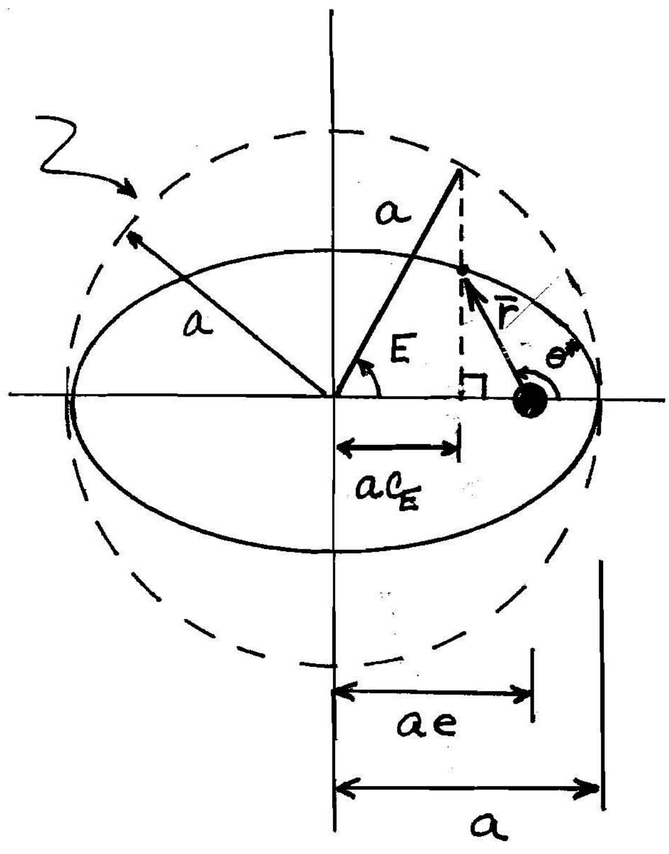
Eccentric Anomaly E

Additional variable defined for ellipse



$$a\cos E = ae - r\cos(180^\circ - \theta^*)$$
$$= ae + r\cos\theta^*$$

$$\boxed{r = \frac{p}{1+e\cos\theta^*} \rightarrow \cos\theta^* = \frac{p}{re} - \frac{1}{e}}$$

$$\cos E = \frac{ae + r\cos\theta^*}{a}$$
$$= e + \frac{r}{a}\cos\theta^*$$

$$\boxed{\cos E = \frac{a-r}{ae}}$$

OR

E obviously related to θ^*. How??

Previously $r\cos\theta^* = a\cos E - ae$

Identity $\cos 2\alpha = 2\cos^2\alpha - 1$

$$r\left(2\cos^2\left(\frac{\theta^*}{2}\right) - 1\right) = a\cos E - ae$$

$$2r\cos^2\left(\frac{\theta^*}{2}\right) = a\cos E - ae + a(1 - e\cos E)$$

$$= (a - ae)\cos E + (a - ae)$$

$$2r\cos^2\left(\frac{\theta^*}{2}\right) = a(1-e)(1+\cos E)$$

Identity $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$r\left(1 - 2\sin^2\left(\frac{\theta^*}{2}\right)\right) = a\cos E - ae$$

$$-2r\sin^2\left(\frac{\theta^*}{2}\right) = a\cos E - ae - a(1 - e\cos E)$$

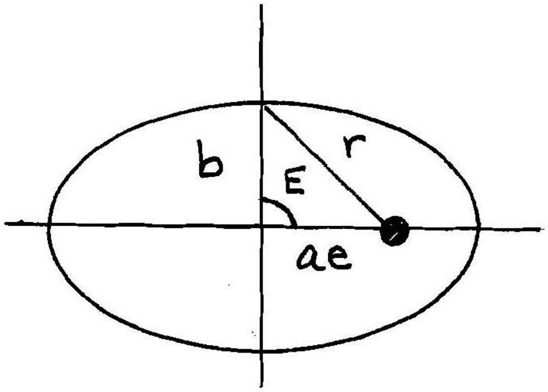
$$= (a + ae)\cos E - (a + ae)$$

$$-2r\sin^2\left(\frac{\theta^*}{2}\right) = a(1+e)(\cos E - 1)$$

$$\frac{-2r\sin^2\left(\theta^*/2\right)}{2r\cos^2\left(\theta^*/2\right)} = \frac{a(1+e)(\cos E - 1)}{a(1-e)(1+\cos E)}$$

Identity: $\tan^2\frac{E}{2} = \frac{1-\cos E}{1+\cos E}$

Note:



At $E = 90^\circ$

$$r = a(1 - e\cos 90^\circ) = a$$

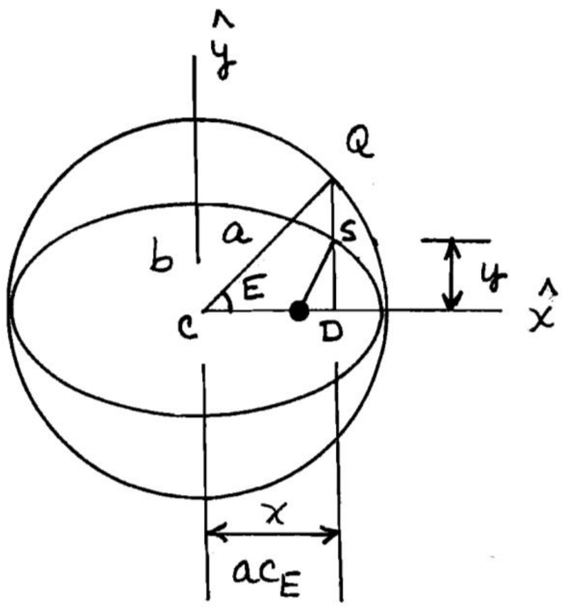
$$b^2 = r^2 - a^2 e^2$$

$$b^2 = a^2(1 - e^2)$$

$$\boxed{b = a\sqrt{1 - e^2}}$$

$$r = \frac{p}{1 + e\cos\theta^*} \rightarrow \cos\theta^* = \frac{p}{re} - \frac{1}{e} = \frac{a(1-e^2)}{ae} - \frac{1}{e} = -e$$

$\Longrightarrow$ $\boxed{}$



$$\boxed{CD = a \cos E}$$

Let x, y be point S on the ellipse

→ measured from <u>center</u> C

SD?

Equation of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{(\text{CD})^2}{a^2} + \frac{(\text{SD})^2}{b^2} = 1$$

$$\frac{a^2 \cos^2 E}{a^2} + \frac{(\text{SD})^2}{b^2} = 1$$

$$(\text{SD})^2 = b^2 \left[1 - \frac{a^2 \cos^2 E}{a^2} \right]$$

All of these fundamental relationships are useful but one of the most important reasons to introduce E is to obtain a <u>relation between position and time</u>

How? Various approaches – consider one

Kepler's Equation (Relation between position and time)

Begin with some relationships that are already known

$$p = \frac{h^2}{\mu}$$

$$r = \frac{p}{1 + e\cos\theta^*}$$

$$h = r^2\dot{\theta} = r^2\frac{d\theta}{dt}$$

Combine to eliminate *r* and *h*

$$h = r^2\frac{d\theta}{dt}$$

$$\sqrt{\mu p} = \frac{p^2}{\left(1 + e\cos\theta^*\right)^2}\frac{d\theta}{dt}$$

Rearrange

$$\boxed{\sqrt{\frac{\mu}{p^3}}dt = \frac{d\theta}{\left(1 + e\cos\theta^*\right)^2}}$$

Need to integrate to get a useful relationship for time as a function of θ^*. Direct integration is nontrivial which is why *t* was always eliminated previously. However, it is possible to use eccentric anomaly E and the relationship

$$r = a\,(1 - e\cos E)$$

Note: introducing E implies that the result will only apply to elliptical orbits.

To use E to relate position and time:

1) Relate equation for r

$$\cos E = \frac{a-r}{ae} \qquad \text{I}$$

2) Differentiate I and rearrange

$$\dot{r} = ae\dot{E}\sin E \qquad \text{II}$$

3) Given

$$\mathcal{E} = \frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Multiply by $\dfrac{2r^2a}{\mu}$

$$\frac{ar^2\dot{r}^2}{\mu} = \frac{-ar^4\dot{\theta}^2}{\mu} + 2ra - r^2$$

But $\qquad r^4\dot{\theta}^2 = h^2 = \mu p = \mu a\left(1-e^2\right)$

$$\therefore \quad \frac{ar^2\dot{r}^2}{\mu} = a^2e^2 - \left(a-r\right)^2 \qquad \text{III}$$

4) Square I

$$a^2e^2\cos^2 E = \left(a-r\right)^2$$

Combine with II, III

$$\frac{ar^2}{\mu}\left[a^2e^2\dot{E}^2\sin^2 E\right] = a^2e^2 - a^2e^2\cos^2 E$$

$$\Rightarrow \qquad r\dot{E} = \sqrt{\frac{\mu}{a}}$$

5) Rewrite

$$r\,dE = \sqrt{\frac{\mu}{a}}\,dt$$

$$r = a\left(1-e\cos E\right)$$

$$\sqrt{\frac{\mu}{a}}\,dt = a\left(1-e\cos E\right)dE$$

6) Integrate (now easy)

Corresponding relationship for a hyperbolic orbit

Can you derive this?

Define $$\mathbf{n} = \sqrt{\frac{\boldsymbol{\mu}}{\mathbf{a}^3}}$$

$$\boldsymbol{M} = \boldsymbol{n}\left(\boldsymbol{t} - \boldsymbol{t}_{\boldsymbol{p}}\right)$$

Short equation but <u>transcendental</u>

Given time (M), cannot solve for E in closed form

Solution usually obtained iteratively

<u>Solution of Kepler's Equation</u>

Because it is frequently required, the solution of Kepler's equation is of great interest. Consider the equation as written

$$M = E - e\sin E$$

By differentiation

$$dM = \left(1 - e\cos E\right)dE$$

Integration between limits 0 and *t*

$$\int_0^{E_t} dE = \int_0^{M_t} \frac{dM}{1 - e\cos E}$$

An expansion using Fourier series and noting that the period of the function is 2π has coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} dE = 1$$

$$\begin{aligned} a_m &= \frac{1}{\pi}\int_0^{2\pi} \cos\left\{m\left(E - e\sin E\right)\right\} dE \\ &= 2\, J_m(me) \end{aligned}$$

J_m is a Bessel function of the first kind of order *m*. For calculation,

$$J_m(me) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{me}{2}\right)^{2n+m}}{n!\left(n+m\right)!}$$

So that an explicit formula for eccentric anomaly is given by

$$E = M + 2\sum_{m=1}^{\infty} \frac{1}{m} J_m(me)\sin(mM)$$

Numerically, at times a few terms can be used to start the solution and then with the first approximation, E_n, continue by a Newton procedure,

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n}, \qquad n = 1, 2, \ldots, p$$

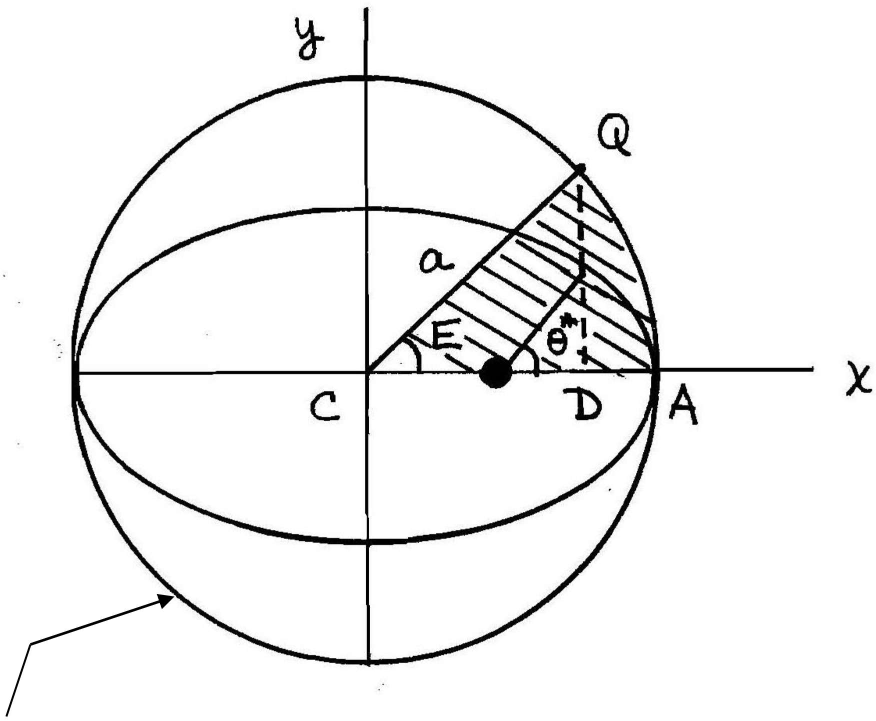
Until E_p no longer varies significantly.

Hyperbolic Anomaly H

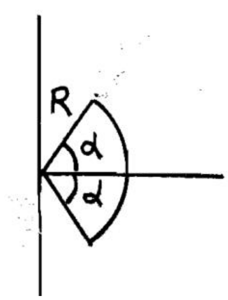
Need analog to ***E*** for hyperbola ← NOT an angle

To accomplish, note that ***E*** actually represents an area

Recall → the definition for ***E*** was based on concept of auxiliary circle
→ now define an area as a sector of that circle



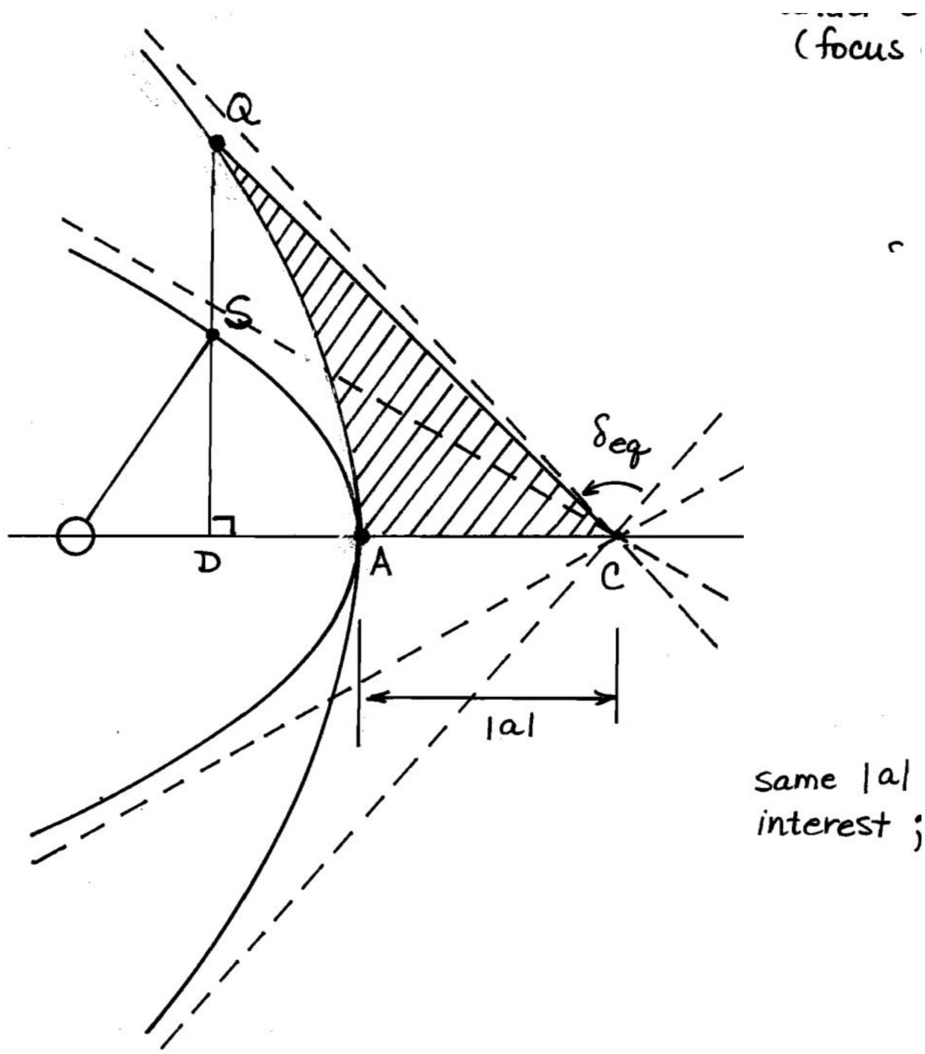
Reference geometric shape is circle



Area of a sector of a circle $= R^2\,\alpha$

Area QCA $= R^2\left(\frac{E}{2}\right) = a^2\,\frac{E}{2}$

Corresponding situation for hyperbola:
Based on reference geometric shape –

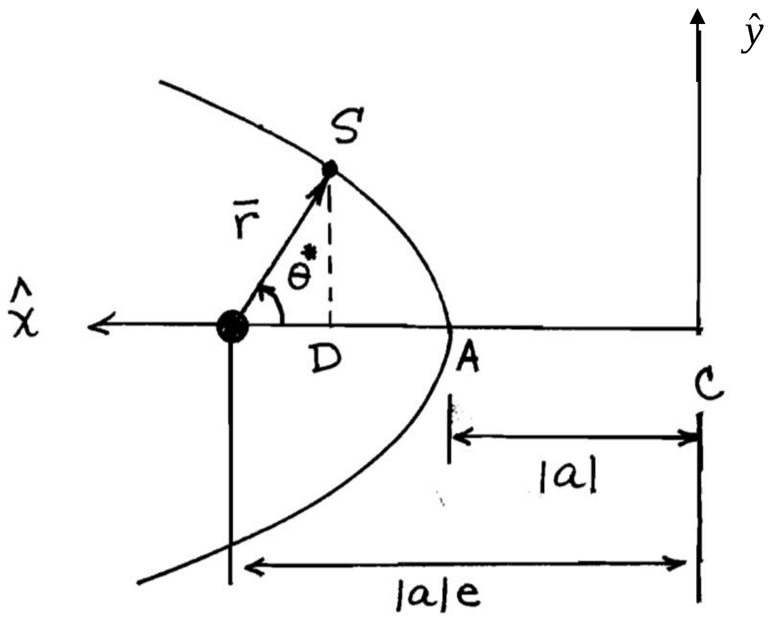


consid
eration
(focus
differs)

Same $|a|$ as orbit of interest; same pericenter; actually different focus

Define hyperbolic anomaly $\boldsymbol{H}$ in terms of relationship to area $\boldsymbol{CQA}$

Useful relationship in terms of ***H***



u = 2 (area ***ASC***)

Point S:

$\left.\begin{aligned} \boldsymbol{x} &= \cosh \boldsymbol{u} \\ \boldsymbol{y} &= \sinh \boldsymbol{u} \end{aligned}\right\}$ for unit $|a|$

Equation for point on hyperbola measured from the <u>center</u>

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{(CD)^2}{a^2} - \frac{(SD)^2}{b^2} = 1$$

$$CD = |a| \cosh H$$

$$SD = |b| \sinh H$$

Useful to obtain other relationships

Note:

$$|a|\, e - r \cos\theta^* = CD$$

$$|a|\, e - r \cos\theta^* = |a| \cosh H$$

$$|a|\, e - r\left(\frac{p}{re} - \frac{1}{e}\right) = |a| \cosh H$$

$$|a|\, e - \frac{r|a|\left(e^2 - 1\right)}{r\, e} + \frac{r}{e} = |a| \cosh H$$

$$\frac{|a| + r}{e} = |a| \cosh H$$

Again, solution iterative!

Parabolic Orbits and Barker's Equation

$$r = \frac{p}{1+\cos\theta^*} = \frac{p}{2}\left(1+\tan^2\frac{\theta^*}{2}\right)$$

$$h = r^2\dot{\theta} = r^2\frac{d\theta}{dt} = \sqrt{\mu p}$$

$$\sqrt{\frac{\mu}{p^3}}\,dt = \frac{d\theta}{\left(1+\cos\theta^*\right)^2} = \frac{1}{4}\left(1+\tan^2\frac{\theta^*}{2}\right)^2 d\theta$$

$$4\sqrt{\frac{\mu}{p^3}}\,dt = \left(1+\tan^2\frac{\theta^*}{2}\right)^2 d\theta$$

$$4\sqrt{\frac{\mu}{p^3}}\,dt = \sec^4\frac{\theta^*}{2}\,d\theta$$

$\Downarrow$

Integrate

$\longleftarrow$ Barker's Eqn

(Barker prepared extensive tables of solutions in the 18th century.)

Define $B = 3\sqrt{\frac{\mu}{p^3}}\left(t - t_p\right)$

$$\Downarrow$$

$$\tan\frac{\theta^*}{2} = \left(B + \sqrt{1 + B^2}\right)^{\frac{1}{3}} - \left(B + \sqrt{1 + B^2}\right)^{-\frac{1}{3}}$$

(from Jerome Cardan method of solving cubic equations 1545 AD)

Now better methods!