P57b

For each of the following functions $f\colon \mathcal{D} \mapsto \mathbb{R}$ determine whether a minimum and/or an infimum of $f(\mathcal{D})$ exists and explain why or why not Weierstrass's theorem applies:

i) $\mathcal{D} = (-1, 1)$, $f(x) = x^2$.

ii) $\mathcal{D} = (1, 2]$, $f(x) = \dfrac{1}{1-x}$.

iii) $\mathcal{D} = [0, 1]$, $f(0) = 0, f(x) = 1, x \in (0, 1]$.

i) $D=(-1,1)$, $f(x)=x^2$.

D is not closed, so Weierstrass's theorem does not apply

But it is clear that $f(x)>0$ $\forall x\neq 0$ and $f(0)=0$.

Hence $x^*=0$ is the unique global minimizer

(ii) $D=(1,2]$ $f(x)=\dfrac{1}{1-x}$

D is not closed, so Weierstrass's theorem does not apply

Note that $f(D)=(-\infty,1]$. Since $f(D)$ is not bounded from below, there is no minimum.

(iii) $D=[0,1]$, $f(0)=0$, $f(x)=1$, $x\in(0,1]$

Since $f(D)$ is bounded from below, an infimum exists. Also the infimum belongs to $f(D)$ so a minimum exist, $x^*=0$

Weierstrass's theorem does not apply since the function f is not continuous, although D is compact

PG06

Determine $\mathrm{vcone}(\mathcal{D},(x_0,y_0))$ for the following sets $\mathcal{D}\subset\mathbb{R}^2$ and $(x_0,y_0)\in\mathcal{D}$:

i) $\mathcal{D}=\{(x,y)\colon y\geq 0\}$, $(x_0,y_0)=(4,0)$.

ii) $\mathcal{D}=\{(x,y)\colon x^2+y^2\leq 1\}$, $(x_0,y_0)=(1,0)$.

iii) $\mathcal{D}=\{(x,y)\colon x^2+y^2=1\}$, $(x_0,y_0)=(1,0)$.

iv) $\mathcal{D}=\{(x,y)\colon y\geq x^2\}$, $(x_0,y_0)=(3,9)$.

i) $D = \{(x,y) : y \geq 0\}$ and $(x_0,y_0) = (4,0)$

$vcone(D,(4,0)) = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : \xi_2 \geq 0, \xi_1 \in \mathbb{R}\}$

ii) $D = \{(x,y) : x^2+y^2 \leq 1\}$ and $(x_0,y_0) = (1,0)$

$vcone(D,(1,0)) = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : \xi_1 < 0, \xi_2 \in \mathbb{R}\} \cup \{(0,0)\}$

iii) $D = \{(x,y) : x^2+y^2 = 1\}$ and $(x_0,y_0) = (1,0)$

$vcone(D,(1,0)) = \{(0,0)\}$

iv) $D = \{(x,y) : y \geq x^2\}$ and $(x_0,y_0) = (3,9)$

$vcone(D,(3,9)) = \{(\xi_1,\xi_2) \in \mathbb{R}^2 : \xi_2 > 6\xi_1\} \cup \{(0,0)\}$

Let $f\colon \mathbb{R}^2 \mapsto \mathbb{R}$ be given by $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. Evaluate $\mathrm{D}_+ f((0,0); (\xi_1, \xi_2))$.

$$f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$$

$$D_+ f((0,0); (\xi_1, \xi_2)) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \left[f((0,0) + \alpha(\xi_1, \xi_2)) - f(0,0) \right]$$

$$= \lim_{\alpha \downarrow 0} \frac{1}{\alpha} \sqrt{\alpha^2 \xi_1^2 + \alpha^2 \xi_2^2} \qquad \alpha > 0$$

so

$$D_+ f((0,0); (\xi_1, \xi_2)) = \lim_{\alpha \downarrow 0} \frac{\alpha}{\alpha} \sqrt{\xi_1^2 + \xi_2^2} = \sqrt{\xi_1^2 + \xi_2^2}$$

P68

Minimize the function $f : \mathcal{D} \to \mathbb{R}$

$$f(x_1, x_2) = x_1^3 + x_2^3$$

where $\mathcal{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0,\ x_2 \geq 0\}$.

Note that since $x_1 \geq 0$ and $x_2 \geq 0$ it follows that

$$x_1^3 \geq 0 \quad \text{and} \quad x_2^3 \geq 0$$

and

$$f(x_1, x_2) \geq 0 \quad \forall \ (x_1, x_2) \in D$$

Furthermore, $f((0,0)) = 0$ Hence $x^* = (0,0)$ is the global minimizer of f.

P95

Assuming steady level flight with a quadratic drag polar, consider the propulsive thrust given by

$$T = \frac{1}{2}\rho V^2 S C_{D_{\text{par}}} + \frac{KW^2}{\frac{1}{2}\rho V^2 S},$$

where ρ is air density, V is aircraft velocity, $C_{D_{\text{par}}}$ is the zero-lift (parasitic) drag coefficient, K is the drag polar constant, and S is wing surface area. The drag coefficient C_D is given by the drag polar

$$C_D = C_{D_{\text{par}}} + KC_L^2,$$

the lift coefficient is

$$C_L = \frac{W}{\frac{1}{2}\rho V^2 S},$$

and the lift-to-drag ratio is C_L/C_D. Consider the problem of finding the aircraft velocity V that minimizes the thrust T. Determine whether this problem is convex, and find all local and global minimizers and the corresponding values of T, C_L, C_D, and C_L/C_D.

The problem can be formulated as

$$\min_{V>0} f(V), \qquad f(V) = \frac{1}{2}\rho V^2 S C_{D_{par}} + \frac{kW^2}{\frac{1}{2}\rho V^2 S}$$

Let's show first that $f(V)$ is convex in V. Note that $D = \{V \in \mathbb{R} : V > 0\}$ is a convex set

$$f''(V) = \rho S C_{D_{par}} + 6\frac{kW^2}{\frac{1}{2}\rho V^4 S} > 0 \qquad \forall V \in D$$

So f is strictly convex in V

Furthermore, $\lim_{V \to \infty} f(V) = +\infty$ and f has a unique minimizer that can be found by setting

$$f'(V) = 0$$

$$f'(V) = \rho V S C_{D_{par}} - 2\frac{kW^2}{\frac{1}{2}\rho V^3 S} = 0$$

or $$V^4 = \frac{4kW^2}{\rho^2 S^2 C_{D_{par}}} \Rightarrow V = \sqrt[4]{\frac{4kW^2}{\rho^2 S^2 C_{D_{par}}}}$$

Then,

$$T = \frac{1}{2}\rho \sqrt{\frac{4kW^2}{\rho^2 S^2 C_{D_{par}}}}\, S C_{D_{par}}$$

$$+ \frac{kW^2}{\frac{1}{2}\rho \sqrt{\frac{4kW^2}{\rho^2 S^2 C_{D_{par}}}}\, S}$$

or

$$T = \sqrt{k C_{D_{par}}}\, W + \sqrt{k C_{D_{par}}}\, W$$

$$= 2\sqrt{k C_{D_{par}}}\, W$$

or

$$C_L = \frac{W}{\frac{1}{2}\rho \sqrt{\frac{4kW^2}{\rho^2 S^2 C_{D_{par}}}}\, S} = \sqrt{\frac{C_{D_{par}}}{k}}$$

$$C_D = C_{D_{par}} + kC_L^2 = 2\, C_{D_{par}}$$

and finally,

$$\frac{C_L}{C_D} = \frac{\sqrt{\frac{C_{D_{par}}}{k}}}{2\, C_{D_{par}}} = \frac{1}{2}\sqrt{\frac{1}{k C_{D_{par}}}}$$

P71

Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(y, z) = (z - py^2)(z - qy^2)$$

where $0 < p < q$.

(a) Show that $x_0 = (0, 0)$ is a local minimizer of f along every line that passes through $(0, 0)$, that is, for all $h \in \mathbb{R}^2$, the function $g(a) = f(x_0 + ah)$ is minimized by $a = 0$.

(b) Show that $f'(x_0) = 0$.

(c) Show that x_0 is not a local minimizer of f. (Hint: If $p < m < q$, then $f(y, my^2) < 0$ for $y \neq 0$ while $f(0, 0) = 0$.)

(d) Plot the function to illustrate why despite the fact that x_0 is the minimizer along every direction, it is not a local minimizer.

i) Take $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Along this direction,

$$g(\alpha) = f(x_0 + \alpha\xi) = (\alpha\xi_2 - p\alpha^2\xi_1^2)(\alpha\xi_2 - q\,\alpha^2\xi_1^2)$$

$$g'(\alpha) = \alpha(2\xi_2^2 + 4\alpha^2 pq\,\xi_1^4 - 3\alpha(p+q)\xi_2\xi_1^2)$$

$$g''(\alpha) = 2\;{}_2^2 + 12\alpha^2 pq\,\xi_1^4 - 6\alpha(p+q)\xi_2\xi_1^2$$

Hence $g'(0) = 0$ and $g''(0) = 2\xi_2^2 > 0$

Hence $x_0 = (0,0)$ is a local minimizer for any $\xi \in \mathbb{R}^2$

ii) $f'(y,z) = [-2y(p+q)z + 4pqy^3 \quad 2z - (p+q)y^2]$

Hence $f'(0,0) = [0 \quad 0]$

iii)

$$f\left(y, \frac{p+q}{2}y\right) = \left(\frac{p+q}{2}y^2 - py^2\right)\left(\frac{p+q}{2}y^2 - qy^2\right)$$

$$= -y^4\left(\frac{p-q}{2}\right)^2 < 0 \qquad \left(m = \frac{p+q}{2}\right)$$

Hence, if we follow the curve $(y, \frac{p+q}{2}y^2)$ from $(0,0)$, the value of f gets strictly negative and thus $(0,0)$ is not a local minimizer