Lecture: Background – Eigenvalues

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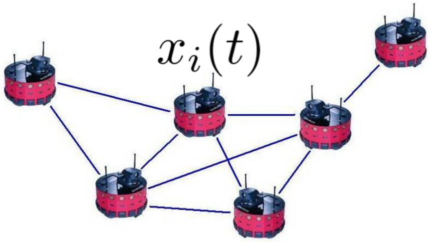
Review Distributed Consensus



agent's dynamics: $x_i(t+1) = u_i$

distributed control: $u_i = f_i(x_j(t), j \in \mathcal{N}_i)$

- ✓ **Objective:** $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$
- ✓ **Update:** $x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$

w_{ij} : the weight assigned by agent *i* to agent *j*

Consensus Goals	**Choices of Weights**
x^* is an unknown constant	$w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases}$ $\sum_{j=1}^{m} w_{ij} = 1$
x^* is the global average $\frac{1}{m}\sum_{i=1}^{m} x_i(0)$	$w_{ij} = \begin{cases} \min\{\frac{1}{d_i}, \frac{1}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, & \text{otherwise.} \end{cases}$
x^* is a specific convex combination $\sum_{i=1}^{m} \gamma_i x_i(0)$	$w_{ij} = \begin{cases} \frac{1}{\gamma_i}\min\{\frac{\gamma_i}{d_i}, \frac{\gamma_j}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, & \text{otherwise.} \end{cases}$

- ✓ **Analysis:** $x(t+1) = Ax(t)$

$A \in \mathbb{R}^{m \times m}$ with entries $A_{ij} = w_{ij}$

$$x(t) = A^t x(0)$$

Eigenvalues

Eigenvalues of a square matrix $M \in \mathbb{R}^{n \times n}$

(Right) Eigenvector: $Mv = \lambda v \quad v \neq 0$

Left-Eigenvector: $w' M = \lambda w' \quad w \neq 0$

- M has a number of *n* eigenvalues
  - ✓ which might be **complex** numbers even M is real.
  - ✓ which might have **repeated** values.

  The number of repeated times of an eigenvalue is called its ***algebraic multiplicity***.

- The ***spectrum*** of M is the set of all its eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$

- The ***spectral radius*** of M is the maximum magnitude of its eigenvalues.

- The sum of all eigenvalues is equal to the trace of M: $\sum_{i=1}^{n} \lambda_i = \text{trace } (M)$

- The product of all eigenvalues is equal to the determinant of M: $\prod_{i=1}^{n} \lambda_i = \det \ (M)$

- For an eigenvalue λ of *M*, the union of 0 and all eigenvectors for λ is called its ***eigen-space***.
  - ✓ The dimension of *an eigenvalue's eigen-sub-space* is called its ***geometric multiplicity***.

    the solution subspace to $Mx = \lambda x$ $\quad (M - \lambda I)x = 0$

    the null space (kernel) of $M - \lambda I$

    - ***Algebraic multiplicity*** is the number of times that an eigenvalue appears.
  - ✓ An eigenvalue's algebraic multiplicity may not be equal to its geometric multiplicity.

algebraic multiplicity $\geq$ **geometric multiplicity**

$$M = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

eigenvalues: 1, 1

Algebraic multiplicity of eigenvalue 1 is 2

Geometric multiplicity of eigenvalue 1 is
the dimension of null space of

$$M - 1 * I_2 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

which is 1

- The **characteristic polynomial** of M is which is a polynomial equation about λ.

$$\det\ (\lambda I - M) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

  - ✓ Eigenvalues of M are roots of its characteristic polynomials.
  - ✓ **Cayley-Hamilton Theorem.** $M^n + c_{n-1}M^{n-1} + \cdots + c_1 M + c_0 I = 0$

try by yourself

$$M = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \det\ (\lambda I - M) = \lambda^2 - 5\lambda - 2$$

$$M^2 - 5M - 2I =$$

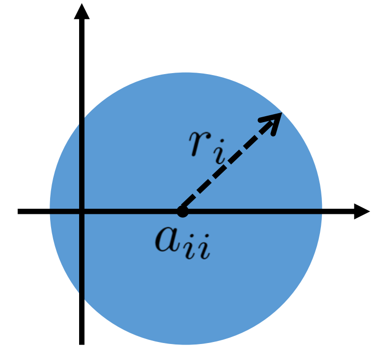
Question: Suppose a 2*2 matrix M has two eigenvalues 1 and 2.
Compute: $M^2 - 3M$

- **Weinstein-Aronaszajn Identity.** $\det\ (I_m + AB) = \det\ (I_n + BA)$

Compute: $\det\ \left(I_5 + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}\right)$

- **Gershgorin Circle Theorem:** For **any** square matrix $A \in \mathbb{C}^{n\times n}$, let λ denote any of its eigenvalue. Then there exists i such that

$$|\lambda - a_{ii}| \le r_i \quad \text{where} \quad r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|$$



Gershgorin Disk

In other words, any eigenvalue must lies in at least one Gershgorin Circle.

Gershgorin Circle Theorem provides a way to roughly estimate eigenvalues of any square matrices from their entries.

✓ Example 1: Eigenvalues

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix}$$

disk 1: $|\lambda - 1| \le 9$

disk 2: $|\lambda - 6| \le 20$

disk 3: $|\lambda - 4| \le 10$

disk 4: $|\lambda - 9| \le 21$

✓ Example 2: **Stochastic matrix**

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/2 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$A\mathbf{1} = \mathbf{1}$ 1 is an eigenvalue;

- 1 is the largest eigenvalue in magnitude ???

In other words, for any eigenvalue λ , one has $|\lambda| \le 1$

prove it by yourself

$$|\lambda| - a_{ii} \le |\lambda - a_{ii}| \le \sum_{j=1, j\neq i}^{n} |a_{ij}| = 1 - a_{ii} \qquad |\lambda| \le 1$$

- ***Lemma:*** Nonzero eigenvalues of *AB* are the same as those of *BA*.

$$A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times m}$$
$$AB \neq BA$$

Proof: Let λ denote any of non-zero eigenvalue of *AB.* Then

$$\det\,(\lambda I_m - AB) = 0 \quad \det\,(I_m - \frac{1}{\lambda}AB) = 0 \quad \det\,(I_n - \frac{1}{\lambda}BA) = 0 \quad \det\,(\lambda I_n - BA) = 0$$

Then λ is also an eigenvalue of *BA.* Similarly, any nonzero eigenvalue of BA is also an eigenvalue of AB.

Example 1: Try this in Matlab by using two random matrices

Example 2: (A Past PhD Qualify Exam Problem). Compute eigenvalues of

$$M = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} e & \pi & \sqrt{2} & \sqrt{3} \end{bmatrix})$$

- ***Lemma:*** Inner product of a left eigenvector and a right eigenvector corresponding to different eigenvalues of the same matrix is 0, namely, $\lambda \neq \bar{\lambda} \rightarrow w'v = 0.$

Proof: Since v is a right eigenvector for λ , one has

$$w'Mv = \lambda w'v$$

Since w is a left eigenvector for λ , one has

$$w'Mv = \bar{\lambda} w'v$$

$$\left.\begin{array}{l} w'Mv = \lambda w'v \\ w'Mv = \bar{\lambda} w'v \end{array}\right\} (\lambda - \bar{\lambda})w'v = 0$$

$$\left.\begin{array}{l} \lambda \neq \bar{\lambda} \\ (\lambda - \bar{\lambda})w'v = 0 \end{array}\right\} w'v = 0$$

Eigenvalues for Symmetric Matrix $M = M'$

- For a symmetric matrix, all its eigenvalues are real $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

  and with corresponding orthonormal eigenvectors $v_1, v_2, ..., v_n$

$$Mv_i = \lambda_i v_i, \quad i = 1, 2, ..., n \qquad v_i' v_j = 0, \quad j \neq i \qquad v_i' v_i = 1$$

Thus these eigenvectors form an **orthonormal basis** for $\mathbb{R}^n = \text{span}\{v_1, v_2, ..., v_n\}$

Eigenspace Decomposition

✓ Analysis for convergence of $x(t+1) = Mx(t)$, where M is symmetric.

$$x(0) = k_1 v_1 + k_2 v_2 + \cdots + k_n v_n \xrightarrow[x(0)'v_i \;=\; k_i]{} \sum_{i=1}^{n} (v_i' x(0))\, v_i$$

$$x(t) = M^t x(0) = \sum_{i=1}^{n} (v_i' x(0)) M^t v_i = \sum_{i=1}^{n} (v_i' x(0)) \lambda_i^t v_i$$

$$\lim_{t \to \infty} x(t) = \begin{cases} 0 & \text{If all eigenvalues are with magnitude strictly less than 1.} \\ \textbf{unbounded} & \text{If there exists one eigenvalue with magnitude larger than 1.} \\ = \sum_{i=1}^{q} (v_i' x(0)) v_i & \text{If } \begin{array}{l} \lambda_1 = \lambda_2 = \cdots = \lambda_q = 1 \\ |\lambda_i| < 1, i = q+1, q+2, ..., n \end{array} \end{cases}$$

- **Min-Max Theorem**: If M is symmetric, (or Hermitian in complex case)

$$\lambda_{\min} x'x \leq x'Mx \leq \lambda_{\max} x'x$$

quadratic form

- For unit vectors x, one has

$$\lambda_{\min} \leq x'Mx \leq \lambda_{\max}$$

- This is especially helpful in proving exponential convergence using **Lyapunov functions**.

Example: For gradient-based distributed formation control, which will be talked about in later lectures, one has

System dynamics: $\dot{x} = -R(x)'e$

Error dynamics: $\dot{e} = -2R(x)R'(x)e$

Lyapunov function: $V = \frac{1}{4}e'e$

$$\dot{V} = \frac{1}{2}e'\dot{e} = -e'R(x)R(x)'e$$

min-max theorem $\leq -\gamma e'e \;= -\gamma V$

$$\gamma = \lambda_{\min}(R(x)R(x)')$$

Singular Values

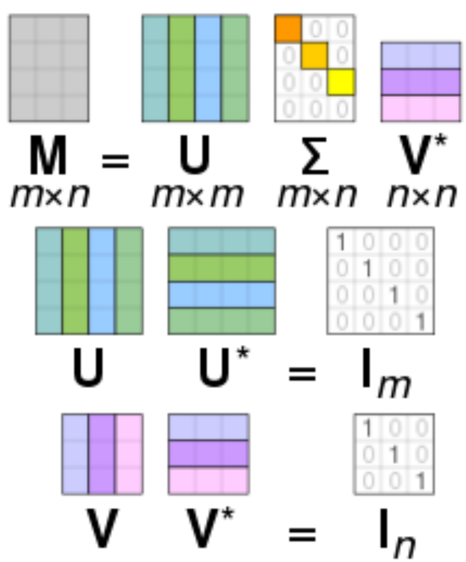
- **Singular values** of a matrix $M \in \mathbb{C}^{m \times n}$ are the square root of eigenvalues of $M^* M$
- **Singular Value Decomposition** of a matrix $M \in \mathbb{C}^{m \times n}$ is a factorization of the form

$$M = U \Sigma V^*$$

U,V: unitary matrix $UU^* = I, VV^* = I$

$\sum$:rectangular diagonal matrix with diagonal entries equal to singular values of M

- Figure Explanation



- How to achieve SVD?

Matlab: [U,S, V]=svd (M)

Let's try M=[1,0,0,0,2;0,0,3,0,0;0,0,0,0,0;0,2,0,0,0] in Matlab

- SVD has extensive applications such as total least squares problem in regression, low-rank matrix approximation, signal processing, and so on.

Research Topic: Distributed Algorithm for SVD?

Summary

- **Cayley-Hamilton Theorem.** $M^n + c_{n-1}M^{n-1} + \cdots + c_1 M + c_0 I = 0$
- **Gershgorin Circle Theorem:** For **any** square matrix $A \in \mathbb{R}^{n\times n}$, let λ denote any of its eigenvalue.

  Then there exists i such that

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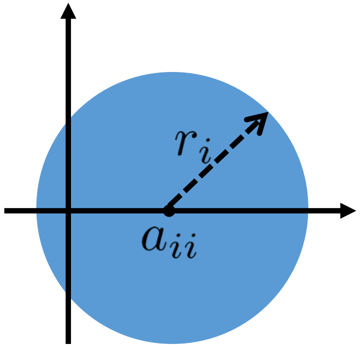

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- **Min-Max Theorem**: If M is symmetric, $\lambda_{\min} x'x \le x'Mx \le \lambda_{\max} x'x$