College of Engineering
Daniel Guggenheim School of Aerospace Engineering

Fall2022 AE6230: Structural Dynamics

Homework 3

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I Problem One

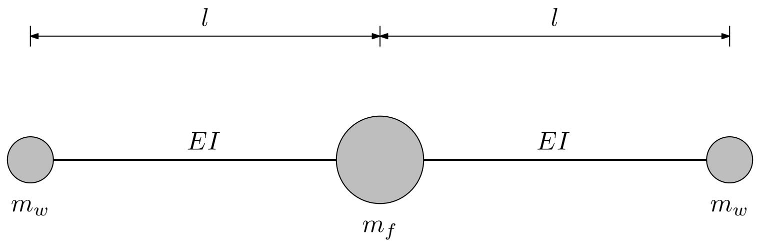


Figure 1: Schematic of an aircraft undergoing out-of-plane (vertical) bending vibrations in free flight.

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Half-wing mass	m_w	750 kg
Fuselage mass	m_f	$5m_w$
Wing semispan	l	10 m
Wing out-of-plane bending stiffness	EI	5×10^6 Nm^2

Figure 1 shows a simplified model for the out-of-plane (vertical) bending vibrations of a free-flying aircraft. The aircraft inertia is modeled by a concentrated mass m_f at the fuselage centerline and two concentrated masses m_w at the wing tips. The elasticity of each half wing is modeled by a beam of negligible mass with out-of-plane bending stiffness EI and length l, which behaves as a spring $k = 3EI/l^3$. The aircraft motion is described in terms of the vertical translations of the left, center, and right masses, denoted by $h_{wl}(t)$, $h_f(t)$, and $h_{wr}(t)$, respectively. These translations are positive upward and measured from the undeformed configuration of the aircraft in Fig. 1. Assuming small-amplitude vibrations and neglecting gravity, answer the following questions:

1. Considering the equations of motion

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_w & 0 \\ 0 & 0 & m_w \end{bmatrix} \begin{Bmatrix} \ddot{h}_f \\ \ddot{h}_{wl} \\ \ddot{h}_{wr} \end{Bmatrix} + \frac{3EI}{l^3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} h_f \\ h_{wl} \\ h_{wr} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{I.1}$$

   evaluate the natural frequencies for the parameters in Table 1 (in ascending order)

2. Evaluate the corresponding mode shapes normalized to have unit maximum displacement.

3. Plot the mode shapes from Question 2 and interpret their meaning.

4. Evaluate the inverse of the modal matrix $\mathbf{U}$ for the assumed mode shape normalization. (Note that the assumed mode shape normalization yields non-unit modal mass.)

5. Assuming that a wind gust causes the initial conditions.

$$\mathbf{q}(0) = \mathbf{q}_0 = \begin{Bmatrix} 0.5 \\ 0.0 \\ 0.0 \end{Bmatrix} \qquad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 = 0 \tag{I.2}$$

   determine the initial conditions for the modal equations.

6. Write the analytical expression of the damped free response in the form.

$$\mathbf{q}(t) = \mathbf{U}\boldsymbol{\eta}(t) \tag{I.3}$$

   considering the modal viscous damping factors $\zeta_1 = 0, \zeta_2 = \zeta_3 = 0.04$

7. Plot the components of $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ for $0 \leq t \leq 20$ s
8. Explain the results from Question 7 (motivate the contribution from each mode).

Solution

Question (1)

For a undamped system, we know that response shows a synchronous motion, and therefore can be represented as

$$\mathbf{q}(t) = \mathbf{u}f(t) = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix} f(t) \tag{I.4}$$

where $\mathbf{q}(t) = \begin{bmatrix} h_f & h_{wl} & h_{wr} \end{bmatrix}^\top$. Let (I.1) be $\mathbf{M}\ddot{h}(t) + \mathbf{K}h(t) = 0$ then if we plug in (I.4) we have

$$\mathbf{M}\mathbf{u}\ddot{f}(t) + \mathbf{K}\mathbf{u}f(t) = 0. \tag{I.5}$$

Now if we premultiply the above with $\mathbf{u}^\top$ we have

$$\begin{gathered} \mathbf{u}^\top\mathbf{M}\mathbf{u}\ddot{f}(t) + \mathbf{u}^\top\mathbf{K}\mathbf{u}f(t) = 0 \\ \ddot{f}(t) + \underbrace{\frac{\mathbf{u}^\top\mathbf{K}\mathbf{u}}{\mathbf{u}^\top\mathbf{M}\mathbf{u}}}_{\lambda} f(t) = 0. \end{gathered}$$

This gives us the next two equations that the synchronous motion satisfies

$$\ddot{f}(t) + \lambda f(t) = 0 \tag{I.6}$$

$$(\mathbf{K} - \lambda\mathbf{M})\mathbf{u} = 0 \tag{I.7}$$

(I.7) is a generalized eigenvalue problem and the value of λ is the square of the natural frequency and $\mathbf{u}$ is the eigenvector corresponding to each eigenvalue. Thus we can find the natural frequency by solving the eigenvalue problem of

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{u} = 0 \quad \Longrightarrow \quad \det(\mathbf{K} - \omega^2\mathbf{M}) = 0.$$

With MATLAB we can easily solve this eigenvalue problem and find the eigenvalues with `eig(K,M)` which gives us the following natural frequencies

$$\boxed{\omega_1 = 0, \quad \omega_2 = 2\sqrt{5} \approx 4.4721, \quad \omega_3 = 2\sqrt{7} \approx 5.2915} \tag{I.8}$$

Question (2)

In MATLAB along with the eigenvalues we have found the eigenvectors

$$\hat{\mathbf{u}}_1 = \begin{bmatrix} 0.0138 \\ 0.0138 \\ 0.0138 \end{bmatrix}, \quad \hat{\mathbf{u}}_2 = \begin{bmatrix} 0.0000 \\ -0.0258 \\ 0.0258 \end{bmatrix}, \quad \hat{\mathbf{u}}_3 = \begin{bmatrix} 0.0087 \\ -0.0218 \\ -0.0218 \end{bmatrix} \tag{I.9}$$

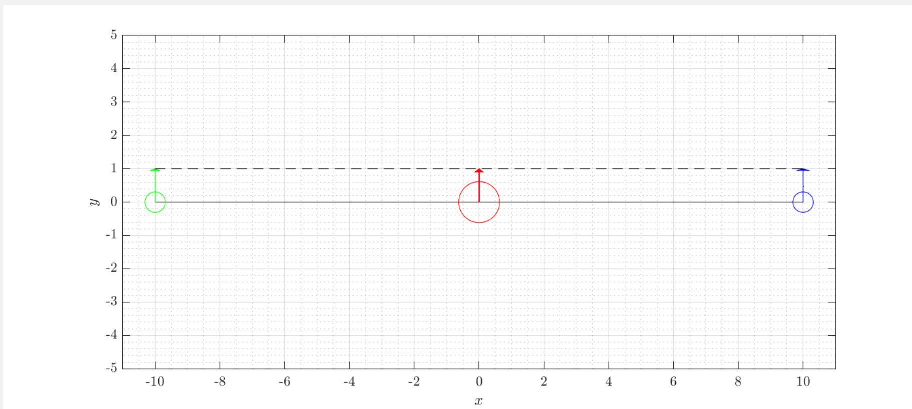
With the operator $\|\cdot\|_\infty$ indicating the L^∞-norm, the eigenvectors or mode shapes can be normalized w.r.t the unit maximum displacement by

$$\mathbf{u}_i = \frac{\hat{\mathbf{u}}_i}{\|\hat{\mathbf{u}}_i\|_\infty}.$$

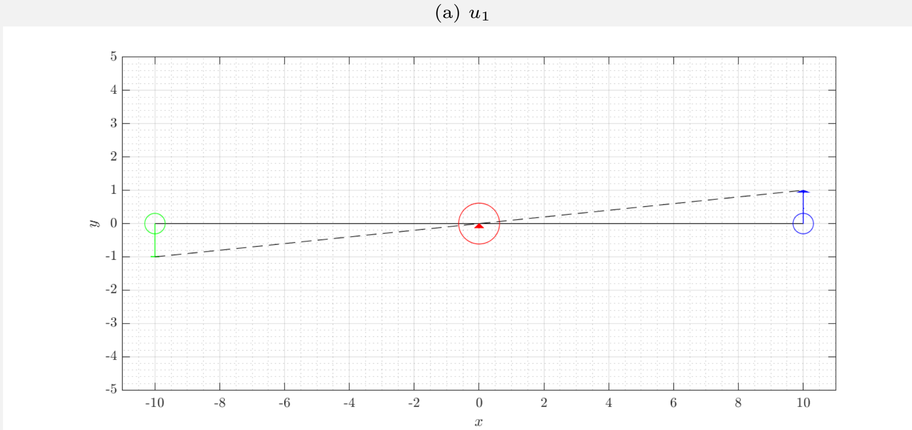
This gives us the following normalized orthogonal bases

$$\boxed{\mathbf{u}_1 = \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0.0000 \\ -1.0000 \\ 1.0000 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0.4000 \\ -1.0000 \\ -1.0000 \end{bmatrix}} \tag{I.10}$$

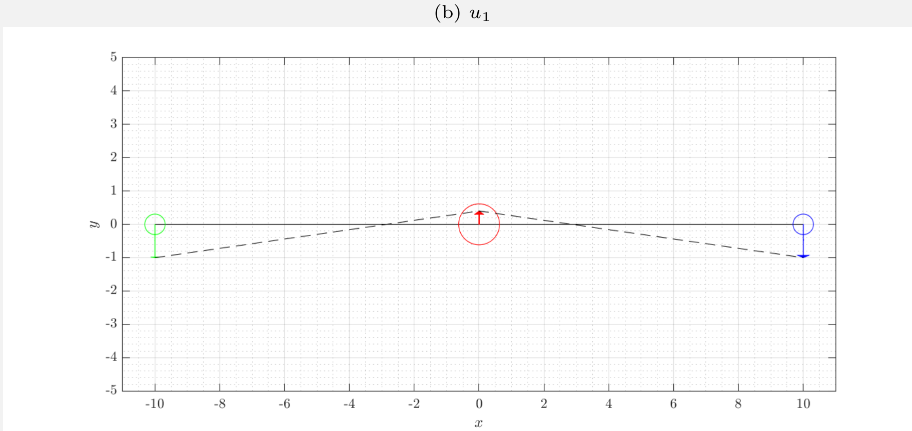
Question (3)



(a) u_1



(b) u_1



(c) u_1

Figure 2: Mode shapes or eigenvector of the aircraft system.

For a modal analysis, we can simplify the MDOF system into multiple SDOF systems using the mode shape and corresponding natural motion. The mode shape or eigenvectors, in particular, represent the direction of the displacement for the corresponding mode. In this case, we can see that the displacement for each mode will be in $\hat{\mathbf{u}}_1$ (red), $\hat{\mathbf{u}}_2$ (green), and $\hat{\mathbf{u}}_3$ (blue). For the first mode we see that the wings and fuselage move together. In the second one the fuselage is not moving and the wings are moving in the opposite directions. For the third mode the wings and fuselage are moving the opposite directions.

Question (4)

We simply take the derivative of the matrix $\mathbf{U} = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix}$ which is

$$\boxed{\mathbf{U}^{-1} = \begin{bmatrix} 0.7143 & 0.1429 & 0.1429 \\ 0.0000 & -0.5000 & 0.5000 \\ 0.7143 & -0.3571 & -0.3571 \end{bmatrix}.} \tag{I.11}$$

Question (5)

By combining the model shapes and the modal functions while paying attention to the first mode that has a frequency of 0, we have

$$\mathbf{q}(t) = \begin{bmatrix} h_f(t) \\ h_{wl}(t) \\ h_{wr}(t) \end{bmatrix} = \mathbf{u}_1 \underbrace{A_1 t + B_1}_{\eta_1(t)} + \mathbf{u}_2 \underbrace{A_2 \cos(\omega_2 t - \phi_2)}_{\eta_2(t)} + \mathbf{u}_3 \underbrace{A_3 \cos(\omega_3 t - \phi_3)}_{\eta_3(t)} \tag{I.12}$$

Thus,

$$\mathbf{q}(t) = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix} \begin{bmatrix} \eta_1(t) \\ \eta_2(t) \\ \eta_3(t) \end{bmatrix} = \mathbf{U}\boldsymbol{\eta}(t), \tag{I.13}$$

and

$$\dot{\mathbf{q}}(t) = \mathbf{u}_1 A_1 - \mathbf{u}_2 A_2 \omega_2 \sin(\omega_2 t - \phi_2) - \mathbf{u}_3 A_3 \omega_3 \sin(\omega_3 t - \phi_3). \tag{I.14}$$

Now plugging in the values to the two equations above we have six equations and six unknowns, and are able to find A_i and ϕ_i values. Or simply you can compute the IC of the modal functions by $\boldsymbol{\eta}(0) = \mathbf{U}^{-1}\mathbf{q}(0)$ and $\dot{\boldsymbol{\eta}}(0) = \mathbf{U}^{-1}\dot{\mathbf{q}}(0)$. Thus, we obtain

$$A_1 = 0, \quad B_1 = \frac{5}{14}, \quad A_2 = 0, \quad \phi_2 = 0, \quad A_3 = \frac{5}{14}, \quad \phi_3 = 0.$$

$$\eta_1(t) = \frac{5}{14} \approx 0.3571 \tag{I.15}$$

$$\eta_2(t) = 0 \tag{I.16}$$

$$\eta_3(t) = \frac{5}{14}\cos(2\sqrt{7}t) \approx 0.3571\cos(5.2915t) \tag{I.17}$$

$$\dot{\eta}_1(0) = \dot{\eta}_2(0) = \dot{\eta}_3(0) = 0. \tag{I.18}$$

Hence,

$$\boxed{\begin{aligned} \eta_1(0) &= \frac{5}{14} \approx 0.3571 \\ \eta_2(0) &= 0 \\ \eta_3(0) &= \frac{5}{14} \approx 0.3571 \\ \dot{\boldsymbol{\eta}}(0) &= \mathbf{0}. \end{aligned}} \tag{I.19}$$

Question (6)

With a damped free response using the modal viscous damping factors, we rewrite the $\eta_i(t)$ function as

$$\eta_i(t) = e^{-\zeta_i\omega_i t}\left(\eta_{0_i}\cos\omega_{d_i}t + \frac{\dot{\eta}_{0_i} + \zeta_i\omega_i\eta_{0_i}}{\omega_{d_i}}\sin\omega_{d_i}t\right) \tag{I.20}$$

where $\omega_{d_i} = \omega_i\sqrt{1-\zeta_i^2}$, $\eta_{0_i} = \eta_i(0)$, and $\dot{\eta}_{0_i} = \dot{\eta}(0)$. Since we have all the values required for (I.20), we can compute the damped free response (code in Appendix IV.i) to be

$$\begin{aligned}
\eta_1(t) &= \frac{5}{14} \\
\eta_2(t) &= 0 \\
\eta_3(t) &= e^{-\frac{2\sqrt{7}t}{25}}\left(\frac{5\cos\left(\frac{4\sqrt{42}}{5}\right)}{14} + \frac{\sqrt{6}\sin\left(\frac{4\sqrt{42}t}{5}\right)}{168}\right) \\
&\approx e^{-0.2117t}(0.3571\cos(5.185t) + 0.0146\sin(5.185t)).
\end{aligned} \tag{I.21}$$

Then

$$\boxed{\begin{aligned}
q_1(t) &= 0.4e^{-0.2117t}(0.3571\cos(5.185t) + 0.0146\sin(5.185t)) + 0.3571 \\
q_2(t) &= 0.3571 - e^{-0.2117t}(0.3571\cos(5.185t) + 0.0146\sin(5.185t)) \\
q_3(t) &= 0.3571 - e^{-0.2117t}(0.3571\cos(5.185t) + 0.0146\sin(5.185t))
\end{aligned}} \tag{I.22}$$

Question (7)

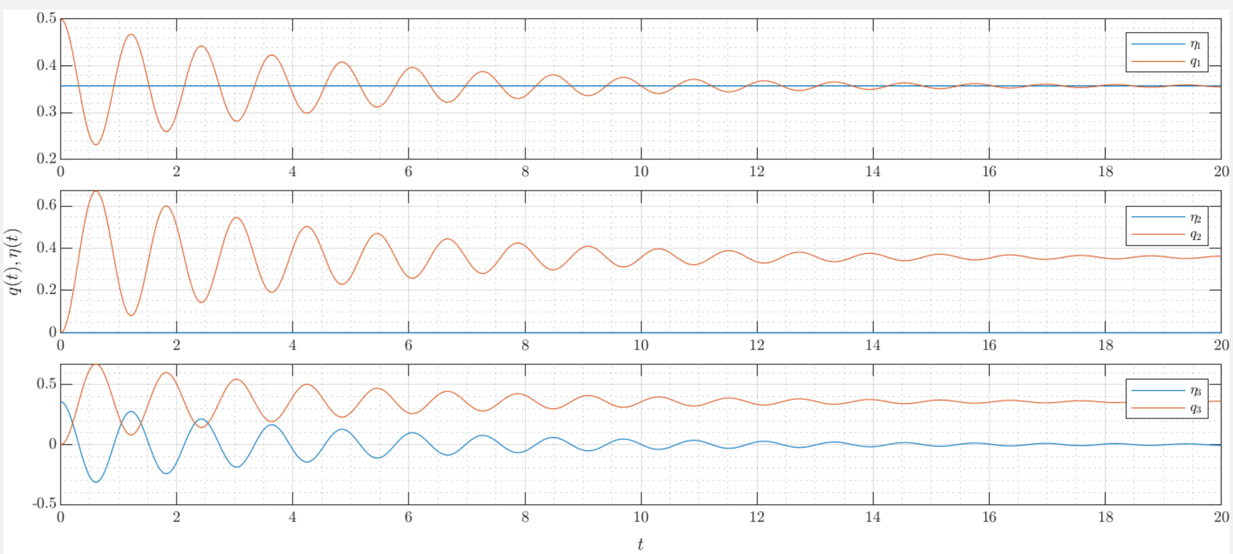


Figure 3: Responses $q(t)$ and modal functions $\eta(t)$.

Question (8)

The first mode is governed by the initial condition of the fuselage moved by the gust of wind. You can observe that the after being displaced by the wind the fuselage oscillates and eventually converges to a vertical distance of 5/14. Since the wings do not have any initial displacements there motion relies on or is dependent on the motion of the fuselage. As can be seen in Figure 3, after the fuselage is displaced, both of the wings oscillate with the same response regardless of different modal function and eventually converges or reaches an equilibrium state at a vertical distance of 5/14 which is the same as the fuselage. This result agrees with our intuition of how when the fuselage is displaced the wings would oscillate and eventually come to the same vertical position of the fuselage.

II Problem Two

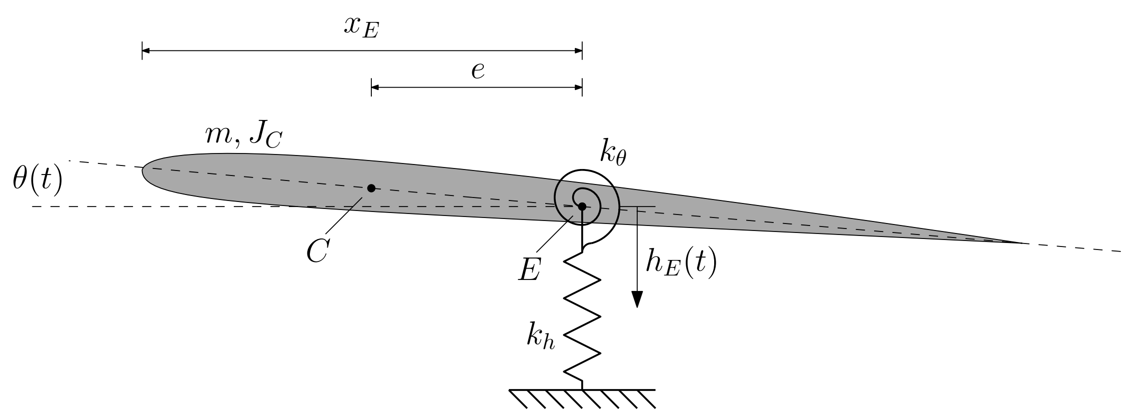


Figure 4: Schematic of typical section model.

Table 2: Parameter values for Problem 2.

Parameter	Symbol	Value
Mass	m	10 kg
Moment of inertia about E	J_E	0.08 kg·m^2
Chord	c	0.2 m
Offset of C from E (positive as in Fig. 4)	e	$-0.2c$
Position of E along the chord (positive as in Fig. 4)	x_E	$0.4c$
Translational spring stiffness	k_h	1000 N/m
Rotational spring stiffness	k_θ	200 Nm/rad

Consider the typical section model in Fig. 4, which is an abstraction for the cross section of a wing undergoing out-of-plane (vertical) bending and torsion. The typical section is subject to the excitation

$$\mathbf{Q}(t) = \mathbf{Q}_0 \sin \omega_0 t \tag{II.1}$$

with $Q_{0_1} = -10$ N, $Q_{0_2} = 1.5$ Nm, and $\omega_0 = 15$ rad/s. The modal mass and stiffness matrices along with the natural frequencies and mode shapes (normalized to have unit modal mass) can be computed using the script `AE6230_Fall2022_L17_MDOF_Free_TypicalSection.m` available in Canvas. Damping effects are captured by the modal viscous damping factors $\zeta_1 = \zeta_2 = 0.02$. Assuming small-amplitude vibrations and neglecting gravity, answer the following questions:

1. Determine the modal excitation $\mathbf{N}(t)$.
2. Considering the frequency response functions $H_1(\omega)$ and $H_2(\omega)$ associated with the modal coordinates $\boldsymbol{\eta}(t)$
   (a) Evaluate their magnitudes at the excitation frequency ω_0.
   (b) Evaluate their phase delays at that frequency.
3. Write the analytical expression of the damped steady-state response in the form of Eq. (I.3).
4. Plot the components of $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ for $0 \leq t \leq 2$ s.
5. Explain the results from Question 4 (motivate the contribution from each mode).

Solution

Question (1)

From the Parallel Axis Theorem, we know that the MoI about C can be represented as $J_C = J_E - m(e/cos\theta)^2 \approx J_E - me^2$. Then the EOM of this problem becomes

$$\underbrace{\begin{bmatrix} m & -me \\ -me & J_E \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \ddot{h}_E(t) \\ \ddot{\theta}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix}}_{\mathbf{K}} \begin{bmatrix} h_E(t) \\ \theta(t) \end{bmatrix} = \underbrace{\begin{bmatrix} Q_{0_1} \\ Q_{0_2} \end{bmatrix}}_{\mathbf{Q}_0} \sin \omega_0 t. \tag{II.2}$$

Now we know that the modal shape matrix normalized by unit modal mass is

$$\mathbf{U} = \begin{bmatrix} -0.3136 & -0.1633 \\ -0.0648 & 3.9523 \end{bmatrix}.$$

Then the modal excitation becomes

$$\boxed{\mathbf{N}(t) = \mathbf{U}^\top \mathbf{Q}_0 \sin \omega_0 t = \begin{bmatrix} 3.0388 \\ 7.5612 \end{bmatrix} \sin(15t)} \tag{II.3}$$

Question (2)

The modal equations for this damped forced response takes the form of

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i + \omega_i^2\eta_i(t) = N_i(t).$$

The modal coordinates/equations can be expressed as

$$\eta_i(t) = \mathbf{u}_i^\top \mathbf{Q}_0 |H_i(\omega_0)| \sin\left[\omega_0 t - \theta_i(\omega_0)\right]. \tag{II.4}$$

The magnitude and phase can be found to be

$$|H_i(\omega_0)| = \left[(\omega_i^2 - \omega_0^2)^2 + (2\zeta_i\omega_0)^2\right]^{-0.5} \tag{II.5}$$

$$\theta_i(\omega_0) = \arctan\left(\frac{2\zeta_i\omega_0}{\omega_i^2 - \omega_0^2}\right) \tag{II.6}$$

Since, we know the natural frequencies for each mode, $\omega_1 = 9.9589$ and $\omega_2 = 56.1322$ we can compute the amplitudes and phase angles easily.

$$\boxed{\begin{aligned} |H_1| &= 0.0079 & \theta_1 &= 3.1368 \\ |H_2| &= 3.4178\text{e–}4 & \theta_2 &= 2.0507\text{e–}4 \end{aligned}} \tag{II.7}$$

Question (3)

The modal equations, $\eta(t)$ are

$$\eta_i(t) = \mathbf{u}_i^\top \mathbf{Q}_0 |H_i| \sin(\omega_0 t - \theta_i). \tag{II.8}$$

and the damped steady-state response becomes

$$\mathbf{q}(t) = \sum_{i=1}^{2} \mathbf{u}_i \eta_i(t). \tag{II.9}$$

$$\boxed{q(t) = \left[\begin{array}{c} -0.007574\,\sin\left(15.0\,t - 3.137\right) - 0.000422\,\sin\left(15.0\,t - 0.0002051\right) \\ 0.01021\,\sin\left(15.0\,t - 0.0002051\right) - 0.001564\,\sin\left(15.0\,t - 3.137\right) \end{array}\right]} \tag{II.10}$$

Question (4)

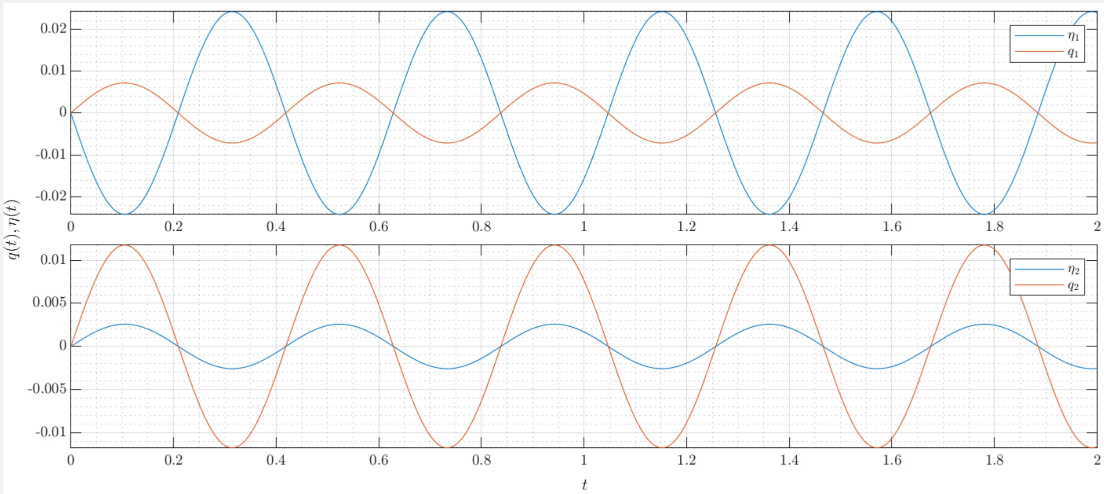


Figure 5: Responses $q(t)$ and modal functions $\eta(t)$.

This was plotted with the code in subsection IV.ii.

Question (5)

In Figure 5, the components of $\eta(t)$ and response $q(t)$ are visualized as blue and red curves respectively. For the first mode we can see that the modal coordinate oscillates with a larger amplitude than the actual response. Whereas in the second mode, the response is larger than the modal coordinate. This shows that the contribution of the second mode is larger than the contribution of the first mode for the modal analysis given the external force and damping of the system.

III Problem Three

Consider the same typical section model as in Problem 2. The model experiences the step excitation

$$\mathbf{Q}(t) = \mathbf{Q}_0 \mathbf{u}(t) \tag{III.1}$$

with $Q_{0_1} = -10$, $Q_{0_2} = 1.5$ Nm, and zero initial conditions. Damping is captured by the proportional model

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K} \tag{III.2}$$

where $\alpha = 1.0$ s^{-1} and $\beta = 1 \times 10^{-5}$ s. Answer the following questions:

1. Evaluate the modal viscous damping factors ζ_1 and ζ_2;
2. Evaluate the damped frequencies ω_{d_1} and ω_{d_2};
3. Write the analytical expression of the damped response in the form of Eq. I.3;
4. Plot the components of $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ for $0 \leq t \leq 10$ s;
5. Explain the results from Question 4 (motivate the contribution from each mode);
6. Obtain the results from Question 4 for $e = -0.05c$ and explain any qualitative changes.

Solution

Question (1)

The viscous damping factors are found by $\zeta_i = (\alpha + \beta\omega_i^2)/2\omega_i$ where $\omega_1 = 9.9589$ and $\omega_2 = 56.1322$, and therefore

$$\boxed{\begin{aligned} \zeta_1 &= 0.0503 \\ \zeta_2 &= 0.0092 \end{aligned}} \tag{III.3}$$

Question (2)

The damped frequencies are simply $\omega_{d_i} = \omega_i\sqrt{1-\zeta_i^2}$, and thus

$$\boxed{\omega_{d_1} = 9.9464, \qquad \omega_{d_2} = 56.1298} \tag{III.4}$$

Question (3)

For a damped forced response we know that the modal response becomes

$$\mathbf{q}(t) = \sum_{i=1}^{2} \mathbf{u}_i \left(\mathbf{u}_i^\top \int_0^t \mathbf{Q}(t) h_i(t-\tau) d\tau \right)$$

where

$$h_i(t) = \frac{e^{-\zeta_i\omega_i t}}{\omega_{d_i}} \sin\omega_{d_i}(t)$$

The equation above can be simplified to

$$\mathbf{q}(t) = \sum_{i=1}^{2} \mathbf{u}_i \left(\mathbf{u}_i^\top \mathbf{Q}_0 \int_0^t u(t) h_i(t-\tau) d\tau \right) = \sum_{i=1}^{2} \mathbf{u}_i \mathbf{u}_i^\top \mathbf{Q}_0 P(t) \tag{III.5}$$

where

$$P(t) = \frac{1}{\omega_i^2}\left[1 - e^{-\zeta_i \omega_i t}\left(\cos \omega_{d_i} t + \frac{\zeta_i}{\sqrt{1-\zeta_i^2}} \sin \omega_{d_i} t\right)\right]$$

Therefore, if we compute this in MATLAB we have

$$\mathbf{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \tag{III.6}$$

where

$$\begin{aligned} q_1(t) = \big(&0.009633\mathrm{e}^{-0.5005t}(\cos(9.946t) + 0.05032\sin(9.946t)) \\ &+0.0003919\mathrm{e}^{-0.5158t}(\cos(56.13t) + 0.009189\sin(56.13t)) - 0.01002\big) \end{aligned}$$

$$\begin{aligned} q_2(t) = \big(&0.00199\mathrm{e}^{-0.5005t}(\cos(9.946t) + 0.05032\sin(9.946t)) \\ &-0.009485\mathrm{e}^{-0.5158t}(\cos(56.13t) + 0.009189\sin(56.13t)) + 0.007496\big) \end{aligned}$$

Or expressed using the modal functions $\eta_i(t)$ as

$$\boxed{\mathbf{q}(t) = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix} \begin{bmatrix} 0.03072 - 0.03072\mathrm{e}^{-0.5005t}(\cos(9.946t) + 0.05032\sin(9.946t)) \\ 0.0024 - 0.0024\mathrm{e}^{-0.5158t}(\cos(56.13t) + 0.009189\sin(56.13t)) \end{bmatrix}} \tag{III.7}$$

Question (4)

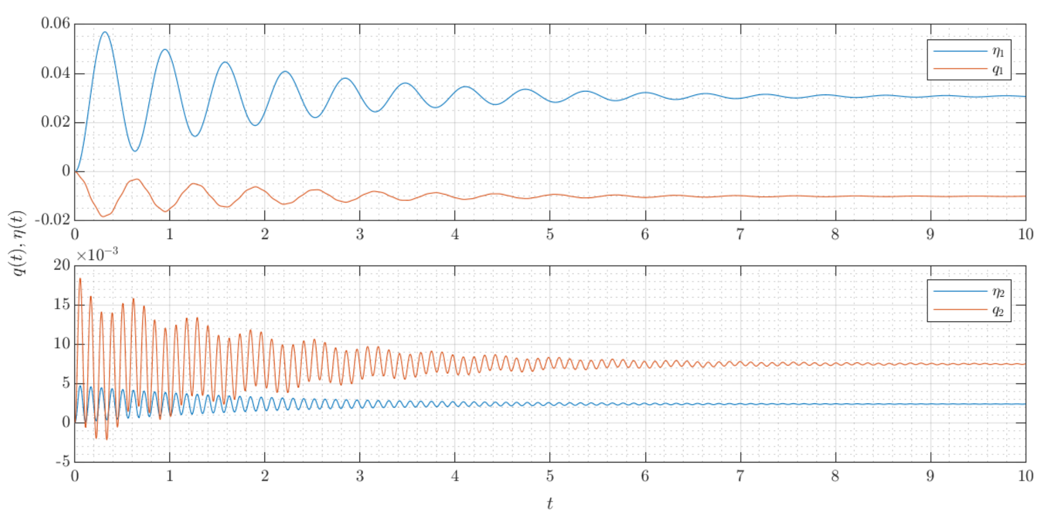


Figure 6: $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ responses for the damped forced MDOF system.

Question (5)

From the modal function η we can see that the vertical displacement corresponding to the first mode oscillates with a lower frequency compared to the second mode and the displacement is always negative. This agrees with how the first element of the modal shapes are both negative. The motion of θ oscillates with a much higher frequency but in a very small angle (which agrees with the small angle approximation). For the second mode the response $q(t)$ seems to be larger than the modal response η meaning that the second mode's contribution is larger for this response compared to the first mode where the response is smaller than the modal response as observed in Figure 6.

Question (6)

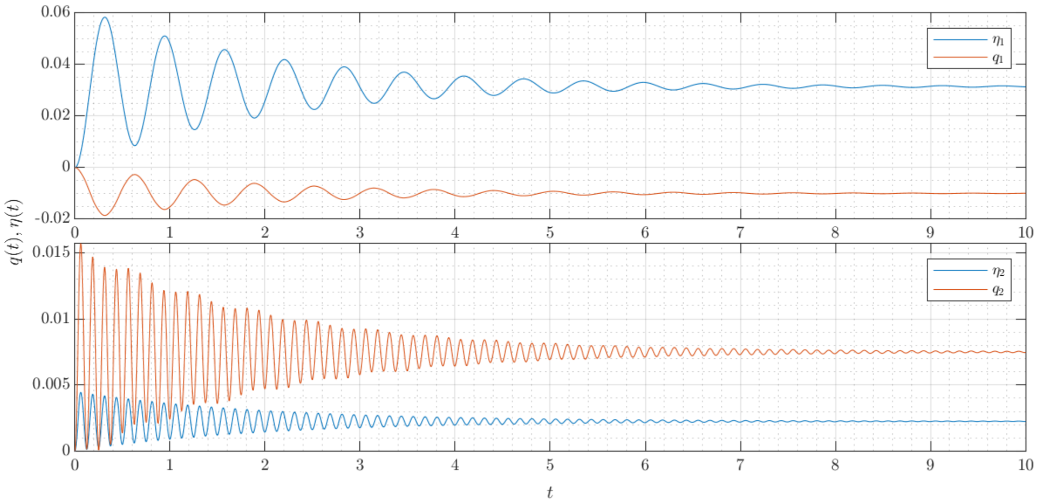


Figure 7: $\mathbf{q}(t)$ and $\boldsymbol{\eta}(t)$ responses for the damped forced MDOF system.

For this we can see a large increase of the amplitude of the second modal response and $q_2(t)$. This agrees to the fact that the moment becomes larger as the distance of e increases and becomes farther from the center of mass. By increasing the e distance we can see that the contribution of the second mode becomes larger.

IV MATLAB Code

IV.i Problem 1

```matlab
% AE6230 HW3 Problem 1
% Author: Tomoki Koike

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
sympref('FloatingPointOutput', false);  % fractions in symbolic

%% Setup
mw = 750;  % [kg]
mf = 5*mw;
l = 10;  % [m]
EI = 5e+6;  % [N-m^2]

%% (a)
% System matrices
M = diag([mf mw mw]);
K = 3*EI/l^3 * [2 -1 -1; -1 1 0; -1 0 1];
[m,n] = size(M);

% Compute the eigenvalues and the eigenvectors
[Uhat,Lambda] = eig(K,M);
omega = sqrt(diag(Lambda));
omega(1) = 0;

%% (b)
U = zeros(size(Uhat));

for i = 1:n
  U(:,i) = Uhat(:,i)/norm(Uhat(:,i),"inf");
end

%% (c)

fig = figure(Renderer="painters",Position=[60 60 900 400]);
  plot([-l l],[0 0],'-k')
  hold on; grid on; grid minor; box on;
  plot(0,0,'or',MarkerSize=30)
  plot(-l,0,'og',MarkerSize=15)
  plot(l,0,'ob',MarkerSize=15)
  arrow([0 0],[0 U(1,1)],'EdgeColor', 'r', 'FaceColor', 'r')
  arrow([-l 0],[-l U(2,1)],'EdgeColor', 'g', 'FaceColor', 'g')
  arrow([l 0],[l U(3,1)],'EdgeColor', 'b', 'FaceColor', 'b')
  plot([-l 0 l], [U(2,1) U(1,1) U(3,1)], '--k')
  hold off;
  xlabel('$x$')
  ylabel('$y$')
  xlim([-l-1,l+1])
```

```
  ylim([-5,5])
saveas(fig,"plots/p1/modeshape1.png")

fig = figure(Renderer="painters",Position=[60 60 900 400]);
  plot([-l l],[0 0],'-k')
  hold on; grid on; grid minor; box on;
  plot(0,0,'or',MarkerSize=30)
  plot(-l,0,'og',MarkerSize=15)
  plot(l,0,'ob',MarkerSize=15)
  arrow([0 0],[0 U(1,2)],'EdgeColor', 'r', 'FaceColor', 'r')
  arrow([-l 0],[-l U(2,2)],'EdgeColor', 'g', 'FaceColor', 'g')
  arrow([l 0],[l U(3,2)],'EdgeColor', 'b', 'FaceColor', 'b')
  plot([-l 0 l], [U(2,2) U(1,2) U(3,2)], '--k')
  hold off;
  xlabel('$x$')
  ylabel('$y$')
  xlim([-l-1,l+1])
  ylim([-5,5])
saveas(fig,"plots/p1/modeshape2.png")

fig = figure(Renderer="painters",Position=[60 60 900 400]);
  plot([-l l],[0 0],'-k')
  hold on; grid on; grid minor; box on;
  plot(0,0,'or',MarkerSize=30)
  plot(-l,0,'og',MarkerSize=15)
  plot(l,0,'ob',MarkerSize=15)
  arrow([0 0],[0 U(1,3)],'EdgeColor', 'r', 'FaceColor', 'r')
  arrow([-l 0],[-l U(2,3)],'EdgeColor', 'g', 'FaceColor', 'g')
  arrow([l 0],[l U(3,3)],'EdgeColor', 'b', 'FaceColor', 'b')
  plot([-l 0 l], [U(2,3) U(1,3) U(3,3)], '--k')
  hold off;
  xlabel('$x$')
  ylabel('$y$')
  xlim([-l-1,l+1])
  ylim([-5,5])
saveas(fig,"plots/p1/modeshape3.png")

%% (4)
Uinv = inv(U);

%% (5)
q0 = [0.5;0;0];
qdot0 = [0;0;0];
eta0 = Uinv * q0;
eta0(2) = 0;
etadot0 = Uinv * qdot0;

% Verify
syms t positive real
syms A_1 A_c2 A_c3 B_1 A_s2 A_s3 real

eta1(t) = A_1 * t + B_1;
eta2(t) = A_c2 * cos(omega(2)*t) + A_s2 * sin(omega(2)*t);
eta3(t) = A_c3 * cos(omega(3)*t) + A_s3 * sin(omega(3)*t);
```

```matlab
q(t) = U(:,1)*eta1 + U(:,2)*eta2 + U(:,3)*eta3;
qdot(t) = diff(q,t);
eqn1 = q(0) == [0.5; 0; 0];
eqn2 = qdot(0) == [0; 0; 0];
ic_sol = solve([eqn1 eqn2], [A_1 A_c2 A_c3 B_1 A_s2 A_s3]);

Ac2 = sign(ic_sol.A_c2)*sqrt(ic_sol.A_c2^2+ic_sol.A_s2^2);
phi2 = atan2(-ic_sol.A_s2, ic_sol.A_c2);
Ac3 = sign(ic_sol.A_c3)*sqrt(ic_sol.A_c3^2+ic_sol.A_s3^2);
phi3 = atan2(-ic_sol.A_s3, ic_sol.A_c3);

%% (6)
zeta = [0; 0.04; 0.04];
omegad = omega .* sqrt(1 - zeta);
eta = @(t,z,w,wd,e0,ed0) exp(-z*w*t)*(e0*cos(wd*t)+(ed0+z*w*e0)/wd*sin(wd*t));
eta1(t) = 5/14;
eta2(t) = eta(t,zeta(2),omega(2),omegad(2),eta0(2),etadot0(2));
eta3(t) = eta(t,zeta(3),omega(3),omegad(3),eta0(3),etadot0(3));

% q1(t) = U(1,1)*eta1 + U(1,2)*eta2 + U(1,3)*eta3;
% q2(t) = U(2,1)*eta1 + U(2,2)*eta2 + U(2,3)*eta3;
% q3(t) = U(3,1)*eta1 + U(3,2)*eta2 + U(3,3)*eta3;
q(t) = U(:,1)*eta1 + U(:,2)*eta2 + U(:,3)*eta3;
q = matlabFunction(q);

%% (7)
tspan = 0:0.01:20;
qval = q(tspan);
fig = figure(Renderer="painters",Position=[60 60 900 400]);
t = tiledlayout(3,1,TileSpacing="tight",Padding="tight");
  nexttile(1);
  plot(tspan,eta1(tspan),DisplayName="$\eta_1$")
  hold on; grid on; grid minor; box on;
  plot(tspan,qval(1,:),DisplayName="$q_1$")
  hold off; legend;

  nexttile(2);
  plot(tspan,eta2(tspan),DisplayName="$\eta_2$")
  hold on; grid on; grid minor; box on;
  plot(tspan,qval(2,:),DisplayName="$q_2$")
  hold off; legend;

  nexttile(3);
  plot(tspan,eta3(tspan),DisplayName="$\eta_3$")
  hold on; grid on; grid minor; box on;
  plot(tspan,qval(3,:),DisplayName="$q_3$")
  hold off; legend;

xlabel(t,'$t$','FontSize',10,'Interpreter','latex')
ylabel(t,'$q(t), \eta(t)$','FontSize',10,'Interpreter','latex')
saveas(fig,"plots/p1/response.png")
```

IV.ii Problem 2 3

```matlab
% AE6230 HW3 Problem 2 & 3
% Author: Tomoki Koike
% Reference: Dr. Cristina Riso "AE6230_Fall2022_L17_MDOF_Free_TypicalSection.m"

%% Housekeeping commands
clear; close all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
sympref('FloatingPointOutput', false);  % fractions in symbolic

%% Setup
m = 10.0;  % [kg]
J_E = 0.08;  % [kg-m^2]
c = 0.2;  % [m]
e = -0.2*c;  % offset of C from E (m, positive ahead) [m]
x_E = 0.4*c;  % position of E from the LE (m, positive backward) [m]
k_h = 1000.0;  % [N/m]
k_theta = 200.0;  % [N-m/rad]

% initial displacements (m and rad)
q0 = [-0.02; deg2rad(5.0)];

% initial velocities (m/s and rad/s)
qdot0 = zeros(2,1);

M = [m -m*e; -m*e J_E];  % inertia matrix
K = [k_h 0.0; 0.0 k_theta];  % stiffness matrix
[~,n] = size(M);

% Excitations
Q0 = [-10; 1.5];
omega0 = 15;  % [rad/s]
syms t real positive
Q = Q0 * sin(omega0 * t);

% modal viscous damping factors
zeta = [0.02; 0.02];


%% Problem 2
%% (1)
% compute eigenvalues and eigenvectors
[U, Lambda] = eig(K,M);
lambda = diag(Lambda);

% compute natural frequencies
omega = sqrt(lambda);

% Eigenvector normalization to have unit modal mass
for i = 1:n
    U(:,i) = U(:,i)/sqrt(U(:,i)'*M*U(:,i));
```

```
end

% modal mass matrix (must be an identity matrix with this normalization)
Mbar = U'*M*U;

% modal stiffness matrix (must be the Omega2 matrix with this normalization)
Kbar = U'*K*U;

% modal matrix inverse
Uinv = U'*M;

% initial modal displacements
eta0 = Uinv*q0;

% initial modal velocities
etadot0 = Uinv*qdot0;

% Compute the modal excitation
N(t) = U.' * Q;

%% (2)
Hamp = @(wi,w0,zi) 1/sqrt((wi^2 - w0^2)^2 + 4*zi^2*w0^2);
phang = @(wi,w0,zi) atan2(2*zi*w0, wi^2 - w0^2);

H1 = Hamp(omega(1),omega0,zeta(1));
H2 = Hamp(omega(2),omega0,zeta(2));
theta1 = phang(omega(1),omega0,zeta(1));
theta2 = phang(omega(2),omega0,zeta(2));

%% (3)
eta1(t) = U(:,1).' * Q0 * H1 * sin(omega0*t - theta1);
eta2(t) = U(:,2).' * Q0 * H2 * sin(omega0*t - theta2);
q(t) = U(:,1)*eta1 + U(:,2)*eta2;
q = matlabFunction(q);

%% (4)
tspan = 0:0.001:2;
qval = q(tspan);
fig = figure(Renderer="painters",Position=[60 60 900 400]);
tile = tiledlayout(2,1,TileSpacing="tight",Padding="tight");
  nexttile(1);
  plot(tspan,eta1(tspan),DisplayName="$\eta_1$")
  hold on; grid on; grid minor; box on;
  plot(tspan,qval(1,:),DisplayName="$q_1$")
  hold off; legend;

  nexttile(2);
  plot(tspan,eta2(tspan),DisplayName="$\eta_2$")
  hold on; grid on; grid minor; box on;
  plot(tspan,qval(2,:),DisplayName="$q_2$")
  hold off; legend;

xlabel(tile,'$t$','FontSize',10,'Interpreter','latex')
ylabel(tile,'$q(t), \eta(t)$','FontSize',10,'Interpreter','latex')
```

```
saveas(fig,"plots/p2/response.png")

%% Problem 3
%% (1)
alpha = 1.0;
beta = 1e-5;
zeta = (alpha + beta*omega.^2)/2./omega;

%% (2)
omegad = omega .* sqrt(1 - zeta.^2);

%% (3)
Pt = @(t,w,wd,z) (1 - exp(-z*w*t)*(cos(wd*t) + z/sqrt(1-z^2)*sin(wd*t)))/wd^2;
eta21(t) = U(:,1).'*Q0*Pt(t,omega(1),omegad(1),zeta(1));
eta22(t) = U(:,2).'*Q0*Pt(t,omega(2),omegad(2),zeta(2));
q2(t) = U(:,1)*eta21(t) + U(:,2)*eta22(t);
q2 = matlabFunction(q2);

%% (4)
tspan = 0:0.001:10;
q2val = q2(tspan);
fig = figure(Renderer="painters",Position=[60 60 900 400]);
tile = tiledlayout(2,1,TileSpacing="tight");
  nexttile(1);
  plot(tspan,eta21(tspan),DisplayName="$\eta_1$")
  hold on; grid on; grid minor; box on;
  plot(tspan,q2val(1,:),DisplayName="$q_1$")
  hold off; legend;

  nexttile(2);
  plot(tspan,eta22(tspan),DisplayName="$\eta_2$")
  hold on; grid on; grid minor; box on;
  plot(tspan,q2val(2,:),DisplayName="$q_2$")
  hold off; legend;

xlabel(tile,'$t$','FontSize',10,'Interpreter','latex')
ylabel(tile,'$q(t), \eta(t)$','FontSize',10,'Interpreter','latex')
saveas(fig,"plots/p3/response1.png")

%% (5)
e = -0.05*c;  % offset of C from E (m, positive ahead) [m]

M = [m -m*e; -m*e J_E];  % inertia matrix
K = [k_h 0.0; 0.0 k_theta];  % stiffness matrix

% compute eigenvalues and eigenvectors
[U, Lambda] = eig(K,M);
lambda = diag(Lambda);

% compute natural frequencies
omega = sqrt(lambda);

% Eigenvector normalization to have unit modal mass
for i = 1:n
```

```
    U(:,i) = U(:,i)/sqrt(U(:,i)'*M*U(:,i));
end

% modal mass matrix (must be an identity matrix with this normalization)
Mbar = U'*M*U;

% modal stiffness matrix (must be the Omega2 matrix with this normalization)
Kbar = U'*K*U;

% modal matrix inverse
Uinv = U'*M;

% initial modal displacements
eta0 = Uinv*q0;

% initial modal velocities
etadot0 = Uinv*qdot0;

zeta = (alpha + beta*omega.^2)/2./omega;
omegad = omega .* sqrt(1 - zeta.^2);

eta31(t) = U(:,1).'*Q0*Pt(t,omega(1),omegad(1),zeta(1));
eta32(t) = U(:,2).'*Q0*Pt(t,omega(2),omegad(2),zeta(2));
q3(t) = U(:,1)*eta31(t) + U(:,2)*eta32(t);
q3 = matlabFunction(q3);

q3val = q3(tspan);
fig = figure(Renderer="painters",Position=[60 60 900 400]);
tile = tiledlayout(2,1,TileSpacing="tight");
  nexttile(1);
  plot(tspan,eta31(tspan),DisplayName="$\eta_1$")
  hold on; grid on; grid minor; box on;
  plot(tspan,q3val(1,:),DisplayName="$q_1$")
  hold off; legend;

  nexttile(2);
  plot(tspan,eta32(tspan),DisplayName="$\eta_2$")
  hold on; grid on; grid minor; box on;
  plot(tspan,q3val(2,:),DisplayName="$q_2$")
  hold off; legend;

xlabel(tile,'$t$','FontSize',10,'Interpreter','latex')
ylabel(tile,'$q(t), \eta(t)$','FontSize',10,'Interpreter','latex')
saveas(fig,"plots/p3/response2.png")
```