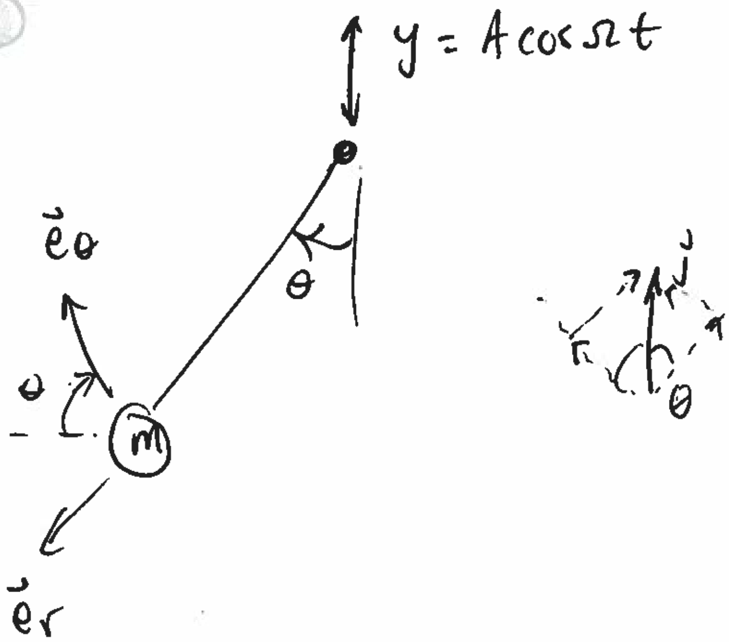
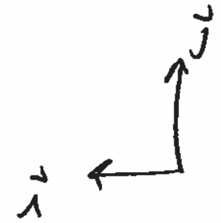
Forced Pendulum $\rightarrow$ vertical excitation





$\vec{j} = -\cos\theta\,\vec{e}_r + \sin\theta\,\vec{e}_\theta$

$$\vec{v} = l\dot{\theta}\,\vec{e}_\theta + \dot{y}\,\vec{j} = (l\dot{\theta} - A\Omega\sin\Omega t\sin\theta)\,\vec{e}_\theta - A\Omega\sin\Omega t\cos\theta\,\vec{e}_r$$

$$V = mg[l(1-\cos\theta) + y(t)]$$

$$L = T - V = \frac{1}{2}m(l\dot{\theta} - A\Omega\sin\Omega t\sin\theta)^2 + \frac{1}{2}mA^2\Omega^2\sin^2\Omega t\cos^2\theta - mgl(1-\cos\theta) - mgA\cos\Omega t$$

$$= \frac{1}{2}mA^2\Omega^2\sin^2\Omega t + \frac{1}{2}ml^2\dot{\theta}^2 - mA\Omega l\sin\Omega t\sin\theta\,\dot{\theta} - mgl(1-\cos\theta) - mgA\cos\Omega t$$

$$\frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta} - mlA\Omega\sin\Omega t\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta} - mlA\Omega^2\cos\Omega t\sin\theta - mlA\Omega\dot{\theta}\sin\Omega t\cos\theta$$

$$\frac{\partial L}{\partial\theta} = -mlA\Omega\dot{\theta}\sin\Omega t\cos\theta - mgl\sin\theta$$

$$m\ell^2\ddot{\theta} - m\ell A\Omega^2\cos\Omega t\sin\theta + mg\ell\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \left(\frac{g}{\ell} - \frac{A\Omega^2\cos\Omega t}{\ell}\right)\sin\theta = 0}$$

Introduce Ordering

$$\theta \to \epsilon\theta$$

$$\sin\theta \approx \theta - \frac{1}{6}\theta^3$$

$$A \to \epsilon A$$

$$\epsilon\ddot{\theta} + \left(\frac{g}{\ell} - \frac{\epsilon A\Omega^2\cos\Omega t}{\ell}\right)\left(\epsilon\theta - \frac{1}{6}\epsilon^3\theta^3\right) = 0$$

$$\ddot{\theta} + \left(\frac{g}{\ell} - \frac{\epsilon A\Omega^2\cos\Omega t}{\ell}\right)\left(\theta - \frac{1}{6}\epsilon^2\theta^3\right) = 0$$

Keep to $O(\epsilon)$ only

$$\boxed{\ddot{\theta} + \frac{g}{\ell}\theta - \frac{\epsilon A\Omega^2}{\ell}\cos\Omega t\,\theta = 0}$$

or

$$\ddot{\theta} + \underbrace{\frac{1}{\ell}\left(g - \epsilon A\Omega^2\cos\Omega t\right)}_{\text{parametric excitation}}\theta = 0$$