College of Engineering
School of Aeronautics and Astronautics

AAE 532
Orbital Mechanics

PS 3
Gravitational Equation of Motion and Conical Equations

Author:
Tomoki Koike

Supervisor:
K. C. Howell

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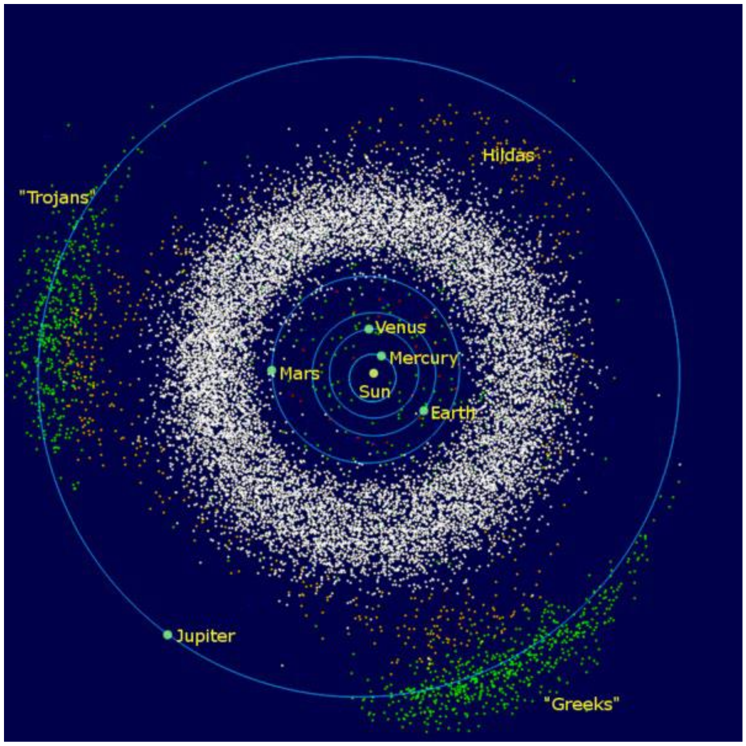
Purdue University
West Lafayette, Indiana

Problem 1: A mission concept was proposed recently to deliver a spacecraft to one or more of the Trojan asteroids, e.g. 8317 Eurysaces. Of course, as seen in the figure, there are many, many asteroids. As they move along their orbital path, the 'green' Trojans remain in the same relative locations with respect to the Sun and Jupiter. To develop a trajectory for the mission, it is also necessary to understand the path of the asteroid.



The asteroids of the inner solar system and Jupiter

Jupiter trojans | Asteroid belt
Hilda asteroids | Orbits of planets

Note that the Jupiter Trojans are divided into two groups: The Greek camp (in front of Jupiter) and the Trojan camp (trailing behind Jupiter).

(a) Return to the small body database and check the orbit of the asteroid Eurysaces. Use a view to "Look at Sun" and "Look from Above". Is Eurysaces in the Greek camp or the Trojan camp? The database lists the orbital period of Eurysaces in years. Compare it to the periods of Jupiter in its orbit relative to the Sun. Take 3 images as Jupiter moves through its orbit relative to the Sun. Use a start date of 9/18/20 and select 3 other dates along Jupiter's orbit. From the images, measure approximately the angle between the lines Sun-Jupiter and Sun-Eurysaces. How much does the angle change over your three dates?

The orbit diagram with the view of "Look at Sun" and "Look from Above" from the small body database is the following.

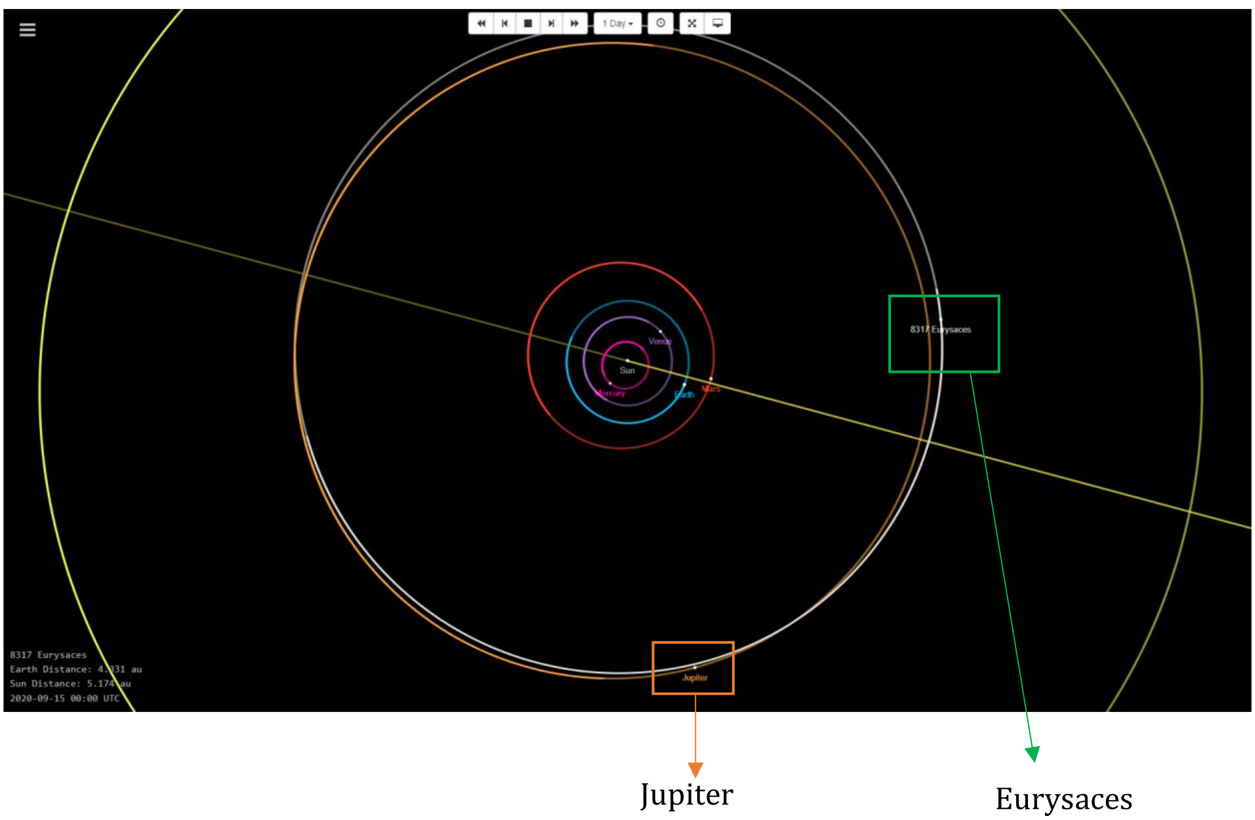


Figure 1: Above view of Eurysaces orbit

The Greek Camp is located in the L_4 Lagrangian Point, 60^o ahead of Jupiter. Whereas the Trojan Camp is located in the L_5 Lagrangian Point, 60^o behind Jupiter. From the positions of the Sun and other planets, we can see that Eurysaces is ahead of Jupiter and Eurysaces belongs to the Greek Camp.

Table 1: Orbital periods of Eurysaces and Jupiter

Orbital Period of Eurysaces (years)	Orbital Period of Jupiter (years)
12.21	11.87

Comparing the periods of Eurysaces and Jupiter, we can see that Eurysaces takes approximately one fourth of a year (3 months) longer to orbit the Sun.

Next, we have selected 3 dates to obtain images of Eurysaces orbit along with Jupiter. The 3 dates are 9/18/20, 3/18/21, 9/18/23.

9/18/20:

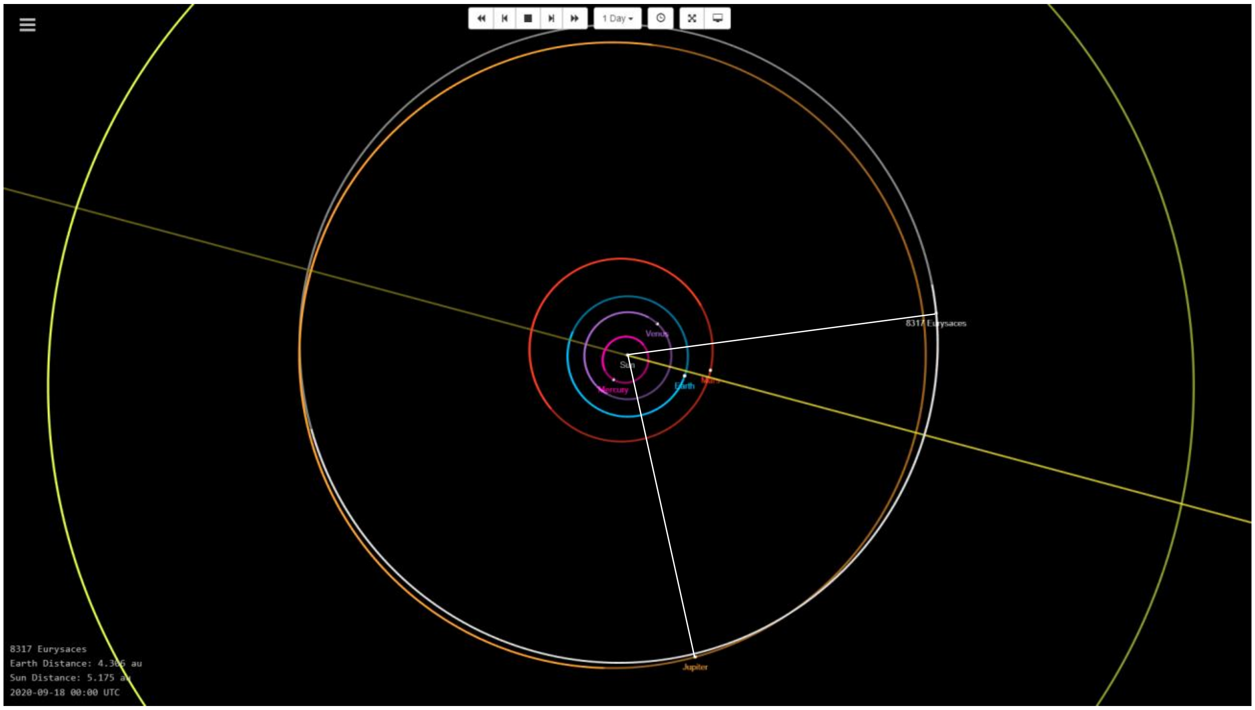


Figure 2: 9/18/20 Eurysaces orbit

3/18/21:

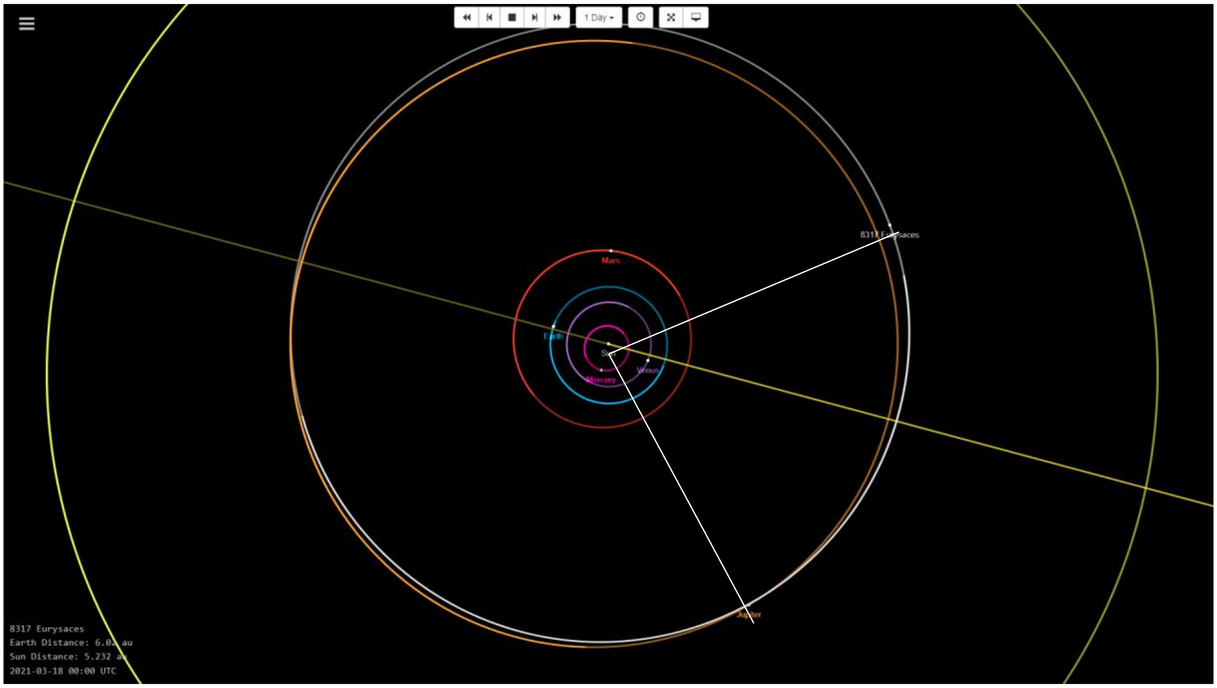


Figure 3: 3/18/21 Eurysaces orbit

9/18/23:

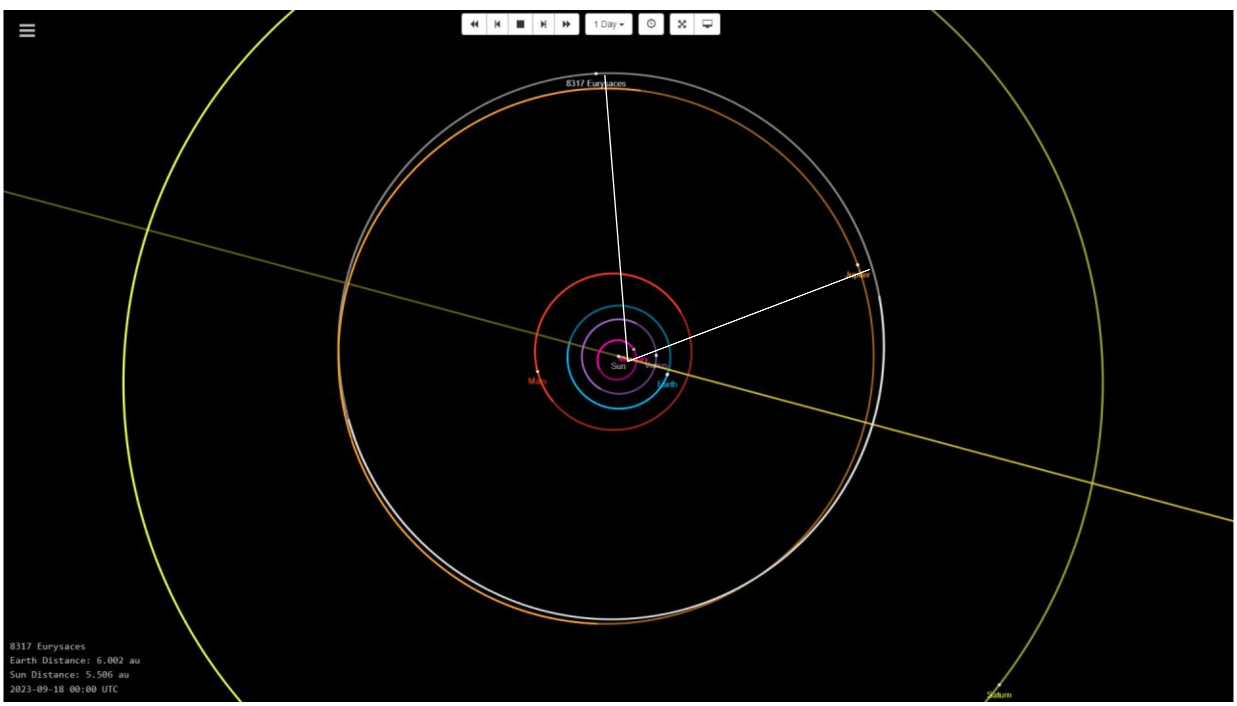


Figure 4: 9/18/23 Eurysaces orbit

By drawing lines in Microsoft Word, we can see the width and height from the "shape format" tools. From the width and height, we can calculate the angle of the line using the inverse of tangent. Then adding the angles for the lines connecting the Sun-Eurysaces and the Sun-Jupiter will give us the angle we are looking for. In the table below, S-E indicates the line for the Sun and Eurysaces, S-J indicates the line for the Sun and Jupiter, and E-S-J indicates the two lines combined. And w is for width, h is for height, and θ is for angle. For the third date, the angles for the lines is taken with respect to the y-axis.

Example Calculation for the angle:

For 9/18/20,

$$\theta_{S-E} = arctan\left(\frac{0.21}{1.6}\right) \times \frac{180^o}{\pi} = 7.4773\ deg$$

$$\theta_{S-\mathrm{J}} = arctan\left(\frac{1.58}{0.35}\right) \times \frac{180^o}{\pi} = 77.510\ deg$$

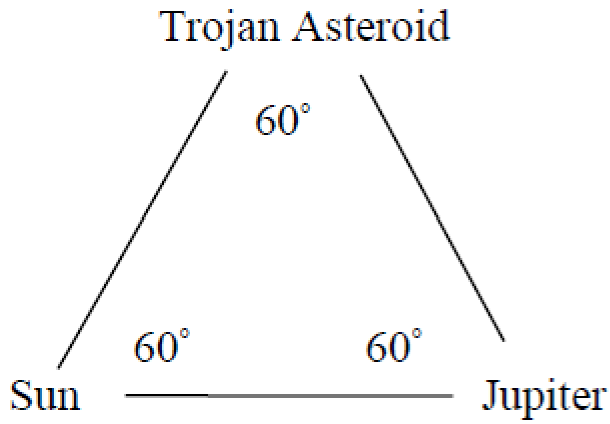
$$\angle E - S - J = \theta_{S-E} + \theta_{S-J} = 84.987\ deg$$

Table 2: Angles for selected 3 dates

	S-E			S-J			
Date	h (in)	w (in)	θ (deg)	h (in)	w (in)	θ (deg)	$\angle E - S - J$ (deg)
9/18/20	0.21	1.6	7.4773	1.58	0.35	77.510	84.987
3/18/21	0.64	1.51	22.969	1.4	0.76	61.504	84.474
9/18/23	1.49	0.12	4.6045	0.48	1.26	69.146	73.750

The last column of the table indicates the angle for the two lines for 3 different dates. The first 2 dates have almost identical angles; however, the third date has an angle approximately 10 degrees smaller than the rest. Overall, the angles do not seem to be changing with a large degree.

(b) For a preliminary assessment, the positions of a sample Trojan asteroid (e.g. Eurysaces), the Sun, and Jupiter can be modeled as located at the vertices of an equilateral triangle as envisioned below.



Assume that the distance between the Sun and Jupiter is equal to the semi-major axis of Jupiter's orbit. Let the mass of the asteroid be assumed as $\mu = 75km^3/sec^2$. Consider the net acceleration on the asteroid.

(i) Write the expression for the acceleration of the asteroid relative to the Sun where Jupiter is a perturbing body. [Note that the definition of a set of unit vectors is necessary.] Write this expression in the form $\ddot{\bar{r}}_{\odot\to asteroid}$ = (sum of terms). Label each of the following terms in this expression: dominant term, direct perturbing terms, indirect perturbing terms. Determine the magnitude and direction of each of the terms in the expression, as well as the net perturbing accelerations for each body and the total net acceleration.

"asteroid" will be shortened as "a" and Jupiter will be "♃" in variable subscripts hereafter.

From Notes D2, we can express the acceleration term of the asteroid relative to the Sun as

$$\boxed{\ddot{\bar{r}}_{\odot a} = \underbrace{-\frac{G(m_a+m_\odot)}{r_{\odot a}^3}\bar{r}_{\odot a}}_{\text{dominant}} + Gm_{♃}\left(\underbrace{\frac{\bar{r}_{a♃}}{r_{a♃}^3}}_{\text{direct}} - \underbrace{\frac{\bar{r}_{\odot ♃}}{r_{\odot ♃}^3}}_{\text{indirect}}\right)}$$

From the NASA database and Supplemental document we know the following constants. Also, we are considering the Jupiter – Sun – Trojan Asteroid system to be shaped as an equilateral triangle (implies that all distances are equal).

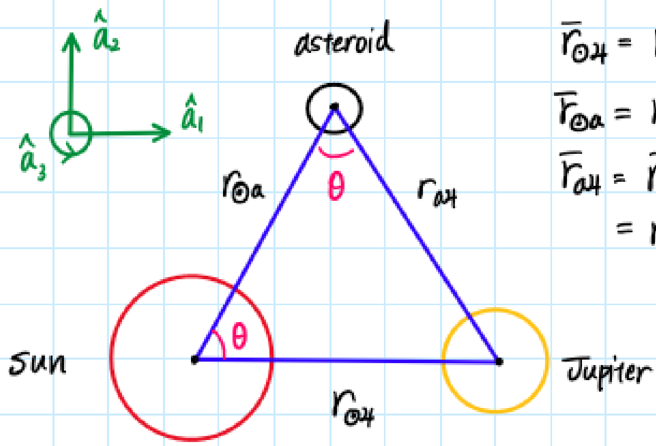
$$r_{\odot\text{♃}} = r_{\odot a} = r_{a\text{♃}} = 778279959 \text{ km}$$

$$Gm_a = \mu_a = 75 \text{ km}^3/s^2$$

$$Gm_\odot = \mu_\odot = 132712440017.99 \text{ km}^3/s^2$$

$$Gm_{\text{♃}} = \mu_{\text{♃}} = 126712767.8578 \text{ km}^3/s^2$$

$$\theta = 60°$$



$$\bar{r}_{\odot\text{♃}} = r_{\odot\text{♃}}\,\hat{a}_1$$

$$\bar{r}_{\odot a} = r_{\odot a}(\cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2)$$

$$\bar{r}_{a\text{♃}} = \bar{r}_{\odot\text{♃}} - \bar{r}_{\odot a} = r_{\odot\text{♃}}[(1-\cos\theta)\hat{a}_1 - \sin\theta\,\hat{a}_2]$$

plug the constants and vector values into the expression, and we get

$$\boxed{\underbrace{\ddot{\bar{r}}_{\odot a}}_{\text{total net acceleration}} = \underbrace{-\frac{G(m_a+m_\odot)}{r_{\odot a}^3}(r_{\odot a})(\cos\theta\,\hat{a}_1+\sin\theta\,\hat{a}_2)}_{\text{dominant term}} + Gm_{\text{♃}}\underbrace{\left\{\underbrace{\frac{r_{a\text{♃}}[(1-\cos\theta)\hat{a}_1 - \sin\theta\,\hat{a}_2]}{r_{a\text{♃}}^3}}_{\text{direct perturbing term}} - \underbrace{\frac{r_{\odot\text{♃}}\hat{a}_1}{r_{\odot\text{♃}}^3}}_{\text{indirect perturbing term}}\right\}}_{\text{net perturbing term}}}$$

Using MATLAB (code in Appendix), take the norm to get the magnitude and divide the vector by it to obtain the direction for each term. The result is tabulated below.

Sample Calculation (for dominant term):

dominant term:

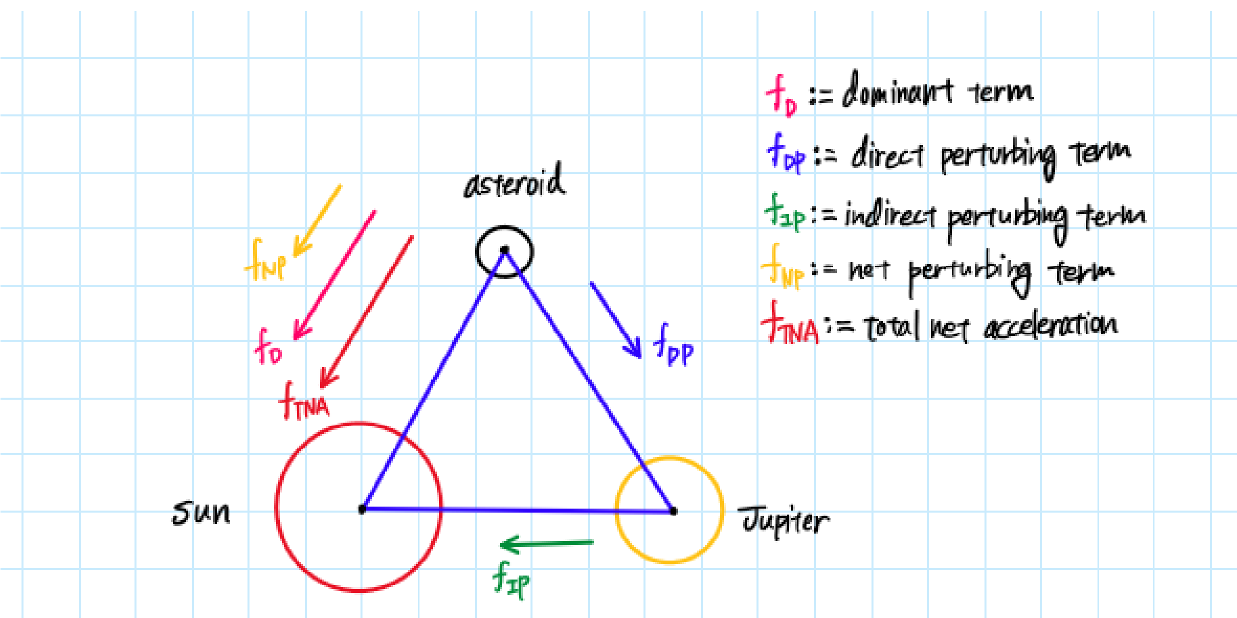
$$-\frac{G(m_a+m_\odot)}{r_{\odot a}^3}(r_{\odot a})(\cos\theta\,\hat{a}_1+\sin\theta\,\hat{a}_2)$$

$$\Rightarrow \text{magnitude}: \frac{G(m_a+m_\odot)}{r_{\odot a}^2} = 2.1910\times10^{-7}\ \text{kN}$$

$$\text{direction}: -\cos60^\circ\,\hat{a}_1-\sin60^\circ\,\hat{a}_2 = -0.5000\,\hat{a}_1-0.8660\,\hat{a}_2$$

Table 3: Gravity acceleration terms for system w.r.t the Sun

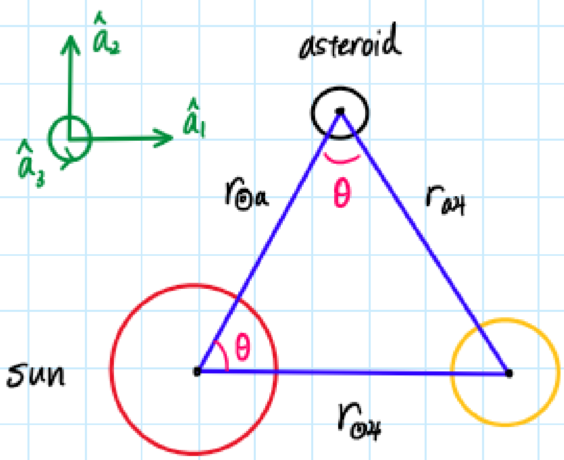
term	*magnitude [kN]*	*direction*	
dominant term	2.1910×10^{-7}	$-0.5000\hat{a}_1-0.8660\hat{a}_2$	asteroid to Sun
direct perturbing term	2.0919×10^{-10}	$0.5000\hat{a}_1-0.8660\hat{a}_2$	asteroid to Jupiter
indirect perturbing term	2.0919×10^{-10}	$-1\hat{a}_1$	Jupiter to Sun
net perturbing term	2.0919×10^{-10}	$-0.5000\hat{a}_1-0.8660\hat{a}_2$	asteroid to Sun
total net acceleration	2.1931×10^{-7}	$-0.5000\hat{a}_1-0.8660\hat{a}_2$	asteroid to Sun



(ii) Re-formulate the problem and write the expression for the acceleration of the asteroid relative to Jupiter. Again, determine the magnitude and direction of each of the terms in the expression, as well as the net perturbing accelerations for each body and the total net acceleration.

Reformulating the previous problem to the acceleration of the asteroid with respect to Jupiter is straightforward.

$$\boxed{\ddot{\bar{r}}_{4a} = \underbrace{-\frac{G(m_a+m_4)}{r_{4a}^3}\bar{r}_{4a}}_{\text{dominant}} + Gm_\odot\left(\underbrace{\frac{\bar{r}_{a\odot}}{r_{a\odot}^3}}_{\text{direct}} - \underbrace{\frac{\bar{r}_{4\odot}}{r_{4\odot}^3}}_{\text{indirect}}\right)}$$



$$\bar{r}_{4a} = r_{a4}(-\cos\theta\,\hat{a}_1 + \sin\theta\,\hat{a}_2)$$

$$\bar{r}_{4\odot} = r_{\odot 4}(-\hat{a}_1)$$

$$\bar{r}_{a\odot} = \bar{r}_{4\odot} - \bar{r}_{4a}$$

$$= r_{\odot 4}\left[(-1+\cos\theta)\hat{a}_1 - \sin\theta\,\hat{a}_2\right]$$

plugging in the constants and vector values give the following

$$\underbrace{\ddot{\vec{r}}_{4a}}_{\text{total net acceleration}} = \underbrace{-\frac{G(m_a+m_4)}{r_{4a}^3}(r_{4a})(-\cos\theta\,\hat{a}_1+\sin\theta\,\hat{a}_2)}_{\text{dominant term}}$$

$$\underbrace{+\,Gm_\odot\left\{\underbrace{\frac{r_{a\odot}\left[(-1+\cos\theta)\hat{a}_1-\sin\theta\,\hat{a}_2\right]}{r_{a\odot}^3}}_{\text{direct perturbing term}} - \underbrace{\frac{-r_{4\odot}\,\hat{a}_1}{r_{4\odot}^3}}_{\text{indirect perturbing term}}\right\}}_{\text{net perturbing term}}$$

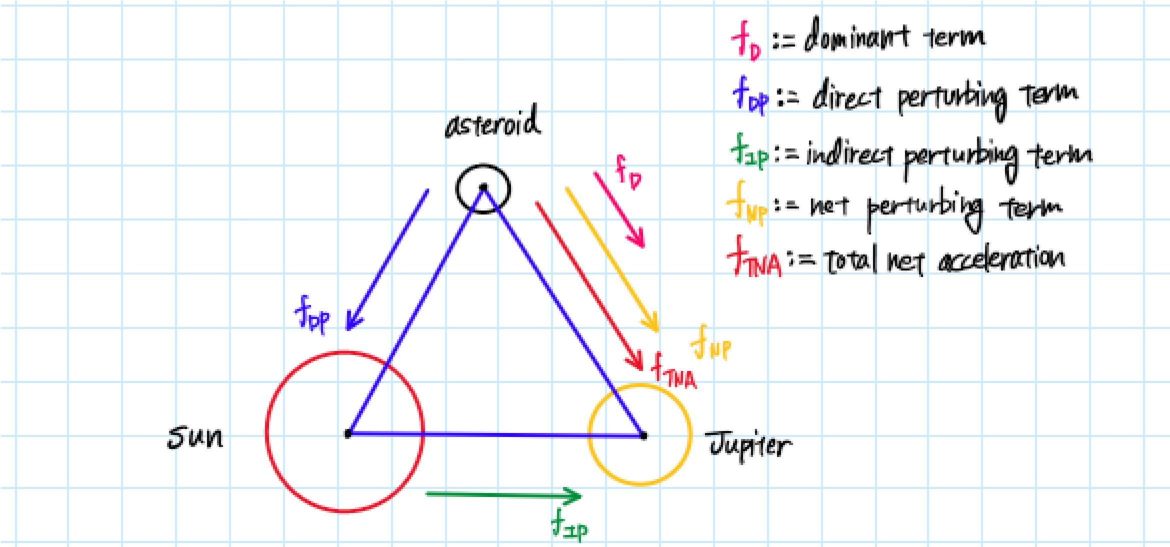


Table 4: Gravity acceleration term for system w.r.t Jupiter

term	*magnitude [kN]*	*direction*	
dominant term	2.0919×10^{-10}	$0.5000\hat{a}_1 - 0.8660\hat{a}_2$	asteroid to Jupiter
direct perturbing term	2.1910×10^{-7}	-$0.5000\hat{a}_1 - 0.8660\hat{a}_2$	asteroid to Sun
indirect perturbing term	2.1910×10^{-7}	$1\hat{a}_1$	Sun to Jupiter
net perturbing term	2.1910×10^{-7}	$0.5000\hat{a}_1 - 0.8660\hat{a}_2$	asteroid to Jupiter
total net acceleration	2.1931×10^{-7}	$0.5000\hat{a}_1 - 0.8660\hat{a}_2$	asteroid to Jupiter

(iii) Which term is the largest in each formulation? How does the magnitude of the dominant term in each formulation compare?

For the formulation where the asteroid is relative to the Sun, the largest term is the dominant term in the direction from the asteroid to the Sun. Whereas for the formulation where the asteroid is relative to Jupiter, the largest terms are the direct and indirect perturbing terms.

The dominant term in the first formulation is greater than the dominant term of the second formulation and equal to the direct and indirect perturbing terms for the second formulation.

(iv) Determine the net perturbing acceleration in each case. Which has the largest impact in each formulation, the Sun or Jupiter?

From the table above, we know that the net perturbing accelerations for the two accelerations are

Asteroid w.r.t the Sun:

$$2.0919 \times 10^{-10}(-0.5000\hat{a}_1 - 0.8660\hat{a}_2)(kN)$$

Asteroid w.r.t Jupiter

$$2.1910 \times 10^{-7}(0.5000\hat{a}_1 - 0.8660\hat{a}_2)(kN)$$

For the first formulation the Sun has the largest impact since the direction of the net perturbing acceleration is toward the Sun. On the other hand, Jupiter has the largest impact in the second formulation since the net perturbing acceleration vector is in the direction of Jupiter.

(v) Is the net total acceleration on the asteroid the same in each formulation? Should it be the same? Why or why not?

The magnitude of the total net acceleration for each formulation is the same but the directions are different. For the first formulation the total net acceleration is in the direction of the Sun and the second formulation is directed towards Jupiter. The magnitudes should be the same since the distances between each bodies are the same from the system being a equilateral triangle, and also because the in an equilateral triangle the angles between each vector is equally 60 degrees.

(vi) Which formulation is correct? Why?

Both formulations are correct. This is because, we know that the Trojan asteroids have an unchanging position relative to the Sun and Jupiter since they are in the L_4 and L_5 Lagrangian Points. The two formulations verify that the acceleration towards the Sun and Jupiter cancel each other out to have the Trojan asteroid be in a stationary position relative to the two bodies.

(vii) From the results here, is it reasonable to model the motion of the asteroid as a two-body problem, i.e., Sun-asteroid or Jupiter-asteroid? Why?

As discussed, we cannot model the problem as a two-body problem of Sun-asteroid or Jupiter-asteroid. We must validate that the asteroid is at an equilibrium position – Lagrangian Point – by solving the three-body problem relative to the Sun and Jupiter.

Problem 2: An Introductory Manual (Intro_Manual_F20 GMATR2020a with OpenFrames) for the General Mission Analysis Tool (GMAT) software is posted under GMAT on Brightspace. GMAT is open source and is easily downloaded. To obtain some practice using GMAT, step through the manual carefully. Complete all the steps and view the final orbits.

(a) Now use an Epoch of 18 Sept 2020. Produce a satellite orbit with a 'semi-major axis' of 60,000 km, 'eccentricity' of 0.7, and an 'inclination' of 45^o. Note that you are using an <u>Earth point mass</u> model. (In the Resource tree, for the 'LowEarthProp' propagator, replace Gravity Model with 'JGM-2' but set degree and order to zero to render a point mass model. Under 'DefaultOtbitView' it will be a cleaner image if you do NOT enable the constellations.) Plot images from GMAT. Also print the summary of the orbit details from the "Report" option. Under the Resources tree, right click on 'Output'. You are offered the opportunity to add a report file with numerical data from the simulation. The report will appear in the Output tree. From the output data, determine,
   (i) Radius of closest approach of Rad, Peri.
   (ii) Radius at farthest excursion or Rad. Apo.
   (iii) Energy
   (iv) Semi-major axis
   (v) Semi-latus rectum
   (vi) Angular momentum
   (vii) The Cartesian components of position and velocity at the initial time.

   (Do not submit the entire output report; cut-and-paste the sections with the required data into a Word document. Highlight the requested quantities.)

The reference frame of this is the "EarthMJ2000Eqn" which is an Earth equator inertial system based on IAU-1976/FK5 theory with 1980 update to nutation.

The image of the simulation is the following

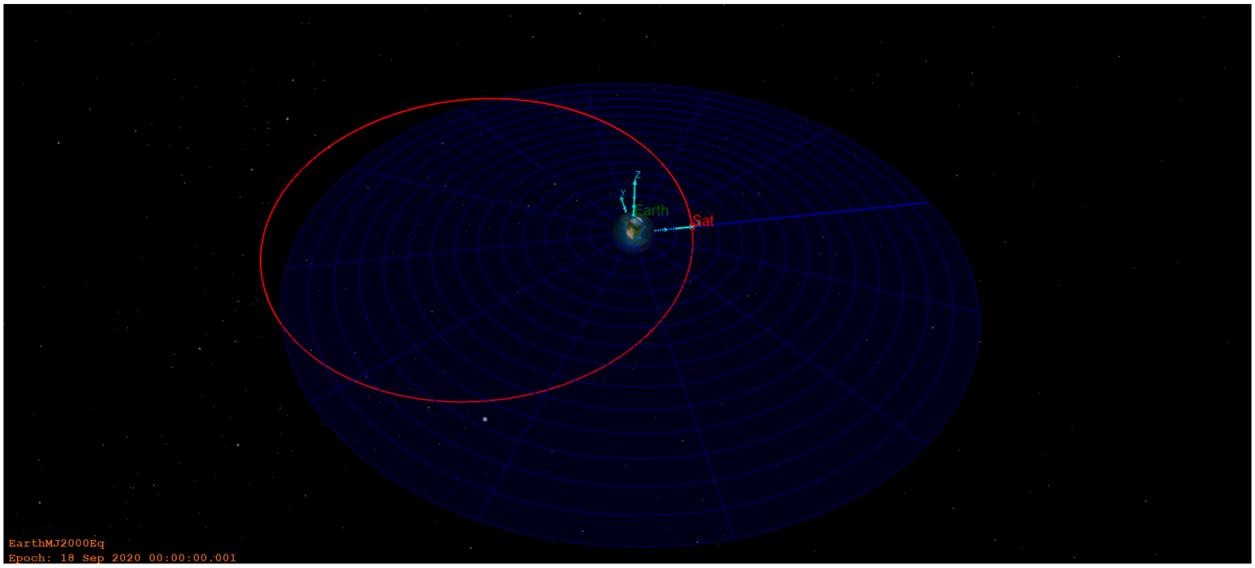


Figure 5: ORbit frame view of satellite from above

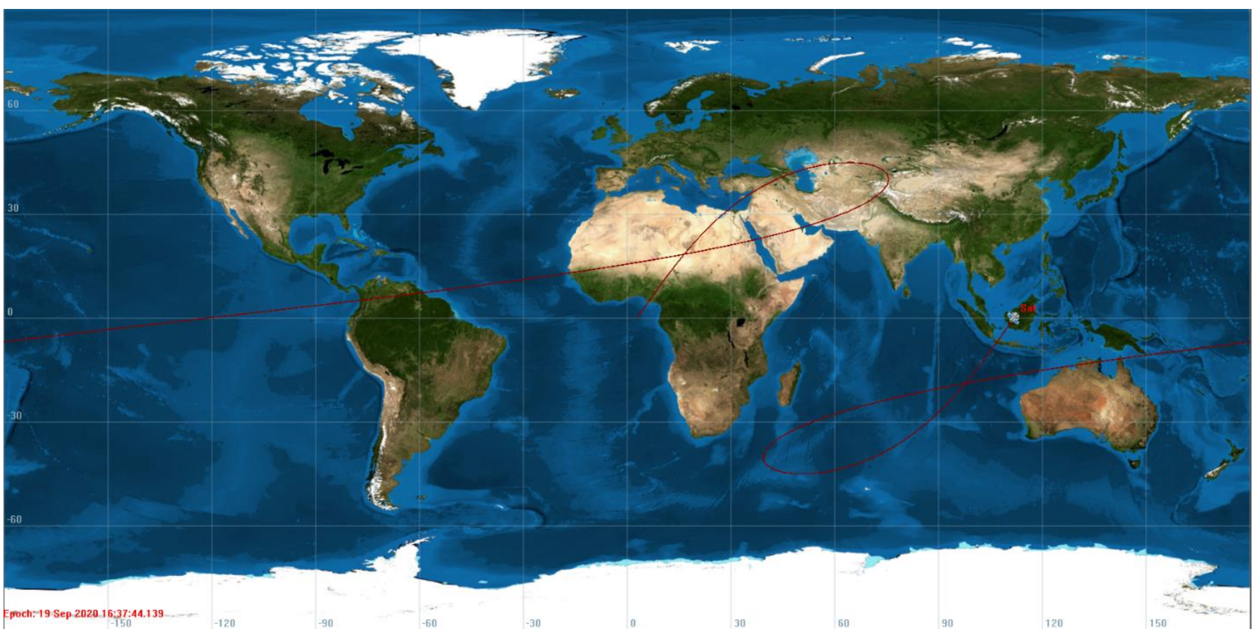


Figure 6: Satellite orbit mapped onto flat earth view

Now taking the report exported from GMAT, we take the average of each column or get the initial values from the data to get the following results from MATLAB. (The report's data is in the Appendix.)

Table 5: Results from the report data

periapsis [km]	18000
apoapsis [km]	102000
energy [kJ]	-3.3217
semi-major axis [km]	60000
semi-latus rectum [km]	30600
H_x **[** $kg \cdot km^2/s$**]**	$-1.1229 \times 10^{-11} \sim 0$
H_y **[** $kg \cdot km^2/s$**]**	-78093
H_z **[** $kg \cdot km^2/s$**]**	78093
$X\|_{t=0}$ **[km]**	18000
$\mathrm{Y}\|_{t=0}$ **[km]**	0
$Z\|_{t=0}$ **[km]**	0
$V_x\|_{t=0}$ **[km/s]**	0
$V_\mathrm{y}\|_{t=0}$ **[km/s]**	4.3385
$V_\mathrm{z}\|_{t=0}$ **[km/s]**	4.3385

(b) Given the orbit in the scenario in part (a), set the inclination to zero. Add a second spacecraft with an orbit of a different color. With the same eccentricity and zero inclination, try a different semi-major axis, i.e., 40,000 km. Add a third satellite with a = 75,000 km.
Repeat the exercise for a = 60,000 km and three eccentricities, i.e., e = 0.2, 0.65, 0.88 In each case, hold the inclination fixed at 45^o. [Note that some combinations go below the radius of the Earth…they can be excluded.]
Repeat the exercise for e = 0.65 and three semi-major axes, i.e., a = 20,000 km, 35,000 km, 75,000 km.
You should have three plots from GMAT; use a view that is looking down on the orbit plane for variations in semi-major axis and eccentricity.

For a = 40,000; 60,000; 75,000

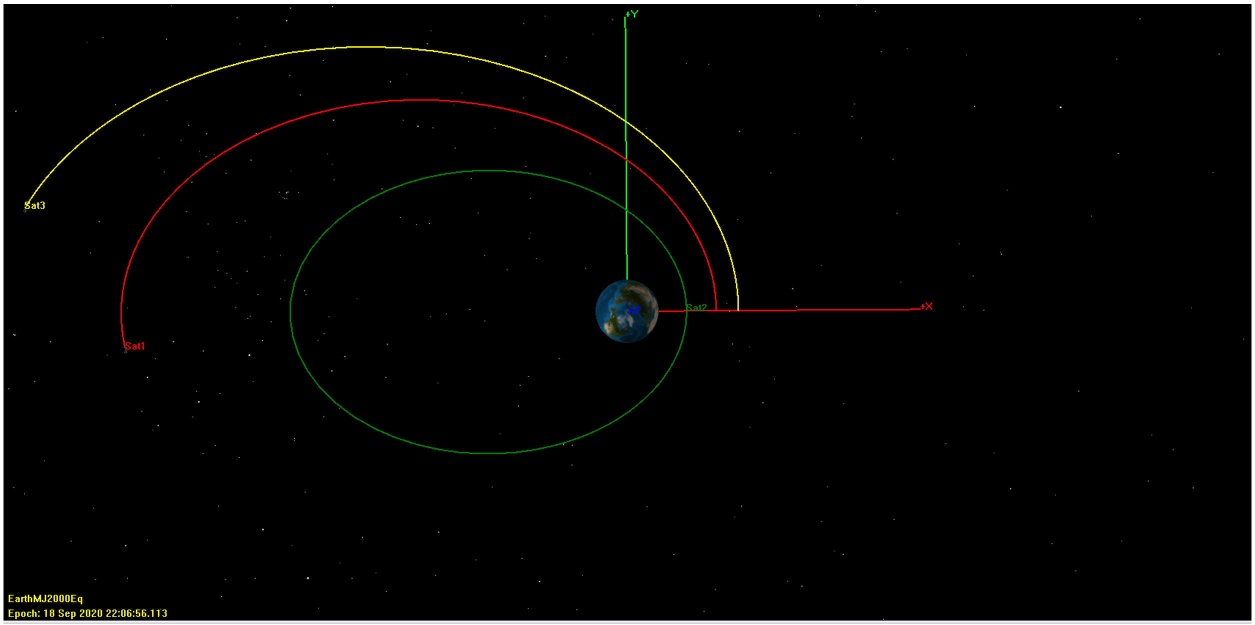


Figure 7: 3 satellite orbit for a = 40,000; 60,000; 75,000

For a = 60,000 and e = 0.2; 0.65; 0.88 and i = 45^o

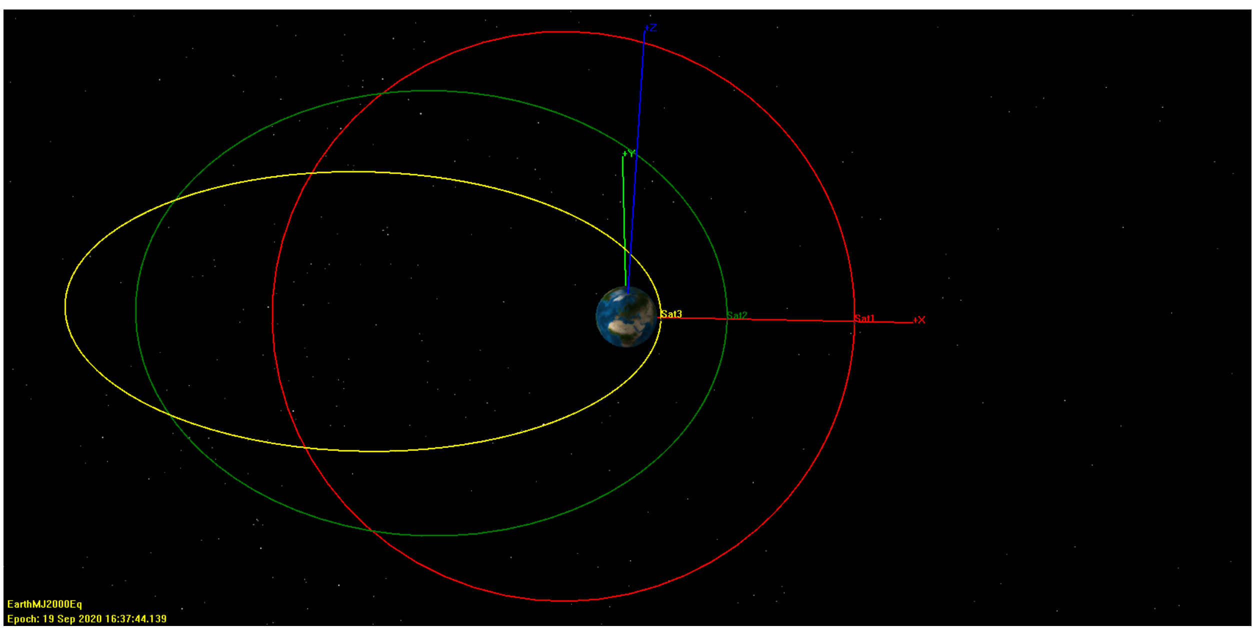


Figure 8: 3 satellite orbit for a = 60,000 and e = 0.2; 0.65; 0.88 and i = 45 deg

For a = 20,000; 35,000; 75,000 and e = 0.65 and i = 45^o

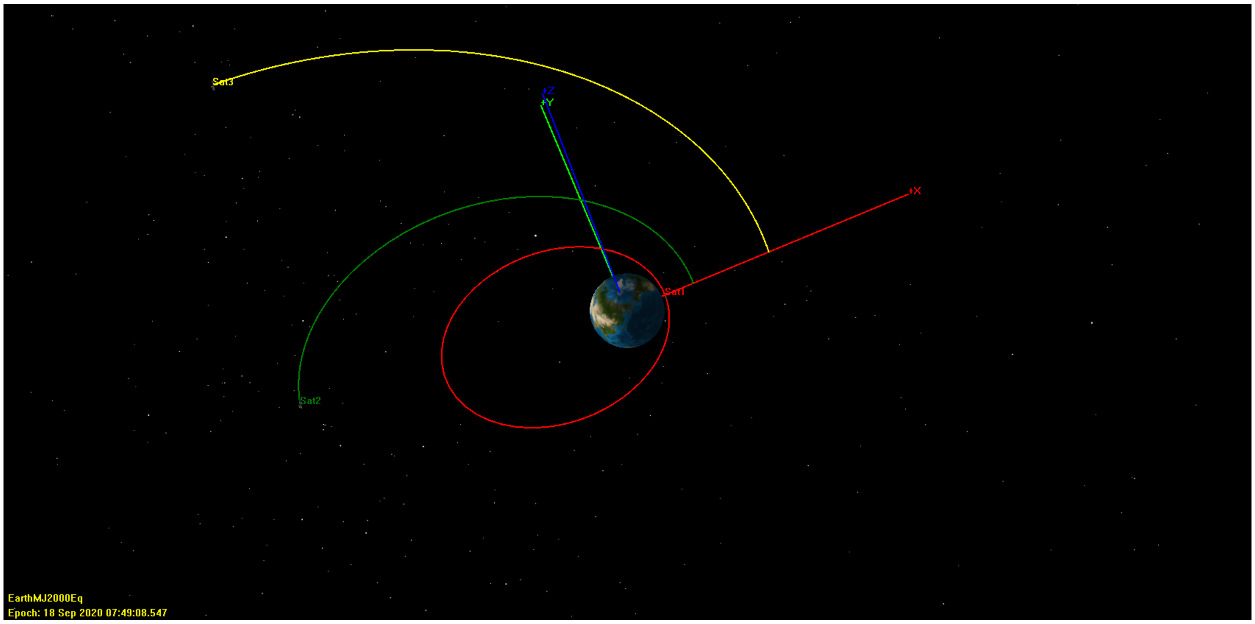
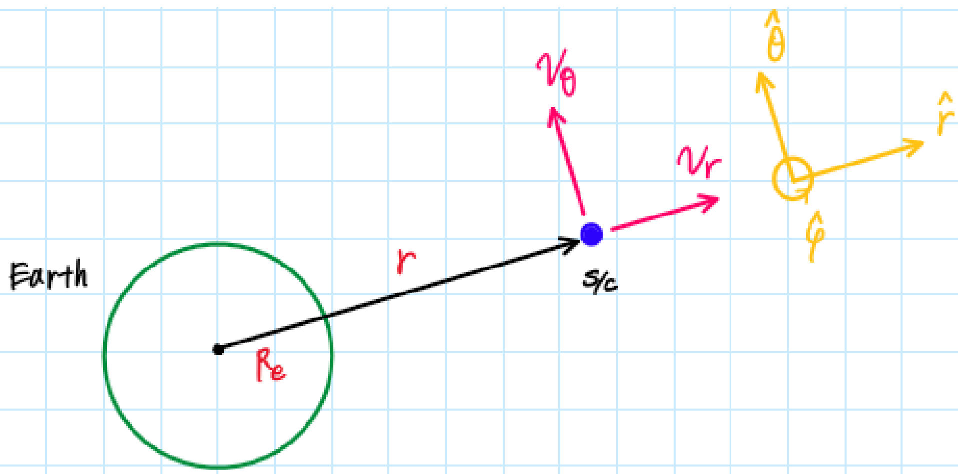


Figure 9: 3 satellite orbit for a = 20,000; 35,000; 75,000 and e = 0.65 and i = 45 deg

Problem 3: Consider only the <u>relative two-body</u> problem (Earth and spacecraft). An Earth-orbiting vehicle is tracked from ground stations; the spacecraft mass is 600 kg. at a certain instant (t_o), the following position and velocity information is obtained relative to an inertial observer:

- Altitude = 8560 km
- Radial component of relative velocity = +2.11 km/s
- Transverse component of relative velocity = 4.89 km/s
- (our tracking system is perfect, so these states are completely without error)

(a) Compute the total system angular momentum $\bar{C}_3$. Specific angular momentum, total kinetic energy for the system, total energy C_4, specific energy, areal velocity.



The constants are

$$m_{s/c} = 600\,\text{kg} \qquad r = 8560\ \text{km}$$
$$m_e = 5.9722\times10^{24}\,\text{kg} \qquad R_e = 6371\ \text{km}$$
$$v_r = +2.11\ \text{km/s} \qquad v_\theta = 4.89\ \text{km/s}$$

From Notes E2, we know that $\bar{C}_3$ is derived as

$$\bar{C}_3 = m_1\left(\frac{-m_2}{m_1+m_2}\bar{r} \times \frac{-m_2}{m_1+m_2}\dot{\bar{r}}\right) + m_2\left(\frac{m_1}{m_1+m_2}\bar{r} \times \frac{m_1}{m_1+m_2}\dot{\bar{r}}\right)$$

where $m_1 := m_e$, $m_2 := m_{s/c}$, $\bar{r} = (R_e + r)\hat{r}$

$\dot{\bar{r}} = v_r \hat{r} + v_\theta \hat{\theta}$

Compute this using MATLAB, and we get

$$\bar{C}_3 = \frac{m_1 m_2}{m_1 + m_2}(\bar{r} \times \dot{\bar{r}}) = \frac{m_e m_{s/c}}{m_e + m_{s/c}}\left[(R_e + r)\hat{r} \times (v_r \hat{r} + v_\theta \hat{\theta})\right]$$

$$\bar{C}_3 = \frac{m_e m_{s/c}}{m_e + m_{s/c}}(R_e + r) v_\theta \hat{\varphi}$$

$$\boxed{\bar{C}_3 = 4.3808\text{e}+6 \ \text{kg} \cdot \text{km}^2/\text{s} \ \hat{\varphi}}$$

The specific angular momentum is (from notes E3)

$$|\bar{h}| = |\bar{r} \times \dot{\bar{r}}| = |(R_e + r) v_\theta \hat{\varphi}|$$

$$\boxed{h = 7.3013\text{e}+3 \ \text{km}^2/\text{s}}$$

The center of mass of this system is

$$\bar{r}_{cm} = \frac{m_e \times 0 + m_{s/c}(R_e + r)}{m_e + m_{s/c}} = 1.5001\text{e}-18 \approx 0$$

It is fair to say that the center of mass is at the center of the earth.

Thus, the total kinetic energy becomes (from notes E6)

$$T = \frac{1}{2}\frac{m_e m_{s/c}}{m_e + m_{s/c}}(\dot{\bar{r}} \cdot \dot{\bar{r}})$$

$$\boxed{T = 8.5093\text{e}+3 \ \text{kJ}}$$

The potential energy is (from notes E6)

$$U = \frac{1}{2} G \frac{m_e m_{s/c}}{R_e + r} + \frac{1}{2} G \frac{m_{s/c} m_e}{R_e + r} = \frac{G m_e m_{s/c}}{R_e + r}$$

$$U = 1.6018 e+4 \text{ kJ}$$

Then,

$$\boxed{C_4 = T - U = -7.5085 e+3 \text{ J}}$$

The specific energy is

$$\mathcal{E} = \frac{|\bar{v}|^2}{2} - \frac{\mu}{R} = \frac{|v_r \hat{r} + v_\theta \hat{\theta}|^2}{2} - \frac{G(m_e + m_{s/c})}{R_e + r}$$

$$\boxed{\mathcal{E} = -12.5142 \ \text{km}^2/\text{s}^2}$$

Lastly, the area velocity is

$$\boxed{\dot{A} = \frac{h}{2} = 3.6506 e+3 \ \text{km}^2/\text{s}}$$

(b) What is the value of the coefficient by which to multiply C_4 to obtain specific energy?

From notes E7

$$C_4 \frac{m_e + m_{s/c}}{m_e m_{s/c}} = \mathcal{E}$$

$$\frac{m_e + m_{s/c}}{m_e m_{s/c}} = 1.6667e-3$$

$$C_4 \times 1.6667e-3 = -12.5142$$

matches with our previous answers

so $\boxed{\frac{m_e + m_{s/c}}{m_e m_{s/c}} = 1.6667e-3}$

(c) Within the context of the relative two-body problem, determine the following orbital characteristics: $p, e, a, \mathbf{P}(period), \gamma, \theta^*$. Write the position vector in terms of the inertial unit vectors $\hat{e}$and $\hat{p}$.

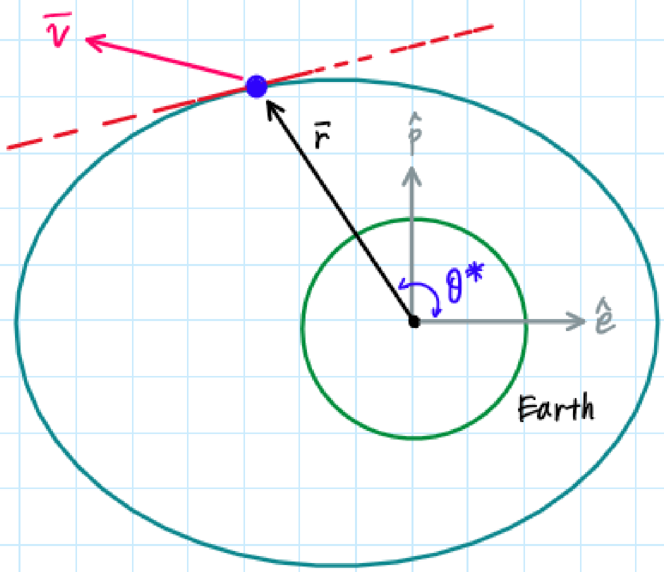
- $\boxed{p = \frac{h^2}{\mu} = 1.3374e+4 \text{ km}}$
- $\boxed{e = \sqrt{1 + \frac{2\mathcal{E}h^2}{\mu^2}} = 0.4003}$
- $\boxed{a = \frac{p}{1-e^2} = 1.5926e+4 \text{ km}}$
- $\boxed{\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu}} = 2.0002e+4 \text{ s}}$
- $\boxed{\gamma = \arctan\left(\frac{v_r}{v_\theta}\right) = 23.3399^\circ}$

- $1 + e\cos\theta^* = \dfrac{p}{R_e + r}$

$$\cos\theta^* = \frac{1}{e}\left(\frac{p}{R_e + r} - 1\right)$$

$$\theta^* = \arccos\left[\frac{1}{e}\left(\frac{p}{R_e + r} - 1\right)\right]$$

$$\boxed{\theta^* = 105.1012^\circ}$$



$$\bar{r} = (R_e + r)\left(\cos\theta^*\,\hat{e} + \sin\theta^*\,\hat{p}\right)$$

$$\boxed{\bar{r} = \left(-3.8899e{+}3\,\hat{e} + 1.4415e{+}4\,\hat{p}\right)\ \text{km}}$$

(d) Compare this relative velocity to the circular relative velocity at this altitude.

The circular relative velocity is

$$v_c = \sqrt{\frac{\mu}{R_e + r}} = \sqrt{\frac{G(m_e + m_{s/c})}{R_e + r}}$$

$$v_c = 5.1668 \text{ km/s}$$

The actual relative velocity is

$$|\bar{v}| = 5.3258 \text{ km/s}$$

Thus, this is theoretically plausible since

$$\boxed{|\bar{v}| = 5.3258 \text{ km/s} < 7.3070 \text{ km/s} = \sqrt{2}\, v_c}$$

Appendix

GMAT Report Data (first 10 rows for all columns)

Sat_Earth_RadPer	Sat_Earth_RadApo	Sat_Earth_Energy	Sat_Earth_SMA	Sat_Earth_SemilatusRectum	Sat_EarthMJ2000Eq_HX	Sat_EarthMJ2000Eq_HY
18000	102000	-3.321670346	60000	30600	0	-78093.44886
18000	102000	-3.321670346	60000	30600	0	-78093.44886
18000	102000	-3.321670346	60000	30600	0	-78093.44886
18000	102000	-3.321670346	60000	30600	2.84217E-14	-78093.44886
18000	102000	-3.321670346	60000	30600	1.13687E-13	-78093.44886
18000	102000	-3.321670346	60000	30600	0	-78093.44886
18000	102000	-3.321670346	60000	30600	0	-78093.44886
18000	102000	-3.321670346	60000	30600	-3.63798E-12	-78093.44886
18000	102000	-3.321670346	60000	30600	-3.63798E-12	-78093.44886
18000	102000	-3.321670346	60000	30600	-3.63798E-12	-78093.44886

Sat_EarthMJ2000Eq_HZ	Sat_EarthMJ2000Eq_X	Sat_EarthMJ2000Eq_Y	Sat_EarthMJ2000Eq_Z	Sat_EarthMJ2000Eq_VX	Sat_EarthMJ2000Eq_VY	Sat_EarthMJ2000Eq_VZ
78093.44886	18000	0	0	0	4.338524937	4.338524937
78093.44886	17999.99938	4.338524887	4.338524887	-0.001230248	4.338524788	4.338524788
78093.44886	17999.99114	16.46696425	16.46696425	-0.004669432	4.338522801	4.338522801
78093.44886	17999.90935	52.6683154	52.6683154	-0.014934771	4.338503087	4.338503087
78093.44886	17999.12077	164.0252508	164.0252508	-0.04651002	4.338313024	4.338313024
78093.44886	17991.82967	499.9951827	499.9951827	-0.14173546	4.336556274	4.336556274
78093.44886	17918.65208	1577.199729	1577.199729	-0.44582624	4.318979547	4.318979547
78093.44886	17678.59895	3131.85742	3131.85742	-0.877118107	4.262014213	4.262014213
78093.44886	17258.88477	4747.427397	4747.427397	-1.308486108	4.164898661	4.164898661
78093.44886	16658.12985	6371.975339	6371.975339	-1.717248424	4.03113584	4.03113584

MATLAB Code

Problem 1 code

```
%% AAE 532 HW 3 Problem 1
% Tomoki Koike
close all; clear all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Fall\AAE532\matlab\outputs\ps3';
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
format longG;
```

```
% Set constants
```

```
planet_consts = setup_planetary_constants();  % Function that sets up all the constants in the table
sun = planet_consts.sun;  % structure of sun
jupiter = planet_consts.jupiter;  % structure of jupiter
G = 6.6743015e-20;  % Gravitational constant [km^3/kg/s^2]
```

```
% (a)
h1_SE = 0.21; w1_SE = 1.6; theta1_SE = atand(h1_SE/w1_SE);
h1_SJ = 1.58; w1_SJ = 0.35; theta1_SJ = atand(h1_SJ/w1_SJ);
ang1_ESJ = theta1_SE + theta1_SJ;

h2_SE = 0.64; w2_SE = 1.51; theta2_SE = atand(h2_SE/w2_SE);
h2_SJ = 1.4; w2_SJ = 0.76; theta2_SJ = atand(h2_SJ/w2_SJ);
ang2_ESJ = theta2_SE + theta2_SJ;

h3_SE = 1.49; w3_SE = 0.12; theta3_SE = atand(w3_SE/h3_SE);
h3_SJ = 0.48; w3_SJ = 1.26; theta3_SJ = atand(w3_SJ/h3_SJ);
ang3_ESJ = theta3_SE + theta3_SJ;
```

```
% (b)(i)
% Distances s=Sun, j=Jupiter, a=asteroid
r_sj = jupiter.smao; r_sa = r_sj; r_aj = r_sj;
% Gravitational parameters
mu_a = 75; mu_s = sun.gp; mu_j = jupiter.gp;  % [km^3/s^2]
theta = 60;  % [deg]
% Postion vectors relative to the Sun, r-vector=rv
rv_sj = r_sj * [1, 0];
rv_sa = r_sa * [cosd(theta), sind(theta)];
rv_aj = rv_sj - rv_sa;
```

```
% Get the terms
% D=dominant, DP=direct perturbing, IP=indirect perturbing,
% NP=net perturbing, TNA=total net acceleration
f_D = -(mu_a + mu_s) / r_sa^3 * rv_sa;
f_D_mag = norm(f_D);
f_D_dir = f_D / f_D_mag;

f_DP = mu_j / r_aj^3 * rv_aj;
f_DP_mag = norm(f_DP);
f_DP_dir = f_DP / f_DP_mag;

f_IP = -mu_j / r_sj^3 * rv_sj;
f_IP_mag = norm(f_IP);
f_IP_dir = f_IP / f_IP_mag;

f_NP = f_DP + f_IP;
f_NP_mag = norm(f_NP);
f_NP_dir = f_NP / f_NP_mag;

f_TNA = f_DP + f_IP + f_D;
f_TNA_mag = norm(f_TNA);
f_TNA_dir = f_TNA / f_TNA_mag;
```

```
% (b)(ii)
% Postion vectors relative to the Sun, r-vector=rv
rv_ja = r_sj * [-cosd(theta), sind(theta)];
rv_js = r_sa * [-1, 0];
rv_as = rv_js - rv_ja;
```

```
% Get the terms
% D=dominant, DP=direct perturbing, IP=indirect perturbing,
% NP=net perturbing, TNA=total net acceleration
f_D = -(mu_a + mu_j) / r_sa^3 * rv_ja;
f_D_mag = norm(f_D);
f_D_dir = f_D / f_D_mag;

f_DP = mu_s / r_aj^3 * rv_as;
f_DP_mag = norm(f_DP);
f_DP_dir = f_DP / f_DP_mag;

f_IP = -mu_s / r_sj^3 * rv_js;
f_IP_mag = norm(f_IP);
f_IP_dir = f_IP / f_IP_mag;

f_NP = f_DP + f_IP;
f_NP_mag = norm(f_NP);
f_NP_dir = f_NP / f_NP_mag;

f_TNA = f_DP + f_IP + f_D;
f_TNA_mag = norm(f_TNA);
f_TNA_dir = f_TNA / f_TNA_mag;
```

Problem 2 code

```
%% AAE 532 HW 2 Problem 2
% Tomoki Koike
close all; clear all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Fall\AAE532\matlab\outputs\ps3';
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
format longG;
```

```
% (a)
T = readtable('inputs\ps3\reportFile_a.txt');
```

```
% Get the periapsis by taking the average of the column
r_p = mean(T.Sat_Earth_RadPer);
r_a = mean(T.Sat_Earth_RadApo);
E = mean(T.Sat_Earth_Energy);
sma = mean(T.Sat_Earth_SMA);
slr = mean(T.Sat_Earth_SemilatusRectum);
H_x = mean(T.Sat_EarthMJ2000Eq_HX);
```

```
H_y = mean(T.Sat_EarthMJ2000Eq_HY);
H_z = mean(T.Sat_EarthMJ2000Eq_HZ);
Xi = T.Sat_EarthMJ2000Eq_X(1);
Yi = T.Sat_EarthMJ2000Eq_Y(1);
Zi = T.Sat_EarthMJ2000Eq_Z(1);
VXi = T.Sat_EarthMJ2000Eq_VX(1);
VYi = T.Sat_EarthMJ2000Eq_VY(1);
VZi = T.Sat_EarthMJ2000Eq_VZ(1);
```

```
writetable(T(1:10,:), fullfile(fdir, 'reportDataTable_p2a.xlsx'), 'Sheet', 1);
```

Problem 3 code

```
%% AAE 532 HW 2 Problem 3
% Tomoki Koike
close all; clear all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Fall\AAE532\matlab\outputs\ps3';
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
format shortEng;
```

```
% Define constants
m_sc = 600;  % [kg]
m_e = 5.9722e24;  % [kg]
r = 8560;  % altitude [km]
Re = 6371;  % radius of the Earth [km]
vr = 2.11;  % radial velocity of the s/c [km/s]
vtheta = 4.89;  % transverse velocity of the s/c [km/s]
rv_sc = [Re+r, 0, 0];  % position vector of s/c (r-hat, theta-hat. phi-hat)
vv_sc = [vr, vtheta, 0];  % velocity vector of s/c (r-hat, theta-hat, phi-hat)
G = 6.6743015e-20;  % Gravitational constant [km^3/kg/s^2]
```

```
% (a)
r_cm = (m_e*0 + m_sc*(Re+r))/(m_e + m_sc);
C3 = m_e*m_sc/(m_e + m_sc) * cross(rv_sc, vv_sc);
h = norm(cross(rv_sc, vv_sc));
T = 0.5*m_sc*m_e/(m_e + m_sc)*(vv_sc*vv_sc')
U = G*(m_e * m_sc)/(Re + r)
C4 = T - U
E_sp = norm(vv_sc)^2/2 - G*(m_sc + m_e)/(Re + r)
Av = h/2;
```

```
% (b)
coeff = (m_e + m_sc)/m_e/m_sc;
```

```
% (c)
mu = G*(m_sc + m_e);
p = h^2/mu;
e = sqrt(1 + 2 * E_sp * h^2 / mu^2);
a = p / (1 - e^2);
```

```
period = 2*pi*sqrt(a^3/mu);
gamma = atand(vr/vtheta);
theta_star = acosd(e^(-1) * (p/(Re + r) - 1));
rv_inertial = (Re + r) * [cosd(theta_star), sind(theta_star)];
```

```
% (d)
vc = sqrt(mu / (Re + r));
norm(vv_sc);
```