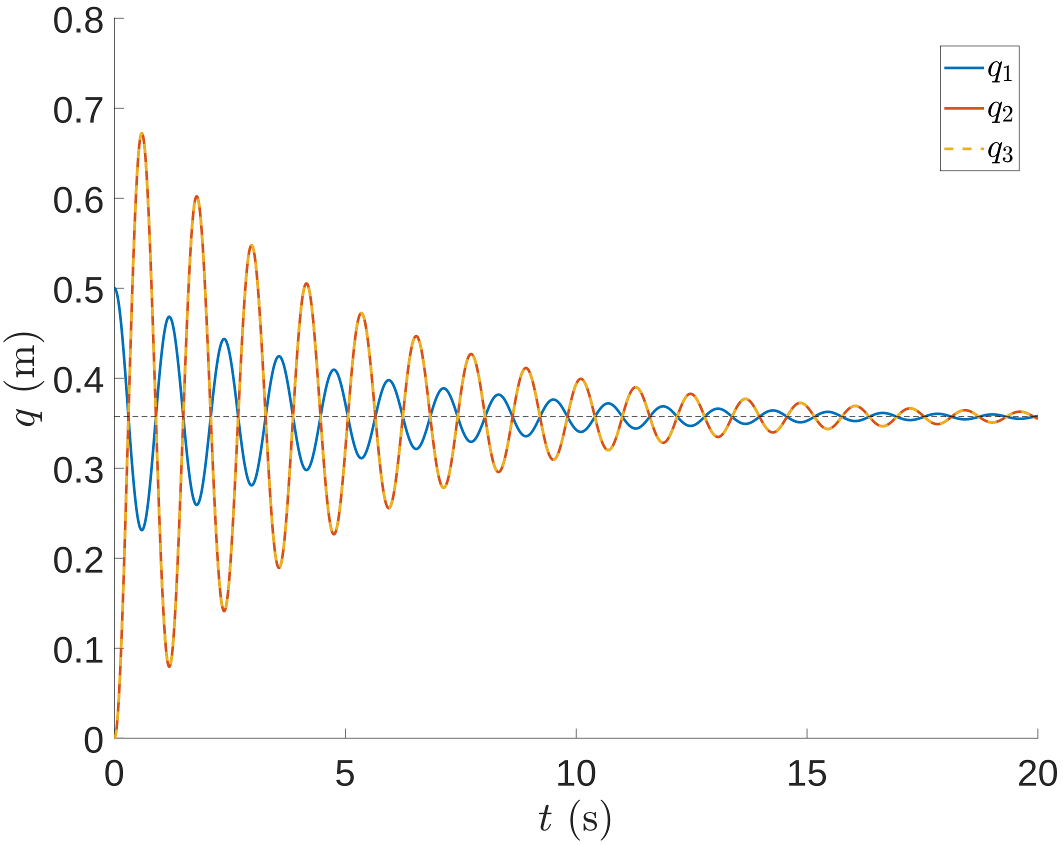
The time histories of the modal coordinates in Fig. 4(a) show the participation of each mode in the responses of the original coordinates $\mathbf{q}(t)$. The first modal coordinate (participation of rigid-body mode) achieves a constant value $\eta_{0_1} = 0.3571$, the second modal coordinate (participation of anti-symmetric elastic mode) is zero, and the third modal coordinate (participation of symmetric elastic mode) undergoes a damped oscillation about zero starting from $\eta_{0_3} = -0.3571$. These trends are expected based on the initial conditions.

The time histories of the original coordinates in Fig. 4(b) show the overall effect of the three modes combined through the modal coordinates. In this problem, the original coordinates are the displacement of the fuselage, left half-wing, and right half-wing masses, respectively. All masses oscillate about the steady-state solution given by the rigid-body term $\mathbf{u}_1 \eta_{1_0}$. The fuselage mass has an initial position of 0.5 m while the wing masses are initially at rest, as required by the initial conditions. The wing masses undergo the same oscillation due to the problem symmetry (geometric, inertial, and of the initial conditions). All masses oscillate with a single frequency ω_3 because the only oscillatory contribution to the response comes from the third mode (symmetric elastic mode). The fuselage and wing masses also displace in opposite directions with respect to the steady-state solution, and the weighted average of their positions (center of mass) is equal to 0.3571 m (rigid-body displacement). In other words, the elastic motion does not change the center of mass position, which is a consequence of the rigid-elastic orthogonality conditions.

The rigid-elastic orthogonality conditions are used in flight dynamics of flexible aircraft to achieve inertial decoupling between rigid-body and elastic degrees of freedom by choosing the mean axes as the body axes. With this choice, the equations of motion are only coupled through the aerodynamics.



(a) Modal coordinates



(b) Original coordinates

Figure 4: Time histories for Problem 1 Question 7.

Problem 2 Solution – 30 points

The modal analysis of the system gives the natural frequencies

$$\omega_1 = 9.9589\,\text{rad/s} \qquad \omega_2 = 56.1322\,\text{rad/s} \tag{19}$$

and the mode shapes

$$\mathbf{u}_1 = \begin{Bmatrix} -0.3136\,\text{m} \\ -0.0648\,\text{rad} \end{Bmatrix} \text{kg}^{-1/2}\text{m}^{-1} \qquad \begin{Bmatrix} -0.1633\,\text{m} \\ 3.9523\,\text{rad} \end{Bmatrix} \text{kg}^{-1/2}\text{m}^{-1} \tag{20}$$

chosen to have unit modal masses. The normalization yields the units of the mode shapes in Eq. (20) as the modal masses based on the non-normalized mode shapes have units of kg·m^2. The first mode shape primarily involves the plunge motion of the typical section with small pitch about the elastic center. The second mode shape primarily involves pitch motion about the elastic center with some plunge.

Question 1 – 5 points

The modal excitation is given by

$$\mathbf{N}(t) = \mathbf{U}^T\mathbf{Q}_0 \sin\omega_0 t = \mathbf{N}_0 \sin\omega_0 t \tag{21}$$

with

$$\mathbf{N}_0 = \mathbf{U}^T\mathbf{Q}_0 = \begin{Bmatrix} 3.0388 \\ 7.5612 \end{Bmatrix} \text{kg}^{1/2}\text{m/s}^2 \tag{22}$$

The excitation has a larger projection onto the second mode shape.

Question 2 – 5 points (magnitudes) + 5 points (phases)

In the case of unit modal masses, the frequency response function for the ith modal coordinate can be written as

$$|H_i(\omega)| = \frac{1}{\sqrt{\left(\omega_i^2 - \omega^2\right)^2 + (2\zeta_i\omega_i\omega)^2}} \tag{23}$$

Evaluating Eq. (23) for $i = 1, 2$ and $\omega = \omega_0$ gives

$$|H_1(\omega_0)| = 0.0079\,(\text{rad/s})^{-2} \qquad |H_2(\omega_0)| = 0.0003\,(\text{rad/s})^{-2} \tag{24}$$

The magnitude $|H_1(\omega_0)|$ is larger than $|H_2(\omega_0)|$ (by a factor of more than 20) because the excitation frequency ω_0 is much closer to the first natural frequency ω_1.

The phase delay of the frequency response function for the ith modal coordinate is

$$\phi_i(\omega) = \tan^{-1}\left(\frac{2\zeta_i\omega_i\omega}{\omega_i^2 - \omega^2}\right) \tag{25}$$

Evaluating Eq. (25) for $i = 1, 2$ and $\omega = \omega_0$ gives

$$\phi_1(\omega_0) = 3.0941\,\text{rad} = 177.2810\,\text{deg} \qquad \phi_2(\omega_0) = 0.0115\,\text{rad} = 0.6595\,\text{deg} \tag{26}$$

The first modal coordinate lags the excitation by about π, while the second modal coordinate is almost in phase with the excitation. This is expected because $\omega_1 < \omega_0 < \omega_2$ such that the excitation frequency ω_0 is "on the right" of the resonance condition for the modal frequency response function $H_1(\omega_0)$ and "on the left" of that for $H_2(\omega_0)$. In the absence of damping, the phase delays would be π and zero, respectively. The phase difference between the two modes is also about π.

Note that we must evaluate Eq. (25) using the inverse tangent function that can deal with complex numbers, not the regular inverse tangent function, to get the correct phase delay for all modal coordinates. The regular inverse tangent function gives the correct phase delay only for the second modal coordinates, which lies between $-\pi/2$ and $\pi/2$.

Question 3 – 5 points

The steady-state harmonic response of the system is given by

$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} N_{0_1} |H_1(\omega)| \sin[\omega_0 t - \phi_1(\omega)] \\ N_{0_2} |H_2(\omega)| \sin[\omega_0 t - \phi_2(\omega)] \end{Bmatrix} \tag{27}$$

where

$$N_{0_1} |H_1(\omega)| = 0.0241\,\mathrm{kg}^{1/2}\,\mathrm{m} \qquad N_{0_2} |H_2(\omega)| = 0.0026\,\mathrm{kg}^{1/2}\,\mathrm{m} \tag{28}$$

We can also write

$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} U_{11}\, N_{0_1} |H_1(\omega)| \sin[\omega_0 t - \phi_1(\omega)] + U_{12}\, N_{0_2} |H_2(\omega)| \sin[\omega_0 t - \phi_2(\omega)] \\ U_{21}\, N_{0_1} |H_1(\omega)| \sin[\omega_0 t - \phi_1(\omega)] + U_{22}\, N_{0_2} |H_2(\omega)| \sin[\omega_0 t - \phi_2(\omega)] \end{Bmatrix} \tag{29}$$

where

$$\begin{aligned} U_{11}\, N_{0_1} |H_1(\omega)| &= -0.0076\,\mathrm{m} \qquad & U_{12}\, N_{0_1} |H_1(\omega)| &= -4.2192 \times 10^{-4}\,\mathrm{m} \\ U_{12}\, N_{0_1} |H_1(\omega)| &= -0.0016\,\mathrm{rad} \qquad & U_{12}\, N_{0_1} |H_1(\omega)| &= 0.0102\,\mathrm{rad} \end{aligned} \tag{30}$$

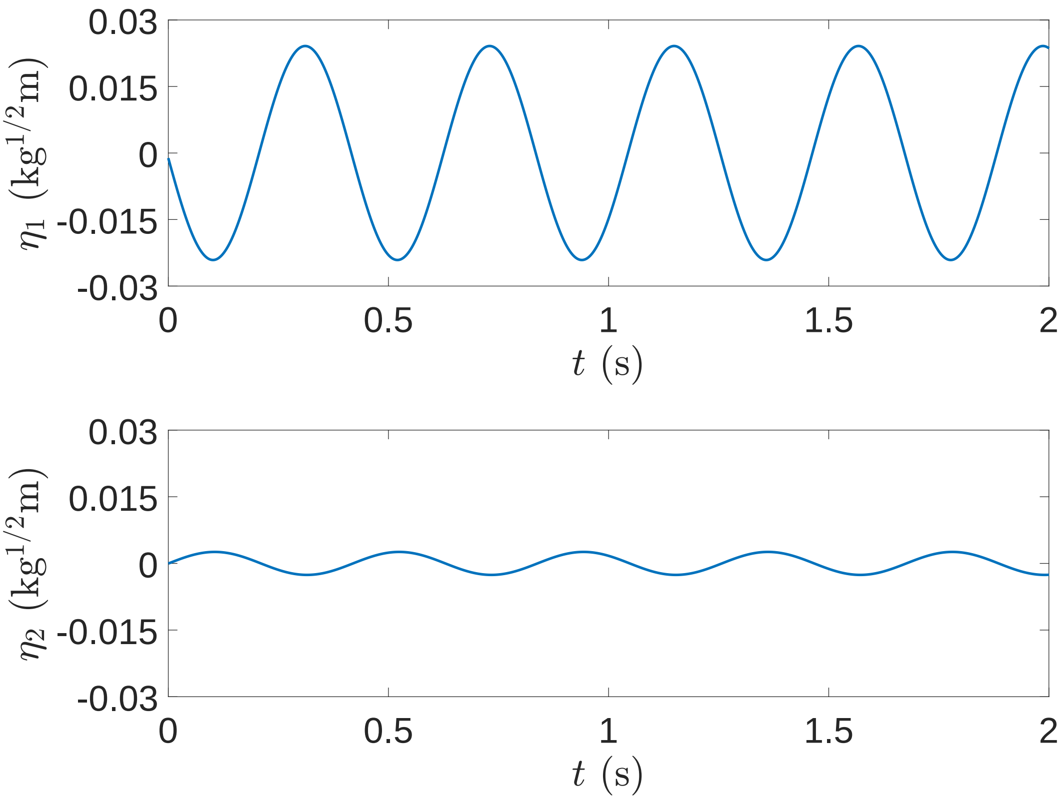
Question 4 – 5 points

Figure (5) shows the steady-state responses of η_i and q_i $(i = 1, 2, 3)$. Although the responses span the time interval $0 \le t \le 2$ s, these represent the responses after a long enough time that the transient terms (homogeneous solution) have vanished. The provided script verifies the steady-state responses against time marching, which shows the effect of the transient terms assuming zero initial conditions.

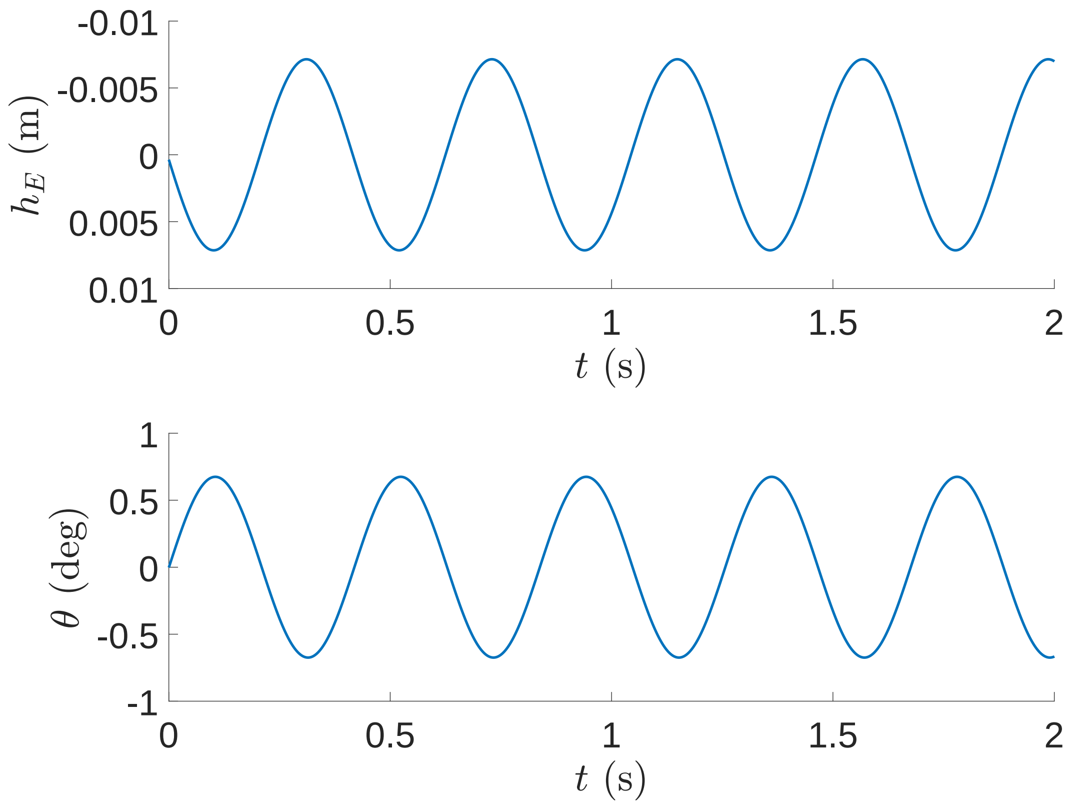
Question 5 – 5 points

The steady-state responses of all coordinates involve only the excitation frequency, as expected. The time histories in Fig. 5(a) show that the first modal coordinate is almost in opposition of phase with the excitation and its amplitude is one order of magnitude larger compared with the second modal coordinate, which is almost in phase with the excitation. These characteristics are expected because the excitation frequency ω_0 is such that $\omega_1 < \omega_0 < \omega_2$ and it closer to ω_1 than ω_2. The first and second modal coordinates also have a phase difference of almost π, which would be exactly π in the absence of damping.

Figure 5(b) shows the original coordinates (plunge and pitch degrees of freedom). The plunge time history is shown with an inverted axis because vertical displacements are positive downward. The contributions from the two modes are subtracted from each other for the plunge degree of freedom, while they are summed for the pitch degree of freedom [see Eqs. (26), (29), (30)]. This is because the multiplication by the modal matrix introduces a π phase shift between the modal contributions for the pitch degree of freedom (the pitch components of the two mode shapes have opposite signs). The plunge degree of freedom is dominated by the contribution from the first mode, while the pitch degree of freedom by the second mode. The motion is such that the typical section pitches up when it plunges down because the phase difference between the modal coordinates is practically canceled out when combining them through the modal matrix.



(a) Modal coordinates



(b) Original coordinates

Figure 5: Time histories for Problem 2 Question 4.

Problem 3 Solution – 30 points

The natural frequencies and mode shapes for Questions 1 through 5 are the same as in Problem 2. The excitation vector also has the same projection onto the mode shapes as in Eq. (22), though the excitation time dependency is that of a step function now.

Question 1 – 5 points

The modal viscous damping factors can be obtained from

$$\overline{c}_i = 2\zeta_i \omega_i = \alpha + \beta \omega_i^2 \qquad (i = 1, 2) \tag{31}$$

where $\overline{c}_i$ is the ith diagonal element of the modal viscous damping matrix

$$\overline{\mathbf{C}} = \mathbf{U}^T \mathbf{C} \mathbf{U} = \mathbf{U}^T \left(\alpha \mathbf{M} + \beta \mathbf{K}\right) \mathbf{U} = \alpha \mathbf{I} + \beta \mathbf{\Omega}^2 \tag{32}$$

Equation (31) gives

$$\zeta_i = \frac{1}{2}\left(\frac{\alpha}{\omega_i} + \beta \omega_i\right) \qquad (i = 1, 2) \tag{33}$$

Evaluating for $i = 1, 2$ gives

$$\zeta_1 = 0.0503 \qquad \zeta_2 = 0.0092 \tag{34}$$

Question 2 – 5 points

The damped natural frequencies are given by

$$\omega_{d_i} = \omega_i \sqrt{1 - \zeta_i^2} \qquad (i = 1, 2) \tag{35}$$

Evaluating for $i = 1, 2$ gives

$$\omega_{d_1} = 9.9464\,\mathrm{rad/s} \qquad \omega_{d_2} = 56.1298\,\mathrm{rad/s} \tag{36}$$

Question 3 – 5 points

The forced response of the system is given by

$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} \eta_1(t) \\ \eta_2(t) \end{Bmatrix} = \mathbf{U} \begin{Bmatrix} N_{0_1}\, S_1(t) \\ N_{0_2}\, S_2(t) \end{Bmatrix} \tag{37}$$

where

$$S_i(t) = \frac{1}{\omega_i^2}\left[1 - e^{-\zeta_i \omega_i t}\left(\cos \omega_{d_i} t + \frac{\zeta_i \omega_i}{\omega_{d_i}} \sin \omega_{d_i} t\right)\right] \tag{38}$$

is the step response function for the ith modal coordinate for the case of unit modal mass. We can also write

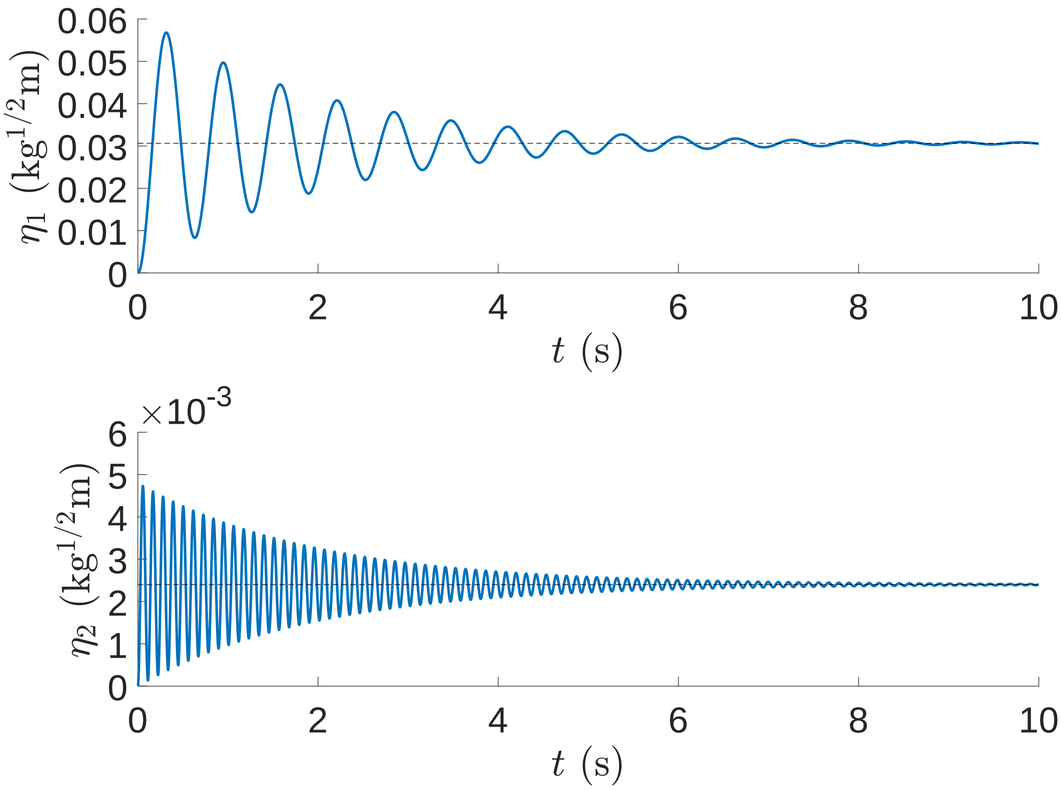
$$\mathbf{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix} = \begin{Bmatrix} U_{11}\, N_{0_1}\, S_1(t) + U_{12}\, N_{0_2}\, S_2(t) \\ U_{21}\, N_{0_1}\, S_1(t) + U_{22}\, N_{0_2}\, S_2(t) \end{Bmatrix} \tag{39}$$

Question 4 – 5 points

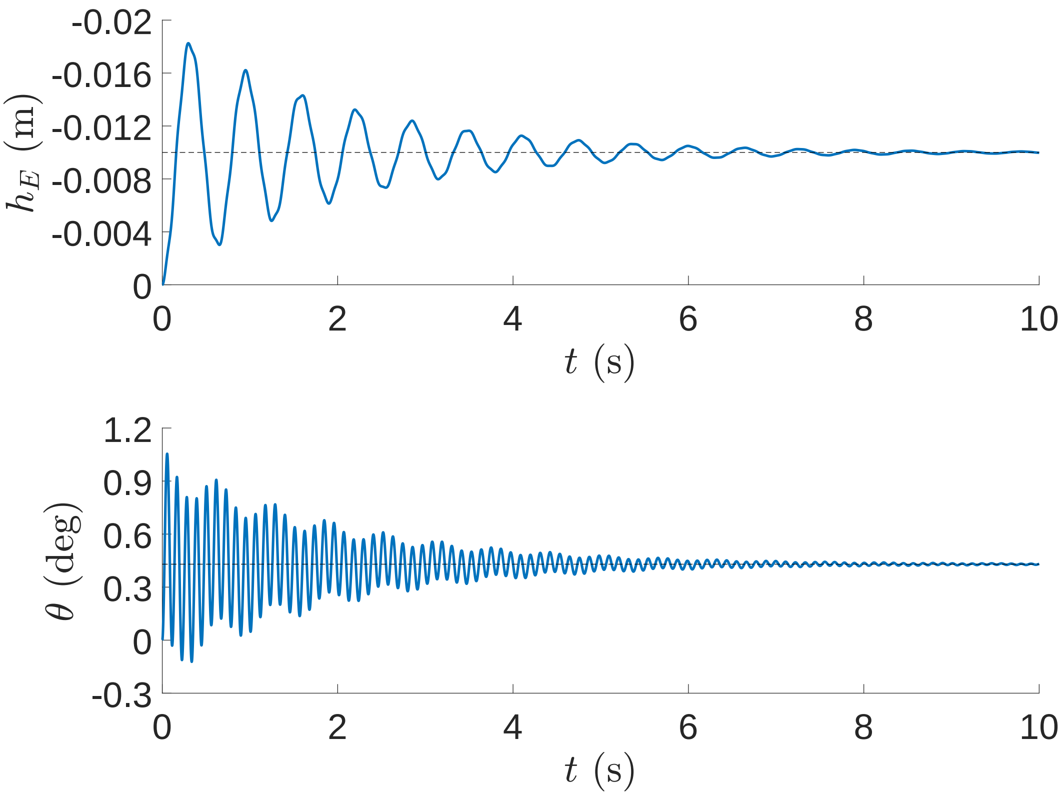
Figure (6) shows the time histories of η_i and q_i $(i = 1, 2, 3)$.

Question 5 – 5 points

The time histories in Fig. 6(a) show that the amplitude of the first modal coordinate is one order of magnitude that of the second one, like in Problem 2. Although the excitation has a larger projection onto the second mode shape, the second natural frequency squared at the denominator of the step response function is also larger. Each modal coordinate oscillates about its static value N_{0_i}/ω_i^2 with the corresponding damped frequency ω_i. The rate of decay is approximately the same because the term $\zeta_i \omega_i$ in the exponential function in the step response functions achieves the same value of about 0.5 for each mode (slightly larger for the second mode).



(a) Modal coordinates



(b) Original coordinates

Figure 6: Time histories for Problem 3 Question 4.

Figure 5(b) shows the original coordinates (plunge and pitch degrees of freedom). Each time history involves contributions from each mode, evidenced by the presence of two frequencies. The non-zero offset between the typical section center of mass and the point where the springs are attached results in coupled plunge-pitch modes, which are both active in the response. Like in Problem 2, the plunge degree of freedom mainly involves the first mode, while the pitch degree of freedom mainly involves the second mode. This is because the pitch contribution in the first mode is small compared with the plunge contribution, and the plunge contribution in the second mode is small compared with the one from pitch as shown by the eigenvectors in Eq. (20).

Note that the time step for plotting the results must be small enough to resolve the highest frequency involved in the response, which is about 56 rad/s in this problem. This means that the time step should be of the order of 0.005 s or smaller (assuming to have at least 20 points within the fastest oscillation). Knowing the natural frequencies of a structure and which modes are active in the response is important for choosing the time step not only for visualization purposes, but also (and more importantly) when using direct time integration.

Question 6 – 5 points

If we set $e = -0.05c$, the modal scenario changes. The modal analysis gives

$$\omega_1 = 9.9974\,\mathrm{rad/s} \qquad \omega_2 = 50.3286\,\mathrm{rad/s} \tag{40}$$

and

$$\mathbf{u}_1 = \begin{Bmatrix} -0.3161\,\mathrm{m} \\ -0.0165\,\mathrm{rad} \end{Bmatrix} \mathrm{kg}^{-1/2}\mathrm{m}^{-1} \qquad \begin{Bmatrix} -0.0370\,\mathrm{m} \\ 3.5578\,\mathrm{rad} \end{Bmatrix} \mathrm{kg}^{-1/2}\mathrm{m}^{-1} \tag{41}$$

The lower e (in magnitude) mainly reduces the second natural frequency and the plunge-pitch coupling in the modal matrix. This is expected considering that the typical section center of mass shifts closer to the attachment point of the springs (abstraction for the elastic axis), approaching the condition of full inertial and elastic decoupling. For completeness, we have

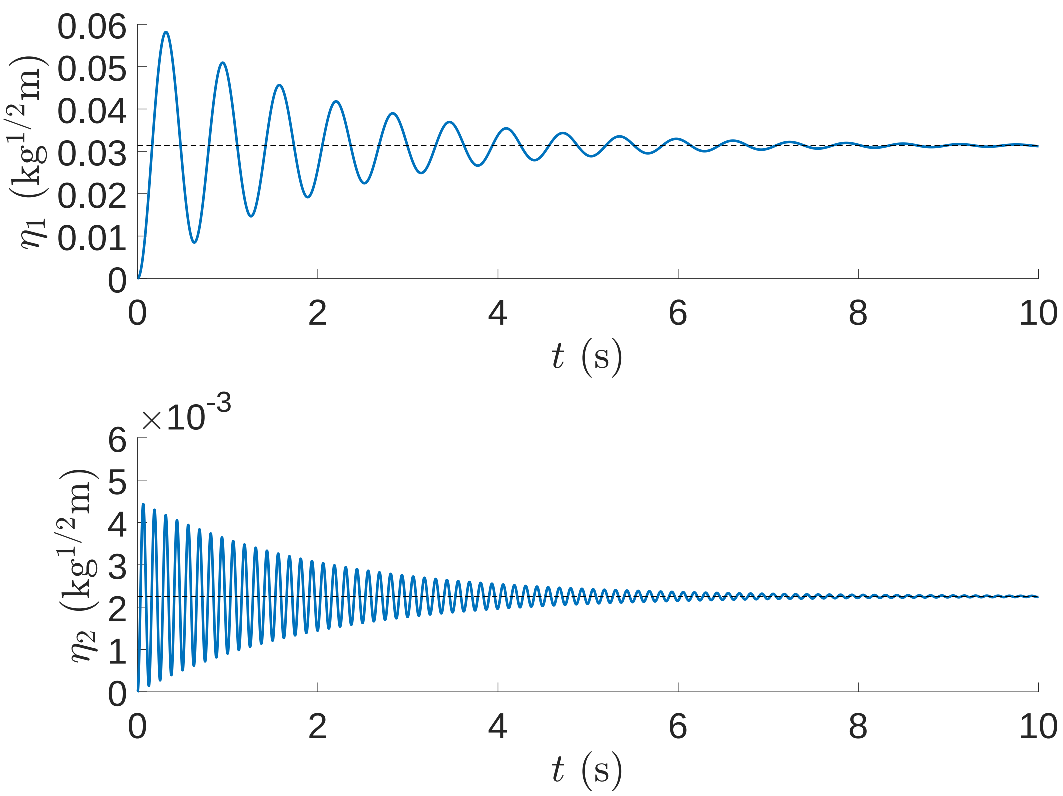
$$\zeta_1 = 0.0501 \qquad \zeta_2 = 0.0102 \tag{42}$$

and

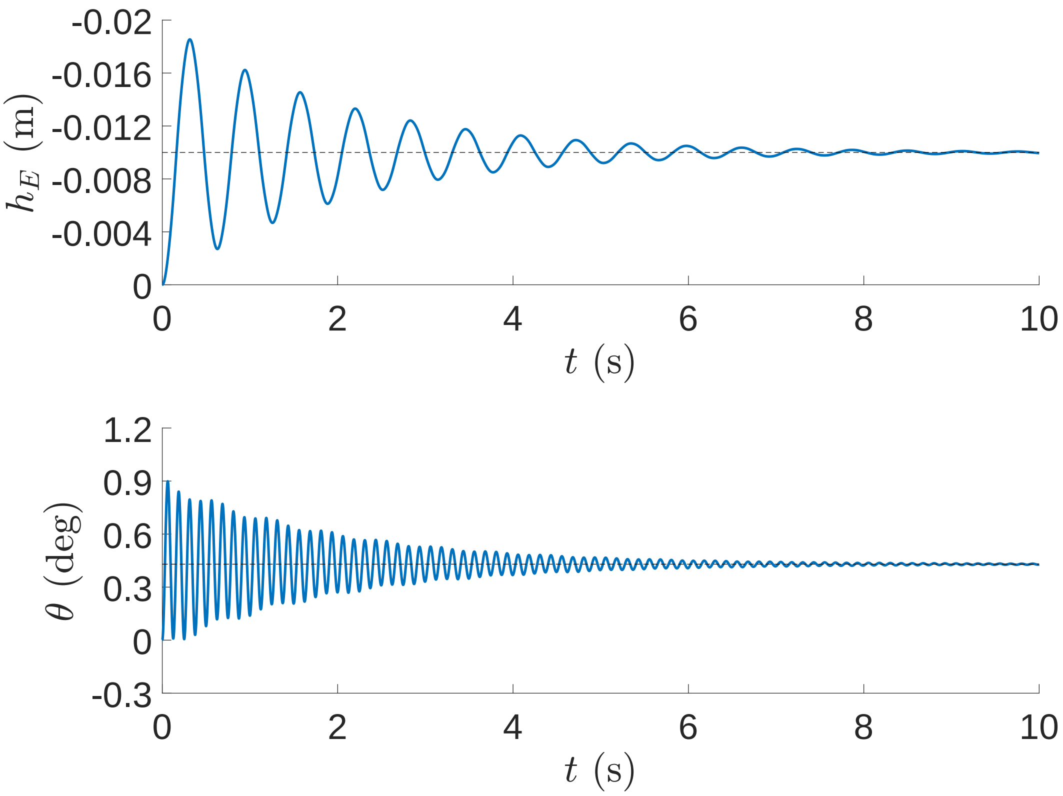
$$\omega_{d_1} = 9.9849\,\mathrm{rad/s} \qquad \omega_{d_2} = 50.3259\,\mathrm{rad/s} \tag{43}$$

With these remarks in mind, we can look at Fig. (7), which shows the time histories of η_i and q_i $(i = 1, 2, 3)$ for $e = -0.05c$. Figure (8) compares the responses for the two values of e. The time histories for the plunge and pitch degrees of freedom show a lower degree of coupling, more closely resembling single-frequency oscillations. In fact, the plunge and pitch degrees of freedom are modal coordinates for the system when $e = 0$. The plunge amplitude is comparable to the previous case, the pitch amplitude is lower. Based on Fig. (8), this can be attributed to the reduced contribution from the lower-frequency oscillation.

A takeaway from this question is that if a structure is modified, its dynamic characteristics (natural frequencies, mode shapes, dynamic response) must be reevaluated. This is one of the (many) complications associated with bringing structural dynamics considerations in design exploration and optimization, where one has to analyze a large number of designs and many load cases per design. For gradient-based optimization, another complication arises from the need for differentiating the eigenvalue problem with respect to the design variables. Situations where modal decoupling does not hold pose additional challenges. For aerospace structures in the linear regime, two examples of situations where modal decoupling does not hold are if there is aerodynamic loading (aeroelasticity) and if damping levels and/or modal density are large (relevant to launch vehicles). The situation gets worse if nonlinear effects play a role and the superposition principle does not longer hold, which prevents from using the tools covered in this class.

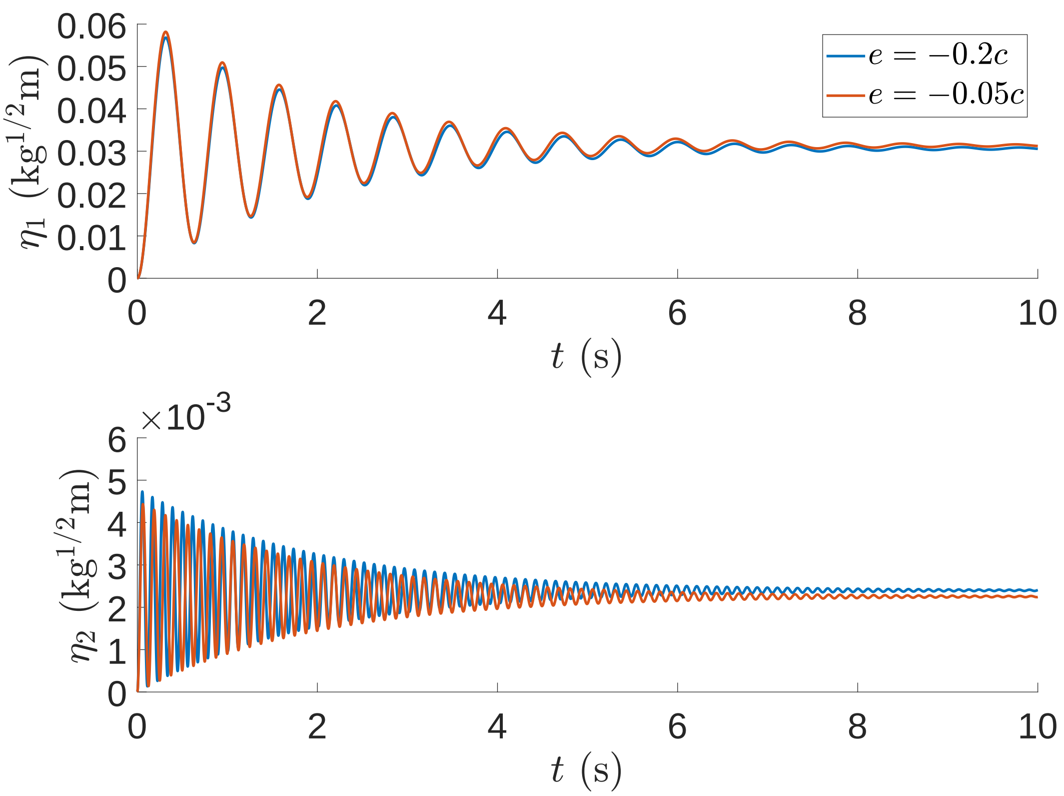


(a) Modal coordinates

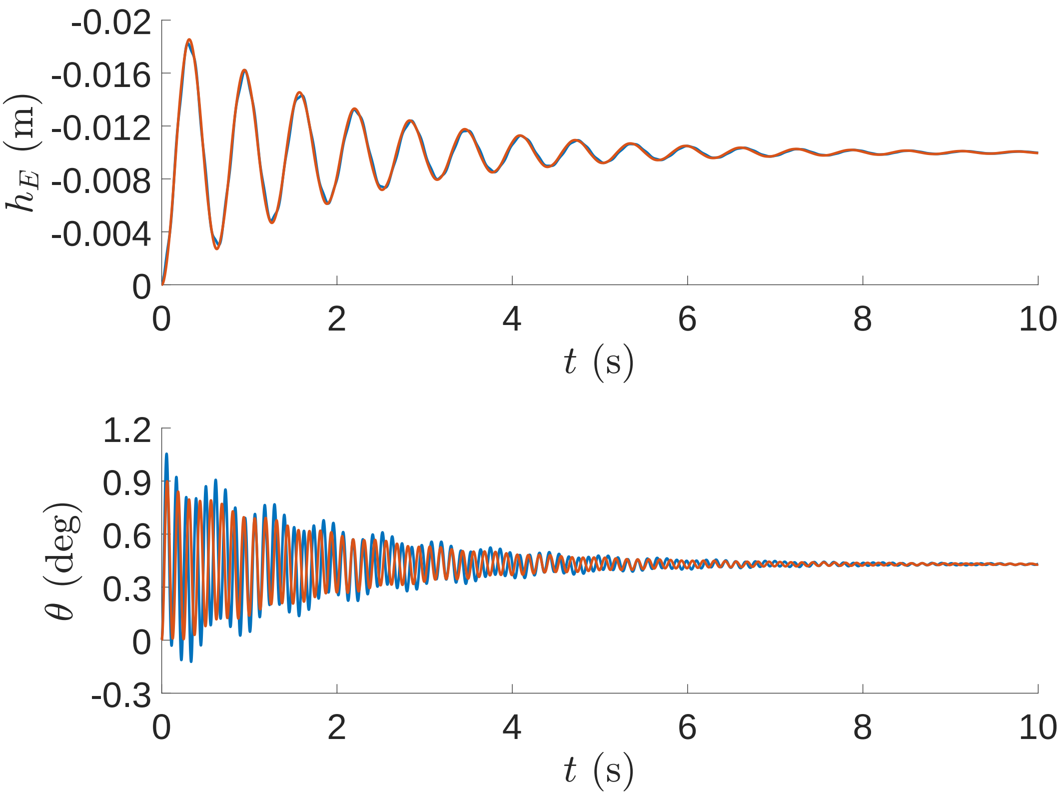


(b) Original coordinates

Figure 7: Time histories for Problem 3 Question 6.



(a) Modal coordinates



(b) Original coordinates

Figure 8: Comparison of time histories for Problem 3.