College of Engineering
School of Aerospace Engineering

ME 6444: Nonlinear Systems

HW5

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Problem 1

(Averaging - Autonomous System) Consider the nonlinear system

$$\ddot{x} + \epsilon\left(x^2 + \dot{x}^2 - 4\right)\dot{x} + x = 0.$$

a. Use the method of averaging to find a periodic solution (i.e., limit cycle) for this system. Report the amplitude and phase of the limit cycle you find - expect dependence on θ in the amplitude. You can assume an amplitude a_0 at a solution phase corresponding to $\theta = 0$.

b. Find the period of the limit cycle.

c. Generate a phase plane (using Mpale, Mathematica, Matlab, etc.) to verify the limit cycle's existence.

Solution:
(a) For this problem, $h(x, \dot{x}) = x^2 + \dot{x}^2 - 4$ where $\epsilon \ll 1$. The if we assume the solution for x to be

$$x(t) = a\cos(t + \phi) = a\cos\theta$$

we have

$$\begin{aligned} h(x, \dot{x}) &= \left(a^2\cos^2\theta + a^2\sin^2\theta - 4\right)(-a\sin\theta) \\ &= -a^3\cos^2\theta\sin\theta - a^3\sin^3\theta + 4a\sin\theta. \end{aligned}$$

From the averaging equations we know that

$$\begin{aligned} \dot{a} &= \frac{\epsilon}{2\pi}\int_0^{2\pi}\left(-a^3\cos^2\theta\sin\theta - a^3\sin^3\theta + 4a\sin\theta\right)\sin\theta d\theta \\ &= \frac{\epsilon}{2\pi}\int_0^{2\pi}\left(-a^3\cos^2\theta\sin^2\theta - a^3\sin^4\theta + 4a\sin^2\theta\right)d\theta \end{aligned}$$

since

$$\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta = \frac{\pi}{4}$$
$$\int_0^{2\pi}\sin^2\theta d\theta = \pi$$
$$\int_0^{2\pi}\sin^4\theta d\theta = \frac{3\pi}{4}$$

we have

$$\dot{a} = 2\epsilon a - \frac{\epsilon}{2}a^3 + O(\epsilon^2).$$

To solve this analytically, let $b = a^{-2}$, then

$$\dot{b} = -2a^{-3}\dot{a} \quad \longrightarrow \quad \dot{a} = -\frac{1}{2}a^3 b.$$

If we plug this back into the nonlinear ODE we get

$$\begin{aligned} -\frac{1}{2}a^3\dot{b} - 2\epsilon a &= -\frac{\epsilon}{2}a^3 \\ \therefore \dot{b} + 4\epsilon b &= \epsilon. \end{aligned}$$

This nonhomogeneous linear ODE can be solved by

$$\begin{aligned} b(t) &= e^{\int -4\epsilon t dt}\left[\int \epsilon e^{\int 4\epsilon t dt} dt + C\right] \\ &= \frac{1}{4} + Ce^{-4\epsilon t}. \end{aligned}$$

If we apply the initial condition of $a(0) = a_0$ we get

$$b(t) = \frac{1}{4} + \left(\frac{1}{a_0^2} - \frac{1}{4}\right)e^{-4\epsilon t}.$$

And therefore,

$$a(t) = \sqrt{\frac{1}{\frac{1}{4} + \left(\frac{1}{a_0^2} - \frac{1}{4}\right)e^{-4\epsilon t}}} + O(\epsilon^2).$$

Additionally,

$$\begin{aligned} \dot{\phi} &= \frac{\epsilon}{2\pi}\int_0^{2\pi}\left(-a^3\cos^2\theta\sin\theta - a^3\sin^3\theta + 4a\sin\theta\right)\cos\theta d\theta \\ &= \frac{\epsilon}{2\pi}\int_0^{2\pi}\left(-a^3\cos^3\theta\sin\theta - a^3\cos\theta\sin^3\theta + 4a\cos\theta\sin\theta\right)d\theta \\ &= 0 \end{aligned}$$

and thus,

$$\phi(t) = \phi_0 + O(\epsilon^2).$$

Finally, we have the following approximated expression

$$x(t) = \sqrt{\frac{1}{\frac{1}{4} + \left(\frac{1}{a_0^2} - \frac{1}{4}\right)e^{-4\epsilon t}}}\cos\left(t + \phi_0\right) + O(\epsilon^2).$$

If we assume, $\dot{x} = 0$

$$0 = -a_0 \sin \phi_0$$
$$\therefore \phi_0 = 0.$$

Hence,

$$\boxed{x(t) = \sqrt{\frac{1}{\frac{1}{4} + \left(\frac{1}{a_0^2} - \frac{1}{4}\right) e^{-4\epsilon t}}} \cos(t) + O(\epsilon^2).}$$

(b) Since $\omega = 1$ for this approximation, the period T of the limit cycle is

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi.}$$

(c) The phase plane of this system looks as follows.

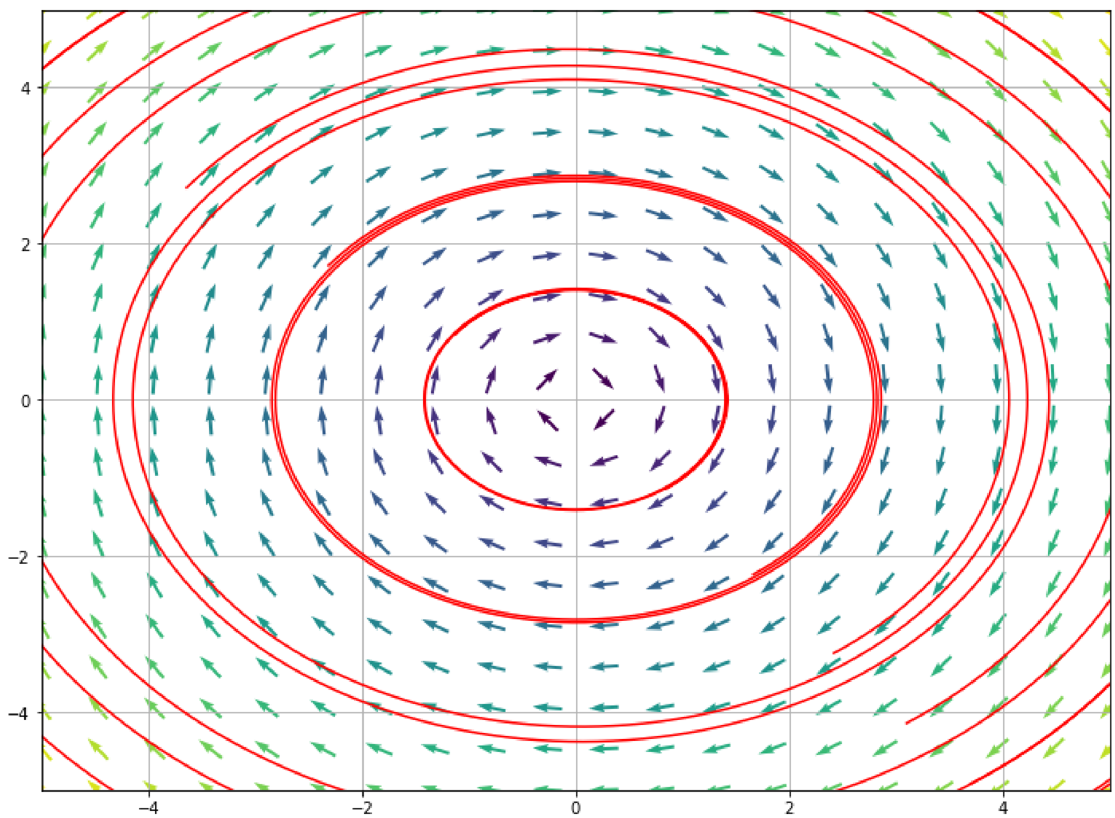

Figure 1: Problem 1 phase plane

Observing Figure 1 we can tell that the system does indeed have a limit cycle.

Problem 2

(Lindstedt-Poincaré and Multiple Scales - Autonomous System) Consider Rayleigh's equation:

$$\ddot{x} + \epsilon \left(\frac{1}{3}\dot{x}^3 - \dot{x} \right) + x = 0$$

with initial conditions $x(0) = a$ and $\dot{x}(0) = 0$. Carry out a **first-order** approximation as follows:

a. Use Lindstedt-Poincaré method to find an approximate solution for $x(t)$.

b. Use the Multiple Scales approach to find an approximate solution for $x(t)$.

c. Generate a phase plane (using Maple, Mathematica, Matlab, etc.) to verify the limit cycle's existence.

Solution:
(a) (A second order approximation is done in my case) First we introduce a dimesionless time $\tau = \omega t$ which gives us

$$\frac{d}{dt} = \omega \frac{d}{d\tau} \qquad \frac{d^2}{dt^2} = \omega^2 \frac{d^2}{d\tau^2}$$

and also we expand the term ω

$$\omega = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots$$

as well as $x(t)$

$$x(t) = x_0 + \epsilon x_1(\tau) + \epsilon^2 x_2(t) + \epsilon^3 x_3(t) + \cdots .$$

Then if we substitute these into the EOM we obtain the

$$\begin{aligned}(\omega_0 + \epsilon\omega_1 + \cdots)^2 & \frac{d^2}{d\tau^2}(\epsilon x_1(\tau) + \epsilon^2 x_2(t) + \cdots) \\ & + (\epsilon x_1(\tau) + \epsilon^2 x_2(t) + \cdots) + \frac{\epsilon}{3}(\omega_0 + \epsilon\omega_1 + \cdots)^3 \left(\frac{d}{d\tau}(\epsilon x_1(\tau) + \epsilon^2 x_2(t) + \cdots) \right)^3 \\ & - \epsilon(\omega_0 + \epsilon\omega_1 + \cdots)\frac{d}{d\tau}(\epsilon x_1(\tau) + \epsilon^2 x_2(t) + \cdots) = 0\end{aligned}$$

We expand this equation and then collect the terms for the ϵ's.

$$\begin{cases} \epsilon^0 : & \omega_0^2 \frac{d^2}{d\tau^2}x_0(\tau) + x_0(\tau) = 0 \\ \epsilon^1 : & \frac{\omega_0^3\left(\frac{d}{d\tau}x_0(\tau)\right)^3}{3} + \omega_0^2\frac{d^2}{d\tau^2}x_1(\tau) + 2\omega_0\omega_1\frac{d^2}{d\tau^2}x_0(\tau) - \omega_0\frac{d}{d\tau}x_0(\tau) + x_1(\tau) = 0 \\ \epsilon^2 : & \omega_0^3\left(\frac{d}{d\tau}x_0(\tau)\right)^2\frac{d}{d\tau}x_1(\tau) + \omega_0^2\omega_1\left(\frac{d}{d\tau}x_0(\tau)\right)^3 + \omega_0^2\frac{d^2}{d\tau^2}x_2(\tau) + 2\omega_0\omega_1\frac{d^2}{d\tau^2}x_1(\tau) \\ & \quad + 2\omega_0\omega_2\frac{d^2}{d\tau^2}x_0(\tau) - \omega_0\frac{d}{d\tau}x_1(\tau) + \omega_1^2\frac{d^2}{d\tau^2}x_0(\tau) - \omega_1\frac{d}{d\tau}x_0(\tau) + x_2(\tau) = 0 \end{cases}$$

By solving the equation for ϵ^0 we have

$$x_0(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

but for convenience we modify this into the polar form

$$x_0 = a\cos(\tau + \beta).$$

Now if we update ϵ^1 you will get

$$\begin{aligned}\omega_0^2 \frac{d^2}{d\tau^2} x_1(\tau) + x_1(\tau) = {} & \frac{a^3\omega_0^3 \sin(\beta+\tau)}{4} - \frac{a^3\omega_0^3 \sin(3\beta + 3\tau)}{12} \\ & + 2a\omega_0\omega_1 \cos(\beta+\tau) - a\omega_0 \sin(\beta+\tau)\end{aligned}$$

to remove the secular term, we set $\omega_1 = 0$ and we must have

$$a = \pm\frac{2}{\omega_0},$$

and thus we let $a = 2/\omega_0$. Then we solve for the particular solution of $x_1(t)$. This gives us

$$x_{1p}(\tau) = \frac{2\sin(3\beta + 3\tau)}{3\left(9\omega_0^2 - 1\right)}$$

using this we update the x_2 equation which gives

$$\begin{aligned}\omega_0^2 \frac{d^2}{d\tau^2} x_2(\tau) + x_2(\tau) = {} & \frac{2\omega_0 \cos(3\beta + 3\tau)}{9\omega_0^2 - 1} + \frac{a^2\omega_0^3 \cos(\beta+\tau)}{18\omega_0^2 - 2} \\ & + \frac{a^2\omega_0^3 \cos(5\beta + 5\tau)}{18\omega_0^2 - 2} - \frac{a^2\omega_0^3 \cos(3\beta+3\tau)}{9\omega_0^2 - 1} + 2a\omega_0\omega_2 \cos(\beta+\tau)\end{aligned}$$

To remove the secular terms we let

$$\omega_2 = -\frac{a\omega_0^2}{36\omega_0^2 - 4},$$

and then we find the particular solution of $x_2(\tau)$ which is

$$x_{2p}(\tau) = -\frac{2\omega_0 \cos(5\beta + 5\tau)}{225\omega_0^4 - 34\omega_0^2 + 1} + \frac{2\omega_0 \cos(3\beta + 3\tau)}{81\omega_0^4 - 18\omega_0^2 + 1}$$

and we reconstitute the equation to find the solution for $O(\epsilon^3)$

$$\begin{cases} x & = x_0 + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3) \\ \omega & = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + O(\epsilon^3) \\ \tau & = \omega t \end{cases}$$

From the problem and the initial conditions we know the values for the following parameters

$$a = 2, \qquad \omega_0 = 1, \qquad \beta = 0,$$

and if we plug this in and reconstitute $x(t)$ we have the solution of

$$\boxed{\begin{aligned} x(t) = 2\cos\left(t\left(1-\frac{\epsilon^2}{16}\right)\right) + \frac{\epsilon\sin\left(3t\left(1-\frac{\epsilon^2}{16}\right)\right)}{12} \\ + \epsilon^2\left(\frac{\cos\left(3t\left(1-\frac{\epsilon^2}{16}\right)\right)}{32} - \frac{\cos\left(5t\left(1-\frac{\epsilon^2}{16}\right)\right)}{96}\right) \\ \because \omega = 1-\frac{\epsilon^2}{16} \end{aligned}}$$

(b) (For the multiple scale method we will implement a first order approximation due to its complexity with multi variate differential equations) The scales are defined as

$$T_0 = t, \qquad T_1 = \epsilon t$$

where T_0 is greatly faster than T_1. The time derivative is then defined to be

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial T_0}\frac{\partial T_0}{\partial t} + \frac{\partial}{\partial T_1}\frac{\partial T_1}{\partial t} + \cdots \\ &= \frac{\partial}{\partial T_0} + \epsilon\frac{\partial}{\partial T_1} + \cdots \\ \frac{d^2}{dt^2} &= \frac{\partial^2}{\partial T_0^2} + 2\epsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2\frac{\partial^2}{\partial T_1^2} + \cdots \end{aligned}$$

Here we abbreviate the differential expressions using the time operators

$$D_0 = \frac{\partial}{\partial T_0}, \qquad D_1 = \frac{\partial}{\partial T_1}.$$

Then the EOM becomes

$$D_0^2 x + 2\epsilon D_0 D_1 x + \epsilon^2 D_1 x + \epsilon\left[\frac{1}{3}\left(D_0 x + \epsilon D_1 x\right)^3 + \left(D_0 x + \epsilon D_1 x\right)\right] + x = 0$$

with the initial conditions of

$$x(0) = a, \qquad \frac{\partial x}{\partial T_0} + \epsilon\frac{\partial x}{\partial T_1} = 0 \qquad \text{for} \quad T_0 = T_1 = 0$$

Now seeking for a asymptotic approximation for x of the form

$$x(t) \equiv x(T_0, T_1; \epsilon) \approx x_o(T_0, T_1) + \epsilon x(T_0, T_1)$$

Substituting this into the EOM above we get (skip this step since the equation is very long) and then we collect the epsilon terms by their order

$$\begin{cases} O(1): & D_0^2 x_0 + x_0 = 0 \\ O(\epsilon): & D_0^2 x_1 + x_1 = -D_0 x_0 - 2D_0 D_1 x_0 - \dfrac{1}{3} D_0^3 x_0^3 \end{cases}$$

The respective intial conditions for x and $\dot{x}$ are given by

$$x_0 = a, \qquad \frac{\partial x_0}{\partial T_0} = 0 \qquad \text{for} \quad T_0 = T_1 = 0$$
$$x_1 = 0, \qquad \frac{\partial x_1}{\partial T_0} = -\frac{\partial x_0}{\partial T_1} \qquad \text{for} \quad T_0 = T_1 = 0$$

Since T_0 and T_1 are being treated as an independent variable temporarily, the differential equation above are partial differential equations for a function of x_0 of two variables T_0 and T_1. However, since no derivatives with respect to T_1 appear in the equation of $O(1)$ collected for ϵ this equation can be regarded as instead as an ordinary differential equation for a function of T_0 regarding T_1 as an auxiliary parameter. Thus, the general solution for x_0 can be expressed as the following

$$x_0 = A_0(T_1)\cos T_0 + B_0(T_1)\sin T_0$$

in which the coefficients for the sine and consine terms can be found using the initial conditions

$$A_0(0) = a \qquad \text{and} \qquad B_0(0) = 0$$

If we take the derivative with respect to T_0 we have

$$D_0 x_0 = -A_0(T_1)\sin T_0 + B_0(T_1)\cos T_0$$

and

$$D_0 D_1 x_0 = \frac{\partial}{\partial T_1}\left(\frac{\partial x_0}{\partial T_0}\right) = -D_1 A_0 \sin T_0 + D_1 B_0 \cos T_0$$

Substituting these into the equation collected from the EOM with respect to $O(\epsilon)$ we obtain the following relationship

$$\begin{aligned} D_0^2 x_1 + x_1 = \frac{A_0^3 \sin(T_0)}{4} &- \frac{A_0^3 \sin(3T_0)}{12} - \frac{A_0^2 B_0 \cos(T_0)}{4} + \frac{A_0^2 B_0 \cos(3T_0)}{4} \\ &+ \frac{A_0 B_0^2 \sin(T_0)}{4} + \frac{A_0 B_0^2 \sin(3T_0)}{4} + 2D_1 A_0 \sin(T_0) + A_0 \sin(T_0) - \frac{B_0^3 \cos(T_0)}{4} \\ &- \frac{B_0^3 \cos(3T_0)}{12} - 2D_1 B_0 \cos(T_0) - B_0 \cos(T_0) \end{aligned}$$

To have the secular terms vanish we let

$$2D_1 A_0 + A_0 + \frac{1}{4}A_0^3 + \frac{1}{4}A_0 B_0^2 = 0$$
$$2D_1 B_0 + B_0 + \frac{1}{4}B_0^3 + \frac{1}{4}A_0^2 B_0 = 0$$

Now if we solve these ordinary differential equations we have

$$A_0 = 2\sqrt{\frac{-\sqrt{e^{C_2}}e^{\frac{T_1}{2}} + \sqrt{e^{C_2}}e^{C_2+\frac{3T_1}{2}} + e^{C_2+T_1} - 1}{-2C_1^2e^{C_2+T_1} + C_1^2e^{2C_2+2T_1} + C_1^2 - 2e^{C_2+T_1} + e^{2C_2+2T_1} + 1}}$$

$$B_0 = 2C_1\sqrt{\frac{-\sqrt{e^{C_2}}e^{\frac{T_1}{2}} + \sqrt{e^{C_2}}e^{C_2+\frac{3T_1}{2}} + e^{C_2+T_1} - 1}{-2C_1^2e^{C_2+T_1} + C_1^2e^{2C_2+2T_1} + C_1^2 - 2e^{C_2+T_1} + e^{2C_2+2T_1} + 1}}$$

Applying the initial conditions $a_0 = a$ and $b_0 = 0$ of we have

$$A_0 = 2\sqrt{\frac{-e^{\frac{T_1}{2}} + e^{\frac{3T_1}{2}} + e^{T_1} - 1}{-2a^2e^{T_1} + a^2e^{2T_1} + a^2 - 2e^{T_1} + e^{2T_1} + 1}}$$

$$B_0 = aA_0$$

Then we finally have

$$\boxed{\begin{aligned} x(t) = 2&\sqrt{\frac{-e^{\frac{\epsilon t}{2}} + e^{\frac{3\epsilon t}{2}} + e^{\epsilon t} - 1}{-2a^2e^{\epsilon t} + a^2e^{2\epsilon t} + a^2 - 2e^{\epsilon t} + e^{2\epsilon t} + 1}}\cos t \\ &+ 2a\sqrt{\frac{-e^{\frac{\epsilon t}{2}} + e^{\frac{3\epsilon t}{2}} + e^{\epsilon t} - 1}{-2a^2e^{\epsilon t} + a^2e^{2\epsilon t} + a^2 - 2e^{\epsilon t} + e^{2\epsilon t} + 1}}\sin t + O(\epsilon^2). \end{aligned}}$$

(c) The following phase plane shows that there exists a limit cycle

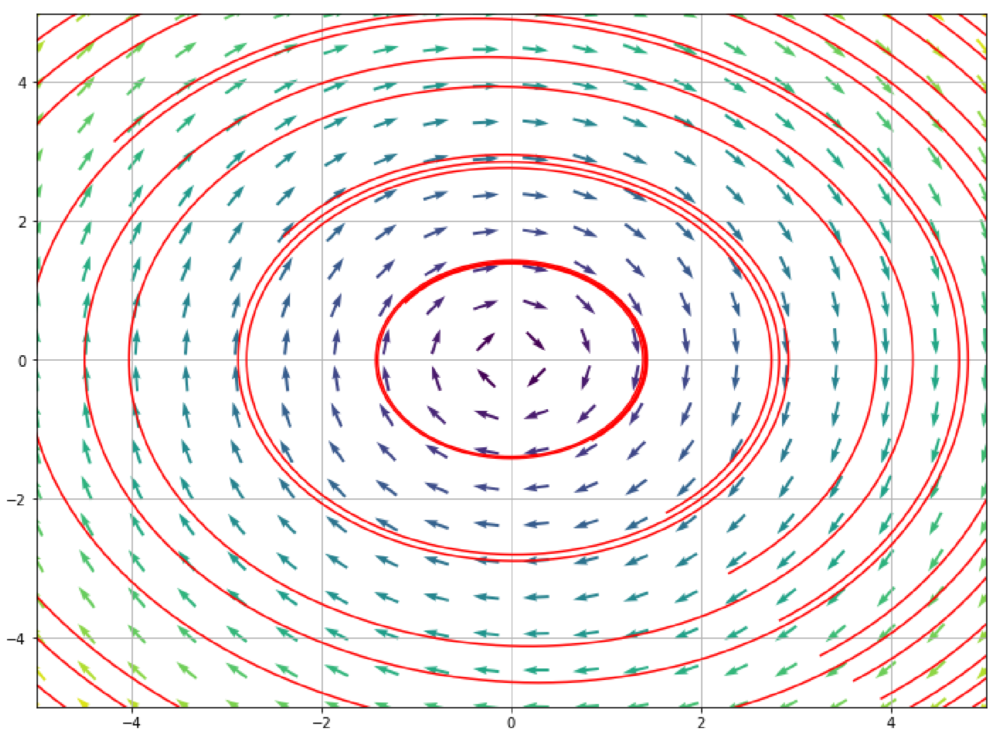

Figure 2: Phase plane of problem 2

Appendix

3.1 Problem 1: Python Code

```
# We plot the phase plane of this system to check the limit cycle

import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp, DOP853
from typing import List

# System
def nlsys(t, x, epsilon):
    return [x[1], -epsilon*(x[0]**2 + x[1]**2 - 4)*x[1] - x[0]]

def solve_diffeq(func, t, tspan, ic, parameters={}, algorithm='DOP853',
↪ stepsize=np.inf):
    return solve_ivp(fun=func, t_span=tspan, t_eval=t, y0=ic, method=algorithm,
                     args=tuple(parameters.values()), atol=1e-8, rtol=1e-5,
                     ↪ max_step=stepsize)

def phasePlane(x1, x2, func, params):
    X1, X2 = np.meshgrid(x1, x2)  # create grid
    u, v = np.zeros(X1.shape), np.zeros(X2.shape)
    NI, NJ = X1.shape
    for i in range(NI):
        for j in range(NJ):
            x = X1[i, j]
            y = X2[i, j]
            dx = func(0, (x, y), *params.values())  # compute values on grid
            u[i, j] = dx[0]
            v[i, j] = dx[1]
    M = np.hypot(u, v)
    u /= M
    v /= M
    return X1, X2, u, v, M

def DEplot(sys: object, tspan: tuple, x0: List[List[float]],
           x: np.ndarray, y: np.ndarray, params: dict):
    if len(tspan) != 3:
        raise Exception('tspan should be tuple of size 3: (min, max, number of
        ↪ points).')
```

```
    # Set up the figure the way we want it to look
    plt.figure(figsize=(12, 9))

    X1, X2, dx1, dx2, M = phasePlane(
        x, y, sys, params
    )

    # Quiver plot
    plt.quiver(X1, X2, dx1, dx2, M, scale=None, pivot='mid')
    plt.grid()

    t1 = np.linspace(0, tspan[0], tspan[2])
    t2 = np.linspace(0, tspan[1], tspan[2])
    if min(tspan) < 0:
        t_span1 = (np.max(t1), np.min(t1))
    else:
        t_span1 = (np.min(t1), np.max(t1))
    t_span2 = (np.min(t2), np.max(t2))
    for x0i in x0:
        sol1 = solve_diffeq(sys, t1, t_span1, x0i, params)
        plt.plot(sol1.y[0, :], sol1.y[1, :], '-r')
        sol2 = solve_diffeq(sys, t2, t_span2, x0i, params)
        plt.plot(sol2.y[0, :], sol2.y[1, :], '-r')

    plt.xlim([np.min(x), np.max(x)])
    plt.ylim([np.min(y), np.max(y)])
    plt.show()


x10 = np.arange(0, 10, 1)
x20 = np.arange(0, 10, 1)
x0 = np.stack((x10, x20), axis=-1)

p = {'epsilon': 0.001}

x1 = np.linspace(-5, 5, 20)
x2 = np.linspace(-5, 5, 20)

DEplot(nlsys, (-8, 8, 1000), x0, x1, x2, p)
```

3.2 Problem 2: Python Code

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp, DOP853
from typing import List

# Generate phase plane of Rayleigh's equation to confirm limit cycle

def rayleigh(t, x, e):
    return [x[1], -x[0] - e*(x[1]**3 / 3 - x[1])]

def solve_diffeq(func, t, tspan, ic, parameters={}, algorithm='DOP853', stepsize=np.inf):
    return solve_ivp(fun=func, t_span=tspan, t_eval=t, y0=ic, method=algorithm,
                     args=tuple(parameters.values()), atol=1e-8, rtol=1e-5, max_step=stepsize)

def phasePlane(x1, x2, func, params):
    X1, X2 = np.meshgrid(x1, x2)  # create grid
    u, v = np.zeros(X1.shape), np.zeros(X2.shape)
    NI, NJ = X1.shape
    for i in range(NI):
        for j in range(NJ):
            x = X1[i, j]
            y = X2[i, j]
            dx = func(0, (x, y), *params.values())  # compute values on grid
            u[i, j] = dx[0]
            v[i, j] = dx[1]
    M = np.hypot(u, v)
    u /= M
    v /= M
    return X1, X2, u, v, M

def DEplot(sys: object, tspan: tuple, x0: List[List[float]],
           x: np.ndarray, y: np.ndarray, params: dict):
    if len(tspan) != 3:
        raise Exception('tspan should be tuple of size 3: (min, max, number of points).')
    # Set up the figure the way we want it to look
    plt.figure(figsize=(12, 9))

    X1, X2, dx1, dx2, M = phasePlane(
```

```python
            x, y, sys, params
        )

    # Quiver plot
    plt.quiver(X1, X2, dx1, dx2, M, scale=None, pivot='mid')
    plt.grid()

    t1 = np.linspace(0, tspan[0], tspan[2])
    t2 = np.linspace(0, tspan[1], tspan[2])
    if min(tspan) < 0:
        t_span1 = (np.max(t1), np.min(t1))
    else:
        t_span1 = (np.min(t1), np.max(t1))
    t_span2 = (np.min(t2), np.max(t2))
    for x0i in x0:
        sol1 = solve_diffeq(sys, t1, t_span1, x0i, params)
        plt.plot(sol1.y[0, :], sol1.y[1, :], '-r')
        sol2 = solve_diffeq(sys, t2, t_span2, x0i, params)
        plt.plot(sol2.y[0, :], sol2.y[1, :], '-r')

    plt.xlim([np.min(x), np.max(x)])
    plt.ylim([np.min(y), np.max(y)])
    plt.show()

x10 = np.arange(0, 10, 1)
x20 = np.arange(0, 10, 1)
x0 = np.stack((x10, x20), axis=-1)

p = {'e': 0.01}

x1 = np.linspace(-5, 5, 20)
x2 = np.linspace(-5, 5, 20)

DEplot(rayleigh, (-8, 8, 1000), x0, x1, x2, p)
```