Problem Set 2 Solution

AAE 532: Orbital Mechanics
MWF: 11:30-12:30
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Contents

Problem 1

After a complex route to the vicinity of the Moon, the two identical - the original - ARTEMIS spacecraft (P1 and P2) arrived in the lunar vicinity on August 23 and October 22, 2010, respectively. The spacecraft eventually inserted into lunar orbits on June 27 and July 17, 2011. The trajectories in the lunar vicinity prior to lunar orbit insertion were influenced significantly bt the gravity fields of other bodies, particularly the Earth and the Sun. The P1 path from arrival in the Moon vicinity to the lunar insertion point appears in the images below. Note that it is far from a familiar elliptical orbit. Define a system that is comprised of five particles. The law of gravity between each pair is the familiar inverse square law. Obviously, the planets are not truly aligned simultaneuosly, but assume that the Sun, spacecraft (s/c), and other bodies are collinear and positioned as indicated below:

Earth – Moon – s/c – Sun – Jupiter

Assume that a single spacecraft is instantaneously located along the transfer path such that the distance between the s/c and the Moon is 140,000km. The total mass of each ARTEMIS spacecraft is about 130kg. The distances of other planets from the Sun are assumed to be equal to the semi-major axis as listed in the Table of Constants located under Supplementary Documents on Brightspace.

(a) Locate the center of mass of the 5-particle system. Identify it on a sketch. Add unit vectors and appropriate position vectors.

(b) Write the vector differential equation for the motion of the s/c with respect to the center of mass, i.e., $\bar{r}_i = \bar{r}_{s/c}$. You should obtain an expression for the accelerations on the s/c, i.e., $\ddot{\bar{r}}_{s/c}$ = (sum of 4 terms). Assuming the alignment above, compute the accelerations on the s/c due to each of the other bodies. Include the directions. Which body produces the largest acceleration on the s/c? smallest? What is the descending order? Net acceleration in km/sec^2?

(c) Compare the relative size of the acceleration terms (gravitational forces) and their directions on the s/c. Is the order of influence what you expected? Which gravity term dominates? Do the acceleration terms seem consistent with your expectations?

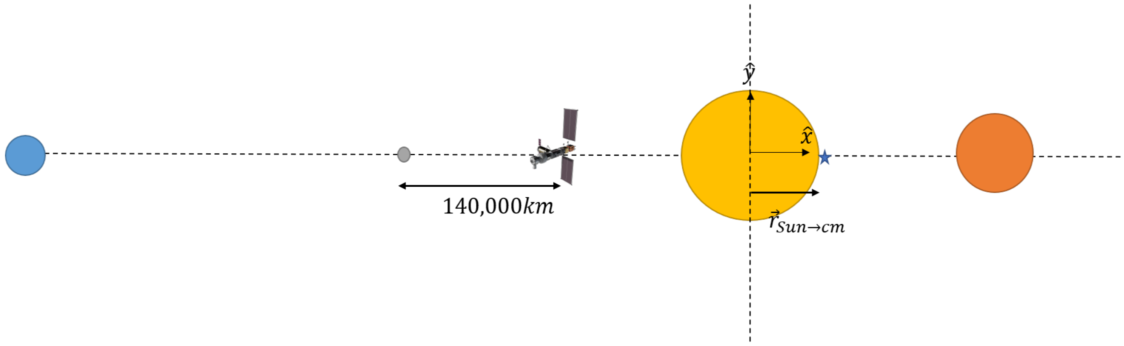

Figure 1: 5 Particles diagram

Problem 1 Solution

(a)

From Figure 1, all celestial bodies and the spacecraft are assumed to be collinear. If we put axes like in the Figure 1, we can write position vectors of bodies with respect to the Sun. And remember that the distance from the Sun is assumed to be the semi major axis.

$$\begin{aligned}
\bar{r}_{\odot\oplus} &= -149597898\ km\ \hat{x} \\
\bar{r}_{\odot☾} &= -149213498\ km\ \hat{x} \\
\bar{r}_{\odot s/c} &= -149073498\ km\ \hat{x} \\
\bar{r}_{\odot\odot} &= 0\ km\ \hat{x} \\
\bar{r}_{\odot ♃} &= 778279959\ km\ \hat{x}
\end{aligned}$$

Then we can get the position of center of mass with respect to the Sun by the following equation:

$$\bar{r}_{\odot c.m.} = \frac{\sum_{i=1}^{5} m_i \bar{r}_{\odot i}}{\sum_{i=1}^{5} m_i} = \frac{G\sum_{i=1}^{5} m_i \bar{r}_{\odot i}}{G\sum_{i=1}^{5} m_i} = \frac{\sum_{i=1}^{5} \mu_i \bar{r}_{\odot i}}{\sum_{i=1}^{5} \mu_i} \tag{1}$$

Since the table of constants provide the gravitational parameter $\mu = Gm$ instead of the mass itself, we want to match the units either by consistently using μ_i, or sticking to m_i. We will use μ_i by acquiring $\mu_{s/c}$ by:

$$\mu_{s/c} = m_{s/c}G = 130\ kg \times 6.67384\ e-20\ km^3/kg/s^2 = 8.675992\ e-18\ km^3/s^2$$

Plug all the values into Equation (1):

$$\begin{aligned}
\bar{r}_{\odot c.m.} &= \frac{\mu_\oplus \bar{r}_{\odot\oplus} + \mu_☾ \bar{r}_{\odot☾} + \mu_{s/c}\bar{r}_{\odot s/c} + \mu_\odot \bar{r}_{\odot\odot} + \mu_♃ \bar{r}_{\odot ♃}}{\mu_\oplus + \mu_☾ + \mu_{s/c} + \mu_\odot + \mu_♃} \\
&\approx 741930\ km\ \hat{x}
\end{aligned}$$

Note that the Sun's equatorial radius is about 695990km, so the center of mass of the 5 particle system would be right outside the solar sphere. We can now get position vectors of each body from the center of mass:

$$
\begin{aligned}
\bar{r}_{\oplus} &= \bar{r}_{\odot\oplus} - \bar{r}_{\odot c.m.} = -150339828\ km\ \hat{x} \\
\bar{r}_{☾} &= \bar{r}_{\odot ☾} - \bar{r}_{\odot c.m.} = -149955428\ km\ \hat{x} \\
\bar{r}_{s/c} &= \bar{r}_{\odot s/c} - \bar{r}_{\odot c.m.} = -149815428\ km\ \hat{x} \\
\bar{r}_{\odot} &= \bar{r}_{\odot\odot} - \bar{r}_{\odot c.m.} = -741930\ km\ \hat{x} \\
\bar{r}_{♃} &= \bar{r}_{\odot ♃} - \bar{r}_{\odot c.m.} = 777538029\ km\ \hat{x}
\end{aligned}
$$

(b)

The center of mass of the 5 particle system is non-accelerating from the conservation of linear momentum. So, we can apply the Netwon's second law to the vectors represented with the center of mass:

$$
\begin{aligned}
& i = s/c \\
& \bar{r}_{c.m.\to s/c} = \bar{r}_{s/c} \\
& \text{Newton's second law} \Rightarrow \\
& m_{s/c}\ddot{\bar{r}}_{s/c} = \bar{F}_T = \sum_{j=1,\ j\neq i}^{5} \left(-\frac{Gm_i m_j}{r_{ji}^3}\bar{r}_{ji}\right) = \sum_{j=1,\ j\neq s/c}^{5} \left(-\frac{Gm_{s/c} m_j}{r_{js/c}^3}\bar{r}_{js/c}\right) \Rightarrow \\
& \boxed{\ddot{\bar{r}}_{s/c}} = \sum_{j=1,\ j\neq s/c}^{5} \left(-\frac{Gm_j}{r_{js/c}^3}\bar{r}_{js/c}\right) = \sum_{j=1,\ j\neq s/c}^{5} \left(-\frac{\mu_j}{r_{js/c}^3}\bar{r}_{js/c}\right) \\
& \boxed{= -\frac{\mu_{\oplus}}{r_{\oplus s/c}^3}\bar{r}_{\oplus s/c} - \frac{\mu_{☾}}{r_{☾\ s/c}^3}\bar{r}_{☾\ s/c} - \frac{\mu_{\odot}}{r_{\odot s/c}^3}\bar{r}_{\odot s/c} - \frac{\mu_{♃}}{r_{♃ s/c}^3}\bar{r}_{♃ s/c}}
\end{aligned}
$$

Getting the spacecraft position vectors relative to each body:

$$\bar{r}_{\oplus s/c} = \bar{r}_{s/c} - \bar{r}_{\oplus} = -149815428\ km\ \hat{x} + 150339828\ km\ \hat{x} = 524400\ km\ \hat{x} \tag{2}$$

$$\bar{r}_{☾\ s/c} = \bar{r}_{s/c} - \bar{r}_{☾} = -149815428\ km\ \hat{x} + 149955428\ km\ \hat{x} = 140000\ km\ \hat{x} \tag{3}$$

$$\bar{r}_{\odot s/c} = \bar{r}_{s/c} - \bar{r}_{\odot} = -149815428\ km\ \hat{x} + 741930\ km\ \hat{x} = -149073498\ km\ \hat{x} \tag{4}$$

$$\bar{r}_{♃ s/c} = \bar{r}_{s/c} - \bar{r}_{♃} = -149815428\ km\ \hat{x} - 777538029\ km\ \hat{x} = -927353457\ km\ \hat{x} \tag{5}$$

Plug in to the above equation to get the acceleration due to each body.

$$
\boxed{
\begin{aligned}
\frac{\bar{F}_{\oplus}}{m_{s/c}} &= -\frac{\mu_{\oplus}}{r_{\oplus s/c}^3}\bar{r}_{\oplus s/c} = -1.4494\ e-6\ km/s^2\ \hat{x} \\
\frac{\bar{F}_{☾}}{m_{s/c}} &= -\frac{\mu_{☾}}{r_{☾\ s/c}^3}\bar{r}_{☾\ s/c} = -2.5014\ e-7\ km/s^2\ \hat{x} \\
\frac{\bar{F}_{\odot}}{m_{s/c}} &= -\frac{\mu_{\odot}}{r_{\odot s/c}^3}\bar{r}_{\odot s/c} = 5.9719\ e-6\ km/s^2\ \hat{x} \\
\frac{\bar{F}_{♃}}{m_{s/c}} &= -\frac{\mu_{♃}}{r_{♃ s/c}^3}\bar{r}_{♃ s/c} = 1.4734\ e-10\ km/s^2\ \hat{x}
\end{aligned}
}
$$

We can see that $\boxed{\text{Sun produces the largest acceleration}}$, $\boxed{\text{Jupiter produces the smallest acceleration}}$. $\boxed{\text{The descending order is Sun-Earth-Moon-Jupiter}}$. The net acceleration is:

$$\boxed{\begin{aligned}\ddot{\bar{r}}_{s/c} &= -\frac{\mu_\oplus}{r^3_{\oplus s/c}}\bar{r}_{\oplus s/c} - \frac{\mu_{☾}}{r^3_{☾\, s/c}}\bar{r}_{☾\, s/c} - \frac{\mu_\odot}{r^3_{\odot s/c}}\bar{r}_{\odot s/c} - \frac{\mu_{♃}}{r^3_{♃ s/c}}\bar{r}_{♃ s/c} \\ &= 4.2724\ e-6\ km/s^2\ \hat{x}\end{aligned}}$$

Also note that we can flip the sign of Equation (5) by using position vectors originating from the spacecraft. Just make sure that the bodies are pulling the spacecraft.

$$\begin{aligned}\ddot{\bar{r}}_{s/c} &= -\frac{\mu_\oplus}{r^3_{\oplus s/c}}\bar{r}_{\oplus s/c} - \frac{\mu_{☾}}{r^3_{☾\, s/c}}\bar{r}_{☾\, s/c} - \frac{\mu_\odot}{r^3_{\odot s/c}}\bar{r}_{\odot s/c} - \frac{\mu_{♃}}{r^3_{♃ s/c}}\bar{r}_{♃ s/c} \\ &= \frac{\mu_\oplus}{r^3_{s/c\oplus}}\bar{r}_{s/c\oplus} + \frac{\mu_{☾}}{r^3_{s/c☾}}\bar{r}_{s/c☾} + \frac{\mu_\odot}{r^3_{s/c\odot}}\bar{r}_{s/c\odot} + \frac{\mu_{♃}}{r^3_{s/c♃}}\bar{r}_{s/c♃}\end{aligned}$$

(c)

There are many ways we could look at the relative size of acceleration terms. Here we are going to look at the relative size of acceleration from each body to the net acceleration. Note that the relative size of acceleration from the Sun is over 100%. This is possible since the acceleration from the Earth and the Moon are in the opposite direction of acceleration from the Sun. The magnitude of net acceleration is smaller than the acceleration solely coming from the Sun.

$$\boxed{\begin{aligned}&\left|\frac{-\frac{\mu_\oplus}{r^3_{\oplus s/c}}\bar{r}_{\oplus s/c}}{\ddot{\bar{r}}_{s/c}}\right| = 33.93\% \\ &\left|\frac{-\frac{\mu_{☾}}{r^3_{☾\, s/c}}\bar{r}_{☾\, s/c}}{\ddot{\bar{r}}_{s/c}}\right| = 5.85\% \\ &\left|\frac{-\frac{\mu_\odot}{r^3_{\odot s/c}}\bar{r}_{\odot s/c}}{\ddot{\bar{r}}_{s/c}}\right| = 139.78\% \\ &\left|\frac{-\frac{\mu_{♃}}{r^3_{♃ s/c}}\bar{r}_{♃ s/c}}{\ddot{\bar{r}}_{s/c}}\right| = 0.0034\%\end{aligned}}$$

And we can see that the signs of the acceleration from each body are:

$$\boxed{\begin{aligned}&\text{Earth: Negative } \hat{x} \\ &\text{Moon: Negative } \hat{x} \\ &\text{Sun: Positive } \hat{x} \\ &\text{Jupiter: Positive } \hat{x}\end{aligned}}$$

The directions/signs of the acceleration match our intuition, as all bodies on the left to the spacecraft are pulling the spacecraft and thus negative acceleration, and vice versa for the bodies on the right to the spacecraft. Even though the Sun is really far away from the spacecraft, it exerts the biggest force on the spacecraft among the four celestial bodies.

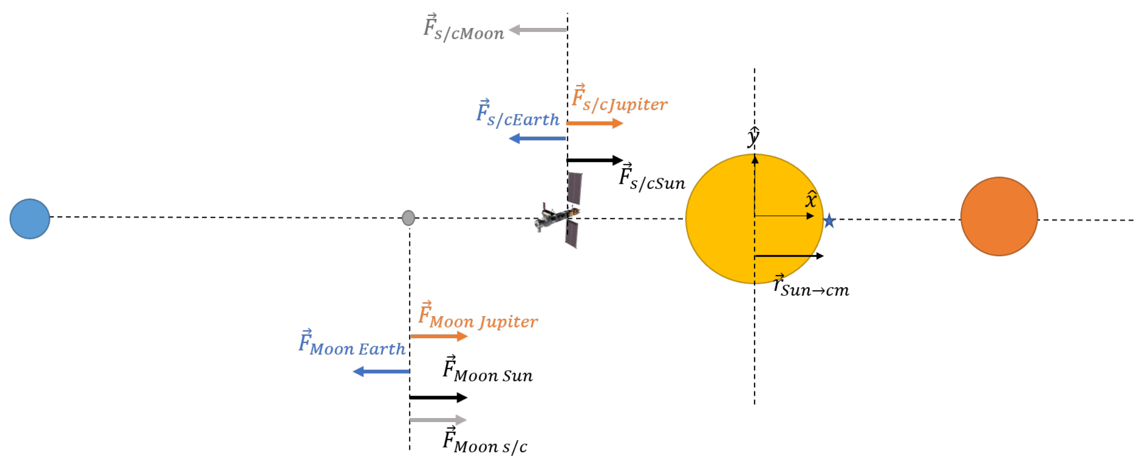

Figure 2: Force exerted on the spacecraft and the Moon [For Problem 2 (a)]

Problem 2

Return to the system in Problem 1.

(a) Write the expression for the acceleration of the spacecraft relative to the Moon due to the gravity of the Earth, Moon, Sun and Jupiter. Write this expression in the form $\ddot{\bar{r}}_{☾\to s/c}$ = (sum of terms). Label each of the following terms in this expression: dominant term, direct perturbing terms, indirect perturbing terms, net perturbing term.

(b) Compute the magnitude and direction of each of the terms in your expression for $\ddot{\bar{r}}_{☾\to s/c}$, as well as the net perturbing accelerations for each body and the total net acceleration. Include the directions. Does the 'dominant' Moon term also possess the largest magnitude? If not, which body contributes the largest individual magnitude terms? Is the largest magnitude term a direct or indirect term? Compare the magnitude and direction of the dominant acceleration and the net perturbing acceleration from each 'perturbing' body. Which 'perturbing' body has the largest impact? If the indirect perturbing acceleration terms are neglected, compare the magnitude of the dominant and the direct perturbing accelerations. Which body would have the largest impact? Do the indirect perturbing term matter?

(c) Assume that all the perturbing bodies are neglected. Given the values of the acceleration terms at this instant along the path, is it reasonable to model the motion of the spacecraft using a two-body problem (i.e., only Moon and s/c)? If a two-body model is not adequate, is there a three-body system that may provide a more reasonable model for the spacecraft motion, e.g., Moon/sc/Earth or Moon/sc/Sun? Would you rather use a four-body model? Which bodis? Why?

(d) Recast the problem and write the expression for the acceleration of the spacecraft relative to the Earth due to the gravity of the Earth, Moon, Sun and Jupiter. now, evaluate the dominant acceleration and each perturbing acceleration term. Is this model equally valid? Observing the terms, compare the values of the dominant acceleration and the net perturbing acceleration from each additional body. Given the assessment in (b), will any of your conclusions change with the formulation? Which model for the spacecraft motion is correct (a) or (d)? Why?

Problem 2 Solution

(a)

Now we seek to describe the dynamic evolution of the **relative** position vector $\ddot{\vec{r}}_{☾\, s/c}$. Since the basepoint of this vector (namely the Moon) is moving and accelerating, we cannot apply Newton's second law directly and must employ the relative formulation. First apply Newton's second law to both the spacecraft and the Moon:

$$\ddot{\bar{r}}_{s/c} = \frac{\mu_{\oplus}}{r^3_{s/c\oplus}}\bar{r}_{s/c\oplus} + \frac{\mu_{☾}}{r^3_{s/c☾}}\bar{r}_{s/c☾} + \frac{\mu_{\odot}}{r^3_{s/c\odot}}\bar{r}_{s/c\odot} + \frac{\mu_{♃}}{r^3_{s/c♃}}\bar{r}_{s/c♃} \tag{6}$$

$$\ddot{\bar{r}}_{☾} = \frac{\mu_{☾}}{r^3_{☾\,\oplus}}\bar{r}_{☾\,\oplus} + \frac{\mu_{☾}}{r^3_{☾\,s/c}}\bar{r}_{☾\,s/c} + \frac{\mu_{\odot}}{r^3_{☾\,\odot}}\bar{r}_{☾\,\odot} + \frac{\mu_{♃}}{r^3_{☾\,♃}}\bar{r}_{☾\,♃} \tag{7}$$

And then subtract Equation (7) from (6) to get:

$$\ddot{\vec{r}}_{☾\,s/c} + \underbrace{\frac{G(m_{s/c} + m_{☾})}{r^3_{☾\,s/c}}\vec{r}_{☾\,s/c}}_{dominant} = G\sum_{\substack{j=1\\ j\neq s/c,☾}}^{N} m_j\left(\underbrace{\frac{\vec{r}_{s/cj}}{r^3_{s/cj}}}_{dir.\ perturbing} - \underbrace{\frac{\vec{r}_{☾\,j}}{r^3_{☾\,j}}}_{indir.\ perturbing}\right) \tag{8}$$

where the red term is referred to as the dominant acceleration, purple is the direct perturbing acceleration, and blue is the indirect perturbing acceleration. Since, we seek the differential equation of the relative vector, we can update our free-body diagram so that the forces on the Moon (initial point of position vector) and the forces on the spacecraft (terminating point of position vector) are present.

(b)

$$
\begin{aligned}
-\frac{G(m_{s/c}+m_{☾})}{r^3_{☾\, s/c}}\vec{r}_{☾\, s/c} &= -2.5014\times 10^{-7}\ \hat{x}\ \ \text{km/s}^2 \quad \text{dominant acceleration}\\
Gm_{\oplus}\frac{\vec{r}_{s/c\oplus}}{r^3_{s/c\oplus}} &= -1.4495\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Earth direct perturbing}\\
Gm_{\oplus}\frac{\vec{r}_{☾\,\oplus}}{r^3_{☾\,\oplus}} &= -2.6976\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Earth indirect perturbing}\\
Gm_{\oplus}\frac{\vec{r}_{s/c\oplus}}{r^3_{s/c\oplus}} - Gm_{\oplus}\frac{\vec{r}_{☾\,\oplus}}{r^3_{☾\,\oplus}} &= 1.2481\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Earth net perturbing}\\
Gm_{\odot}\frac{\vec{r}_{s/c\odot}}{r^3_{s/c\odot}} &= 5.9719\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Sun direct perturbing}\\
Gm_{\odot}\frac{\vec{r}_{☾\,\odot}}{r^3_{☾\,\odot}} &= 5.9606\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Sun indirect perturbing}\\
Gm_{\odot}\frac{\vec{r}_{s/c\odot}}{r^3_{s/c\odot}} - Gm_{\odot}\frac{\vec{r}_{☾\,\odot}}{r^3_{☾\,\odot}} &= 1.1201\times 10^{8}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Sun net perturbing}\\
Gm_{♃}\frac{\vec{r}_{s/c♃}}{r^3_{s/c♃}} &= 1.4734\times 10^{-10}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Jupiter direct perturbing}\\
Gm_{♃}\frac{\vec{r}_{☾\,♃}}{r^3_{☾\,♃}} &= 1.4730\times 10^{-10}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Jupiter indirect perturbing}\\
Gm_{♃}\frac{\vec{r}_{s/c♃}}{r^3_{s/c♃}} - Gm_{♃}\frac{\vec{r}_{☾\,♃}}{r^3_{☾\,♃}} &= 4.4478\times 10^{-14}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Jupiter net perturbing}\\
\ddot{\vec{r}}_{☾\, s/c(Pert)} &= 1.2593\times 10^{-6}\ \ \text{km/s}^2 \quad \text{Total perturbing acceleration}\\
\ddot{\vec{r}}_{☾\, s/c} &= 1.0091\times 10^{-6}\ \ \text{km/s}^2 \quad \text{Total acceleration}
\end{aligned}
$$

Note that positive $\hat{x}$ direction is to the right, and negative $\hat{x}$ is to the left in the Figure 1. The 'dominant' acceleration from the Moon is not the largest. We can see that the largest magnitude term comes from the Sun from, the direct term on the spacecraft. Comparing the magnitude between the dominant acceleration and the net perturbing acceleration, the descending order is:

$$\boxed{\text{Earth(perturbing}, +\hat{x})/\text{Moon(dominant}, -\hat{x})/\text{Sun(perturbing}, +\hat{x})/\text{Jupiter(perturbing}, +\hat{x})}$$

The largest net perturbing acceleration comes from the Earth. If we only compare the direct perturbing acceleration, the order changes to:

$$\boxed{\text{Sun(perturbing}, +\hat{x})/\text{Earth(perturbing}, -+\hat{x})/\text{Moon(dominant}, -\hat{x})/\text{Jupiter(perturbing}, +\hat{x})}$$

In this case, the Sun has the largest impact. And we can see that the direct perturbing accelerations from the Sun and the Earth are bigger than the 'dominant' acceleration from the Moon. Comparing the two results, we can see that the indirect perturbing terms matter.

(c)

Intuitively, 140,000km is a long distance from the Moon to approximate it as a two-body problem. Looking at the individual acceleration terms, the net perturbing acceleration from the Earth is greater than the 'dominant' acceleration. So, it seems two-body is not adequate for this problem. If we should choose the third body to incorporate to the approximation, we should choose the Earth. And then if we move on to four-body model, the Sun comes next in line for the largest perturbing acceleration and we must incorporate the Sun to the model. The level of approximation would depend on the problem that we are looking at. If we are looking for a higher-fidelity trajectory, the results from the four-body model will provide better results compared to three-body model. This depends on the relative size of the perturbing accelerations too. Since the acceleration from Jupiter is relatively small compared to the rest of the bodies, including will Jupiter would result in more accurate result but that might not be noticeable in a short amount of time.

(d)

More practice with Newton's second law! First apply Newton's second law to both the spacecraft and the Earth:

$$\ddot{\bar{r}}_{\oplus} = \frac{\mu_{\oplus}}{r^3_{\oplus☾}}\bar{r}_{\oplus☾} + \frac{\mu_{\oplus}}{r^3_{\oplus s/c}}\bar{r}_{\oplus s/c} + \frac{\mu_{\odot}}{r^3_{\oplus\odot}}\bar{r}_{\oplus\odot} + \frac{\mu_{♃}}{r^3_{\oplus ♃}}\bar{r}_{\oplus ♃} \tag{10}$$

And then subtract Equation (10) from (6) to get:

$$\ddot{\vec{r}}_{\oplus s/c} + \underbrace{\frac{G(m_{s/c}+m_{\oplus})}{r^3_{\oplus s/c}}\vec{r}_{\oplus s/c}}_{dominant} = G\sum_{\substack{j=1\\ j\neq s/c,\oplus}}^{N} m_j\left(\underbrace{\frac{\vec{r}_{s/cj}}{r^3_{s/cj}}}_{dir.\ perturbing} - \underbrace{\frac{\vec{r}_{\oplus j}}{r^3_{\oplus j}}}_{indir.\ perturbing}\right) \tag{11}$$

$$\boxed{\begin{aligned}
-\frac{G(m_{s/c}+m_{\oplus})}{r^3_{\oplus s/c}}\vec{r}_{\oplus s/c} &= -1.4495\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{dominant acceleration}\\
Gm_{\oplus}\frac{\vec{r}_{s/c☾}}{r^3_{s/c☾}} &= -2.5014\times 10^{-7}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Moon direct perturbing}\\
Gm_{☾}\frac{\vec{r}_{\oplus☾}}{r^3_{\oplus☾}} &= 3.3180\times 10^{-8}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Moon indirect perturbing}\\
Gm_{\oplus}\frac{\vec{r}_{s/c☾}}{r^3_{s/c☾}} - Gm_{☾}\frac{\vec{r}_{\oplus☾}}{r^3_{\oplus☾}} &= -2.8332\times 10^{-7}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Moon net perturbing}\\
Gm_{\odot}\frac{\vec{r}_{s/c\odot}}{r^3_{s/c\odot}} &= 5.9719\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Sun direct perturbing}\\
Gm_{\odot}\frac{\vec{r}_{\oplus\odot}}{r^3_{\oplus\odot}} &= 5.9301\times 10^{-6}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Sun indirect perturbing}\\
Gm_{\odot}\frac{\vec{r}_{s/c\odot}}{r^3_{s/c\odot}} - Gm_{\odot}\frac{\vec{r}_{\oplus\odot}}{r^3_{\oplus\odot}} &= 4.1794\times 10^{8}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Sun net perturbing}\\
Gm_{♃}\frac{\vec{r}_{s/c♃}}{r^3_{s/c♃}} &= 1.4734\times 10^{-10}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Jupiter direct perturbing}\\
Gm_{♃}\frac{\vec{r}_{\oplus♃}}{r^3_{\oplus♃}} &= 1.4718\times 10^{-10}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Jupiter indirect perturbing}\\
Gm_{♃}\frac{\vec{r}_{s/c♃}}{r^3_{s/c♃}} - Gm_{♃}\frac{\vec{r}_{\oplus♃}}{r^3_{\oplus♃}} &= 1.6650\times 10^{-13}\ \hat{x}\ \ \text{km/s}^2 \quad \text{Jupiter net perturbing}\\
\ddot{\vec{r}}_{\oplus s/c(Pert)} &= -2.4153\times 10^{-7}\ \ \text{km/s}^2 \quad \text{Total perturbing acceleration}\\
\ddot{\vec{r}}_{\oplus s/c} &= -1.6910\times 10^{-6}\ \ \text{km/s}^2 \quad \text{Total acceleration}
\end{aligned}}$$

Mathematically, the two models relative to the Moon and to the Earth are equally valid. Comparing the magnitude between the dominant acceleration and the net perturbing acceleration, the descending order is:

$$\boxed{\text{Earth(dominant}, -\hat{x})/\text{Moon(perturbing}, -\hat{x})/\text{Sun(perturbing}, +\hat{x})/\text{Jupiter(perturbing}, +\hat{x})}$$

In this case, the 'dominant' term from the Earth actually is the largest magnitude among all. Even though both formulations of equations are correct, the latter might be preferred if we are going to use two-body approximation of the dynamics. In (a), the 'dominant' acceleration from the Moon was smaller than the perturbing force from the Earth. But in the new formulation, the 'dominant' acceleration form the Earth is actually the largest. If we use a two-body approximation for the central body, the new formulation with the Earth as the center will produce better results.

Problem 3

The dwarf planet Pluto has 5 known moons. The largest moon, Charon, is nearly half the size of Pluto and is the largest known moon in comparison to its parent body. Assume the Pluto-Charon system is modeled as an isolated two-body problem for the motion of Charon relative to Pluto due to the mutual gravity. Ignore all other forces.

(a) Sketch the system and define appropriate unit vectors; let $\hat{\imath}, \hat{\jmath}, \hat{k}$ be an inertial set of unit vectors such that $\hat{\imath}$ is parallel to $\bar{r}_{PC}$ at the initial time. Locate the center of mass and define position vectors for each object with respect to the fixed center of mass. Is the cm outside the radius of Pluto?

(b) Let $\bar{r}_{PC} = \bar{r}$ be the relative position of Charon with respect to Pluto. Write the kinematic expressions for the relative position and velocity for the motion of Charon relative to Pluto, that is, $\bar{r}, \bar{v}$ in terms of rotating unit vectors, $\hat{r}, \hat{\theta}$. At $t = 0$, the inertial velocities are known such that $\dot{\bar{r}}_C = 0.211319 km/s \hat{\jmath}$ and $\dot{\bar{r}}_P = -0.025717 km/s \hat{\jmath}$. Determine angular velocity for the motion of Charon relative to Pluto.

(c) Determine the system linear momentum; use this result to compute the velocity of the system center of mass. Does this result make sense?

(d) Determine the constant $\bar{C}_3$ for this system. What are the correct units?

(e) Evaluate the energy constant C_4; of course, include the units!

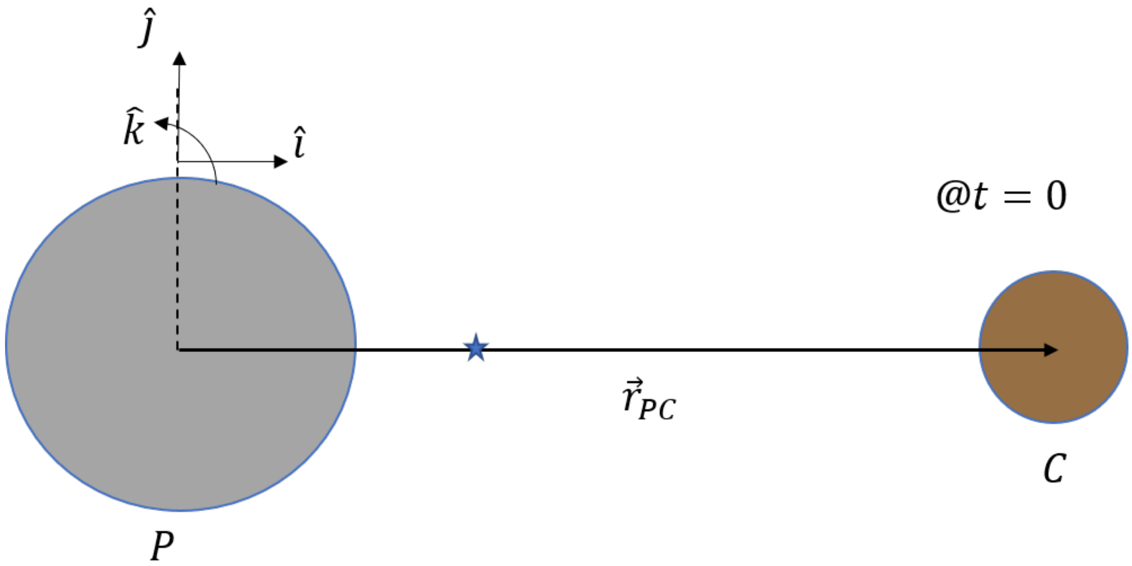

Figure 3: Pluto Charon Diagram

Problem 3 Solution

(a)

First, the center of mass with respect to Pluto can be computed by:

$$\bar{r}_{PP} = 0\ \hat{i}$$
$$\bar{r}_{PC} = 19596\ km\ \hat{i}$$
$$\boxed{\bar{r}_{Pc.m.} = \frac{\sum_{i=1}^{2} \mu_i \bar{r}_{Pi}}{\sum_{i=1}^{2} \mu_i} = 2126.392\ km\ \hat{i}}$$

The position vectors with respect to the center of mass are:

$$\boxed{\begin{aligned} \bar{r}_{c.m.P} &= \bar{r}_{PP} - \bar{r}_{Pc.m.} = -2126.392\ km\ \hat{i} \\ \bar{r}_{c.m.C} &= \bar{r}_{PC} - \bar{r}_{Pc.m.} = 17469.608\ km\ \hat{i} \end{aligned}}$$

$\boxed{\text{And the center of mass is outside the radius of Pluto.}}$

(b)

The inertial velocity is related to the rotating velocity by transport theorem (basic kinematic equation):

$$\frac{{}^{I}d\bar{r}}{dt} = \frac{{}^{R}d\bar{r}}{dt} +^{I} \bar{\omega}^{R} \times \bar{r} = \dot{r}\hat{r} + (\dot{\theta}\hat{k} + r\hat{r}) = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

A number of observations are in order:

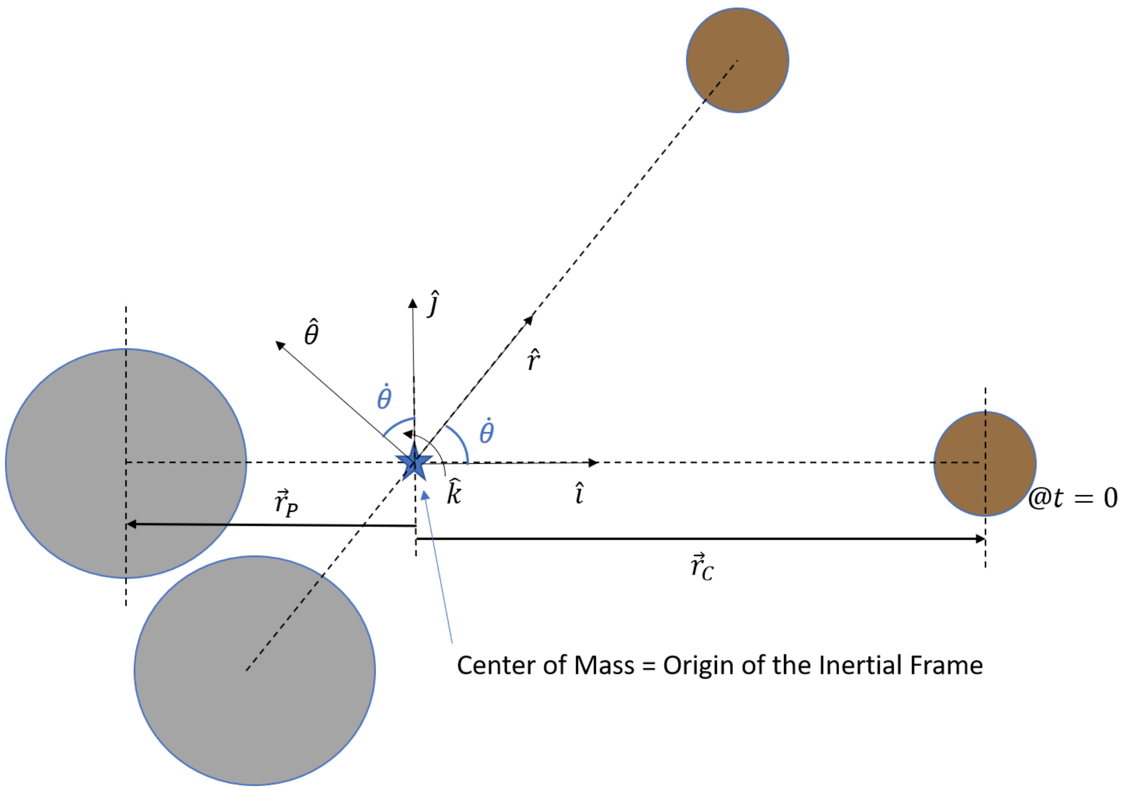

Figure 4: Inertial frame and rotating frame

1. Firstly, the vectors employed in the expression above are *relative vectors*, indicating the dynamical behavior of the relative position vector $\vec{r}_{PS}$ over time.
2. Secondly, the velocity components in the rotating frame are expressed in terms of the radial velocity $\dot{r}$ and transverse velocity $r\dot{\theta}$. The magnitude of the velocity expressed in the rotating frame must be equal to the magnitude of the velocity in the standard frame $\hat{\imath} - \hat{\jmath} - \hat{k}$.
3. Thirdly, the inertial velocities are intially give to be in the $\hat{\jmath}$ frame, indicating that there is no radial velocity at that given time (thus $\dot{r} = 0$). We can use this fact to solve directly for angular velocity $\dot{\theta}$ which are the same for both primaries relative to the center of mass.

Using the given inertial velocity at the initial time:

$$\text{at } t = 0, \;\; \hat{\imath} = \hat{r}, \; \hat{\jmath} = \hat{\theta}$$

$$^I\dot{\vec{r}} = ^I\dot{\vec{r}}_C - ^I\dot{\vec{r}}_P = 0.211319 \; km/s \; \hat{\jmath} + 0.025717 \; km/s \; \hat{\jmath}$$

$$= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} = \dot{r}\hat{\imath} + r\dot{\theta}\hat{\jmath} \Rightarrow$$

$$r\dot{\theta} = 0.211319 \; km/s \; + 0.025717 \; km/s = 0.237036 km/s$$

$$\boxed{\dot{\theta} = \frac{0.237036 km/s}{19596 km} = 1.2096 \; e - 5 \; rad/s}$$

(c)

Note that the the origin of the inertial frame coincides with the center of mass of the two-particle system. This means that since we do not have any external forces, the linear momentum of the system should be zero. However when we compute the linear momentum, it is a vector with a "small" non-zero number in $\hat{j}$ direction.

$$\underbrace{G \cdot \bar{C}_2}_{G\cdot(\text{linear momentum})} = \sum_{i=1}^{2} G m_i \dot{\bar{r}}_i = \sum_{i=1}^{2} \mu_i \dot{\bar{r}}_i = \mu_P {}^I\dot{\bar{r}}_P + \mu_C {}^I\dot{\bar{r}}_C = 0.00456\ km^4/s^3\ \hat{j}$$

or divide by G to get the linear momentum $\Rightarrow$

$$\underbrace{\bar{C}_2}_{\text{linear momentum}} = 6.8345\ e + 16\ kg \cdot km/s\ \hat{j}$$

And the linear momentum of a system is linked to the velocity of the center of mass by the following equation:

$$\sum_{i=1}^{2} m_i {}^I\dot{\bar{r}}_{c.m.} = \sum_{i=1}^{2} m_i \dot{\bar{r}}_i \Rightarrow$$

$${}^I\dot{\bar{r}}_{c.m.} = \frac{\sum_{i=1}^{2} m_i \dot{\bar{r}}_i}{\sum_{i=1}^{2} m_i} = 4\ e - 6\ km/s\hat{j} \approx \bar{0}$$

Without the external force, we know the center of mass should be stationary. But the computed velocity is a non-zero number, which we could interpret as a numerical error. We notice that the magnitude of this value is 4e-6, the order of which is the same to the last digit of the numbers given for the inertial velocities. We can thus deduce that this non-zero number is coming from the numerical error (limits on the significant digits that we have) associated with given numbers. If they were represented with simple ratios, the linear momentum $\bar{C}_2$ as well as the velocity of the center or mass would be equal to zero vectors. In short,

$$\boxed{\begin{aligned} \underbrace{\bar{C}_2}_{\text{linear momentum}} &= \bar{0} \\ {}^I\dot{\bar{r}}_{c.m.} = {}^I\bar{v}_{c.m.} &= \bar{0} \end{aligned}}$$

(d)

$$\begin{aligned} \boxed{\bar{C}_3} &= \sum_{i=1}^{2} m_i(\bar{r}_i \times^I \dot{\bar{r}}_i) = \sum_{i=1}^{2} \frac{\mu_i}{G}(\bar{r}_i \times^I \dot{\bar{r}}_i) \\ &= \frac{\mu_P}{G}(\bar{r}_P \times {}^I\dot{\bar{r}}_P) + \frac{\mu_C}{G}(\bar{r}_C \times {}^I\dot{\bar{r}}_C) \\ &= \frac{\mu_P}{G}(-2126.392\ km\ \hat{\imath} \times (-0.025721\ km/s\ \hat{j})) + \frac{\mu_C}{G}(17469.608\ km\ \hat{\imath} \times (0.211314\ km/s\ \hat{j})) \\ &\boxed{= 7.4134\ e + 24\ kg \cdot km^2/s\ \hat{k}} \end{aligned}$$

The unit should be $kg \cdot km^2/s$. And note that $\bar{C}_3$ is a vector, so we should include the direction as well.

(e)

$$
\begin{aligned}
C_4 = T - U &= \sum_{i=1}^{N} m_i \left(\frac{1}{2} {}^I\dot{\vec{r}}_i \cdot {}^I\dot{\vec{r}}_i \right) - \frac{1}{2} G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{r_{ij}} \\
&= \frac{1}{2} m_P ({}^I\dot{\vec{r}}_P \cdot {}^I\dot{\vec{r}}_P) + \frac{1}{2} m_C ({}^I\dot{\vec{r}}_C \cdot {}^I\dot{\vec{r}}_C) - G \frac{m_P m_C}{r_{PC}} \\
&= \boxed{-4.4841\ e + 19 \ \ \text{kg*km}^2/\text{s}^2}
\end{aligned}
$$

We can interpret this negative energy value such that Charon will continue to orbit around Pluto in this configuration and won't escape without additional forces. More insight into orbital energy will follow in later classes!