College of Engineering
School of Aerospace Engineering

ME 6444: Nonlinear Systems

HW4

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Problem 1

(30 points) Discrete Nonlinear Modelling, Hamiltonian, and Phase Plane Analysis

(a) Use Lagrange's equations to verify that the spinning shaft pictured is governed by

$$\frac{1}{4}m\ddot{\theta} - \frac{1}{4}m\dot{\theta}\tan\theta - \frac{1}{4}m\Omega^2\tan\theta + \frac{1}{2}k\frac{(1-\cos\theta)}{\cos\theta}\tan\theta = 0,$$

where Ω is an imposed <u>constant</u> rotational spin. Assume mass-less links and an unstretched spring when θ is zero. Neglect gravitational potential energy (small compared to stored elastic energy).

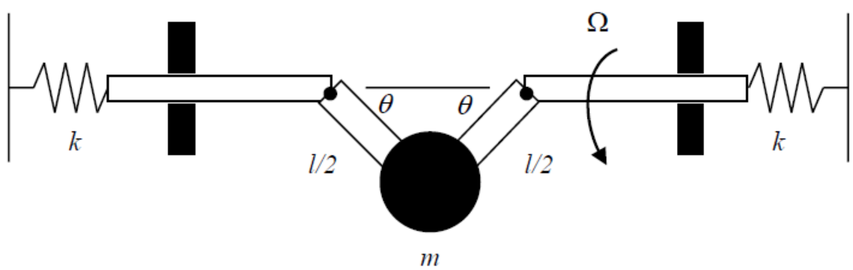

Figure 1: Spinning shaft system

(b) Sketch the local trajectories in phase plane about each fixed point assuming $\Omega > 0$.

(c) Show that the total energy $E = T + V$ is <u>not</u> conserved, and instead that the Hamiltonian $H = L - \frac{\partial L}{\partial \dot{\theta}}\dot{\theta}$ is conserved.

(d) Using, H, determine the stability of the fixed points.

Solution:

(a) First we derive the kinematics of this system to compute the position and velocity of the mass, m. From Figure 2 we define the reference frames, e-frame and a-frame, and the position vector of the point mass can be expressed as

$$\vec{r}_{OA} = \frac{l}{2}\sin\theta\hat{a}_1.$$

Then the velocity vector with respect to the e-frame becomes

$$\begin{aligned} {}^e\vec{v}_{OA} &= {}^e\frac{d\vec{r}_{OA}}{dt} \\ &= {}^a\frac{d}{dt}\left(\frac{l}{2}\sin\theta\hat{a}_1\right) + \Omega\hat{e}_1 \times \frac{l}{2}\sin\theta\hat{a}_1 \\ &= \frac{l}{2}\dot{\theta}\cos\theta\hat{a}_1 + \frac{l}{2}\Omega\sin\theta\hat{a}_2. \end{aligned}$$

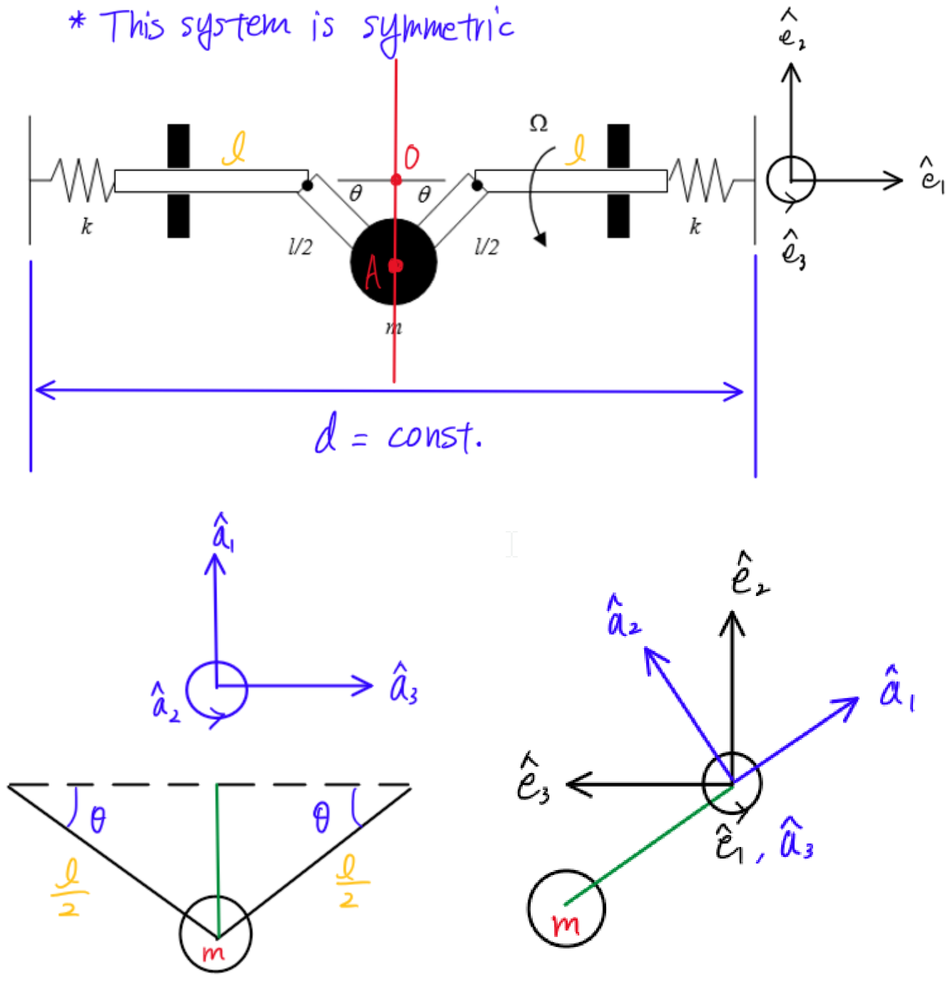

Figure 2: Sktech of diagram from front, top, and side view with reference frames

Then the kinetic energy becomes

$$\begin{aligned}
T &= \frac{1}{2}m({}^e\vec{v}_{OA} \cdot {}^e\vec{v}_{OA}) \\
&= \frac{1}{8}ml^2\dot{\theta}^2\cos^2\theta + \frac{1}{8}ml^2\Omega^2\sin^2\theta.
\end{aligned}$$

On the other hand, the potential energy is governed by the stored elastic energy and we can ignore the gravitational potential energy since it is small compared to the stored elastic energy.

$$\begin{aligned}
V &= \frac{1}{2}k\left(\frac{l}{2}(1-\cos\theta)\right)^2 = \frac{1}{2}k\left(\frac{l}{2}(1-\cos\theta)\right)^2 \\
&= \frac{1}{4}kl^2(1-\cos\theta)^2
\end{aligned}$$

Since the Lagrangian is $L = T - V$ we can compute

$$\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}} &= \frac{1}{4}ml^2\dot{\theta}\cos^2\theta \\
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= \frac{1}{4}ml^2\ddot{\theta}\cos^2\theta - \frac{1}{2}ml^2\dot{\theta}^2\cos\theta\sin\theta \\
\frac{\partial L}{\partial \theta} &= -\frac{1}{4}ml^2\dot{\theta}^2\cos\theta\sin\theta + \frac{1}{4}ml^2\Omega^2\sin\theta\cos\theta - \frac{1}{2}kl^2(1-\cos\theta)\sin\theta.
\end{aligned}$$

Thus from

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

we have

$$\frac{1}{4}ml^2\ddot{\theta}\cos^2\theta - \frac{1}{2}ml^2\dot{\theta}^2\cos\theta\sin\theta + \frac{1}{4}ml^2\dot{\theta}^2\cos\theta\sin\theta - \frac{1}{4}ml^2\Omega^2\sin\theta\cos\theta + \frac{1}{2}kl^2(1-\cos\theta)\sin\theta = 0$$

$$\frac{1}{4}ml^2\ddot{\theta}\cos^2\theta - \frac{1}{4}ml^2\dot{\theta}^2\cos\theta\sin\theta - \frac{1}{4}ml^2\Omega^2\sin\theta\cos\theta + \frac{1}{2}kl^2(1-\cos\theta)\sin\theta = 0$$

Hence,

$$\boxed{\frac{1}{4}m\ddot{\theta} - \frac{1}{4}m\dot{\theta}^2\tan\theta - \frac{1}{4}m\Omega^2\tan\theta + \frac{1}{2}k\frac{(1-\cos\theta)}{\cos\theta}\tan\theta = 0.}$$

(b) If we rewrite the given EOM with $x_1 = \theta$ and $x_2 = \dot{\theta}$ we have

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= x_2^2\tan x_1 + \Omega^2\tan x_1 - \frac{2k(1-\cos x_1)}{m\cos x_1}\tan x_1
\end{aligned}$$

Then the fixed point or equilibrium points can be found by solving for $\dot{x_1} = 0$ and $\dot{x_2} = 0$.

$$\begin{aligned}
x_2 = 0 &\longrightarrow x_{2e} = 0 \\
0 &= \Omega^2\tan x_1 - \frac{2k(1-\cos x_1)}{m\cos x_1}\tan x_1 \\
0 &= \tan x - 1\left(\Omega^2 - \frac{2k}{m}\frac{(1-\cos x_1)}{\cos x_1}\right).
\end{aligned}$$

Which gives,

$$\begin{aligned}
x_{1e} &= n\pi \qquad n = 0, 1, 2, ... \\
&\mathit{or} \\
x_{1e} &= \arccos\left(\frac{2k}{2k + m\Omega^2}\right)
\end{aligned}$$

and

$$x_{2e} = 0.$$

If we declare the system matrix A to be

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we know that

$$a = 0,$$

The elements a, b, c, d are

$$\begin{aligned} a &= \frac{\partial \dot{x}_1}{\partial x_1} = 0 \\ b &= \frac{\partial \dot{x}_1}{\partial x_2} = 1 \\ c &= \frac{\partial \dot{x}_2}{\partial x_1} = x_2^2 \sec^2 x_1 + \Omega^2 \sec^2 x_1 - \frac{2k}{m} \sin x_1 \sec x_1 \tan x_1 \\ &\qquad - \frac{2k}{m}(1 - \cos x_1) \sec x_1 \tan^2 x_1 - \frac{2k}{m}(1 - \cos x_1) \sec^3 x_1 \\ d &= \frac{\partial \dot{x}_2}{\partial x_2} = 2x_2 \tan x_1 \end{aligned}$$

Now if we plug in the equilibrium points into this matrix we can analyze the local trajectories in the phase plane. For the equilibrium point $(n\pi, 0)$ we have

$$A = \begin{cases} \begin{bmatrix} 0 & 1 \\ \Omega^2 & 0 \end{bmatrix} & \text{if } n \text{ is even} \\ \\ \begin{bmatrix} 0 & 1 \\ \dfrac{4k}{m} + \Omega^2 & 0 \end{bmatrix} & \text{if } n \text{ is odd} \end{cases}$$

For the case where n is even the eigenvalues and eigenvectors are

$$\lambda_1 = \Omega, \qquad \lambda_2 = -\Omega, \qquad v = \left\{ \begin{bmatrix} \frac{1}{\Omega} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\Omega} \\ 1 \end{bmatrix} \right\}.$$

This shows that this equilibrium point is a saddle point.

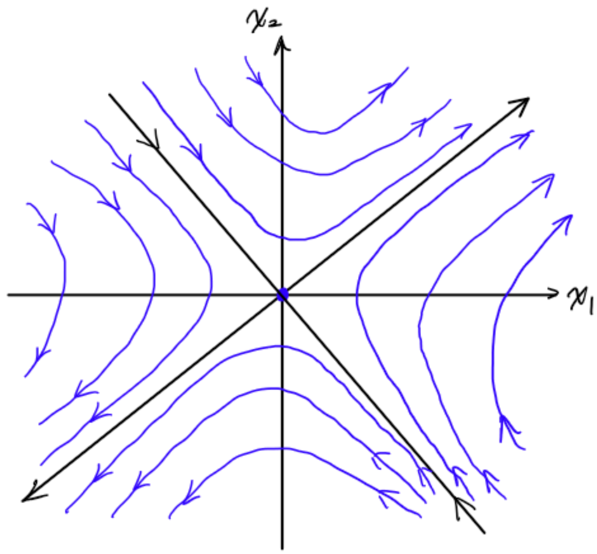

Figure 3: Saddle point equilibrium point and local trajectories

Similarly, when n is odd we have

$$\lambda_1 = \frac{\sqrt{m(m\Omega^2+4k)}}{m\Omega^2+4k}, \qquad \lambda_2 = -\frac{\sqrt{m(m\Omega^2+4k)}}{m\Omega^2+4k},$$

$$v = \left\{ \begin{bmatrix} \frac{\sqrt{m(m\Omega^2+4k)}}{m\Omega^2+4k} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{m(m\Omega^2+4k)}}{m\Omega^2+4k} \\ 1 \end{bmatrix} \right\}$$

This is also a saddle point, and therefore, the local trajectories of this equilibrium point is the same as Figure 3.

Next, if we plug in the equilibrium point of

$$\left(\arccos\left(\frac{2k}{2k+m\Omega^2}\right), 0\right)$$

we have the A matrix of

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{0.2500\,\Omega^2\,(\Omega^4\,m^2+6\,\Omega^2\,k\,m+8\,k^2)}{k^2} & 0 \end{bmatrix}$$

and the eigenvalues and eigenvectors become

$$\lambda_1 = -\frac{\Omega\,\sqrt{-(m\,\Omega^2+2\,k)\,(m\,\Omega^2+4\,k)}}{2k}, \qquad \lambda_2 = \frac{0.5000\,\Omega\,\sqrt{-(m\,\Omega^2+2\,k)\,(m\,\Omega^2+4\,k)}}{2k}$$

$$v = \left\{ \begin{bmatrix} \frac{2\,\Omega\,k\,\sqrt{-(m\,\Omega^2+2\,k)\,(m\,\Omega^2+4\,k)}}{\Omega^6\,m^2+6\,\Omega^4\,k\,m+8\,\Omega^2\,k^2} \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2\,\Omega\,k\,\sqrt{-(m\,\Omega^2+2\,k)\,(m\,\Omega^2+4\,k)}}{\Omega^6\,m^2+6\,\Omega^4\,k\,m+8\,\Omega^2\,k^2} \\ 1 \end{bmatrix} \right\}$$

For this equilibrium point, we have pure imaginary eigenvalues and a geometric multiplicity of 2. Thus, it is a center which is sketched as follows.

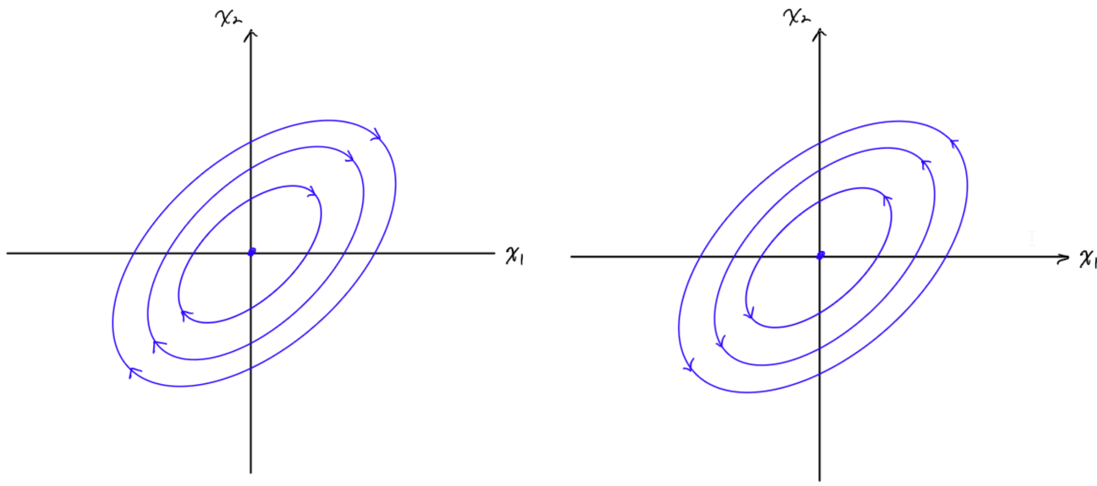

Figure 4: Center equilibrium point

Since, the direction of circulation for the center depends on the constants Ω, k. and m it could be circulating in either direction.

(c) We know that the Hamiltonian is

$$
\begin{aligned}
H &= L - \frac{L}{\dot{\theta}}\dot{\theta} \\
&= \frac{1}{8}ml^2\dot{\theta}^2\cos^2\theta + \frac{1}{8}ml^2\Omega^2\sin^2\theta - \frac{1}{4}kl^2(1-\cos\theta)^2 - \dot{\theta}\left(\frac{1}{4}ml^2\dot{\theta}\cos^2\theta\right) \\
&= -\frac{1}{8}ml^2\dot{\theta}^2\cos^2\theta + \frac{1}{8}ml^2\Omega^2\sin^2\theta - \frac{1}{4}kl^2(1-\cos\theta)^2.
\end{aligned}
$$

Then, we compute

$$
\begin{aligned}
\frac{\partial H}{\partial \dot{\theta}} &= -\frac{1}{4}ml^2\dot{\theta}\cos^2\theta \\
\frac{\partial H}{\partial \theta} &= \frac{1}{4}ml^2\dot{\theta}^2\cos\theta\sin\theta + \frac{1}{4}ml^2\Omega^2\sin\theta\cos\theta - \frac{1}{2}kl^2(1-\cos\theta)\sin\theta.
\end{aligned}
$$

Next we calculate

$$\begin{aligned}
\frac{dH}{dt} &= \frac{\partial H}{\partial \dot{\theta}}\frac{\partial \dot{\theta}}{\partial t} + \frac{\partial H}{\partial \theta}\frac{\partial \theta}{\partial t} \\
&= \left(-\frac{1}{4}ml^2\dot{\theta}\cos^2\theta\right)\ddot{\theta} + \left(\frac{1}{4}ml^2\dot{\theta}^2\cos\theta\sin\theta + \frac{1}{4}ml^2\Omega^2\sin\theta\cos\theta - \frac{1}{2}kl^2(1-\cos\theta)\sin\theta\right)\dot{\theta} \\
&= -\frac{\dot{\theta}}{l^2\cos^2\theta}\underbrace{\left(\frac{1}{4}m\ddot{\theta} - \frac{1}{4}m\dot{\theta}^2\tan\theta - \frac{1}{4}m\Omega^2\tan\theta + \frac{1}{2}k\frac{(1-\cos\theta)}{\cos\theta}\tan\theta\right)}_{\text{EOM}=0} \\
&= 0
\end{aligned}$$

Hence, we can conclude that

$$\boxed{\frac{dH}{dt} = 0.}$$

This posits that the Hamiltonian is conserved.

Whereas, the total energy is

$$E = T + V = \frac{1}{8}ml^2\dot{\theta}^2\cos^2\theta + \frac{1}{8}ml^2\Omega^2\sin^2\theta + \frac{1}{4}kl^2(1-\cos\theta)^2.$$

This equation is not the same as the Hamiltonian, which implies that the total energy of this system E is not conserved.

(d) From the Hamiltonian we can find the elements of the system matrix A

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where

$$\begin{aligned}
a &= \frac{1}{2}\,l^2\,m\,\dot{\theta}\,\cos(\theta)\,\sin(\theta) \\
b &= -\frac{1}{4}\,l^2\,m\cos(\theta)^2 \\
c &= \frac{1}{2}\,k\,l^2\sin(\theta)^2 - \frac{1}{4}\,\Omega^2\,l^2\,m\cos(\theta)^2 + \frac{1}{4}\,\Omega^2\,l^2\,m\sin(\theta)^2 - \frac{1}{4}\,l^2\,m\,\dot{\theta}^2\cos(\theta)^2 \\
&\quad - \frac{1}{2}\,k\,l^2\,\cos(\theta)\,(\cos(\theta)-1) + \frac{1}{4}\,l^2\,m\,\dot{\theta}^2\sin(\theta)^2 \\
d &= -\frac{1}{2}\,l^2\,m\,\dot{\theta}\,\cos(\theta)\,\sin(\theta)
\end{aligned}$$

and then

$$
\begin{aligned}
p &= a + d = 0 \qquad \text{identically zero} \\
q &= ad - bc \\
&= \frac{1}{4}\, l^2\, m \cos{(\theta)}^2\, \left(\frac{1}{2}\, k\, l^2 \sin{(\theta)}^2 - \frac{1}{4}\, \Omega^2\, l^2\, m \cos{(\theta)}^2 + \frac{1}{4}\, \Omega^2\, l^2\, m \sin{(\theta)}^2 \right. \\
&\qquad \left. - \frac{1}{4}\, l^2\, m\, \dot{\theta}^2 \cos{(\theta)}^2 - \frac{1}{2}\, k\, l^2\, \cos{(\theta)}\, \left(\cos{(\theta)} - 1\right) + \frac{1}{4}\, l^2\, m\, \dot{\theta}^2 \sin{(\theta)}^2 \right) \\
&\qquad\qquad - \frac{1}{4}\, l^4\, m^2\, \dot{\theta}^2 \cos{(\theta)}^2 \sin{(\theta)}^2
\end{aligned}
$$

For the equilibrium point $(n\pi, 0)$, q becomes

$$
q = \begin{cases} -\frac{1}{4}\, l^2\, m\, \left(\frac{1}{4}\, m\, \Omega^2\, l^2 + k\, l^2\right) < 0 & \text{if } n \text{ is odd} \\ -\frac{1}{16}\, \Omega^2\, l^4\, m^2 < 0 & \text{if } n \text{ is even} \end{cases}
$$

Thus, the equilibrium point of $(n\pi, 0)$ is an unstable saddle. Then for the equilibrium point of $\left(\arccos\left(\frac{2k}{2k + m\Omega^2}\right), 0\right)$ we have

$$
q = \frac{\Omega^2\, k^2\, l^4\, m^2\, \left(m\, \Omega^2 + 4\, k\right)}{4\, {\left(m\, \Omega^2 + 2\, k\right)}^3} > 0.
$$

Hence, this equilibrium point is a stable center from the Hamiltonian stability analysis.

Problem 2

(20 points) Phase Plane with Stability Analysis

Determine the characteristic (type of fixed point and stability) of all fixed points associated with each equation of motion below. Plot the phase plane for each.

(a) $\ddot{u} + 2\mu\dot{u} + u + u^3 = 0$

(b) $\ddot{u} + 2\mu\dot{u} + u - u^3 = 0$

(c) $\ddot{u} + 2\mu\dot{u} - u + u^3 = 0$

(d) $\ddot{u} + 2\mu\dot{u} - u - u^3 = 0$

In all cases treat the damping coefficient μ as greater than zero.

Solution:
(a) Let $x_1 = u$ and $x_2 = \dot{u}$, then system can be expressed in the form of

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2\mu x_2 - x_1 - x_1^3 \end{cases}$$

The elements a, b, c, d are

$$\begin{aligned} a &= \frac{\partial \dot{x}_1}{\partial x_1} = 0 \\ b &= \frac{\partial \dot{x}_1}{\partial x_2} = 1 \\ c &= \frac{\partial \dot{x}_2}{\partial x_1} = -3x_1^2 - 1 \\ d &= \frac{\partial \dot{x}_2}{\partial x_2} = -2\mu \end{aligned}$$

which gives

$$A = \begin{bmatrix} 0 & 1 \\ -3x_1^2 - 1 & -2\mu \end{bmatrix}.$$

The equilibrium points for this system is $x_{1e} = x_{2e} = 0$. At this point the system matrix A becomes

$$A_e = \begin{bmatrix} 0 & 1 \\ -1 & -2\mu \end{bmatrix}.$$

and the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu + \sqrt{\mu^2 - 1}, \qquad \lambda_2 = -\mu - \sqrt{\mu^2 - 1}$$

$$v_1 = \begin{bmatrix} -\mu - \sqrt{\mu^2 - 1} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -\mu + \sqrt{\mu^2 - 1} \\ 1 \end{bmatrix}$$

Thus, the equilibrium points are

for (0, 0)
$\rightarrow$ if $\mu > 1$: a stable node
$\rightarrow$ if $\mu = 1$: a stable singular node
$\rightarrow$ if $0 < \mu < 1$: a stable spiral/focus

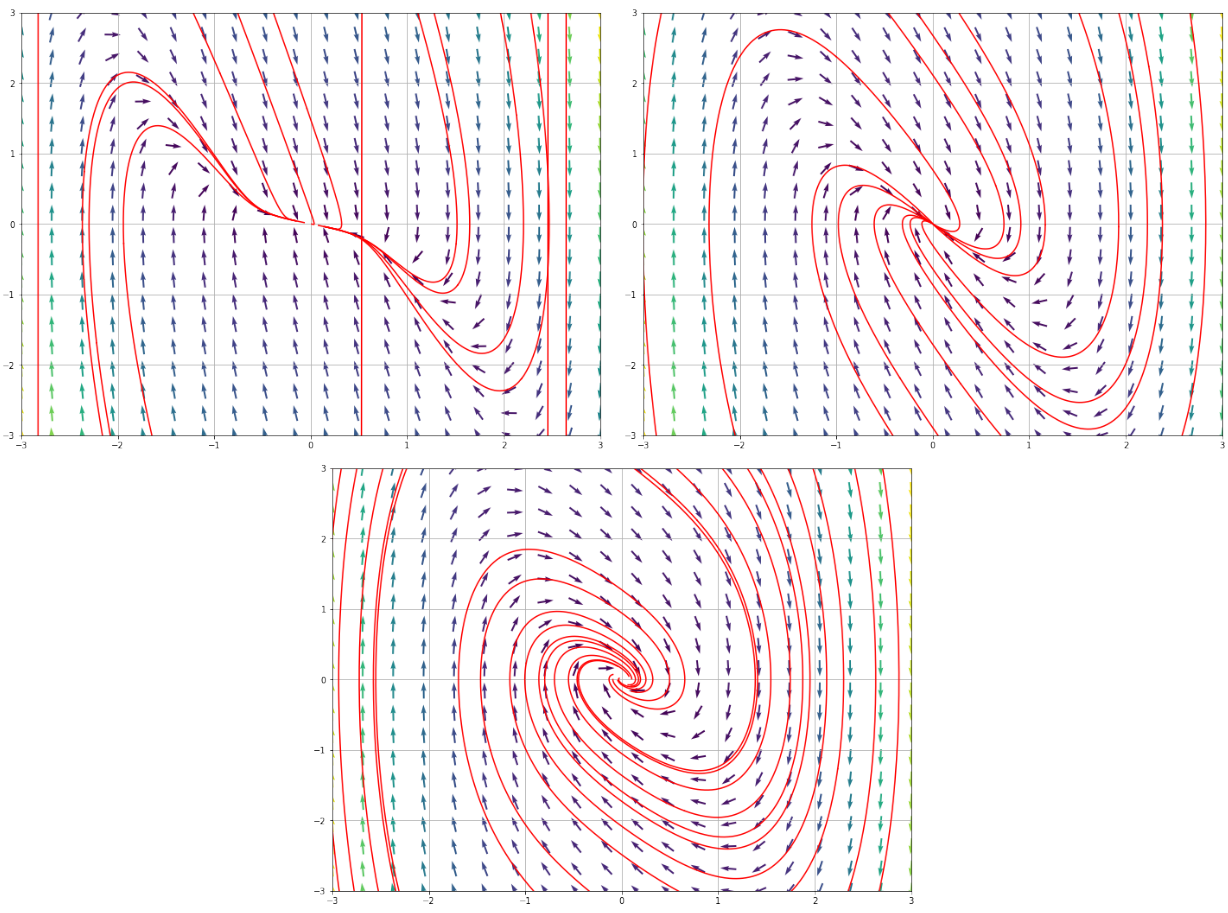

Figure 5: Phase plane for (a) $\mu > 1$ (top left), $\mu = 1$ (top right), $0 < \mu < 1$ (bottom)

(b) Let $x_1 = u$ and $x_2 = \dot{u}$, then system can be expressed in the form of

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2\mu x_2 - x_1 + x_1^3 \end{cases}$$

The elements a, b, c, d are

$$\begin{aligned} a &= \frac{\partial \dot{x}_1}{\partial x_1} = 0 \\ b &= \frac{\partial \dot{x}_1}{\partial x_2} = 1 \\ c &= \frac{\partial \dot{x}_2}{\partial x_1} = 3x_1^2 - 1 \\ d &= \frac{\partial \dot{x}_2}{\partial x_2} = -2\mu \end{aligned}$$

which gives

$$A = \begin{bmatrix} 0 & 1 \\ 3x_1^2 - 1 & -2\mu \end{bmatrix}.$$

The equilibrium points for this system is $(x_{1e}, x_{2e}) = (-1, 0), (0, 0), (1, 0)$. At these points the system matrix A becomes

$$A|_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ 2 & -2\mu \end{bmatrix}$$

$$A|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -2\mu \end{bmatrix}$$

$$A|_{(1,0)} = \begin{bmatrix} 0 & 1 \\ 2 & -2\mu \end{bmatrix}.$$

For equilibrium point $(-1, 0)$ the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu - \sqrt{\mu^2 + 2}, \qquad \lambda_2 = -\mu + \sqrt{\mu^2 + 2}$$

$$v_1 = \begin{bmatrix} \frac{1}{2}\mu - \frac{1}{2}\sqrt{\mu^2 + 2} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \frac{1}{2}\mu + \frac{1}{2}\sqrt{\mu^2 + 2} \\ 1 \end{bmatrix}.$$

For equilibrium point $(0, 0)$ the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu + \sqrt{\mu^2 - 1}, \qquad \lambda_2 = -\mu - \sqrt{\mu^2 - 1}$$

$$v_1 = \begin{bmatrix} -\mu - \sqrt{\mu^2 - 1} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -\mu + \sqrt{\mu^2 - 1} \\ 1 \end{bmatrix}$$

For equilibrium point $(1, 0)$ the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu - \sqrt{\mu^2 + 2}, \qquad \lambda_2 = -\mu + \sqrt{\mu^2 + 2}$$

$$v_1 = \begin{bmatrix} \frac{1}{2}\mu - \frac{1}{2}\sqrt{\mu^2 + 2} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \frac{1}{2}\mu + \frac{1}{2}\sqrt{\mu^2 + 2} \\ 1 \end{bmatrix}.$$

Thus, the equilibrium points are

for (0, 0)
$\rightarrow$ if $\mu > 1$: a stable node
$\rightarrow$ if $\mu = 1$: a stable singular node
$\rightarrow$ if $0 < \mu < 1$: a stable spiral/focus

for (-1, 0) and (1, 0)
$\rightarrow$ it is an unstable saddle point

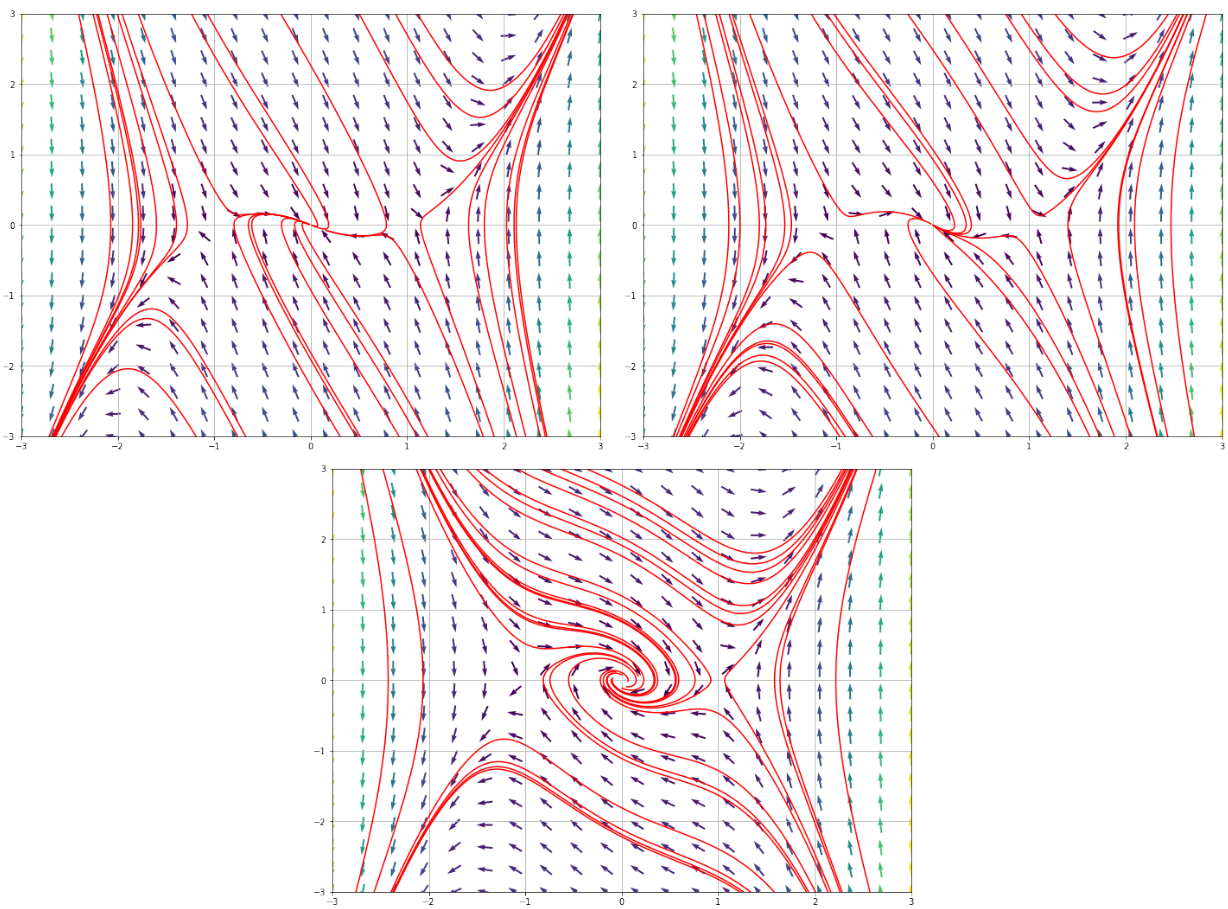

Figure 6: Phase plane for (b) $\mu > 1$ (top left), $\mu = 1$ (top right), $0 < \mu < 1$ (bottom)

(c) Let $x_1 = u$ and $x_2 = \dot{u}$, then system can be expressed in the form of

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2\mu x_2 + x_1 - x_1^3 \end{cases}$$

The elements a, b, c, d are

$$\begin{aligned} a &= \frac{\partial \dot{x}_1}{\partial x_1} = 0 \\ b &= \frac{\partial \dot{x}_1}{\partial x_2} = 1 \\ c &= \frac{\partial \dot{x}_2}{\partial x_1} = -3x_1^2 + 1 \\ d &= \frac{\partial \dot{x}_2}{\partial x_2} = -2\mu \end{aligned}$$

which gives

$$A = \begin{bmatrix} 0 & 1 \\ -3x_1^2 + 1 & -2\mu \end{bmatrix}.$$

The equilibrium points for this system is $(x_{1e}, x_{2e}) = (-1, 0), (0, 0), (1, 0)$. At these points the system matrix A becomes

$$A|_{(-1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -2\mu \end{bmatrix}$$

$$A|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -2\mu \end{bmatrix}$$

$$A|_{(1,0)} = \begin{bmatrix} 0 & 1 \\ -2 & -2\mu \end{bmatrix}.$$

For equilibrium point $(-1, 0)$ the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu - \sqrt{\mu^2 - 2}, \qquad \lambda_2 = -\mu + \sqrt{\mu^2 - 2}$$

$$v_1 = \begin{bmatrix} -\frac{1}{2}\mu + \frac{1}{2}\sqrt{\mu^2 - 2} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -\frac{1}{2}\mu - \frac{1}{2}\sqrt{\mu^2 - 2} \\ 1 \end{bmatrix}.$$

For equilibrium point $(0, 0)$ the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu + \sqrt{\mu^2 + 1}, \qquad \lambda_2 = -\mu - \sqrt{\mu^2 + 1}$$

$$v_1 = \begin{bmatrix} \mu - \sqrt{\mu^2 + 1} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \mu + \sqrt{\mu^2 + 1} \\ 1 \end{bmatrix}$$

For equilibrium point $(1, 0)$ the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu - \sqrt{\mu^2 - 2}, \qquad \lambda_2 = -\mu + \sqrt{\mu^2 - 2}$$

$$v_1 = \begin{bmatrix} -\frac{1}{2}\mu + \frac{1}{2}\sqrt{\mu^2 + 2} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} -\frac{1}{2}\mu - \frac{1}{2}\sqrt{\mu^2 + 2} \\ 1 \end{bmatrix}.$$

Thus, the equilibrium points are

for (-1, 0) and (1, 0)
$\rightarrow$ if $\mu > \sqrt{2}$: a stable node
$\rightarrow$ if $\mu = \sqrt{2}$: a stable singular node
$\rightarrow$ if $0 < \mu < \sqrt{2}$: a stable spiral/focus

for (0, 0)
$\rightarrow$ it is an unstable saddle point

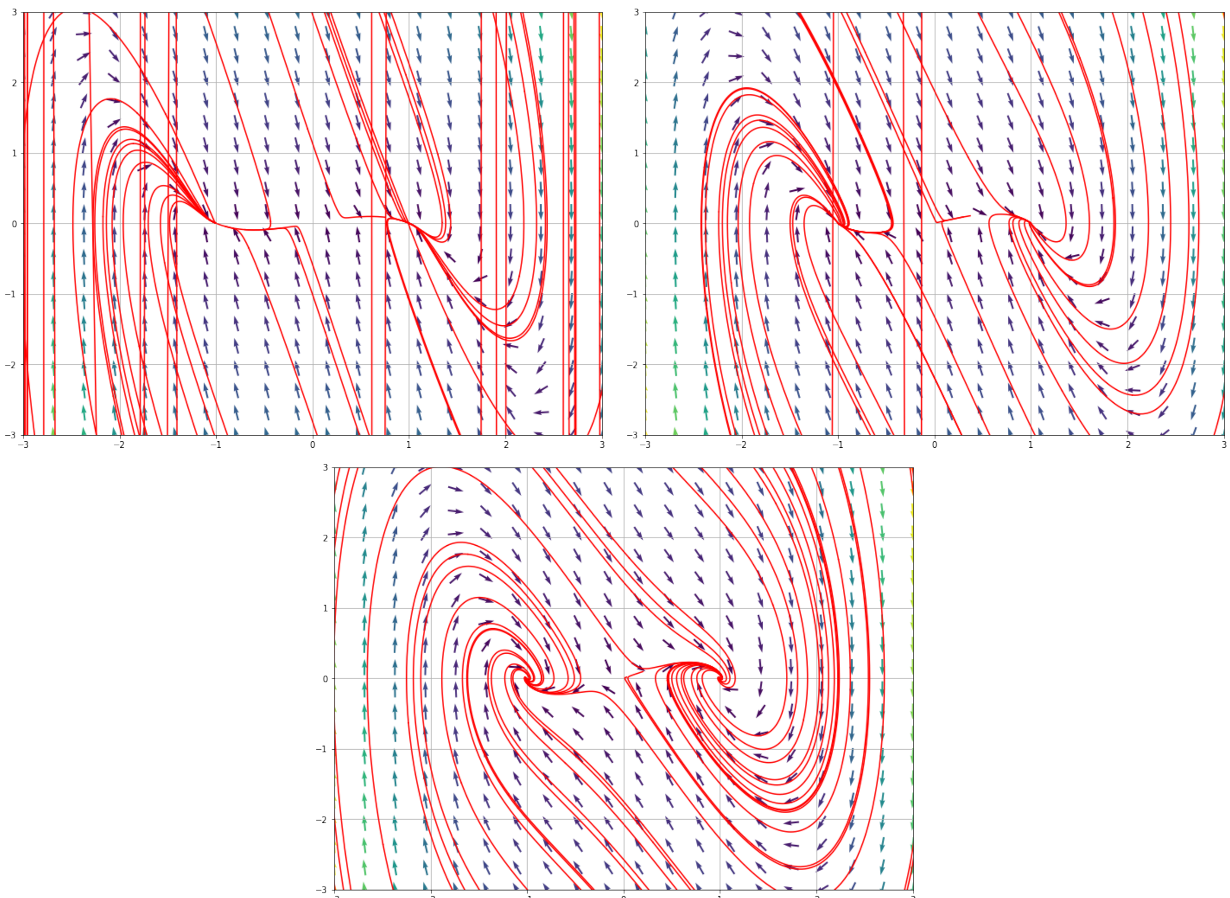

Figure 7: Phase plane for (c) $\mu > \sqrt{2}$ (top left), $\mu = \sqrt{2}$ (top right), $0 < \mu < \sqrt{2}$ (bottom)

(d) Let $x_1 = u$ and $x_2 = \dot{u}$, then system can be expressed in the form of

$$\begin{cases} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2\mu x_2 + x_1 + x_1^3 \end{cases}$$

The elements a, b, c, d are

$$a = \frac{\partial \dot{x}_1}{\partial x_1} = 0$$
$$b = \frac{\partial \dot{x}_1}{\partial x_2} = 1$$
$$c = \frac{\partial \dot{x}_2}{\partial x_1} = 3x_1^2 + 1$$
$$d = \frac{\partial \dot{x}_2}{\partial x_2} = -2\mu$$

which gives

$$A = \begin{bmatrix} 0 & 1 \\ 3x_1^2 + 1 & -2\mu \end{bmatrix}.$$

The equilibrium points for this system is $x_{1e} = x_{2e} = 0$. At this point the system matrix A becomes

$$A_e = \begin{bmatrix} 0 & 1 \\ 1 & -2\mu \end{bmatrix}.$$

and the eigenvalues and eigenvectors become

$$\lambda_1 = -\mu - \sqrt{\mu^2 + 1}, \qquad \lambda_2 = -\mu + \sqrt{\mu^2 + 1}$$

$$v_1 = \begin{bmatrix} \mu - \sqrt{\mu^2 - 1} \\ 1 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} \mu + \sqrt{\mu^2 - 1} \\ 1 \end{bmatrix}$$

Thus, the equilibrium points are

for (0, 0)
$\rightarrow$ it is an unstable saddle point

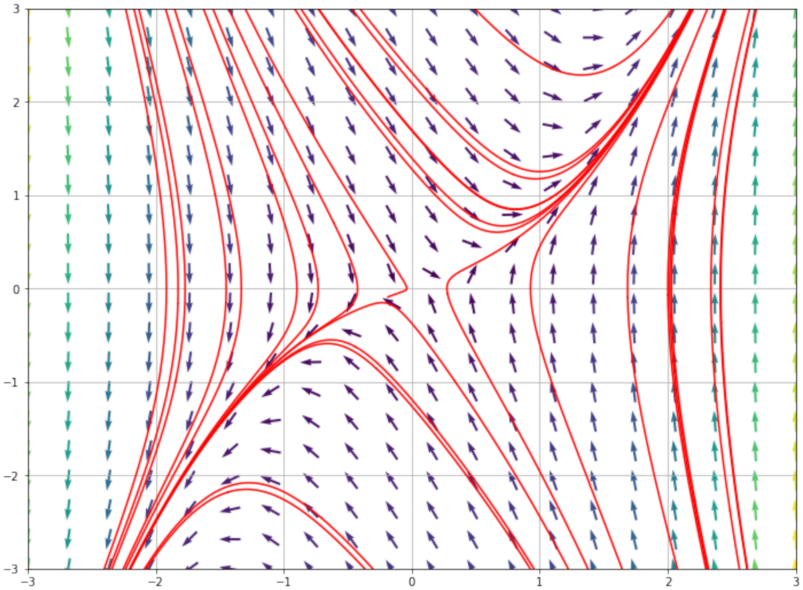

Figure 8: Phase plane for (d) $\mu = 0.8$ (μ could be any positive value)

Appendix

3.1 Problem 1: MATLAB Code

```
% AE6444 HW4 Problem 1 MATLAB
% Tomoki Koike
close all; clear all; clc;
%%
% Define the system
syms x_1 x_2 Omega k m n l theta theta_dot
assume(n, {'integer', 'positive'});

T = 1/8*m*l^2*theta_dot^2*cos(theta)^2 + 1/8*m*l^2*Omega^2*sin(theta)^2;
V = 1/4*k*l^2 * (1 - cos(theta))^2;
L = T - V;

x1dot = x_2
x2dot = x_2^2 * tan(x_1) + Omega^2 * tan(x_1) - 2*k/m*(1-cos(x_1))*sec(x_1)*tan(x_1);
%%
% System matrix: A
a = diff(x1dot, x_1);
b = diff(x1dot, x_2);
c = diff(x2dot, x_1)
d = diff(x2dot, x_2)
A = [a, b; c, d];
%%
% Equilibrium points
x1e_1 = n*pi;
x1e_2 = acos(2*k / (2*k + m*Omega^2));
x2e = 0;
%%
% First possible solution
A1 = subs(A, [x_1, x_2], [x1e_1, x2e]);
A1_odd = subs(A1, n, 1)  % odd n value
A1_even = subs(A1, n, 2)  % even n value
%%
% Eigenvalues for first possible solution
[v_even, d_even] = eig(A1_even)
[v_odd, d_odd] = eig(A1_odd)
%%
% Second possible solution
A2 = simplify(subs(A, [x_1, x_2], [x1e_2, x2e]))
%%
```

```
% Eigenvalues for second possible solution
[v2, d2] = eig(A2)
%%
% Hamiltonian

H = L - diff(L, theta_dot)*theta_dot;
%%
a_h = diff(diff(H, theta), theta_dot)
b_h = diff(diff(H, theta_dot), theta_dot)
c_h = -diff(diff(H, theta), theta)
d_h = -diff(diff(H, theta_dot), theta)
%%
p = a_h + d_h
q = a_h * d_h - b_h * c_h
%%
% (n\pi, 0)
subs(q, [theta, theta_dot], [1*pi, 0])
subs(q, [theta, theta_dot], [2*pi, 0])

% (<>, 0)
simplify(subs(q, [theta, theta_dot], [x1e_2, 0]))
```

3.2 Problem 2: MATLAB Code

```
% AE6444 HW4 Problem 2 MATLAB
% Tomoki Koike
close all; clear all; clc;
%%
syms mu u(t) x_1 x_2
assume(mu, 'positive');

% System
x1dot = x_2;
x2dot = -2*mu*x_2 - x_1 - x_1^3;

% System matrix: A
a = diff(x1dot, x_1);
b = diff(x1dot, x_2);
c = diff(x2dot, x_1)
d = diff(x2dot, x_2)
A = [a, b; c, d]

% Equilibrium points
```

```
x1e = 0;
x2e = 0;

% A matrix for equilibrium point
A1 = subs(A, [x_1 x_2], [x1e x2e])

% Eigenvalues and eigenvectors
[V, D] = eig(A1)
%%
% System
x1dot = x_2;
x2dot = -2*mu*x_2 - x_1 + x_1^3;

% System matrix: A
a = diff(x1dot, x_1);
b = diff(x1dot, x_2);
c = diff(x2dot, x_1)
d = diff(x2dot, x_2)
A = [a, b; c, d]

% Equilibrium points
x1e_1 = -1;
x2e_1 = 0;
x1e_2 = 0;
x2e_2 = 0;
x1e_3 = 1;
x2e_3 = 0;

% A matrix for equilibrium point
A1 = subs(A, [x_1 x_2], [x1e_1 x2e_1])
A2 = subs(A, [x_1 x_2], [x1e_2 x2e_2])
A3 = subs(A, [x_1 x_2], [x1e_3 x2e_3])

% Eigenvalues and eigenvectors
[V1, D1] = eig(A1)
[V2, D2] = eig(A2)
[V3, D3] = eig(A3)
%%
% System
x1dot = x_2;
x2dot = -2*mu*x_2 + x_1 - x_1^3;

% System matrix: A
a = diff(x1dot, x_1);
b = diff(x1dot, x_2);
```

```
c = diff(x2dot, x_1)
d = diff(x2dot, x_2)
A = [a, b; c, d]

% Equilibrium points
x1e_1 = -1;
x2e_1 = 0;
x1e_2 = 0;
x2e_2 = 0;
x1e_3 = 1;
x2e_3 = 0;

% A matrix for equilibrium point
A1 = subs(A, [x_1 x_2], [x1e_1 x2e_1])
A2 = subs(A, [x_1 x_2], [x1e_2 x2e_2])
A3 = subs(A, [x_1 x_2], [x1e_3 x2e_3])

% Eigenvalues and eigenvectors
[V1, D1] = eig(A1)
[V2, D2] = eig(A2)
[V3, D3] = eig(A3)
%%
% System
x1dot = x_2;
x2dot = -2*mu*x_2 + x_1 + x_1^3;

% System matrix: A
a = diff(x1dot, x_1);
b = diff(x1dot, x_2);
c = diff(x2dot, x_1)
d = diff(x2dot, x_2)
A = [a, b; c, d]

% Equilibrium points
x1e = 0;
x2e = 0;

% A matrix for equilibrium point
A1 = subs(A, [x_1 x_2], [x1e x2e])

% Eigenvalues and eigenvectors
[V, D] = eig(A1)
```

3.3 Problem 2: Python Code

```python
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import solve_ivp, DOP853
from typing import List

# System ODE
def duff1(t, x, mu):
    return [x[1], -2*mu*x[1] - x[0] - x[0]**3]

def duff2(t, x, mu):
    return [x[1], -2*mu*x[1] - x[0] + x[0]**3]

def duff3(t, x, mu):
    return [x[1], -2*mu*x[1] + x[0] - x[0]**3]

def duff4(t, x, mu):
    return [x[1], -2*mu*x[1] + x[0] + x[0]**3]

def solve_diffeq(func, t, tspan, ic, parameters={}, algorithm='DOP853', stepsize=np.inf):
    return solve_ivp(fun=func, t_span=tspan, t_eval=t, y0=ic, method=algorithm,
                     args=tuple(parameters.values()), atol=1e-8, rtol=1e-5, max_step=stepsize)

def phasePlane(x1, x2, func, params):
    X1, X2 = np.meshgrid(x1, x2)  # create grid
    u, v = np.zeros(X1.shape), np.zeros(X2.shape)
    NI, NJ = X1.shape
    for i in range(NI):
        for j in range(NJ):
            x = X1[i, j]
            y = X2[i, j]
            dx = func(0, (x, y), *params.values())  # compute values on grid
            u[i, j] = dx[0]
            v[i, j] = dx[1]
    M = np.hypot(u, v)
    u /= M
    v /= M
    return X1, X2, u, v, M

def DEplot(sys: object, tspan: tuple, x0: List[List[float]],
```

```python
            x: np.ndarray, y: np.ndarray, params: dict):
    if len(tspan) != 3:
        raise Exception('tspan should be tuple of size 3: (min, max, number of
        ↪  points).')
    # Set up the figure the way we want it to look
    plt.figure(figsize=(12, 9))

    X1, X2, dx1, dx2, M = phasePlane(
        x, y, sys, params
    )

    # Quiver plot
    plt.quiver(X1, X2, dx1, dx2, M, scale=None, pivot='mid')
    plt.grid()

    t1 = np.linspace(0, tspan[0], tspan[2])
    t2 = np.linspace(0, tspan[1], tspan[2])
    if min(tspan) < 0:
        t_span1 = (np.max(t1), np.min(t1))
    else:
        t_span1 = (np.min(t1), np.max(t1))
    t_span2 = (np.min(t2), np.max(t2))
    for x0i in x0:
        sol1 = solve_diffeq(sys, t1, t_span1, x0i, params)
        plt.plot(sol1.y[0, :], sol1.y[1, :], '-r')
        sol2 = solve_diffeq(sys, t2, t_span2, x0i, params)
        plt.plot(sol2.y[0, :], sol2.y[1, :], '-r')

    plt.xlim([np.min(x), np.max(x)])
    plt.ylim([np.min(y), np.max(y)])
    plt.show()

# mu > 1
# A stable node

x0 = np.random.uniform(-2.5, 2.5, (10, 2))

p = {'mu': 2}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff1, (-6, 6, 1000), x0, x1, x2, p)

```

```
# mu = 1
# A stable singular node

x0 = np.random.uniform(-2.5, 2.5, (10, 2))

p = {'mu': 1}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff1, (-6, 6, 1000), x0, x1, x2, p)

# 0 < mu < 1
# A stable spiral/focus

x0 = np.random.uniform(-2.5, 2.5, (10, 2))

p = {'mu': 0.5}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff1, (-6, 6, 1000), x0, x1, x2, p)

# mu > 1
# (0, 0) -> a stable node
# (-1, 0), (1, 0) -> an unstable saddle point

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': 1.2}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff2, (-6, 6, 1000), x0, x1, x2, p)

# mu = 1
# (0, 0) -> a stable singular node
# (-1, 0), (1, 0) -> an unstable saddle point

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': 1}
```

```python

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff2, (-6, 6, 1000), x0, x1, x2, p)

# 0 < mu < 1
# (0, 0) -> a stable spiral/focus
# (-1, 0), (1, 0) -> an unstable saddle point

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': 0.35}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff2, (-6, 6, 1000), x0, x1, x2, p)

# mu > sqrt(2)
# (0, 0) -> an unstable saddle point
# (-1, 0), (1, 0) -> a stable node

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': 2}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff3, (-6, 6, 1000), x0, x1, x2, p)

# mu = sqrt(2)
# (0, 0) -> an unstable saddle point
# (-1, 0), (1, 0) -> a stable singular node

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': np.sqrt(2)}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff3, (-6, 6, 1000), x0, x1, x2, p)
```

```python

# 0 < mu < sqrt(2)
# (0, 0) -> an unstable saddle point
# (-1, 0), (1, 0) -> a stable sprial/focus

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': 0.8}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff3, (-6, 6, 1000), x0, x1, x2, p)

# An unstable saddle point

x0 = np.random.uniform(-2.5, 2.5, (30, 2))

p = {'mu': 0.8}

x1 = np.linspace(-3, 3, 20)
x2 = np.linspace(-3, 3, 20)

DEplot(duff4, (-6, 6, 1000), x0, x1, x2, p)
```