AAE 666 Homework 1 Solution

January 25, 2021

Exercise 1

$$2\ddot{q}_1 + \ddot{q}_2 + sinq_1 = 0$$
$$\ddot{q}_1 + 2\ddot{q}_2 + sinq_2 = 0$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} -sinq_1 \\ -sinq_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -sinq_1 \\ -sinq_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -sinq_1 \\ -sinq_2 \end{bmatrix}$$
$$\ddot{q}_1 = -\frac{2}{3} sinq_1 + \frac{1}{3} sinq_2$$
$$\ddot{q}_2 = \frac{1}{3} sinq_1 - \frac{1}{3} sinq_2$$

Define state variables:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$
$$x_4 = \dot{q}_2$$

This yields

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{2}{3} \sin x_1 + \frac{1}{3} \sin x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{1}{3} \sin x_1 - \frac{1}{3} \sin x_3$$

Exercise 2

$$\ddot{q}_1 + \dot{q}_2 + q_1^3 = 0$$
$$\dot{q}_1 + \dot{q}_2 + q_2^3 = 0$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -q_1^3 \\ -\dot{q}_1 - q_2^3 \end{bmatrix}$$
$$\ddot{q}_1 = -q_1^3 + \dot{q}_1 + q_2^3$$
$$\dot{q}_2 = -\dot{q}_1 - q_2^3$$

Define state variables:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$

This yields

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1^3 + x_2 + x_3^3$$
$$\dot{x}_3 = -x_2 - x_3^3$$

Exercise 3

$$\ddot{q}_1 + q_1 + 2\dot{q}_2 = 0$$
$$\ddot{q}_1 + \dot{q}_2 + q_2 = 0$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$
$$\begin{bmatrix} \ddot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -q_1 \\ -q_2 \end{bmatrix}$$
$$\ddot{q}_1 = q_1 - 2q_2$$
$$\dot{q}_2 = -q_1 + q_2$$

Define state variables:

$$x_1 = q_1$$
$$x_2 = \dot{q}_1$$
$$x_3 = q_2$$

This yields

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1 - 2x_3 \\
\dot{x}_3 &= -x_1 + x_3
\end{aligned}$$

Exercise 4

$$q_1(k+2) + q_1(k) + 2q_2(k+1) = 0$$
$$q_1(k+2) + q_1(k+1) + q_2(k) = 0$$
$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(k+2) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} -q_1(k) \\ -q_1(k+1) - q_2(k) \end{bmatrix}$$
$$\begin{bmatrix} q_1(k+2) \\ q_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -q_1(k) \\ -q_1(k+1) - q_2(k) \end{bmatrix}$$
$$\begin{bmatrix} q_1(k+2) \\ q_2(k+1) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -q_1(k) \\ -q_1(k+1) - q_2(k) \end{bmatrix}$$
$$q_1(k+2) = -q_1(k+1) - q_2(k)$$
$$q_2(k+1) = -\frac{1}{2}q_1(k) + \frac{1}{2}q_1(k+1) + \frac{1}{2}q_2(k)$$

Define state variables:

$$\begin{aligned}
x_1(k) &= q_1(k) \\
x_2(k) &= q_1(k+1) \\
x_3(k) &= q_2(k)
\end{aligned}$$

Fianlly:

$$\begin{aligned}
x_1(k+1) &= x_2(k) \\
x_2(k+1) &= -x_2(k) - x_3(k) \\
x_3(k+1) &= -\frac{1}{2}x_1(k) + \frac{1}{2}x_2(k) + \frac{1}{2}x_3(k)
\end{aligned}$$

Exercise 5

When the system is at its equilibrium, it means for all k

$$x(k+1) = x(k)$$

This means when $x = x^e$

$$\begin{aligned}
x(k+1) - x(k) &= -\frac{g(x(k))}{g'(x(k))} \\
0 &= -\frac{g(x^e)}{g'(x^e)}
\end{aligned}$$

And this is only true if $g(x^e) = 0$.

Exercise 6

The first nonlienar system is given by:

$$\dot{x} = -\alpha \text{sgm}(x)$$

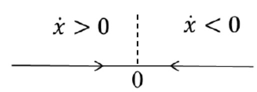

Figure 1: Exercise 6 Plot

Exercise 7

$$\dot{x} = x^2(x^2 - 1) = x^2(x-1)(x+1)$$

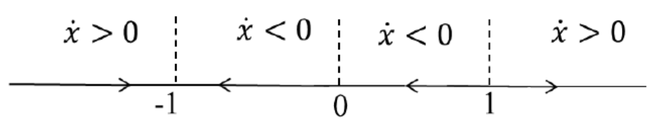

Figure 2: Exercise 7 Plot

Exercise 8

$$\dot{x} = -x^3$$

Two conditions:
(1): At equalibrium:

$$x_e = 0$$

(2):

$$-\frac{1}{3x^3} dx = dt$$

Integrate and obtain:

$$\frac{1}{2x^2} - C = t$$

For positive solution:

$$x(t) = \frac{1}{\sqrt{2t + 2C}}$$

where C can be written as a function of initial x:

$$x_0 = \frac{1}{\sqrt{2C}}$$

Exercise 9

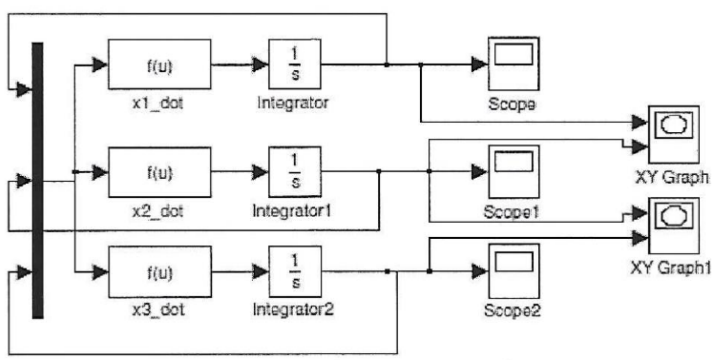

Figure 3: Exercise 9 Simulink Model

The difference in time history started after 40 seconds for all states. For the 60 seconds simulated, the history differs mostly in the frequency. The amplitude of the response remain roughly the same.

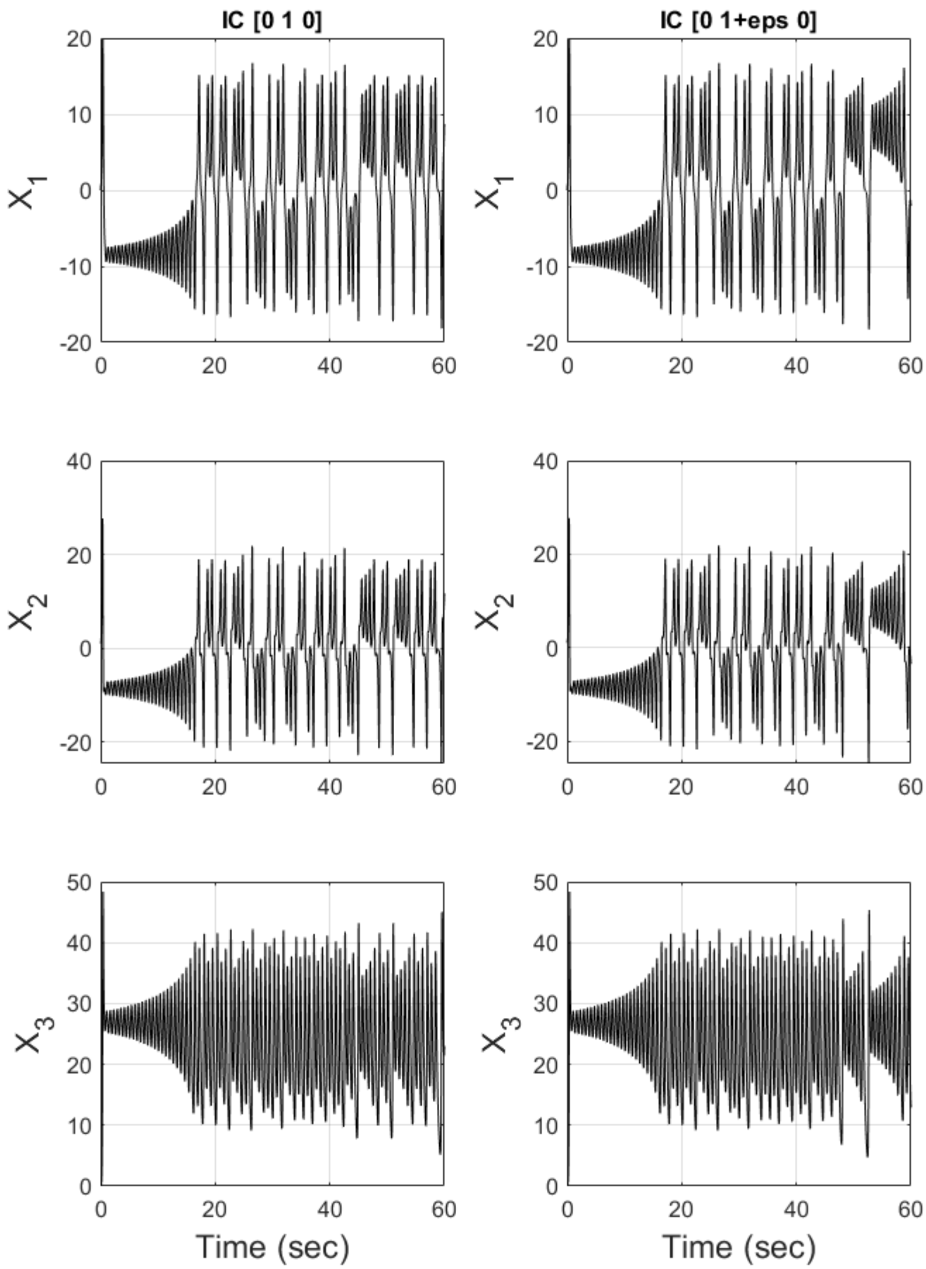

Figure 4: Exercise 9 Simulation Results