Transfer Orbits: Lambert Arcs

Two approaches to mission planning:

(a) Given the transfer orbit → initial and final positions are specified; relate to the time of flight

(b) Given the initial (departure) and final (target) points → determine the orbit that passes through the points

Transfer Orbit Design
(special class of boundary value problem)

1. **Geometrical relationships**

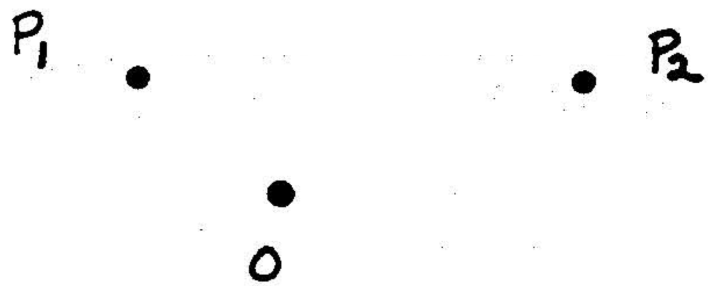
2. **Analytical Relationships**

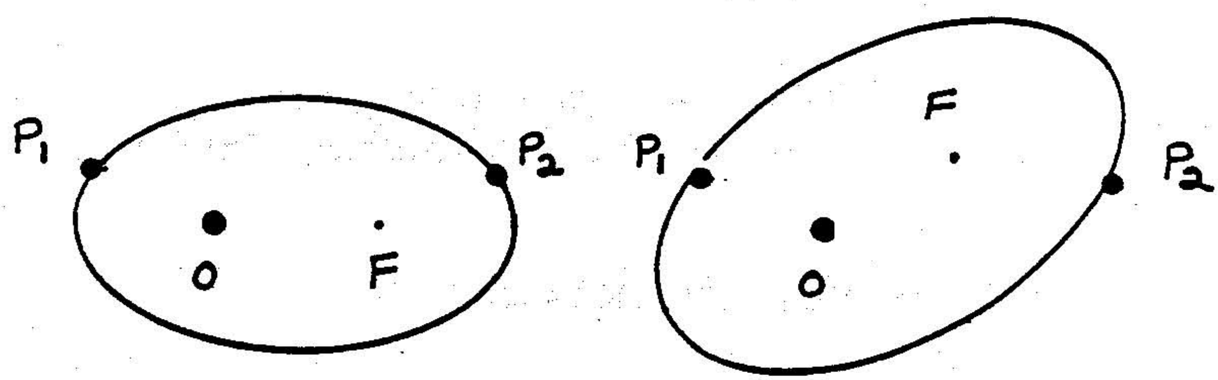
3. **Lambert's Theorem**

<u>Geometrical Relationships</u>: Ellipse

Given two fixed points P_1, P_2; center of force at point O

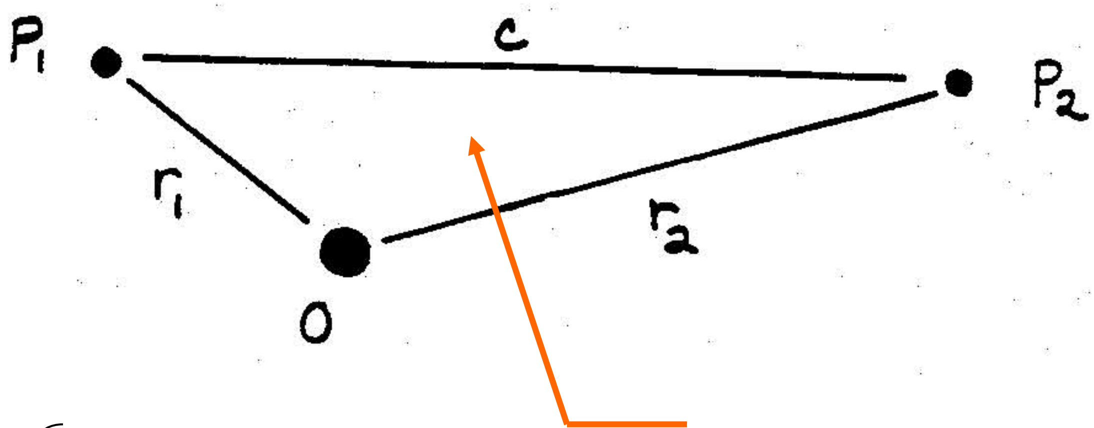
Find: ellipse with focus at point O that connects P_1, P_2





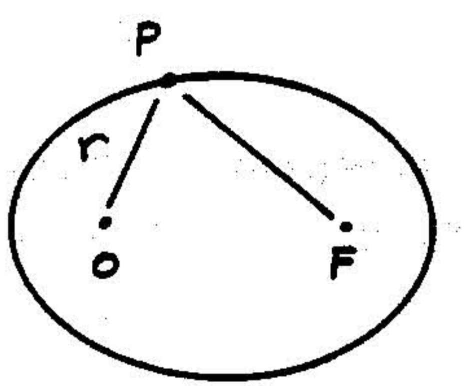
If F is not specified ⟹

Thus, find the locus of all possible F locations
Pick one of the F sites and the ellipse is determined



Let $\begin{cases} OP_1 = r_1 \\ OP_2 = r_2 \\ P_1P_2 = c \end{cases}$

Since P_1 and P_2 must both lie on the same ellipse, F must be selected such that



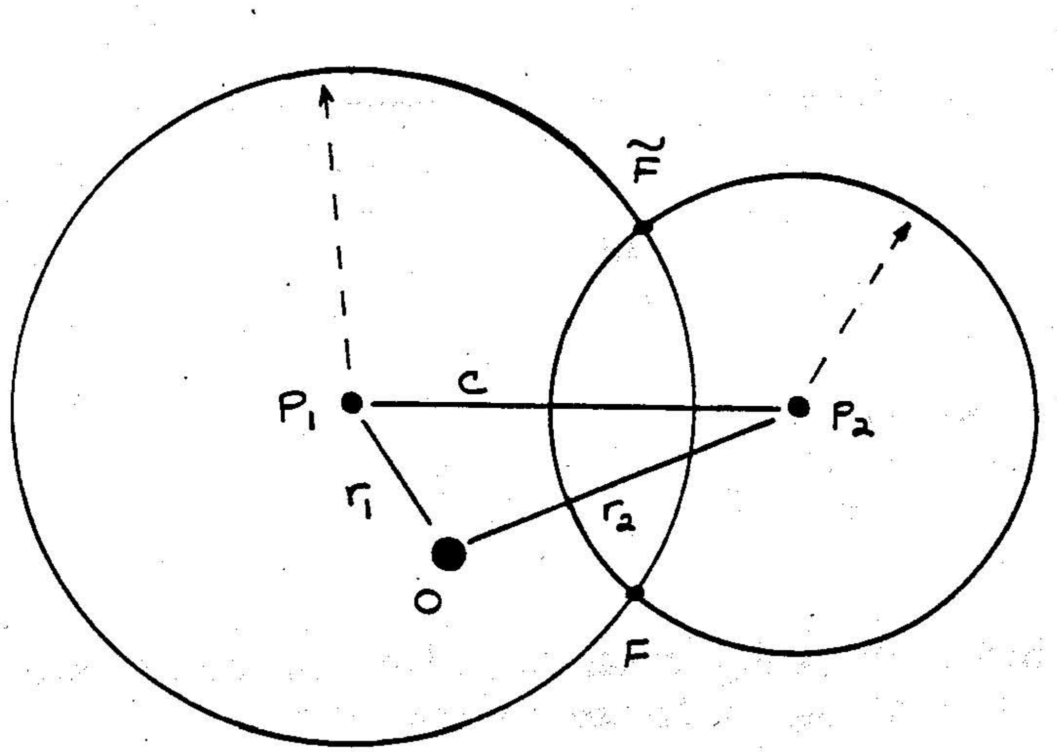
$$\overbrace{OP_1} + P_1F = 2a = \overbrace{OP_2} + P_2F$$

(always true for an ellipse)

OR

For ellipse with major axis $2a$, point F determined as the intersection of two circles centered at P_1 and P_2 with radii $2a - r_1$ and $2a - r_2$

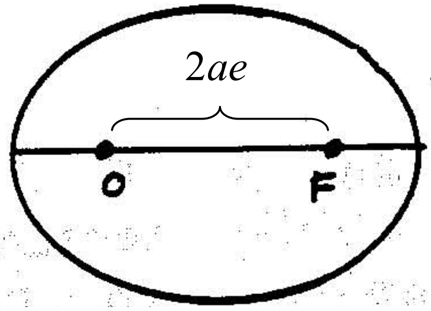
$$\left.\begin{aligned} P_1F &= 2a - r_1 \\ P_2F &= 2a - r_2 \end{aligned}\right\}$$



For a given "*a*" two possible intersection points $\longrightarrow$

Closest to *O* $\longrightarrow$

Given "*a*" $\Longrightarrow$ distance between foci *O* and *F* = 2*ae*



$\therefore \tilde{F}$ associated with

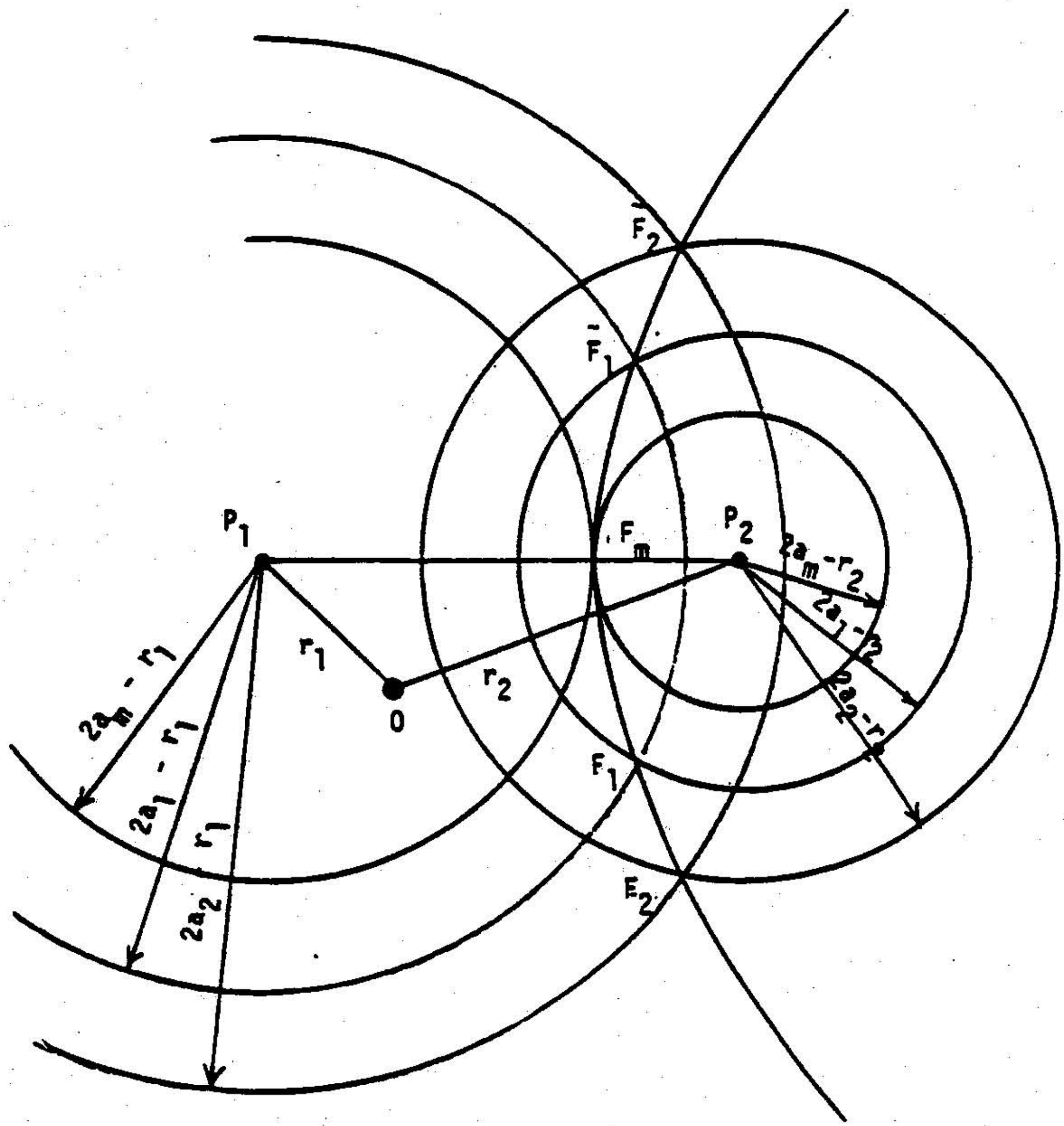
Choose 3 different values of "a"

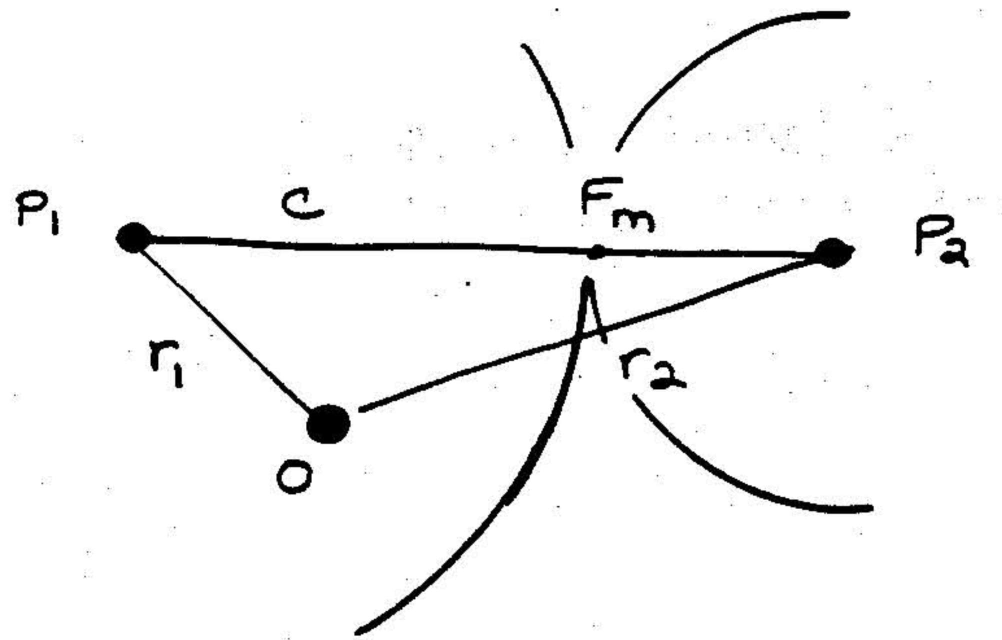
$\Rightarrow$

Note: there is a smallest value of "a" (a_m) below which there is <u>no ellipse</u> that connects P_1 and P_2 because the circles do not intersect

$a = a_m$ ➡

➡





$$\left(2a_m - r_1\right) + \left(2a_m - r_2\right) = c$$

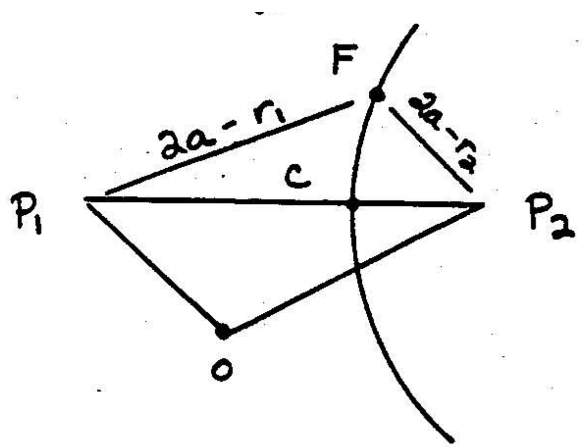
$$4a_m = r_1 + r_2 + c \qquad \text{OR}$$

→ F_m defines <u>minimum energy</u> elliptic path from P_1 to P_2

$$\left(\boldsymbol{\mathcal{E}} = -\frac{\mu}{2a_m} \quad \text{when } a_m \text{ small as possible, } \boldsymbol{\mathcal{E}} \text{ is min} \right)$$

Note: choosing different values of "a", produces pairs of vacant foci ($F, \tilde{F}$)

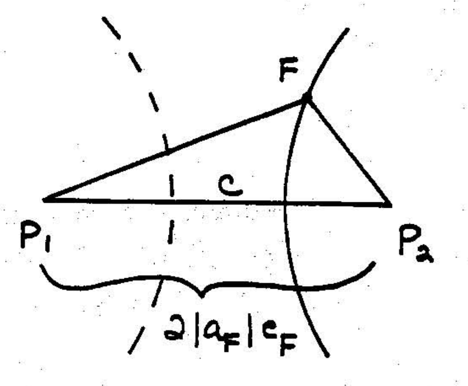
Sketch curve through all vacant foci F's
What does curve look like?



Equations for circles $\begin{cases} P_1F = 2a - r_1 \\ P_2F = 2a - r_2 \end{cases}$

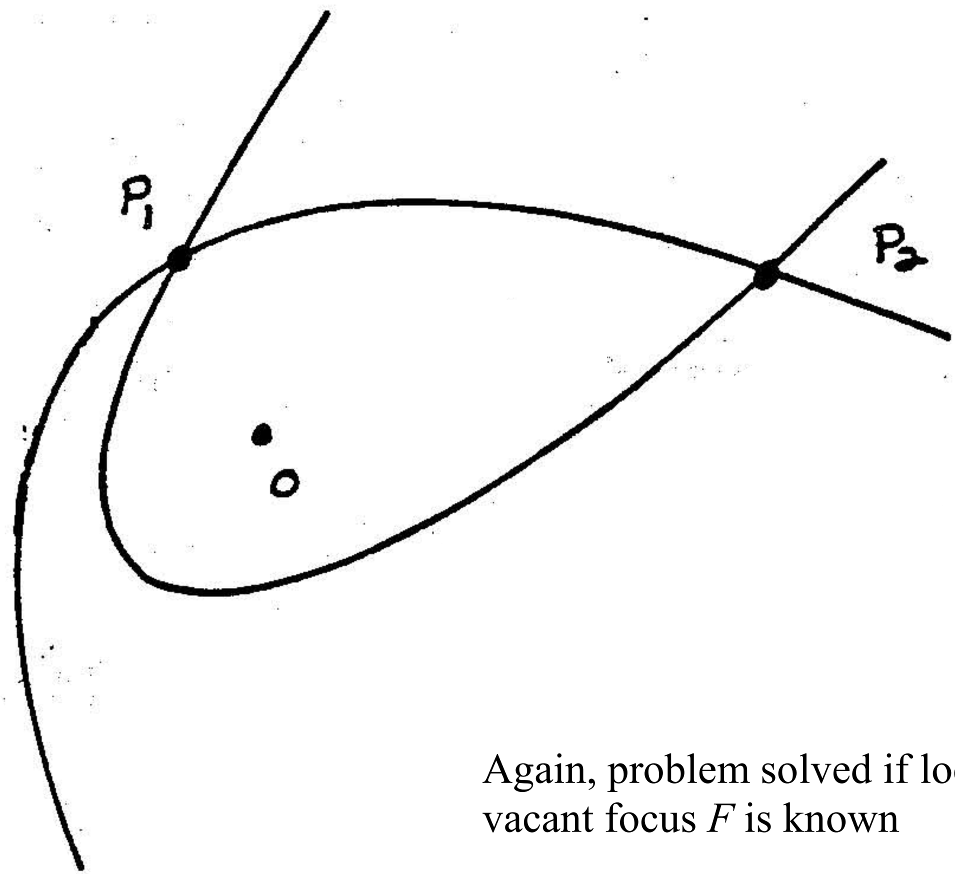
Subtract equations

⟹ Equation of a hyperbola: F is point on hyperbola
P_1, P_2 are foci
constant on right side: $2|a_F|$



<u>Geometrical Relationships</u>: Hyperbola

Given two fixed points P_1, P_2; center of force at point O
Find: hyperbola with focus at point O that connects P_1, P_2



Again, problem solved if location of vacant focus *F* is known

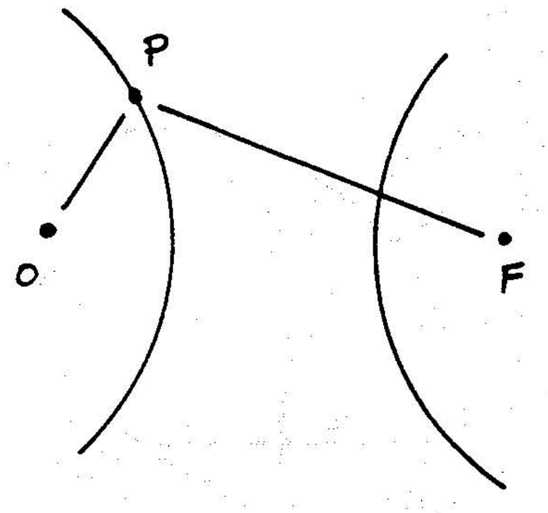
Since P_1 and P_2 must both lie on the same hyperbola, F must be selected such that

$$P_1F - \overbrace{OP_1} = 2|a| = P_2F - \overbrace{OP_2}$$

always true for hyperbola

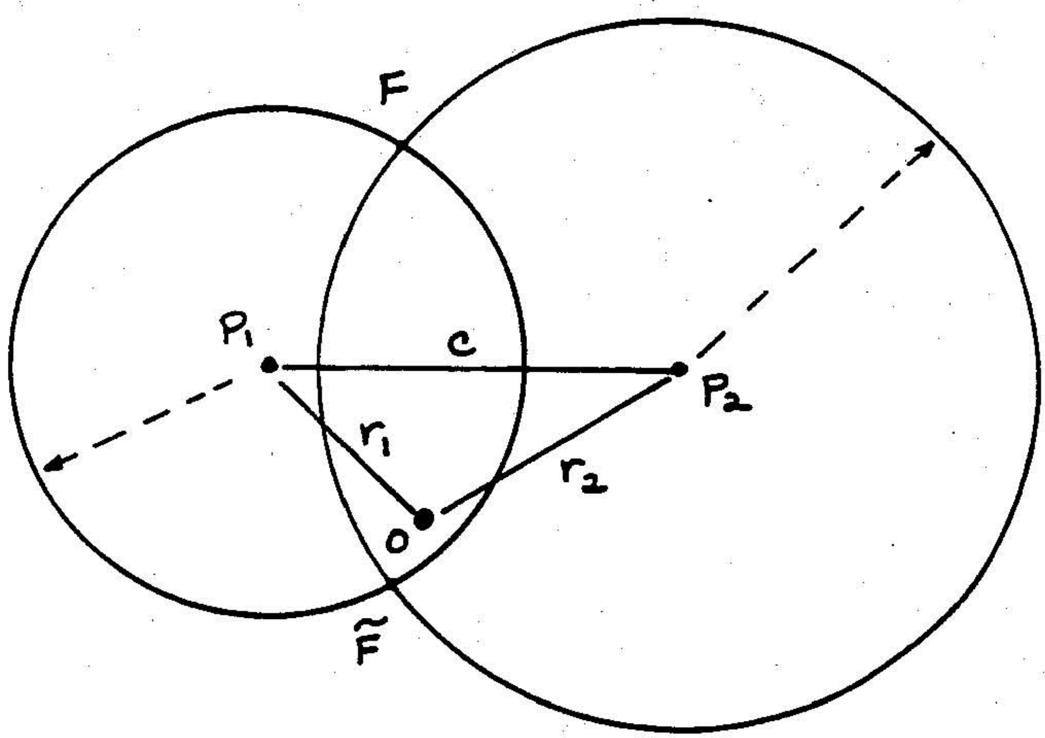
OR

$$P_1F = 2|a| + r_1$$
$$P_2F = 2|a| + r_2$$



For hyperbola, with major axis $2|a|$, point F determined as the intersection of two circles centered at P_1 and P_2 with radii $2|a| + r_1$ and $2|a| + r_2$

$$\left.\begin{array}{l} P_1F = 2|a| + r_1 \\ P_2F = 2|a| + r_2 \end{array}\right\}$$

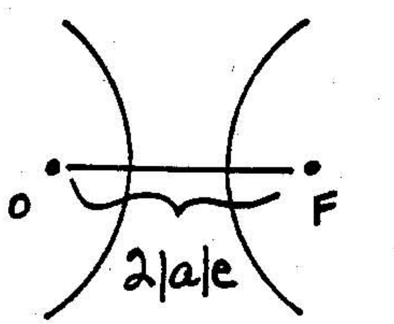


For a given $|a|$, two possible intersection points

→ 2 possible hyperbolic paths between P_1 and P_2

$F, \quad \tilde{F}$

Given $|a|$ ⟹ distance between foci O and $F = 2|a|e$



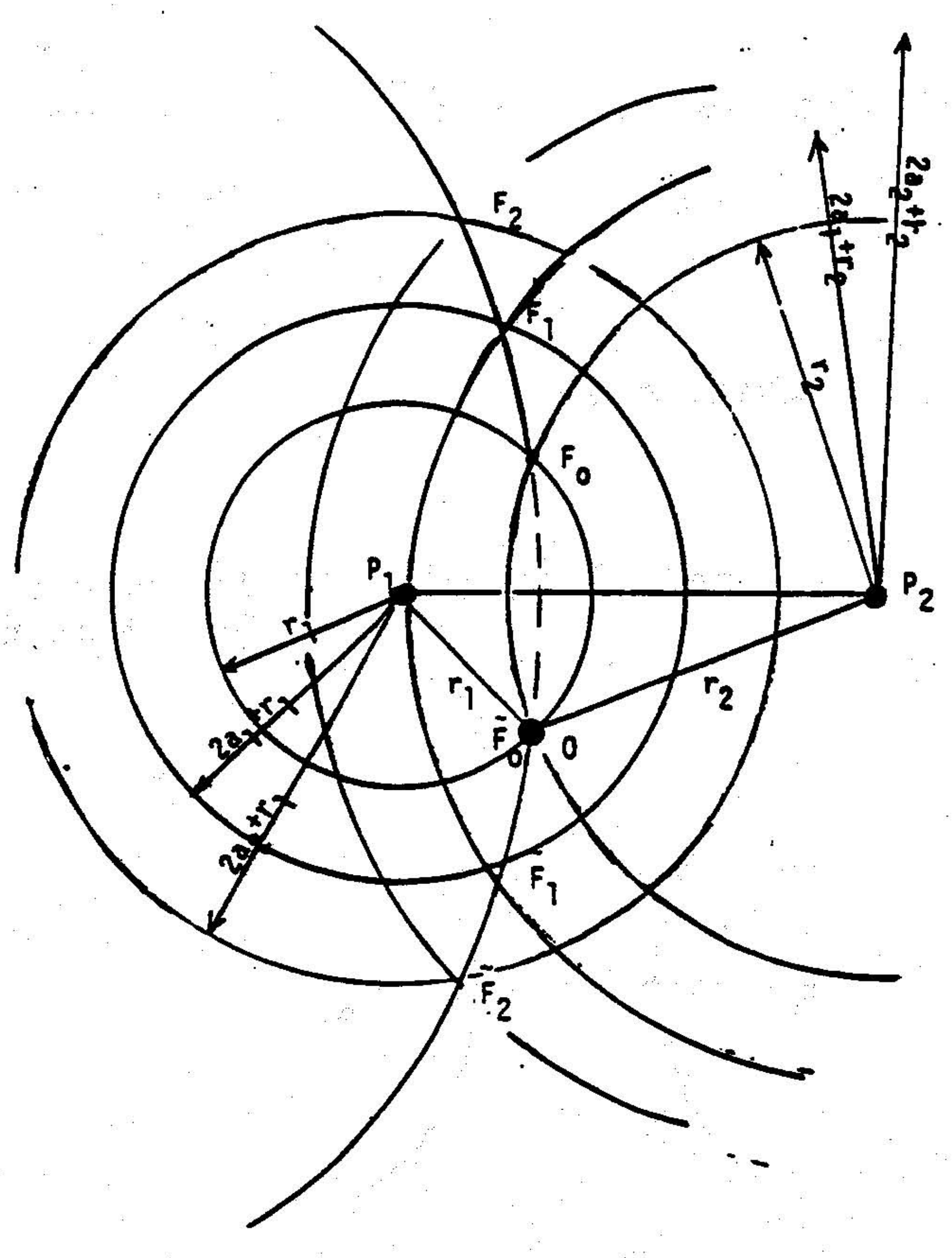
$\therefore F$ associated with $\Big\{$

Choose 3 different values of $|a|$

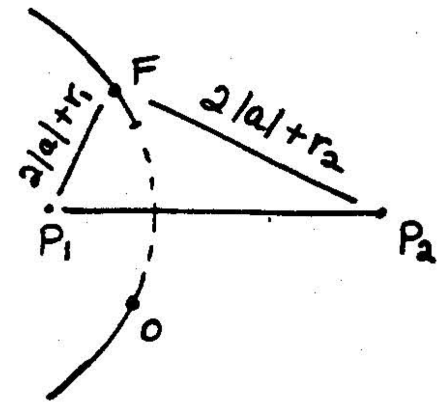
$\Rightarrow$ as $|a|$ gets smaller, circles shrink

Note: smallest value of $|a|$ that is possible is

(then circles have radii r_1 and r_2) $\Longrightarrow$



Note: Now sketch a curve through all vacant F's
What does the curve look like?

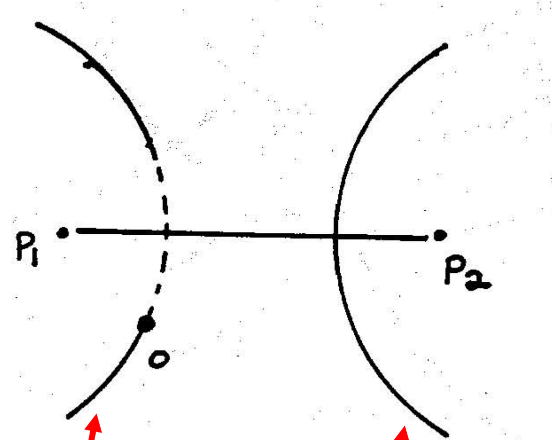


Locus of vacant foci **is** branch of a hyperbola

Equations for circles $\begin{cases} P_1F = 2a + r_1 \\ P_2F = 2a + r_2 \end{cases}$

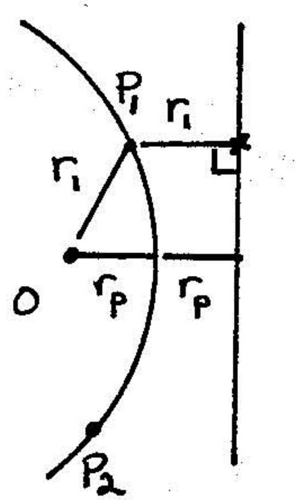
Subtract equations $P_2F - P_1F = \underbrace{r_2 - r_1}$ unknown is F again!

Equation of a hyperbola: other branch of **same** hyperbola
P_1, P_2 are foci
constant on right side: $2|a_F|$

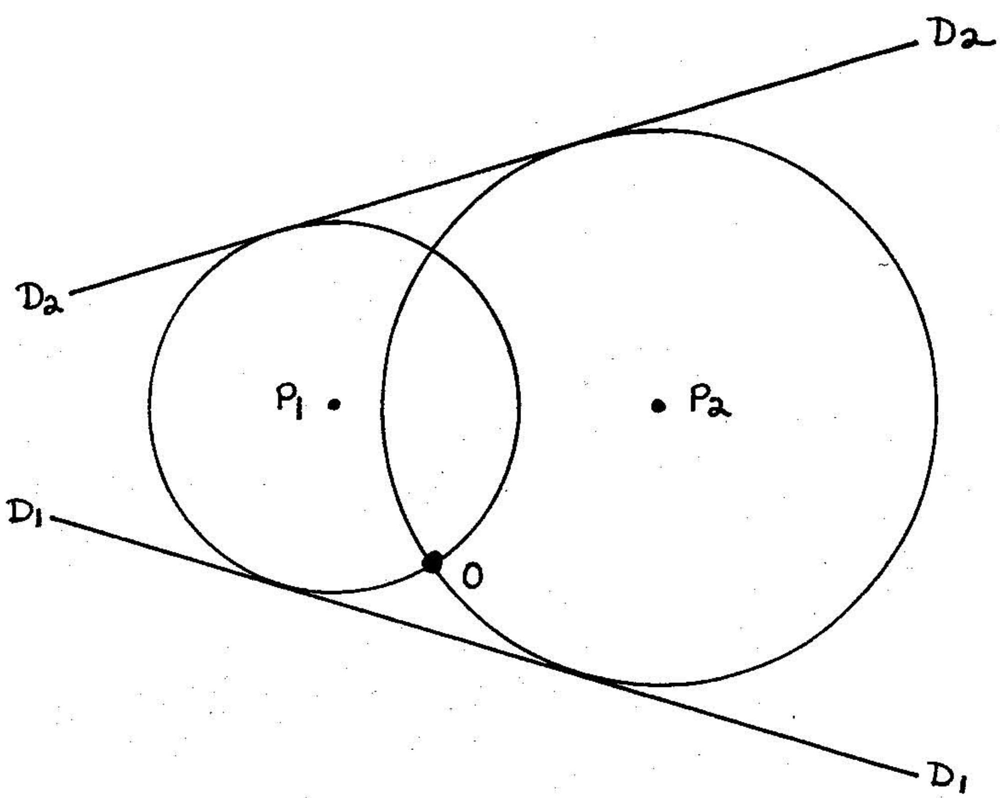


Geometrical Relationships: Parabola

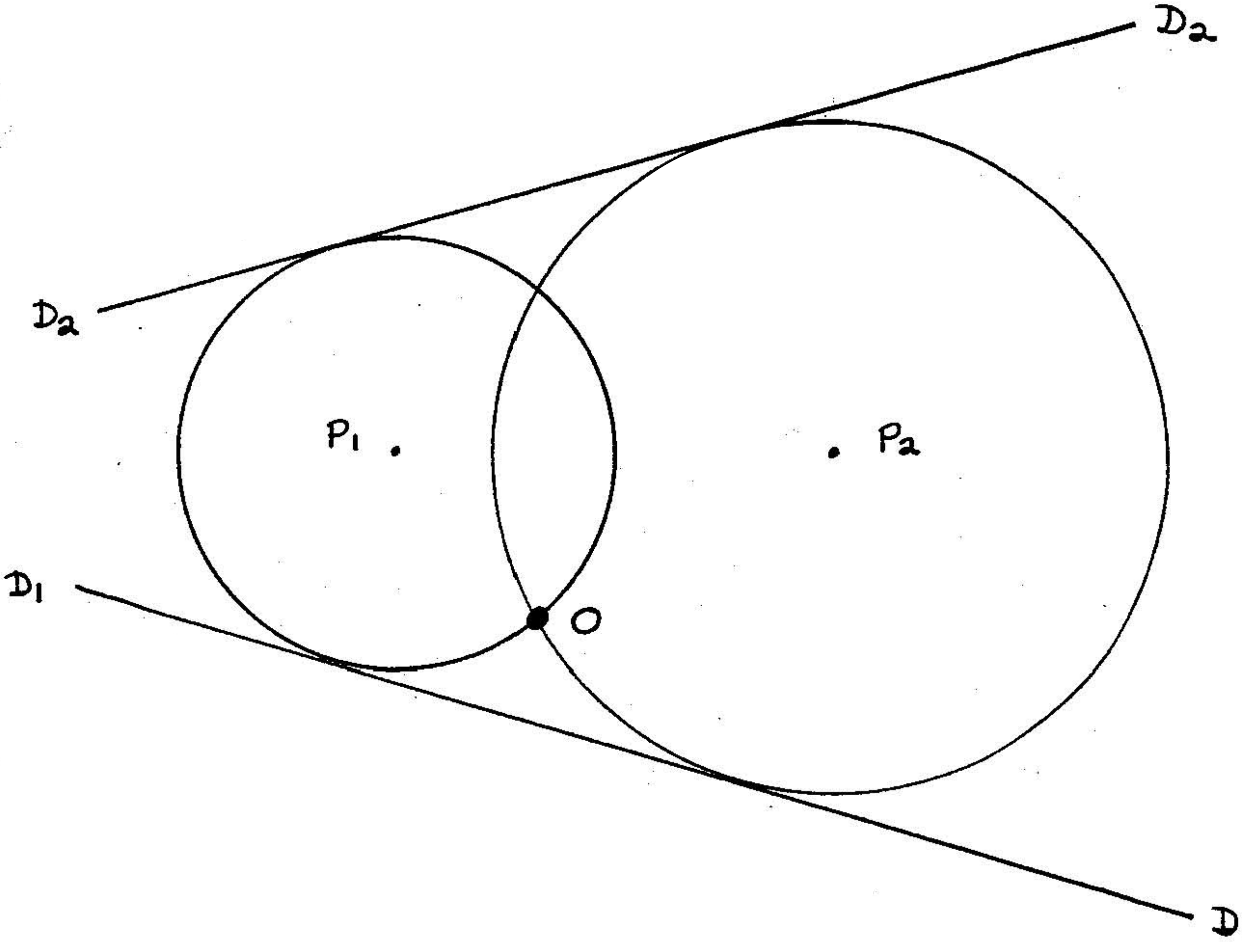
Only two possible parabolas ← $a = \infty$; F at ∞



Definition of parabola:
OP = distance to perpendicular intersection with directrix



To construct parabolas: requires normals *N* and vertices *V*

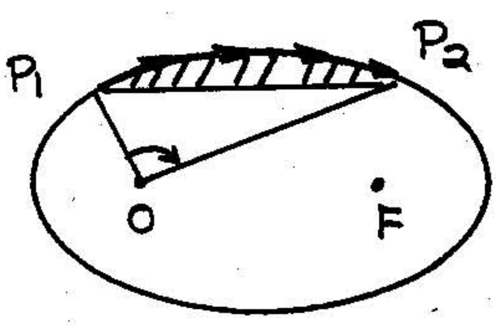


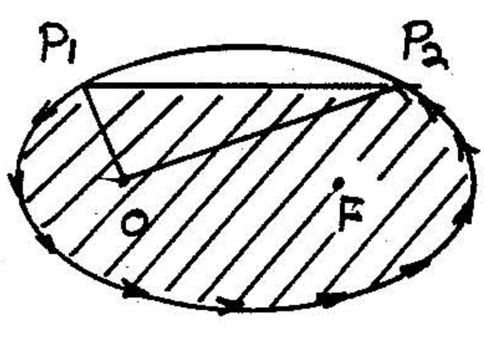
<u>Geometrical Relationships</u>: Summary

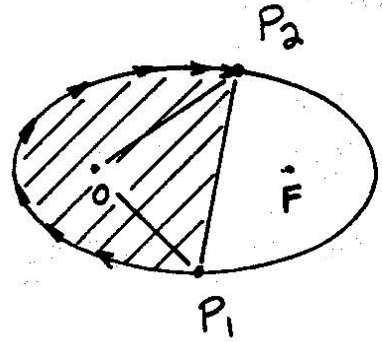
Once F is selected or otherwise identified, particular conic section is known

Necessary to define a method to categorize or classify transfers

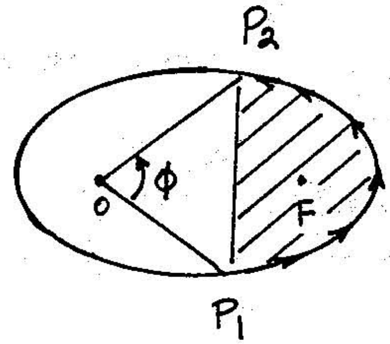
Legend:	A –	Ellipse (F NOT between chord and arc)
	B –	Ellipse (F between chord and arc)
	H –	Hyperbola
	1 –	Transfer Angle $< 180^\circ$
	2 –	Transfer Angle $> 180^\circ$



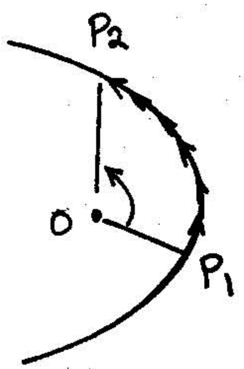




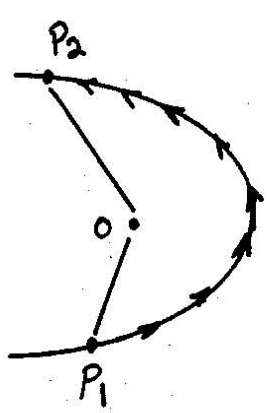
P2
O
F
P1



P2
O
ϕ
F
P1



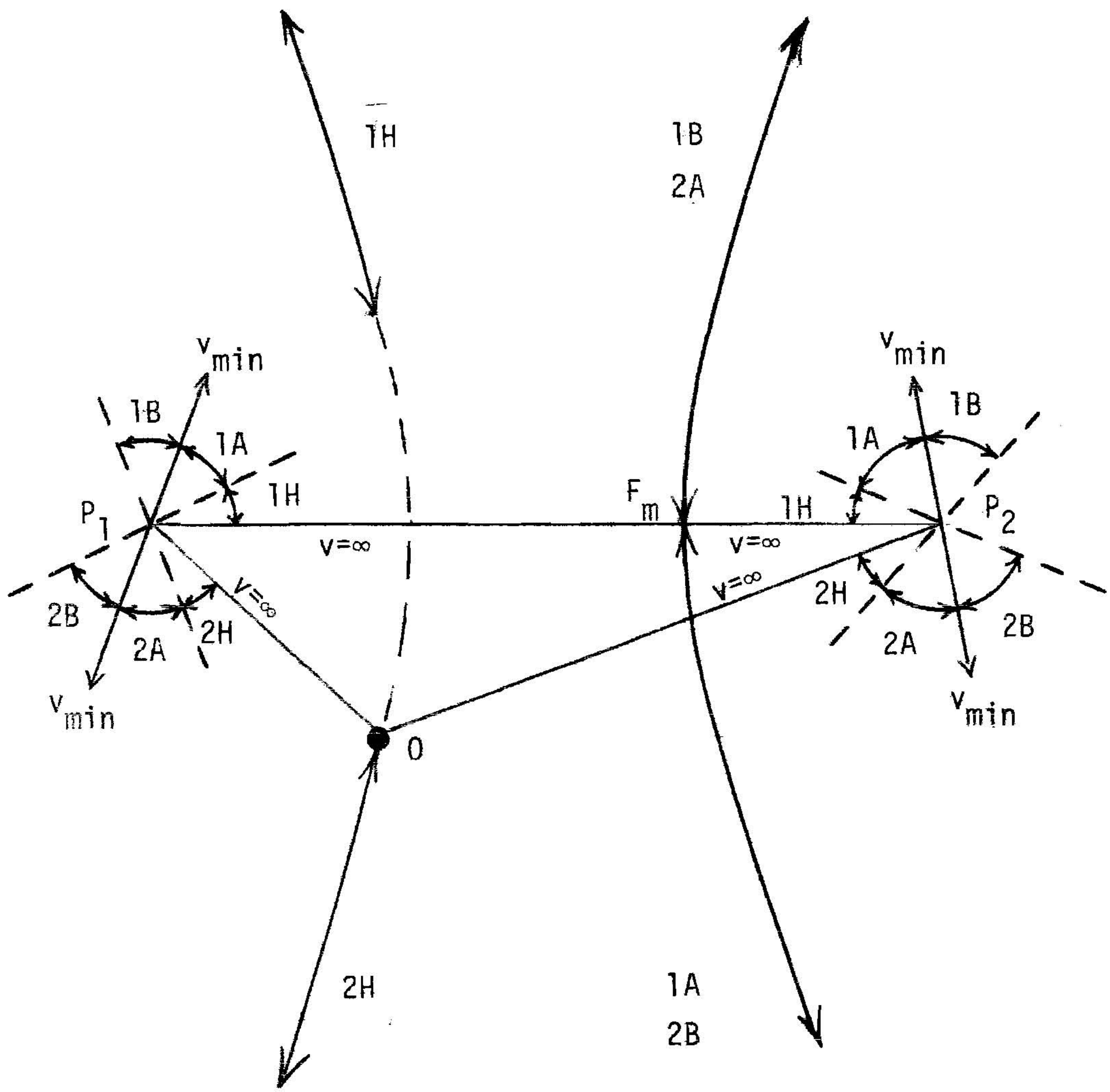
P2
O
P1



P2
O
P1

Various Orbits Between Two Points P_1, P_2

Locus of Vacant Focus F



Legend:

A - Ellipse (F not between chord and focus)

B - Ellipse (F between chord and focus)

H - Hyperbola

1 - Transfer Angle < 180°

2 - Transfer Angle > 180°

We may suppose $r_2 \geq r_1$.