College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 3
State Space Representation, Linearization, and Transfer Functions

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Exercise 1 Obtain the transfer function of the following system

$$\ddot{q}(t) + 4\dot{q} + q(t) = u(t) + \dot{u}(t-T)$$
$$y(t) = q(t)$$

where the constant $T > 0$ represents a constant time delay.

$$\ddot{q} + 4\dot{q} + q = u + \dot{u}(t-T) \quad \cdots \text{①}$$
$$y = q \quad \cdots \text{②}$$

Assume this to be a SISO and zero IC problem.
Take the Laplace transform of ①

$$\mathcal{L} \rightarrow s^2\hat{q}(s) - \cancel{s q(0)}^{0} - \cancel{\dot{q}(0)}^{0} + 4s\hat{q}(s) - \cancel{4q(0)}^{0} + \hat{q}(s) = \hat{u}(s) + e^{-sT}(s\hat{u}(s) - \cancel{u(0)}^{0})$$

$$s^2\hat{q}(s) + 4s\hat{q}(s) + \hat{q}(s) = \hat{u}(s) + se^{-sT}\hat{u}(s)$$

$$\therefore \quad \frac{\hat{q}(s)}{\hat{u}(s)} = \frac{1 + se^{-sT}}{s^2 + 4s + 1}$$

since $\quad y(t) = q(t) \Longleftrightarrow \hat{y}(s) = \hat{q}(s)$

$$\boxed{\therefore \quad G(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{1 + se^{-sT}}{s^2 + 4s + 1}}$$

Exercise 2 The two pendulum cart system. Unless otherwise specified, from now on, we will consider the two pendulum cart system as an input-output system with input u and output y described by

$$\begin{aligned}
(m_0+m_1+m_2)\ddot{y} - m_1 l_1\cos\theta_1\,\ddot{\theta}_1 - m_2 l_2\cos\theta_2\,\ddot{\theta}_2 &+ m_1 l_1 \sin\theta_1\dot{\theta}_1^2 + m_2 l_2\sin\theta_2\dot{\theta}_2^2 &= u\\
-m_1 l_1\cos\theta_1\,\ddot{y} + m_1 l_1^2\ddot{\theta}_1 &+ m_1 l_1 g\sin\theta_1 &= 0\\
-m_2 l_2\cos\theta_2\,\ddot{y} + m_2 l_2^2\ddot{\theta}_2 &+ m_2 l_2 g\sin\theta_2 &= 0
\end{aligned}$$

(a) For what constant values of u does the system have equilibrium states?

(b) Consider the equilibrium configurations defined by $u^e = 0$ and

$$\begin{aligned}
E1: &\quad (y^e, \theta_1^e, \theta_2^e) = (0, 0, 0)\\
E2: &\quad (y^e, \theta_1^e, \theta_2^e) = (0, \pi, \pi)
\end{aligned}$$

Using MATLAB, obtain the A, B, C, D matrices for state space representations of the linearizations corresponding to the following combinations of parameters and equilibrium conditions:

L1	P1	E1
L2	P1	E2
L3	P4	E1
L4	P4	E2

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

(a) Solve the given system equations for $\ddot{y}$, $\ddot{\theta}_1$, and $\ddot{\theta}_2$ using MATLAB (code in Appendix) we get

$$\ddot{y} = \frac{-m_1 l_1 \sin(\theta_1)\dot{\theta}_1^2 - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^2}{m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)}$$

$$- \frac{m_1 g \sin(\theta_1)\cos(\theta_1) + m_2 g \sin(\theta_2)\cos(\theta_2)}{m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)}$$

$$+ \frac{u}{m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)}$$

$$\ddot{\theta}_1 = - \frac{m_1 l_1 \cos(\theta_1)\sin(\theta_1)\dot{\theta}_1^2 + m_2 l_2 \cos(\theta_1)\sin(\theta_2)\dot{\theta}_2^2}{l_1\left[m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)\right]}$$

$$+ \frac{m_2 g\left[\sin(\theta_1)\cos^2(\theta_2) - \cos(\theta_1)\sin(\theta_2)\cos(\theta_2)\right]}{l_1\left[m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)\right]}$$

$$- \frac{(m_0 + m_1 + m_2) g \sin(\theta_1)}{l_1\left[m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)\right]}$$

$$+ \frac{u\cos(\theta_1)}{l_1\left[m_0 + m_1 + m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)\right]}$$

$$\ddot{\theta}_2 = -\frac{m_1 l_1 \cos(\theta_2)\sin(\theta_1)\dot{\theta}_1^2 + m_2 l_2 \cos(\theta_2)\sin(\theta_2)\dot{\theta}_2^2}{l_2[m_0+m_1+m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)]}$$

$$+\frac{m_1 g[\sin(\theta_2)\cos^2(\theta_1) - \cos(\theta_2)\sin(\theta_1)\cos(\theta_1)]}{l_2[m_0+m_1+m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)]}$$

$$-\frac{(m_0+m_1+m_2)g\sin(\theta_2)}{l_2[m_0+m_1+m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)]}$$

$$+\frac{u\cos(\theta_2)}{l_2[m_0+m_1+m_2 - m_1\cos^2(\theta_1) - m_2\cos^2(\theta_2)]}$$

if u is a constant and $(\theta_1, \theta_2) = (0, 0), (\pi, \pi)$

$$\ddot{y} = \frac{u}{m_0+m_1+m_2 - m_1\cancel{\cos^2(\theta_1)}^{\to 1} - m_2\cancel{\cos^2(\theta_2)}^{\to 1}} = -\frac{u}{m_0}$$

$$\ddot{\theta}_1 = \frac{u\cancel{\cos(\theta_1)}^{\to 1}}{l_1[m_0+m_1+m_2 - m_1\cancel{\cos^2(\theta_1)}^{\to 1} - m_2\cancel{\cos^2(\theta_2)}^{\to 1}]} = \frac{u}{m_0 l_1}$$

$$\ddot{\theta}_2 = \frac{u\cancel{\cos(\theta_2)}^{\to 1}}{l_1[m_0+m_1+m_2 - m_1\cancel{\cos^2(\theta_1)}^{\to 1} - m_2\cancel{\cos^2(\theta_2)}^{\to 1}]} = \frac{u}{m_0 l_2}$$

Thus, to have an equilibrium state $\boxed{u^e = 0}$

Solved the complicated algebra using the following MATLAB code

```
% (a)
syms m_0 m_1 m_2 l_1 l_2 y_ddot theta_ddot_1 theta_ddot_2 theta_dot_1 theta_dot_2
theta_1 theta_2 u g
eqn1 = (m_0+m_1+m_2)*y_ddot-m_1*l_1*cos(theta_1)*theta_ddot_1-
m_2*l_2*cos(theta_2)*theta_ddot_2...
    +m_1*l_1*sin(theta_1)*theta_dot_1^2+m_2*l_2*sin(theta_2)*theta_dot_2^2 == u
eqn2 = -m_1*l_1*cos(theta_1)*y_ddot+m_1*l_1^2*theta_ddot_1+m_1*l_1*g*sin(theta_1)
== 0
eqn3 = -m_2*l_2*cos(theta_2)*y_ddot+m_2*l_2^2*theta_ddot_2+m_2*l_2*g*sin(theta_2)
== 0

eqns = [eqn1, eqn2, eqn3];
S = solve(eqns, [y_ddot, theta_ddot_1, theta_ddot_2]);

S.y_ddot
S.theta_ddot_1
S.theta_ddot_2
```

The trim() command for the Simulink model on the next page gives the following results

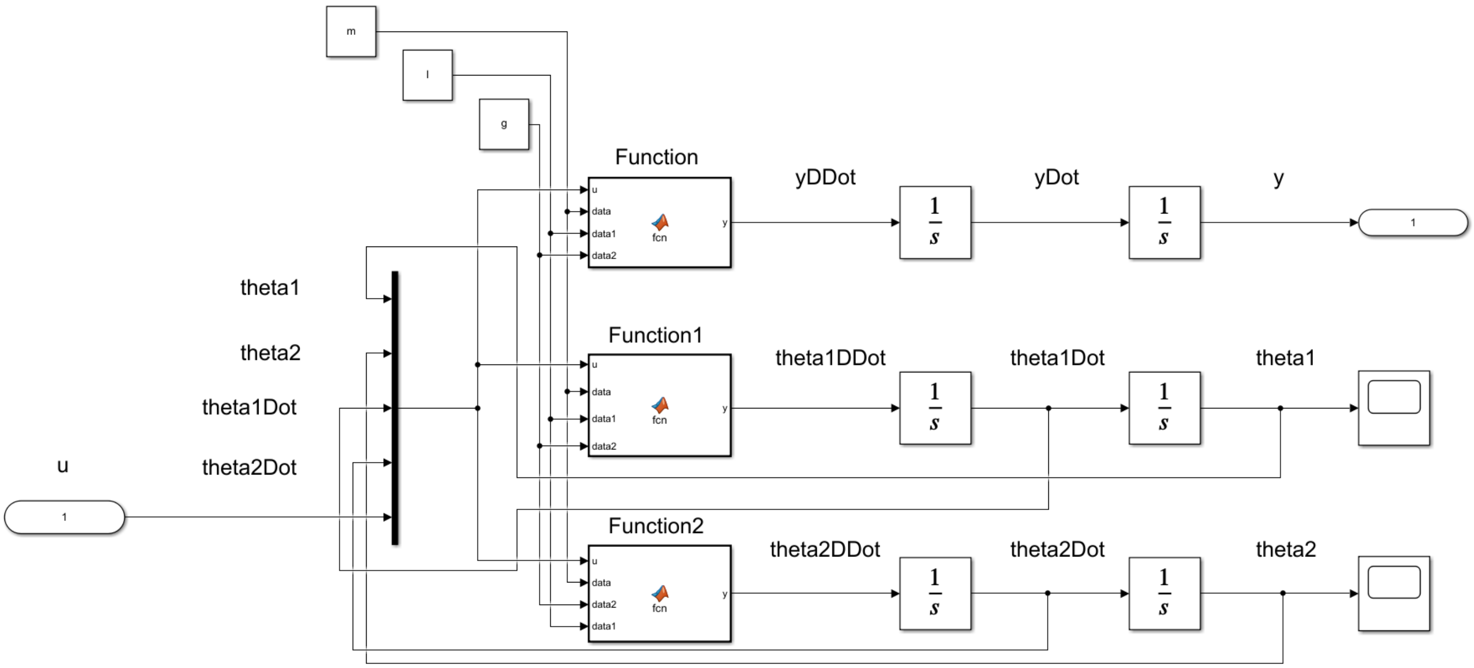
```
m = [2, 1, 1]; l = [1, 1]; g = 1;
[x,u,y,dx] = trim('hw3_p2_type2');
```

x = 6×1 0 3.1416 3.1416 0 0 0	dx = 6×1 10^-15 × 0 0 0 0.1225 -0.2449 -0.2449
u = 0	y = 0

This verifies our results.

(b)

First, we made a Simulink model of the system



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;
```

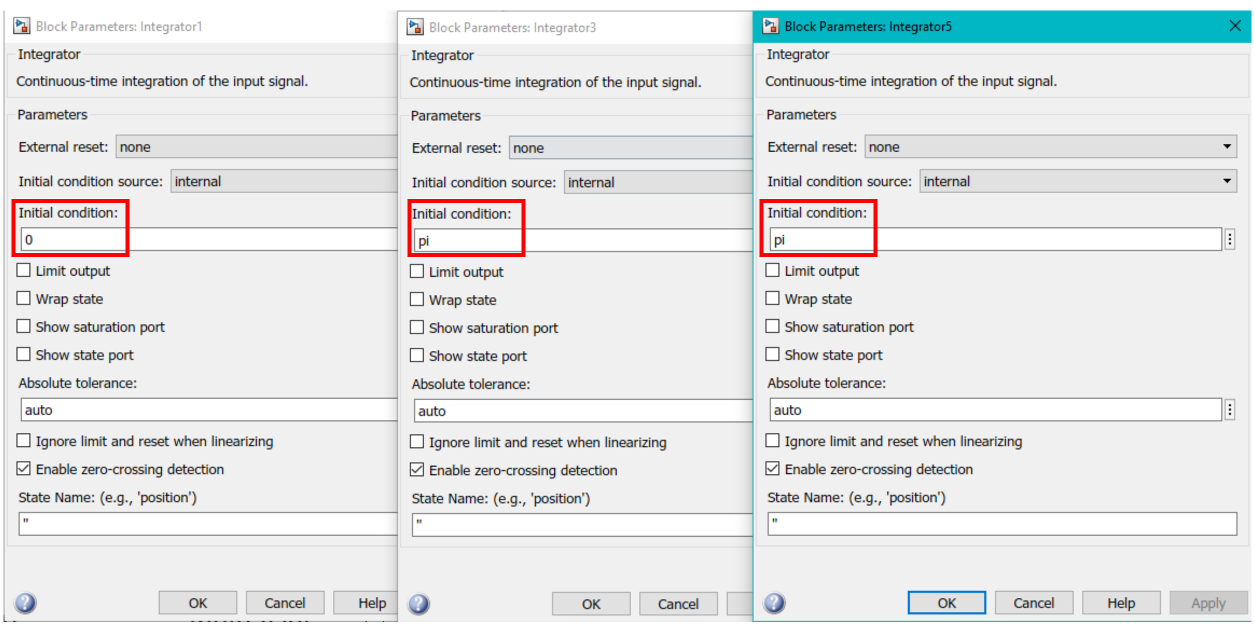
```
    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



Then finally, by running the following code we can get the state space realization for the linearized system.

```
% (b) Linearizing using simulink
% Set the global variables for the sFunction used in the simulink model
% L1 & L2
m = [2, 1, 1]; l = [1, 1]; g = 1;
[A, B, C, D] = linmod('hw3_p2_type2')

% L3 & L4
m = [2, 1, 1]; l = [1, 0.5]; g = 1;
[A, B, C, D] = linmod('hw3_p2_type2')
```

L1:

```
A = 6×6
     0         0         0    1.0000         0         0
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
     0   -0.5000   -0.5000         0         0         0
     0   -1.5000   -0.5000         0         0         0
     0   -0.5000   -1.5000         0         0         0

B = 6×1
     0
     0
     0
0.5000
0.5000
0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

L2:

```
A = 6×6
     0         0         0    1.0000         0         0
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
     0   -0.5000   -0.5000         0         0         0
     0    1.5000    0.5000         0         0         0
     0    0.5000    1.5000         0         0         0

B = 6×1
      0
      0
      0
 0.5000
-0.5000
-0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

L3:

A = 6×6

0	0	0	1.0000	0	0
0	0	0	0	1.0000	0
0	0	0	0	0	1.0000
0	-0.5000	-0.5000	0	0	0
0	-1.5000	-0.5000	0	0	0
0	-1.0000	-3.0000	0	0	0

B = 6×1

0
0
0
0.5000
0.5000
1.0000

C = 1×6

1	0	0	0	0	0

D = 0

L4:

A = 6×6

0	0	0	1.0000	0	0
0	0	0	0	1.0000	0
0	0	0	0	0	1.0000
0	-0.5000	-0.5000	0	0	0
0	1.5000	0.5000	0	0	0
0	1.0000	3.0000	0	0	0

B = 6×1

0
0
0
0.5000
-0.5000
-1.0000

C = 1×6

1	0	0	0	0	0

D = 0

Exercise 3 **Poles and zeros of the two pendulum cart system.** Using MATLAB, obtain the poles and zeros for $L1$-$L4$.

Convert the A, B, C, D matrices to state space using `ss('sys'),` and then convert it to a transfer function using `tf('sys)`. Then, using `zero('sys')` and `pole('sys')` we obtain the zeros and poles.

```
sys_ss = ss(A, B, C, D);
sys_tf = tf(sys_ss);
zeros = zero(sys_tf)
poles = pole(sys_tf)
```

L1:

```
zeros = 4×1 complex
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
   0.0000 + 1.0000i
   0.0000 - 1.0000i
```

```
poles = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

L2:

```
zeros = 4×1 complex
  -1.0000 + 0.0000i
  -1.0000 - 0.0000i
   1.0000 + 0.0000i
   1.0000 + 0.0000i
```

```
poles = 6×1
         0
         0
   -1.4142
   -1.0000
    1.4142
    1.0000
```

L3:

```
zeros = 4×1 complex
  -0.0000 + 1.4142i
  -0.0000 - 1.4142i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

```
poles = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.8113i
   0.0000 - 1.8113i
  -0.0000 + 1.1042i
  -0.0000 - 1.1042i
```

L4:

```
zeros = 4×1
   -1.4142
   -1.0000
    1.4142
    1.0000
```

```
poles = 6×1
         0
         0
   -1.8113
   -1.1042
    1.8113
    1.1042
```

Exercise 4 Obtain a state space realization of the transfer function,

$$\hat{G}(s) = \frac{s^2+4s+4}{s^2+3s+2}.$$

Is your realization minimal?

$$\hat{G}(s) = \frac{s^2+4s+4}{s^2+3s+2} = \frac{(s+2)^2}{(s+2)(s+1)}$$

$$= \frac{s+2}{s+1} = \frac{(s+1)+1}{s+1}$$

$$= \frac{1}{s+1} + 1$$

$$\boxed{A=-1 \quad B=1 \quad C=1 \quad D=1}$$

This is the minimal realization

Exercise 5 Obtain a state space representation of the following transfer function.

$$\hat{G}(s) = \begin{pmatrix} \dfrac{s^2+1}{s^2-1} \\[2ex] \dfrac{2}{s^2+1} \end{pmatrix}$$

$$\hat{G}(s) = \begin{pmatrix} \dfrac{s^2+1}{s^2-1} \\ \dfrac{2}{s^2+1} \end{pmatrix} = \begin{pmatrix} \dfrac{2}{s^2-1} \\ \dfrac{2}{s^2+1} \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The lowest common denominator is

$$(s^2-1)(s^2+1) = s^4 - 1 := d(s)$$

and $\hat{G} = \frac{1}{d} N$ where

$$N = \begin{pmatrix} 2(s^2+1) \\ 2(s^2-1) \end{pmatrix} = \begin{pmatrix} 2s^2+2 \\ 2s^2-2 \end{pmatrix}$$

$$= s^3 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + s^2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Hence, a controllable realization of $\hat{G}$ is given by

$$\boxed{A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 2 & 0 & 2 & 0 \\ -2 & 0 & 2 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

Therefore the state space representation becomes

$$\boxed{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 + u \end{pmatrix} \quad \begin{aligned} y_1 &= 2x_1 + 2x_3 + u \\ y_2 &= -2x_1 + 2x_3 \end{aligned}}$$

Exercise 6 Obtain a state space realization of the transfer function,

$$\hat{G}(s) = \begin{pmatrix} \frac{s^2}{s^2-4} & \frac{s}{s-2} \\ \frac{1}{s+2} & -\frac{1}{s} \end{pmatrix}.$$

Within $\hat{G}(s)$

$$\frac{s^2}{s^2-4} = \frac{(s^2-4)+4}{s^2-4} = \frac{4}{s^2-4} + 1$$

and

$$\frac{s}{s-2} = \frac{(s-2)+2}{s-2} = \frac{2}{s-2} + 1$$

thus,

$$\hat{G}(s) = \begin{pmatrix} \frac{4}{s^2-4} & \frac{2}{s-2} \\ \frac{1}{s+2} & -\frac{1}{s} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

The Lowest Common Denominator is

$$d(s) = (s^2-4)s = s^3 - 4s$$

and $\hat{G} = \frac{1}{d} N$ where

$$N(s) = \begin{pmatrix} 4s & 2s(s+2) \\ s(s-2) & -(s^2-4) \end{pmatrix}$$

$$= \begin{pmatrix} 4s & 2s^2+4s \\ s^2-2s & -s^2+4 \end{pmatrix}$$

$$\therefore N(s) = s^2\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} + s\begin{pmatrix} 4 & 4 \\ -2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

Thus, the state space realization is given by

$$\boxed{A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 4 & 4 & 0 & 2 \\ 0 & 4 & -2 & 0 & 1 & -1 \end{pmatrix} \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}$$

Exercise 7 (a) Obtain a state space realization of the following single-input single-output system.

$$\ddot{y} - 3\dot{y} - 4y = \ddot{u} - 2\dot{u} - 8u$$

(b) Is your realization minimal?

(a)

Take the Laplace transform of the equation, and assume zero ICs.

$$s^2\hat{y}(s) - 3s\hat{y}(s) - 4\hat{y}(s) = s^2\hat{u}(s) - 2s\hat{u}(s) - 8\hat{u}(s)$$

$$\therefore \hat{G}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s^2 - 2s - 8}{s^2 - 3s - 4}$$

This becomes

$$\hat{G}(s) = \frac{(s-4)(s+2)}{(s-4)(s+1)} = \frac{s+2}{s+1}$$

$$\hat{G}(s) = \frac{(s+1)+1}{s+1} = \frac{1}{s+1} + 1$$

Thus, the state space realization is given by

$$\boxed{A = -1 \quad B = 1 \quad C = 1 \quad D = 1}$$

(b)

The realization in (a) is the lowest dimension of realization, Hence, is the minimal realization.

Exercise 8 Obtain a state space realization of the following input-output system.

$$\begin{aligned}\dot{y}_1 + y_2 &= \dot{u}_2 + u_1 \\ \dot{y}_2 + y_1 &= \dot{u}_1 + u_2\end{aligned}$$

Take the Laplace transform of the 2 equations assuming zero ICs

$$\begin{cases} s\hat{y}_1(s) + \hat{y}_2(s) = s\hat{u}_2(s) + \hat{u}_1(s) & \cdots (1) \\ s\hat{y}_2(s) + \hat{y}_1(s) = s\hat{u}_1(s) + \hat{u}_2(s) & \cdots (2) \end{cases}$$

$$\Rightarrow \begin{pmatrix} s & 1 \\ 1 & s \end{pmatrix}\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

solve this for the outputs

$$\begin{pmatrix} \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \frac{1}{s^2-1}\begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix}\begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

$$= \frac{1}{s^2-1}\begin{pmatrix} 0 & s^2-1 \\ s^2-1 & 0 \end{pmatrix}\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

Hence, the state space realization is given by

$$\boxed{A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Exercise 9 Obtain a linearized state space description of the following system about $u(t) \equiv 0$ and $q(t) \equiv 0$.

$$\ddot{q} + \sin q = u + \dot{u}$$
$$y + y^3 = q + \cos u$$

Let $x_1 := q$, $x_2 := \dot{q}$, $\dot{x}_2 = x_3 := \ddot{q}$

$u_1 := u$, $\dot{u}_1 = u_2 := \dot{u}$, $y_1 := y$

Rewrite the equations as

$$\begin{cases} \dot{x}_2 = -\sin x_1 + u_1 + \dot{u}_1 & \cdots ① \\ y_1 + y_1^3 = x_1 + \cos u - 1 & \cdots ② \end{cases}$$

We are given from the instructions that

$$\boxed{u_{1e} = 0} \quad \& \quad \boxed{x_{1e} = 0}$$

Since, $\dot{x}_1 = x_2 \longrightarrow \boxed{x_{2e} = 0}$

Define as,

$$x_1 = x_{1e} + \delta x_1 , \quad x_2 = x_{2e} + \delta x_2$$
$$u_1 = u_{1e} + \delta u_1 , \quad y_1 = y_{1e} + \delta y_1$$

We want to know y_{1e}, so from eqn ②

$$y_{1e} + y_{1e}^3 = \cancel{x_{1e}}^{\,0} + \cos \cancel{u_{1e}}^{\,1} - 1$$

$$y_{1e} + y_{1e}^3 = 1 - 1 = 0$$

$$y_{1e}(1 + y_e^2) = 0$$

$$\therefore \boxed{y_{1e} = 0}$$

Also, from equ ①

$$0 = -\cancel{\sin x_{1e}}^{\,0} + \cancel{u_{1e}}^{\,0} + \dot{u}_{1e}$$

$$\therefore \boxed{\dot{u}_{1e} = 0}$$

Now linearize equations ① & ②

let,
$$\left.\begin{aligned} f_1(x_1, u_1, \dot{u}_1) &= -\sin x_1 + u_1 + \dot{u}_1 \\ f_2(x_1, u_1) &= x_1 + \cos u_1 - 1 \end{aligned}\right\}$$ Taylor expansion

①: $$\delta \dot{x}_2 = \cancel{f_1(x_1, u_1, \dot{u}_1)}^{\,0} + (-\cancel{\cos x_{1e}}^{\,1})\delta x_1 + \delta u_1 + \delta \dot{u}_1$$

②: $$\delta y_1 + 3\cancel{y_{1e}^2}^{\,0}\delta y_1 = \delta x_1 + (-\cancel{\sin u_{1e}}^{\,0})\delta u_1$$

Hence, we get

$$\begin{cases} \delta \dot{x}_2 = -\delta x_1 + \delta u_1 + \delta \dot{u}_1 \\ \delta y_1 = \delta x_1 \end{cases}$$

Laplace transformation is (zero ICs)

$$s^2 \delta\hat{q} + \delta\hat{q} = s\hat{u} + \delta\hat{u}$$

$$(s^2+1)\delta\hat{q} = (s+1)\delta\hat{u}$$

$$\hat{G}(s) = \frac{\delta\hat{q}}{\delta\hat{u}} = \frac{s+1}{s^2+1}$$

Hence, the state space realization is given by

$$\boxed{A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \end{pmatrix} \quad D = 0}$$