AE 6230 – HW2: Response of SDOF Systems and Equations of MDOF Systems

Out: September 29, 2022; **Due:** October 9, 2022 by 11:59 PM ET in Canvas

Guidelines

- Read each question carefully before doing any work;
- If you find yourself doing pages of math, pause and consider if there is an easier approach;
- You can consult any relevant materials;
- You can discuss solution approaches with others, but your submission must be your own work;
- If you have doubts, please ask questions in class, during office hours, and/or Piazza (no questions via email);
- The solution to each question should concisely and clearly show the steps;
- Simplify your results as much as possible;
- Box the final answer for each question;
- Submit any code with the solution (but remember to also submit all relevant plots).

Problem 1 – 40 points

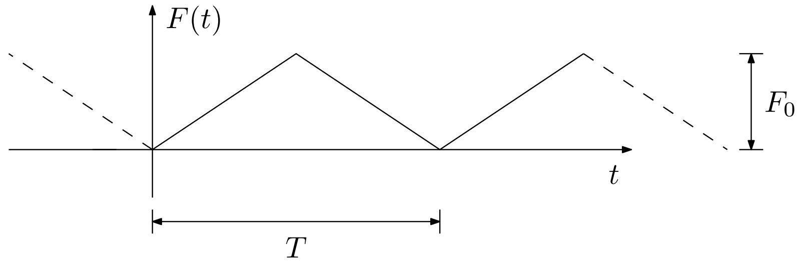


Figure 1: Periodic excitation applied to a single-degree-of-freedom system.

Consider a single-degree-of-freedom system subject to the periodic excitation in Fig. 1, with parameters given in Table 1. Answer the following questions:

1. Determine the expressions of the coefficients of the Fourier series representation of $F(t)$

$$F(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos p\omega_0 t + \sum_{p=1}^{\infty} b_p \sin p\omega_0 t \tag{1}$$

2. Plot the discrete frequency spectrum associated with Eq. (1)

$$c_p = \sqrt{a_p^2 + b_p^2} \quad \text{vs.} \quad p \tag{2}$$

   for $p = 0, \dots, 12$;

3. Determine how many terms must be kept in Eq. (1) such that the highest-order harmonic has an amplitude below $0.05F_0$, $0.025F_0$, and $0.005F_0$;

4. Plot the truncated Fourier series representations of $F(t)$ identified via the convergence study in Question 3 against the true function in Fig. 1 for $t \in [0, T]$;

5. Determine the expression of the steady-state response of the system $x(t)$ subject to $F(t)$;

6. Plot the discrete frequency spectrum for $x(t)$ (that is, the amplitude of each harmonic) for $p = 0, \dots, 12$;

7. Plot $x(t)$ for each truncated Fourier series representation of $F(t)$ identified in Question 3 for $t \in [0, T]$;

8. Motivate the trends observed in the plots for Questions 2, 4, 6, 7 for increasing p.

Guidelines:

- **Questions 1 and 5:** do not substitute the values of the parameters for these questions;
- **Question 3:** you can solve this analytically or numerically (or both).

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Excitation peak value	F_0	1 N
Excitation period	T	0.2 s
Natural frequency	ω_n	$5\omega_0$
Viscous damping factor	ζ	0.05
Stiffness constant	k	10 N/m

Problem 2 – 35 points

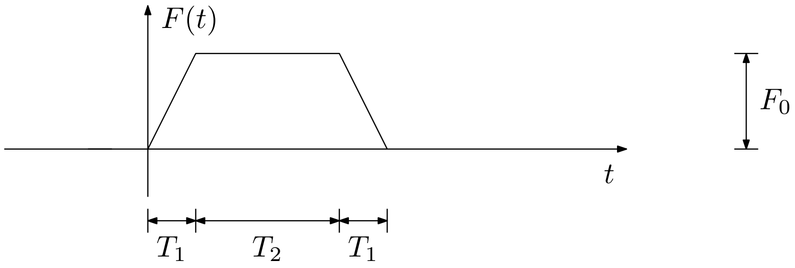


Figure 2: Trapezoidal input applied to a single-degree-of-freedom system.

Consider a single-degree-of-freedom system subject to the input in Fig. 2, with parameters in Table 2. Assuming the system at rest for $t \leq 0$ and neglecting damping, answer the following questions:

1. Using the convolution integral, show that the response for $0 \leq t \leq T_1$ is given by

$$x(t) = \frac{x_s}{T_1 \omega_n} (\omega_n t - \sin \omega_n t) \tag{3}$$

   where $\omega_n = 2\pi/T_n$ is the natural frequency of the system and $x_s = F_0/k$ is the response for a static input having the same amplitude as the trapezoidal input in Fig. 2;

2. Considering the other time intervals $T_1 \leq t \leq T_1 + T_2$, $T_1 + T_2 \leq t \leq 2T_1 + T_2$, and $t \geq 2T_1 + T_2$
   (a) Explain the approach you pursue to determine $x(t)$;
   (b) Derive the expression of $x(t)$ specialized to each time interval;

3. Plot $x(t)/x_s$ for $T_1 = 0.1T_n, 0.5T_n, T_n, 1.5T_n, 2T_n, 2.5T_n$ for $t \in [0, 1.5]$ s and $x(t)/x_s \in [-2, 2]$;

4. Determine the maximum value of $x(t)/x_s$ in the time interval $T_1 \leq t \leq T_1 + T_2$ as a function of T_1/T_n;

5. Plot the result from Question 4 for $T_1/T_n \in [0, 4]$;

6. Discuss the trends in the results for Question 3 and 5.

Guidelines:

- **Question 4:** you can use the plots from Question 3 to check the results for this question.

Table 2: Parameter values for Problem 2.

Parameter	Symbol	Value
Time length of constant input	T_2	0.5 s
Natural frequency of the system	ω_n	20π rad/s

Problem 3 – 25 points

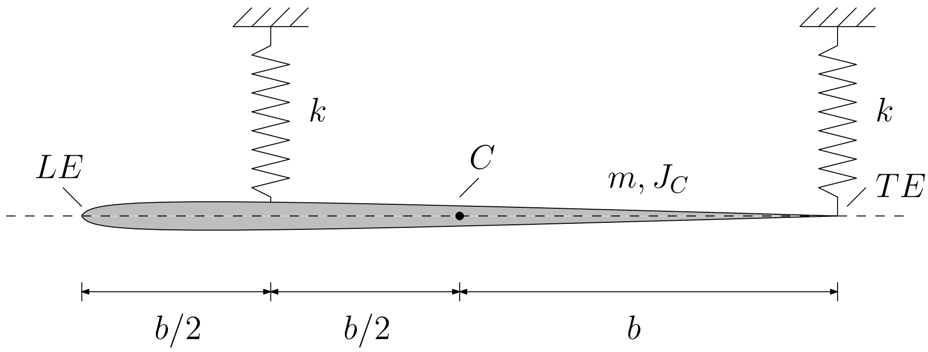


Figure 3: Schematic of wind-tunnel wing model undergoing plunge and pitch vibrations.

Consider a rigid wing mounted in a wind-tunnel test section (Fig. 3). The wing undergoes plunge (vertical translation) and pitch vibrations, which are restrained by two springs attached to the quarter-chord and trailing-edge points as shown in Fig. 3. The wing has mass m and pitch moment of inertia J_C about the center of mass C, located at the half-chord point. The chord has length $2b$ and the two springs both have spring constant k. The motion is described by choosing the vertical translations of the leading-edge and trailing-edge points, denoted by $h_{LE}(t)$ and $h_{TE}(t)$, as the generalized coordinates. The traslations are assumed to be positive in the upward direction and are measured from the horizontal configuration in shown Fig. 3. Neglecting gravity and assuming small-amplitude motions, answer the following questions:

1. Write the kinetic and potential energies of the system as functions of $h_{LE}(t)$ and $h_{TE}(t)$;
2. Derive the equations of motion in the matrix form
$$\mathbf{M}\begin{Bmatrix}\ddot{h}_{LE}(t)\\ \ddot{h}_{TE}(t)\end{Bmatrix}+\mathbf{K}\begin{Bmatrix}h_{LE}(t)\\ h_{TE}(t)\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix} \tag{4}$$
using Lagrange's equations;
3. Define a coordinate transformation that results in inertial decoupling (but not necessarily elastic decoupling) and derive the corresponding transformation matrix $\mathbf{T}$;
4. Obtain the new mass and stiffness matrices based on the transformation matrix from Question 3;
5. Derive the equations of motion using the Newtonian approach based on the free-body diagram for the system (to be included in the solution) and compare the results with Question 4.

Guidelines:

- **Question 2:** show the steps in the process, not only the final form of the matrices;
- **Question 4:** you can verify the results by obtaining the new matrices directly from Lagrange's equations.

Problem 1 Solution – 40 points

Question 1 – 5 points

Because $F(t)$ in Fig. 1 is an even function of time, Eq. (1) reduces to the Fourier cosine series

$$F(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} a_p \cos p\omega_0 t \tag{5}$$

The constant term is

$$a_0 = F_0 \tag{6}$$

This can be derived from Fig. 1 because

$$a_0 = \frac{2}{T} \int_0^T F(t) dt \tag{7}$$

is twice the average of $F(t)$ over a period, which is $F_0/2$. Additionally, because $F(t)$ and $\cos p\omega_0$ are symmetric about the half period, the a_p coefficients can be computed considering half the period and multiplying the result by two. After these considerations, the task of computing the Fourier coefficients reduces to evaluating the integrals

$$a_p = \frac{4}{T} \int_0^{T/2} F(t) \cos p\omega_0 t dt \qquad (p = 1, \ldots, \infty) \tag{8}$$

with

$$F(t) = \frac{2F_0}{T} t \qquad 0 \le t \le T/2 \tag{9}$$

Evaluating Eq. (8) by parts (or using a table of integrals or a symbolic manipulator) gives

$$\begin{aligned}
a_n &= \frac{4}{T} \int_0^{T/2} F(t) \cos p\omega_0 t dt \\
&= \frac{8F_0}{T^2} \int_0^{T/2} t \cos p\omega_0 t dt \\
&= \frac{8F_0}{T^2} \left[\frac{t \sin p\omega_0 t}{p\omega_0} \bigg|_0^{T/2} - \frac{1}{p\omega_0} \int_0^{T/2} \sin p\omega_0 t dt \right] \\
&= \frac{8F_0}{T^2 (p\omega_0)^2} \cos p\omega_0 t \bigg|_0^{T/2} \\
&= \frac{2F_0}{(p\pi)^2} (\cos p\pi - 1) \\
&= \begin{cases} -\dfrac{4F_0}{(p\pi)^2} & p = 1, 3, \ldots, \infty \\ 0 & p = 2, 4, \ldots, \infty \end{cases}
\end{aligned} \tag{10}$$

Using Eqs. (6) and (10), we obtain

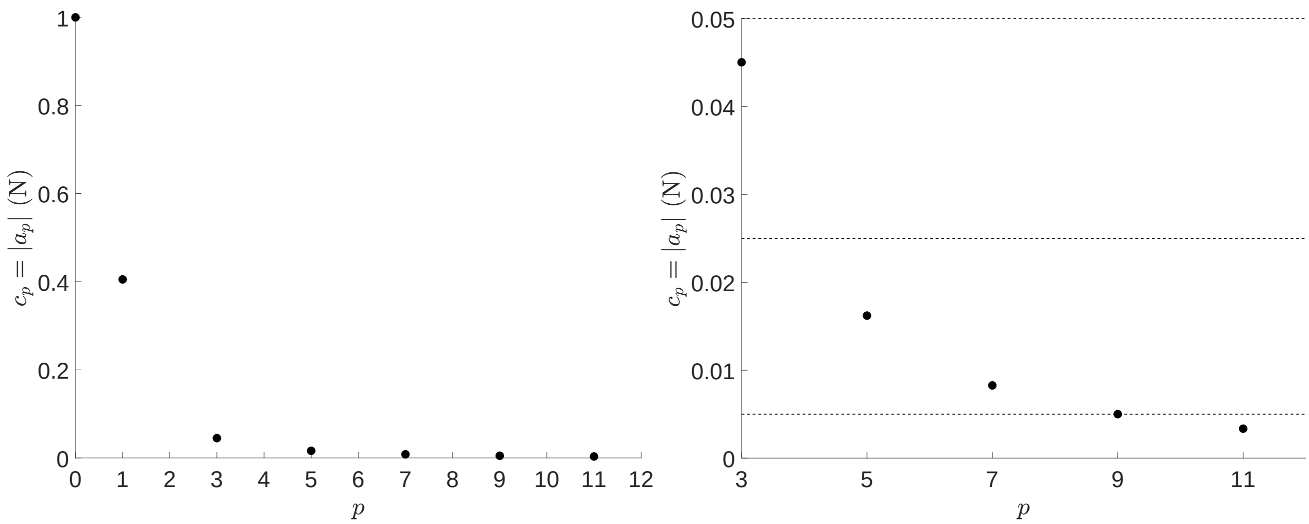
$$F(t) = \frac{F_0}{2} - \frac{4F_0}{\pi^2} \sum_{p=1,3,\ldots}^{\infty} \frac{\cos p\omega_0 t}{p^2} \tag{11}$$

The harmonics associated with even multiples of the fundamental frequency do not contribute to the Fourier cosine series and the amplitude of the other harmonics is inversely proportional to p^2.

Question 2 – 5 points

Figure 4 shows the discrete frequency spectrum

$$c_p = |a_p| \tag{12}$$



(a) $p = 0, \ldots, 12$ (b) Zoom

Figure 4: Discrete frequency spectrum of $F(t)$ for Problem 1 Question 2.

for the parameters in Table 1 and $p = 0, \ldots, 12$. Figure 4(a) includes the requested values of p while Fig. 4(b) shows a zoom for $p = 3, \ldots, 12$. The plotted value for $p = 0$ is a_0 for consistency with the definition of c_p, though the amplitude of the zeroth-order term in the Fourier cosine series is $a_0/2$ (which could have been plotted instead). The markers for $p = 2, 4, \ldots, 12$ are not plotted as they correspond to harmonics with zero amplitude, which do not contribute to Eq. (5). In this case, the discrete frequency spectrum is given by $c_p = |a_p|$ because the Fourier series in Eq. (1) reduces to Eq. (5). In general, one must consider a_p and b_p both to define c_p.

Question 3 – 5 points

To answer this question, it is convenient to set $p = 2r - 1$ ($r = 1, \ldots, \infty$) to count only harmonics associated with odd multiples of the fundamental frequency, as the others do not contribute to Eq. (5). The minimum number N of *non-zero* terms (excluding the zeroth-order term) to be retained such that the highest-order harmonic has an amplitude below $0.05F_0$, $0.025F_0$, and $0.005F_0$ can be found by solving

$$|a_{2N-1}| = \varepsilon F_0 \tag{13}$$

where $\varepsilon = 0.05, 0.025, 0.005$. Substituting Eq. (10) gives

$$\frac{4F_0}{\pi^2(2N-1)^2} = \varepsilon F_0 \tag{14}$$

and

$$N \geq \frac{1}{2} + \frac{1}{\pi\sqrt{\varepsilon}} \tag{15}$$

After having rounded to the next integer, Eq. (15) gives $N = 2, 3, 6$. This corresponds to keeping $N + 1 = 3, 4$, and 7 *non-zero* terms in the Fourier cosine series, also counting the zeroth-order term, and to truncating the Fourier series to $p = 3, 5, 11$. Table 3 shows the amplitudes of the harmonics for $p = 1, 3, \ldots, 11$ normalized by $F_0 = 1$ N. The results confirm that keeping 3, 4, and 7 *non-zero* terms in the Fourier cosine series (including the zeroth-order term) satisfies the convergence criterion. The table also confirms that the amplitude of the coefficients decreases like $1/p^2$ for increasing p, for instance $|a_3|/|a_1| = 1/9$.

Question 4 – 5 points

Figure 5 shows the truncated Fourier series representations of the excitation determined in Question 3. The approximations to $F(t)$ are shown in separate plots to highlight their comparison with the true function.

Table 3: Values of $c_p/F_0 = |a_p|/F_0$ for $p = 1, 3, \ldots, 11$.

$N+1$	p	$\lvert a_p\rvert/F_0$
2	1	0.4053
3	3	0.0450
4	5	0.0162
5	7	0.0083
6	9	0.0050
7	11	0.0033

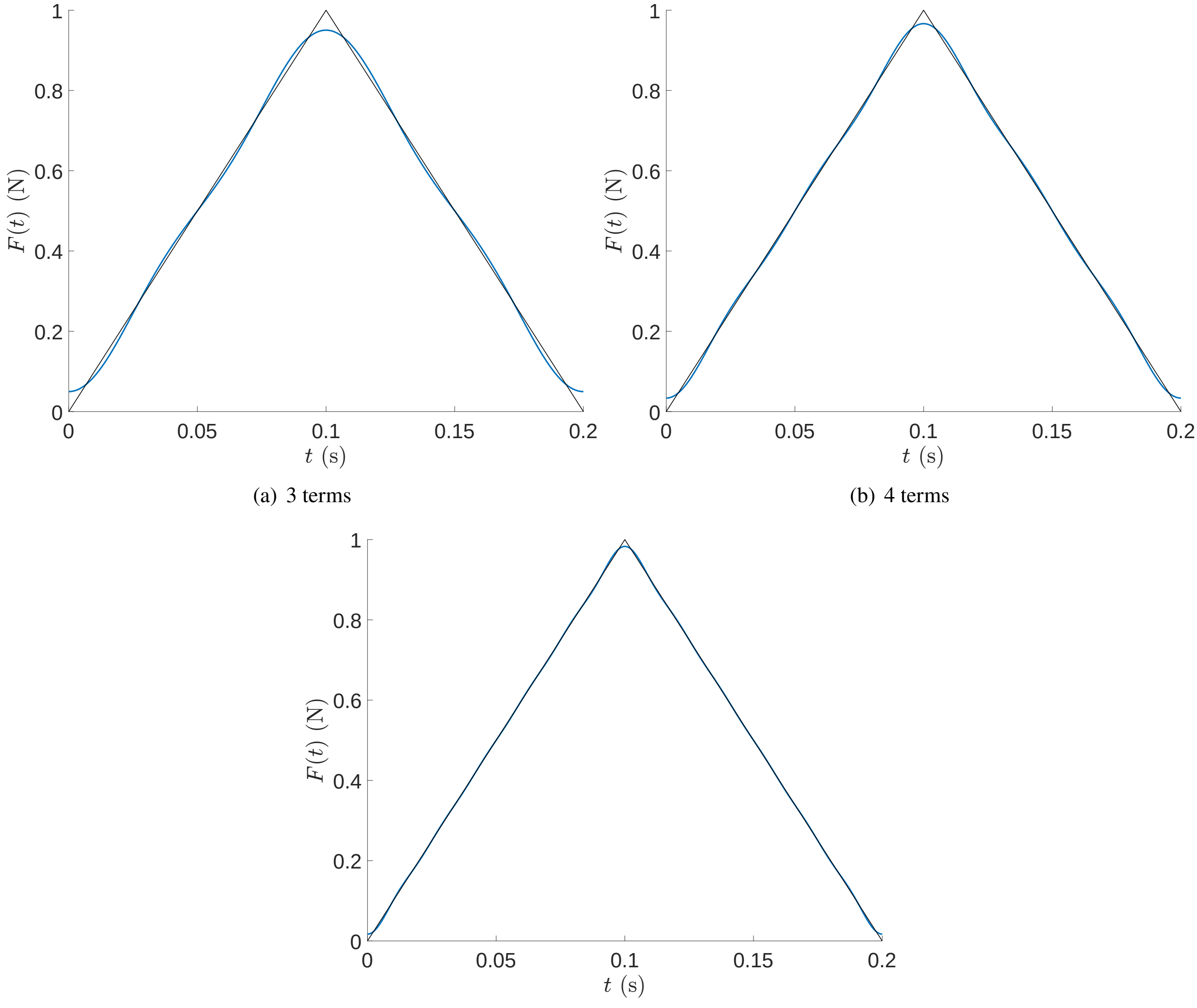
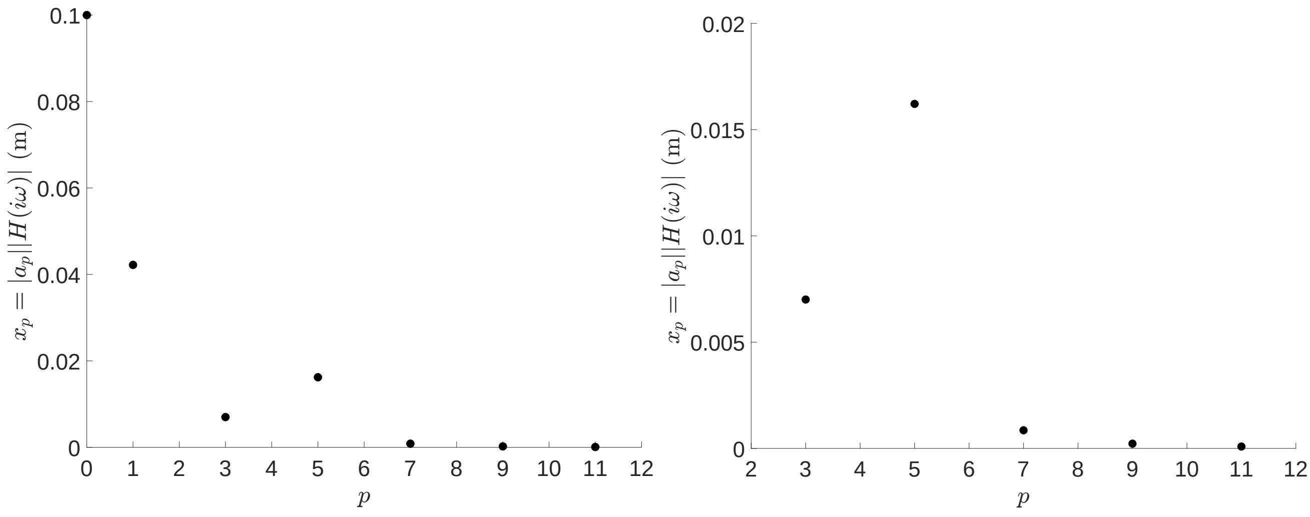


Figure 5: Truncated Fourier series representations of the excitation for Problem 1 Question 4.



(a) $p = 0, \ldots, 12$

(b) Zoom

Figure 6: Discrete frequency spectrum of $x(t)$ for Problem 1 Question 6.

Question 5 – 5 points

The steady state-response is given by

$$x(t) = \frac{F_0}{2k} - \frac{4F_0}{\pi^2} \sum_{p=1,3,\ldots}^{\infty} \frac{1}{p^2} |H(i\omega_p)| \cos\left[\omega_p t - \theta(\omega_p)\right] \tag{16}$$

where $\omega_p = p\omega_0$, $\omega_n = 5\omega_0$, and

$$|H(i\omega_p)| = \frac{1}{k} \frac{1}{\sqrt{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}} \Bigg|_{\omega=\omega_p} \tag{17}$$

$$\theta(\omega) = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2}\right) \Bigg|_{\omega=\omega_p} \tag{18}$$

For numerical evaluation, Eq. (16) is limited to 3, 4, and 7 *non-zero* terms (counting the zeroth-order term) based on the results from Question 3. Note that $\theta(\omega)$ in Eq. (18) must be computed considering the four-quadrant inverse tangent function (or by manually checking the quadrant and adjusting the phase). The standard inverse tangent function would only give the correct angle for $\omega_p \leq \omega_n$.

Question 6 – 5 points

Figure 6 shows the discrete frequency spectrum of the steady-state response for the parameters in Table 1 and $p = 0, \ldots, 12$. The discrete frequency spectrum is computed as

$$x_p = |a_p||H(i\omega_p)| \tag{19}$$

for consistency with Eq. (12). However, the amplitude of zeroth-order term is $a_0/(2k)$. The markers for $p = 2, 4, \ldots$ are omitted because the corresponding harmonics do not contribute to Eq. (16).

Question 7 – 5 points

Figure 6 shows the steady-state responses associated with the convergence study in Question 3.

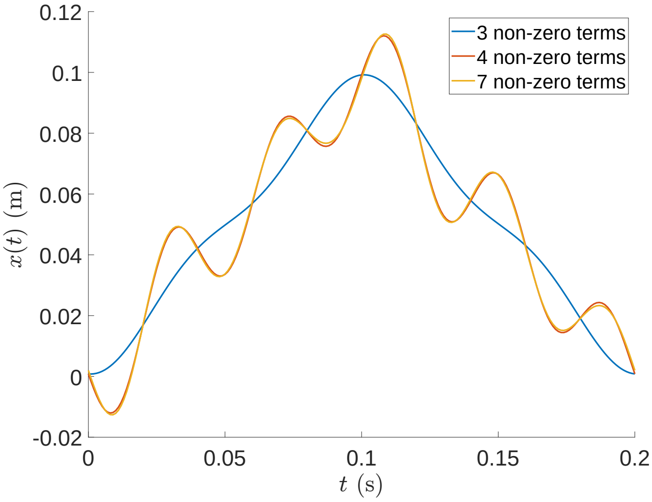


Figure 7: Steady-state responses for Problem 1 Question 7.

Question 8 – 5 points

Discussion:

- **Discrete frequency spectrum of the excitation:** Figure 4 shows the expected trend with p based on Eq. (10). The amplitude of higher-order harmonics decays like $1/p^2$, meaning that there is a decreasing return in accuracy when one adds terms associated with higher values of p to the Fourier cosine series.

- **Representation of the excitation:** the trend in Figure 4 is reflected in Fig. 5. As the number of retained terms increases, the Fourier cosine series globally approximates the true excitation more accurately. However, the convergence is slow at the sharp edges of the excitation. This is an issue of the Fourier series, like the occurrence of localized high-frequency oscillations around discontinuities. To more accurately capture $F(t)$ at those non-smooth points, we should use a convergence criterion based on a local error metric instead of the global criterion used in this problem.

- **Discrete frequency spectrum of the steady-state response:** Figure 6 shows a non-monotonic trend, differently from Fig. 4. This is because $|x_p| = |a_p||H(i\omega_p)|$ with $\omega_p = p\omega_0$. The quantity $|a_p|$ decays like $1/p^2$, but the frequency response function has a peak close to $\omega = \omega_n = 5\omega_0 = \omega_5$. Because the harmonic for $p = 5$ is associated with a non-zero $|a_p|$, the discrete frequency spectrum of the steady-state response has a peak at $p = 5$. The amplitude of harmonics for values of $p \geq 5$ decreases monotonically for increasing p as it is the product of two terms that both decrease with p.

- **Discrete frequency spectrum of the steady-state response:** the steady-state response is practically converged once we consider four *non-zero* terms ($p = 0, 1, 3, 5$) in the Fourier series. This is expected based on Fig. 6. The amplitude of harmonics associated with $p > 5$ (more than four terms) is much smaller than those of the lower-frequency harmonics and decreases monotonically for increasing p. Retaining the term $p = 5$ is key to accurately capturing the response because $\omega_5 = \omega_n$. While this problem only required a convergence study based on the excitation, one should also take into account other aspects such as the natural frequency of the system and its frequency response function when assessing how many terms to retain in the periodic response.

Problem 2 Solution – 35 points

Question 1 – 5 points

In the interval $0 \le t \le T_1$, the excitation in Fig. 3 is a ramp with magnitude F_0/T_1:

$$F(t) = \frac{F_0}{T_1}t \qquad 0 \le t \le T_1 \tag{20}$$

Thus, we obtain the response to a unit ramp an multiply it by F_0/T_1. Using the convolution integral, the response to the unit ramp with zero initial conditions is written as

$$x(t) = \int_0^t (t-\tau)\, h(\tau) d\tau \tag{21}$$

where

$$h(\tau) = \frac{\sin \omega_n \tau}{m\omega_n} \tag{22}$$

is the unit impulse response of an undamped (second-order) single-degree-of-freedom system, valid for $t \ge 0$. The convolution integral is written in the form

$$x(t) = \int_0^t F(t-\tau) h(\tau) d\tau \tag{23}$$

because it makes the integration easier. The integral of the unit impulse response is the unit step response

$$S(\tau) = \int_0^t h(\tau) d\tau = \frac{1-\cos\omega_n t}{m\omega_n^2} = \frac{1-\cos\omega_n t}{k} \tag{24}$$

valid for $t \ge 0$. Integrating Eq. (20) by parts gives

$$\begin{aligned}
x(t) &= \int_0^t (t-\tau)\, h(\tau) d\tau && (25)\\
&= tS(\tau)\Big|_0^t - \int_0^t \tau h(\tau) d\tau && (26)\\
&= tS(\tau)\Big|_0^t - \tau S(\tau)\Big|_0^t + \int_0^t S(\tau) d\tau && (27)\\
&= \int_0^t S(\tau) d\tau && (28)\\
&= \frac{1}{k}\int_0^t (1-\cos\omega_n t)\, d\tau && (29)\\
&= \frac{t\omega_n - \sin\omega_n t}{k\omega_n} && (30)
\end{aligned}$$

valid for $t \ge 0$. Multiplying by F_0/T_1 gives

$$x(t) = \frac{F_0}{T_1}\frac{t\omega_n - \sin\omega_n t}{k\omega_n} = \frac{x_s}{T_1\omega_n}\left(t\omega_n - \sin\omega_n t\right) \tag{31}$$

Question 2 – 5 points (approach) + 5 points (expression of the response)

The excitation in Fig. 2 can be interpreted as the superposition of four ramps with different slopes and delays, as shown in Fig. 8. Thus, we can obtain the response by superimposing responses of the form in Eq. (31) with appropriate multiplying factors and delays (because the system is linear and time invariant)[1]. Let us define

$$x_1(t) = \frac{x_s}{T_1\omega_n}\left(t\omega_n - \sin\omega_n t\right) \tag{32}$$

[1]This question could have been solved in other ways.

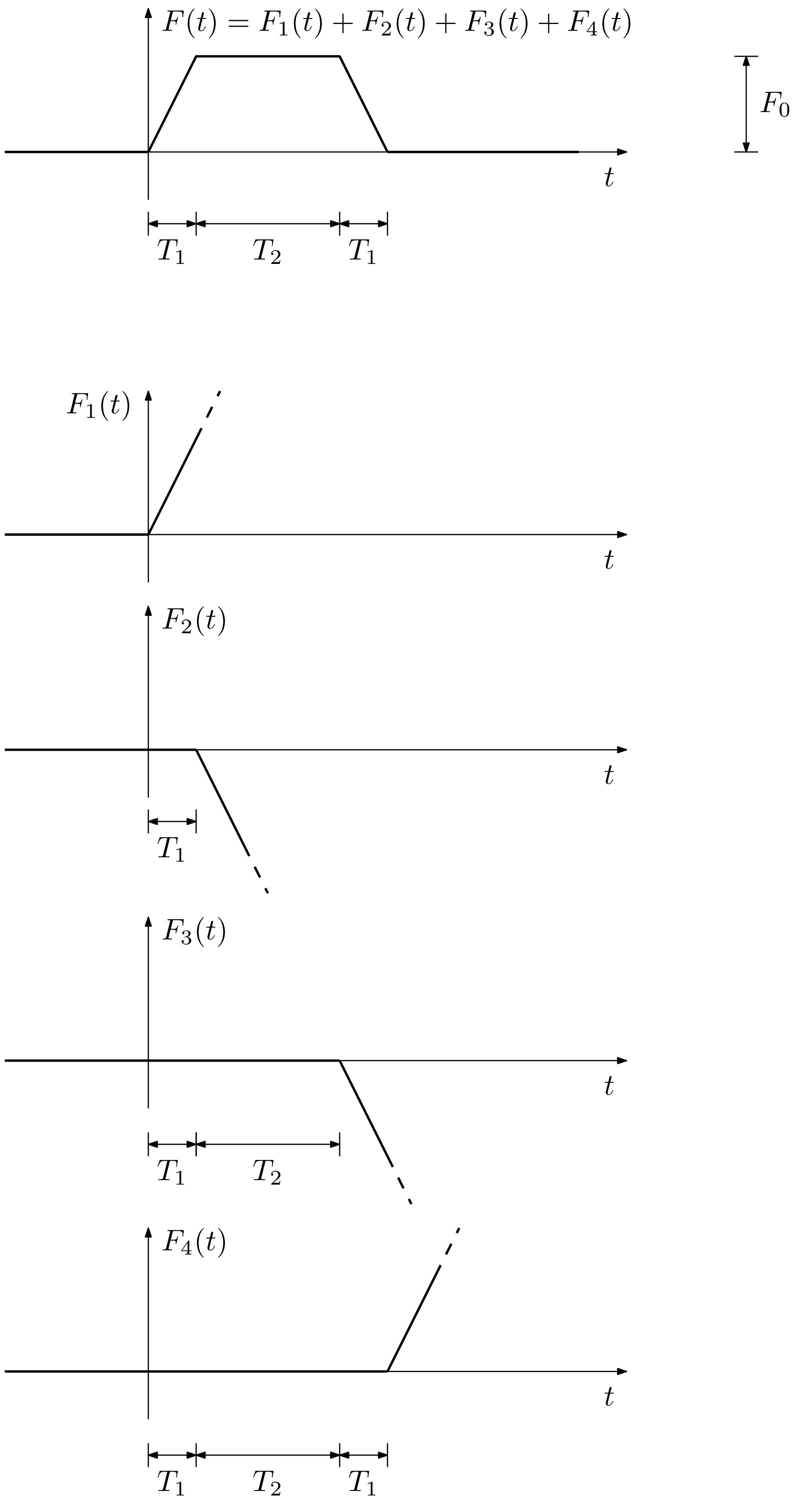


Figure 8: Decomposition of the excitation for Problem 2.

valid for $t \geq 0$. This is the response to $F_1(t)$ in Fig. 8, which coincides with the response to $F(t)$ for $0 \leq t \leq T_1$. The response to $F(t)$ can be written as

$$x(t) = x_1(t)\mathrm{u}(t) + x_1(t-T_1)\mathrm{u}(t-T_1) + x_1(t-T_1-T_2)\mathrm{u}(t-T_1-T_2) + x_1(t-2T_1-T_2)\mathrm{u}(t-2T_1-T_2) \quad (33)$$

where the unit step function $\mathrm{u}(t)$ indicates that the function is zero for $t < 0$. The expressions of the response in the different time intervals can be obtained by specializing the delayed unit step inputs:

$$x(t) = \begin{cases} \dfrac{x_s}{T_1\omega_n}(t\omega_n - \sin\omega_n t) & 0 \leq t \leq T_1 \\ x_s + \dfrac{x_s}{T_1\omega_n}[\sin\omega_n(t-T_1) - \sin\omega_n t] & T_1 \leq t \leq T_1 + T_2 \\ x_s\left(2 + \dfrac{T_2}{T_1}\right) + \dfrac{x_s}{T_1\omega_n}[-\omega_n t + \sin\omega_n(t-T_1-T_2) + \sin\omega_n(t-T_1) - \sin\omega_n t] & T_1 + T_2 \leq t \leq 2T_1 + T_2 \\ \dfrac{x_s}{T_1\omega_n}[\sin\omega_n(t-T_1-T_2) + \sin\omega_n(t-T_1) - \sin\omega_n t - \sin\omega_n(t-2T_1-T_2)] & t \geq 2T_1 + T2 \end{cases} \quad (34)$$

Question 3 – 5 points

Figure 9 shows the response for ramp lengths equal to $0.1T_n$, $0.5T_n$, T_n, $1.5T_n$, $2T_n$, $2.5T_n$. The dashed lines from the bottom to the top indicate $x = 0$ (initial equilibrium), the static response $x = x_s$, and the maximum oscillation amplitude during the pulse obtained in the following question, respectively. Note that the system does not oscillate if ramp lasts an integer multiple of the period of the undamped motion T_n, while the system oscillates about the static response x_s in the other cases.

Question 4 – 5 points

Using Eq. (34), the normalized response for $T_1 \leq t \leq T_2$ is recast as[2]

$$\begin{aligned} \frac{x(t)}{x_s} &= 1 + \frac{1}{T_1\omega_n}[\sin\omega_n(t-T_1) - \sin\omega_n t] \\ &= 1 - \frac{2}{T_1\omega_n}\sin\left(\frac{\omega_n T_1}{2}\right)\cos\omega_n\left(t - \frac{T_1}{2}\right) \\ &= 1 - \frac{T_n}{\pi T_1}\sin\left(\frac{\pi T_1}{T_n}\right)\cos\omega_n\left(t - \frac{T_1}{2}\right) \end{aligned} \quad (35)$$

When $\sin(\pi T_1/T_n) \geq 0$, the maximum value of $x(t)/x_s$ is

$$\frac{x_{max}}{x_s} = 1 + \frac{T_n}{\pi T_1}\sin\left(\frac{\pi T_1}{T_n}\right) \quad (36)$$

for $\cos\omega_n(t - T_1/2) = -1$. When $\sin(\pi T_1/T_n) < 0$, the maximum possible value of $x(t)/x_s$ is

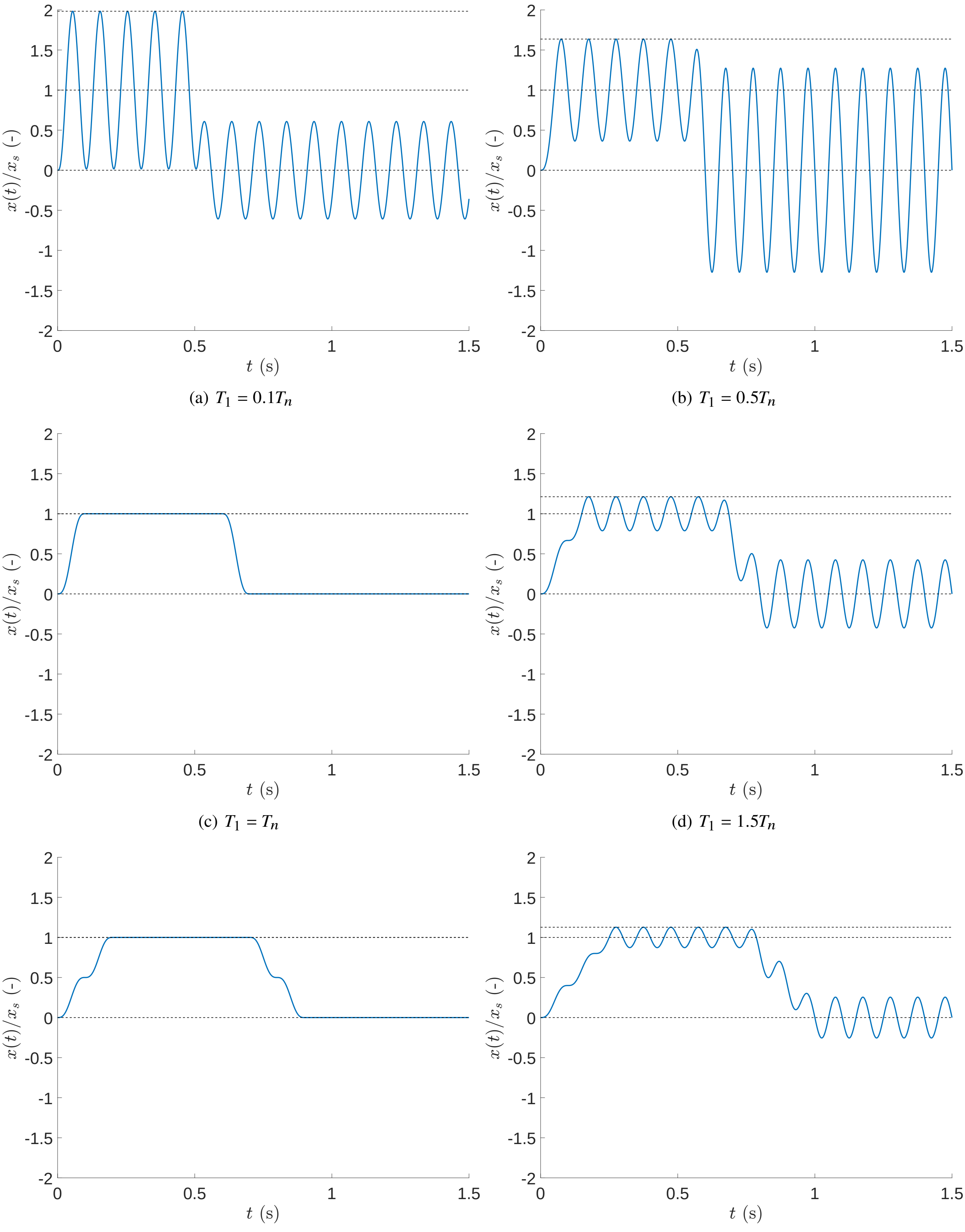
$$\frac{x_{max}}{x_s} = 1 - \frac{T_n}{\pi T_1}\sin\left(\frac{\pi T_1}{T_n}\right) \quad (37)$$

for $\cos\omega_n(t - T_1/2) = 1$. Combining the two cases:

$$\frac{x_{max}}{x_s} = 1 + \frac{T_n}{\pi T_1}\left|\sin\left(\frac{\pi T_1}{T_n}\right)\right| \quad (38)$$

These relations hold assuming T_2 is long enough for the response to peak, which is verified here ($T_2 = 5T_n$).

[2] This question could have been solved in other ways.



(a) $T_1 = 0.1T_n$

(b) $T_1 = 0.5T_n$

(c) $T_1 = T_n$

(d) $T_1 = 1.5T_n$

(e) $T_1 = 2T_n$

(f) $T_1 = 2.5T_n$

Figure 9: Responses for Problem 2 Question 3.

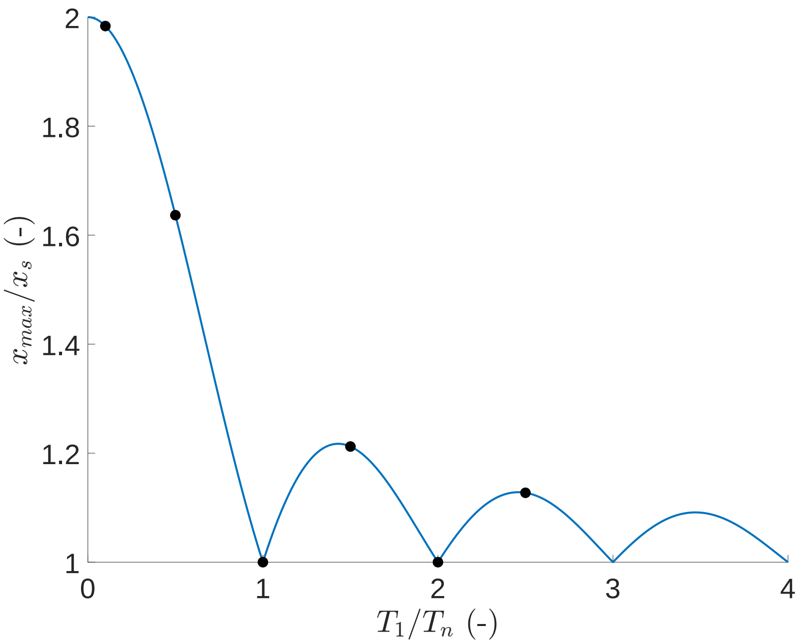


Figure 10: Maximum amplitude during the pulse for Problem 2 Question 4.

Question 5 – 5 points

Figure 10 shows the maximum oscillation amplitude during the pulse given by Eq. (38). The markers indicate the cases in Fig. 9. A plot like the one in Fig. 10 is called *response spectrum* and shows the maximum value of the response as a function of the natural frequency ω_n normalized by a characteristic frequency of the excitation. Even if the excitation is not oscillatory in this problem, we can define a characteristic frequency as $2\pi/T_1$. With more mathematical manipulations, a similar plot could be obtained for the time interval $t \geq 2T_1 + T_2$ or for the response as a whole. Response spectrum plots are used to study the system response to excitations that have a "shock" nature, that is, they excite the system suddenly and for a short time. They are often used in the design process of vibrating structures.

Question 6 – 5 points

The results in Fig. 10 explain the trend in Fig. 9. When the ramp length approaches zero, the maximum oscillation amplitude during the pulse tends to twice the static response, as expected for a step input. As the ramp length increases from 0 to T_1, the maximum oscillation amplitude decreases until the system does not oscillate when $T_1/T_n = 1$. As the ramp length increases further, there is an amplification of oscillations followed by a decrease until the system does not oscillate when $T_1/T_n = 2$. This pattern is repeated each time the ramp length equals a multiple of the period of the undamped motion. However, the amplification reduces as the ramp length increases because the excitation is "slow" compared with the natural frequency of the system.

One way to explain the behavior in Fig. 10 is to consider the response $x(t)$ and velocity $\dot{x}(t)$ at $t = T_1$:

$$x(T_1) = \frac{x_s}{T_1\omega_n}(T_1\omega_n - \sin\omega_n T_1) \qquad \dot{x}(T_1) = \frac{x_s}{T_1}(1 - \cos\omega_n T_1) \tag{39}$$

Setting $T_1 = T_n$ gives

$$x(T_n) = \frac{x_s}{2\pi}(2\pi - \sin 2\pi) = x_s \qquad \dot{x}(T_n) = \frac{x_s}{2\pi/\omega_n}(1 - \cos 2\pi) = 0 \tag{40}$$

The same result is obtained for $T_1 = 2T_n, 3T_n, \ldots$. This means that when the ramp length is equal to the period of the undamped motion or an integer multiple, the system achieves the static response x_s with zero velocity at the end of the ramp. This results in no oscillations once the excitation becomes constant. Other values of T_1 cause the system to reach a different amplitude with non-zero velocity at the end of the ramp, resulting in the oscillations about $x = x_s$. A similar behavior can be observed in the last time interval, $t \geq 2T_1 + T_2$.

A second way to explain the behavior in Fig. 10 is that the maximum of the oscillation must occur at times where the velocity $\dot{x}(t)$ vanishes. At the same time, we expect the response during the pulse to oscillate about an average value $x = x_s$. Because the condition of zero velocity is verified at the end of the ramp when $T_1 = T_n, 2T_n, \ldots$ with the response being equal to x_s, the system keeps that value during the entire pulse.

Finally, another way to explain the behavior in Fig. 10 is based on the normalized response in Eq. (35). When $T_1 = T_n$, we have

$$\sin \omega_n (t - T_1) = \sin \omega_n (t - T_n) = \sin(\omega_n t - 2\pi) = \sin \omega_n t \tag{41}$$

In this situation, the two sine functions in Eq. (35) cancel each other, such that the response is constant and equal to x_s. The same considerations apply if T_1 is a multiple of T_n.

Problem 3 Solution – 25 points

Question 1 – 5 points

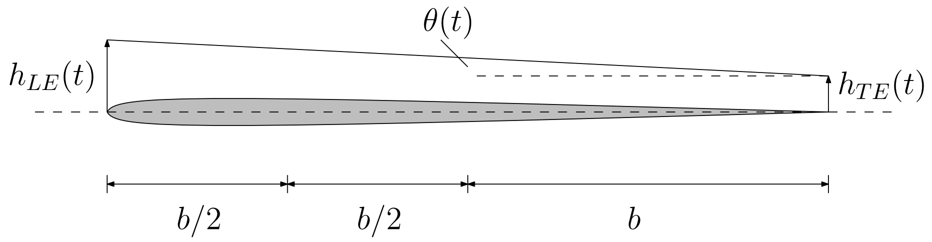


Figure 11: Kinematics of the wind-tunnel wing model for Problem 3.

To determine the kinetic energy of the system, we must write the vertical displacement of the center of mass $h_C(t)$, located at the half-chord point, and the pitch angle $\theta(t)$ as functions of the chosen degrees of freedom. Assuming small-amplitude displacements (and rotations):

$$h_C(t) = \frac{h_{LE}(t) + h_{TE}(t)}{2} \qquad \theta(t) = \frac{h_{LE}(t) - h_{TE}(t)}{2b} \tag{42}$$

These relations are obtained considering the kinematics in Fig. 11 where the pitch angle is assumed to be clockwise positive, that is, $\theta(t) > 0$ for $h_{LE}(t) > h_{TE}(t)$. For brevity, the dependencies on time are omitted from now on. Using Eq. (42), the kinetic energy is

$$T = \frac{1}{2} m \dot{h}_C^2 + \frac{1}{2} J_C \dot{\theta}^2 = \frac{m}{8} \left(\dot{h}_{LE} + \dot{h}_{TE}\right)^2 + \frac{J_C}{8b^2} \left(\dot{h}_{LE} - \dot{h}_{TE}\right)^2 \tag{43}$$

To determine the potential elastic energy, we must write the vertical displacement of the quarter-chord point as a function of the chosen degrees of freedom. This is necessary because one of the two springs connects to the wing at that point (Fig. 3). Again assuming small-amplitude motions:

$$h_{QC} = h_{LE} - \frac{b}{2}\theta = h_{LE} - \frac{h_{LE} - h_{TE}}{4} = \frac{3h_{LE} + h_{TE}}{4} \tag{44}$$

Using this result, the elastic energy is

$$U = \frac{1}{2} k h_{TE}^2 + \frac{1}{2} k h_{QC}^2 = \frac{k}{2} h_{TE}^2 + \frac{k}{32} \left(3h_{LE} + h_{TE}\right)^2 \tag{45}$$

Question 2 – 5 points

The first Lagrange's equation is given by

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{h}_{LE}} + \frac{\partial U}{\partial h_{LE}} &= \frac{m}{4}\left(\ddot{h}_{LE} + \ddot{h}_{TE}\right) + \frac{J_C}{4b^2}\left(\ddot{h}_{LE} - \ddot{h}_{TE}\right) + \frac{3k}{16}\left(3h_{LE} + h_{TE}\right) \\ &= \frac{1}{4}\left(m + \frac{J_C}{b^2}\right)\ddot{h}_{LE} + \frac{1}{4}\left(m - \frac{J_C}{b^2}\right)\ddot{h}_{TE} + \frac{3k}{16}\left(3h_{LE} + h_{TE}\right) \end{aligned} \tag{46}$$

The second Lagrange's equation is given by

$$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial \dot{h}_{TE}} + \frac{\partial U}{\partial h_{TE}} &= \frac{m}{4}\left(\ddot{h}_{LE} + \ddot{h}_{TE}\right) - \frac{J_C}{4b^2}\left(\ddot{h}_{LE} - \ddot{h}_{TE}\right) + k h_{TE} + \frac{k}{16}\left(3h_{LE} + h_{TE}\right) \\ &= \frac{1}{4}\left(m - \frac{J_C}{b^2}\right)\ddot{h}_{LE} + \frac{1}{4}\left(m + \frac{J_C}{b^2}\right)\ddot{h}_{TE} + \frac{k}{16}\left(3h_{LE} + 17h_{TE}\right) \end{aligned} \tag{47}$$

Recasting the terms in matrix form gives

$$\frac{1}{4}\begin{bmatrix} m + \frac{J_C}{b^2} & m - \frac{J_C}{b^2} \\ m - \frac{J_C}{b^2} & m + \frac{J_C}{b^2} \end{bmatrix} \begin{Bmatrix} \ddot{h}_{LE} \\ \ddot{h}_{TE} \end{Bmatrix} + \frac{k}{16}\begin{bmatrix} 9 & 3 \\ 3 & 17 \end{bmatrix} \begin{Bmatrix} h_{LE} \\ h_{TE} \end{Bmatrix} \tag{48}$$

where the mass matrix is

$$\mathbf{M} = \frac{1}{4}\begin{bmatrix} m + \dfrac{J_C}{b^2} & m - \dfrac{J_C}{b^2} \\ m - \dfrac{J_C}{b^2} & m + \dfrac{J_C}{b^2} \end{bmatrix} \tag{49}$$

and the stiffness matrix is

$$\mathbf{K} = \frac{k}{16}\begin{bmatrix} 9 & 3 \\ 3 & 17 \end{bmatrix} \tag{50}$$

The mass and stiffness matrices are symmetric and full.

Question 3 – 5 points

To decouple the equations inertially, we choose the translation of the center of mass h_C and the pitch angle θ as the new coordinates. In fact, these coordinates allow us to write the kinetic energy as the sum of a purely translational and a purely rotational contribution each involving only one coordinate.

To find the transformation matrix that defines the coordinate change, we write the original coordinates h_{LE} and h_{TE} as a function of the new ones:

$$h_{LE} = h_C + b\theta \qquad h_{TE} = h_C - b\theta \tag{51}$$

This gives

$$\begin{Bmatrix} h_{LE} \\ h_{TE} \end{Bmatrix} = \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix}\begin{Bmatrix} h_C \\ \theta \end{Bmatrix} = \mathbf{T}\begin{Bmatrix} h_C \\ \theta \end{Bmatrix} \tag{52}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} \tag{53}$$

is the transformation matrix we are looking for.

Question 4 – 5 points

The mass and stiffness matrices for the new coordinates h_C and θ are given by

$$\mathbf{M}' = \mathbf{T}^T\mathbf{M}\mathbf{T} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix}\begin{bmatrix} m + \dfrac{J_C}{b^2} & m - \dfrac{J_C}{b^2} \\ m - \dfrac{J_C}{b^2} & m + \dfrac{J_C}{b^2} \end{bmatrix}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix}\begin{bmatrix} m & \dfrac{J_C}{b} \\ m & -\dfrac{J_C}{b} \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix} \tag{54}$$

$$\mathbf{K}' = \mathbf{T}^T\mathbf{K}\mathbf{T} = \frac{k}{16}\begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix}\begin{bmatrix} 9 & 3 \\ 3 & 17 \end{bmatrix}\begin{bmatrix} 1 & b \\ 1 & -b \end{bmatrix} = \frac{k}{8}\begin{bmatrix} 1 & 1 \\ b & -b \end{bmatrix}\begin{bmatrix} 6 & 3b \\ 10 & -7b \end{bmatrix} = k\begin{bmatrix} 2 & -\dfrac{b}{2} \\ -\dfrac{b}{2} & \dfrac{5b^2}{4} \end{bmatrix} \tag{55}$$

Another way to obtain the matrices is to write the kinetic and elastic energies of the system in terms of the new coordinates and derive the corresponding Lagrange's equations. This alternative approach can be used to verify the results. As expected, the new mass matrix is diagonal, but the stiffness matrix is not. To decouple the equations elastically (but not inertially), we should choose the translations of the points where the springs connect to the wing as the coordinates.

Question 5 – 5 points

The equations of motion obtained using the Lagrangian approach by choosing h_C and θ as the coordinates must match those obtained using the Newtonian approach. The free-body diagram of the system is sketched in Fig. 12,

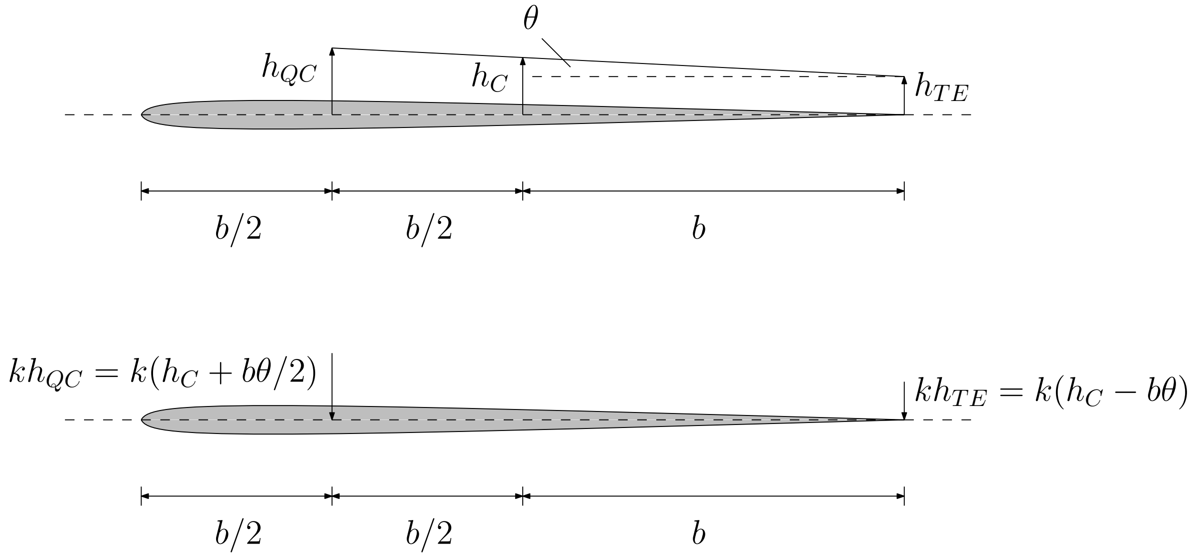


Figure 12: Free-body diagram for the wind-tunnel wing model for Problem 3.

where dependencies on time are omitted. The equations of motion for the plunge and pitch motions read

$$\begin{aligned} m\ddot{h}_C &= -kh_{CQ} - kh_{TE} \\ &= -k\left(h_C + \frac{b\theta}{2}\right) - k\left(h_C - b\theta\right) \\ &= -2kh_C + \frac{kb}{2}\theta \end{aligned} \tag{56}$$

$$\begin{aligned} J_C\ddot{\theta} &= -kh_{CQ}\frac{b}{2} + kh_{TE}b \\ &= -\frac{kb}{2}\left(h_C + \frac{b\theta}{2}\right) + kb\left(h_C - b\theta\right) \\ &= \frac{kb}{2}h_C - \frac{5kb^2}{4}\theta \end{aligned} \tag{57}$$

Bringing the terms on the left-hand side gives

$$\begin{aligned} m\ddot{h}_C + 2kh_C - \frac{kb}{2}\theta = 0 \\ J_C\ddot{\theta} - \frac{kb}{2}h_C + \frac{5kb^2}{4}\theta = 0 \end{aligned} \tag{58}$$

or

$$\begin{bmatrix} m & 0 \\ 0 & J_C \end{bmatrix} \begin{Bmatrix} \ddot{h}_{LE} \\ \ddot{h}_{TE} \end{Bmatrix} + k \begin{bmatrix} 2 & -\dfrac{b}{2} \\ -\dfrac{b}{2} & \dfrac{5b^2}{4} \end{bmatrix} \begin{Bmatrix} h_{LE} \\ h_{TE} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{59}$$

This is the same result obtained in Question 4.