College of Engineering
School of Aerospace Engineering

ISYE 7750: Mathematical Foundations of Machine Learning

Homework 1

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September 4, 2022

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I Problem One

Visualizing quadratic functions and linear algebra review: 20 points For $x \in \mathbb{R}^d$, consider the function $f(x) = \mathbf{x}^T \mathbf{A}^T \mathbf{x} + \mathbf{b}^T \mathbf{x} + c$ for a square and symmetric matrix $\mathbf{A} \in \mathbb{R}^{d \times d}$, vector $b \in \mathbb{R}^d$ and scalar c.

(a) Write the gradient of f. Suppose the matrix $\mathbf{A}$ is invertible. Is there a unique solution to $\nabla f(x) = 0$, and if so, what is it?

(b) You have been given an `iPython` notebook with starter code to generate 3D plots and contour plots of the functions. Fill in the requisite lines of code and provide both 3D and contour plots of f for the following settings of $(\mathbf{A}, \mathbf{b}, c)$:

- $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{b} = 0, \quad c = 0.$
- $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{b} = 0, \quad c = 0.$
- $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, \quad \mathbf{b} = 0, \quad c = 0.$
- $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{b} = 0, \quad c = 0.$
- $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{b} = 0, \quad c = 0.$

Hopefully this gives you some idea of the geometry of these functions. Comment on the contour plots that you observe in all cases and how they relate the respective 3D plots.
Note: If you prefer to use your own code to generate these plots instead of the provided starter code, that is fine.

(c) Show that we have the linear representation $f(\mathbf{x}) - \sum_{j=1}^{M} \alpha_j p_j(\mathbf{x})$ where $p_1, ..., p_M$ are all the monomials in $(x_1, ..., x_d)$ of degree at most 2. In particular, for each such monomial write down the coefficient that multiplies it.
Takeaway: This and the previous part should convince you that there are quite a few shapes taken by functions that can be represented as quadratic polynomials in two variables.

(d) For the fifth example (non-diagonal $\mathbf{A}$), use a package to compute its eigen-decomposition. Compare the orthogonal matrix returned by this eigen-decomposition to what you see on the corresponding contour plot. Is $\mathbf{A}$ positive semidefinite (PSD)?

(e) Now return to the original problem in dimension d with general $(\mathbf{A}, \mathbf{b}, c)$. If $\mathbf{A}$ is PSD and invertible, i.e., positive definite, argue that $\hat{\mathbf{x}} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}$ is the unique global minimum of f.
Hint: There are several ways to do this, but one way is to write the function as $f(\mathbf{x}) = (\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{A} (\mathbf{x} - \hat{\mathbf{x}}) +$ other terms and argue from there.

Solution:

(a) Since the $\mathbf{A}$ matrix is a symmetric matrix the gradient of f becomes

$$\nabla f(x) = 2\mathbf{A}\mathbf{x} + \mathbf{b}.$$

If we equate this result with 0 we have

$$\begin{aligned} 2\mathbf{A}\mathbf{x} + \mathbf{b} &= 0 \\ \mathbf{x} &= -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}. \end{aligned}$$

This is possible because $\mathbf{A}$ is invertible, and this solution is unique.

(b) The plots are as follows.

Case 1:

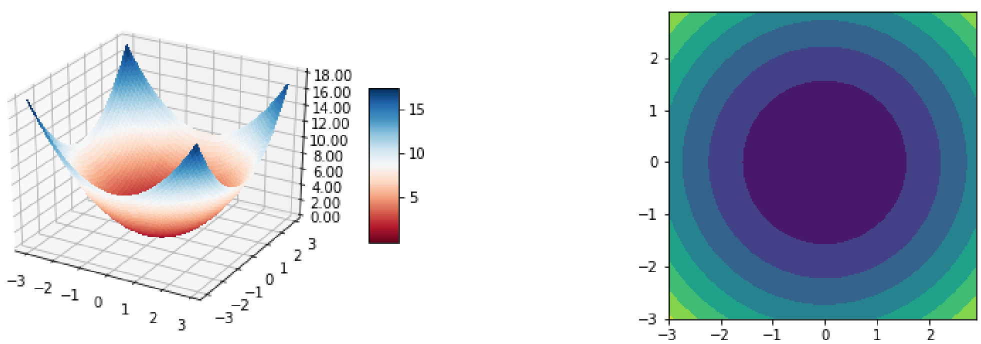

Figure 1: Case 1: 3D and contour plots

Case 2:

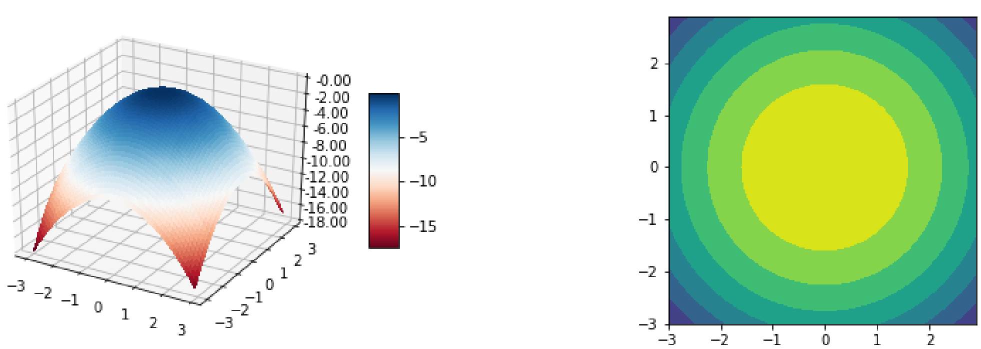

Figure 2: Case 2: 3D and contour plots

Case 3:

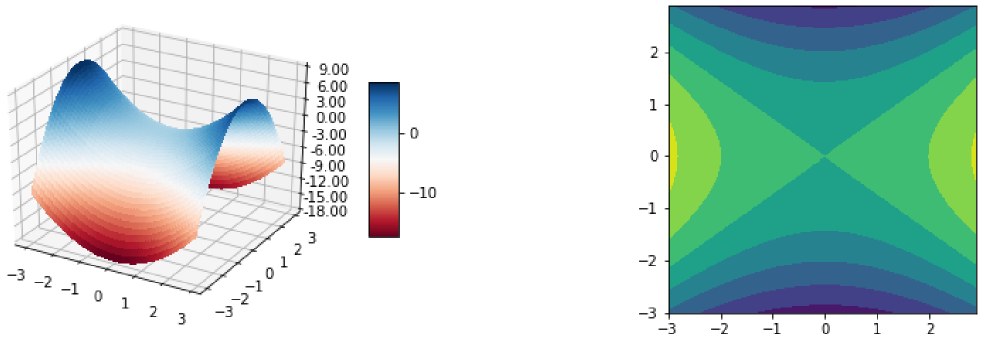

Figure 3: Case 3: 3D and contour plots

Case 4:

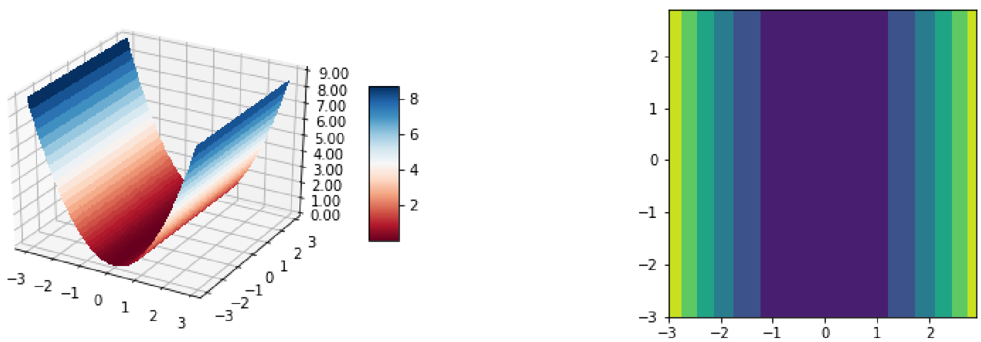

Figure 4: Case 4: 3D and contour plots

Case 5:

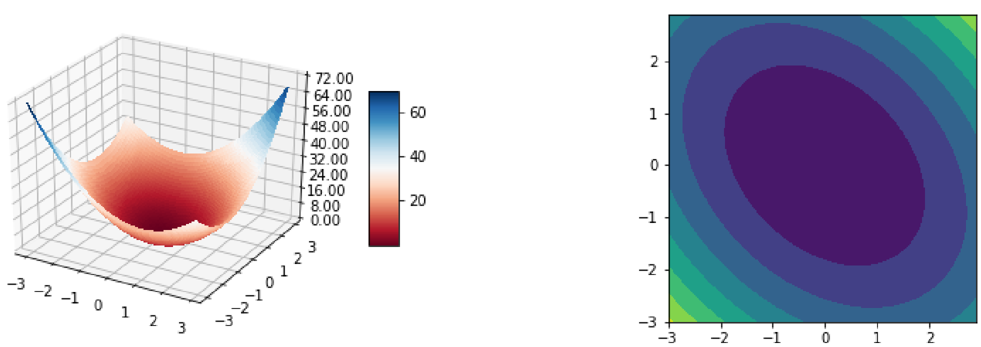

Figure 5: Case 5: 3D and contour plots

```python
def z_func(x1,x2, A): #A here is expected to be a two-by-two multidimensional array
    # Version 1
    X = vstack([x1.flatten(), x2.flatten()])
    f_x = dot(dot(X.T, A), X)
    f_x = diag(f_x).reshape(x1.shape)
    return f_x
```

For all contour plots we can observe that it has a light green color for locations corresponding to high gradients in the 3D plot and dark blue color for low gradients. Thus, the color transitions from light green to dark blue as the values become smaller or approach the minimal value.

(c) Since we know that for some two matrices $\mathbf{E}$ and $\mathbf{F}$ with a reasonable dimension of $d \times d$

$$(\mathbf{E} \cdot \mathbf{F})_i = \sum_{k=1}^{d} E_{jk} \cdot F_{ki}.$$

Then

$$\left(\mathbf{x}^T \cdot \mathbf{A} \cdot \mathbf{x}\right) = \sum_{i=1}^{d} \sum_{j=1}^{d} x_i A_{ij} x_j.$$

Now we can represent the quadratic equation as

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{d} \sum_{j=1}^{d} x_i A_{ij} x_j + \sum_{i=1}^{d} b_i x_i + c \\ &= \sum_{j=1}^{M} \alpha_j p_j(\mathbf{x}). \end{aligned}$$

Thus, this shows that the linear representation are monomials with degree at most 2. Now since we can rewrite the expression above as

$$\begin{aligned} f(\mathbf{x}) &= \sum_{i=1}^{d} \sum_{j=1}^{d} x_i A_{ij} x_j + \sum_{i=1}^{d} b_i x_i + c \\ &= \sum_{i=1}^{d} A_{ii} x_i^2 + \sum_{i \neq j}^{d} A_{ij} x_i x_j, \end{aligned}$$

We can say that the coefficients are

$$\alpha_j = \begin{cases} A_{ii} & \text{for } x_i^2 \\ 2A_{ij} & \text{for } x_i x_j, \quad i \neq j \\ b_i & \text{for } x_i \\ c & \text{for } 1 \end{cases}.$$

(d) From the eigendecomposition we have

$$\lambda_1 = 4 \quad \lambda_2 = 2$$

$$v_1 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}, \quad U = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

The orthogonal matrix U has the eigenvectors v_1 and v_2, and if we look at the contour plot we can see that the ellipse's axes are in the direction of these eigenvectors. Also, since matrix A is symmetric and has all positive eigenvalues, we know that the matrix A is PSD.

(e) If $\hat{\mathbf{x}} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}$, then we have that

$$f(x) = (\mathbf{x} - \hat{\mathbf{x}})^T \mathbf{A}(\mathbf{x} - \hat{\mathbf{x}}) + (c - \hat{\mathbf{x}}^T A \hat{\mathbf{x}}).$$

Now since $\mathbf{A}$ is PSD, i.e. $\mathbf{y}^T\mathbf{A}\mathbf{y} \leq 0$. This inequality holds if and only if $y = 0$. Hence, $f(x)$ obtains its minimum when $\mathbf{x} - \hat{\mathbf{x}} = 0$ is true. This leads to the fact that the unique global minimum is attained for

$$\mathbf{x} = \hat{\mathbf{x}} = -\frac{1}{2}\mathbf{A}^{-1}\mathbf{b}. \quad \blacksquare$$

II Problem Two

Monomial basis functions and invertibility: 20 points

(a) In class, we claimed that given a dataset $\{x_i, y_i\}_{i=1}^n$ where $x_i, y_i \in \mathbb{R}$ were distinct across i, there is a *unique* polynomial of degree $n-1$ that interpolates these points. Without proving uniqueness, we showed that one such polynomial interpolator of degree $n-1$ was given by the Lagrange polynomial

$$p(x) = \sum_{k=1}^{n} y_k \cdot \prod_{\substack{1 \le j \le n \\ j \ne k}} \frac{x - x_j}{x_k - x_j}.$$

In parts (a)-(e), you will prove the *uniqueness* claim for every finite n, provided $d = 1$. Construct the square matrix

$$\mathbf{\Phi} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

Argue formally that the Lagrange polynomial $p(x)$ is the *unique* polynomial of degree $n-1$ that interpolates the points $\{x_i, y_i\}_{i=1}^n$ *if and only if* $\mathbf{\Phi}$ is invertible.

(b) Next, show that if $n = 2$ and $x_1 \ne x_2$, then there is a unique interpolating polynomial of degree 1 that interpolates these points.

(c) We will not try to set up a general way to attack the invertibility of $\mathbf{\Phi}$ by calculating its determinant. In particular, we will form another matrix $\mathbf{\Phi}'$. Let M_j denote the j-th column of a matrix $\mathbf{M}$ and execute the following loop:

$$\{\mathbf{\Phi}' = \mathbf{\Phi}_{n-j} - x_1 \cdot \mathbf{\Phi}_{n-j-1} : \; j = 0, ..., n-1\}$$

Use the convention that $\Phi_0 = 0$. Argue that $\det(\mathbf{\Phi}) = \det(\mathbf{\Phi}')$.

(d) Show that $\det(\mathbf{\Phi}) = (x_2 - x_1) \times (x_3 - x_1) \times \cdots \times (x_n - x_1) \times \det(\bar{\mathbf{\Phi}})$, where we have defined the $(n-1) \times (n-1)$ matrix

$$\bar{\mathbf{\Phi}} = \begin{bmatrix} 1 & x_2 & x_2^2 & \cdots & x_2^{n-2} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-2} \end{bmatrix}.$$

(e) Proceed iteratively and produce an expression for $\det(\mathbf{\Phi})$. Conclude that $\mathbf{\Phi}$ is invertible if and only if the points $\{x_i\}_{i=1}^n$ are distinct.
Note: The situation when $x_i \in \mathbb{R}^d$ is significantly more delicate; indeed, uniqueness of the points does not suffice. However, if the points are sufficiently generic (say $\{x_i\}_{i=1}^n$ are chosen i.i.d. from a standard Gaussian distribution $\mathcal{N}(0, \mathbb{I})$) then it can be shown that $\mathbf{\Phi}$ is indeed invertible. Bonus credit for proving this, but note that it is a challenge problem.

(f) The remaining parts are unrelated to the previous parts. We will now argue that the number of monomials of degree less than or equal to l in d variables can be explicitly calculated. First argue that the number of such monomails can be counted by placing d stars and $l-1$ bars in a sequence and counting the number of unique configurations that result when these are permuted.

(g) How many unique configurations are there above? Your expression should be explicit, and a function of the pair (d, l).

Solution:
(a) Let

$$l_k(x) = \prod_{\substack{1 \le j \le n \\ j \ne k}} \frac{x - x_j}{x_k - x_j}.$$

From the Vandermonde matrix ($\mathbf{\Phi}$) and Lagrange polynomial we know that

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} l_1(x_1) & 0 & \cdots & 0 \\ 0 & l_2(x_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & l_n(x_n) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{II.1}$$

Now if Φ is invertible this means that there is a unique set of coeffcients $a_i = (a_0, a_1, ..., a_n)$ which deduces that fact that there is a unique polynomial of degree $n - 1$ for the Lagrange polynomial as well. To prove the sufficient condition, if we let that the Lagrange polynomial is unique, this means that there is a unique polynomial of

$$p(x) = \sum_{k=1}^{n} y_k \cdot l_k(x).$$

If this is unique this means that in Equation (II.1), the coefficients $a_i = (a_0, a_1, ..., a_n)$ should have a unique set. Thus,

$$\begin{aligned} \mathbf{\Phi a} &= \mathbf{y} \\ \mathbf{a} &= \mathbf{\Phi}^{-1}\mathbf{y}. \end{aligned}$$

Hence, $\mathbf{\Phi}$ should be invertible.

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(b) This seems to be the first step for a proof of induction. If $n = 2$ the Vandermonde matrix becomes

$$\mathbf{\Phi}_2 = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$$

If we take the determinant of this we have

$$\det(\mathbf{\Phi}_2) = x_2 - x_1 \ne 0 \qquad \because\ x_1 \ne x_2$$

Now from part (a) we know that if the Vandermonde matrix is invertible there exists a unique interpolating polynomial.

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(c) The matrix $\mathbf{\Phi}'$ is

$$\mathbf{\Phi}' = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & x_2^2 - x_1 x_2 & \cdots & x_2^{n-1} - x_1 x_2^{n-2} \\ 1 & x_3 - x_1 & x_3^2 - x_1 x_3 & \cdots & x_3^{n-1} - x_1 x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & x_n^2 - x_1 x_n & \cdots & x_n^{n-1} - x_1 x_n^{n-2} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n-2} \\ 1 & x_3 - x_1 & (x_3 - x_1)x_3 & \cdots & (x_3 - x_1)x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & (x_n - x_1)x_n & \cdots & (x_n - x_1)x_n^{n-2} \end{bmatrix}. \tag{II.2}$$

Since, this matrix $\mathbf{\Phi}'$ is made just by a column operation of $\mathbf{\Phi}$ and the determinant remains the same with a column operation. Thus, $\det(\mathbf{\Phi}) = \det(\mathbf{\Phi}')$.

(d) Continuing on (c) we have

$$\det(\mathbf{\Phi}') = \det\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n-2} \\ 1 & x_3 - x_1 & (x_3 - x_1)x_3 & \cdots & (x_3 - x_1)x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n - x_1 & (x_n - x_1)x_n & \cdots & (x_n - x_1)x_n^{n-2} \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n-2} \\ 0 & x_3 - x_1 & (x_3 - x_1)x_3 & \cdots & (x_3 - x_1)x_3^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_n - x_1 & (x_n - x_1)x_n & \cdots & (x_n - x_1)x_n^{n-2} \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} x_2 - x_1 & (x_2 - x_1)x_2 & \cdots & (x_2 - x_1)x_2^{n-2} \\ x_3 - x_1 & (x_3 - x_1)x_3 & \cdots & (x_3 - x_1)x_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_n - x_1 & (x_n - x_1)x_n & \cdots & (x_n - x_1)x_n^{n-2} \end{bmatrix}\right) \quad \text{(reduce order)}$$
$$= \det\left(\begin{bmatrix} x_2 - x_1 & 0 & \cdots & 0 \\ 0 & x_3 - x_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n - x_1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{bmatrix}\right) \quad \text{(refactor)}$$

Then since the determinant of a diagonal matrix is the product of the diagonal elements we have

$$\det(\mathbf{\Phi}') = \prod_{j=2}^{n}(x_j - x_1) \det\left(\begin{bmatrix} 1 & x_2 & \cdots & x_2^{n-2} \\ 1 & x_3 & \cdots & x_3^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{bmatrix}\right) = \prod_{j=2}^{n}(x_j - x_1) \times \det(\bar{\mathbf{\Phi}}).$$

■

(e) Now to conclude the proof by induction, we use the inductive hypothesis and attain

$$\det(\mathbf{\Phi}) = \det(\mathbf{\Phi}') = \prod_{j=2}^{n}(x_j - x_1) \cdot \prod_{2 \le i < j \le n}(x_j - x_i) = \prod_{1 \le i < j \le n}(x_j - x_i). \tag{II.3}$$

Now using the result of (II.3), if $\{x_i\}_{i=1}^n$ are all distinct that means that all of $(x_j - x_i) \neq 0$ and the determinant is nonzero. This means that the Vandermonde matrix is invertible. For the sufficient condition, if the $\mathbf{\Phi}$ is invertible this implies that the matrix is non-singular, and therefore, the determinant cannot be 0. Thus, from (II.3) there can be no value of $x_j = x_i$ in the set $\{x_i\}$.

■

III Problem Three

Splines and linear equations: 20 points

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a second-order spline defined by the overlap of five B-splines:

$$f(x) = \sum_{k=0}^{4} \alpha_k b_2(x-k),$$

where $b_2(x)$ is defined as on page 11 of the Fall 20 notes:

$$b_2(x) = \begin{cases} \frac{(x+\frac{3}{2})^2}{2} & -\frac{3}{2} \le x \le -\frac{1}{2} \\ -x^2 + \frac{3}{4} & -\frac{1}{2} \le x \le \frac{1}{2} \\ \frac{(x-\frac{3}{2})}{2} & \frac{1}{2} \le x \le \frac{3}{2} \\ 0 & \frac{3}{2} \le |x| \end{cases}.$$

(a) Write a function which takes $\alpha = \{\alpha_0, ..., \alpha_4\}$ and $\mathbf{x}$ as an input and returns samples of $f(\mathbf{x})$ at the locations specified in the vector $\mathbf{x}$. Turn in a plot of $f(\mathbf{x})$ for $\alpha = \{2, 1, -1, 3, -1\}$. Sample x densely enough (by specifying sufficiently many points in $\mathbf{x}$) so that your plots look like a smooth function.

(b) Suppose

$$f(0) = 2, \quad f(1) = 2, \quad f(2) = -5, \quad f(3) = -5, \quad f(4) = -2.$$

What are the corresponding α_k? (Hint: you will have to construct a system of equations then solve it. You don't necessarily have to do this by hand.)

(c) To generalize this, suppose that f is now a superposition of N B-splines, with

$$f(x) = \sum_{n=0}^{N-1} \alpha_n b_2(x-n).$$

Describe how to construct the $N \times N$ matrix that maps the coefficients α to the N samples $f(0), ..., f(N-1)$. That is, find a matrix $\mathbf{A}$ such that

$$\begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N-1) \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{bmatrix}$$

(d) Argue that matrix $\mathbf{A}$ from part (c) is invertible for all values of N. (Hint: Observer that the matrix $\mathbf{A}$ is banded and has large values on diagonal. How might you use this to argue invertibility?)

(e) To take this even further, suppose that

$$f(x) = \sum_{n=-\infty}^{\infty} \alpha_n b_2(x-n),$$

so f is described by the (possibly infinite) sequence of numbers $\{\alpha_n\}_{n\in\mathbb{Z}}$. Show that there is a convolution operator that maps the sequence $\{\alpha_n\}$ to the sequence $\{f(n)\}$.

That is, find a sequence of numbers $\{h_n\}_{n\in\mathbb{Z}}$ such that

$$f(n) = \sum_{l=-\infty}^{\infty} h_l \alpha_{n-l}.$$

Solution:

(a) The plot of this bspline interpolation is as follows.

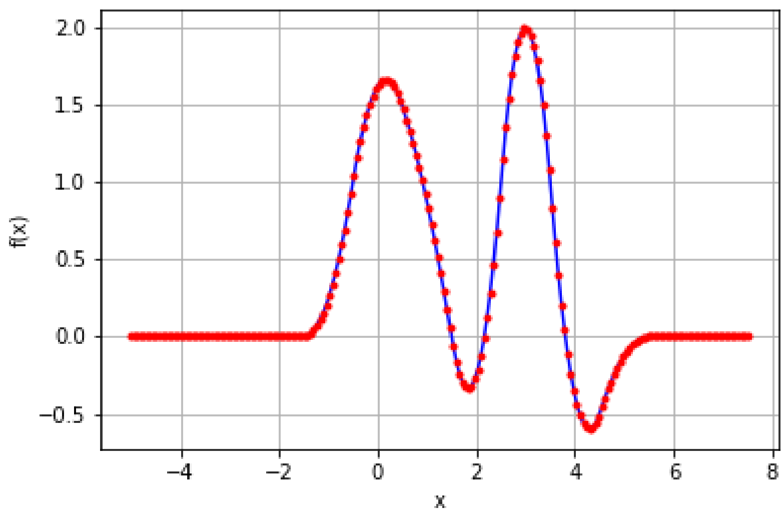


Figure 6: Problem (a) trajectory of bspline interpolation.

```
import numpy as np
import matplotlib.pyplot as plt

def b2(x):
  if -1.5 <= x < -0.5:
    return (x + 1.5)**2/2
  elif -0.5 <= x < 0.5:
    return -x**2 + 0.75
  elif 0.5 <= x <= 1.5:
    return (x - 1.5)**2/2
  else:
    return 0


def bspline(A, X):
  fx = []
  for x in X:
    tmp = 0
    for k, a in enumerate(A):
      tmp += a * b2(x - k)
    fx.append(tmp)
  return np.array(fx)
```

```python

pts = np.linspace(-5, 7.5, 200)
alpha = [2, 1, -1, 3, -1]
res = bspline(alpha, pts)

# Plotting
fig = plt.plot()
plt.plot(pts, res, ".-", c='blue', mfc='red', mec='red')
plt.xlabel("x")
plt.ylabel("f(x)")
plt.grid()
```

(b) Let the given points of x are $x_i = \{0, 1, 2, 3, 4\}$ where $i \in [0, 4]$. To get the coefficients for this polynomial we have to solve the following system of equations for the α values

$$\begin{bmatrix} b_2(x_0) & b_2(x_0-1) & b_2(x_0-2) & b_2(x_0-3) & b_2(x_0-4) \\ b_2(x_1) & b_2(x_1-1) & b_2(x_1-2) & b_2(x_1-3) & b_2(x_1-4) \\ b_2(x_2) & b_2(x_2-1) & b_2(x_2-2) & b_2(x_2-3) & b_2(x_2-4) \\ b_2(x_3) & b_2(x_3-1) & b_2(x_3-2) & b_2(x_3-3) & b_2(x_3-4) \\ b_2(x_4) & b_2(x_4-1) & b_2(x_4-2) & b_2(x_4-3) & b_2(x_4-4) \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ f(x_4) \end{bmatrix}.$$

We solve this using python code

```python
fi = np.array([2, 2, -5, -5, -2])
pts = np.array([0,1,2,3,4])

B = np.zeros((5,5))
for i, x in enumerate(pts):
  for j in range(5):
    B[i, j] = b2(x - j)

alpha = np.linalg.inv(B) @ fi
```

Which gives us

$$\alpha_i = \begin{bmatrix} 2.1044733 & 3.37316017 & -6.34343434 & -5.31255411 & -1.78124098 \end{bmatrix}^T$$

and the plots is as follows

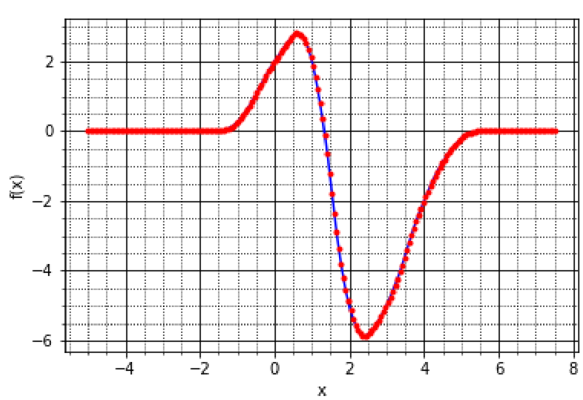


Figure 7: Problem (b) trajectory of bspline interpolation.

(c) From we the previous problem we can generalize the matrix $\mathbf{A}$ to be

$$\mathbf{A} = \begin{bmatrix} b_2(0) & b_2(0-1) & b_2(0-2) & \cdots & b_2(0-(N-1)) \\ b_2(1) & b_2(1-1) & b_2(1-2) & \cdots & b_2(1-(N-1)) \\ b_2(2) & b_2(2-1) & b_2(2-2) & \cdots & b_2(2-(N-1)) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_2(N-1) & b_2((N-1)-1) & b_2((N-1)-2) & \cdots & b_2((N-1)-(N-1)) \end{bmatrix}$$

(d) If we simplify the matrix we have

$$\mathbf{A} = \begin{bmatrix} b_2(0) & b_2(-1) & 0 & 0 & \cdots & 0 \\ b_2(1) & b_2(0) & b_2(-1) & 0 & \cdots & 0 \\ 0 & b_2(1) & b_2(0) & b_2(-1) & \cdots & 0 \\ 0 & 0 & b_2(1) & b_2(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b_2(0) \end{bmatrix}$$

This is a Toeplitz matrix, and since

$$|b_2(0)| = \frac{3}{4} \geq \frac{1}{2} = \frac{1}{4} + \frac{1}{4} = |b_2(1)| + |b_2(-1)|$$

This matrix is diagonally dominant. Additionally, because the diagonal entries are strictly positive for this Toeplitz matrix, this matrix is positive definite. Finally, positive definite matrices are invertible. Thus, this matrix $\mathbf{A}$ is invertible.

■

IV Problem Four

Fun with norms and inner products: 20 points

(a) For parts (a-e), $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$. Prove that $|\langle \mathbf{x}, \mathbf{y} \rangle| \le \|\mathbf{x}\|_\infty \cdot \|\mathbf{y}\|_1$.

(b) Prove that $\|\mathbf{x}\|_1 \le \sqrt{N} \cdot \|\mathbf{x}\|_2$. (Hint: Cauchy–Schwarz)

(c) Let B_2 be the unit ball for the ℓ_2 norm in $\mathbb{R}^N$. Show that

$$\max_{\mathbf{x} \in B_2} \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{y}\|_2.$$

Describe the vector $\mathbf{x}$ (as a function of $\mathbf{y}$) which achieves the maximum.

(d) Let B_∞ be the unit ball for the ℓ_∞ norm in $\mathbb{R}^N$. Show that

$$\max_{\mathbf{x} \in B_\infty} \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{y}\|_1.$$

Describe the vector $\mathbf{x}$ (as a function of $\mathbf{y}$) which achieves the maximum.

(e) Let B_1 be the unit ball for the ℓ_1 norm in $\mathbb{R}^N$. Show that

$$\max_{\mathbf{x} \in B_1} \langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{y}\|_\infty.$$

Describe the vector $\mathbf{x}$ (as a function of $\mathbf{y}$) which achieves the maximum.

(f) For parts (f-i), suppose you are given an $N \times N$ matrix $\mathbf{Q}$, and set

$$\langle \mathbf{x}, \mathbf{y} \rangle_Q = \mathbf{y}^\top \mathbf{Q} \mathbf{x},$$

for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. Prove that if $\mathbf{Q}$ has an entry along its diagonal that is nonpositive, then $\langle \cdot, \cdot \rangle_Q$ cannot be a valid inner product on $\mathbb{R}^N$.

(g) Prove that if $\mathbf{Q}$ is not symmetric, then $\langle \cdot, \cdot \rangle_Q$ cannot be valid inner product on $\mathbb{R}^N$.

(h) Recall that $\mathbf{Q}$ is *symmetric positive definite* if it is symmetric and obeys

$$\mathbf{x}^\top \mathbf{Q} \mathbf{x} > 0, \quad \text{for all} \;\; \mathbf{x} \in \mathbb{R}^N, \; \mathbf{x} \neq 0.$$

Prove that $\langle \cdot, \cdot \rangle_Q$ is a valid inner product on $\mathbb{R}^N$ if and only if $\mathbf{Q}$ is symmetric positive definite.

(i) Define the norm on $\mathbb{R}^2$

$$\|\mathbf{x}\| = \|\mathbf{A}\mathbf{x}\|_2, \quad \mathbf{A} = \begin{bmatrix} 3 & 3 \\ -1/2 & 1/2 \end{bmatrix}.$$

Find $\mathbf{Q}$ so that $\langle \cdot, \cdot \rangle_Q$ induces this norm.

Solution:
(a) (Theorem) **Hölder's Inequality**: Let $1 \leq p \leq \infty$ and q is dual, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. For any two sequence $\{x_j\}_{j=1}^{\infty}$, $\{y_j\}_{j=1}^{\infty} \in \mathbb{R}$ or $\mathbb{C}$ we have:

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \cdot \|y\|_q = \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{\frac{1}{p}} \left(\sum_{j=1}^{\infty} |y_j|^q\right)^{\frac{1}{q}} \tag{IV.1}$$

The proof is as follows.
(i) In a particular case of $\|x\|_p = \|y\|_q = 1$

$$\sum_{j=1}^{\infty} |x_j y_j| \leq \sum_{j=1}^{\infty} \left(\frac{|x_j|^p}{p} + \frac{|y_j|^q}{q}\right) = \frac{1}{p}\sum |x_j|^p + \frac{1}{q}\sum |y_j|^q = \frac{1}{p} + \frac{1}{q} = 1.$$

(ii) For a general case,

$$a_j = \frac{x_j}{\left(\sum |x_j|^p\right)^{\frac{1}{p}}}, \quad b_j = \frac{y_j}{\left(\sum |y_j|^q\right)^{\frac{1}{q}}}$$

where $\sum |a_j|^p = \sum |b_j|^q = 1$. Now $\{a_j\}$ and $\{b_j\}$ satisfy part (i) and

$$\sum |a_j b_j| \leq 1$$
$$\sum \frac{|x_j y_j|}{\left(\sum |x_j|^p\right)^{\frac{1}{p}} \left(\sum |y_j|^q\right)^{\frac{1}{q}}} \leq 1$$
$$\therefore \sum_{j=1}^{\infty} |x_j y_j| \leq \|x\|_p \cdot \|y\|_q$$

Now that we have proven Hölder's Inequality, we can use this to prove this problem. Since $p = 1$ and $q = \infty$ suffice the duality condition, from Hölder's Inequality (IV.1) we can directly say that

$$|\langle x, y\rangle| = \sum |x_j y_j| \leq \|x_j\|_\infty \cdot \|y_j\|_1$$

■

(b) From Cauchy-Schwartz,

$$\|x\|_1 = \sum_{j=1}^{N} |x_j| = \sum_{j=1}^{N} |x_j| \cdot 1 \leq \left(\sum_{j=1}^{N} |x_j|^2\right)^{\frac{1}{2}} \left(\sum_{j=1}^{N} 1^2\right)^{\frac{1}{2}} = \sqrt{N}\, \|x\|_2 .$$

■

(c) From Cauchy-Schwartz we have

$$\langle \mathbf{x}, \mathbf{y}\rangle \leq \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$$

Since, $\mathbf{x} \in B_2$ of a unit circle $max\, \|\mathbf{x}\|_2 = 1$, and therefore

$$\langle \mathbf{x}, \mathbf{y}\rangle \leq \|\mathbf{y}\|_2$$

Hence,

$$\max\langle x, y\rangle = \|\mathbf{y}\|_2 .$$

■

Now, the vector $\mathbf{x}$ can be written in terms of $\mathbf{y}$ for the maximum on the boundary of the unit ball ∂B_2 as

$$\mathbf{x} = \frac{\mathbf{y}}{\|\mathbf{y}\|_2}.$$

(d) From (a) we know that

$$|\langle x, y\rangle| = \sum |x_j y_j| \leq \|x_j\|_\infty \cdot \|y_j\|_1$$

Thus, if $\mathbf{x} \in B_\infty$

$$\max\langle x, y\rangle = 1 \cdot \|y\|_1 .$$

When this maximum to be true we need the vector $\mathbf{x}$ to be no the unit circle thus,

$$\mathbf{x} = \frac{\mathbf{y}}{\|y\|_\infty}.$$

(e) From (a) we know that

$$|\langle x, y\rangle| = \sum |x_j y_j| \leq \|x_j\|_\infty \cdot \|y_j\|_1$$

Thus, if $\mathbf{x} \in B_1$

$$\max\langle x, y\rangle = 1 \cdot \|y\| |\infty.$$

When this maximum to be true we need the vector $\mathbf{x}$ to be no the unit circle thus,

$$\mathbf{x} = \frac{\mathbf{y}}{\|y\|_1}.$$

(f) From the definition of inner products we know that $\langle x, y\rangle \geq 0$. Thus,

$$\langle x, y\rangle_{\mathbf{Q}} = \mathbf{y}^T\mathbf{Q}\mathbf{x} > 0 \quad \text{if } \mathbf{x}, \mathbf{y} \text{ are nonzero.}$$

This implies that the matrix $\mathbf{Q}$ is positive definite and such matrix must have positive diagonal entries.

■

(g) Let $(e_1, ..., e_n)$ be a basis for $\mathbb{R}^n$, and $\mathbf{x} = \{x_i\}_{i=1}^n$ and $\mathbf{y} = \{y_i\}_{i=1}^n$ such that:

$$\mathbf{x} = x_1e_1 + \cdots + x_ne_n, \qquad \mathbf{y} = y_1e_1 + \cdots + y_ne_n.$$

Then

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{y}\rangle &= \langle x_1e_1 + \cdots + x_ne_n,\ y_1e_1 + \cdots + y_ne_n\rangle \\
&= \sum_{i,j=1}^n x_iy_j\langle e_i, e_j\rangle \\
&= \sum_{i,j=1}^n x_ia_{ij}y_j
\end{aligned}$$

Where $a_{ij} = \langle e_i, e_j \rangle$. Now if you let $\mathbf{Q}$ to be the matrix whose (i, j)-th entry is a_{ij}, then the above becomes

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{Q} \mathbf{y}$$

Finally, $a_{ij} = \langle e_i, e_j \rangle = \langle e_j, e_i \rangle = a_{ji}$, so $\mathbf{Q}$ is symmetric.

(h) From the previous problems (f) and (g) we have proven the sufficient condition. To prove the necessary condition let $\mathbf{Q}$ be symmetric positive definite. We can first prove that $\langle x, y \rangle = \langle y, x \rangle$. We show

$$\langle x, y \rangle = \mathbf{y}^T \mathbf{Q} \mathbf{x} = (\mathbf{y}^T \mathbf{Q} \mathbf{x})^T = \mathbf{x}^T \mathbf{Q}^T \mathbf{y} = \mathbf{x}^T \mathbf{Q} \mathbf{y} = \langle y, x \rangle,$$

since $\mathbf{Q}$ is symmetric. Now since $\mathbf{Q}$ is symmetric and postiive definite, there exists an invertible matrix $\mathbf{P}$ with $\mathbf{P}^{-1} = \mathbf{P}^T$, such that $\mathbf{Q} = \mathbf{P}\mathbf{D}\mathbf{P}^T$, where $\mathbf{D}$ is the diagonal matrix of eigenvalues λ_i of $\mathbf{Q}$ such that $\lambda_i > 0 \ \forall i$. Then

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{Q} \mathbf{x} = \mathbf{x}^T \mathbf{P}\mathbf{D}\mathbf{P}^T \mathbf{x} = (\mathbf{P}^T \mathbf{x})^T \mathbf{D} \mathbf{P}^T \mathbf{x} = \mathbf{y}\mathbf{D}\mathbf{y}^T$$

where $\mathbf{y} = \mathbf{P}^T \mathbf{x}$. If $\mathbf{y} = a_1 y_1 + \cdots + a_n y_n$ and you calculate this out, you should get:

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{y}\mathbf{D}\mathbf{y}^T = \lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 \geq 0.$$

Furthermore, if $\langle x, x \rangle = 0$, then $\lambda_1 a_1^2 + \cdots + \lambda_n a_n^2 = 0$ but then $a_1 = \cdots = a_n = 0$, but then $\mathbf{y} = 0$, and so $\mathbf{x} = 0$.
Finally, since $\langle x, y \rangle = \mathbf{y}^T \mathbf{Q} \mathbf{x}$ this is linear by nature. Hence, $\langle \cdot, \cdot \rangle$ satisfies all requirements of the inner product if $\mathbf{Q}$ is symmetric positive definite.

(i) Let $\mathbf{x} = [x_1, x_2]^T$, then

$$\mathbf{A}\mathbf{x} = \begin{bmatrix} 3x_1 + 3x_2 \\ -\frac{x_1}{2} + \frac{x_2}{2} \end{bmatrix}$$

and from $\|\mathbf{x}\|_2 = \|\mathbf{A}\mathbf{x}\|_2$ we have

$$\begin{aligned} x_1^2 + x_2^2 &= 9(x_1 + x_2)^2 + \frac{1}{4}(-x_1 + x_2)^2 \\ 0 &= 33x_1^2 + 33x_2^2 + 70x_1 x_2 \\ 0 &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 33 & 35 \\ 35 & 33 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

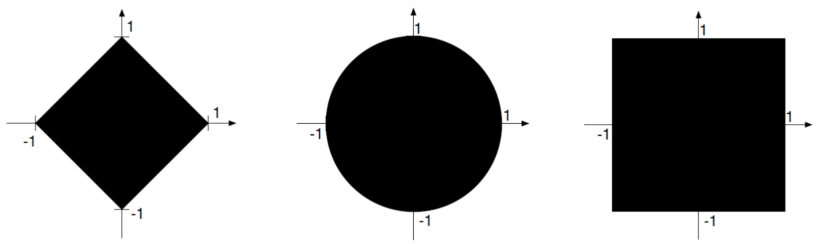
Hence,

$$\mathbf{Q} = \begin{bmatrix} 33 & 35 \\ 35 & 33 \end{bmatrix}$$

V Problem Five

Visualizing norm balls: 20 points

One way to visualize a norm in $\mathbb{R}^2$ is by its *unit ball*, the set of all vectors such that $\|\mathbf{x}\| \leq 1$. For example, we have seen that the unit balls for the ℓ_1, ℓ_2, and ℓ_∞ norms look like:



$B_1 = \{\mathbf{x} : \|\mathbf{x}\|_1 \leq 1\}$ $\quad$ $B_2 = \{\mathbf{x} : \|\mathbf{x}\|_2 \leq 1\}$ $\quad$ $B_\infty = \{\mathbf{x} : \|\mathbf{x}\|_\infty \leq 1\}$

Given an appropriate subset of the plane, $B \subset \mathbb{R}^2$, it might be possible to define a corresponding norm using

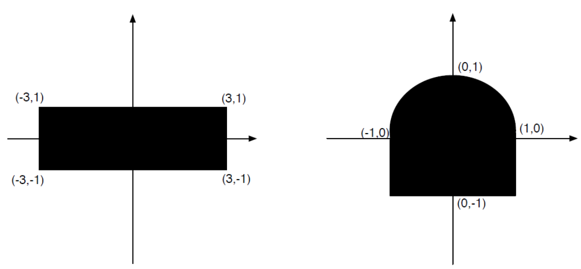
$$\|\mathbf{x}\|_B = \text{minimum value } r \geq 0 \text{ such that } \mathbf{x} \in rB, \tag{V.1}$$

where rB is just a scaling of the set B:

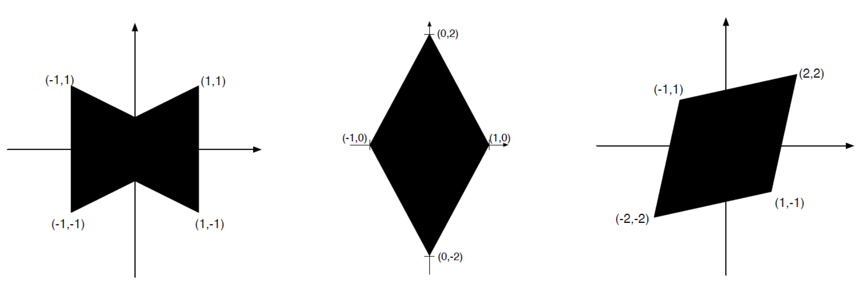
$$\mathbf{x} \in B \quad \Rightarrow \quad r \cdot \mathbf{x} \in rB.$$

(a) Let $\mathbf{x} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. For $p = 1, 2, \infty$, find $r = \|\mathbf{x}\|_p$, and sketch $\mathbf{x}$ and rB_p (use different axes for each of the three values of p).

(b) Consider the 5 shapes below.



B_a $\quad$ B_b



B_c $\quad$ B_d $\quad$ B_e

Explain why $\|\cdot\|_{B_b}$ and $\|\cdot\|_{B_c}$ are **not** valid norms. The most convincing way to do this is to find vectors for which one of the three properties of a valid norm are violated.

(c) Give a concrete method for computing $\|\mathbf{x}\|_{B_a}$, $\|\mathbf{x}\|_{B_d}$, and $\|\mathbf{x}\|_{B_e}$ for any given vector $\mathbf{x}$. (For example: for B_1, which corresponds to the ℓ_1 norm, we would write $\|\mathbf{x}\|_1 = |x_1| + |x_2|$.) Using you expressions, show that these are indeed valid norms.

Solution:
(a) First we compute each norm

$$r_1 = \|\mathbf{x}\|_1 = 4 + 2 = 6$$
$$r_2 = \|\mathbf{x}\|_2 = \sqrt{4^2 + 2^2} = 2\sqrt{5}$$
$$r_\infty = \|\mathbf{x}\|_\infty = 4$$

Hence, the plots are as follows.

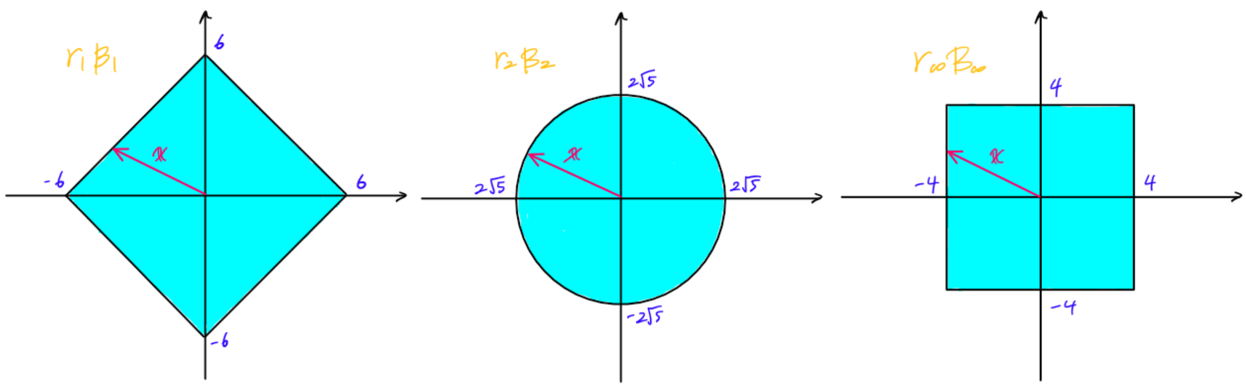


Figure 8: Sketch of $r_1 B_1$, $r_2 B_2$, $r_\infty B_\infty$.

(b) From the homogeneity property of the norm, $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$, we know that the unit ball should be symmetric about the origin. This rules out B_b. Another defining axiom of the norm is the triangular inequality which implies convexity. B_c is a concave shape and thus violates the convexity of a norm. So B_c is also invalid as a norm.
(c) The method for B_a would be using a scaling factor within the ∞-norm

$$\|\mathbf{x}\|_{B_a} = \max(3|x_1|, |x_2|).$$

This is positive, and it suffices homogeneity since

$$\|c\mathbf{x}\|_{B_a} = \max(3|cx_1|, |cx_2|) = c \max(3|x_1|, |x_2|).$$

Finally,

$$\begin{aligned}\|\mathbf{x} + \mathbf{y}\|_{B_a} &= \max(3|x_1 + y_1|, |x_2 + y_2|) \leq \max(3|x_1| + 3|y_1|, |x_2| + |y_2|) \\ &= \max(3|x_1|, |x_2|) + \max(3|y_1|, |y_2|) = \|\mathbf{x}\|_{B_a} + \|\mathbf{y}\|_{B_a}.\end{aligned}$$

The method for B_d is

$$\|\mathbf{x}\|_{B_d} = |x_1| + 2|x_2|.$$

This is positive, and it suffices homogeneity since

$$\|c\mathbf{x}\|_{B_d} = |cx_1| + 2|cx_2| = c|x_1| + 2|c||x_2|.$$

Finally,

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_{B_d} &= |x_1+y_1| + 2|x_2+y_2| \leq |x_1| + |y_1| + 2|x_2| + 2|y_2| \\ &= \|\mathbf{x}\|_{B_d} + \|\mathbf{y}\|_{B_d}.\end{aligned}$$

Finally the method for B_e is

VI Problem Six

Bonus: 5 points
What do you expect to learn from this class? Please be honest and as detailed as you would like. Note that while the theme of the class is theoretical and we will stick to it, but we may adjust our coverage slightly if there is enough interest to learn a particular topic. Are you struggling with any particular concept so far and do you have any suggestions for the course staff?

Solution:
In my research, I am most likely going to use scientific machine learning, and I am required to understand the fundamentals of machine learning in order to work on it as well as the skills to actually code a machine learning application using such as Pytorch, TensorFlow, etc. Aside from the technical skills, this class will definitely teach me the mathematics that I need to understand applications of machine learning. So far it has been a good refresher for linear algebra but I am hoping to get into things I have not yet encountered.
The only thing I am struggling so far is my time management between the class, hw, research, and other work that I have to do. I am quite concerned whether or not the homework are all going to be as lengthy as this one. I would prefer about 3 problems max if there are this many subproblems for each problem.