A-AE 567 Final Homework Spring 2021

Your work must be neat and easy to read.

Place your final answer on the answer sheet

You must work alone.

Due 11:00PM Tuesday May 4 by Gradescope.

NAME:

Problem 1. Consider the strictly positive matrix

$$T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Using this matrix consider the inner product on $\mathbb{C}^3$ defined by

$$(x, y)_T = (Tx, y) = y'Tx \qquad (x \in \mathbb{C}^3 \text{ and } y \in \mathbb{C}^3)$$

where $'$ denotes the complex conjugate transpose. The norm of this inner product is given by $\|x\|_T = \sqrt{(Tx, x)}$. Let

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Solve the following optimization problem

$$\begin{aligned} \delta &= \min\left\{\|e_1 - \alpha e_2 - \beta e_3\|_T^2 : \alpha \in \mathbb{C} \text{ and } \beta \in \mathbb{C}\right\} \\ &= \min\left\{\big(T(e_1 - \alpha e_2 - \beta e_3), (e_1 - \alpha e_2 - \beta e_3)\big) : \alpha \in \mathbb{C} \text{ and } \beta \in \mathbb{C}\right\} \end{aligned}$$

In other words, find δ and scalars a and b such that

$$\|e_1 - a e_2 - b e_3\|_T^2 = \delta = \min\left\{\|e_1 - \alpha e_2 - \beta e_3\|_T^2 : \alpha \in \mathbb{C} \text{ and } \beta \in \mathbb{C}\right\}$$

Problem 2. Consider the discrete time system

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} e^{-\frac{n}{50}} & 1 \\ 2 & \cos(\frac{n}{50}) \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} e^{-\frac{n}{50}}\cos(\frac{n}{50}) \\ 1 \end{bmatrix} u(n)$$

$$y(n) = \begin{bmatrix} 1 + e^{-\frac{n}{50}} & 2 + \sin(\frac{n}{50}) \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \Big(1 + \frac{1}{2}\sin(\frac{n}{50})\Big) v(n)$$

where u and v are independent Gaussian white noise processes. The initial conditions $x(0) = 0$ and $\widehat{x}(0) = 0$. To generate u and v in Matlab, set

rng(1000); u = randn(1,20);

rng(2000); v = randn(1,20);

Let $\mathcal{M}_n = \text{span}\{y(j)\}_0^n$. Find the following

(i) $P_{\mathcal{M}_{n-1}} x_1(n)$ for n= 8,9,10.

(ii) $P_{\mathcal{M}_n} x_2(n)$ for n= 8,9,10.

(Note the indices on the state $x_1(n)$ in Part (i), and $x_2(n)$ in Part (ii).) Be careful Matlab does not have a zero index. So for example, in Matlab

$$A(0) = A\{1\} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A(1) = A\{2\} = \begin{bmatrix} e^{-\frac{1}{50}} & 1 \\ 2 & \cos(\frac{1}{50}) \end{bmatrix} \quad \text{ect.}$$

Problem 3. Let $\mathbf{x}$ be a mean zero, variance one, Gaussian random variable, and $\{\mathbf{v}_n\}_0^\infty$ be a mean zero, variance one, Gaussian white noise process, which is independent of $\mathbf{x}$. Consider the discrete time random process $\mathbf{y}_n$ defined by

$$\mathbf{y}_n = \mathbf{x} + \mathbf{v}_n$$

where $n \geq 0$ is a positive integer.

(i) Find best estimate for $\mathbf{x}$ given $\{\mathbf{y}_j\}_{j=0}^{n-1}$, that is, find

$$\widehat{\mathbf{x}}_n = E(\mathbf{x}|\mathbf{y}_0, \mathbf{y}_1, \cdots, \mathbf{y}_{n-1}) = P_{\mathcal{M}_{n-1}}\mathbf{x}$$

where $\mathcal{M}_{n-1} = \text{span}\{\mathbf{y}_j\}_{j=0}^{n-1}$.

(ii) Find the error σ_n in your estimate, that is,

$$\sigma_n^2 = E(\mathbf{x} - \widehat{\mathbf{x}}_n)^2$$

Problem 4. Consider the unstable state space system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -0.5 & 30 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 5 \end{bmatrix} w + \begin{bmatrix} 0 & 0 \\ \frac{1}{4} & 0 \\ 0 & 0 \\ 0 & \frac{1}{10} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y_1 = x_1 + \frac{v_1}{4}$$

$$y_2 = x_3 + \frac{v_2}{20}$$

where u_1, u_2, v_1 and v_2 are all independent white noise processes. Moreover, w is the input. Assume that all the initial conditions are zero. Design a feedback controller $w = -K\widehat{x}$ based on the steady state Kalman filter such that $|x_1(t)| \leq 1$ and $|x_3(t)| \leq .35$. Your state feedback gain $K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$ must satisfy $|k_j| \leq 25$ for $j = 1, 2, 3, 4$. Simulate your controller in Simulink for 30 seconds. Hand the graphs from your Simulink program for:

(i) The state x_1 and its estimate $\widehat{x}_1$ on the same graph.

(ii) The state x_2 and its estimate $\widehat{x}_2$ on the same graph.

(iii) The state x_3 and its estimate $\widehat{x}_3$ on the same graph.

(iv) The state x_4 and its estimate $\widehat{x}_4$ on the same graph.

(v) Hand in your gain K.

On the band limited white nose generators, set the seed for u_1, u_2, v_1 and v_2 respectively to, 23341, 23342, 23343 and 23344.

Answer sheet NAME:

Problem 1:

(i) $a =$

(ii) $b =$

(iii) $\delta =$

Problem 2:

(i) $P_{\mathcal{M}_{n-1}} x_1(n)$ for $n = 8, 9, 10$:

(ii) $P_{\mathcal{M}_n} x_2(n)$ for $n = 8, 9, 10$:

Answer sheet NAME:

Problem 3 Part (i) The best estimate for $\mathbf{x}$ given $\{\mathbf{y}_j\}_{j=0}^{n-1}$ is given by

$$\widehat{\mathbf{x}}_n =$$

Part (ii) The error σ_n is given by

$$\sigma_n^2 = E(\mathbf{x} - \widehat{\mathbf{x}}_n)^2 =$$

Answer sheet   NAME:

Problem 4: Your value for K.

Attach the four graphs of $x_j(t)$ and $\widehat{x}_j(t)$ for $j = 1, 2, 3, 4$ and $0 \leq t \leq 30$ here.