College of Engineering
School of Aeronautics and Astronautics

AAE 421
Flight Dynamics and Controls

HW 2
Aerodynamics, Longitudinal Stability, & Linearization

Author:
Tomoki Koike

Supervisor:
Ran Dai

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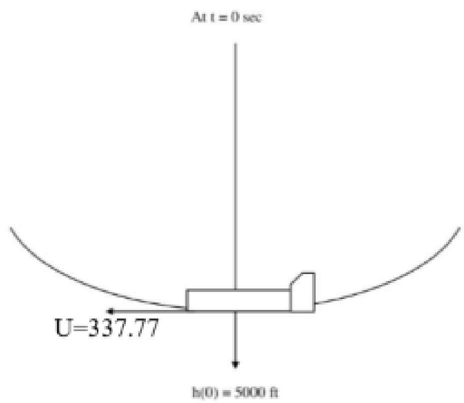
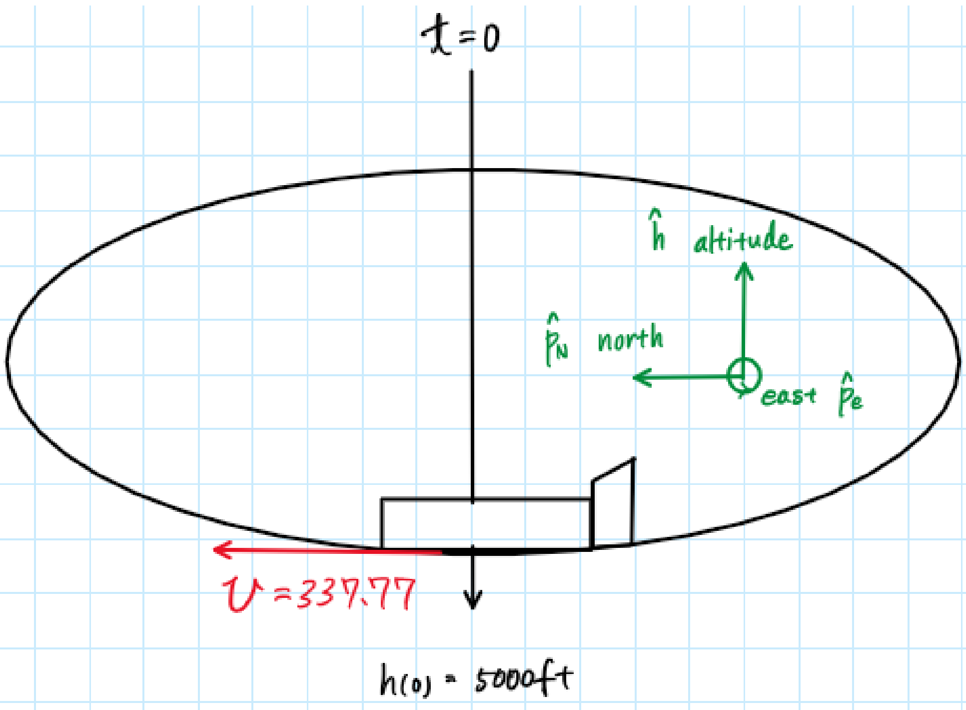
Purdue University
West Lafayette, Indiana

Problem 1. (15pts)

An aircraft is flying straight and level at a constant velocity of 337. 77 ft/sec, and then performs a symmetric pull up such that $\dot{\theta} = 0.05$ rad/s=constant. Assume the aircraft's x-axis is aligned with the flight path throughout the motion and that at $t = 0$, the aircraft's location in North-East-Altitude coordinate is $p_N = 0$, $p_E = 0$, and $h = 5000$ ft. Find the position coordinates (p_N, p_E, h) at $t = 5$ sec. Assume $\Psi = 0$.





The initial position based on the North-East-Altitude coordinates is $(0, 0, 5000)$ ft

$\bar{V}_B = v\hat{i}$, bank angle $= 0°$ $\because$ symmetric pull up

pitch angle $= \dot{\theta}t$

yaw angle $= 0°$

$$\bar{V}_I = R_3(-\psi)R_2(-\theta)R_1(-\varphi)\bar{V}_B = R_3(0)R_2(\dot{\theta}t)R_1(0)\bar{V}_B$$

$$\begin{pmatrix}\hat{p}_N \\ \hat{p}_E \\ -\hat{h}\end{pmatrix} = \begin{bmatrix}\cos(0) & -\sin(0) & 0 \\ \sin(0) & \cos(0) & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\dot{\theta}t & 0 & \sin\dot{\theta}t \\ 0 & 1 & 0 \\ -\sin\dot{\theta}t & 0 & \cos\dot{\theta}t\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & \cos(0) & -\sin(0) \\ 0 & \sin(0) & \cos(0)\end{bmatrix}\begin{bmatrix}v \\ 0 \\ 0\end{bmatrix}$$

$$= \begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\dot{\theta}t & 0 & \sin\dot{\theta}t \\ 0 & 1 & 0 \\ -\sin\dot{\theta}t & 0 & \cos\dot{\theta}t\end{bmatrix}\begin{bmatrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}v \\ 0 \\ 0\end{bmatrix}$$

$$= \begin{bmatrix}\cos\dot{\theta}t & 0 & \sin\dot{\theta}t \\ 0 & 1 & 0 \\ -\sin\dot{\theta}t & 0 & \cos\dot{\theta}t\end{bmatrix}\begin{bmatrix}v \\ 0 \\ 0\end{bmatrix} = v\cos\dot{\theta}t\,\hat{p}_N - v\sin\dot{\theta}t\,(-\hat{h})$$

$$\longrightarrow \bar{V}_I(t) = v\cos\dot{\theta}t\,\hat{p}_N + v\sin\dot{\theta}t\,\hat{h}$$

$$\bar{R}_I(t) = \int\left(v\cos\dot{\theta}t\,\hat{p}_N + v\sin\dot{\theta}t\,\hat{h}\right)dt + C_1\hat{p}_N + C_2\hat{h}$$

$$= \frac{v}{\dot{\theta}}\sin\dot{\theta}t\,\hat{p}_N + C_1\hat{p}_N - \frac{v}{\dot{\theta}}\cos\dot{\theta}t\,\hat{h} + C_2\hat{h}$$

$$= \left(\frac{v}{\dot{\theta}}\sin\dot{\theta}t + C_1\right)\hat{p}_N + \left(-\frac{v}{\dot{\theta}}\cos\dot{\theta}t + C_2\right)\hat{h}$$

$$\bar{R}_I(0) = 0\,\hat{p}_N + h_0\,\hat{h}$$

$$\therefore \frac{v}{\dot{\theta}}\cancelto{0}{\sin 0} + C_1 = 0 \longrightarrow C_1 = 0$$

$$-\frac{v}{\dot{\theta}}\cos 0 + C_2 = h_0 \longrightarrow C_2 = h_0 + \frac{v}{\dot{\theta}}$$

$$\therefore \quad \vec{R}_1(5) = \frac{(337.77\,ft/s)}{(0.05\,rad/s)}\sin(0.25\,rad)\,\hat{p}_N$$

$$+\left(5000\,ft + \frac{(337.77\,ft/s)}{(0.05\,rad/s)} - \frac{(337.77\,ft/s)}{(0.05\,rad/s)}\cos(0.25\,rad)\right)\hat{h}$$

$$= \left(1671.31\,\hat{p}_N + 5210.01\,\hat{h}\right) ft$$

$$\boxed{@\ t = 5 \quad (p_N, p_E, h) = (1671.31,\ 0,\ 5210.01)\ ft}$$

Problem 2. (10pts)

The aircraft velocity vector expressed in the Earth-fixed reference frame is
$\bar{V}_I = U\boldsymbol{I} + V\boldsymbol{J} + W\boldsymbol{K} = 6.6637\boldsymbol{I} + 289.1164\boldsymbol{J} - 407.8815\boldsymbol{K}$ (ft/sec)
and in the aircraft fixed body reference frame it is given by
$\bar{V}_b = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} = 497.7939\boldsymbol{i} + 17.4497\boldsymbol{j} + 43.5513\boldsymbol{k}$ (ft/sec)
Find the attitude of the aircraft in terms of its Euler angles (Ψ, Θ, Φ). Is your answer unique?

From the lecture notes, we know that

$$\begin{bmatrix} U \\ V \\ W \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} U \\ V \\ W \end{bmatrix}}_{V_I} = \underbrace{\begin{bmatrix} C_\theta C_\psi & S_\varphi S_\theta C_\psi - C_\varphi S_\psi & C_\varphi S_\theta C_\psi + S_\varphi S_\psi \\ C_\theta S_\psi & S_\varphi S_\theta S_\psi + C_\varphi S_\psi & C_\varphi S_\theta S_\psi - S_\varphi C_\psi \\ -S_\theta & S_\varphi C_\theta & C_\varphi C_\theta \end{bmatrix}}_{R_{\psi\theta\varphi}} \underbrace{\begin{bmatrix} u \\ v \\ w \end{bmatrix}}_{V_B}$$

$S :=$ sine $\quad C :=$ Cosine

Then using the following code implementing the Global Optimization Toolbox in MATLAB we can get the Euler Angles

```
clear all; close all; clc;

V_i = [6.6637; 289.1164; -407.8815];  % [ft/s]
V_b = [497.7939; 17.4497; 43.5513];  % [ft/s]

% Global optimization
start = [0,0,0];
lb = [0,0,0]; ub = [2*pi, 2*pi, 2*pi];
A = []; b = [];
Aeq = []; beq = [];
objective = @(param) objfunc(param, V_i, V_b);
[angles, fval] = patternsearch(objective,start,A,b,Aeq,beq,lb,ub);

psi_opt=angles(1)-2*pi
theta_opt=angles(2)
```

```
phi_opt=angles(3)-2*pi
rad2deg(psi_opt)
rad2deg(theta_opt)
rad2deg(phi_opt)

% Objective function for the global optimization
function res = objfunc(param, V_i, V_b)
    psi = param(1); theta = param(2); phi = param(3);
    R_psi = [cos(psi), -sin(psi), 0;
                 sin(psi),  cos(psi), 0;
                        0,         0, 1];
    R_theta = [ cos(theta), 0, sin(theta);
                         0, 1,          0;
               -sin(theta), 0, cos(theta)];
    R_phi = [1,        0,         0;
             0, cos(phi), -sin(phi);
             0, sin(phi),  cos(phi)];
    temp = R_psi * R_theta * R_phi * V_b - V_i;
    res = norm(temp);
end
```

This gives us the following angles

Euler Angle	*radian*	*degree*
Ψ	-1.4667	-84.0359
Θ	2.1250	121.7535
Φ	-1.8623	-106.7615

Then validate that this is correct with the following code

```
function V_i = Vrot_BF2IF(psi, theta, phi, V_b)
    %{
       NAME:     ROT_BF2IF
       AUTHOR:   TOMOKI KOIKE
       INPUTS:   (1) psi:    THE YAW ANGLE (AROUND THE BODY Z-AXIS) IN
                             RADIANS
                 (2) theta:  THE PITCH ANGLE (AROUND THE BODY Y-AXIS) IN
                             RADIANS
                 (3) phi:    THE ROLL/BANK ANGLE (AROUND THE BODY X-AXIS) IN
                             RADIANS
                 (4) V_b:    THE VELOCITY VECTOR IN THE BODY FRAME 3x1
       OUTPUTS:  (1) V_i:    THE VELOCITY VECTOR IN THE INERTIAL FRAME 3x1
       DESCRIPTION: CONVERTS THE VELOCITY WITH RESPECT TO THE BODY FRAME TO
                    THE VELOCITY WITH RESPECT TO THE INERTIAL FRAME.
    %}

    [rows, cols] = size(V_b);
    if rows == 3 && cols == 1
```

```
    elseif rows == 1 && cols == 3
        V_b = reshape(V_b, [3, 1]);
    else
        error('Incorrect dimensions for the velocity vector');
    end

    R_psi = [cos(psi), -sin(psi), 0;
             sin(psi),  cos(psi), 0;
                    0,         0, 1];
    R_theta = [ cos(theta), 0, sin(theta);
                         0, 1,          0;
               -sin(theta), 0, cos(theta)];
    R_phi = [1,        0,         0;
             0, cos(phi), -sin(phi);
             0, sin(phi),  cos(phi)];
    V_i = R_psi * R_theta * R_phi * V_b;
end

% Validate optimization
V_i_validation = Vrot_BF2IF(psi_opt, theta_opt, phi_opt, V_b)
```

The validation gives us the following velocity vector for the inertial frame

```
V_i_validation = 3×1
    6.6639
  289.1167
 -407.8812
```

This is approximately the same as the values given in the problem statement. Thus, we confirmed that our Euler angles are correct.

Now, for the Euler angles, we can add 360 degrees to all the angles, and we will still get the same outcome. Thus, the Euler angle that we have obtained are not unique.

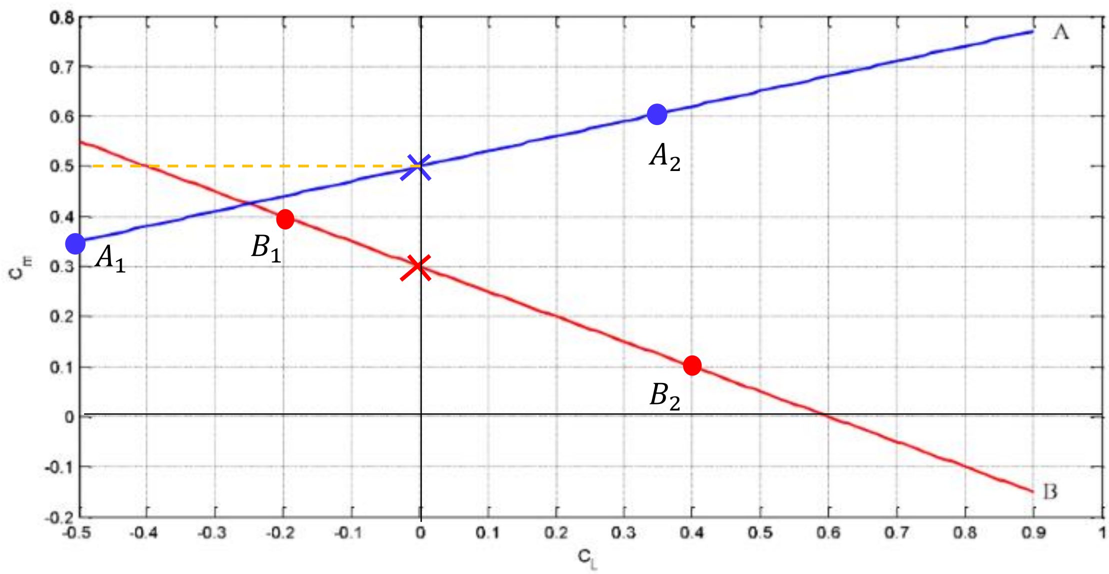
Problem 3 (10pts)

For the C_L and C_m relationship shown in the following plots

(1) Find the linear expressions of C_m in term of C_L for line A and B, respectively.

(2) To obtain a trim condition, which line should be selected?

(3) Assuming $\frac{x_{cg}}{\bar{c}}$=0.6, how to relocate the a.c center ($\frac{x_{ac}}{\bar{c}}$) to obtain a new $C_{L,trim}$=0.8?



(1) The 4 points $A_1,\ A_2,\ B_1, and\ B_2$ have the following values

Point	C_L	C_m
A_1	-0.50	0.35
A_2	0.34	0.60
B_1	-0.20	0.40
B_2	0.40	0.10

The slope for each line A and B becomes

$$\frac{dC_m}{dC_L}^A = \frac{0.60 - 0.35}{0.34 - (-0.50)} = 0.298$$

$$\frac{dC_m}{dC_L}^B = \frac{0.10 - 0.40}{0.40 - (-0.20)} = -0.500$$

The y-intercepts for each line A and B is (indicated as a cross in the graph) 0.50 and 0.30 respectively. Thus, the equations for line A and B are given by

Answers:

$$\boxed{\begin{aligned} A:\quad C_m &= 0.298C_L + 0.50 \\ B:\quad C_m &= -0.500C_L + 0.30 \end{aligned}}$$

(b) Since the condition for longitudinal static stability is

$$\frac{\partial C_m}{\partial \alpha} = C_{m_\alpha} < 0$$

Since $C_L = C_{L_\alpha}(\alpha - \alpha_0)$ with $C_{L_\alpha} > 0$, then an equivalent condition for longitudinal static stability is that

$$\frac{\partial C_m}{\partial C_L} < 0$$

Thus, line B is the line that should be selected to obtain a trim condition.

(c) From the relation

$$C_{m_\alpha} = -\frac{x_{cm} - x_{ac}}{\bar{c}} C_{L_\alpha}$$

$$\frac{\partial C_m}{\partial \alpha} = \left(\frac{-x_{cm}}{\bar{c}} + \frac{x_{ac}}{\bar{c}}\right)\frac{\partial C_L}{\partial \alpha}$$

Then we have,

$$\frac{\partial C_m}{\partial C_L} = \frac{-x_{cm}}{\bar{c}} + \frac{x_{ac}}{\bar{c}}$$

Hence, if we change x_{ac} then $\partial C_m / \partial C_L$ changes. This means that we want to change the slope of line B to have $C_{L,trim} = 0.8$. For this the new slope becomes

$$\frac{\partial C_m}{\partial C_L} = -\frac{C_{m0}}{C_{L,trim}} = -\frac{0.30}{0.80} = -0.375$$

Hence, where $x_{cg}/\bar{c} = 0.6$

$$-0.375 = -0.6 + \frac{x_{ac}}{\bar{c}}$$

$$\boxed{\left(\frac{x_{ac}}{\bar{c}}\right)_{new} = 0.225}$$

The original location of the aerodynamic center was

$$-0.500 = -0.6 + \frac{x_{ac}}{\bar{c}}$$

$$\left(\frac{x_{ac}}{\bar{c}}\right)_{original} = 0.1$$

Hence,

$$\Delta\left(\frac{x_{ac}}{\bar{c}}\right) = \left(\frac{x_{ac}}{\bar{c}}\right)_{new} - \left(\frac{x_{ac}}{\bar{c}}\right)_{original} = 0.125$$

We have to relocate the aerodynamic center by 0.125 from the original position for line B.

Problem 4. (15pts)

Wind tunnel test on a full-scale flying wing yielded the following database

Angle of Attack, deg	C_L	$C_{m_{cg}}$
8.0	0.64	-0.014
5.0	0.40	0.010
2.0	0.16	0.034
-3.0	-0.24	0.074

The configuration c.g is located 0.58 ft from the leading edge of the chord and the chord length is 2.6 ft.

(a) Estimate the configuration lift curve slope
(b) Is the configuration, as tested, statically stable? Explain your answer.
(c) Estimate values for C_m at the aerodynamic center and aerodynamic center location.
(d) Can this configuration, as tested, be trimmed in a steady glide condition? Explain your answer.
(e) If the answer to part (d) is yes, then estimate values for trim angle of attack and trim lift coefficient

$$\alpha = [-0.0524, 0.0349, 0.0873, 0.1396]\ (rad)$$

(a) .

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_1 = \frac{0.16-(-0.24)}{0.0349-(-0.0524)} = \frac{0.40}{0.0873} \cong 4.5819$$

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_2 = \frac{0.40-0.16}{0.0873-0.0349} = \frac{0.24}{0.0524} \cong 4.5802$$

$$\left(\frac{\partial C_L}{\partial \alpha}\right)_3 = \frac{0.64-0.40}{0.1396-0.0873} = \frac{0.24}{0.0523} \cong 4.5889$$

Then,

$$\text{mean}\left(\left(\frac{\partial C_L}{\partial \alpha}\right)_i\right) = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\partial C_L}{\partial \alpha}\right)_i$$

$$= \frac{4.5819 + 4.5802 + 4.5889}{3}$$

$$\frac{\partial C_L}{\partial \alpha} \cong 4.5837$$

Next,

$$C_L = \frac{\partial C_L}{\partial \alpha}\alpha + b \qquad b := \text{y-intercept}$$

$$b_1 = C_{L_1} - \frac{\partial C_L}{\partial \alpha}\alpha_1 = -0.24 - (4.5837)(-0.0524)$$
$$= 0.1971\,e{-5}$$

likewise,
$$b_2 = C_{L_2} - \frac{\partial C_L}{\partial \alpha}\alpha_2 = -0.1314\,e{-5}$$

$$b_3 = C_{L_3} - \frac{\partial C_L}{\partial \alpha}\alpha_3 = -0.3285\,e{-5}$$

$$b_4 = C_{L_4} - \frac{\partial C_L}{\partial \alpha}\alpha_4 = -0.5225\,e{-5}$$

$$b = \text{mean}(b_i) = -1.9708\,e{-6} \approx 0$$

Hence,

$$\boxed{\begin{array}{ll} C_L = 4.5837\,\alpha & \text{in radians} \\ C_L = 0.0800\,\alpha & \text{in degrees} \end{array}}$$

Verify this using MATLAB

```
clear all; close all; clc;

% (a)
alpha_deg = [-3.0, 2.0, 5.0, 8.0];  % angle of attack [deg]
alpha_rad = deg2rad(alpha_deg);  % angle of attack array in radians [rad]
Cl = [-0.24, 0.16, 0.40, 0.64];  % lift coefficients
Cm_cg = [0.074, 0.034, 0.010, -0.014];  % pitch moment coefficients w.r.t the cg
x_cg = 0.58;  % [ft] from the leading edge of the chord

% y-intercept estitmation
slope = 4.5837;
bs = []
for i = 1:numel(alpha_rad)
    b = Cl(i) - slope*alpha_rad(i);
    bs = [bs, b];
end
bs
b_mean = mean(bs)
```

```
% Linear fit for the lift coefficient over angle of attack (deg)
p = polyfit(alpha_deg, Cl, 1)
Cl_fit = polyval(p, alpha_deg)
Cl_resid = Cl - Cl_fit
SSresid = sum(Cl_resid.^2)
SStotal = (length(Cl)-1) * var(Cl)
rsq = 1 - SSresid/SStotal
rsq_adj = 1 - SSresid/SStotal * (length(Cl)-1)/(length(Cl)-length(p))

% Linear fit for the lift coefficient over angle of attack (rad)
p = polyfit(alpha_rad, Cl, 1)
Cl_fit = polyval(p, alpha_rad)
Cl_resid = Cl - Cl_fit
SSresid = sum(Cl_resid.^2)
SStotal = (length(Cl)-1) * var(Cl)
rsq = 1 - SSresid/SStotal
rsq_adj = 1 - SSresid/SStotal * (length(Cl)-1)/(length(Cl)-length(p))
```

The result is identical to the calculations done by hand.

	slope	*y-intercept*
radians	4.5837	~0
degrees	0.0800	~0
		$R^2 \sim 1$

(b).

Compute the pitch moment gradient with respect to the angle of attack for the given data using MATLAB

```
dCm_da = [];
for i = 1:numel(alpha_rad)-1
    dCm_da_i = (Cm_cg(i+1) - Cm_cg(i)) / (alpha_rad(i+1) - alpha_rad(i));
    dCm_da = [dCm_da, dCm_da_i];
end
dCm_da
dCm_da_mean = mean(dCm_da)
```

$$\left(\frac{\Delta C_{m_{cg}}}{\Delta\alpha}\right)_i = \frac{C_{M_{cg,i+1}} - C_{M_{cg,i}}}{\alpha_{i+1} - \alpha_i} = [-0.4584, -0.4584, -0.4584]$$

Thus,

$$\frac{\Delta C_{M_{cg}}}{\Delta\alpha} = -0.4584 < 0$$

The pitch moment gradient with respect to the angle of attack is negative meaning that the configuration is statically stable.

(c) .

From the equation of C_L that we have computed in part (a) we know that the lift coefficient is zero when the angle of attack, α is zero. Thus, we interpolate the pitching moment values for angle of attack of zero. This pitching moment is the moment at the aerodynamic center.

$$C_{m_{ac}} = \frac{0.034 - 0.074}{2.0 - (-3.0)}\left[0 - (-3.0)\right] + 0.074$$

$$C_{m_{ac}} = 0.0500$$

MATLAB also gives us the same value

```
% Interpolation
Cm_ac = interp1(alpha_deg, Cm_cg, 0);
```

```
Cm_ac = 0.0500
```

Hence,

$$C_{m_{ac}} = 0.0500$$

Then, to find x_{ac} we use the following two values

$$\begin{pmatrix} \dfrac{\Delta C_L}{\Delta \alpha} & = 4.5837 \\ \dfrac{\Delta C_{m_{cg}}}{\Delta \alpha} & = -0.4584 \end{pmatrix}$$

We are also given that

$$\frac{x_{cg}}{\bar{c}} = \frac{2.6 - 0.58}{2.6} \cong 0.7769$$

We can plug these values into the following equation

$$C_{M_\alpha} = -\frac{x_{cg} - x_{ac}}{\bar{c}} C_{L_\alpha}$$

$$x_{cg} - x_{ac} = -\frac{C_{M_\alpha}}{C_{L_\alpha}} \bar{c}$$

$$\frac{x_{ac}}{\bar{c}} = \frac{C_{M_\alpha}}{C_{L_\alpha}} + \frac{x_{cg}}{\bar{c}} = \frac{-0.4584}{4.5837} + 0.7769 = 0.6769$$

Hence,

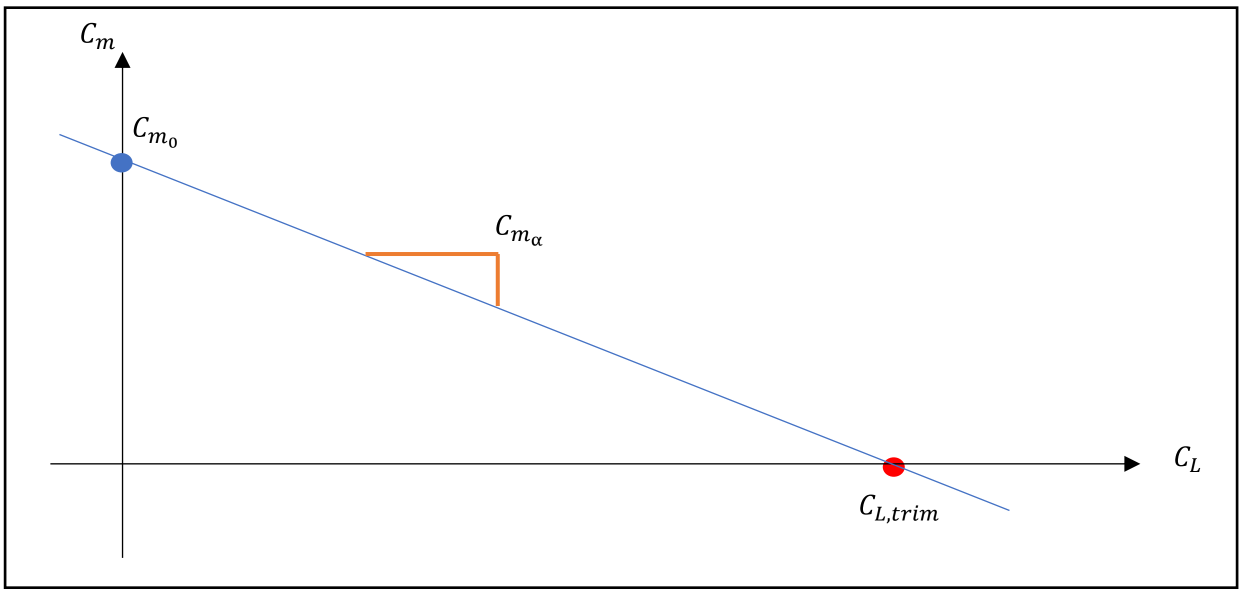
$$x_{ac} = 0.6739 \times 2.6\, ft = 1.7599\, ft \quad w.r.t \;\; the \; trailing \; edge$$

Thus, from the leading edge

$$\boxed{x_{ac} = 2.6 - 1.7599 = 0.8401\, ft}$$

(d) .

Since, $C_{m_\alpha} < 0$ and $C_{m_0} > 0$, the graph of the pitching moment becomes like the follow



Thus, we can tell that this configuration is possible to have a trim condition for a steady level flight.

(e) .

From interpolation, we can find the angle where the pitching moment is zero.

$$\boxed{\alpha_{trim} = \frac{8.0 - 5.0}{-0.014 - 0.010}(0 - 0.010) + 5.0 \; = \; 6.25 \, deg}$$

This agrees with the MATLAB results

```
% Interpolation to find the trim angle of attack
alpha_trim = interp1(Cm_cg, alpha_deg, 0);
```

```
alpha_trim = 6.2500
```

Then interpolate the lift coefficient based on this trim angle of attack

$$\boxed{C_{L,trim} = \frac{0.64 - 0.40}{8.0 - 5.0}(6.25 - 5.0) + 0.40 = \; 0.50}$$

This is congruent with the results computed by MATLAB

```
% Interpolation to find the trim lift coefficient
Cl_trim = interp1(alpha_deg, Cl, alpha_trim);
```

```
Cl_trim = 0.5000
```

Problem 5. (10pts)

Consider the following nonlinear 2nd-order system

$$\ddot{y} + a_1 \dot{y}^2 + \frac{a_0}{y^2} = f$$

where a_0 and a_1 are constant, and $a_0 > 0$.

(1) For a constant input $f = f_0 > 0$, determine the equilibrium points of the system
(2) Obtain the linearized equations of the system at the equilibrium points
(3) Express the linearized model in state equations, choosing $x_1 = \delta y$, $x_2 = \delta \dot{y}$, and $u = \delta f$

(1) .

$$\ddot{y} = f_0 - \frac{a_0}{y^2} - a_1 \dot{y}^2$$

Now, we know from $\frac{dy}{dt} = \dot{y} \longleftrightarrow \dot{y}_e = 0$
And from the equation above

$$0 = f_0 - \frac{a_0}{y_e^2} - a_1 \cancel{\dot{y}_e^2}^{\,0}$$

$$\longleftrightarrow y_e = \pm \sqrt{\frac{a_0}{f_0}}$$

Thus, if

$$\boxed{x_e = \begin{bmatrix} y_e \\ \dot{y}_e \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{a_0}{f_0}} \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -\sqrt{\frac{a_0}{f_0}} \\ 0 \end{bmatrix} \qquad u_e = f_0}$$

(2).

When,

$$y_e = \sqrt{\frac{a_0}{f_0}}$$

$$y = y_e + \delta y = \sqrt{\frac{a_0}{f_0}} + \delta y, \quad \dot{y} = \dot{y}_e + \delta\dot{y} = \delta\dot{y}$$

$$\longrightarrow \ddot{y} = \delta\ddot{y}$$

Plug these into the equation

say, $f(y, \dot{y}, f) = f - \frac{a_0}{y^2} - a_1\dot{y}^2$ and use Taylor expansion

$$\delta\ddot{y} = \frac{\partial}{\partial y} f(y,\dot{y},f) + \frac{\partial}{\partial\dot{y}} f(y,\dot{y},f)\delta\dot{y} + \frac{\partial}{\partial f} f(y,\dot{y},f)\delta f$$

$$\delta\ddot{y} = -(-2)\frac{a_0}{y_e^3}\delta y - \cancel{2a_1\dot{y}_e\delta\dot{y}}^{\,0} + \delta f$$

$$\delta\ddot{y} = \frac{2a_0}{y_e^3}\delta y + \delta f$$

$$\longrightarrow y_e^3 = \frac{a_0}{f_0}\sqrt{\frac{a_0}{f_0}}$$

$$\therefore \delta\ddot{y} = 2a_0 \cdot \frac{f_0}{a_0}\sqrt{\frac{f_0}{a_0}}\delta y + \delta f = 2f_0\sqrt{\frac{f_0}{a_0}}\delta y + \delta f$$

$$\boxed{\delta\ddot{y} = 2f_0\sqrt{\frac{f_0}{a_0}}\delta y + \delta f}$$

When,

$$y_e = -\sqrt{\frac{a_0}{f_0}}$$

Plug $y_e = -\sqrt{\frac{a_0}{f_0}}$ into ①, then

$$\delta\ddot{y} = 2a_0\left(-\frac{f_0}{a_0}\sqrt{\frac{f_0}{a_0}}\right)\delta y + \delta f$$

$$\boxed{\delta\ddot{y} = -2f_0\sqrt{\frac{f_0}{a_0}}\,\delta y + \delta f}$$

(3).

if, $x_1 := \delta y \quad x_2 := \delta\dot{y} \quad u := \delta f$

$$\delta\ddot{y} = 2f_0\sqrt{\frac{f_0}{a_0}}\,\delta y \quad \text{or} \quad \delta\ddot{y} = -2f_0\sqrt{\frac{f_0}{a_0}}\,\delta y$$

$$\longrightarrow \quad \dot{x}_2 = 2f_0\sqrt{\frac{f_0}{a_0}}\,x_1 + u \qquad \dot{x}_2 = -2f_0\sqrt{\frac{f_0}{a_0}}\,x_1 + u$$

$$\dot{x}_1 = x_2 \qquad \dot{x}_1 = x_2$$

Hence, the state space representation is given by,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 2f_0\sqrt{\frac{f_0}{a_0}}\,x_1 + u \end{pmatrix}$$

or

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -2f_0\sqrt{\frac{f_0}{a_0}}\,x_1 + u \end{pmatrix}$$

Problem 6. (10 pts)

consider an airplane in constant-altitude, straight-line flight. The velocity equation is

$$\dot{V} = T - \frac{1}{2}kV^2$$

where the second term represents aerodynamic drag, and assume k = constant, and T is the engine thrust acceleration. Treat T as the control (input). Let V^* be a given constant cruise speed. Obtain the linearized differential equation for the velocity around V^*.

From the equation

$$\dot{V} = T - \frac{1}{2}kV^2$$

Set the LHS to 0

$$0 = T - \frac{1}{2}kV_e^2$$

$$V_e = \pm\sqrt{\frac{2T}{k}}$$

$$\rightarrow \quad V^* = V_e = \pm\sqrt{\frac{2T}{k}}$$

let $\quad V = V_e + \delta V \quad \& \quad \dot{V} = \dot{V}_e + \delta\dot{V} = \delta\dot{V}$

$$\delta\dot{V} = \delta T - \frac{1}{2}k(V_e + \delta V)^2$$

$$\delta\dot{V} = \delta T - \frac{kV_e^2}{2}\left(1 + \frac{\delta V}{V_e}\right)^2$$

$$\left(1 + \frac{\delta V}{V_e}\right)^2 \approx 1 + \frac{2\delta V}{V_e} \quad (\because \delta V \ll V_e)$$

$$\delta\dot{V} = \delta T - \frac{kV_e^2}{2}\left(1 + \frac{2\delta V}{V_e}\right)$$

$$\delta\dot{V} = \delta T - \frac{kV_e^2}{2} - kV_e\delta V$$

if $\quad V_e = V^* = \sqrt{\frac{2T}{k}}$

$$\delta\dot{V} = \delta T - \cancel{\frac{k}{2}\cdot\frac{2T}{k}}^{\,0} - k\sqrt{\frac{2T}{k}}\,\delta V$$

$$\boxed{\delta\dot{V} = \delta T - \sqrt{2Tk}\,\delta V}$$

if $\quad V_e = V^* = -\sqrt{\frac{2T}{k}}$

$$\delta\dot{V} = T - \frac{k}{2}\frac{2T}{k} + k\sqrt{\frac{2T}{k}}\,\delta V$$

$$\boxed{\delta\dot{V} = \delta T + \sqrt{2Tk}\,\delta V}$$

Problem 7. (15pts)

From the nonlinear flight dynamics model, derive the following linear perturbation equations for Y force

$$m(\dot{v} + u_0 r) = \Delta Y + mg\cos(\theta_0)\phi \ .$$

And moments

$$\Delta L = I_{xx}\dot{p} - I_{xz}\dot{r}$$

$$\Delta M = I_{yy}\dot{q}$$

$$\Delta N = -I_{xz}\dot{p} + I_{zz}\dot{r}$$

Show all steps.

The equations we need are

$$\begin{cases} Y + mg\cos\Theta\sin\Phi = m(\dot{V} + RU - PW) \\ L = I_{xx}\dot{P} - I_{xz}(\dot{R} + PQ) - (I_{yy} - I_{zz})QR \\ M = I_{yy}\dot{Q} - I_{zx}(R^2 - P^2) - (I_{zz} - I_{xx})PR \\ N = I_{zz}\dot{R} - I_{zx}(\dot{P} - QR) - (I_{xx} - I_{yy})PQ \end{cases}$$

Trim conditions are

$$\begin{aligned} &X_0 - mg\sin\theta_0 = 0 \qquad && L_0 = 0 \qquad && \psi_0 = 0 \\ &Y_0 = 0 && M_0 = 0 && \varphi_0 = 0 \\ &Z_0 + mg\cos\theta_0 = 0 && N_0 = 0 && \\ & && P_0 = 0 && \\ &V_0 = 0 && q_0 = 0 && \\ &W_0 = 0 && r_0 = 0 && \end{aligned}$$

(i) $$Y + mg\cos\Theta\sin\Phi = m(\dot{V} + RU - PW)$$

$$Y = m(\dot{V} + RU - PW) - mg\cos\Theta\sin\Phi$$

let, $f(\dot{V}, R, U, P, W, \theta, \varphi)$

Use Taylor expansion

$$y = \left(\frac{\partial}{\partial \dot{V}} f\right)\dot{v} + \left(\frac{\partial}{\partial R} f\right) r + \left(\frac{\partial}{\partial U} f\right) u + \left(\frac{\partial}{\partial P} f\right) p + \left(\frac{\partial}{\partial W} f\right) w + \left(\frac{\partial}{\partial \Theta} f\right)\theta + \left(\frac{\partial}{\partial \Phi} f\right)\varphi$$

$$y = m\dot{v} + mU_0 r + m\cancel{r_0}^{0} u - m\cancel{W_0}^{0} p - m\cancel{p_0}^{0} w + mg\sin\cancel{\theta_0}^{0}\sin\varphi_0\,\theta - mg\cos\theta_0\cancel{\cos\varphi_0}^{1}\varphi$$

$$y = m(\dot{v} + U_0 r) - mg\cos(\theta_0)\varphi$$

let $\Delta Y = y$

$$\therefore \quad \boxed{m(\dot{v} + U_0 r) = \Delta Y + mg\cos(\theta_0)\varphi \quad \text{q.e.d}}$$

(ii)

$$L = I_{xx}\dot{P} - I_{xz}(\dot{R} + PQ) - (I_{yy} - I_{zz})QR$$

let $f(\dot{P}, \dot{R}, P, Q, R)$

$$\Delta L = \left(\frac{\partial}{\partial \dot{P}} f\right)\dot{p} + \left(\frac{\partial}{\partial \dot{R}} f\right)\dot{r} + \left(\frac{\partial}{\partial P} f\right)p + \left(\frac{\partial}{\partial Q} f\right)q + \left(\frac{\partial}{\partial R} f\right)r$$

$$= I_{xx}\dot{p} - I_{xz}\dot{r} - I_{xz}\cancel{q_0}^{0} p + \left[-I_{xz}\cancel{p_0}^{0} - (I_{yy} - I_{zz})\cancel{r_0}^{0}\right] q - (I_{yy} - I_{zz})\cancel{q_0}^{0} r$$

$$\boxed{\therefore \Delta L = I_{xx}\dot{p} - I_{xz}\dot{r} \quad \text{q.e.d}}$$

(iii)

$$M = I_{yy}\dot{Q} - I_{zx}(R^2 - P^2) - (I_{zz} - I_{xx})PR$$

let the RHS be equal to $f(\dot{Q}, R, P)$ do Taylor expansion

$$\Delta M = \left(\frac{\partial}{\partial \dot{Q}} f\right)\dot{q} + \left(\frac{\partial}{\partial R} f\right)r + \left(\frac{\partial}{\partial P} f\right)p$$

$$= I_{yy}\dot{q} - 2I_{zx}\cancel{r_0}^{0} r - (I_{zz} - I_{xx})\cancel{p_0}^{0} r + 2I_{zx}\cancel{p_0}^{0} p - (I_{zz} - I_{xx})\cancel{r_0}^{0} p$$

$$\boxed{\therefore \Delta M = I_{yy}\dot{q} \quad \text{q.e.d}}$$

(iv) $$N = I_{zz}\dot{R} - I_{zx}(\dot{P} - QR) - (I_{xx} - I_{yy})PQ$$

let the RHS be $f(\dot{R}, \dot{P}, P, Q, R)$, then Taylor expansion

$$\Delta N = \left(\frac{\partial}{\partial \dot{R}} f\right)\dot{r} + \left(\frac{\partial}{\partial \dot{P}} f\right)\dot{p} + \left(\frac{\partial}{\partial P} f\right)p + \left(\frac{\partial}{\partial Q} f\right)q + \left(\frac{\partial}{\partial R} f\right)r$$

$$= I_{zz}\dot{r} - I_{zx}\dot{p} - (I_{xx} - I_{yy})\cancel{q_0}^{0}p$$

$$+ I_{zx}\cancel{r_0}^{0}q - (I_{xx} - I_{yy})\cancel{P_0}^{0}q + I_{zx}\cancel{q_0}^{0}r$$

$$\boxed{\therefore \Delta N = -I_{zx}\dot{p} + I_{zz}\dot{r}}$$

q.e.d

Problem 8. (15 pts)

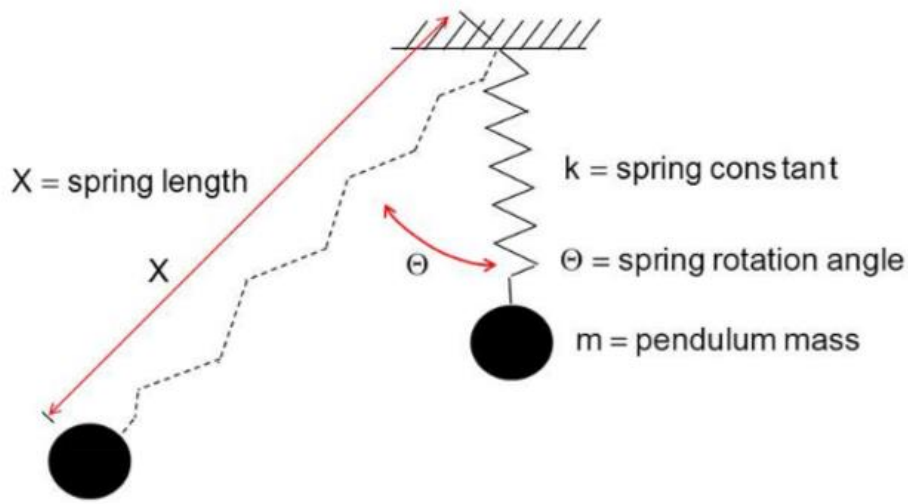
Consider the 2-degree-of-freedom spring mass pendulum shown below (All motion is in the plane of the picture shown). The nonlinear equations of motion are given by

$$\begin{aligned} (1) \quad & \ddot{X} + \frac{k}{m}(X - L) - g\cos(\Theta) - X\dot{\Theta}^2 &= 0 \\ (2) \quad & X^2\ddot{\Theta} + gX\sin(\Theta) + 2\dot{\Theta}X\dot{X} &= 0 \end{aligned}$$

where L is the original spring length.

Linearize the equations of motion for this system. Let the reference condition be the equilibrium (no motion) state for the pendulum mass. In particular

(a) Define a set of perturbation variables
(b) Substitute the results of Part (a) into the equations of motion
(c) Expand the equations and discard appropriate terms (show the terms that are to be discarded)



First we will look for the equilibrium values
From equation (1)

$$\cancelto{0}{\ddot{X}} = -\frac{k}{m}(X-L) + g\cos\Theta + \cancelto{0}{X\dot{\Theta}^2}$$

$$\longrightarrow \quad 0 = -\frac{k}{m}(X_e - L) + g\cos\Theta_e + 0$$

From equation (2)

$$\cancelto{0}{X^2\ddot{\Theta}} = -gX\sin\Theta - \cancelto{0}{2\dot{\Theta}X\dot{X}}$$

$$0 = -gX\sin\Theta_e$$

$$\therefore \ \Theta_e = 0$$

Then, from above

$$\frac{k}{m}(X_e - L) = g$$

$$X_e = \frac{mg}{k} + L = \frac{mg + kL}{k}$$

Now, let

$$\boxed{\begin{aligned} X &= X_e + x = \frac{mg+kL}{k} + x \\ \Theta &= \Theta_e + \theta = \theta \\ \text{where} \quad & x := \delta X, \ \dot{x} := \delta\dot{x}, \ \ddot{x} := \delta\ddot{x} \\ & \theta := \delta\Theta, \ \dot{\theta} := \delta\dot{\Theta}, \ \ddot{\theta} := \delta\ddot{\Theta} \end{aligned}}$$

plug these into the original equations based on the equilibrium state

$$(1) \quad \cancelto{0}{\ddot{X}_e} + \delta\ddot{X} = -\frac{k}{m}(X_e + \delta X - L) + g\cos(\cancelto{0}{\Theta_e} + \delta\Theta)$$
$$+ (X_e + \delta X)(\cancelto{0}{\dot{\Theta}_e} + \delta\dot{\Theta})^2$$

$$\ddot{x} = -\frac{k}{m}\left(\frac{mg+kL}{k} + x - L\right) + g\cos\theta + \cancel{\left(\frac{mg+kL}{k} + x\right)\dot{\theta}^2} \rightarrow 0$$

$$\ddot{x} = -g - \cancel{\frac{kL}{m}}^{\,0} - \frac{k}{m}x + \cancel{\frac{kL}{m}}^{\,0} + g\cos\cancel{\theta}^{\,1}$$

$$\boxed{\ddot{x} = -\frac{k}{m}x}$$

(2) $(X_e + \delta X)^2(\cancel{\ddot{\Theta}_e}^{\,0} + \delta\ddot{\Theta})$

$= -g(X_e + \delta X)\sin(\cancel{\Theta}^{\,0} + \delta\Theta) - 2(\cancel{\dot{\Theta}_e}^{\,0} + \delta\dot{\Theta})(X_e + \delta X)(\cancel{\dot{X}_e}^{\,0} + \delta\dot{X})$

$\rightarrow$ $(X_e + \delta X)^2\,\delta\ddot{\Theta}$

$= -g(X_e + \delta X)\sin\delta\Theta - 2\delta\dot{\Theta}(X_e + \delta X)\delta\dot{X}$

$\rightarrow$ $\ddot{\theta}$

$= -g(X_e + x)\sin\theta\,(X_e + x)^{-2} - 2\dot{\theta}(X_e + x)\dot{x}\,(X_e + x)^{-2}$

$\rightarrow$ $\ddot{\theta} = -g\sin\theta\,(X_e + x)^{-1} - 2\dot{\theta}\dot{x}(X_e + x)^{-1}$

$\because$ $(X_e + x)^{-1} = \frac{1}{X_e}\left(1 + \frac{x}{X_e}\right)^{-1} \approx \frac{1}{X_e}\left(1 - \frac{x}{X_e}\right)$

$\therefore$ $x \ll X_e$

$\rightarrow$ $\ddot{\theta} = -g\,\cancel{\sin\theta}^{\,\theta} \cdot \frac{1}{X_e}\left(1 - \frac{x}{X_e}\right) - 2\dot{\theta}\dot{x} \cdot \frac{1}{X_e}\left(1 - \frac{x}{X_e}\right)$

$\ddot{\theta} = -\frac{g}{X_e}\theta + \cancel{\frac{g}{X_e^2}x\theta}^{\,\sim 0} - \cancel{\frac{2}{X_e}\dot{\theta}\dot{x}}^{\,\sim 0} + \cancel{\frac{2}{X_e^2}\dot{\theta}\dot{x}x}^{\,\sim 0}$

$\ddot{\theta} = -g \cdot \frac{k}{mg + kL}\theta$

$$\boxed{\ddot{\theta} = -\frac{kg}{mg + kL}\theta}$$