College of Engineering

School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW6

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Table of Contents

Exercise 1

Prove the following result: Suppose there exists a positive-definite symmetric matrix P and a positive scalar α which satisfy

$$PA_1 + A_1^T P + 2\alpha P \leq 0$$
$$PA_2 + A_2^T P + 2\alpha P \leq 0$$

where $A_1 := A_0 + a\Delta A$ and $A_2 := A_0 + b\Delta A$. Then system

$$\dot{x} = A(x)x$$
$$A(x) = A_0 + \psi(x)\Delta A$$
$$a \leq \psi(x) \leq b$$

is globally exponentially stable about the origin with rate of convergence α.

Solution:
As a candidate Lyapunov function for GES we consider $V(x) = x^T P x$. Then,

$$\begin{aligned}
\dot{V} &= \dot{x}^T P x + x^T P \dot{x} \\
&= 2x^T P \dot{x} \\
&= 2x^T P(A_0 + \psi(x)\Delta A)x \\
&= 2x^T P A_0 x + 2x^T P \psi(x) \Delta A x \\
&= 2x^T (PA_0)x + 2x^T (P\Delta A) x \psi(x).
\end{aligned}$$

Now the upper bound of $\psi(x)$ can be a or b so

$$\begin{aligned}
\dot{V} &= 2x^T (PA_0)x + 2x^T (P\Delta A) x \psi(x) \\
&= 2x^T P(A_1 - a\Delta A)x + 2x^T (P\Delta A) x a \\
&= 2x^T (PA_1)x - 2ax^T (P\Delta A)x + 2ax^T (P\Delta A)x \\
&= 2x^T (PA_1)x \\
&= x^T (PA_1 + A_1^T P)x.
\end{aligned}$$

Similarly, if $\psi(x) = b$

$$\dot{V} = x^T (PA_2 + A_2^T P)x.$$

We are given that

$$PA_1 + A_1^T P + 2\alpha P \leq 0$$
$$PA_2 + A_2^T P + 2\alpha P \leq 0$$

and if this holds true for some positive-definite symmetric matrix P and a positive scalar α, we can say

$$PA_1 + A_1^T P \leq -2\alpha_1 P$$
$$PA_2 + A_2^T P \leq -2\alpha_2 P.$$

Thus, we obtain

$$\dot{V} = \begin{cases} x^T(PA_1 + A_1^T P)x \leq -2\alpha_1 x^T P x \\ x^T(PA_2 + A_2^T P)x \leq -2\alpha_2 x^T P x. \end{cases}$$

Hence,

$$\dot{V} = -2\alpha V.$$

This guarantees the system is GES with rate α.

Exercise 2

What is the supremal value of $\gamma > 0$ for which Theorem 16 guarantees that the following system is guaranteed to be stable about the origin?

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + \gamma e^{-x_1^2} x_2 \\ \dot{x}_2 &= -x_1 - 3x_2 - \gamma e^{-x_1^2} x_1\end{aligned}$$

Theorem 16 *Suppose there exists a positive-definite symmetric matrix P which satisfies the following linear matrix inequalities:*

$$\begin{aligned}PA_1 + A_1^T P < 0 \\ PA_2 + A_2^T P < 0\end{aligned}$$

Then system

$$\begin{aligned}\dot{x} &= A(x)x \\ A(x) &= A_0 + \psi(x)\Delta A \\ a \leq \psi&(x) \leq b\end{aligned}$$

is globally exponentially stable (GES) about the origin with Lyapunov matrix P.

Solution:
From the given system equation, we know that

$$A_0 = \begin{bmatrix} -2 & 1 \\ -1 & -3 \end{bmatrix}, \qquad \Delta A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\psi(x) = \gamma e^{-x_1^2} \quad .$$

Because of the negative exponential, we have

$$0 \leq \gamma e^{-x_1^2} \leq \gamma \quad .$$

Hence, $a = 0$ and $b = \gamma$,

$$\begin{aligned}A_1 &:= A_0 + a\Delta A = A_0 \\ A_2 &:= A_0 + b\Delta A = A_0 + \gamma\Delta A \quad .\end{aligned}$$

From what we have defined, we can formulate LMI for the problem

$$\begin{aligned}PA_1 + A_1^T P &< 0 \\ PA_2 + A_2^T P &< 0 \\ P &> I.\end{aligned}$$

To solve this we run the LMI Toolbox commands in MATLAB and run it inside a loop where the gamma value is incremented by a small value (e.g. 0.0001). For each iteration we check if the following condition

$$V_1 = PA_1 + A_1^T P$$
$$V_2 = PA_2 + A_2^T P$$

then

$$V_1 = V_1^T$$
$$V_2 = V_2^T$$

and

$$\begin{array}{ll} \text{all} & eig(V_1) < 0 \\ \text{all} & eig(V_2) < 0 \end{array} \quad .$$

Or this could be done by simply checking the `tfeas` to be less than 0.

The MATLAB code is as follows.

```
% AAE 666 HW6 Exercise 2
% Tomoki Koike
close all; clear all; clc;
%%
echo off;
%gamma=1;
gamma=2.6;
A0 = [-2 1; -1 -3];
DelA = [0 1; -1 0];
while true
    % Quadratic stability LMI of the problem
    A1 = A0;
    A2 = A0 + gamma*DelA;

    % Setup LMI
    setlmis([]);
    % Equation 1
    p=lmivar(1, [2,1]);
    lmi1=newlmi;
    lmiterm([lmi1,1,1,p],1,A1,'s');
    % Equation 2
    lmi2=newlmi;
    lmiterm([lmi2,1,1,p],1,A2,'s');
```

```
    % Equation 3
    Plmi= newlmi;
    lmiterm([-Plmi,1,1,p],1,1);
    lmiterm([Plmi,1,1,0],1);
    % Configure for solver
    lmis = getlmis;
    % Results
    [tfeas, xfeas] = feasp(lmis);
    P = dec2mat(lmis,xfeas,p);

    v1 = P*A1 + A1'*P;
    v2 = P*A2 + A2'*P;
    % Check symmetry
    cp11 = issymmetric(v1);
    cp12 = issymmetric(v2);
    cp1 = cp11 & cp12;
    % Check negative eigenvalues
    cp21 = all(eig(v1) < 0);
    cp22 = all(eig(v2) < 0);
    cp2 = cp21 & cp22;
    % Check if the conditions are negative definite
    if ~(cp1 && cp2)
        break;
    end
    % Increment gamma value
    gamma = gamma + 0.0001;
end
```

The supremal γ value gave the following results for the LMI

$$\gamma_{max} = \infty$$

The code keeps on running infinitely and we cannot find a supremal value.

Exercise 3

Consider the pendulum system example of 119 with $\gamma = 1$. Obtain the largest rate of exponential convergence that can be obtained using the results of Exercise 31 and LMI toolbox.

Example 119 *Inverted pendulum under linear feedback.* Consider an inverted pendulum under linear control described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -2x_1 - x_2 + \gamma \sin x_1 \\ \gamma &> 0 \end{aligned}$$

This system can be described by

$$\begin{aligned} \dot{x} &= A(x)x \\ A(x) &= A_0 + \psi(x)\Delta A \\ a &\leq \psi(x) \leq b \end{aligned}$$

with

$$A_0 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \qquad \Delta A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

and

$$\psi(x) = \begin{cases} \gamma \sin x_1 / x_1 & \text{if} \quad x_1 \neq 0 \\ \gamma & \text{if} \quad x_1 = 0 \end{cases}$$

Since $|\sin x_1| \leq |x_1|$, we have

$$-\gamma \leq \psi(x) \leq \gamma.$$

hence $a = -\gamma$ and $b = \gamma$.

Solution:
To solve for the possible largest rate of convergence, α we have to change the MATLAB LMI code accordingly to

$$\begin{aligned} PA_1 + A_1^T P + 2\alpha P &\leq 0 \\ PA_2 + A_2^T P + 2\alpha P &\leq 0 \\ P &> I \quad . \end{aligned}$$

And then we will iterate over the code to find the α using the following conditions

$$\begin{aligned} V_1 &= PA_1 + A_1^T P + 2\alpha P \\ V_2 &= PA_2 + A_2^T P + 2\alpha P \end{aligned}$$

then

$$V_1 = V_1^T$$
$$V_2 = V_2^T$$

and

$$\begin{array}{ll} \text{all} & eig(V_1) \leq 0 \\ \text{all} & eig(V_2) \leq 0 \end{array} \; .$$

Or this could be done by simply checking the `tfeas` to be less than 0.

The MATLAB code is as follows.

```
% AAE 666 HW6 Exercise 3
% Tomoki Koike
close all; clear all; clc;
%%
echo off;
gamma=1;
A0 = [0 1; -2 -1];
DelA = [0 0; 1 0];
alpha = 0.1
while true
    % Quadratic stability LMI of the problem
    A1 = A0 - gamma*DelA;
    A2 = A0 + gamma*DelA;

    % Setup LMI
    setlmis([]);
    % Equation 1
    p=lmivar(1, [2,1]);
    lmi1=newlmi;
    lmiterm([lmi1,1,1,p],1,A1,'s');
    lmiterm([lmi1,1,1,p],2*alpha,eye(2));
    % Equation 2
    lmi2=newlmi;
    lmiterm([lmi2,1,1,p],1,A2,'s');
    lmiterm([lmi2,1,1,p],2*alpha,eye(2));
    % Equation 3
    Plmi= newlmi;
    lmiterm([-Plmi,1,1,p],1,1);
    lmiterm([Plmi,1,1,0],1);
    % Configure for solver
    lmis = getlmis;
```

```
    % Results
    [tfeas, xfeas] = feasp(lmis);
    P = dec2mat(lmis,xfeas,p);

    v1 = P*A1 + A1'*P + 2*alpha*P;
    v2 = P*A2 + A2'*P + 2*alpha*P;
    % Check symmetry
    cp11 = issymmetric(v1);
    cp12 = issymmetric(v2);
    cp1 = cp11 & cp12;
    % Check negative eigenvalues
    cp21 = all(eig(v1) <= 0);
    cp22 = all(eig(v2) <= 0);
    cp2 = cp21 & cp22;
    % Check if the conditions are negative semi-definite
    if ~(cp1 && cp2)
        alpha = alpha - 0.0001;
        break;
    end
    % Increment gamma value
    alpha = alpha + 0.0001;
end


setlmis([]);
% Equation 1
p=lmivar(1, [2,1]);
lmi1=newlmi;
lmiterm([lmi1,1,1,p],1,A1,'s');
lmiterm([lmi1,1,1,p],2*alpha,eye(2));
% Equation 2
lmi2=newlmi;
lmiterm([lmi2,1,1,p],1,A2,'s');
lmiterm([lmi2,1,1,p],2*alpha,eye(2));
% Equation 3
Plmi= newlmi;
lmiterm([-Plmi,1,1,p],1,1);
lmiterm([Plmi,1,1,0],1);
% Configure for solver
lmis = getlmis;
% Results
[tfeas, xfeas] = feasp(lmis);
P = dec2mat(lmis,xfeas,p);
```

This MATLAB computation gave us the following result.

The largest convergence rate of

$$\alpha_{max} = 0.1220$$

where `tfeas` $= -0.0014$ and corresponding Lyapunov matrix is

$$P = \begin{pmatrix} 207.3728 & 51.9072 \\ 51.9072 & 103.6427 \end{pmatrix} .$$

We can verify this by using the `gevp` command that solves a generalized eigenvalue minimization problem. The MATLAB code is as follows.

```
close all; clear all; clc;
gamma=1;
A0 = [0 1; -2 -1];
DelA = [0 0; 1 0];

% Quadratic stability LMI of the problem
A1 = A0 - gamma*DelA;
A2 = A0 + gamma*DelA;

setlmis([]);
p = lmivar(1,[2 1]);

lmiterm([1 1 1 0], 1);  % P > I : I
lmiterm([-1 1 1 p], 1, 1);  % P > I : P
lmiterm([2 1 1 p], A1, 1, 's');  % LFC#1 {lhs)
lmiterm([-2 1 1 p], 2, 1);  % LFC#1 (rhs)
lmiterm([3 1 1 p], A2, 1, 's');  % LFC#2 {lhs)
lmiterm([-3 1 1 p], 2, 1);  % LFC#2 (rhs)
lmis = getlmis;

% Results
[alpha, popt] = gevp(lmis,2);
```

This code gives us

$$\lambda = -0.1220$$

which means

$$\alpha_{max} = -\lambda = 0.1220$$

This agrees with results obtain from the iterative approach we took in the first MATLAB code. Thus the α_{max} value is correct.

Exercise 4

Consider the double inverted pendulum described by

$$\ddot{\theta}_1 + 2\dot{\theta}_1 - \dot{\theta}_2 + 2k\theta_1 - k\theta_2 - \sin\theta_1 = 0$$
$$\ddot{\theta}_2 - \dot{\theta}_1 + \dot{\theta}_2 - k\theta_1 + k\theta_2 - \sin\theta_2 = 0$$

Using the results of Theorem 17, obtain a value of the spring constant k which guarantees that this system is globally exponentially stable about the zero solution.

Theorem 17 *Suppose there exists a positive-definite symmetric matrix P which satisfies the following linear matrix inequalities:*

$$PA + A^T P < 0 \qquad \forall A \text{ in } \mathcal{A}$$

Then system

$$\dot{x} = A(x)x$$
$$A(x) = A_0 + \psi_1(x)\Delta A_1 + +\psi_l(x)\Delta A_l$$
$$a_i \leq \psi_i(x) \leq b_i$$

is globally exponentially stable about the origin with Lyapunov matrix P.

Solution:
The given system equations can be modified as

$$\ddot{\theta}_1 = -2\dot{\theta}_1 + \dot{\theta}_2 - 2k\theta_1 + k\theta_2 + \sin\theta_1$$
$$\ddot{\theta}_2 = \dot{\theta}_1 - \dot{\theta}_2 + k\theta_1 - k\theta_2 + \sin\theta_2$$

In space-state representation it becomes

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin\theta_1 \\ \sin\theta_2 \end{bmatrix}$$

Now if we define $x_1 := \theta_1$, $x_2 := \theta_2$, $x_3 := \dot{\theta}_1$, and $x_4 := \dot{\theta}_2$, we can rewrite this as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \ddot{x}_3 \\ \ddot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin x_1 \\ \sin x_2 \end{bmatrix}.$$

We structure the nonlinearity to be

$$\psi_1(x) = \sin x_1$$
$$\psi_2(x) = \sin x_2$$

and since

$$-1 \leq \sin x_1 \leq 1$$
$$-1 \leq \sin x_2 \leq 1$$

we can write the extreme matrices to be

$$\begin{aligned} A_1 &= A_0 - \Delta A_1 - \Delta A_2 \\ A_2 &= A_0 - \Delta A_1 + \Delta A_2 \\ A_3 &= A_0 + \Delta A_1 - \Delta A_2 \\ A_4 &= A_0 + \Delta A_1 + \Delta A_2 \end{aligned}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2k & k & -2 & 1 \\ k & -k & 1 & -1 \end{bmatrix}, \quad \Delta A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \Delta A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now we can find k value that guarantees stability by solving the following LMI problem

$$\begin{aligned} PA_1 + A_1^T P &< 0 \\ PA_2 + A_2^T P &< 0 \\ PA_3 + A_3^T P &< 0 \\ PA_4 + A_4^T P &< 0 \\ P &> I \end{aligned}$$

The following MATLAB code runs an iterative process to find the possible k value.

```
% AAE 666 HW6 Exercise 4
% Tomoki Koike
close all; clear all; clc;
%%
echo off;
%k = 1;
k = 17.9;
while true
    % Quadratic stability LMI of the problem
    A0 = [   0  0  1  0;
```

```
          0  0  0  1;
       −2*k  k −2  1;
          k −k  1 −1];
DelA1 = [0 0 0 0;
         0 0 0 0;
         1 0 0 0;
         0 0 0 0];
DelA2 = [0 0 0 0;
         0 0 0 0;
         0 0 0 0;
         0 1 0 0];
A1 = A0 − DelA1 − DelA2;
A2 = A0 − DelA1 + DelA2;
A3 = A0 + DelA1 − DelA2;
A4 = A0 + DelA1 + DelA2;

% Setup LMI
setlmis([]);
% P matrix
p=lmivar(1, [4,1]);
% Equation 1
lmi1=newlmi;
lmiterm([lmi1,1,1,p],1,A1,'s');
% Equation 2
lmi2=newlmi;
lmiterm([lmi2,1,1,p],1,A2,'s');
% Equation 3
lmi2=newlmi;
lmiterm([lmi2,1,1,p],1,A3,'s');
% Equation 4
lmi2=newlmi;
lmiterm([lmi2,1,1,p],1,A4,'s');
% Equation 5
Plmi= newlmi;
lmiterm([−Plmi,1,1,p],1,1);
lmiterm([Plmi,1,1,0],1);
% Configure for solver
lmis = getlmis;
% Results
[tfeas, xfeas] = feasp(lmis);
P = dec2mat(lmis,xfeas,p);

v1 = P*A1 + A1'*P;
v2 = P*A2 + A2'*P;
v3 = P*A3 + A3'*P;
```

```
    v4 = P*A4 + A4'*P;
    % Check symmetry
    cp11 = issymmetric(v1);
    cp12 = issymmetric(v2);
    cp13 = issymmetric(v3);
    cp14 = issymmetric(v4);
    cp1 = cp11 & cp12 & cp13 & cp14;
    % Check negative eigenvalues
    cp21 = all(eig(v1) < 0);
    cp22 = all(eig(v2) < 0);
    cp23 = all(eig(v3) < 0);
    cp24 = all(eig(v4) < 0);
    cp2 = cp21 & cp22 & cp23 & cp24;
    % Check tfeas to be negative
    cp3 = tfeas < 0;
    % Check if the conditions are negative definite
    cp = cp1 & cp2 & cp3
    % Check if the conditions are negative definite
    if cp
        break;
    end
    % Increment gamma value
    %k = k + 0.1;
    k = k + 0.0001;
end
```

As a result, we obtain the minimal spring constant k that guarantees that this system is GES about the zero solution to be

$$k = 18.0398$$

with a corresponding P matrix of

$$P = \begin{bmatrix} 7.1787\text{e}{+}03 & -1.9663\text{e}{+}03 & 53.2057 & 35.5866 \\ -1.9663\text{e}{+}03 & 5.2124\text{e}{+}03 & 35.5862 & 88.7925 \\ 53.2057 & 35.5862 & 288.9914 & 179.9061 \\ 35.5866 & 88.7925 & 179.9061 & 468.8980 \end{bmatrix}.$$