Transfer Orbits: Lambert Arcs

Transfer Orbit Design
(special class of boundary value problem)

1. **Geometrical relationships**

   Conic paths connecting two points that are fixed in space with focus at the attracting center

2. **Analytical Relationships**

3. **Lambert's Theorem**

Lambert's Theorem

Know a lot about possible orbits connecting two points

But analytical relationships rely on "*a*"

how to get it?

Must somehow select "*a*" ← an additional specification about the transfer path

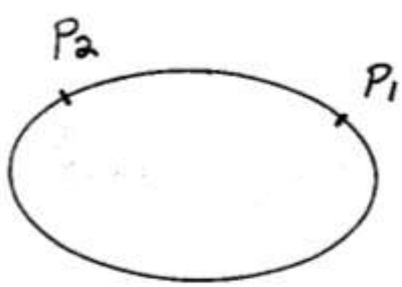
What to specify?

Lambert: conjecture that given I.C.s (r_1, r_2, c)

TOF depends only on "*a*"

i.e., $t = t(a, r_1 + r_2, c)$

(Lagrange actually proved this later)



$$n(t_1 - t_p) = E_1 - e\sin E_1$$
$$n(t_2 - t_p) = E_2 - e\sin E_2$$

Given TOF, this relationship contains unknowns: E_1, E_2, e, a

Must be rewritten in terms of only <u>one</u> unknown → a

HOW?

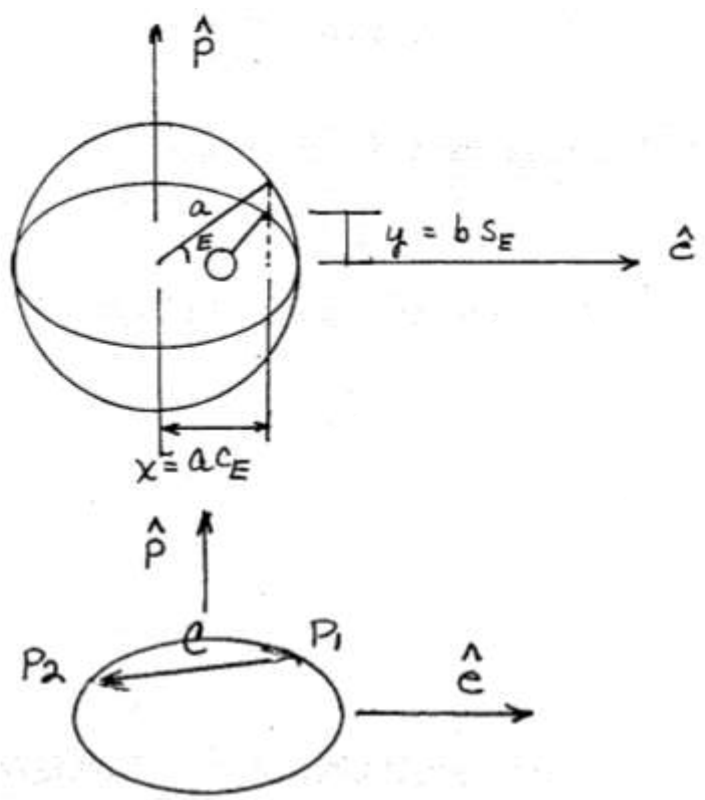
Define:

$$r_1 = a(1 - e\cos E_1) \qquad r_2 = a(1 - e\cos E_2)$$

$$r_1 + r_2 = a[2 - e(\cos E_1 + \cos E_2)]$$

$$= a\left[2 - e\left(2\cos\left(\frac{E_1 + E_2}{2}\right)\cos\left(\frac{E_1 - E_2}{2}\right)\right)\right]$$

$$\boxed{r_1 + r_2 = 2a[1 - e\cos E_p \cos E_M]}$$



measured from center

$$\begin{aligned}\bar{c} &= \bar{p}_2 - \bar{p}_1 \\ &= (x_2 - x_1)\hat{e} + (y_2 - y_1)\hat{p}\end{aligned}$$

$$\begin{aligned}c^2 &= (a\cos E_2 - a\cos E_1)^2 + (b\sin E_2 - b\sin E_1)^2 \\ &= a^2(\cos E_2 - \cos E_1)^2 + a^2(1-e^2)^2(\sin E_2 - \sin E_1)^2\end{aligned}$$

$$c^2 = a^2\left[(\cos E_2 - \cos E_1)^2 + (1-e^2)^2(\sin E_2 - \sin E_1)^2\right]$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\longrightarrow \cos E_2 - \cos E_1 = -2\sin E_p \sin E_M$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\longrightarrow \sin E_2 - \sin E_1 = 2\cos E_p \sin E_M$$

$$\begin{aligned}c^2 &= a^2\left[4\sin^2 E_p \sin^2 E_M + (1-e^2)4\cos^2 E_p \sin^2 E_M\right] \\ &= 4a^2\sin^2 E_M\left(\sin^2 E_p + \cos^2 E_p - e^2\cos^2 E_p\right)\end{aligned}$$

$$\boxed{c^2 = 4a^2\sin^2 E_M\left(1 - e^2\cos^2 E_p\right)}$$

Define $\boxed{\cos\eta = e\cos E_p}$ ok because $e < 1$

$$\Downarrow$$

$$r_1 + r_2 = 2a\left[1 - \cos\eta\cos E_M\right]$$

$$c^2 = 4a^2\sin^2 E_M\left(1 - \cos^2\eta\right)$$

OR

$$c = 2a\sin E_M\sin\eta$$

$$\Downarrow$$

So $r_1 + r_2 + c = 2a - 2a\cos\eta\cos E_M + 2a\sin E_M\sin\eta$

Defined previously

$$s = \frac{1}{2}\left(r_1 + r_2 + c\right) \quad \text{and} \quad \underbrace{\alpha = 2\sin^{-1}\sqrt{\frac{s}{2a}}}_{\sin^2\frac{\alpha}{2} = \frac{s}{2a}}$$

OR

$$r_1 + r_2 + c = 4a\underbrace{\sin^2\left(\frac{\alpha}{2}\right)}_{\left(\frac{1-\cos\alpha}{2}\right)}$$

$$r_1 + r_2 + c = 2a(1 - \cos\alpha)$$

$$r_1 + r_2 + c = 2a\underbrace{\left(1-\cos\eta\cos E_M + \sin E_M \sin\eta\right)}_{1-\cos\left(\eta+E_M\right)} = 2a\left(1-\cos\alpha\right)$$

Also $\quad r_1 + r_2 - c = 2a\left(1-\cos\eta\cos E_M - \sin E_M \sin\eta\right)$

Defined previously $\quad \underbrace{\beta = 2\sin^{-1}\sqrt{\frac{s-c}{2a}}}_{\sin^2\frac{\beta}{2} = \frac{s-c}{2a}}$

OR

$$r_1 + r_2 - c = 4a\underbrace{\sin^2\left(\frac{\beta}{2}\right)}_{\left(\frac{1-\cos\beta}{2}\right)}$$

$$r_1 + r_2 - c = 2a(1-\cos\beta)$$

$$r_1 + r_2 - c = 2a\underbrace{\left(1-\cos\eta\cos E_M - \sin E_M \sin\eta\right)}_{\boxed{1-\cos\left(\eta-E_M\right)}} = 2a\left(1-\cos\beta\right)$$

$$n\left(t_2 - t_1\right) = \left[\left(E_2 - E_1\right) - e\left(\underbrace{\sin E_2 - \sin E_1}_{2\cos\left(\frac{E_2 + E_1}{2}\right)\sin\left(\frac{E_2 - E_1}{2}\right)}\right)\right]$$

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[\left(E_2 - E_1\right) - 2\,\underbrace{e\cos E_p}_{\cos\eta}\,\sin E_M\right]$$

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[2\,E_M - 2\cos\eta\,\sin E_M\right]$$

$$\underline{\underline{\sqrt{\mu}\left(t_2 - t_1\right) = 2a^{3/2}\left[\,E_M - \cos\eta\,\sin E_M\right]}}$$

$$\left\{\begin{array}{l}\text{Note:}\\ \qquad \alpha - \beta = \left(\eta + E_M\right) - \left(\eta - E_M\right) = 2\,E_M \\ \qquad \alpha + \beta = 2\,\eta\end{array}\right\}$$

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[\,\left(\alpha - \beta\right) - \underbrace{2\cos\eta\,\sin E_M}_{\underbrace{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}_{\boxed{\sin\alpha - \sin\beta}}}\right]$$

$$\boxed{\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[\ (\alpha - \beta) - (\sin\alpha - \sin\beta)\right]}$$

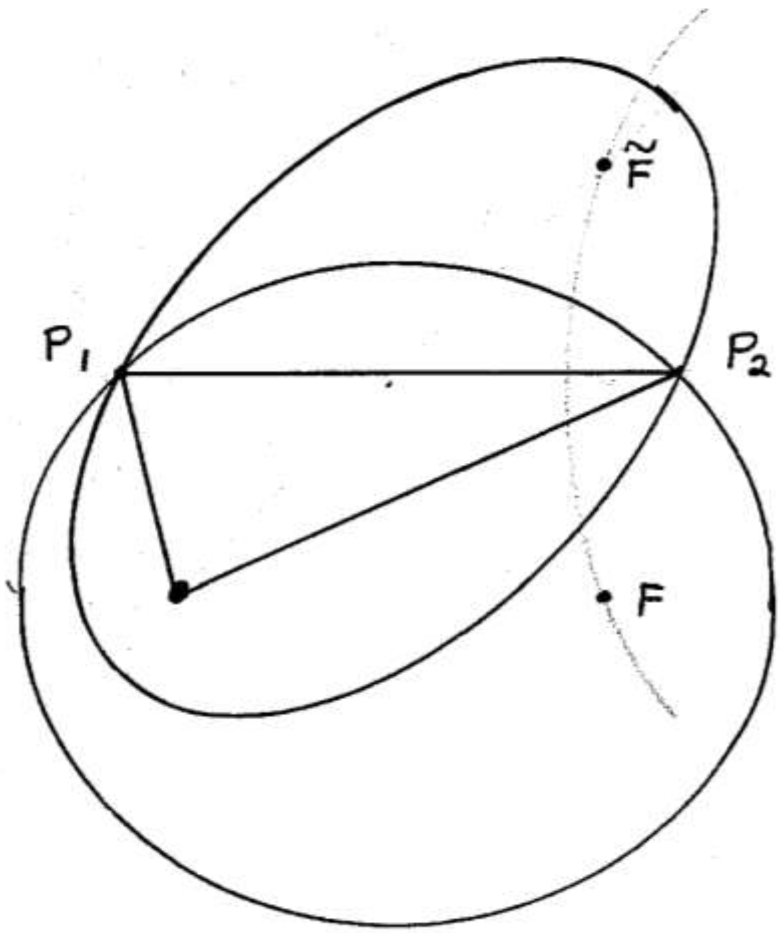
Lambert's Equation

Conjecture by Lambert (1761): time to traverse arc depends only on a and two geometric properties of the space triangle (c, r_1+r_2); Lagrange proves theorem in 1778.

Johann Heinrich Lambert (1728 - 1777)

α, β Quadrant Ambiguities: Elliptic Transfers

For a given space triangle and value "a", there exist four arcs that could serve as the solution:



4 solutions correspond to quadrant ambiguities associated with angles α and β

Principal values α_o, β_o → $0 \le \beta_o \le \alpha_o \le \pi$

Recall derivation of Lambert's Equation

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[\left(E_2 - E_1\right) - e\left(\sin E_2 - \sin E_1\right)\right]$$

$$\Downarrow$$

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[\left(\alpha - \beta\right) - \left(\sin\alpha - \sin\beta\right)\right]$$

$$\left.\begin{aligned} \alpha &= 2\sin^{-1}\sqrt{\frac{s}{2a}} \\ \beta &= 2\sin^{-1}\sqrt{\frac{s-c}{2a}} \end{aligned}\right\} \quad \text{quadrant ambiguities exist}$$

Do α, β have any physical meaning that would help?

$$\left.\begin{aligned} \alpha &= \eta + E_M \\ \beta &= \eta - E_M \end{aligned}\right\} \qquad \begin{aligned} \alpha - \beta &= 2E_M \\ &= 2\left(\frac{E_2 - E_1}{2}\right) \end{aligned}$$

$$\Downarrow$$

$$\boxed{}$$

BUT, generally $\alpha \neq E_2; \quad \beta \neq E_1$

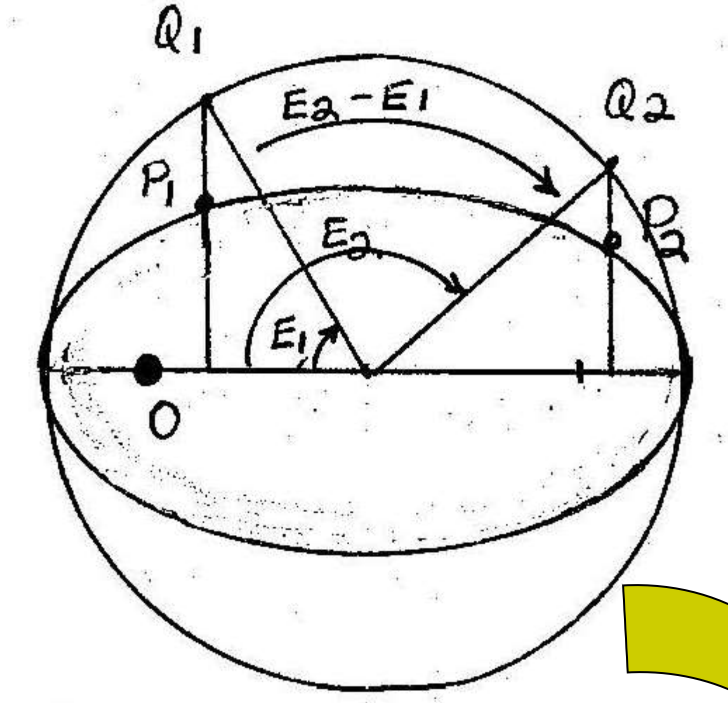
However, all ellipses with the same "*a*" have the same TOF
Useful to choose an equivalent ellipse with the same "*a*"?

Yes $\longrightarrow$ choose a rectilinear ellipse ($e = 1$, $p = 0$)

$$\text{Here} \quad \left\{\begin{aligned} \alpha &= E_2^R \\ \beta &= E_1^R \end{aligned}\right.$$

Use a rectilinear ellipse with the same "*a*" to resolve the quadrant ambiguity issue and provide a geometrical interpretation of α, β

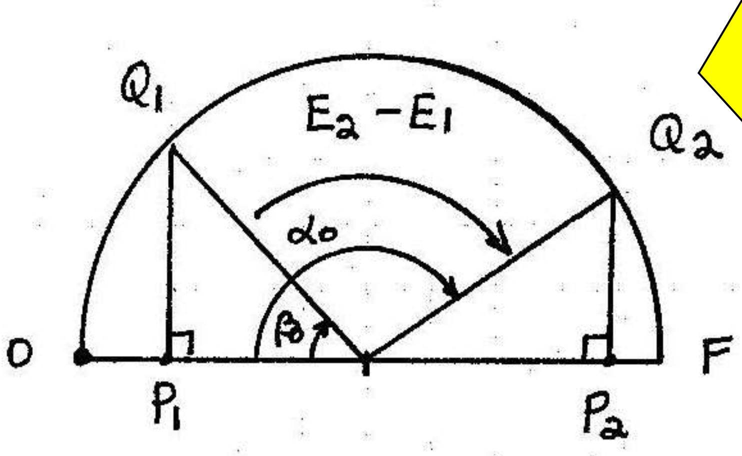
Actual ellipse



Note: Transfer corresponds to what arc on the auxiliary circle?

$$\alpha - \beta = E_2 - E_1$$

To create rectilinear ellipse P_1, P_2 remain in place on chord; O, F move along new ellipses to "shift" locations



Define α_o, β_o consistent with principal value

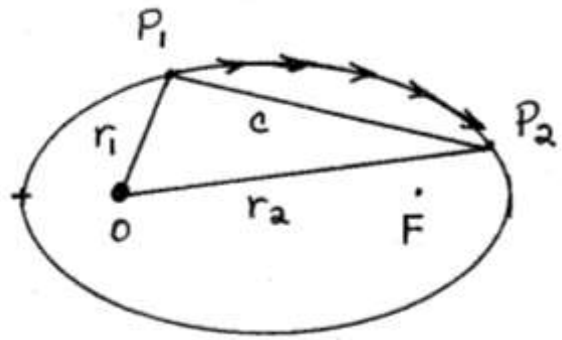
$$\alpha - \beta = E_2 - E_1$$

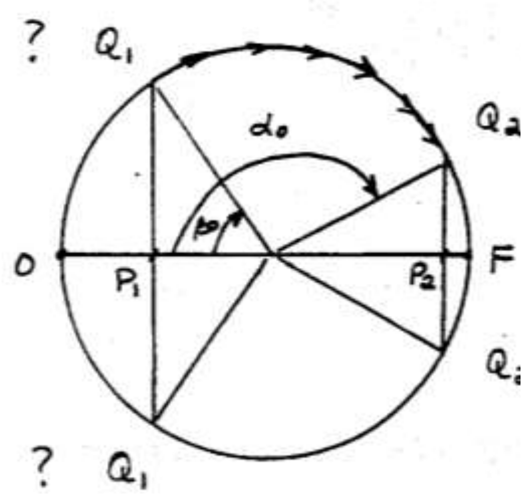
Now consider path for 4 different types of arcs

1A

TA $< 180^\circ$

F is NOT between chord / arc





Transfer follows what arc of the auxiliary circle?

Calculate α_o, β_o ⇨ which of 4 combinations yields correct Q_1, Q_2 ? ***E_1* and *E_2* ?**

Check orbit ⇨ in moving from P_1 to P_2 do you pass through periapsis? apoapsis?

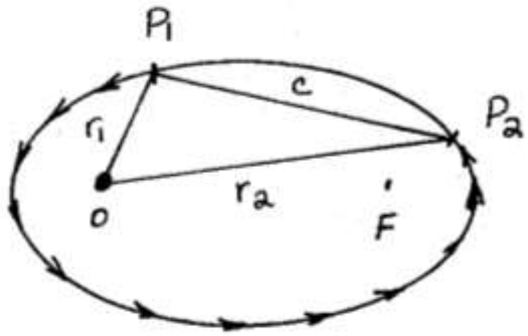
1A $\alpha = \alpha_o \qquad \beta = \beta_o \qquad E_2 - E_1 = \alpha_o - \beta_o$

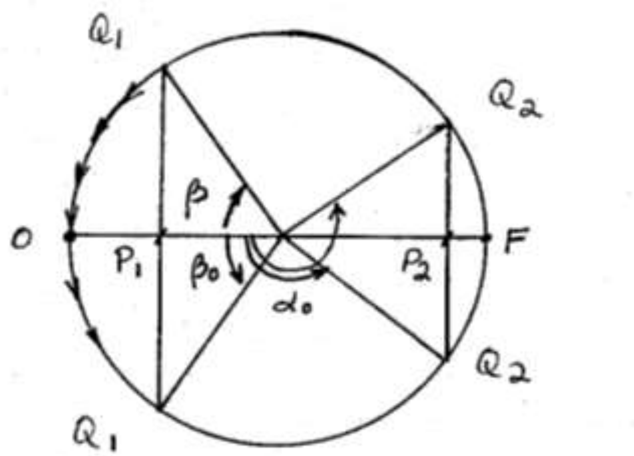
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o)\right]$$

2B

TA > 180°

F is between chord / arc





$$E_2 - E_1 = \alpha - \beta$$
$$= \left[\alpha_o + (\pi - \alpha_o) + (\pi - \alpha_o)\right] - (-\beta_o)$$

2B $\alpha = 2\pi - \alpha_o \qquad \beta = -\beta_o \qquad E_2 - E_1 = 2\pi - \alpha_o + \beta_o$

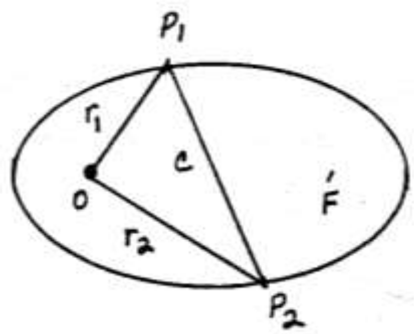
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[\left(2\pi - \alpha_o - \sin(2\pi - \alpha_o)\right) - \left(-\beta_o - \sin(-\beta_o)\right)\right]$$

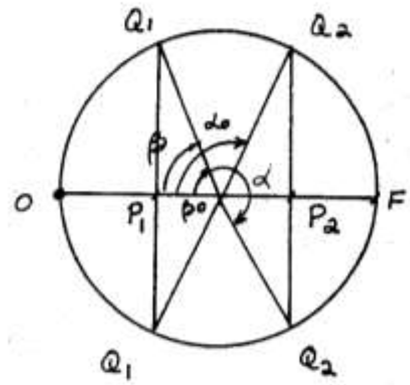
2B $\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[2\pi - (\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o)\right]$

1B

TA $< 180^\circ$

F is between chord / arc





$$E_2 - E_1 = \alpha - \beta$$
$$= \left[\alpha_o + (\pi - \alpha_o) + (\pi - \alpha_o)\right] - \beta_o$$

1B $\boldsymbol{\alpha = 2\pi - \alpha_o}$ $\boldsymbol{\beta = \beta_o}$ $\boldsymbol{E_2 - E_1 = 2\pi - \alpha_o - \beta_o}$

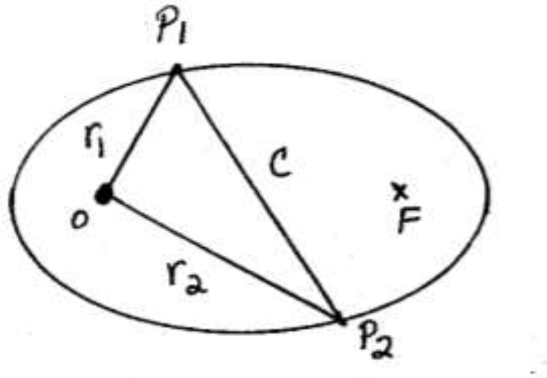
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(2\pi - \alpha_o - \sin(2\pi - \alpha_o)) - (\beta_o - \sin(\beta_o))\right]$$

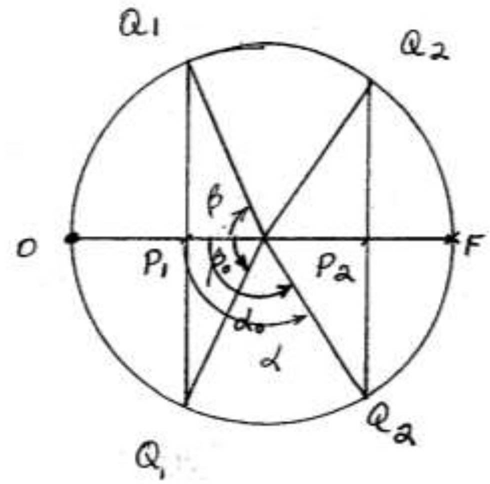
1B $\boldsymbol{\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[2\pi - (\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o)\right]}$

2A

TA > 180°

F is NOT between chord / arc





$$E_2 - E_1 = \alpha - \beta$$
$$= (\alpha_o) - (-\beta_o)$$

2A $\alpha = \alpha_o$ $\beta = -\beta_o$ $E_2 - E_1 = \alpha_o + \beta_o$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha_o - \sin(\alpha_o)) - (-\beta_o - \sin(-\beta_o))\right]$$

2A $\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o)\right]$

α', β' Quadrant Ambiguities: Hyperbolic Transfers

Without the luxury of a visual geometrical technique, straight integration is required for a result

$$\sqrt{\mu}(t_2 - t_1) = |a|^{3/2}\left[(\sinh\alpha' - \alpha') - (\sinh\beta' - \beta')\right]$$

$$H_2 - H_1 = \alpha' - \beta'$$

1H

$$\alpha' = \alpha'_o$$

$$\beta' = \beta'_o$$

$$\sqrt{\mu}(t_2 - t_1) = |a|^{3/2}\left[(\sinh\alpha'_o - \alpha'_o) - (\sinh\beta'_o - \beta'_o)\right]$$

2H

$$\alpha' = \alpha'_o$$

$$\beta' = -\beta'_o$$

$$\sqrt{\mu}(t_2 - t_1) = |a|^{3/2}\left[(\sinh\alpha'_o - \alpha'_o) + (\sinh\beta'_o - \beta'_o)\right]$$

Parabolic Transfers

Used Lambert's TOF theorem to write

$$TPF = TOF\left(a, r_1 + r_2, c\right)$$

⇓

Produced relationships for elliptic and hyperbolic transfers (1A, 1B, 2A, 2B, 1H, 2H)

TOF relationship for parabolic transfer ?

Recall: only TWO possible parabolas that connect points

⇓

TOF determined as limit of other elliptic cases as $a \to \infty$

Parabolic Transfer (Euler's Equation)

$$\mathbf{TOF_1} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} - (s-c)^{3/2}\right]$$

$$\mathbf{TOF_2} = \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} + (s-c)^{3/2}\right]$$

Lambert's Theorem

Time of Flight for Transfer Orbits Between Two Given Positions

ELLIPTIC ORBITS:

$$\sqrt{\frac{\mu}{a^3}}(t_2 - t_1) = \underset{\substack{\uparrow \\ \text{number of complete revolutions}}}{2m\pi} + \begin{cases} (\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o) \\ 2\pi - (\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o) \\ (\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o) \\ 2\pi - (\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o) \end{cases}$$

where

$$c = \text{chord } P_1 P_2$$

$$s = \text{semi-perimeter } \frac{r_1 + r_2 + c}{2}$$

$$\left.\begin{aligned} \alpha &= 2\sin^{-1}\sqrt{\frac{s}{2a}} \\ \beta &= 2\sin^{-1}\sqrt{\frac{s-c}{2a}} \end{aligned}\right\} \alpha_o,\ \beta_o \text{ are principal values}$$

HYPERBOLIC ORBITS:

$$\sqrt{\frac{\mu}{|a|^3}}(t_2 - t_1) = \begin{cases} (\sinh\alpha_o' - \alpha_o') - (\sinh\beta_o' - \beta_o') \\ (\sinh\alpha_o' - \alpha_o') + (\sinh\beta_o' - \beta_o') \end{cases}$$

where $$\left.\begin{aligned} \alpha' &= 2\sinh^{-1}\sqrt{\frac{s}{2|a|}} \\ \beta' &= 2\sinh^{-1}\sqrt{\frac{s-c}{2|a|}} \end{aligned}\right\} \alpha_o',\ \beta_o' \text{ are principal values}$$