



College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 5
Eigenvalues and Vectors of LTI Systems

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Exercise 1

Compute the **eigenvalues** and **eigenvectors** of the matrix

$$A = \begin{pmatrix} 2 & -3 & 0 \\ 2 & -3 & 0 \\ 3 & -5 & 1 \end{pmatrix}$$

From the definition of eigenvalues $\lambda v = Av$ where $v \neq 0$,

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda - 2 & 3 & 0 \\ -2 & \lambda + 3 & 0 \\ -3 & 5 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 2) \begin{vmatrix} \lambda + 3 & 0 \\ 5 & \lambda - 1 \end{vmatrix} - 3 \begin{vmatrix} -2 & 0 \\ -3 & \lambda - 1 \end{vmatrix} = 0$$

$$(\lambda - 2)(\lambda + 3)(\lambda - 1) + 6(\lambda - 1) = 0$$

$$\lambda^3 - \lambda = 0$$

$$\lambda(\lambda - 1)(\lambda + 1) = 0$$

Thus, the eigenvalues are

$$\lambda = -1, 0, 1$$

When $\lambda = -1$

$$\lambda I - A = \begin{pmatrix} -3 & 3 & 0 \\ -2 & 2 & 0 \\ -3 & 5 & -2 \end{pmatrix}$$

For this matrix conduct a Gaussian Elimination.

Cancel the leading column of row 2: $R_2 \rightarrow R_2 - \frac{2}{3}R_1$

$$\sim \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ -3 & 5 & -2 \end{pmatrix}$$

Cancel the leading column of row 3: $R_3 \rightarrow R_3 - R_1$

$$\sim \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -2 \end{pmatrix}$$

Swap the rows 2 and 3: $R_2 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -3 & 3 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Divide row 1 by -3 and row 2 by 2:

$$\sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1: $R_1 \rightarrow R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - x_3 & = & 0 \\ x_2 - x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

When $\lambda = 0$

$$\lambda I - A = \begin{pmatrix} -2 & 3 & 0 \\ -2 & 3 & 0 \\ -3 & 5 & -1 \end{pmatrix}$$

For this matrix conduct a Gaussian Elimination.

Swap rows 1 and 3: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -3 & 5 & -1 \\ -2 & 3 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \rightarrow R_2 - \frac{2}{3}R_1$

$$\sim \begin{pmatrix} -3 & 5 & -1 \\ 0 & -1/3 & 2/3 \\ -2 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \rightarrow R_3 - \frac{2}{3}R_1$

$$\sim \begin{pmatrix} -3 & 5 & -1 \\ 0 & -1/3 & 2/3 \\ 0 & -1/3 & 2/3 \end{pmatrix}$$

Divide row 1 by -3, multiply row 2 and 3 by -3:

$$\sim \begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{pmatrix}$$

Cancel out row 3 with row 2: $R_3 \rightarrow R_3 - R_2$

$$\sim \begin{pmatrix} 1 & -5/3 & 1/3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1: $R_1 \rightarrow R_1 + \frac{5}{3}R_2$

$$\sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - 3x_3 & = & 0 \\ x_2 - 2x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 3/2 \\ 1 \\ 1/2 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

When $\lambda = 1$

$$\lambda I - A = \begin{pmatrix} -1 & 3 & 0 \\ -2 & 4 & 0 \\ -3 & 5 & 0 \end{pmatrix}$$

For this matrix conduct a Gaussian Elimination.

Swap rows 1 and 3: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ -2 & 4 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \rightarrow R_2 - \frac{2}{3}R_1$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ 0 & 2/3 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \rightarrow R_3 - \frac{1}{3}R_1$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ 0 & 2/3 & 0 \\ 0 & 4/3 & 0 \end{pmatrix}$$

Cancel row 3: $R_3 \rightarrow R_3 - 2R_2$

$$\sim \begin{pmatrix} -3 & 5 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Multiply row 2 by 3/2 and then cancel the second column in row 1:

$$\sim \begin{pmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Divide the first row by -3

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 & = & 0 \\ x_2 & = & 0 \end{pmatrix}$$

Let $x_3 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

MATLAB Verification

```
% Exercise 1  
A = [2,-3,0;2,-3,0;3,-5,1];  
[v, d] = eig(A)
```

| | |
|------------------------|------------------|
| $v = 3 \times 3$ | $d = 3 \times 3$ |
| 0 -0.5774 -0.8018 | 1.0000 0 0 |
| 0 -0.5774 -0.5345 | 0 -1.0000 0 |
| 1.0000 -0.5774 -0.2673 | 0 0 0.0000 |

Exercise 2

Compute the **eigenvalues** and **eigenvectors** of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

From the definition of eigenvalues $\lambda v = Av$ where $v \neq 0$,

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & -1 & 0 \\ 0 & 0 & \lambda & -1 \\ -1 & 0 & 0 & \lambda \end{vmatrix} = 0$$

$$\lambda \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = 0$$

$$\lambda \left(\lambda \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} - \begin{vmatrix} 0 & -1 \\ 0 & \lambda \end{vmatrix} \right) - \left(- \begin{vmatrix} 0 & -1 \\ -1 & \lambda \end{vmatrix} \right) = 0$$

$$\lambda^4 - 1 = 0$$

Thus, the eigenvalues are

$$\lambda = -1, 1, -i, i$$

When $\lambda = -1$,

$$\lambda I - A = \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & 0 & -1 \end{pmatrix}$$

Perform Gaussian Elimination on this matrix.

Cancel the leading column of row 4: $R_4 \rightarrow R_4 - R_1$

$$\sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Cancel the second column of row 4: $R_4 \rightarrow R_4 + R_2$

$$\sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

Cancel row 4 with row 3:

$$\sim \begin{pmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1 with row 2:

$$\sim \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 or row 2 with row 3:

$$\sim \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 or row 1 with row 3:

$$\sim \begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply the matrix with -1

$$\sim \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 + x_4 & = & 0 \\ x_2 - x_4 & = & 0 \\ x_3 + x_4 & = & 0 \end{pmatrix}$$

Let $x_4 = s$ be a free variable. Then,

$$x = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

When $\lambda = 1$,

$$\lambda I - A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

Cancel the leading column of row 4 with row 1: $R_4 \rightarrow R_4 + R_1$

$$\sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

Cancel the leading column of row 4 and column 2 of row 1 with row 2: $R_4 + R_2$ & $R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Cancel row 4 with row 3:

$$\sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 of row 1 and 2 with row 3

$$\sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - x_4 = 0 \\ x_2 - x_4 = 0 \\ x_3 - x_4 = 0 \end{pmatrix}$$

Let $x_4 = s$ be a free variable. Then,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

When $\lambda = i$,

$$\lambda I - A = \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ -1 & 0 & 0 & i \end{pmatrix}$$

Cancel the leading column of row 4: $R_4 \rightarrow R_4 - iR_1$

$$\sim \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & i & 0 & i \end{pmatrix}$$

Cancel the leading column of row 4 with row 2: $R_4 \rightarrow R_4 - R_2$

$$\sim \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 1 & i \end{pmatrix}$$

Cancel row 4 with row 3: $R_4 \rightarrow R_4 + iR_3$

$$\sim \begin{pmatrix} i & -1 & 0 & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 2 or row 1 with row 2: $R_1 \rightarrow R_1 - iR_2$

$$\sim \begin{pmatrix} i & 0 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 of row 2 with row 3: $R_2 \rightarrow R_2 - iR_3$

$$\sim \begin{pmatrix} i & 0 & i & 0 \\ 0 & i & 0 & i \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel column 3 of row 1 with row 3: $R_1 \rightarrow R_1 - R_3$

$$\sim \begin{pmatrix} i & 0 & 0 & 1 \\ 0 & i & 0 & i \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiply row 1 by $-i$, divide row 2 by i , and multiply row 3 by $-i$

$$\sim \begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & i \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

From the matrix, expression for $\lambda I - A = 0$ will become

$$\begin{pmatrix} x_1 - ix_4 & = & 0 \\ x_2 + x_4 & = & 0 \\ x_3 + ix_4 & = & 0 \end{pmatrix}$$

Let $x_4 = s$ be a free variable. Then,

$$x = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_3 = \begin{pmatrix} i \\ -1 \\ -i \\ 1 \end{pmatrix}$$

When $\lambda = -i$,

$$v_4 = \overline{v_3} = \begin{pmatrix} -i \\ -1 \\ i \\ 1 \end{pmatrix}$$

MATLAB Verification

% Exercise 2

```
A = [0,1,0,0;0,0,1,0;0,0,0,1;1,0,0,0]
```

```
[v, d] = eig(A)
```

```
v = 4x4 complex
```

```
-0.5000 + 0.0000i    0.0000 - 0.5000i    0.0000 + 0.5000i   -0.5000 + 0.0000i
 0.5000 + 0.0000i    0.5000 + 0.0000i    0.5000 - 0.0000i   -0.5000 + 0.0000i
-0.5000 + 0.0000i   -0.0000 + 0.5000i   -0.0000 - 0.5000i   -0.5000 + 0.0000i
 0.5000 + 0.0000i   -0.5000 + 0.0000i   -0.5000 + 0.0000i   -0.5000 + 0.0000i
```

```
d = 4x4 complex
```

```
-1.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i
 0.0000 + 0.0000i    0.0000 + 1.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i
 0.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 - 1.0000i    0.0000 + 0.0000i
 0.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i    1.0000 + 0.0000i
```

Exercise 3

Determine whether or not the following matrix is nondefective.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

From the definition of eigenvalues $\lambda v = Av$ where $v \neq 0$,

$$\det(\lambda I - A) = 0$$

$$\begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 1 & -1 & \lambda - 1 \end{vmatrix} = 0$$

$$\lambda \begin{vmatrix} \lambda & -1 \\ -1 & \lambda - 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 1 & \lambda - 1 \end{vmatrix} = 0$$

$$\lambda((\lambda)(\lambda - 1) - 1) + 1 = 0$$

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

$$(\lambda + 1)^2(\lambda - 1) = 0$$

Thus, the eigenvalues are

$$\lambda = -1, -1, 1$$

We must be careful because -1 is a repeated eigenvalue.

When $\lambda = 1$,

$$\lambda I - A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix}$$

The reduced echelon form of this matrix is

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Considering $\lambda I - A = 0$, express this with $x = [x_1, x_2, x_3]$,

$$\begin{pmatrix} x_1 - x_3 & = & 0 \\ x_2 - x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$, as a free variable,

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_{1,2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

When $\lambda = -1$,

$$\lambda I - A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

The reduced echelon form of this matrix is

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Considering $\lambda I - A = 0$, express this with $x = [x_1, x_2, x_3]$,

$$\begin{pmatrix} x_1 - x_3 & = & 0 \\ x_2 + x_3 & = & 0 \end{pmatrix}$$

Let $x_2 = s$, as a free variable,

$$x = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} s$$

Thus, the corresponding eigenvector is

$$v_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$$

The algebraic multiplicity, $a.m.$ and geometric multiplicity, $g.m.$ are

$$a.m. = 3 > g.m. = 2$$

Thus, matrix A is **defective**.

MATLAB Verification

% Exercise 3

```
A = [0,1,0; 0,0,1; -1,1,1]
```

```
[v, d] = eig(A)
```

```
ech1 = rref(real(d(1,1))*eye(3) - A);
```

```
ech2 = rref(real(d(3,3))*eye(3) - A);
```

```
v = 3x3 complex
```

```
0.5774 - 0.0000i    0.5774 + 0.0000i   -0.5774 + 0.0000i
0.5774 - 0.0000i    0.5774 + 0.0000i    0.5774 + 0.0000i
0.5774 + 0.0000i    0.5774 + 0.0000i   -0.5774 + 0.0000i
```

```
d = 3x3 complex
```

```
1.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i
0.0000 + 0.0000i    1.0000 - 0.0000i    0.0000 + 0.0000i
0.0000 + 0.0000i    0.0000 + 0.0000i   -1.0000 + 0.0000i
```

```
ech1 = 3x3
```

```
1.0000         0   -1.0000
         0   1.0000   -1.0000
         0         0         0
```

```
ech2 = 3x3
```

```
1         0        -1
         0         1         1
         0         0         0
```

Exercise 4

What is the companion matrix whose eigenvalues are $-1, -2$, and -3 ?

From the eigenvalues, we can find the following polynomial equation

$$p(\lambda) = (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

$$p(\lambda) = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

Thus, the companion matrix becomes

$$C(p) = \begin{pmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{pmatrix}$$

MATLAB Verification

```
% Exercise 4
u = [1, 6, 11, 6];
C = compan(u)
```

```
C = 3x3
    -6    -11    -6
     1     0     0
     0     1     0
```


Exercise 5

What is the real 2 x 2 companion matrix with eigenvalues $1 + j, 1 - j$?

From the eigenvalues, we can find the following polynomial equation

$$\begin{aligned} p(\lambda) &= (\lambda - 1 + j)(\lambda - 1 - j) \\ p(\lambda) &= \lambda^2 - (1 + j + 1 - j)\lambda + (1 + j)(1 - j) \\ p(\lambda) &= \lambda^2 - 2\lambda + 2 \end{aligned}$$

Thus, the companion matrix becomes

$$C(p) = \begin{pmatrix} 0 & -2 \\ 1 & 2 \end{pmatrix}$$

MATLAB Verification

```
% Exercise 5
u = [1, -2, 2];
C = compan(u)
```

```
C = 2x2
     2    -2
     1     0
```

Exercise 6

Suppose A is a real square matrix and the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}$$

are eigenvectors of A corresponding to eigenvalues -1 and 2 and j , respectively. What is the response $x(t)$ of the system $\dot{x} = Ax$ to the following initial conditions.

$$(a) \ x(0) = \begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} \quad (b) \ x(0) = \begin{pmatrix} -1 \\ -2 \\ -4 \\ -8 \end{pmatrix} \quad (c) \ x(0) = \begin{pmatrix} 0 \\ 3 \\ 3 \\ 9 \end{pmatrix} \quad (d) \ x(0) = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

A is a nondefective matrix.

$$x(t) = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} e^{2t} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} e^{jt} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} e^{-jt}$$

$$x(t) = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} + c_3 e^{jt} + c_4 e^{-jt} \\ -c_1 e^{-t} + 2c_2 e^{2t} + jc_3 e^{jt} - jc_4 e^{-jt} \\ c_1 e^{-t} + 4c_2 e^{2t} - c_3 e^{jt} - c_4 e^{-jt} \\ -c_1 e^{-t} + 8c_2 e^{2t} - jc_3 e^{jt} + jc_4 e^{-jt} \end{pmatrix}.$$

Thus, for each (a)~(d) we only have to find the coefficients $c_1 \sim c_4$ that satisfy

$$x_0 = x(0) = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$$

(a).

We must solve the relation

$$\begin{pmatrix} 2 \\ -2 \\ 2 \\ -2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 2 \\ -1 & 2 & j & -j & -2 \\ 1 & 4 & -1 & -1 & 2 \\ -1 & 8 & -j & j & -2 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$x(t) = \begin{pmatrix} 2e^{-t} \\ -2e^{-t} \\ 2e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

(b).

We must solve the relation

$$\begin{pmatrix} -1 \\ -2 \\ -4 \\ -8 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 \\ -1 & 2 & j & -j & -2 \\ 1 & 4 & -1 & -1 & -4 \\ -1 & 8 & -j & j & -8 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$x(t) = \begin{pmatrix} -e^{2t} \\ -2e^{2t} \\ -4e^{2t} \\ -8e^{2t} \end{pmatrix}.$$

(c).

We must solve the relation

$$\begin{pmatrix} 0 \\ 3 \\ 3 \\ 9 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ -1 & 2 & j & -j & 3 \\ 1 & 4 & -1 & -1 & 3 \\ -1 & 8 & -j & j & 9 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence,

$$x(t) = \begin{pmatrix} -e^{-t} + e^{2t} \\ e^{-t} + 2e^{2t} \\ -e^{-t} + 4e^{2t} \\ e^{-t} + 8e^{2t} \end{pmatrix}.$$

(d).

We must solve the relation

$$\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 4 \\ 8 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_4 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}.$$

Therefore, we solve the augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 2 & j & -j & 1 \\ 1 & 4 & -1 & -1 & -1 \\ -1 & 8 & -j & j & -1 \end{pmatrix}.$$

The row reduced echelon form turns out to be

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0.5 - j0.5 \\ 0 & 0 & 0 & 1 & 0.5 + j0.5 \end{pmatrix}.$$

The coefficients become

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0.5 - j0.5 \\ 0.5 + j0.5 \end{pmatrix}.$$

Hence,

$$x(t) = \begin{pmatrix} (0.5 - j0.5)e^{jt} + (0.5 + j0.5)e^{-jt} \\ j(0.5 - j0.5)e^{jt} - j(0.5 + j0.5)e^{-jt} \\ -(0.5 - j0.5)e^{jt} - (0.5 + j0.5)e^{-jt} \\ -j(0.5 - j0.5)e^{jt} + j(0.5 + j0.5)e^{-jt} \end{pmatrix}$$

$$x(t) = \begin{pmatrix} (0.5 - j0.5)e^{jt} + (0.5 + j0.5)e^{-jt} \\ (j0.5 + 0.5)e^{jt} + (-j0.5 + 0.5)e^{-jt} \\ (-0.5 + j0.5)e^{jt} + (-0.5 - j0.5)e^{-jt} \\ (-j0.5 - 0.5)e^{jt} + (j0.5 - 0.5)e^{-jt} \end{pmatrix}$$

$$x(t) = \begin{pmatrix} (0.5 - j0.5)(\cos(t) + j\sin(t)) + (0.5 + j0.5)(\cos(t) - j\sin(t)) \\ (j0.5 + 0.5)(\cos(t) + j\sin(t)) + (-j0.5 + 0.5)(\cos(t) - j\sin(t)) \\ (-0.5 + j0.5)(\cos(t) + j\sin(t)) + (-0.5 - j0.5)(\cos(t) - j\sin(t)) \\ (-j0.5 - 0.5)(\cos(t) + j\sin(t)) + (j0.5 - 0.5)(\cos(t) - j\sin(t)) \end{pmatrix}.$$

Rule out all the imaginary terms

$$x(t) = \begin{pmatrix} 0.5\cos(t) + 0.5\sin(t) + 0.5\cos(t) + 0.5\sin(t) \\ 0.5\cos(t) - 0.5\sin(t) + 0.5\cos(t) - 0.5\sin(t) \\ -0.5\cos(t) - 0.5\sin(t) - 0.5\cos(t) - 0.5\sin(t) \\ -0.5\cos(t) + 0.5\sin(t) - 0.5\cos(t) + 0.5\sin(t) \end{pmatrix}$$

$$x(t) = \begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) - \sin(t) \\ -\cos(t) - \sin(t) \\ -\cos(t) + \sin(t) \end{pmatrix}.$$

MATLAB Verification

% Exercise 6

```
A1 = [1,1,1,1,2; -1,2,1j,-1j,-2; 1,4,-1,-1,2; -1,8,-1j,1j,-2];
E1 = rref(A1)
```

```
A2 = [1,1,1,1,-1; -1,2,1j,-1j,-2; 1,4,-1,-1,-4; -1,8,-1j,1j,-8];
E2 = rref(A2)
```

```
A3 = [1,1,1,1,0; -1,2,1j,-1j,3; 1,4,-1,-1,3; -1,8,-1j,1j,9];
E3 = rref(A3)
```

```
A4 = [1,1,1,1,1; -1,2,1j,-1j,1; 1,4,-1,-1,-1; -1,8,-1j,1j,-1];
E4 = rref(A4)
```

| | |
|---|---|
| <pre>E1 = 4x5 1 0 0 0 2 0 1 0 0 0 0 0 1 0 0 0 0 0 1 0</pre> | <pre>E2 = 4x5 1 0 0 0 0 0 1 0 0 -1 0 0 1 0 0 0 0 0 1 0</pre> |
| <pre>E3 = 4x5 1 0 0 0 -1 0 1 0 0 1 0 0 1 0 0 0 0 0 1 0</pre> | |
| <pre>E4 = 4x5 complex 1.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 1.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 1.0000 + 0.0000i 0.0000 + 0.0000i 0.5000 - 0.5000i 0.0000 + 0.0000i 0.0000 + 0.0000i 0.0000 + 0.0000i 1.0000 + 0.0000i 0.5000 + 0.5000i</pre> | |

Exercise 7

Consider a discrete-time LTI system described by $x(k + 1) = Ax(k)$.

(a) Suppose that the vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

are eigenvectors of A corresponding to eigenvalues -2 and 3 , respectively. What is the response $x(k)$ of the system to the initial condition

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} ?$$

(b) Suppose A is a real matrix and the vector

$$\begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix}$$

is an eigenvector of A corresponding to the eigenvalue $2 + 3j$. What is the response $x(k)$ of the system to the initial condition

$$x(0) = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} ?$$

(a).

Assuming that the matrix A is nondefective, we can express the given initial conditions with the linear function with coefficients $c_1 \sim c_4$. This is because we can say that matrix A is 4×4 and has 4 linearly independent eigenvectors. We do not know the other two eigenvalues and corresponding eigenvectors but we can still express the following.

$$x(k) = c_1(-2)^k \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 3^k \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \lambda_3^k v_3 + c_4 \lambda_4^k v_4.$$

Then,

$$x(0) = c_1(-2)^0 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 3^0 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 \lambda_3^0 v_3 + c_4 \lambda_4^0 v_4$$

$$x(0) = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 v_3 + c_4 v_4$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_3 v_3 + c_4 v_4$$

A combination of $c_1 \sim c_4$ satisfying this relation is

$$\begin{aligned} c_1 &= 0.5 \\ c_2 &= 0.5 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

Thus, the response becomes,

$$x(k) = \frac{(-2)^k}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + \frac{3^k}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$x(k) = \frac{1}{2} \begin{pmatrix} (-2)^k + 3^k \\ -(-2)^k + 3^k \\ (-2)^k + 3^k \\ -(-2)^k + 3^k \end{pmatrix}$$

(b).

Similar to part (a), assume a nondefective real matrix A to have 4 linearly independent eigenvectors. We have 2 known eigenvalues and eigenvectors, so we can express the following,

$$x(k) = c_1(2 + 3j)^k \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2(2 - 3j)^k \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3\lambda_3^k v_3 + c_4\lambda_4^k v_4.$$

Then,

$$x(0) = c_1(2 + 3j)^0 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2(2 - 3j)^0 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3\lambda_3^0 v_3 + c_4\lambda_4^0 v_4$$

$$x(0) = c_1 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3 v_3 + c_4 v_4$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix} + c_3 v_3 + c_4 v_4$$

A combination of $c_1 \sim c_4$ satisfying this relation is

$$\begin{aligned} c_1 &= 0.5 \\ c_2 &= 0.5 \\ c_3 &= 0 \\ c_4 &= 0 \end{aligned}$$

Thus, the response becomes,

$$x(k) = \frac{(2 + 3j)^k}{2} \begin{pmatrix} 1 \\ j \\ -1 \\ -j \end{pmatrix} + \frac{(2 - 3j)^k}{2} \begin{pmatrix} 1 \\ -j \\ -1 \\ j \end{pmatrix}$$

$$x(k) = \begin{pmatrix} (2 + 3j)^k + (2 - 3j)^k \\ j(2 + 3j)^k - j(2 - 3j)^k \\ -(2 + 3j)^k - (2 - 3j)^k \\ -j(2 + 3j)^k + j(2 - 3j)^k \end{pmatrix}$$

Exercise 8

(You may use MATLAB) Recall the 2-pendulum cart system. Consider the equilibrium configurations defined by

$$E1: (y^e, \theta_1^e, \theta_2^e) = (0, 0, 0)$$

$$E2: (y^e, \theta_1^e, \theta_2^e) = (0, \pi, \pi)$$

Consider state space representations of the linearizations corresponding to the following combinations of parameters and equilibrium conditions:

| | | |
|----|----|----|
| L1 | P1 | E1 |
| L2 | P1 | E2 |
| L3 | P2 | E1 |
| L4 | P2 | E2 |
| L5 | P3 | E1 |
| L6 | P3 | E2 |
| L7 | P4 | E1 |
| L8 | P4 | E2 |

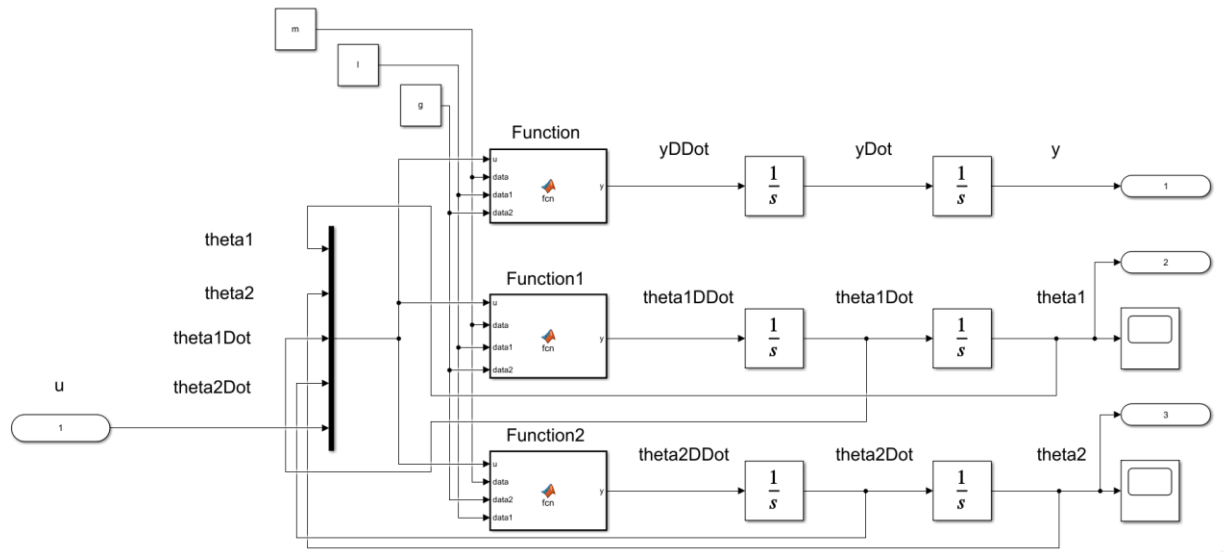
- (a) Determine the eigenvalues of all the linearized systems L1-L8.
- (b) Compare the behavior of the nonlinear system with that of the linearized system for cases L7 and L8. Illustrate your results with time histories of y , θ_1 , and θ_2 .

Defined parameter sets:

| | m_0 | m_1 | m_2 | l_1 | l_2 | g | u |
|----|-------|-------|-------|-------|-------|-----|-----|
| P1 | 2 | 1 | 1 | 1 | 1 | 1 | 0 |
| P2 | 2 | 1 | 1 | 1 | 0.99 | 1 | 0 |
| P3 | 2 | 1 | 0.5 | 1 | 1 | 1 | 0 |
| P4 | 2 | 1 | 1 | 1 | 0.5 | 1 | 0 |

(a).

The Simulink model used for this is shown below,

**Embedded MATLAB Block – Function (code)**

```
function y = fcn(u, data, data1, data2)
%{
    EMBEDDED MATLAB BLOCK FUNCTION
%}

m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
g = data2;

num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4) ...
      - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2)) ...
      + u(5);
den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
y = num / den;
end
```

Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
%{
    EMBEDDED MATLAB BLOCK FUNCTION1
%}

m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
g = data2;

num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4)) ...
      + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2))) ...
      - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

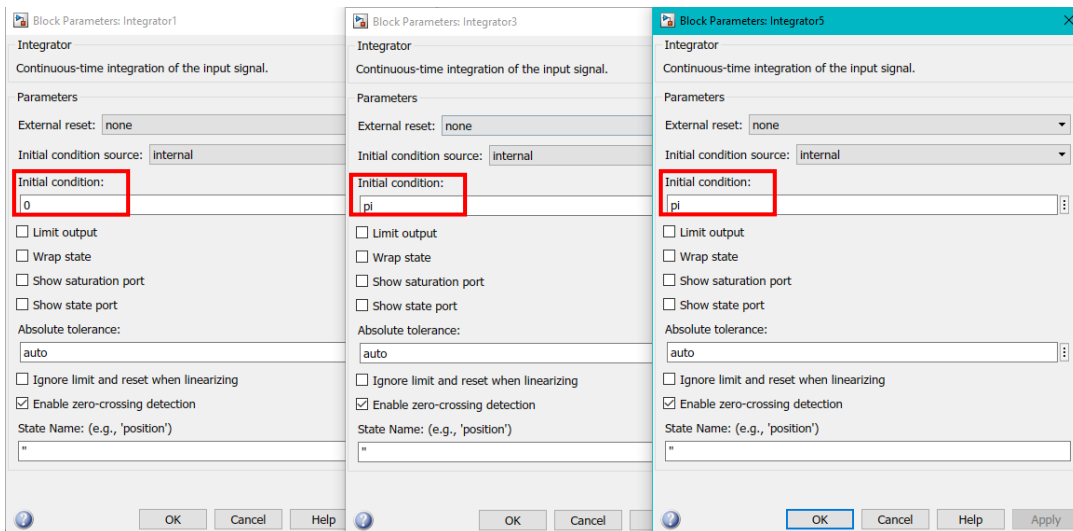
```

function y = fcn(u, data, data2, data1)
%{
    EMBEDDED MATLAB BLOCK FUNCTION2
%}
m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
        + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
        - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end

```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y , θ_1 , and θ_2 correspondingly to y^e , θ_1^e , θ_2^e ; like in the following windows,



The code to run the linearization and eigenvalue computation is the following

```
% (a)
global m l g ye t1e t2e
param_combo = ["L1","L2","L3","L4","L5","L6","L7","L8"];
for i = 1:numel(param_combo)
    define_params(param_combo(i));
    [A, B, C, D] = linmod('db_pend_cart_lin');
    lin_sys(i).Amat = A;
    lin_sys(i).Bmat = B;
    lin_sys(i).Cmat = C;
    lin_sys(i).Dmat = D;
    sys_ss = ss(A, B, C, D); % get the state space system
    sys_tf = tf(sys_ss); % get the transfer function
    lin_sys(i).eigVal = pole(sys_tf); % get the eigenvalues
end
```

```
function define_params(L)
% Function to define parameters
global m l g ye t1e t2e
if L == "L1"
    m = [2,1,1]; l = [1,1]; g = 1; % P1
    ye = 0; t1e = 0; t2e = 0; % E1
elseif L == "L2"
    m = [2,1,1]; l = [1,1]; g = 1; % P1
    ye = 0; t1e = pi; t2e = pi; % E2
elseif L == "L3"
    m = [2,1,1]; l = [1,0.99]; g = 1; % P2
    ye = 0; t1e = 0; t2e = 0; % E1
elseif L == "L4"
    m = [2,1,1]; l = [1,0.99]; g = 1; % P2
    ye = 0; t1e = pi; t2e = pi; % E2
elseif L == "L5"
    m = [2,1,0.5]; l = [1,1]; g = 1; % P3
    ye = 0; t1e = 0; t2e = 0; % E1
elseif L == "L6"
    m = [2,1,0.5]; l = [1,1]; g = 1; % P3
    ye = 0; t1e = pi; t2e = pi; % E2
elseif L == "L7"
    m = [2,1,1]; l = [1,0.5]; g = 1; % P4
    ye = 0; t1e = 0; t2e = 0; % E1
elseif L == "L8"
    m = [2,1,1]; l = [1,0.5]; g = 1; % P4
    ye = 0; t1e = pi; t2e = pi; % E2
else
    print('error: did not match any')
end
end
```

List the state space matrices A, B, C, and D for the linearized system and eigenvalues.

L1:

| | | | |
|----------------|--|----------------|---|
| A = 6×6 | <pre> 0 0 0 1.0000 0 0 0 0 0 0 1.0000 0 0 0 0 0 0 1.0000 0 -0.5000 -0.5000 0 0 0 0 -1.5000 -0.5000 0 0 0 0 -0.5000 -1.5000 0 0 0 </pre> | B = 6×1 | <pre> 0 0 0 0.5000 0.5000 0.5000 </pre> |
| C = 1×6 | <pre> 1 0 0 0 0 0 </pre> | D = 0 | |

```

eigVal = 6×1 complex
0.0000 + 0.0000i
0.0000 + 0.0000i
0.0000 + 1.4142i
0.0000 - 1.4142i
-0.0000 + 1.0000i
-0.0000 - 1.0000i

```

L2:

| | | | |
|----------------|--|----------------|---|
| A = 6×6 | <pre> 0 0 0 1.0000 0 0 0 0 0 0 1.0000 0 0 0 0 0 0 1.0000 0 -0.5000 -0.5000 0 0 0 0 1.5000 0.5000 0 0 0 0 0.5000 1.5000 0 0 0 </pre> | B = 6×1 | <pre> 0 0 0 0.5000 -0.5000 -0.5000 </pre> |
| C = 1×6 | <pre> 1 0 0 0 0 0 </pre> | D = 0 | |

```

eigVal = 6×1
0
0
-1.4142
-1.0000
1.4142
1.0000

```

L3:

| | | | | | | | |
|----------------|---|---------|---------|--------|--------|--------|----------------|
| A = 6×6 | 0 | 0 | 0 | 1.0000 | 0 | 0 | B = 6×1 |
| | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| | 0 | -0.5000 | -0.5000 | 0 | 0 | 0 | 0.5000 |
| | 0 | -1.5000 | -0.5000 | 0 | 0 | 0 | 0.5000 |
| | 0 | -0.5051 | -1.5152 | 0 | 0 | 0 | 0.5051 |
| C = 1×6 | 1 | 0 | 0 | 0 | 0 | 0 | D = 0 |

```
eigVal = 6×1 complex
0.0000 + 0.0000i
0.0000 + 0.0000i
-0.0000 + 1.4178i
-0.0000 - 1.4178i
0.0000 + 1.0025i
0.0000 - 1.0025i
```

L2:

| | | | | | | | |
|----------------|---|---------|---------|--------|--------|--------|----------------|
| A = 6×6 | 0 | 0 | 0 | 1.0000 | 0 | 0 | B = 6×1 |
| | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| | 0 | -0.5000 | -0.5000 | 0 | 0 | 0 | 0.5000 |
| | 0 | 1.5000 | 0.5000 | 0 | 0 | 0 | -0.5000 |
| | 0 | 0.5051 | 1.5152 | 0 | 0 | 0 | -0.5051 |
| C = 1×6 | 1 | 0 | 0 | 0 | 0 | 0 | D = 0 |

```
eigVal = 6×1
0
0
1.4178
1.0025
-1.4178
-1.0025
```

L5:

| | | | | | | | |
|----------------|---|---------|---------|--------|--------|--------|----------------|
| A = 6×6 | 0 | 0 | 0 | 1.0000 | 0 | 0 | B = 6×1 |
| | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| | 0 | -0.5000 | -0.2500 | 0 | 0 | 0 | 0.5000 |
| | 0 | -1.5000 | -0.2500 | 0 | 0 | 0 | 0.5000 |
| | 0 | -0.5000 | -1.2500 | 0 | 0 | 0 | 0.5000 |
| C = 1×6 | 1 | 0 | 0 | 0 | 0 | 0 | D = 0 |

```
eigVal = 6×1 complex
0.0000 + 0.0000i
0.0000 + 0.0000i
-0.0000 + 1.3229i
-0.0000 - 1.3229i
-0.0000 + 1.0000i
-0.0000 - 1.0000i
```

L6:

| | | | | | | | |
|----------------|---|---------|---------|--------|--------|--------|----------------|
| A = 6×6 | 0 | 0 | 0 | 1.0000 | 0 | 0 | B = 6×1 |
| | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| | 0 | -0.5000 | -0.2500 | 0 | 0 | 0 | 0.5000 |
| | 0 | 1.5000 | 0.2500 | 0 | 0 | 0 | -0.5000 |
| | 0 | 0.5000 | 1.2500 | 0 | 0 | 0 | -0.5000 |
| C = 1×6 | 1 | 0 | 0 | 0 | 0 | 0 | D = 0 |

```
eigVal = 6×1
0
0
1.3229
1.0000
-1.3229
-1.0000
```


L7:

| | | | | | | | |
|----------------|---|---------|---------|--------|--------|--------|----------------|
| A = 6×6 | 0 | 0 | 0 | 1.0000 | 0 | 0 | B = 6×1 |
| | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| | 0 | -0.5000 | -0.5000 | 0 | 0 | 0 | 0.5000 |
| | 0 | -1.5000 | -0.5000 | 0 | 0 | 0 | 0.5000 |
| | 0 | -1.0000 | -3.0000 | 0 | 0 | 0 | 1.0000 |
| C = 1×6 | 1 | 0 | 0 | 0 | 0 | 0 | D = 0 |

```
eigVal = 6×1 complex
0.0000 + 0.0000i
0.0000 + 0.0000i
0.0000 + 1.8113i
0.0000 - 1.8113i
-0.0000 + 1.1042i
-0.0000 - 1.1042i
```

L8:

| | | | | | | | |
|----------------|---|---------|---------|--------|--------|--------|----------------|
| A = 6×6 | 0 | 0 | 0 | 1.0000 | 0 | 0 | B = 6×1 |
| | 0 | 0 | 0 | 0 | 1.0000 | 0 | 0 |
| | 0 | 0 | 0 | 0 | 0 | 1.0000 | 0 |
| | 0 | -0.5000 | -0.5000 | 0 | 0 | 0 | 0.5000 |
| | 0 | 1.5000 | 0.5000 | 0 | 0 | 0 | -0.5000 |
| | 0 | 1.0000 | 3.0000 | 0 | 0 | 0 | -1.0000 |
| C = 1×6 | 1 | 0 | 0 | 0 | 0 | 0 | D = 0 |

```
eigVal = 6×1
0
0
-1.8113
-1.1042
1.8113
1.1042
```

(b).

For the non-linear system we used the Simulink model below.

The Embedded MATLAB Blocks have the exact same as the model used in the previous part of this problem.

For the initial condition,

$$\begin{aligned} y &= y^e + \delta y \\ \theta_1 &= \theta_1^e + \delta \theta_1 \\ \theta_2 &= \theta_2^e + \delta \theta_2 \end{aligned}$$

Thus,

$$\begin{aligned} y_i &= y^e + \delta y_i \\ \theta_{1i} &= \theta_1^e + \delta \theta_{1i} \\ \theta_{2i} &= \theta_2^e + \delta \theta_{2i} \end{aligned}$$

If the ICs defined in HW1 Exercise 7 is used,

| | y | θ_1 | θ_2 | \dot{y} | $\dot{\theta}_1$ | $\dot{\theta}_2$ |
|------------|-----|------------|------------|-----------|------------------|------------------|
| <i>IC1</i> | 0 | -10° | 10° | 0 | 0 | 0 |
| <i>IC2</i> | 0 | 10° | 10° | 0 | 0 | 0 |
| <i>IC3</i> | 0 | -90° | 90° | 0 | 0 | 0 |
| <i>IC4</i> | 0 | -90.01° | 90° | 0 | 0 | 0 |
| <i>IC5</i> | 0 | 100° | 100° | 0 | 0 | 0 |
| <i>IC6</i> | 0 | 100.01° | 100° | 0 | 0 | 0 |
| <i>IC7</i> | 0 | 179.99° | 0° | 0 | 0 | 0 |

Say we use IC1 for the initial condition,

For L7, the initial condition $(y^e, \theta_1^e, \theta_2^e) = (0, 0, 0)$

$$(\delta y_i, \delta \theta_{1i}, \delta \theta_{2i}) = \left(0, -10^\circ \cdot \frac{\pi}{180^\circ}, 10^\circ \cdot \frac{\pi}{180^\circ} \right) = (0, -0.1745, 0.1745)$$

Then,

$$(y_i, \theta_{1i}, \theta_{2i}) = (0, -0.1745, 0.1745)$$