Parameters Identification of
The 2 Degrees of Freedom Helicopter Model

AAE364L: Control Systems Lab

This experiment is devoted to the two degrees of freedom helicopter. First we will use system identification techniques to determine the moment of inertia, damping and input gains for the helicopter. In future experiments we will design a PID controller and a LQR controller for the helicopter.

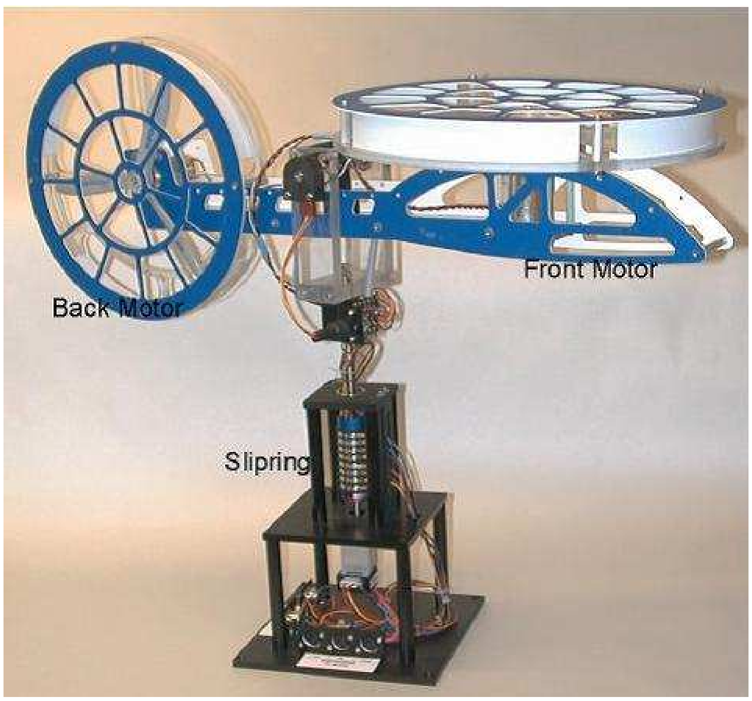

Figure 1: The helicopter

1 The equations of motion

The equations of motions for the Helicopter are given by

$$\begin{aligned}
&J_p\ddot{\theta} + J_y \sin(\theta)\cos(\theta)\dot{\psi}^2 + mg(h\sin(\theta) + R_c\cos(\theta)) + c_p\dot{\theta} \\
&= lk_{pp}v_p + k_{py}v_y \\
\\
&\left(J_y\cos(\theta)^2 + J_{shaft}\right)\ddot{\psi} - 2J_y\cos(\theta)\sin(\theta)\dot{\theta}\dot{\psi} + c_y\dot{\psi} \\
&= lk_{yy}v_y\cos(\theta) + k_{yp}v_p\cos(\theta).
\end{aligned} \tag{1.1}$$

The free body diagram is given in Figure 2 where $\theta = p$ and $\psi = y$. The parameters are defined by

- θ is the pitch angle
- ψ is the yaw angle (corresponding to the fixed vertical axis)
- R_c is the horizontal distance of the center of mass from the pivot point ($R_c > 0$ by design)
- h is the vertical distance of the center of mass from the pivot point ($h > 0$ by design)
- l is the distance from the front (back) propeller axis to the pivot point ($l = 0.184$ m)
- L is the total length of the helicopter ($L = 0.483$ m).
- x is the distance from the small mass to the pivot point ($x = 0.120$ m)
- m_h is the of the helicopter without the small mass ($m_h = 1.17$ kg)
- m_s is small mass added to the helicopter ($m_s = 0.156$ kg)
- $m = m_h + m_s$ is the total mass of the helicopter ($m = 1.326$ kg)
- $m_{motors} = 0.754$ kg is the mass of pitch and yaw propellers, propeller shields and motors
- $m_b = 0.416$ kg is the mass moving about the pitch axis.
- J_p is the moment of inertia of the helicopter relative to the pitch axis
- J_y is the moment of inertia of the helicopter relative to the yaw axis
- $J_{shaft} = 0.0039\ kg \cdot m^2$ is the moment of inertia of metal shaft about yaw axis at end point.
- $c_p = B_p$ is the coefficient of viscous friction corresponding to the pitch axis
- $c_y = B_y$ is the coefficient of viscous friction corresponding to the yaw axis
- v_p is input the voltage to the pitch or front motor
- v_y is input the voltage to the yaw or back motor
- k_{pp} is the gain from the pitch motor to the pitch angle

- k_{py} is the gain from the yaw motor to the pitch angle
- k_{yy} is the gain from the yaw motor to the yaw angle
- k_{yp} is the gain from the pitch motor to the yaw angle.

Throughout this experiment we will assume that $J_p \approx J_y$. Notice that if the yaw motor is turned off ($v_y = 0$) and the pitch motor is running, then the pitch motor will change both the pitch and yaw angles. Similarly, if the pitch motor is turned off ($v_p = 0$) and the yaw motor is running, then the yaw motor will change both the yaw and pitch angles of the helicopter.

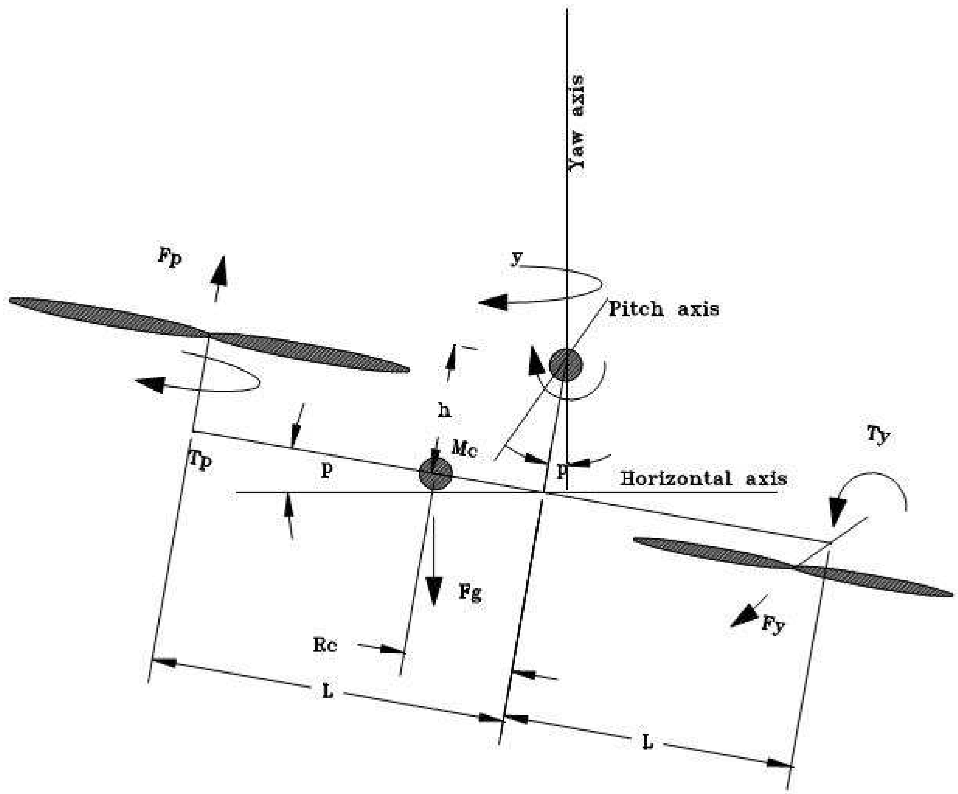

Figure 2: The free body diagram (top view) of the helicopter

REMARK 1.1 *The function from the voltage to the thrust is in reality a nonlinear function. To be precise, the equations of motion are given by*

$$\begin{aligned}
&J_p\ddot{\theta} + J_y \sin(\theta)\cos(\theta)\dot{\psi}^2 + mg(h\sin(\theta) + R_c\cos(\theta)) + c_p\dot{\theta} \\
&= lF_p(v_p) + T_p(v_y) \\
&\left(J_y\cos(\theta)^2 + J_{shaft}\right)\ddot{\psi} - 2J_y\cos(\theta)\sin(\theta)\dot{\theta}\dot{\psi} + c_y\dot{\psi} \\
&= lF_y(v_y)\cos(\theta) + T_y(v_p)\cos(\theta).
\end{aligned} \tag{1.2}$$

Here $F_p(v_p)$ is the force corresponding to the pitch angle due to the voltage applied to the pitch motor, and $T_p(v_y)$ is the torque associated with the pitch angle due to the voltage applied to the yaw motor. Moreover, $F_y(v_y)$ is the force corresponding to the yaw angle due to the voltage applied to the yaw motor, and $T_y(v_p)$ is the torque associated with the yaw angle due to the voltage applied to the pitch motor. It is emphasized that $F_p(v_p)$ and $T_y(v_p)$ are nonlinear functions in the pitch voltage v_p, while $F_y(v_y)$ and $T_p(v_y)$ are nonlinear functions in the yaw voltage v_y. In our experiment we assume that these are all linear functions, that is,

$$\begin{aligned}
F_p(v_p) &= k_{pp}v_p \qquad \textit{and} \qquad T_y(v_p) = k_{yp}v_p \\
F_y(v_y) &= k_{yy}v_y \qquad \textit{and} \qquad T_p(v_y) = k_{py}v_y.
\end{aligned}$$

In fact, the experimental data will show that these functions can be approximated by linear functions.

1.1 The linearized equations of motion

Let $\delta\theta = \theta - \theta_e$ and $\delta\psi = \psi - \psi_e$, be the variations around a fixed pitch angle θ_e and a fixed yaw angle ψ_e. Then the linearized equations of motion for the helicopter around the equilibrium angles θ_e and ψ_e are given by

$$\begin{aligned}
J_p\delta\ddot{\theta} + c_p\delta\dot{\theta} + mg\left(h\cos(\theta_e) - R_c\sin(\theta_e)\right)\delta\theta &= lk_{pp}\delta v_p + k_{py}\delta v_y \\
\left(J_y\cos(\theta_e)^2 + J_{shaft}\right)\delta\ddot{\psi} + c_y\delta\dot{\psi} &= lk_{yy}\left(\cos(\theta_e)\delta v_y - v_{ye}\sin(\theta_e)\delta\theta\right) \\
&\quad + k_{yp}\left(\cos(\theta_e)\delta v_y - v_{pe}\sin(\theta_e)\delta\theta\right) \\
\delta v_p &= v_p - v_{pe} \\
\delta v_y &= v_y - v_{ye}.
\end{aligned} \tag{1.3}$$

As expected, $\delta v_p = v_p - v_{pe}$ is the variation around a fixed input voltage v_{pe} associated with the pitch motor, while $\delta v_y = v_y - v_{ye}$ is the variation around a fixed input voltage v_{ye} associated with the yaw motor.

Notice that in the previous equation $\delta\dot{\psi} = \dot{\psi}$ and $\delta\ddot{\psi} = \ddot{\psi}$. In particular, if the helicopter is in level flight $\theta_e = 0$, then the linearized equations of motion reduce to

$$\begin{aligned}
J_p\ddot{\theta} + c_p\dot{\theta} + mgh\theta &= lk_{pp}\delta v_p + k_{py}\delta v_y \\
(J_p + J_{shaft})\ddot{\psi} + c_y\dot{\psi} &= lk_{yy}\delta v_y + k_{yp}\delta v_p.
\end{aligned} \tag{1.4}$$

2 Part (i): Identification of R_c and h

In this section, we describe a methodology to find the parameters R_c and h which contribute to the gravitational torque acting on the helicopter. In order to reduce the front or pitch motor power requirement, a small mass m_s is attached to the tail of the original helicopter. Hence the total mass of the whole system is now given by $m = m_s + m_h$, where $m_h = 1.17$ kg is the mass of the helicopter. Moving this small mass results in relocation of the center of mass of the total system. We will displace this small mass to find the distances R_c and h.

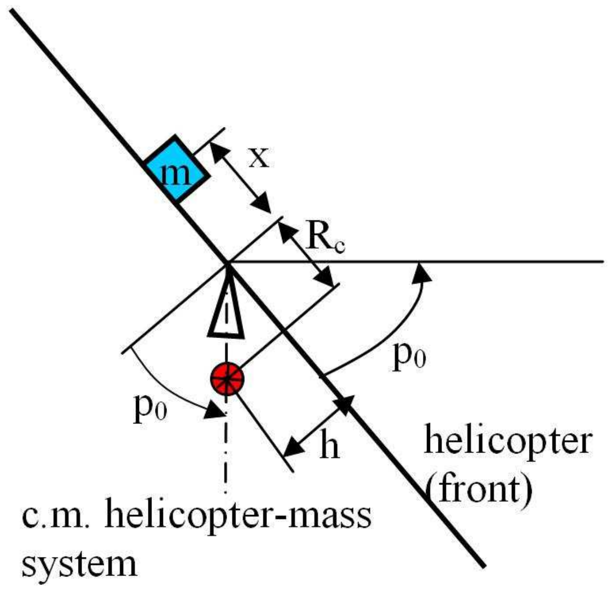

Figure 3: Center of mass when the helicopter is at rest.

When at rest, the helicopter pitches downward, resulting in its center of mass sitting directly under the pivot point; see Figure 3. The pitch angle at this position is denoted with p_0, which is negative. The only information we know about the center of mass now is that it lies somewhere on the vertical axis going through the pivot point.

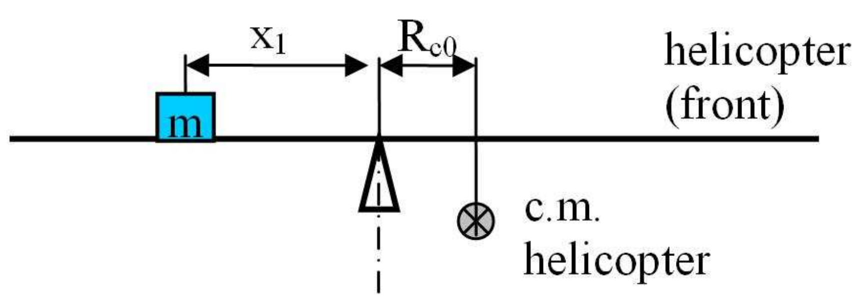

Figure 4: Center of mass of the helicopter with mass m_h and the balancing small mass.

Let R_{c0} be the longitudinal position of the center of mass of the original helicopter with mass m_h, that is, without the small mass. In order to find R_{c0}, we can place this small mass to balance the helicopter in its level flight condition, that is, $\theta = 0$. This results in the center of mass of the whole system lying underneath the pitch pivot; see Figure 4. Let x_1 be the distance of the small mass from the pitch pivot. By the mechanical static equilibrium condition, we arrive at

$$m_h R_{c0} = m_s x_1.$$

If the small mass is shifted back to its original position (distance x from the pitch pivot), then the center of mass of the whole system is now located at distance R_c from the pitch pivot. The center of mass in along the horizonal axis for the helicopter with the small mass at position x_1 is given by

$$R_c = \frac{m_h}{m} R_{c0} - \frac{m_s}{m} x.$$

Using $R_{c0} = m_s x_1 / m_h$, we arrive at

$$R_c = \frac{m_s(x_1 - x)}{m}. \tag{2.1}$$

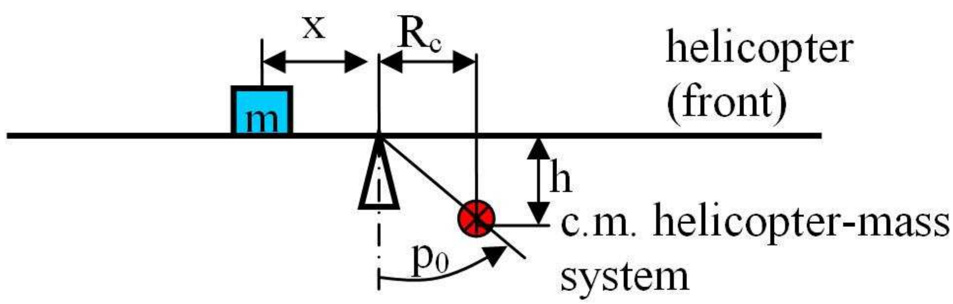

Figure 5: Relocation of center of mass of the system when the small mass is moved.

As stated earlier, the level flight condition is no longer the equilibrium condition since the helicopter pitches downward, with a pitch angle p_0. By consulting Figure 5 (or Figure 3), we see that the static equilibrium condition yields $h/R_c = \tan(\pi/2 - |p_0|) = 1/\tan(|p_0|)$. This readily implies that h can be computed by

$$h = \frac{R_c}{\tan(|p_0|)}. \tag{2.2}$$

REMARK 2.1 *Due to the sensitivity of the model to the parameter h, it is very important to use at least two significant digits for its value. Preferably, use the 'exact' value obtained from (2.2) throughout all your calculation in MATLAB instead of typing the result by hand.*

2.1 The Lab steps to identify R_c and h

(i) Measure the distance x of the small mass at its original position (**closest to the pivot**). Wait until the helicopter stops moving.

(ii) Open MATLAB and change the directory to $Desktop : \backslash AAE364L \backslash Sec\# \backslash ID$. Run the setup file `setup_lab_heli_2d.m`.

(iii) In the MATLAB command window, type: `theta_0 = 0`.

(iv) Open the Simulink model `Heli_ID.mdl`. Make sure both Slider Gains are set to 0. Open the `Scopes` subsystem and, from there, the Pitch scope (theta).

(v) In the scope window, select `Parameters` and set the `Time range` to 20. Under the `History` tab, make sure that `Limit data points` is *not* selected.

(vi) Click `Quarc - Build`. Select `Simulation - Connect to Target`, and `Quarc - Start`.

(vii) Unscrew the bolt from the small mass attached to the helicopter. Move the small mass away from the pivot to balance the helicopter model in the level flight condition. Secure the small mass to the helicopter.

(viii) Wait until the helicopter model settles in equilibrium. Read and record the angle from the pitch scope. This angle is p_0. Click the Stop button. Measure the distance x_1 to the mass at its new position.

(ix) Make sure you have a record of the values: x, x_1, and p_0.

3 Part (ii): Identification of J_p and c_p

In this part of the experiment we will identify the "pitch" moment of inertia J_p and its corresponding damping c_p. According to (1.4), the linearized pitch equation of motion around the level flight condition $\theta_e = 0$ is determined by

$$J_p\ddot{\theta} + c_p\dot{\theta} + mgh\theta = lk_{pp}\left(v_p - v_{pe}\right) + k_{py}\left(v_y - v_{ye}\right).$$

We will identify the parameters J_p and c_p without turning on the front and back motors. To accomplish this will put the helicopter model in level flight by moving the small mass to balance the helicopter around $\theta_e = 0$ as we did before. However, the relocation of this small mass results in a change of the moment of inertia for the helicopter. In this case, the linearized equation of motion for the pitch becomes

$$J_p'\ddot{\theta} + c_p\dot{\theta} + mgh\theta \;\; = \;\; 0,$$

where the new moment of inertia J_p' is determined by

$$J_p' \;\; = \;\; J_p + m_s(x_1^2 - x^2).$$

Recall that x_1 is the position of the small mass which moves the helicopter to level flight condition, that is, $\theta_e = 0$. By providing a small initial conditions $\theta(0) \neq 0$ and $\dot{\theta}(0) = 0$, the solution for the pitch angle $\theta(t)$ for is given by

$$\theta(t) \;\; = \;\; \frac{\theta(0)}{\cos(\varphi)}\, e^{-\sigma t}\cos(\omega t + \varphi)\,, \tag{3.1}$$

where

$$\sigma = \frac{c_p}{2J_p'} \quad \text{and} \quad \omega = \sqrt{\frac{mgh}{J_p'} - \sigma^2}. \tag{3.2}$$

Let us first estimate the damped angular frequency ω from the response θ. This value is easily obtained by counting the number of cycles per time unit and multiplying it by 2π, that is,

$$\omega = 2\pi \frac{\text{number of cycles}}{\text{time}}. \tag{3.3}$$

To find σ from the response θ, choose $n > 1$ consecutive peaks and measure the magnitude $\theta(t_0)$ of the first peak and the magnitude $\theta(t_0 + 2\pi(n-1)/\omega)$ of the last peak. Here t_0 represents the time of the first peak. The numbering of the peaks $k = 1, 2, \cdots, n$ starts with $k = 1$ corresponding to t_0 the first peak. According to (3.1), we have

$$\begin{aligned} \frac{\theta(t_0)}{\theta(t_0 + 2\pi(n-1)/\omega)} &= \frac{e^{-\sigma t_0} \cos(\omega t_0 + \varphi)}{e^{-\sigma(t_0 + 2\pi(n-1)/\omega)} \cos(\omega t_0 + 2\pi(n-1) + \varphi)} \\ &= \frac{e^{\sigma 2\pi(n-1)/\omega} \cos(\omega t_0 + \varphi)}{\cos(\omega t_0 + \varphi)} \\ &= e^{\sigma 2\pi(n-1)/\omega}. \end{aligned}$$

Therefore, σ is

$$\sigma = \frac{\omega}{2\pi(n-1)} \ln\left(\frac{\theta(t_0)}{\theta(t_0 + 2\pi(n-1)/\omega)}\right). \tag{3.4}$$

By rearranging the terms in (3.2), we can compute J_p' as follows

$$J_p' = \frac{mgh}{\omega^2 + \sigma^2}. \tag{3.5}$$

Finally, the parameters J_p and c_p that we are looking for are determined by

$$\begin{aligned} J_p &= J_p' - m_s(x_1^2 - x^2) \\ c_p &= 2\sigma J_p'. \end{aligned} \tag{3.6}$$

3.1 The Lab steps to identify J_p and c_p

(i) Place the small mass m_s at position x_1 to put the helicopter in the level flight condition $\theta(0) = 0$. Verify that `theta_0 = 0` in the MATLAB command window.

(ii) Open the Simulink model `Heli_ID.mdl` and set the simulation time to 59 seconds. Select `Tools - External Mode Control Panel - Signal & Triggering`, and change the duration to 30,000. Make sure that both Slider Gains are set to 0.

(iii) Open the `Scopes` subsystem and, from there, the Average Pitch scope (average theta). **In order to properly record your data, select Parameters. Deselect the *"Limit to 5000 data points"* check-box. Select *Save data to workspace.* Enter an appropriate variable name, in this case *theta.* Make sure the format is "Structure with time".**

(iv) Click `Quarc - Build`. Select `Simulation - Connect to Target`, and `Quarc - Start`. Manually displace the pitch between ± 15 to ± 20 degrees (± 0.3 radians) from the level flight position. Release the helicopter. Wait at least 20 seconds before stopping the experiment.

(v) Save the pitch angle data in MATLAB (`File - Save - Save as Mat file`).

3.2 In your lab report, include the following under Part (ii):

(a) Hand in the plots of the measured pitch angle and the envelope computed from $\theta(0)\,e^{-\sigma t}$ on the same figure. Show only meaningful data. The time the helicopter was released and the initial time of the simulation must coincide. If your plot includes the data before the helicopter model is released, NO CREDIT will be given.

(b) Since J_p' and c_p are calculated from the linear approximation method, they may not represent the actual parameters. Create a Simulink model to simulate the nonlinear pitch equation of motion, using this J_p' and c_p as starting point. Note that since $\dot{\psi} = 0$, you do not have to be concerned about J_y. Moreover, since the small mass is placed to balance the helicopter in level flight, R_c becomes zero. Thus the nonlinear equation of motion you need to simulate is

$$J_p'\ddot{\theta} + c_p\dot{\theta} + mgh\theta = 0\,. \tag{3.7}$$

Adjust J_p' and c_p so that your simulation result more closely matches the experimental data. Hand in this Simulink model in the Appendix.

(c) In a separate figure, plot the simulation of the pitch angle compared with the actual experimental data. Show only meaningful data. The time the helicopter was released and the initial time of the simulation must coincide. If your plot includes the data before the helicopter model is released, NO CREDIT will be given.

4 Part (iii): Identification of J_y and c_y

Recall that the yaw equation of motion is given by the second equation in (1.1). If we can somehow force the pitch angle to be constant at zero, that is $\theta(t) = 0$, without turning the motors on, then this equation reduces to

$$(J_y + J_{shaft})\ddot{\psi} + c_y\dot{\psi} = 0.$$

Notice that this is a linear first order differential equation in $\dot{\psi}$, with initial condition $\dot{\psi}(0) \neq 0$. The solution to this equation is given by

$$\dot{\psi}(t) = \dot{\psi}(0)e^{-\sigma t} \quad \text{where} \quad \sigma = \frac{c_y}{J_y + J_{shaft}}. \tag{4.1}$$

By observing the plot of (4.1), the parameter σ can be easily estimated.

Note that in this experiment J_{shaft} is already known. By obtaining σ alone, we can only estimate the ratio $\frac{c_y}{J_y+J_{shaft}}$ in (4.1). The parameters J_y, c_y, k_{pp}, and k_{py} can be estimated all together when we turn the front and back motor on. However, at this point we will assume that J_y approximately equals to J_p. Therefore we can compute J_y and c_y by

$$\begin{aligned} J_y &\approx J_p \\ c_y &\approx \sigma(J_p + J_{shaft}). \end{aligned} \tag{4.2}$$

4.1 The Lab steps to identify J_y and c_y

(i) Move the small mass back to its original position x (the closest distance to the pivot point). Verify that `theta_0 = 0` in the MATLAB command window.

(ii) Have TA secure the helicopter so that the pitch angle is forced to stay at zero.

(iii) Open the Simulink model `Heli_ID.mdl` and set the simulation time to 59 seconds. Select `Tools - External Mode Control Panel - Signal & Triggering`, and change the duration to 30,000. Make sure both Slider Gains are set to 0.

(iv) Open the `Scopes` subsystem and, from there, the Average Yaw Rate scope (average psi dot). **In order to properly record your data, select Parameters. Deselect the *"Limit to 5000 data points"* check-box. Select *Save data to workspace.* Enter an appropriate variable name, in this case *psi_dot.* Make sure the format is "Structure with time".**

(v) Click `Quarc - Build`. Select `Simulation - Connect to Target`, and `Quarc - Start`. Manually spin the helicopter **clockwise** around the yaw axis (by a single push or tap). Wait at least 20 seconds before stopping the experiment.

(vi) Save the yaw rate data in MATLAB (`File - Save - Save as Mat file`).

(vii) Repeat step (v), but this time spin the helicopter in the **counter-clockwise** direction.

(viii) Save the yaw rate data in MATLAB.

4.2 In your lab report, include the following under Part (iii):

(a) Include the estimated c_y. When fitting the exponential, use only the first 75% of the measured yaw rate, starting from the time the helicopter was pushed. The tail of the measured data is greatly affected by the yaw static friction and should be discarded.

(b) Hand in the plots of yaw rate data and $\dot{\psi}(0)e^{-\sigma t}$ for the clockwise spin on the same figure. On a separate figure, hand in the plots of yaw rate data and $\dot{\psi}(0)e^{-\sigma t}$ for the counter-clockwise spin. Show only meaningful data. Initial times must coincide. If your plot includes the data before the helicopter model is released with initial condition $\dot{\psi}(0)$, NO CREDIT will be given.

5 Part (iv): Identification of k_{pp} and k_{py}

Recall that in the equations of motion for the helicopter, we assumed that the force generated by the pitch and yaw motors is linear in the input voltage; see Remark 1.1. However, before any movement of the helicopter can occur, these forces must overcome the Coulomb or static friction force. Coulomb friction F_c can be viewed as a force or torque acting depending on the velocity $\dot{y}$ and for our purposes is defined by

$$\begin{aligned} F_c(\dot{y}) &= \beta && \text{if } \dot{y} > 0 \\ &\in [\alpha, \beta] && \text{if } \dot{y} = 0 \\ &= \alpha && \text{if } \dot{y} < 0. \end{aligned}$$

If $\dot{y} > 0$, then the force or corresponding torque is given by $\beta > 0$. If $\dot{y} < 0$, then $\alpha < 0$. If the velocity $\dot{y} = 0$, then the force or corresponding torque is somewhere in the interval $[\alpha, \beta]$. Now assume that the input to our system is determined by a servo motor with input voltage v. The force or torque needed to overcome the Coulomb friction is denoted by $f_c(v)$, that is, $F_c(\dot{y}) = f_c(v)$. In terms of voltage this function is given by

$$\begin{aligned} f_c(v) &= \beta && \text{if } v \geq v_+ \\ &= [\alpha, \beta] && \text{if } v_- \leq v \leq v_+ \\ &= \alpha && \text{if } v \leq v_-. \end{aligned}$$

Recall that in our problem, the force due to the input v is linear. So the input force can be viewed as $kv - F_c(\dot{y}) = kv - f_c(v)$. Figure 6 shows the force generated from the motor (dash-dot line), the Coulomb friction force (dash line), and the resultant force (solid line) applied to the helicopter when the motor in turned on with input voltage v.

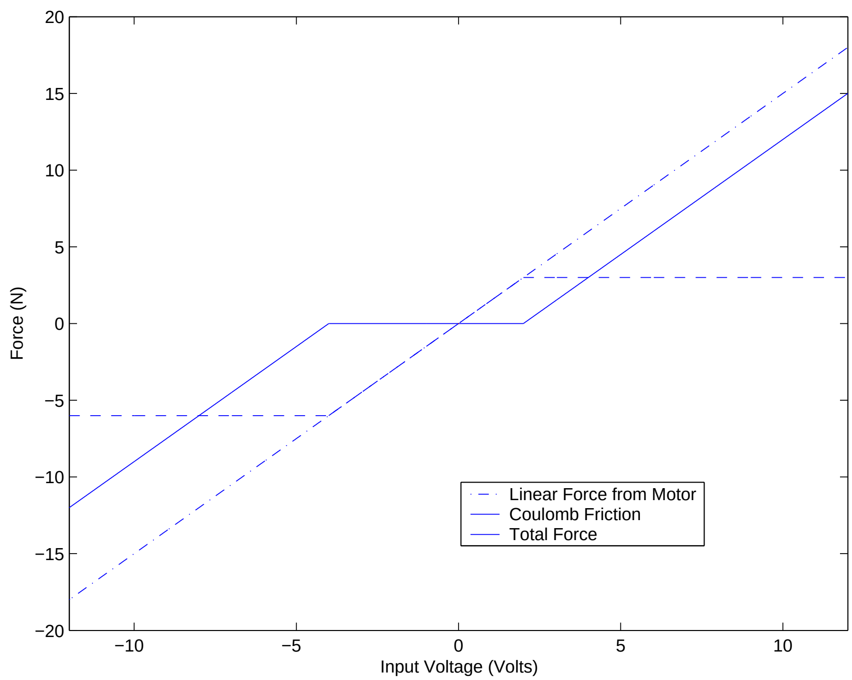

Figure 6: Relations of input voltage and force, linear force from motor is shown in dash-dot line, Coulomb friction is shown in dash line, sum of these two forces is shown in solid line.

If we can somehow keep the helicopter model steady in yaw axis, that is, $\dot{\psi} = 0$, then the pitch equation of motion with Coulomb friction reduces to

$$J_p\ddot{\theta} + c_p\dot{\theta} + mg(h\sin(\theta) + R_c\cos(\theta)) + F_{cp}(\dot{\theta}) = lk_{pp}v_p + k_{py}v_y\,,$$

that is,

$$J_p\ddot{\theta} + c_p\dot{\theta} + mg(h\sin(\theta) + R_c\cos(\theta)) = lk_{pp}v_p + k_{py}v_y - f_{cp}(v_p, v_y)\,.$$

Here $F_{cp}(\dot{\theta})$ is the Coulomb friction corresponding to the pitch angle, and $f_{cp}(v_p, v_y) = F_{cp}(\dot{\theta})$ is the torque exerted by the motor to overcome this Coulomb friction. Observe that $f_{cp}(v_p, v_y)$ is a function of voltages from both the pitch and yaw motor. Without proof, we claim that this equation is stable in our operating conditions, that is, sum of the torques applied from the two motors is not large enough to cause θ to go unbounded. In fact, we claim that $\dot{\theta}(\infty) = 0$ within the input voltage range. Thus when the motors are turned on with constant voltages, this equation reaches steady state where

$$mg(h\sin(\theta(\infty)) + R_c\cos(\theta(\infty))) = lk_{pp}v_p + k_{py}v_y - f_{cp}(v_p, v_y).$$

If only the front motor is on, with a constant voltage v_p, then $v_y = 0$ yields

$$mg(h\sin(\theta(\infty)) + R_c\cos(\theta(\infty))) = lk_{pp}v_p - f_{cp}(v_p, 0). \tag{5.1}$$

If the input voltage v_p is not large enough, then the Coulomb friction has the same magnitude as the torque from the motor, that is, $lk_{pp}v_p = f_{cp}(v_p, 0)$. Hence $\theta(\infty)$ satisfies

$$mg(h\sin(\theta(\infty)) + R_c\cos(\theta(\infty))) = 0.$$

This equation is the same as the static equilibrium condition of the helicopter model. In other words, if the input voltage is not large enough, then the helicopter model does not pitch at all. On the other hand, if the input voltage is large enough to overcome the Coulomb friction torque, then the Coulomb torque remain constant, and thus, we have the linear (affine) relation in the voltage $y = ax + b$ where $y = mg(h\sin(\theta(\infty)) + R_c\cos(\theta(\infty)))$ while $a = lk_{pp}$, $x = v_p$, and $b = -f_{cp}$. By varying the input voltage v_p in the allowable region, we can find the slope of this line. Similarly, if only the back motor is turned on ($v_p = 0$), then in steady state we arrive at

$$mg(h\sin(\theta(\infty)) + R_c\cos(\theta(\infty))) = k_{py}v_y - f_{cp}(0, v_y). \tag{5.2}$$

As before, if v_y is not large enough, then the helicopter model does not pitch. Still by varying the input voltage v_y in the allowable region, we can find the slope of this line. Therefore the parameters k_{pp} and k_{py} can be estimated by

$$\begin{aligned} k_{pp} &= \text{Slope of (5.1)}/l \\ k_{py} &= \text{Slope of (5.2)}. \end{aligned} \tag{5.3}$$

5.1 The Lab steps to identify k_{pp} and k_{py}

(i) Have TA secure the helicopter model so that it cannot yaw. Make sure the mass is at its original position x (closest to the pivot).

(ii) Recall that you have measured the angle p_0, the pitch angle when the helicopter model is initially at rest. In the MATLAB command window, type `theta_0 =` and then type in your numerical value of p_0. Note that **this value must be negative and the units must be in radians**. This will set the zero angle for the level flight position.

(iii) Open the Simulink model `Heli_ID.mdl` and set the simulation time to `inf`.

(iv) From the `Scopes` subsystem, open the scope Average Pitch Angle (average theta). Set the `Time range` to 20 seconds and un-check the `Limit data points` check box.

(v) Click `Quarc - Build`. Select `Simulation - Connect to Target`, and `Quarc - Start`.

(vi) In the Simulink model, make sure the slider gain for the back motor voltage is set to zero. Use the slider gain for the front motor voltage find the minimum input voltage v_+ to overcome the Coulomb friction, that is, find the smallest voltage that makes the helicopter model start pitching.

(vii) Make no less than four measurements between $v_p = v_+$ and $v_p = 11$ Volts. Note the pitch angles for each voltage value.

(viii) Set the slider gain for the front motor voltage to zero. Use the slider gain for the back motor voltage find the minimum input voltage v_+ to overcome the Coulomb friction. Make no less than four measurements between $v_y = v_+$ and $v_y = 12$ Volts. Record the pitch angles for each voltage value.

(ix) Click the Stop button. Make sure you have a record of at least 6 v_p values and their corresponding angles, and at least 6 v_y values and their corresponding angles.

5.2 In your lab report, include the following under Part (iv):

(a) Hand in the plots of your measurements where the x-axis is the input front voltage v_p and y-axis is the corresponding

$$mg(h \sin(p(\infty)) + R_c \cos(p(\infty))).$$

This plot must be discrete. NO CREDIT will be given if you use a continuous line to connect the points. On the same figure, plot the equation (5.1) as a continuous function of the voltage.

(b) Repeat Part (a) with the data obtained using back motor.

6 Part (v): Identification of k_{yp} and k_{yy}

By a similar analysis to that in Section 5, if we can keep the helicopter at the constant pitch angle $\theta = 0$, then the yaw equation of motion reduces to

$$(J_y + J_{shaft})\ddot{\psi} + c_y\dot{\psi} + F_{cy}(\dot{\psi}) = k_{yp}v_p + lk_{yy}v_y\ ,$$

that is,

$$(J_y + J_{shaft})\ddot{\psi} + c_y\dot{\psi} = k_{yp}v_p + lk_{yy}v_y - f_{cy}(v_p, v_y)\,.$$

Here $F_{cy}(\dot{\psi})$ is the Coulomb friction corresponding to the yaw rate, and $f_{cy}(v_p, v_y) = F_{cy}(\dot{\psi})$ is the torque exerted by the motor to overcome this Coulomb friction. Observe that $f_{cy}(v_p, v_y)$ is a function of the voltages from both the pitch and yaw motor. As before, the helicopter will start yawing only when the torque from the motors is large enough. Now assume that only the pitch motor is on with constant voltage and v_p is large enough, then this equation simplified to

$$(J_y + J_{shaft})\ddot{\psi} + c_y\dot{\psi} = k_{yp}v_p - f_{cy}(v_p, 0).$$

Since $(J_y + J_{shaft})$ and c_y are positive, this equation is a stable first order linear differential equation in $\dot{\psi}$. Moreover, in steady state

$$\dot{\psi}(\infty) = \frac{k_{yp}}{c_y}v_p - \frac{f_{cy}}{c_y}. \tag{6.1}$$

If only the back motor is on with constant voltage v_y large enough, then in steady state, we also obtain

$$\dot{\psi}(\infty) = \frac{lk_{yy}}{c_y}v_y - \frac{f_{cy}}{c_y}. \tag{6.2}$$

By varying the front and back input voltage, the parameters k_{yp} and k_{yy} can be estimated by

$$\begin{aligned} k_{yp} &= \text{Slope of (6.1)} \times c_y \\ k_{yy} &= \text{Slope of (6.2)} \times c_y/l. \end{aligned} \tag{6.3}$$

6.1 The Lab steps to identify k_{yp} and k_{yy}

(i) Have TA secure the helicopter so that it keeps the pitch angle constant at zero radians. Make sure the mass is at its original position x (closest to the pivot). In the MATLAB command window, enter `theta_0 = 0` .

(ii) Open the Simulink model `Heli_ID.mdl` and set the simulation time to `inf`. From the `Scopes` subsystem, open the Average Yaw Rate (average psi dot) scope. Set the `Time range` to 120 seconds and de-select the `Limit data points` check box.

(iii) Click `Quarc - Build`. Select `Simulation - Connect to Target`, and `Quarc - Start`.

(iv) In the Simulink model, make sure the slider gain for the yaw motor voltage is set to zero.

(v) Use the slider gain for the pitch motor voltage to find the minimum input voltage v_- and v_+ to overcome the Coulomb friction, that is, find the voltage which make the helicopter model start yawing in each direction.

(vi) Make no less than four measurements between $v_p = -12$ Volts and $v_p = v_-$. Make no less than four measurement between $v_p = v_+$ and $v_p = 12$ Volts. Record the yaw rates for each voltage value.

(vii) Set the slider gain for the pitch motor voltage to zero. Repeat steps (v) and (vi) using the yaw motor instead.

(viii) Click the Stop button. Make sure you have a record of at least 12 v_p values and their corresponding yaw rates, and at least 12 v_y values and their corresponding yaw rates.

6.2 In your lab report, include the following under Part (v):

(a) Hand in the plots of your measurements where the x-axis is the input yaw voltage v_p and y-axis is $\dot{\psi}(\infty)$. This plot must be discrete. NO CREDIT will be given if you use a continuous line to connect the points. On the same figure, plot equation (6.1), using line.

(b) Repeat Part (a) with the data obtained using yaw motor.

7 Part (vi): Obtaining v_{pe} and v_{ye}

When the helicopter is in level flight, the voltage applied to the pitch motor is v_{pe} and the voltage applied to the yaw motor is v_{ye}. Ideally, in this equilibrium attitude, if the Coulomb frictions are not present, the helicopter model may have any yaw angle but the pitch angle θ_e, and the voltages v_{pe}, and v_{ye} must satisfy

$$\begin{aligned} mg(h\sin(\theta_e) + R_c\cos(\theta_e)) &= lk_{pp}v_{pe} + k_{py}v_{ye} \\ k_{yp}v_{pe} + lk_{yy}v_{ye} &= 0 \\ \theta_e &\neq \pm\pi/2. \end{aligned} \tag{7.1}$$

The equilibrium position θ_e and voltages v_{pe} and v_{ye} will be use to linearized the equations of motion of the helicopter model. Moreover, any linear controller designed from this linearized equations of motion is meant to be operating around small perturbation away from this equilibrium position. In particular, since most of the time the helicopter is flown in level flight condition, the equilibrium position where $\theta_e = 0$ is of the most interest.

7.1 The Lab steps to obtain v_{pe} and v_{ye}

(i) Remove all the tools used for securing the helicopter model in constant pitch and yaw position.

(ii) Remove everything not connected to the helicopter model away from the model, far enough so that the air flow is not disrupted.

(iii) In the MATLAB command window, type `theta_0 =` and then type the numerical value of p_0 that you measured in Part (i) – remember, it **must be negative and in radians**.

(iv) In the Simulink model `Heli_ID.mdl`, set the simulation time is set to `inf`.

(v) From the `Scopes` subsystem, open the Pitch Display (theta), Average Pitch Angle, and Average Yaw Rate (psi dot) scopes. Change the `Time range` to 60 seconds. Click `Quarc - Build`. Select `Simulation - Connect to Target`, and `Quarc - Start`.

(vi) In the Simulink model, use the two slider gains for the input voltages to find the pitch and yaw voltage that keep the helicopter model at zero pitch angle and zero yaw rate. You may choose to pitch the helicopter model so that the pitch is close to zero first and then try to reduce its yaw rate. Then fine-tune your pitch v_{pe} and yaw v_{ye} voltages.

(vii) Click the Stop button. Make sure you have a record of v_{pe} and v_{ye} before you leave.

(viii) Close the Simulink model *without* saving changes.