Mathematical Interpretation of Bank Angle and Roll Rate

Assume some path data which consists of the list of nodes [[x,y],...]. From this data, we will calculate the bank angle and the roll rate. First, we must see if this is a differentially flat system to be able to obtain angles and angular velocities (that is, we must show that all relevant variables can be reduced to functions of simply x and y).

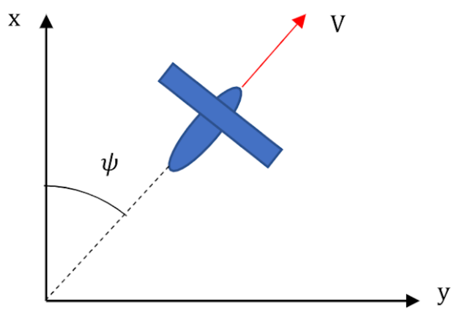

Figure 1: x-y plane positioning of fixed-wing aircraft

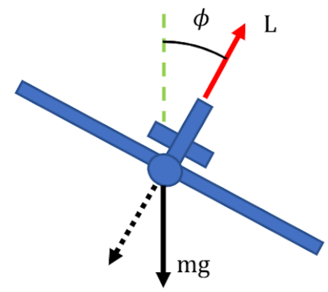

Figure 2: front view of fixed-wing aircraft

We will assume constant velocity for this problem. From Figure 1, we can derive the kinematic relations of position x-y, velocity V, and yaw angle ψ of the fixed-wing aircraft.

$$(1)\dot{x} = Vx = V\cos\psi$$
$$(2)\ \dot{y} = Vy = V\sin\psi$$

These are both results of simple trigonometric relations, given the constant velocity assumption.

From Figure 2, we can setup the force equilibrium equation

$$(3)\ L\cos\Phi = mg$$

This comes from the assumption of 2D (or level) flight.
Also assuming that any banking will be done smoothly as an arc, the centripetal acceleration equation for circular motion can be applied:

$$(4)\ L\sin\Phi = \frac{mV^2}{r}$$

Therefore, dividing equation (3) by equation (4), we find:

$$(5)\ tan\Phi = \frac{V^2}{rg}$$

Additionally, given the assumption of circular motion, distance travelled can be approximated by analyzing the circle travelled.

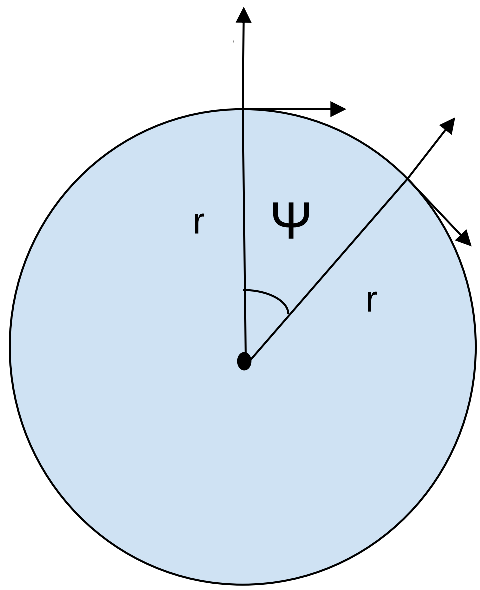

Figure 3: Motion about a Circle

Analyzing Figure 3, it is evident that rotation about some angle Ψ incurs some portion of travel about the circle's circumference. Therefore, we find:

$$(6)\ p = 2\pi r \frac{\psi}{2\pi} = \psi r$$

where p is the position function. Differentiating to determine velocity,

$$(7)\ V = r\dot{\psi}$$

Combining (5) and (7), we see:

$$(8)\ \dot{\psi} = \frac{\mathrm{V}}{\mathrm{r}} = \mathrm{gtan}\Phi$$

Now, since,

$(x,y) \Longrightarrow \Phi = f(x, x,...,y,y,...),\psi = f(x,x,...,y,y,...),\ \psi = f(x,x,...,y,y,...)$

we can say that this system is differentially flat. The objective of the path planning is to minimize the roll rate Φ, and to do so we must first derive the functions for the angles and the angular velocities.
From equations (1) and (2) we can obtain,

$$(9)\ \psi = \arctan\frac{\dot{y}}{\dot{x}}$$

Since we have each node ([x, y]), we can calculate the difference for each neighboring point, or Δx and Δy.

$$(10)\ \psi = \arctan\frac{\dot{y}}{\dot{x}} = \arctan\frac{\Delta y}{\Delta t}\frac{\Delta t}{\Delta x} = \arctan\frac{\Delta y}{\Delta x}$$

Taking the time derivative of (9) using the chain rule,

$$(11)\ \dot{\psi} = \frac{\partial\psi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial t}$$

Then, taking the partial derivatives of (10) for x and y:

$$(12)\ \frac{\partial\psi}{\partial x} = -\frac{\Delta y}{(\Delta x)^2 + (\Delta y)^2}\Delta^2 x$$

$$(13)\ \frac{\partial\psi}{\partial y} = \frac{\Delta x}{(\Delta x)^2 + (\Delta y)^2}\Delta^2 y$$

Thus, substituting equations (1), (2), (9), and (12), and (13) into (11):

$$(14)\ \dot{\psi} = -\frac{\Delta y}{(\Delta x)^2 + (\Delta y)^2}(\Delta^2 x)V\cos[\arctan\frac{\Delta y}{\Delta x}] + \frac{\Delta x}{(\Delta x)^2 + (\Delta y)^2}(\Delta^2 y)V\sin[\arctan\frac{\Delta y}{\Delta x}]$$

Since we can obtain Δx and Δy as the difference between x and y values neighboring nodes,

$$(15)\ \Delta x = x_i + 1 - x_i\,, \Delta y = y_i + 1 - y_i$$

Now, the problem arises for what Δ^2x and Δ^2y will be. Defining *h* to be the non-homogeneous step size, we will define Δ^2x and Δ^2y as

$$(16)\ \Delta^2 x = \hat{x}' = \frac{\Delta x_{i+1} - \Delta x_{i-1}}{2h} + O(h^2)$$

The same holds true for $\Delta^2 y$. Now, in the code, equation (16) will be computed with the following function: np.gradient().

Then, using the yaw rate, we can compute the bank angle. By manipulating equation (8), we retrieve:

$$(17)\ \Phi = \arctan\frac{V\dot{\psi}}{g}$$

Finally, we can compute the roll rate by using the function np.gradient() for the list of bank angles.