E BP

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Solution: Relative Motion of Two Bodies

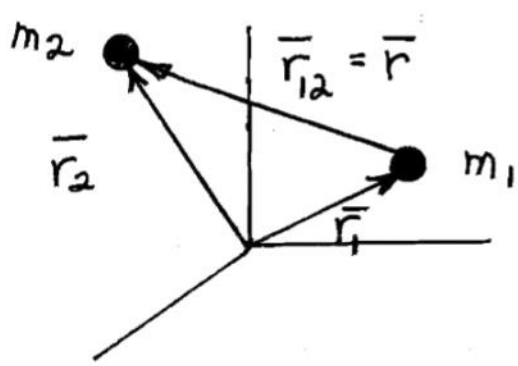
Solve $\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \bar{0}$ $\qquad \bar{r}_{12} = \bar{r}$

Method 1 : **Classical Derivation**

I. Observations from angular momentum

1. n-body problem – angular momentum of system

$$\sum_{i=1}^{n} m_i\left(\bar{r}_i \times \dot{\bar{r}}_i\right) = \bar{C}_3 \quad \text{constant vector}$$



Let $n = 2$

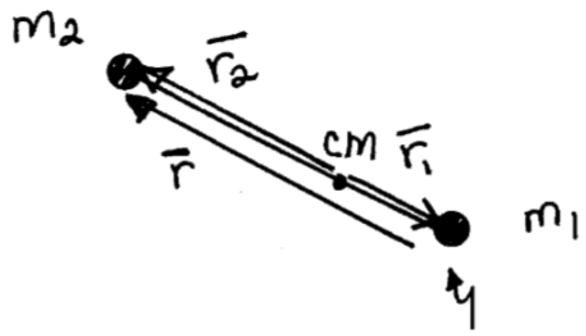
$\bar{C}_3 = m_1(\bar{r}_1 \times \dot{\bar{r}}_1) + m_2(\bar{r}_2 \times \dot{\bar{r}}_2)$

System linear momentum conserved

→ $\bar{v}_{cm}$ = const in mag. & dir.

→ any frame moving w/ CM is inertial

→ in new inertial frame cm appear "fixed",
cm as "inertially fixed base pt"



Use m_1 to locate cm

$$(m_1+m_2)\bar{r}_{cm} = m_2\bar{r}$$

$$\bar{r}_{cm} = \frac{m_2}{m_1+m_2}\bar{r}$$

$$\downarrow$$

$$\bar{r}_1 = -\frac{m_2}{m_1+m_2}\bar{r}$$

$$\bar{r}_2 = \frac{m_1}{m_1+m_2}\bar{r}$$

$$\bar{C}_3 = m_1(\bar{r}_1 \times \dot{\bar{r}}_1) + m_2(\bar{r}_2 \times \dot{\bar{r}}_2)$$

Sub back into equation for $\bar{C}_3$

$$\bar{C}_3 = m_1\left(\frac{-m_2}{m_1+m_2}\bar{r} \times \frac{-m_2}{m_1+m_2}\dot{\bar{r}}\right) + m_2\left(\frac{m_1}{m_1+m_2}\bar{r} \times \frac{m_1}{m_1+m_2}\dot{\bar{r}}\right)$$

$$\bar{C}_3 = \frac{m_1 m_2}{m_1+m_2}(\bar{r} \times \dot{\bar{r}})$$

$$\frac{m_1+m_2}{m_1 m_3}\bar{C}_3 = \bar{r} \times \dot{\bar{r}} = \bar{h}$$

specific angular momentum

const. { mag & dir

Note: $\dot{\bar{r}} = \frac{{}^{i}d\bar{r}}{dt}$ **relative velocity**

$\bar{h}$ always $\perp$ to $\bar{r}$ & $\dot{\bar{r}}$

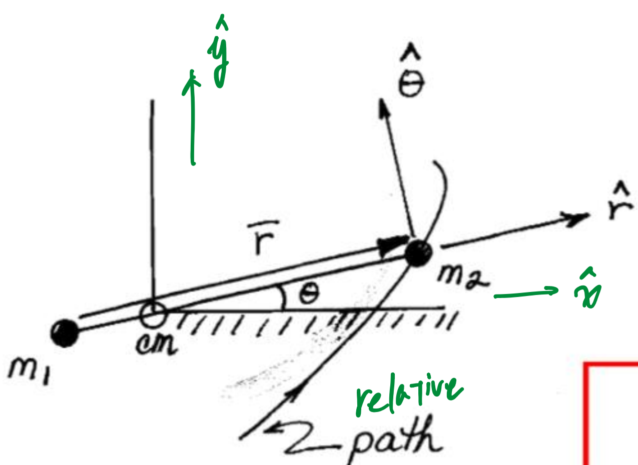
$$\underline{\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = 0}$$

E3

2. $\bar{h} = \bar{r} \times \dot{\bar{r}}$ = constant $\begin{cases} \bar{h} \perp \bar{r}, \dot{\bar{r}} \\ \bar{r}, \dot{\bar{r}} \text{ remain in a plane} \\ \text{motion planar} \end{cases}$ relative motion in 2BP is planar

Invariable plane – plane contains cm & whose normal coincides w/ $\bar{h}$

3. Represent $\bar{h}$ in scalar component / magnitude form



kinematics

$\bar{r} = r\,\hat{r}$

$\dot{\bar{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$

$|\bar{h}| = |\bar{r} \times \dot{\bar{r}}| = r^2\,\dot{\theta}$

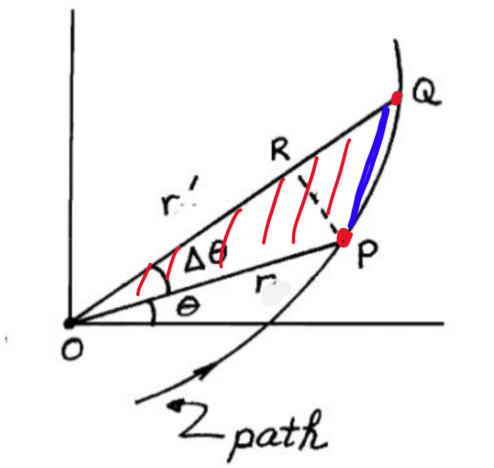
$$\boxed{h = r^2\dot{\theta} \quad \text{const}}$$

tradeoff between r & $\dot{\theta}$ related to areal velocity

Kepler's 3rd law

4. $\boldsymbol{h}$ related to areal velocity

[Kepler III. Line joining planet to Sun sweeps out equal areas in equal times.]



ΔA differential area swept out P→Q

Actually already known from $\boldsymbol{h}$

(Assume motion in a plane)

ΔA represents area of triangle ***OPQ*** swept over by radius vector in interval Δt

$$\text{Area triangle} = \frac{1}{2}\text{ (base) (height)}$$

$$\Delta A = \frac{1}{2}(r')(r\sin\Delta\theta) = \frac{r' r \sin\Delta\theta}{2}$$

$$\frac{\Delta A}{\Delta t} = \frac{r' r}{2}\,\frac{\sin\Delta\theta}{\Delta\theta}\,\frac{\Delta\theta}{\Delta t}$$

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{h}{2}$$

$$\dot{A} = \frac{h}{2} = \text{const.}$$

As $\Delta\theta$ diminishes, ratio of area of triangle to that of sector approaches unity as a limit

$\Rightarrow$ limit of $\dfrac{\sin\Delta\theta}{\Delta\theta}$ is unity

Passing to the limit $\Delta t \rightarrow 0$

II. Observations from energy

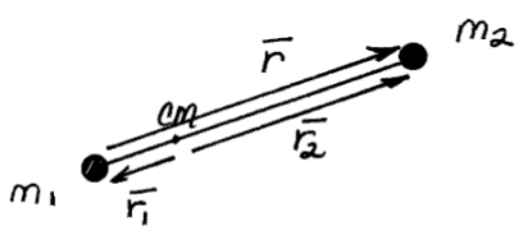
scalar const.

1. gravity field is conservative

$$T - U = C_4$$

KE − Grav pot = const.

2. write equation $T - U = C_4$ in a more convenient form



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$$\bar{r}_1 = \frac{-m_2}{m_1 + m_2}\bar{r} \qquad \dot{\bar{r}}_1 = \frac{-m_2}{m_1 + m_2}\dot{\bar{r}}$$

$$\bar{r}_2 = \frac{m_1}{m_1 + m_2}\bar{r} \qquad \dot{\bar{r}}_2 = \frac{m_1}{m_1 + m_2}\dot{\bar{r}}$$

kinetic energy

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\, \bar{v}_i^P \bullet \bar{v}_i^P$$

$$T = \frac{1}{2}m_1\left(\dot{\bar{r}}_1 \bullet \dot{\bar{r}}_1\right) + \frac{1}{2}m_2\left(\dot{\bar{r}}_2 \bullet \dot{\bar{r}}_2\right)$$

$$T = \frac{1}{2}m_1\left(\frac{-m_2}{m_1 + m_2}\dot{\bar{r}} \bullet \frac{-m_2}{m_1 + m_2}\dot{\bar{r}}\right) + \frac{1}{2}m_2\left(\frac{m_1}{m_1 + m_2}\dot{\bar{r}} \bullet \frac{m_1}{m_1 + m_2}\dot{\bar{r}}\right)$$

$$\boxed{T = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left(\dot{\bar{r}} \cdot \dot{\bar{r}}\right)}$$

$$U=\frac{1}{2}G\sum_{i=1}^{n}\sum_{\substack{j=1\\j\neq i}}^{n}\frac{m_i m_j}{r_{ji}}$$

$n=2$

$$U=\frac{1}{2}G\left(\frac{m_1m_2}{r}+\frac{m_2m_1}{r}\right)=\frac{Gm_1m_2}{r}$$

$$U=\frac{Gm_1m_2}{r}$$

$$T-U=\frac{1}{2}(\dot{\bar{r}}\bullet\dot{\bar{r}})\frac{m_1m_2}{m_1+m_2}-\frac{Gm_1m_2}{r}=C_4$$

Multiply by $\dfrac{m_1+m_2}{m_1m_2}$

$$\frac{1}{2}\underbrace{(\dot{\bar{r}}\bullet\dot{\bar{r}})}_{\bar{v}}-\frac{\overbrace{G(m_1+m_2)}^{\mu}}{r}=C_4\frac{(m_1+m_2)}{m_1m_2}=\xi$$

specific energy

Define $\bar{v}=\dot{\bar{r}}=\frac{{}^{I}d\bar{r}}{dt}$ ⟵ base point moves!!!

relative velocity

$\bar{v}=\dot{\bar{r}}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}$

$v^2=|\dot{\bar{r}}|^2=\dot{r}^2+r^2\dot{\theta}^2$

$$\frac{v^2}{2}-\frac{G(m_1+m_2)}{r}=C_4\frac{(m_1+m_2)}{m_1m_2}=\boldsymbol{\varepsilon}$$ "energy"

Let $\mu=G(m_1+m_2)$

$$\boxed{\xi=\frac{v^2}{2}-\underbrace{\frac{\mu}{r}}_{U'}}$$

what info can be extract from this to get a soln?

General U: $\boldsymbol{\varepsilon}=\dfrac{v^2}{2}-U'$

III. Using known constants ($h; \mathcal{E}$), vector 2nd-order DE

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \bar{0}$$

solve ! $\bar{r}, \dot{\bar{r}}$

has been replaced by two 1st-order scalar differential equations in the dependent variables r, θ

Note: only 2 dependent variables because motion takes place in a plane (polar components simplifies problem)

$$\begin{cases} h = r^2\dot{\theta} \\ \mathcal{E} = \frac{1}{2}v^2 - U' = \frac{1}{2}\underbrace{\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}_{\bar{v}\cdot\bar{v}} - \underbrace{U'}_{\frac{\mu}{r}} \end{cases}$$

$\bar{v} = \dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\bar{v}\cdot\bar{v} = \dot{r}^2 + r^2\dot{\theta}^2$

Solution ?

Expression for ***r*** ?

$$h = r^2 \frac{d\theta}{dt} \quad \Longrightarrow \quad dt = \frac{r^2}{h} d\theta$$ can remove time

$\dot{r}$ $\quad r^2\dot{\theta}^2$

$$\varepsilon = \frac{1}{2}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\} - U'$$

sub for dt

$$2\varepsilon = \left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{d\theta}\frac{h}{r^2}\right)^2 - 2U' \leftarrow -2\frac{\mu}{r}$$

$$= \left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{h^2}{r^2} - 2U'$$

$$\frac{2}{h^2}[\varepsilon + U'] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \longleftarrow$$ differential equation

integrate?

solve $r = r(\theta)$ because t removed

Introduce new variable

$$\varsigma = \frac{1}{r} = r^{-1} \qquad \frac{d\varsigma}{d\theta} = -r^{-2}\frac{dr}{d\theta}$$

$\Downarrow$

$$\frac{2}{h^2}[\varepsilon + U'] = \left(\frac{d\varsigma}{d\theta}\right)^2 + \varsigma^2$$

$\Downarrow$

$\varsigma = \frac{1}{r}$

$$\frac{d\varsigma}{d\theta} = \pm\sqrt{\frac{2}{h^2}(\varepsilon + U') - \varsigma^2}$$

$\frac{\mu}{r}$

3 const: h, μ, ε

$$d\theta = \frac{d\varsigma}{\pm\sqrt{\frac{2}{h^2}(\mathcal{E}+U') - \varsigma^2}}$$ → 3 parameter family of orbits

$\frac{\mu}{r}$

fcn of $\varsigma \rightarrow U' = \frac{\mu}{r} = \mu\varsigma$

$\underbrace{\textbf{Integrate}}_{\text{easily}}$

$$\theta = \cos^{-1}\frac{\varsigma - \frac{\mu}{h^2}}{\sqrt{\frac{\mu^2}{h^4}+\frac{2\mathcal{E}}{h^2}}} + \omega$$

integration const.

measured from a reference

direction "periapsis"

$$\frac{1}{r} = \frac{\mu}{h^2} + \sqrt{\frac{\mu^2}{h^4}+\frac{2\mathcal{E}}{h^2}}\cos(\theta-\omega)$$

$$\boxed{r = \frac{p}{1+e\cos(\theta-\omega)}}$$

Standard polar equation of a conic section referred to the focus as the origin

where $p = \frac{h^2}{\mu}$ semi latus rectum

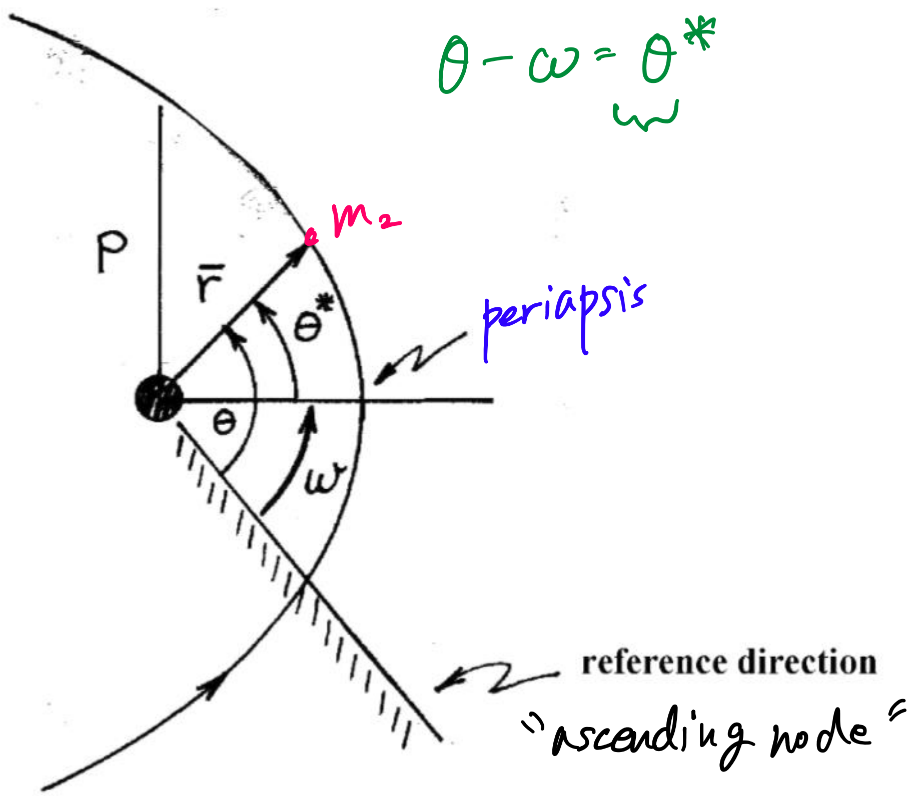
$$e = \sqrt{1+\frac{2\mathcal{E}h^2}{\mu^2}}$$ eccentricity

vectors late 1800s

ellipse, parabola, hyperbola

Conic Section

$\theta - \omega = \theta^*$



$$h = \sqrt{\mu p} \qquad \mathcal{E} = -\frac{\mu^2}{2h^2}\left(1-e^2\right)$$

define $\quad a = \dfrac{p}{\left(1-e^2\right)} \qquad$ semimajor axis

$$\Rightarrow \quad \mathcal{E} = -\frac{\mu}{2a} \quad \Rightarrow \quad -\frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

if path circular $\quad r = a \quad e = 0$

so $\quad v_c = \sqrt{\dfrac{\mu}{r}}$

Method 2 : **Direct Vector Derivation**

$$\ddot{\bar r} + \frac{\mu}{r^3}\bar r = \bar 0 \qquad \text{equation of relative motion for two-body system}$$

$\mu \rightarrow G(m_1+m_2)$

$$\bar h = \bar r \times \dot{\bar r} \qquad \text{constant}$$

mag
direction

Start with some observations concerning vectors

$$\frac{{}^I d\hat r}{dt} = \frac{{}^I d}{dt}\left(\frac{\bar r}{r}\right) = \frac{\dot{\bar r}}{r} - \frac{\bar r}{r^2}\dot r = \frac{r^2\dot{\bar r} - r\dot r\,\bar r}{r^3}$$

$$\left[\text{Note:} \quad \dot{\bar r} = \frac{{}^I d\,\bar r}{dt} \qquad \dot r = \frac{d\,r}{dt}\right]$$

Rewrite

$$\frac{{}^I d\hat r}{dt} = \frac{(\bar r \bullet \bar r)\dot{\bar r} - (\bar r \bullet \dot{\bar r})\bar r}{r^3}$$

$\bar r \cdot \bar r = r^2$

$\bar r \cdot \dot{\bar r} = (r\hat r)\cdot(\dot r\hat r + r\dot\theta\hat\theta)$
$= r\dot r$

Identity: $\bar A \times (\bar B \times \bar C) = (\bar A \bullet \bar C)\,\bar B - (\bar A \bullet \bar B)\,\bar C$

$\bar r \qquad \dot{\bar r} \quad \bar r \qquad (\bar r \bullet \bar r)\dot{\bar r} - (\bar r \bullet \dot{\bar r})\bar r$

$$\frac{{}^I d\,\hat r}{dt} = -\frac{\bar r \times (\bar r \times \dot{\bar r})}{r^3} = \frac{\overbrace{(\bar r \times \dot{\bar r})}^{\bar h} \times \bar r}{r^3}$$

$$= \qquad\qquad = \bar h \times \underbrace{\frac{\bar r}{r^3}}_{\text{sub}}$$

$\ddot{\bar r} + \frac{\mu}{r^3}\bar r = \bar 0 \qquad$ EOM

$\frac{\bar r}{r^3} = -\frac{\ddot{\bar r}}{\mu}$

$$\dot{\hat{r}} = \bar{h} \times \left(-\frac{\ddot{\bar{r}}}{\mu} \right) \quad \text{OR} \quad \dot{\hat{r}} = \frac{\ddot{\bar{r}} \times \bar{h}}{\mu}$$

constant

constant

Integrate once

$$\hat{r} = \frac{\dot{\bar{r}} \times \bar{h}}{\mu} + \text{integration constant} \quad = \frac{\dot{\bar{r}} \times \bar{h}}{\mu} - \bar{e}$$

eccentricity vector

$\bar{e}$ **in** $\left\{ \begin{array}{l} \text{direction of periapsis} \\ \text{oriented via } \omega \end{array} \right.$

r as a scalar (→ conic eqn)

Dot product with $\bar{r}$

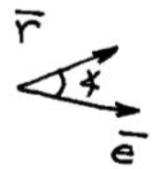
$$\bar{r} \bullet \hat{r} = \bar{r} \bullet \frac{\dot{\bar{r}} \times \bar{h}}{\mu} - \bar{r} \bullet \bar{e}$$

$$\left[\quad \text{Identity} \quad \bar{A} \bullet \bar{B} \times \bar{C} = \bar{C} \bullet \bar{A} \times \bar{B} \right]$$

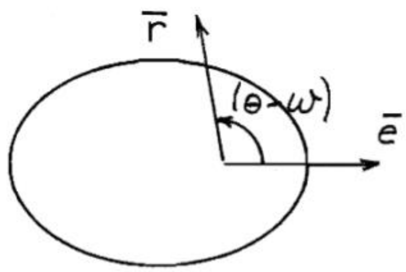
$$r = \frac{\bar{h}}{\mu} \bullet \underbrace{(\bar{r} \times \dot{\bar{r}})}_{\bar{h}} - \bar{r} \bullet \bar{e} \quad = \quad \frac{\bar{h} \cdot \bar{h}}{\mu} - \bar{r} \cdot \bar{e}$$

$$r = \frac{h^2}{\mu} - \bar{r} \bullet \bar{e}$$

$$r + \bar{r} \bullet \bar{e} = h^2/\mu$$



$\bar{r} \cdot \bar{e} = re\cos \measuredangle$



$$\boxed{r + re\cos(\theta - \omega) = h^2/\mu}$$

$$\boxed{r = \frac{h^2/\mu}{1 + e\cos(\theta-\omega)}}$$

conic eqn

Note: $\bar{h}$ → 3 constants
$\bar{h}$ normal to plane of motion
$\bar{e}$ always IN plane of motion
→ $\bar{h}$ and $\bar{e}$ are NOT independent so only represents 5 arbitrary constants

But together $\bar{h}$, $\bar{e}$ determine size, shape, and orientation of conic with respect to focus

6th constant related to <u>time</u>, i.e., orbital position relative to periapsis