College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW1

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Exercise 1

By appropriate definition of state variables, obtain a first order state space description of the following system where q_1 and q_2 are real scalars.

$$\begin{aligned} 2\ddot{q}_1 + \ddot{q}_2 + \sin q_1 &= 0 \\ \ddot{q}_1 + 2\ddot{q}_2 + \sin q_2 &= 0 \end{aligned}$$

Solution:
In order to represent this system as a state space we have to have to manipulate these equations to have either the second order derivative term of q_1 or q_2. So multiply the first equation by 2 and calculate the difference of the 2 equations.

$$\begin{array}{rr} & 4\ddot{q}_1 + 2\ddot{q}_2 + 2\sin q_1 = 0 \\ - & \ddot{q}_1 + 2\ddot{q}_2 + \sin q_2 = 0 \\ \hline & 3\ddot{q}_1 + 2\sin q_1 - \sin q_2 = 0 \end{array}$$

Move the highest order differential term to the LHS and the rest to the RHS.

$$\ddot{q}_1 = -\frac{2}{3}\sin q_1 + \frac{1}{3}\sin q_2$$

Plug this into the first equation and we obtain

$$\begin{aligned} 2\left(-\frac{2}{3}\sin q_1 + \frac{1}{3}\sin q_2\right) + \ddot{q}_2 + \sin q_1 &= 0 \\ \ddot{q}_2 = \frac{1}{3}\sin q_1 - \frac{2}{3}\sin q_2 & \end{aligned}$$

Now represent $x_1 := q_1$, $x_2 := q_2$, $x_3 := \dot{q}_1$, and $x_4 := \dot{q}_2$. Thus, the state space representation becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ -\frac{2}{3}\sin x_1 + \frac{1}{3}\sin x_2 \\ \frac{1}{3}\sin x_1 - \frac{2}{3}\sin x_2 \end{bmatrix}$$

Exercise 2

By appropriate definition of state variables, obtain a first order state space description of the following system where q_1 and q_2 are real scalars.

$$\begin{aligned}\ddot{q}_1 + \dot{q}_2 + q_1^3 = 0\\ \dot{q}_1 + \dot{q}_2 + q_2^3 = 0\end{aligned}$$

Solution:
The highest order differential term for q_1 is 2 and for q_2 is 1. Thus, subtract the second equation from the first one.

$$\begin{array}{rr} & \ddot{q}_1 + \dot{q}_2 + q_1^3 = 0\\ - & \dot{q}_1 + \dot{q}_2 + q_2^3 = 0\\ \hline \multicolumn{2}{c}{\ddot{q}_1 - \dot{q}_1 + q_1^3 - q_2^3 = 0}\end{array}$$

Thus, we obtain

$$\ddot{q}_1 = \dot{q}_1 - q_1^3 + q_2^3$$

And we also know that

$$\dot{q}_2 = -\dot{q}_1 - q_2^3$$

Now represent $x_1 := q_1$, $x_2 := q_2$, and $x_3 := \dot{q}_1$. Thus, the state space representation becomes

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}x_3\\ -\dot{x}_1 - x_2^3\\ x_3 - x_1^3 + x_2^3\end{bmatrix}$$

Exercise 3

By appropriate definition of state variables, obtain a first order state space description of the following system where q_1 and q_2 are real scalars.

$$\begin{aligned} \ddot{q}_1 + q_1 + 2\dot{q}_2 &= 0 \\ \ddot{q}_1 + \dot{q}_2 + q_2 &= 0 \end{aligned}$$

Solution:
The highest order differential term for q_1 is 2 and for q_2 is 1. Thus, subtract the second equation from the first one.

$$\begin{array}{rr} & \ddot{q}_1 + q_1 + 2\dot{q}_2 = 0 \\ - & \ddot{q}_1 + \dot{q}_2 + q_2 = 0 \\ \hline & q_1 + \dot{q}_2 - q_2 = 0 \end{array}$$

Thus, we obtain

$$\dot{q}_2 = -q_1 + q_2$$

Substitute this into the first equation and we obtain

$$\begin{aligned} \ddot{q}_1 + q_1 + 2\left(-q_1 + q_2\right) &= 0 \\ \ddot{q}_1 &= q_1 - 2q_2 \end{aligned}$$

Now represent $x_1 := q_1$, $x_2 := q_2$, and $x_3 := \dot{q}_1$. Thus, the state space representation becomes

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_1 + x_2 \\ x_1 - 2x_2 \end{bmatrix}$$

Exercise 4

By appropriate definition of state variables, obtain a first order state space description of the following system where q_1 and q_2 are real scalars.

$$q_1(k+2) + q_1(k) + 2q_2(k+1) = 0$$
$$q_1(k+2) + q_1(k+1) + q_2(k) = 0$$

Solution:
The highest order term for q_1 is k+2 and for q_2 is k+1. Thus, organize the second equation and we obtain

$$q_1(k+2) = -q_1(k+1) - q_2(k)$$

Substitute this into the first equation and we obtain

$$(-q_1(k+1) - q_2(k)) + q_1(k) + 2q_2(k+1) = 0$$
$$q_2(k+1) = -\frac{1}{2}q_1(k) + \frac{1}{2}q_2(k) + \frac{1}{2}q_1(k+1)$$

Now represent $x_1(k) := q_1(k)$, $x_2(k) := q_2(k)$, and $x_3(k) := q_1(k+1)$. Thus, the state space representation becomes

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} x_3(k) \\ -\frac{1}{2}x_1(k) + \frac{1}{2}x_2(k) + \frac{1}{2}x_3(k) \\ -x_2(k) - x_3(k) \end{bmatrix}$$

Exercise 5

Show that x^e is an equilibrium state of the system

$$x(k+1) = x(k) - \frac{g(x(k))}{g'(x(k))}$$

if and only if $g(x^e) = 0$.

Solution:
To solve for the equilibrium state of this discrete time system, we say that $x^e := x(k+1)$ and $x^e := x(k)$. Substitute this into the given system.

$$x^e = x^e - \frac{g(x^e)}{g'(x^e)}$$
$$\frac{g(x^e)}{g'(x^e)} = 0$$

Now, this relation is only true when

$$\begin{cases} g(x^e) = 0 \\ g'(x^e) \neq 0 \end{cases}$$

Therefore, we can say that x^e is an equilibrium state of this system if and only if $g(x^e) = 0$.

q.e.d

Exercise 6

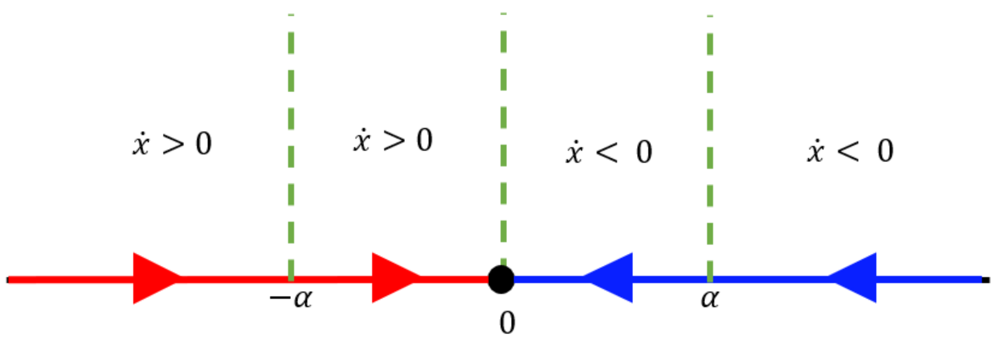
Draw the state portrait of the following nonlinear system

$$\dot{x} = -\alpha sgm(x)$$

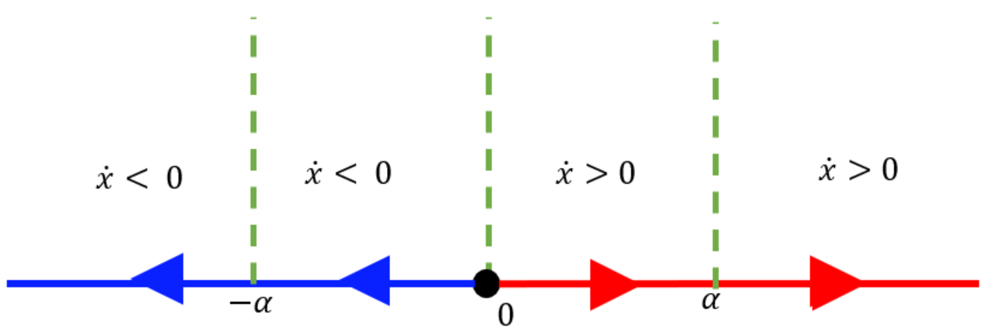
Solution:
This nonlinear function is equivalent to

$$\dot{x} = \begin{cases} -\alpha & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \alpha & \text{if } x < 0 \end{cases}$$

Thus, if $\alpha > 0$ the phase line of this system becomes



and if $\alpha < 0$ the phase line becomes



Exercise 7

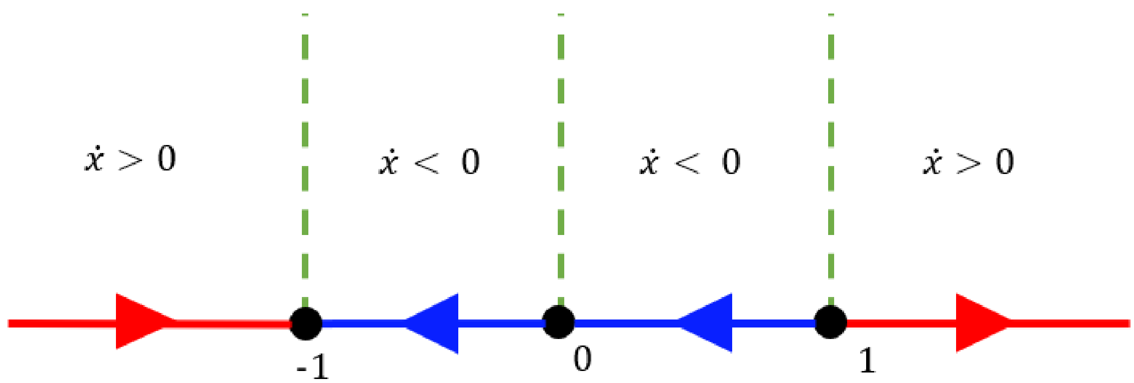
Draw the state portrait of the following nonlinear system

$$\dot{x} = x^4 - x^2$$

Solution:
This nonlinear function is equivalent to

$$\dot{x} = x^2(x+1)(x-1)$$

Thus, the phase line of this system becomes



Exercise 8

Obtain an explicit expression for all solutions of

$$\dot{x} = -x^3$$

Solution:
Solve this differential equation analytically

$$\frac{dx}{dt} = -x^3$$

$$-x^{-3}dx = dt$$

$$\int -x^{-3}dx = \int dt$$

$$\frac{1}{2x^2} + x_0 = t$$

Thus, the explicit solution for this differential equation is

$$\frac{1}{2x^2} + t + x_0 = 0$$

Exercise 9

Consider the Lorenz system deescribed by

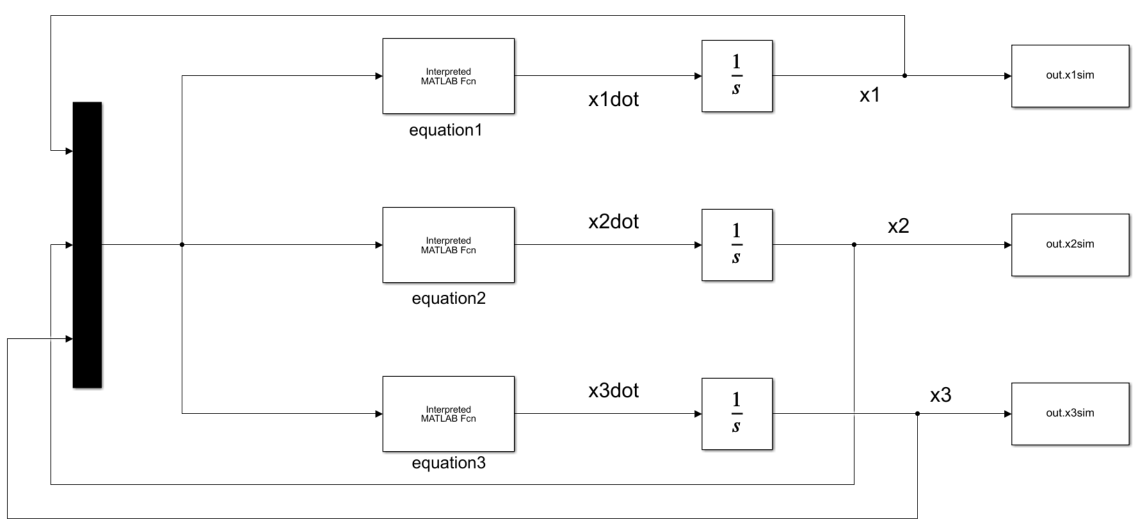
$$
\begin{aligned}
\dot{x}_1 &= \sigma(x_2 - x_1) \\
\dot{x}_2 &= r x_1 - x_2 - x_1 x_3 \\
\dot{x}_3 &= -b x_3 + x_1 x_2
\end{aligned}
$$

with $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$. Simulate this system with initial states

$$
\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 + eps \\ 0 \end{pmatrix}
$$

where eps is the floating point relative accuracy in MATLAB. Comment on your results for the itegration interval [0, 60].

Solution:
To simulate this nonlinear system equation, we first make a SIMULINK model which represents the system. The model is as follows.



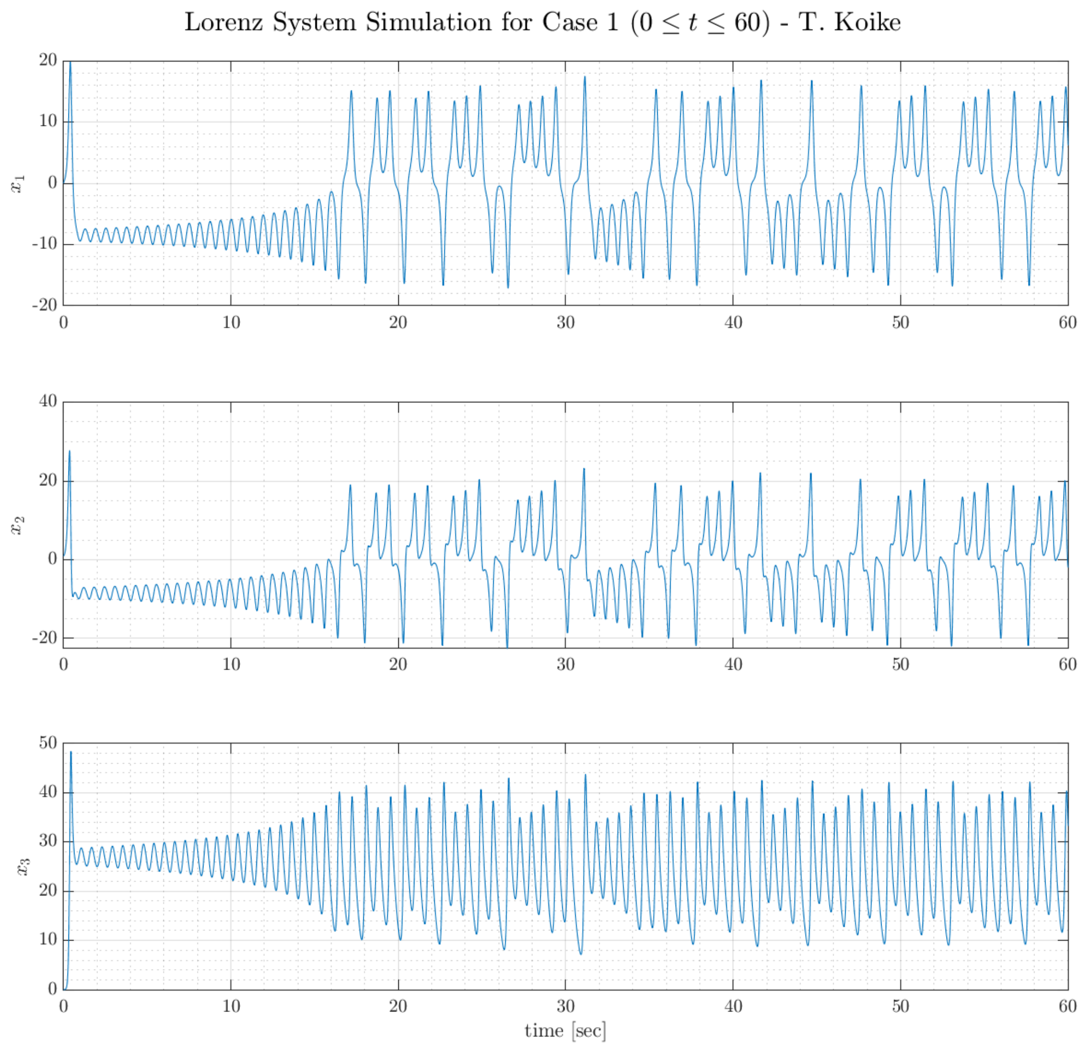
Now, calling the first initial condition set as Case 1 and the second one as Case 2, we simulate the system using the following MATLAB code.

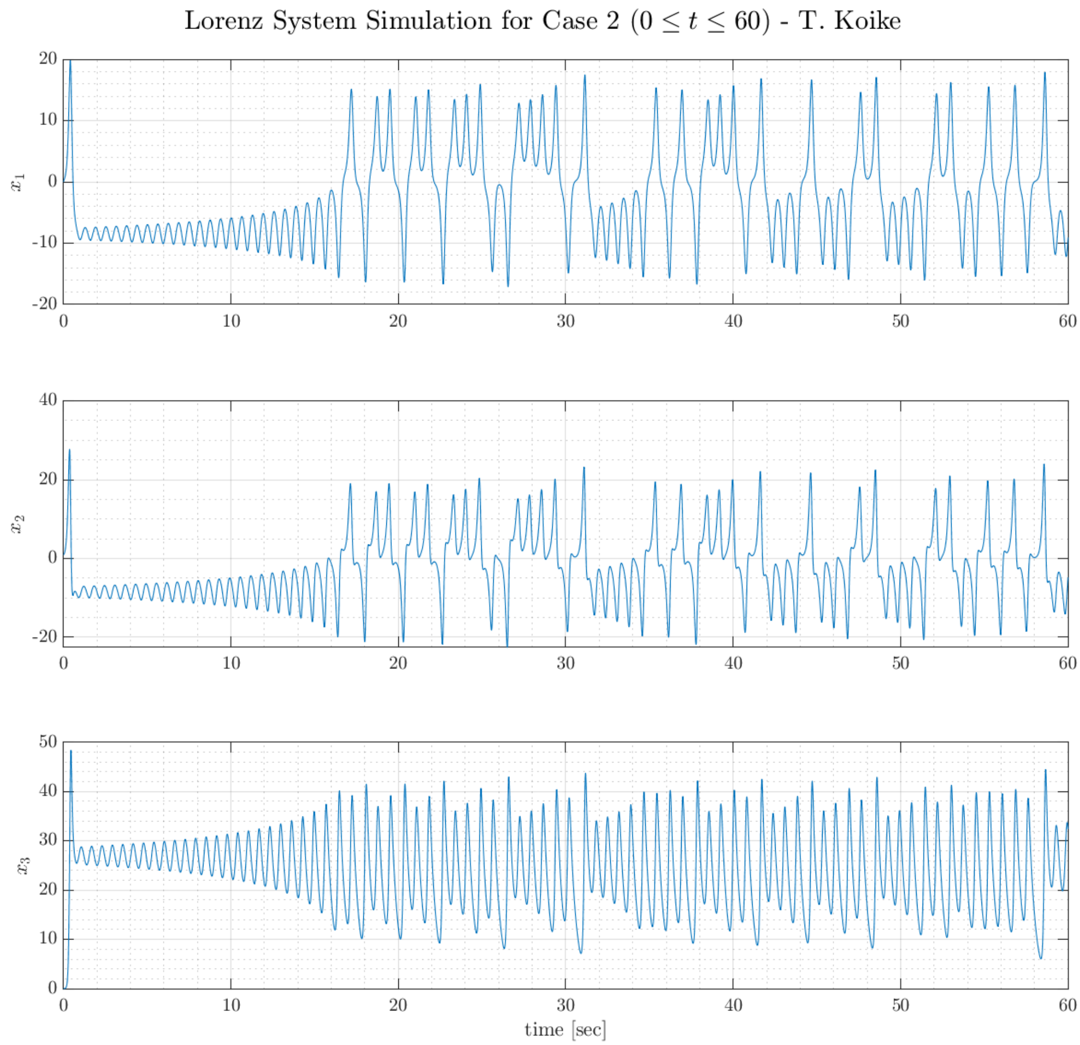
```
close all; clear all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2021-Spring\AAE666\matlab\hw1';
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
```

```
%%
% Constants
sigma = 10;
b = 8/3;
r = 28;

% Case 1
% — Initial conditions
x1_0 = 0;
x2_0 = 1;
x3_0 = 0;

% — Simulate
simout = sim("lorenzSystem.slx");

% — Data rendering
x1 = simout.x1sim.signals.values;
x2 = simout.x2sim.signals.values;
x3 = simout.x3sim.signals.values;
t  = simout.tout;

% — Plot
fig = figure("Renderer","painters","Position",[60 60 900 800]);
    subplot(3,1,1)
    plot(t, x1)
    grid on; grid minor; box on;
    ylabel('$x_1$')
    subplot(3,1,2)
    plot(t, x2)
    grid on; grid minor; box on;
    ylabel('$x_2$')
    subplot(3,1,3)
    plot(t, x3)
    grid on; grid minor; box on;
    ylabel('$x_3$')
    xlabel('time [sec]')
    title_string = 'Lorenz System Simulation for Case 1 ($0\leq t \leq 60$) — T. Koike';
    sgtitle(title_string)
saveas(fig, 'ex9_case1.png')
%%
% Case 2
% — Initial conditions
x1_0 = 0;
x2_0 = 1+eps;
```

```
x3_0 = 0;

% — Simulate
simout = sim("lorenzSystem.slx");

% — Data rendering
x1 = simout.x1sim.signals.values;
x2 = simout.x2sim.signals.values;
x3 = simout.x3sim.signals.values;
t  = simout.tout;

% — Plot
fig = figure("Renderer","painters","Position",[60 60 900 800]);
    subplot(3,1,1)
    plot(t, x1)
    grid on; grid minor; box on;
    ylabel('$x_1$')
    subplot(3,1,2)
    plot(t, x2)
    grid on; grid minor; box on;
    ylabel('$x_2$')
    subplot(3,1,3)
    plot(t, x3)
    grid on; grid minor; box on;
    ylabel('$x_3$')
    xlabel('time [sec]')
    title_string = 'Lorenz System Simulation for Case 2 ($0\leq t \leq 60$) — T. Koike';
    sgtitle(title_string)
saveas(fig, 'ex9_case2.png')
```

The simulation plots for each case is as follows.




Lorenz System Simulation for Case 2 ($0 \leq t \leq 60$) - T. Koike
x_1
x_2
x_3
time [sec]