LL BP

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Transfer Orbits: Lambert Arcs

Transfer Orbit Design
(special class of boundary value problem)

1. **Geometrical relationships**

   Conic paths connecting two points that are fixed in space with focus at the attracting center

2. **Analytical Relationships**

3. **Lambert's Theorem**

Lambert's Theorem

{ simply a different way to write Kepler's Eqn.

Know a lot about possible orbits connecting two points { solution space

But analytical relationships rely on "a"

how to get it?

Must somehow select "a" ← an additional specification about the transfer path

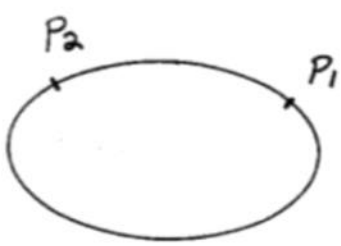
What to specify? From a number of different options → choose TOF

Lambert: conjecture that given I.C.s (r_1, r_2, c)

TOF depends only on "a"

i.e., $t = t(a, r_1 + r_2, c)$

(Lagrange actually proved this later)



$$\underbrace{\begin{aligned} n(t_1 - t_p) &= E_1 - e\sin E_1 \\ n(t_2 - t_p) &= E_2 - e\sin E_2 \end{aligned}}_{\text{subtract}}$$

$$\boxed{(t_2 - t_1) = \text{TOF} = \frac{1}{n}\left[(E_1 - E_1) - e\sin E_2 + e\sin E_1\right]}$$

func of "a"

relate E's, e to r_1, r_2, c?

Given TOF, this relationship contains unknowns: E_1, E_2, e, a

Must be rewritten in terms of only <u>one</u> unknown → a

HOW?

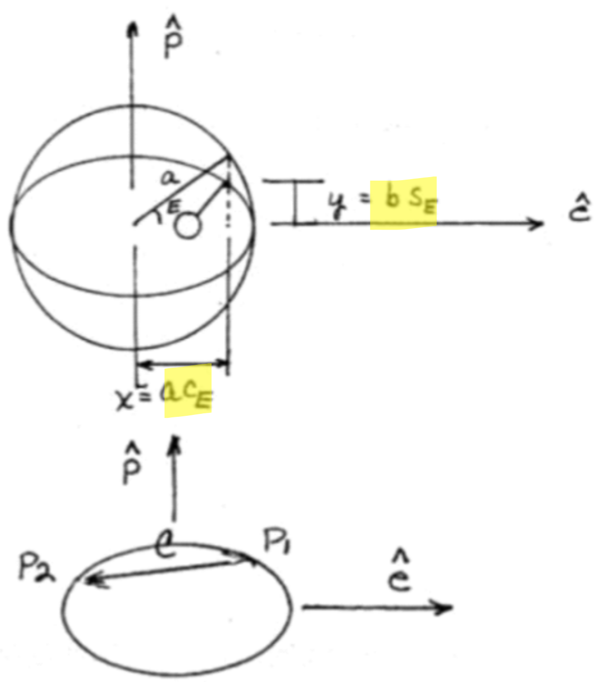
Define: $\boxed{E_p = \frac{E_1 + E_2}{2}}$ $\boxed{E_m = \frac{E_2 - E_1}{2}}$

$$r_1 = a(1 - e\cos E_1) \qquad r_2 = a(1 - e\cos E_2)$$

$$r_1 + r_2 = a[2 - e(\cos E_1 + \cos E_2)]$$

$$= a\left[2 - e\left(2\cos\left(\frac{E_1 + E_2}{2}\right)\cos\left(\frac{E_1 - E_2}{2}\right)\right)\right]$$

$$\boxed{r_1 + r_2 = 2a[1 - e\cos E_p \cos E_M]}$$



measured from center

$$\bar{P}_1 = x_1\hat{e} + y_1\hat{p}$$

$$\bar{P}_2 = x_2\hat{e} + y_2\hat{p}$$

$$\bar{c} = \bar{P}_2 - \bar{P}_1$$

$$\bar{c} = \bar{p}_2 - \bar{p}_1$$
$$= (x_2 - x_1)\hat{e} + (y_2 - y_1)\hat{p}$$
$$c^2 = (a\cos E_2 - a\cos E_1)^2 + (b\sin E_2 - b\sin E_1)^2$$
$$= a^2(\cos E_2 - \cos E_1)^2 + a^2(1-e^2)^2(\sin E_2 - \sin E_1)^2$$
$$c^2 = a^2\left[(\cos E_2 - \cos E_1)^2 + (1-e^2)^2(\sin E_2 - \sin E_1)^2\right]$$

trig identities

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\longrightarrow \cos E_2 - \cos E_1 = -2\sin E_p \sin E_M$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\longrightarrow \sin E_2 - \sin E_1 = 2\cos E_p \sin E_M$$

$$c^2 = a^2\left[4\sin^2 E_p \sin^2 E_M + (1-e^2)4\cos^2 E_p \sin^2 E_M\right]$$
$$= 4a^2\sin^2 E_M\left(\sin^2 E_p + \cos^2 E_p - e^2\cos^2 E_p\right)$$
$$\boxed{c^2 = 4a^2\sin^2 E_M\left(1 - e^2\cos^2 E_p\right)}$$

e is still there!

Define $\boxed{\cos\eta = e\cos E_p}$ ok because $e < 1$

e, E_1, E_2

e, E_p, E_m

$\Downarrow$

η, E_μ

$$r_1 + r_2 = 2a\left[1 - \cos\eta\cos E_M\right]$$

$$c^2 = 4a^2\sin^2 E_M\left(1 - \cos^2\eta\right)$$

OR

$$c = 2a\sin E_M \sin\eta$$

So $r_1 + r_2 + c = 2a - 2a\cos\eta\cos E_M + 2a\sin E_M\sin\eta$

Defined previously

$$\underbrace{s = \frac{1}{2}(r_1 + r_2 + c)}_{\text{semi perimeter}} \quad \text{and} \quad \underbrace{\alpha = 2\sin^{-1}\sqrt{\frac{s}{2a}}}_{\sin^2\frac{\alpha}{2} = \frac{s}{2a}}$$

OR

$$r_1 + r_2 + c = 4a\underbrace{\sin^2\left(\frac{\alpha}{2}\right)}_{\left(\frac{1-\cos\alpha}{2}\right)}$$

$$r_1 + r_2 + c = 2a(1 - \cos\alpha)$$

α

$$r_1 + r_2 + c = 2a\underbrace{\left(1 - \cos\eta \cos E_M + \sin E_M \sin\eta\right)}_{1-\cos(\eta + E_M)} = 2a\left(1 - \cos\alpha\right)$$

$$\boxed{\alpha = \eta + E_M}$$

Also $\quad r_1 + r_2 - c = 2a\left(1 - \cos\eta \cos E_M - \sin E_M \sin\eta\right)$

Defined previously $\quad \underbrace{\beta = 2\sin^{-1}\sqrt{\dfrac{s-c}{2a}}}_{\sin^2\frac{\beta}{2} = \frac{s-c}{2a}}$

OR

$$r_1 + r_2 - c = 4a \underbrace{\sin^2\left(\frac{\beta}{2}\right)}_{\left(\frac{1-\cos\beta}{2}\right)}$$

$$r_1 + r_2 - c = 2a(1 - \cos\beta)$$

$$r_1 + r_2 - c = 2a\underbrace{\left(1 - \cos\eta \cos E_M - \sin E_M \sin\eta\right)}_{\boxed{1-\cos(\eta - E_M)}} = 2a\left(1 - \cos\beta\right)$$

$$\boxed{\beta = \eta - E_M}$$

Kepler's Eqn. $\qquad e, E_1, E_2 \longrightarrow e, \eta, E_M$

Kepler's Eqn. $\qquad e, E_1, E_2 \rightarrow e, \eta, E_M \downarrow \alpha, \beta$

$$n(t_2 - t_1) = \left[(E_2 - E_1) - e \underbrace{(\sin E_2 - \sin E_1)}_{2\cos\left(\frac{E_2+E_1}{2}\right)\sin\left(\frac{E_2-E_1}{2}\right)} \right]$$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(E_2 - E_1) - 2\underbrace{e\cos E_p}_{\cos\eta} \sin E_M \right]$$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[2E_M - 2\cos\eta \sin E_M \right]$$

$$\underline{\sqrt{\mu}(t_2 - t_1) = 2a^{3/2}\left[E_M - \cos\eta \sin E_M \right]}$$

Note:

$$\alpha - \beta = (\eta + E_M) - (\eta - E_M) = 2E_M$$

$$\alpha + \beta = 2\eta$$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - \beta) - \underbrace{\underbrace{2\cos\eta \sin E_M}_{2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)}}_{\boxed{\sin\alpha - \sin\beta}} \right]$$

Kepler's Equation written in α, β

Kepler's Equation written in α, β

$$\boxed{\sqrt{\mu}\underbrace{(t_2 - t_1)}_{\text{TOF}} = a^{3/2}\left[\,(\alpha - \beta) - (\sin\alpha - \sin\beta)\right]}$$

$\alpha = 2\arcsin\sqrt{\frac{s}{2a}}$

$\beta = 2\arcsin\sqrt{\frac{s-c}{2a}}$

Lambert's Equation

quadrant ambiguities

transcendental equation

Conjecture by Lambert (1761): time to traverse arc depends only on a and two geometric properties of the space triangle (c, r_1+r_2); Lagrange proves theorem in 1778.

Johann Heinrich Lambert (1728 - 1777)

$\text{TOF} = \text{TOF}(a\ \&\ \underbrace{\text{space triangle stuff}}_{\text{known } r_1,\, r_2,\, c})$

α, β Quadrant Ambiguities: Elliptic Transfers

For a given space triangle and value "a", there exist <u>four</u> arcs that could serve as the solution:

solve Lambert Eqn $\longrightarrow$ a (4 options)

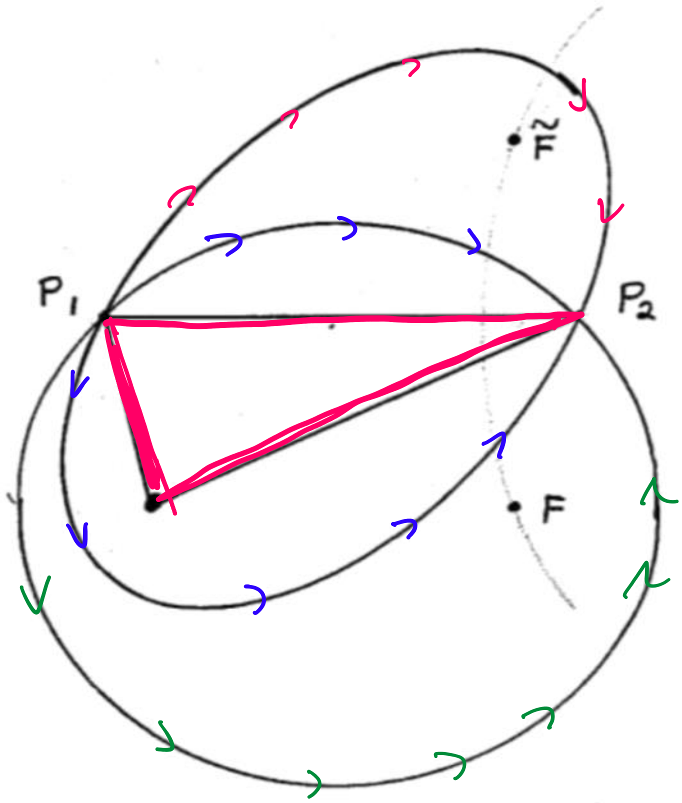
$\downarrow$

2 ellipses (different e)

$\downarrow$

2 directions

1A
1B
2A
2B



4 solutions correspond to quadrant ambiguities associated with angles α and β

Principal values α_o, β_o $\longrightarrow$ $0 \leq \beta_o \leq \alpha_o \leq \pi$

Recall derivation of Lambert's Equation

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[\left(E_2 - E_1\right) - e\left(\sin E_2 - \sin E_1\right)\right]$$

$$\Downarrow$$

$$\sqrt{\mu}\left(t_2 - t_1\right) = a^{3/2}\left[\left(\alpha - \beta\right) - \left(\sin\alpha - \sin\beta\right)\right]$$

$$\left.\begin{aligned} \alpha &= 2\sin^{-1}\sqrt{\frac{s}{2a}} \\ \beta &= 2\sin^{-1}\sqrt{\frac{s-c}{2a}} \end{aligned}\right\} \text{ quadrant ambiguities exist}$$

Do α, β have any physical meaning that would help?

$$\left.\begin{aligned} \alpha &= \eta + E_M \\ \beta &= \eta - E_M \end{aligned}\right\} \qquad \begin{aligned} \alpha - \beta &= 2E_M \\ &= 2\left(\frac{E_2 - E_1}{2}\right) \end{aligned}$$

$$\Downarrow$$

$$\boxed{}$$

BUT, generally $\alpha \neq E_2$; $\beta \neq E_1$

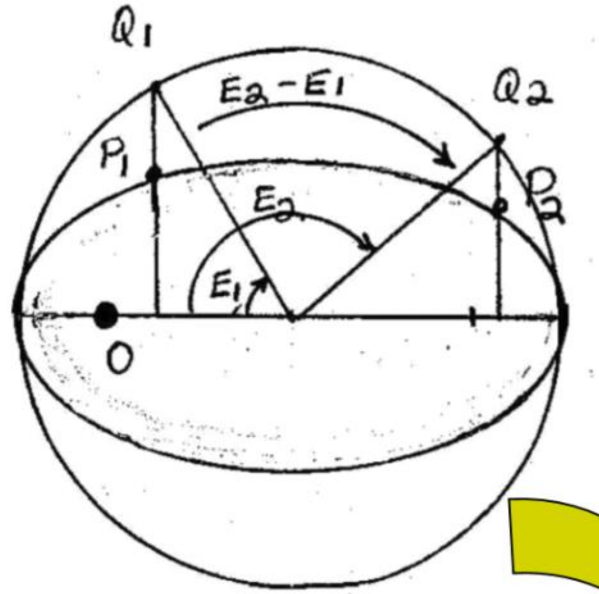
However, all ellipses with the same "a" have the same TOF
Useful to choose an equivalent ellipse with the same "a"?

Yes $\longrightarrow$ choose a rectilinear ellipse ($e = 1, p = 0$)

Here $\left\{\begin{aligned} \alpha &= E_2^R \\ \beta &= E_1^R \end{aligned}\right.$

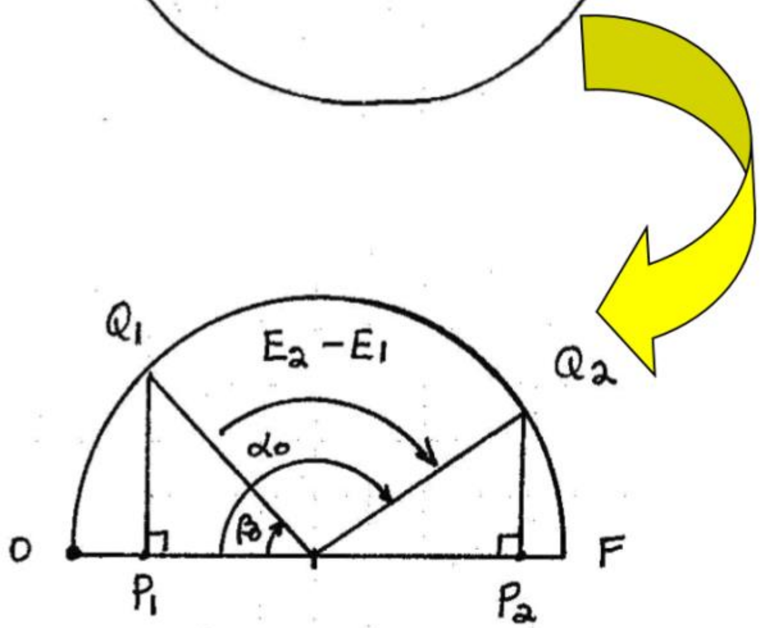
Use a rectilinear ellipse with the same "a" to resolve the quadrant ambiguity issue and provide a geometrical interpretation of α, β

Actual ellipse



Note: Transfer corresponds to what arc on the auxiliary circle?

$$\alpha - \beta = E_2 - E_1$$



To create rectilinear ellipse P_1, P_2 remain in place on chord; O, F move along new ellipses to "shift" locations

Define α_o, β_o consistent with principal value

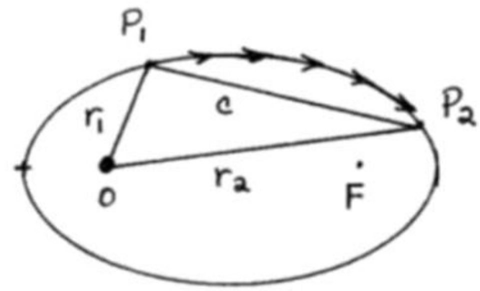
$$\alpha - \beta = E_2 - E_1$$

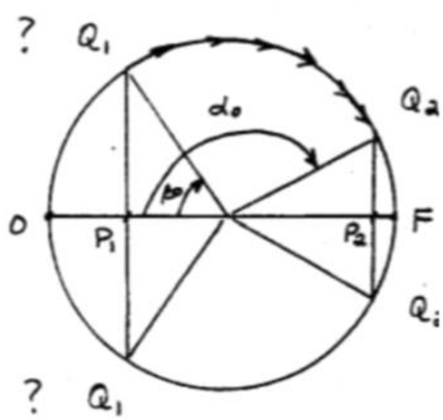
Now consider path for 4 different types of arcs

1A

TA < 180°

F is NOT between chord / arc





uses rectilinear ellipses + auxiliary circle

Transfer follows what arc of the auxiliary circle?

Calculate α_o, β_o ⇒ which of 4 combinations yields correct Q_1, Q_2 ? ***E_1* and *E_2* ?**

Check orbit ⇒ in moving from P_1 to P_2 do you pass through periapsis? apoapsis?

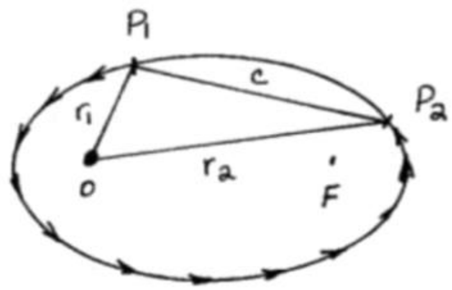
$$\boxed{\begin{gathered}\textbf{1A}\quad \alpha = \alpha_o \qquad \beta = \beta_o \qquad E_2 - E_1 = \alpha_o - \beta_o \\ \sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o)\right]\end{gathered}}$$

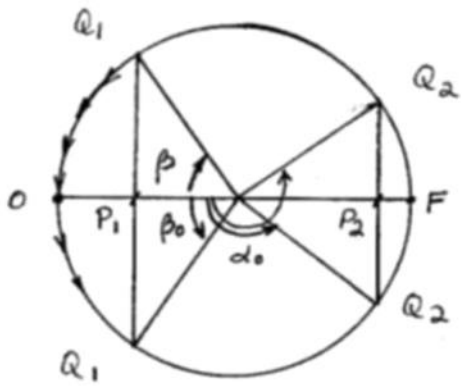
$\alpha_0\ \beta_0$ principle values

2B

TA $> 180^\circ$

F is between chord / arc





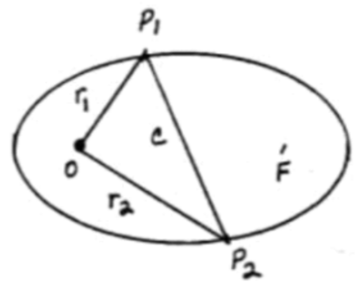
$$E_2 - E_1 = \alpha - \beta$$
$$= \left[\alpha_o + (\pi - \alpha_o) + (\pi - \alpha_o)\right] - (-\beta_o)$$

$$\boxed{\textbf{2B} \quad \alpha = 2\pi - \alpha_o \qquad \beta = -\beta_o \qquad E_2 - E_1 = 2\pi - \alpha_o + \beta_o}$$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - sin\alpha) - (\beta - sin\beta)\right]$$
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[\left(2\pi - \alpha_o - sin(2\pi - \alpha_o)\right) - \left(-\beta_o - sin(-\beta_o)\right)\right]$$

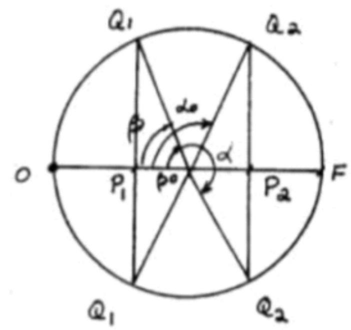
$$\boxed{\textbf{2B} \quad \sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[2\pi - (\alpha_o - sin\,\alpha_o) + (\beta_o - sin\,\beta_o)\right]}$$

1B



TA $< 180^\circ$

F is between chord / arc



$$E_2 - E_1 = \alpha - \beta$$
$$= \left[\alpha_o + (\pi - \alpha_o) + (\pi - \alpha_o)\right] - \beta_o$$

$$\boxed{\text{1B} \quad \alpha = 2\pi - \alpha_o \qquad \beta = \beta_o \qquad E_2 - E_1 = 2\pi - \alpha_o - \beta_o}$$

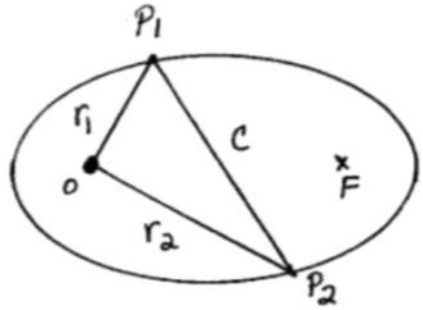
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(2\pi - \alpha_o - \sin(2\pi - \alpha_o)) - (\beta_o - \sin(\beta_o))\right]$$

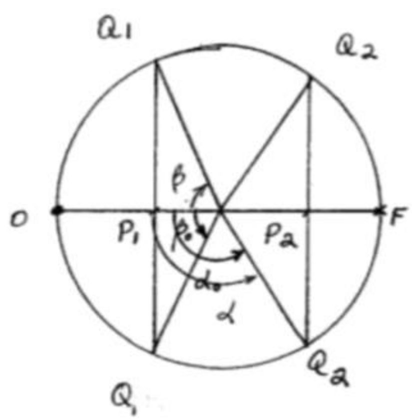
$$\boxed{\text{1B} \quad \sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[2\pi - (\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o)\right]}$$

2A

TA > 180°

F is NOT between chord / arc





$$E_2 - E_1 = \alpha - \beta$$
$$= (\alpha_o) - (-\beta_o)$$

$$\boxed{\textbf{2A} \quad \alpha = \alpha_o \qquad \beta = -\beta_o \qquad E_2 - E_1 = \alpha_o + \beta_o}$$

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$
$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha_o - \sin(\alpha_o)) - (-\beta_o - \sin(-\beta_o))\right]$$

$$\boxed{\textbf{2A} \quad \sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[(\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o)\right]}$$

α', β' Quadrant Ambiguities: Hyperbolic Transfers

Without the luxury of a visual geometrical technique, straight integration is required for a result

$$\sqrt{\mu}(t_2 - t_1) = |a|^{3/2}\left[(\sinh\alpha' - \alpha') - (\sinh\beta' - \beta')\right]$$

$$H_2 - H_1 = \alpha' - \beta'$$

1H

$$\alpha' = \alpha'_o$$
$$\beta' = \beta'_o$$
$$\sqrt{\mu}(t_2 - t_1) = |a|^{3/2}\left[(\sinh\alpha'_o - \alpha'_o) - (\sinh\beta'_o - \beta'_o)\right]$$

2H

$$\alpha' = \alpha'_o$$
$$\beta' = -\beta'_o$$
$$\sqrt{\mu}(t_2 - t_1) = |a|^{3/2}\left[(\sinh\alpha'_o - \alpha'_o) + (\sinh\beta'_o - \beta'_o)\right]$$

Parabolic Transfers

Used Lambert's TOF theorem to write

$$TPF = TOF\left(a, r_1 + r_2, c\right)$$

$$\Downarrow$$

Produced relationships for elliptic and hyperbolic transfers (1A, 1B, 2A, 2B, 1H, 2H)

TOF relationship for <u>parabolic</u> transfer ?

Recall: only TWO possible parabolas that connect points

$$\Downarrow$$

TOF determined as limit of other elliptic cases as $a \to \infty$

Parabolic Transfer (Euler's Equation)

$$\boxed{\begin{aligned} \mathrm{TOF}_1 &= \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} - (s-c)^{3/2}\right] \\ \mathrm{TOF}_2 &= \frac{1}{3}\sqrt{\frac{2}{\mu}}\left[s^{3/2} + (s-c)^{3/2}\right] \end{aligned}}$$

Lambert's Theorem

Time of Flight for Transfer Orbits Between Two Given Positions

ELLIPTIC ORBITS:

$$\sqrt{\frac{\mu}{a^3}}(t_2 - t_1) = \underbrace{2m\pi}_{\text{number of complete revolutions}} + \begin{cases} (\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o) \\ 2\pi - (\alpha_o - \sin\alpha_o) - (\beta_o - \sin\beta_o) \\ (\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o) \\ 2\pi - (\alpha_o - \sin\alpha_o) + (\beta_o - \sin\beta_o) \end{cases}$$

where

$$c = \text{chord } P_1P_2$$

$$s = \text{semi-perimeter } \frac{r_1 + r_2 + c}{2}$$

$$\left.\begin{aligned} \alpha &= 2\sin^{-1}\sqrt{\frac{s}{2a}} \\ \beta &= 2\sin^{-1}\sqrt{\frac{s-c}{2a}} \end{aligned}\right\} \alpha_o,\ \beta_o \text{ are principal values}$$

HYPERBOLIC ORBITS:

$$\sqrt{\frac{\mu}{|a|^3}}(t_2 - t_1) = \begin{cases} (\sinh\alpha'_o - \alpha'_o) - (\sinh\beta'_o - \beta'_o) \\ (\sinh\alpha'_o - \alpha'_o) + (\sinh\beta'_o - \beta'_o) \end{cases}$$

where

$$\left.\begin{aligned} \alpha' &= 2\sinh^{-1}\sqrt{\frac{s}{2|a|}} \\ \beta' &= 2\sinh^{-1}\sqrt{\frac{s-c}{2|a|}} \end{aligned}\right\} \alpha'_o,\ \beta'_o \text{ are principal values}$$