College of Engineering
Daniel Guggenheim School of Aerospace Engineering

AE6210: Advanced Dynamics I

Homework 4

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March 4, 2022

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I Instructions

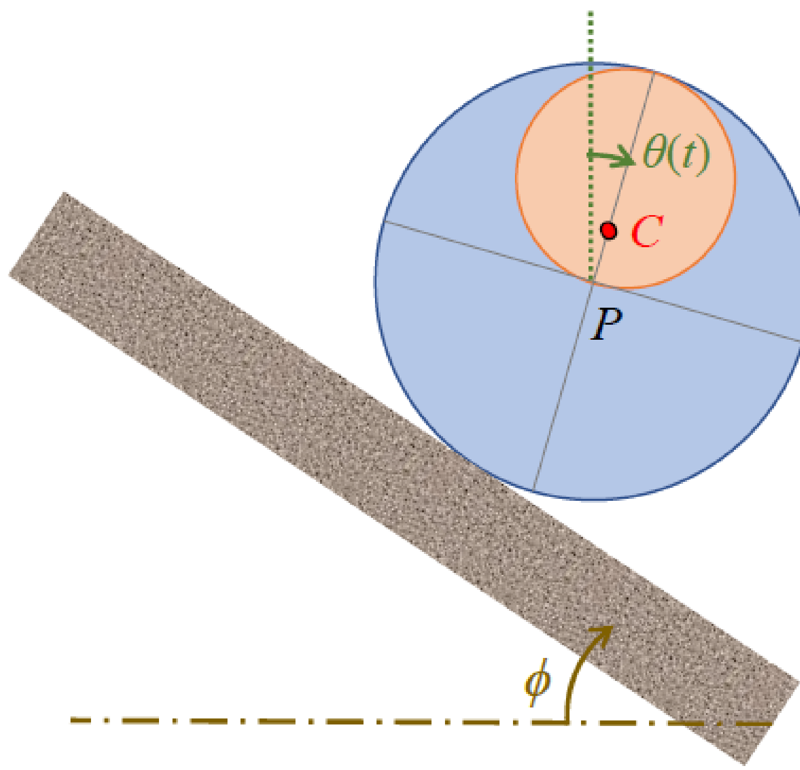

Figure 1: Rolling cylinder diagram.

Consider a circular cylinder of radius R made up of two materials – blue material and red material – on an inclined plane as shown below. The red cylinder has a radius of $R/2$ and the density of the red material is 5 times the density of the blue cylinder. The total mass of the cylinder is m.

II Problem One

Calculate the location of the center of mass (CoM), the moment of inertia (MoI) about the center of mass C, and the moment of inertia about the geometric center P in terms of m and R

Solution:

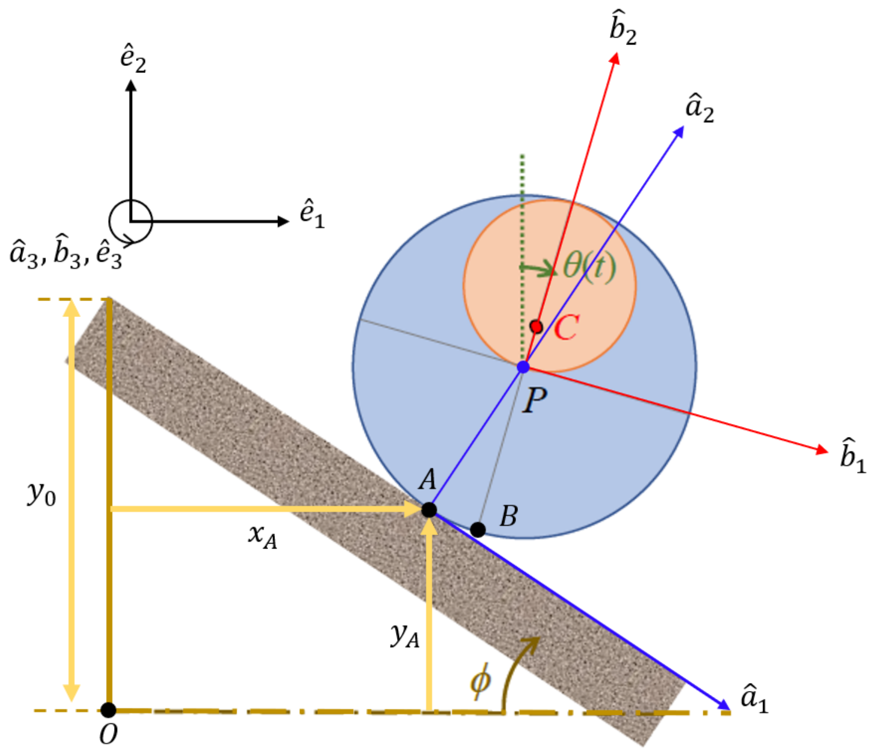

Figure 2: Problem diagram with coordinate systems.

We first define the coordinate systems of the problem as in Figure 2. The A-, B-, and E-frame represent the slope, body, and inertial frame respectively. Point O indicates the contact point of the body with the slope. The angular velocity for this system is then

$$^E\vec{\omega}^B = {^E\vec{\omega}^A} + {^A\vec{\omega}^B} = 0 + \dot{\theta}(-\hat{\mathbf{e}}_3) = -\dot{\theta}\hat{\mathbf{e}}_3 \tag{II.1}$$

The relation between each frame is as follows (shorthand notation for sine and cosine)

$$\begin{bmatrix}\hat{\mathbf{e}}_1\\\hat{\mathbf{e}}_2\\\hat{\mathbf{e}}_3\end{bmatrix} = \begin{bmatrix}c_\phi & s_\phi & 0\\-s_\phi & c_\phi & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{a}}_1\\\hat{\mathbf{a}}_2\\\hat{\mathbf{a}}_3\end{bmatrix} \tag{II.2}$$

$$\begin{bmatrix}\hat{\mathbf{e}}_1\\\hat{\mathbf{e}}_2\\\hat{\mathbf{e}}_3\end{bmatrix} = \begin{bmatrix}c_\theta & s_\theta & 0\\-s_\theta & c_\theta & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{b}}_1\\\hat{\mathbf{b}}_2\\\hat{\mathbf{b}}_3\end{bmatrix} \tag{II.3}$$

Let the CoG for blue and red part of the cylinder located on the line OP be points C_b and C_r respectively. Also let the area of the total cylinder, blue part, and red part be A_t, A_b, and A_r respectively.

$$A_t = \pi R^2, \quad A_b = \frac{3}{4}\pi R^2, \quad A_r = \frac{1}{4}\pi R^2. \tag{II.4}$$

Then the CoG of the blue part becomes

$$\overrightarrow{AC_b} = \frac{A_t R - A_r R/2}{A_b}\hat{\mathbf{b}}_2 = \frac{\pi R^3 - \pi R^3/8}{3\pi R/4}\hat{\mathbf{b}}_2 = \frac{7}{6}R\hat{\mathbf{b}}_2. \tag{II.5}$$

If we rewrite this with respect to point O, then we have

$$\overrightarrow{AC_b} = \overrightarrow{AP} + \overrightarrow{PC_b} = R\hat{\mathbf{a}}_2 + \overrightarrow{BC_b} - \overrightarrow{BP} = R\hat{\mathbf{a}}_2 + \frac{7}{6}R\hat{\mathbf{b}}_2 - R\hat{\mathbf{b}}_2 = R\hat{\mathbf{a}}_2 + \frac{1}{6}R\hat{\mathbf{b}}_2. \tag{II.6}$$

On the other hand, the CoG for the red part is

$$\overrightarrow{AC_r} = \overrightarrow{AP} + \frac{R}{2}\hat{\mathbf{b}}_2 = R\hat{\mathbf{a}}_2 + \frac{R}{2}\hat{\mathbf{b}}_2. \tag{II.7}$$

From (II.2) and (II.3) we know that

$$\begin{bmatrix}\hat{\mathbf{b}}_1\\ \hat{\mathbf{b}}_2\\ \hat{\mathbf{b}}_3\end{bmatrix} = \begin{bmatrix} c_\theta c_\phi + s_\theta s_\phi & c_\theta s_\phi - s_\theta c_\phi & 0\\ s_\theta c_\phi - c_\theta s_\phi & s_\theta s_\phi + c_\theta c_\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{a}}_1\\ \hat{\mathbf{a}}_2\\ \hat{\mathbf{a}}_3\end{bmatrix} = \begin{bmatrix}\sigma_{11} & \sigma_{12} & 0\\ \sigma_{21} & \sigma_{22} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{\mathbf{a}}_1\\ \hat{\mathbf{a}}_2\\ \hat{\mathbf{a}}_3\end{bmatrix}. \tag{II.8}$$

Hence, we can convert (II.6) and (II.7) to the A-frame which becomes

$$\overrightarrow{AC_b} = \frac{R}{6}\sigma_{21}\hat{\mathbf{a}}_1 + \frac{R}{6}\left(\sigma_{22}+6\right)\hat{\mathbf{a}}_2. \tag{II.9}$$

$$\overrightarrow{AC_r} = \frac{R}{2}\sigma_{21}\hat{\mathbf{a}}_1 + \frac{R}{2}(\sigma_{22}+2)\hat{\mathbf{a}}_2. \tag{II.10}$$

Let the densities of the blue part and red part be ρ_b, $\rho_r = 5\rho_b$ respectively, and denote the thickness of the cylinder as l. Then the CoG of the entire cylinder becomes

$$\begin{aligned}\overrightarrow{AC} &= \frac{(\rho_b A_b l)\overrightarrow{AC_b} + (\rho_r A_r l)\overrightarrow{AC_r}}{\rho_b A_b l + \rho_r A_r l} = \frac{A_b\overrightarrow{AC_b} + 5A_r\overrightarrow{AC_r}}{A_b + 5A_r}\\ &= \frac{(3\pi R^2/4)\overrightarrow{AC_b} + 5(\pi R^2/4)\overrightarrow{AC_r}}{3\pi R^2/4 + 5(\pi R^2/4)} = \frac{3\overrightarrow{AC_b} + 5\overrightarrow{AC_r}}{8}\\ &= \frac{3}{8}\left(\frac{R}{6}\sigma_{21}\hat{\mathbf{a}}_1 + \frac{R}{6}\left(\sigma_{22}+6\right)\hat{\mathbf{a}}_2\right) + \frac{5}{8}\left(\frac{R}{2}\sigma_{21}\hat{\mathbf{a}}_1 + \frac{R}{2}(\sigma_{22}+2)\hat{\mathbf{a}}_2\right)\\ &= \frac{3}{8}R\sigma_{21}\hat{\mathbf{a}}_1 + \left(\frac{3}{8}R\sigma_{22} + R\right)\hat{\mathbf{a}}_2\end{aligned}$$

The center of gravity in the A-frame is expressed as

$$\boxed{\overrightarrow{AC} = \frac{3}{8}R\sigma_{21}\hat{\mathbf{a}}_1 + \left(\frac{3}{8}R\sigma_{22} + R\right)\hat{\mathbf{a}}_2 = R\hat{\mathbf{a}}_2 + \frac{3}{8}R\hat{\mathbf{b}}_2.} \tag{II.11}$$

From what we have so far we know that the masses of the blue and red part correspond to

$$m_b = \frac{3}{8}m, \quad m_r = \frac{5}{8}m \tag{II.12}$$

To find the MoI with respect to the CoG, C and geometric center (GC) point, P we will have to find the MoI for the blue part and the red part separately. First, we will obtain the MoIs for the blue part. Since, we are only analyzing this in 2D we can consider the cylinder to be equivalent to a disk. Now, to find the MoI of only the blue part we have to find the MoI for when we have a complete disk of radius R with the blue material and create a hole corresponding to the red part with a radius of $R/2$. The hypothetical disk with radius R consisting of only the blue material would have a mass of $m_b' = m/2$ from the area ratio. Then we can compute the MoI of the blue part with respect to point P to be

$$\begin{aligned}
I_b^P &= \frac{1}{2}m_b'R^2 - \left[\frac{1}{2}\left(m_b' - m_b\right)\left(\frac{R}{2}\right)^2 + \left(m_b' - m_b\right)\left(\frac{R}{2}\right)\right] \\
&= \frac{mR^2}{4} - \left(\frac{mR^2}{64} + \frac{mR^2}{32}\right) \\
&= \frac{5}{64}mR^2.
\end{aligned} \tag{II.13}$$

Next, we find the MoI for the red part. This is straightforward. With respect to point P it is

$$I_r^P = \frac{1}{2}m_r\frac{R^2}{4} + m_r \cdot \frac{R^2}{4} = \frac{5}{64}mR^2 + \frac{5}{32}mR^2 = \frac{15}{64}mR^2. \tag{II.14}$$

Finally, the MoIs for whole body with respect to point P becomes

$$I^P = I_b^P + I_r^P = \frac{5}{64}mR^2 + \frac{15}{64}mR^2 = \frac{5}{16}mR^2 \tag{II.15}$$

The MoI of the body with respect to the CoG can be computed using the parallel axis theorem

$$\begin{aligned}
I^C = I^P - m|\overrightarrow{CP}| &= \frac{5}{16}mR^2 - m\left[\left(\frac{3}{8}R\right)^2\left(\sigma_{21}^2 + \sigma_{22}^2\right)\right] \\
&= \frac{5}{16}mR^2 - m \cdot \frac{9}{64}R^2 = \frac{11}{64}mR^2.
\end{aligned} \tag{II.16}$$

In conclusion, the MoIs are

$$\boxed{I^C = \frac{11}{64}mR^2, \quad I^P = \frac{5}{16}mR^2.} \tag{II.17}$$

III Problem Two

If the surfaces are frictionless, derive the equations of motion for the motion of the cylinder.

Solution:
First, note that

$$\dot{\sigma}_{21} = \dot{\theta}(c_\theta s_\phi - s_\theta c_\phi) = -\dot{\theta}\sigma_{21} \tag{III.1}$$
$$\dot{\sigma}_{22} = \dot{\theta}(c_\theta c_\phi + s_\theta s_\phi) = \dot{\theta}\sigma_{22} \tag{III.2}$$
$$\tag{III.3}$$

and

$$\frac{\partial}{\partial\theta}\sigma_{21} = \sigma_{22} \tag{III.4}$$
$$\frac{\partial}{\partial\theta}\sigma_{22} = -\sigma_{21} \tag{III.5}$$

and let

$$|\overrightarrow{CP}| = \frac{3}{8}R = \gamma R \tag{III.6}$$

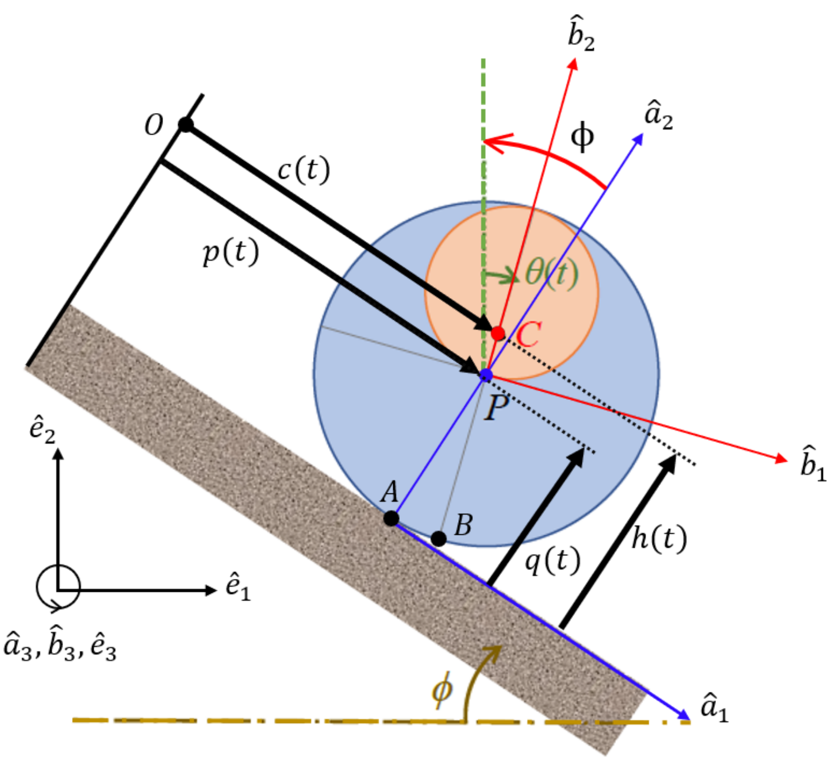

Figure 3: Diagram with generalized coordinate representations.

Defining the generalized coordinates as $c(t)$, $p(t)$, and $\theta(t)$ as above, we will find the equations of motion for when the slope is frictionless using the Lagrange's equation. The kinetic energy of the center of mass C

becomes

$$
\begin{aligned}
T &= \frac{m}{2}(\dot{c}^2 + \dot{h}^2) + \frac{1}{2}I^c\dot{\theta}^2 \\
&= \frac{m}{2}\left(\dot{c}^2 + \gamma^2 R^2 \sigma_{21}^2 \dot{\theta}^2\right) + \frac{1}{2}I^c\dot{\theta}^2 \qquad \because h = \gamma R \sigma_{22} + R.
\end{aligned} \tag{III.7}
$$

Next, the potential energy can be expressed as $U = -mgz$ where z is the vertical coordinate of the center of gravity. Then,

$$
U = -mgz = -mgcs_\phi. \tag{III.8}
$$

The Lagrangian is

$$
L(c,\theta) = T - U = \frac{m}{2}\left(\dot{c}^2 + \gamma^2 R^2 \sigma_{21}^2 \dot{\theta}^2\right) + \frac{1}{2}I^c\dot{\theta}^2 + mgcs_\phi. \tag{III.9}
$$

The first equation corresponding to the generalized coordinate c is obtained in the following procedure,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{c}} &= m\dot{c} \\
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{c}}\right) &= m\ddot{c} \\
\frac{\partial L}{\partial c} &= mgs_\phi,
\end{aligned}
$$

and therefore,

$$
\frac{\partial}{\partial}\left(\frac{\partial L}{\partial \dot{c}}\right) - \frac{\partial L}{\partial c} = 0
$$

$$
\ddot{c} - gs_\phi = 0. \tag{III.10}
$$

Similarly,

$$
\frac{\partial L}{\partial \dot{\theta}} = m\gamma^2 R^2 \sigma_{21}^2 \dot{\theta} + I^c\dot{\theta}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) &= 2m\gamma^2 R^2 \sigma_{21}\dot{\sigma}_{21}\dot{\theta} + m\gamma^2 R^2 \sigma_{21}^2 \ddot{\theta} + I^c\ddot{\theta} \\
&= 2m\gamma^2 R^2 \sigma_{21}\sigma_{22}\dot{\theta}^2 + \left(I^c + m\gamma^2 R^2 \sigma_{21}^2\right)\ddot{\theta},
\end{aligned} \tag{III.11}
$$

and

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= \frac{m}{2}\left(\gamma^2 R^2 \dot{\theta}^2 \frac{\partial}{\partial \theta}\sigma_{21}^2\right) \\
&= m\gamma^2 R^2 \dot{\theta}^2 \sigma_{21}\sigma_{22}.
\end{aligned} \tag{III.12}
$$

Therefore, the equation of motion for $\theta(t)$ becomes

$$
(I^c + m\gamma^2 R^2 \sigma_{21}^2)\ddot{\theta} + m\gamma^2 R^2 \sigma_{21}\sigma_{22}\dot{\theta}^2 = 0 \tag{III.13}
$$

where

$$
\sigma_{21} = s_\theta c_\phi - c_\theta s_\phi, \quad \sigma_{22} = s_\theta s_\phi + c_\theta c_\phi, \quad I^c = \frac{11}{64}mR^2. \tag{III.14}
$$

Now, to rewrite this in terms of point P, we use the relation of $c = p + \gamma R\sigma_{21}$. With this, we have

$$\dot{c} = \dot{p} + \gamma R\dot{\sigma}_{21} = \dot{p} + \gamma R\dot{\theta}\sigma_{22}$$
$$\ddot{c} = \ddot{p} + \gamma R\ddot{\theta}\sigma_{21} - \gamma R\dot{\theta}^2\sigma_{21}$$

Then the Lagrangian (III.9) can be rewritten as

$$\begin{aligned} L(p,\theta) = \frac{m}{2}\left[\left(\dot{p} + \gamma R\sigma_{22}\dot{\theta}\right)^2 + \gamma^2R^2\sigma_{21}^2\dot{\theta}^2\right] + \frac{1}{2}I^c\dot{\theta}^2 + mg(p + \gamma R\sigma_{21})s_\phi = 0 \\ \frac{m}{2}\left(\dot{p}^2 + 2\gamma R\sigma_{22}\dot{p}\dot{\theta} + \gamma^2R^2\dot{\theta}^2\right) + \frac{1}{2}I^c\dot{\theta}^2 + mgs_\phi p + mg\gamma Rs_\phi\sigma_{21} = 0. \end{aligned} \tag{III.15}$$

and hence the equation of motion for the generalized coordinate $p(t)$ becomes

$$\ddot{p} + \gamma R\sigma_{22}\ddot{\theta} - \gamma R\sigma_{21}\dot{\theta}^2 - gs_\phi = 0. \tag{III.16}$$

and for which the equation of motion for $\theta(t)$ in terms of point P becomes

$$(I^c + m\gamma^2R^2)\ddot{\theta} + m\gamma R\sigma_{22}\ddot{p} - mg\gamma Rs_\phi\sigma_{22} = 0. \tag{III.17}$$

In summary, the EOM for CoG becomes

$$\boxed{\begin{aligned} &\ddot{c} - gs_\phi = 0 \\ &(I^c + m\gamma^2R^2\sigma_{21}^2)\ddot{\theta} + m\gamma^2R^2\sigma_{21}\sigma_{22}\dot{\theta}^2 = 0 \end{aligned}} \tag{III.18}$$

On the other hand, for the GC the EOM is

$$\boxed{\begin{aligned} &\ddot{p} + \gamma R\sigma_{22}\ddot{\theta} - \gamma R\sigma_{21}\dot{\theta}^2 - gs_\phi = 0 \\ &(I^c + m\gamma^2R^2)\ddot{\theta} + m\gamma R\sigma_{22}\ddot{p} - mg\gamma Rs_\phi\sigma_{22} = 0 \end{aligned}} \tag{III.19}$$

Also we will consider the possibility of the cylinder/disk not having any contact with the incline due to some bounce. The body will lose contact with the incline when the centrifugal force normal to the incline, which is

$$\begin{aligned} F_c &= m\dot{\vec{\theta}} \times \left(\dot{\vec{\theta}} \times \gamma R\hat{\mathbf{b}}_2\right) \\ &= m\dot{\theta}(-\hat{\mathbf{b}}_3) \times \left(\dot{\theta}\left(-\hat{\mathbf{b}}_3\right) \times \gamma R\hat{\mathbf{b}}_2\right) \\ &= m\gamma R\dot{\theta}^2\hat{\mathbf{b}}_2 = m\gamma R\dot{\theta}^2\left(\sigma_{21}\hat{\mathbf{a}}_1 + \sigma_{22}\hat{\mathbf{a}}_2\right), \end{aligned} \tag{III.20}$$

is greater or equal to the normal component of the total cylinder weight. That is,

$$\begin{aligned} m\gamma R\sigma_{22}\dot{\theta}^2 \geq mgc_\phi \\ \gamma R\sigma_{22}\dot{\theta}^2 - gc_\phi \geq 0. \end{aligned} \tag{III.21}$$

Now when solving the EOM we must consider two variations for the variable $h(t)$ which is as follows.

$$\ddot{h}(t) = \begin{cases} -\gamma R\sigma_{22}\dot{\theta}^2 & \text{if} \quad F_c < N \\ -gc_\phi & \text{if} \quad F_c \geq N \end{cases} \tag{III.22}$$

where N is the normal force from the incline to the body. There is a constraint of $h(t) \geq (1-\gamma)R$. Also, note that the distance $q(t)$ is represented as

$$q(t) = h(t) - \gamma R\sigma_{22}. \tag{III.23}$$

IV Problem Three

Solve the equations for the motion of the cylinder starting from rest for $\theta(0) = 0$ and $\theta(0) = \phi$. Do the points C and/or P have a constant acceleration? Does the cylinder have a non-zero angular accelerations?

Solution:
The result of simulating these EOMs in MATLAB are given below for the given initial conditions. The code is in the Appendix VI.

The states for EOM in terms of the CoG when $\theta(0) = 0$ is

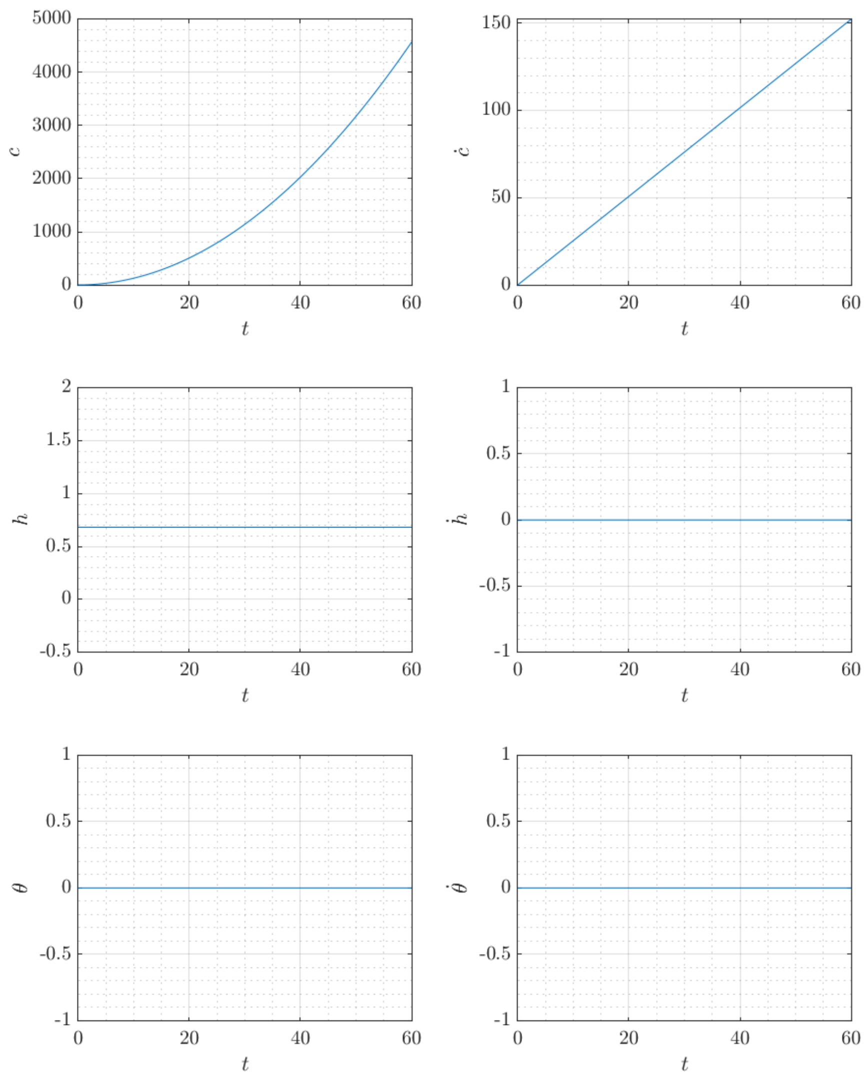

Figure 4: States of EOM in terms of the CoG with $\theta(0) = 0$.

The states for EOM in terms of the CoG when $\theta(0) = \phi$ is

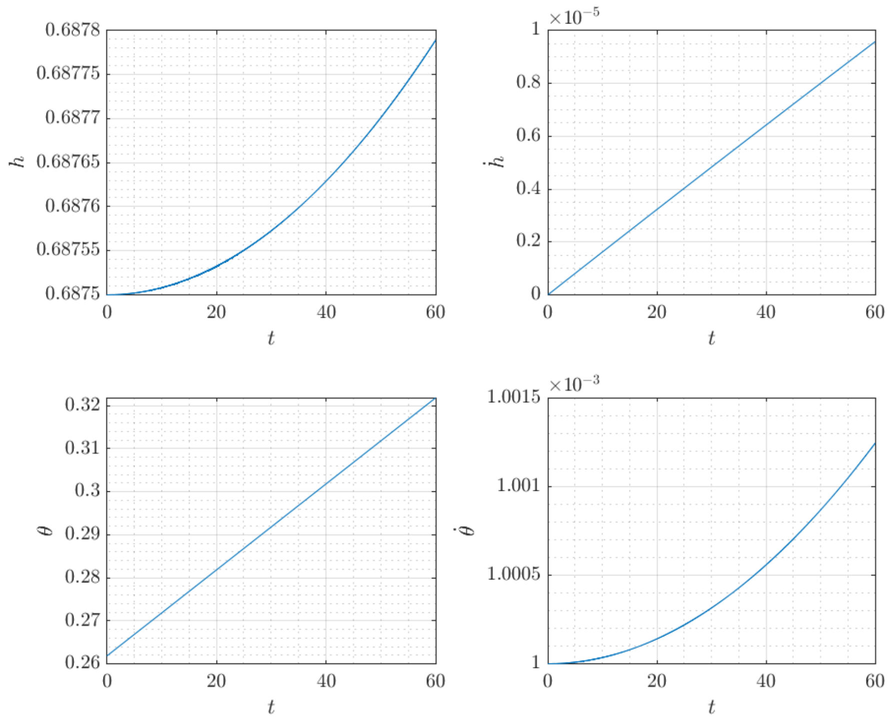

Figure 5: States of EOM in terms of the CoG with $\theta(0) = \phi$.

The states for EOM in terms of the GC when $\theta(0) = 0$ is

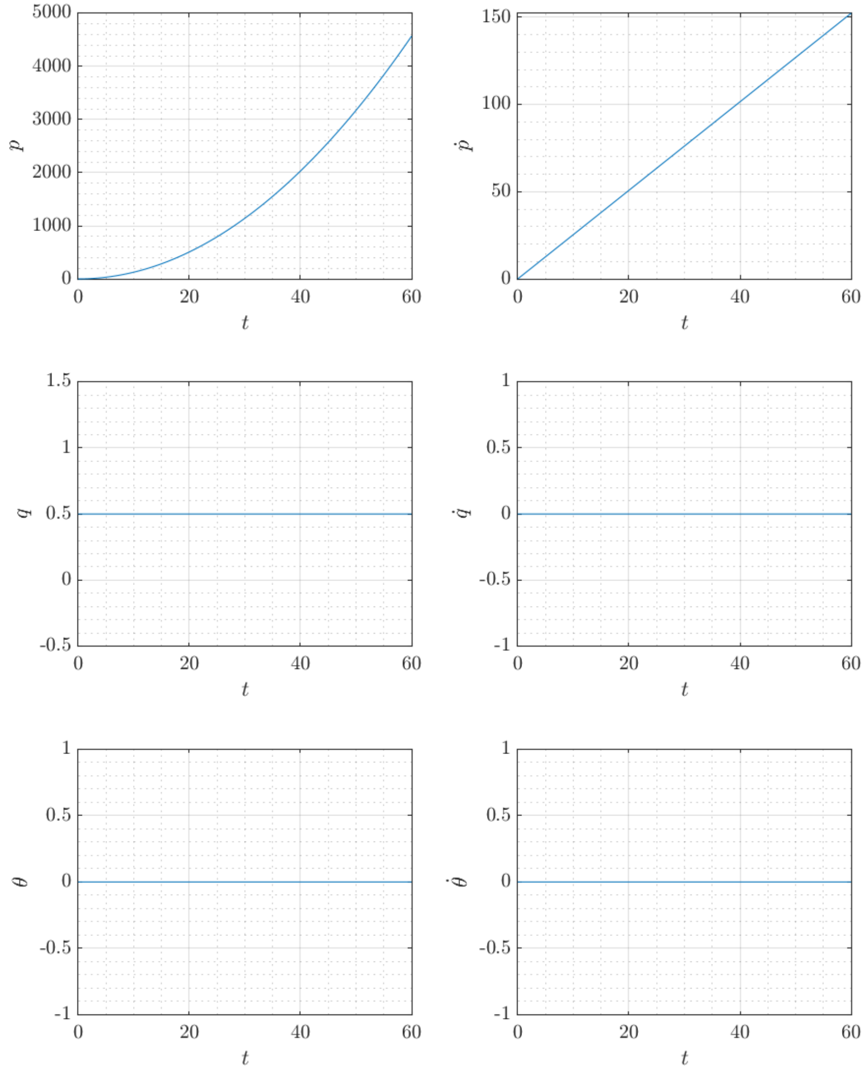

Figure 6: States of EOM in terms of the CoG with $\theta(0) = 0$.

The states for EOM in terms of the GC when $\theta(0) = \phi$ is

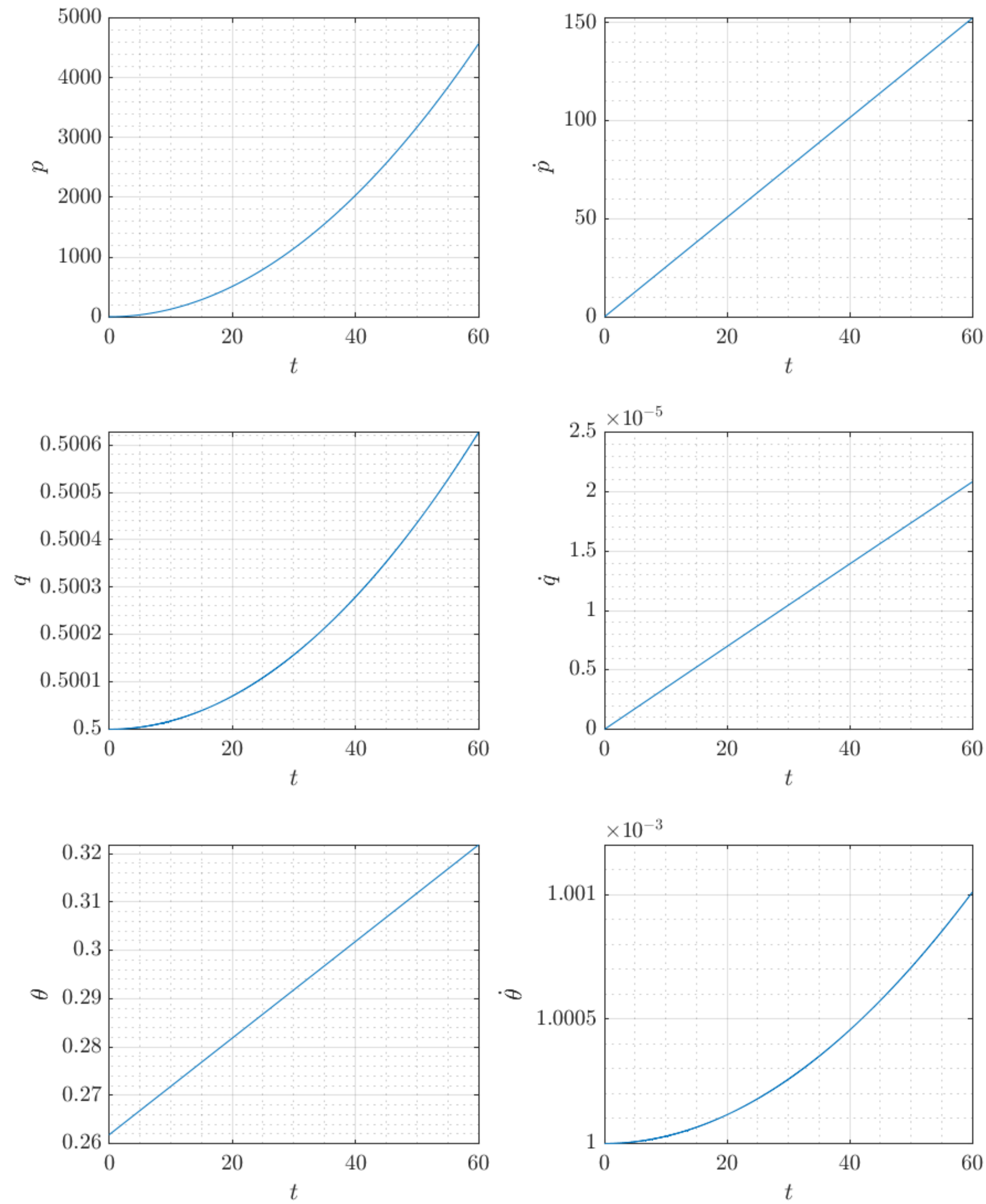

Figure 7: States of EOM in terms of the CoG with $\theta(0) = \phi$.

From the figures, we can see that the acceleration for the CoG and GC are constant and that the angular accelerations are non-zero.

V Problem Four

Now assume that there is sufficient friction so that the cylinder is always rolling when it is in contact with the surface. Derive the equations of motion for the motion of the cylinder. Make sure that the number of equations is equal to the number of unknowns.

Solution:
From Problem III, we know the potential energy U and kinetic energy T in terms of point P is

$$U = -mg(ps_\phi - \gamma R\sigma_{22}) \tag{V.1}$$

$$T = \frac{1}{m}\left[\dot{p}^2 + 2\gamma R\sigma_{22}\dot{p}\dot{\theta} + \gamma^2 R^2(\sigma_{22}^2 - \sigma_{21}^2)\dot{\theta}^2\right]. \tag{V.2}$$

Then the Lagrangian becomes

$$L(p,\dot{p},\theta,\dot{\theta}) = \frac{1}{2}m\dot{p}^2 + m\gamma R\sigma_{22}\dot{p}\dot{\theta} + \frac{1}{2}m\gamma^2R^2(\sigma_{22}^2 - \sigma_{21}^2)\dot{\theta}^2 + \frac{1}{2}I^c\dot{\theta}^2 + mg(ps_\phi - \gamma R\sigma_{22}). \tag{V.3}$$

Now from the rolling constraint, we know that

$$p = R\theta \implies \theta = \frac{p}{R} \implies \dot{\theta} = \frac{\dot{p}}{R} \implies \ddot{\theta} = \frac{\ddot{p}}{R}. \tag{V.4}$$

Then we can rewrite the Lagrangian in terms of only p and $\dot{p}$ which becomes

$$L(p,\dot{p}) = \frac{1}{2}m\dot{p}^2 + 2m\gamma\xi_{22}\dot{p}^2 + \frac{1}{2}m\gamma^2(\xi_{22}^2 - \xi_{21}^2)\dot{p}^2 + \frac{I^c}{2R^2}\dot{p}^2 + mg(ps_\phi - \gamma R\xi_{22}), \tag{V.5}$$

where

$$\xi_{21} = \sin\left(\frac{p}{R}\right)\cos(\phi) - \sin(\phi)\cos\left(\frac{p}{R}\right) \tag{V.6}$$

$$\xi_{22} = \cos\left(\frac{p}{R}\right)\cos(\phi) + \sin\left(\frac{p}{R}\right)\sin(\phi). \tag{V.7}$$

Then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) = \left(m + m\gamma^2 + 2m\gamma\xi_{22} + \frac{I^c}{R^2}\right)\ddot{p} - \frac{2m\gamma\xi_{21}}{R}\dot{p}^2 - \frac{4m\gamma^2\xi_{21}\xi_{22}}{R}\dot{p}^2 \tag{V.8}$$

$$\frac{\partial L}{\partial p} = -\frac{m\gamma\xi_{21}}{R}\dot{p}^2 - \frac{2m\gamma^2\xi_{21}\xi_{22}}{R}\dot{p}^2 + mg\gamma\xi_{21}. \tag{V.9}$$

Then since with no virtual work we have the following EOM

$$\boxed{\left(m + m\gamma^2 + 2m\gamma\xi_{22} + \frac{I^c}{R^2}\right)\ddot{p} - \frac{m\gamma\xi_{21}}{R}\dot{p}^2 - \frac{2m\gamma^2\xi_{21}\xi_{22}}{R}\dot{p}^2 - mg\gamma\xi_{21} = 0.} \tag{V.10}$$

If we simulate this result with some initial velocity we get an interesting plot as follows.

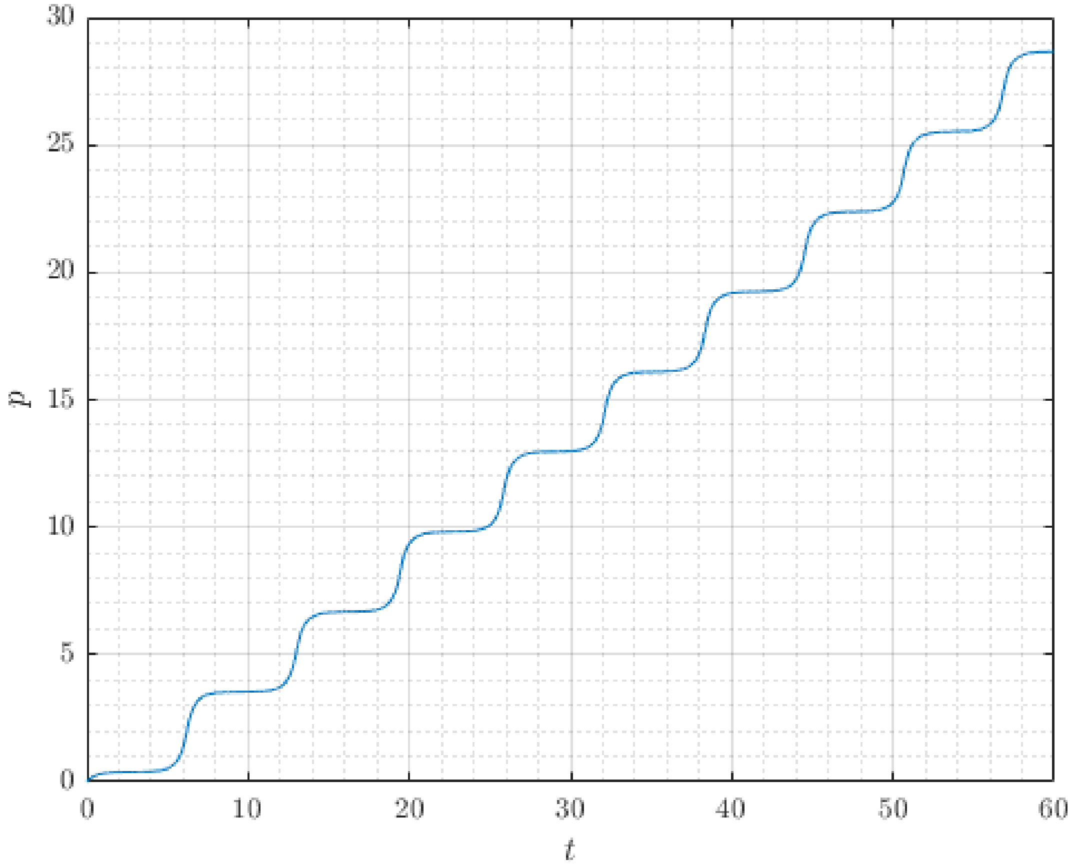

Figure 8: Simulation results of GC when it is rolling.

VI Appendix

MATLAB Code

```
%% AE6210 Advanced dynamics
% Author: Tomoki Koike
%% House keeping commands
close all; clear all; clc;
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

%% Parameters
global m R phi Ic Ip gamma g
m = 1;  % mass [kg]
R = 0.5;  % radius [m]
phi = deg2rad(45);  % slope angle [rad]
Ic = 11/64*m*R^2;  % moment of inertia about the CoG [kg-m^2]
Ip = 5/16*m*R^2;  % moment of inertia about the GC [kg-m^2]
gamma = 3/8;  % scale factor of distance from GC to CoG
g = 9.81;  % gravitational acceleration [m/s^2]

%% No Friction Simulation of CoG

% Simulate setup
tspan = 0:0.01:60;
opts = odeset('RelTol',1e-5,'AbsTol',1e-6);

% First of initial conditions
IC1 = [0; 0; 0; 0; (gamma*R*cos(phi)+R); 0];
[t1,x1] = ode45(@NF_CoG_roll,tspan,IC1,opts);
plot_res(t1,x1,'cog',["nf_cog_ic1_traj.png", "nf_cog_ic1_states.png"])

% Second initial conditions
IC2 = [0; 0; phi; 0.001; (gamma*R+R); 0];
[t2,x2] = ode45(@NF_CoG_roll,tspan,IC2,opts);
plot_res(t2,x2,'cog',["nf_cog_ic2_traj.png", "nf_cog_ic2_states.png"])

% Save data for later use
save nf_cog_data t1 x1 t2 x2;

%% No Friction Simulation of GC

% First of initial conditions
[t1,x1] = ode45(@NF_GC_roll,tspan,IC1,opts);
plot_res(t1,x1,'gc',["nf_gc_ic1_traj.png", "nf_gc_ic1_states.png"])

% Second initial conditions
[t2,x2] = ode45(@NF_GC_roll,tspan,IC2,opts);
plot_res(t2,x2,'gc',["nf_gc_ic2_traj.png", "nf_gc_ic2_states.png"])

% Save data for later use
save nf_gc_data t1 x1 t2 x2;

```

```
%% With friction (rolling) Simulation of CoG

% First of initial conditions
tspan = 0:0.01:60;
opts = odeset('RelTol',1e-5,'AbsTol',1e-6);

theta0 = pi/3;
IC = [0; 0.7781];
% Simulate
[t,x] = ode45(@WF_GC_roll,tspan,IC,opts);
p = x(:,1);
pdot = x(:,2);
theta = p / R;
h = gamma*R*(cos(phi)*cos(theta) + sin(phi)*sin(theta)) + R;

figure(1);
plot(t,p);
grid on; grid minor; box on;
xlabel("$t$")
ylabel("$p$")

%% Functions

% EOM for no friction CoG roll
function dXdt = NF_CoG_roll(t, X)
    global m R phi Ic gamma g
    % x1 = c; x2 = cdot; x3 = theta; x4 = thetadot; x5 = h; x6 = hdot;
    % x1 = X(1);
    x2 = X(2); x3 = X(3); x4 = X(4); x5 = X(5); x6 = X(6);

    % Add bounds for h(t)
    if x5 < (1-gamma)*R
        x5 = (1-gamma)*R;
    end

    sigma21 = sin(x3)*cos(phi) - cos(x3)*sin(phi);
    sigma22 = sin(x3)*sin(phi) - cos(x3)*cos(phi);

    cdot = x2;
    cddot = g*sin(phi);
    thetadot = x4;
    thetaddot = (-m*gamma^2*R^2*sigma21*sigma22*x4^2) / (Ic + m*gamma^2*R^2*sigma21^2);
    hdot = x6;

    % In contact or lose contact
    tol = 1e-4;
    chi = gamma*R*sigma22*x4^2 - g*cos(phi);
    if chi < tol
        hddot = -gamma*R*sigma22*x4^2;
    else
        hddot = -g*cos(phi);
    end

    dXdt = [cdot; cddot; thetadot; thetaddot; hdot; hddot];
```

```
end

% EOM for no friction GC (geometric center) roll
function dXdt = NF_GC_roll(t, X)
    global m R phi Ic gamma g
    % x1 = p; x2 = pdot; x3 = theta; x4 = thetadot
    % x1 = X(1);
    x2 = X(2); x3 = X(3); x4 = X(4); x5 = X(5); x6 = X(6);

    sigma21 = sin(x3)*cos(phi) - cos(x3)*sin(phi);
    sigma22 = sin(x3)*sin(phi) - cos(x3)*cos(phi);

    % Add bounds for h(t)
    if x5 < (1-gamma)*R
        x5 = (1-gamma)*R;
    end

    pdot = x2;
    den = m*R^2*gamma^2 + Ic - m*R^2*gamma^2*sigma22^2;
    pddot = (m*sigma21*R^3*gamma^3*x4^2 - g*m*sin(phi)*R^2*gamma^2*sigma22^2 ...
        + g*m*sin(phi)*R^2*gamma^2 + Ic*sigma21*R*gamma*x4^2 + Ic*g*sin(phi)) / den;
    thetadot = x4;
    thetaddot = -R^2*gamma^2*m*sigma21*sigma22*x4^2 / den;
    hdot = x6;

    % In contact or lose contact
    tol = 1e-4;
    chi = gamma*R*sigma22*x4^2 - g*cos(phi);
    if chi < tol
        hddot = -gamma*R*sigma22*x4^2;
    else
        hddot = -g*cos(phi);
    end

    dXdt = [pdot; pddot; thetadot; thetaddot; hdot; hddot];
end

% EOM with friction
function dXdt = WF_GC_roll(t,X)
    global g m gamma Ic R phi
    x1 = X(1); x2 = X(2);

    xi21 = sin(x1/R)*cos(phi) - sin(phi)*cos(x1/R);
    xi22 = cos(x1/R)*cos(phi) + sin(x1/R)*sin(phi);
    den = m + m*gamma^2 + 2*m*gamma*xi22 + Ic/R^2;

    c1 = m*gamma*xi21/R;
    c2 = 2*m*gamma^2*xi21*xi22/R;
    c3 = m*g*gamma*xi21;

    x1dot = x2;
    x2dot = (c1*x2^2 + c2*x2^2 + c3)/den;

    dXdt = [x1dot; x2dot];
```

```
end

function plot_res(t, x, flag, im_filenames)
    global gamma R phi
    if flag == "cog"
        c = x(:,1);
        cdot = x(:,2);
        theta = x(:,3);
        thetadot = x(:,4);
        h = x(:,5);
        hdot = x(:,6);

        % Trajectory
        fig1 = figure(Renderer="painters");
            plot(c,h); grid on; grid minor; box on;
            xlabel('$c$'); ylabel('$h$');
        % states over t
        fig2 = figure(Renderer="painters", Position=[90 90 650 800]);
        subplot(3,2,1) % c over t
            plot(t,c); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$c$');
        subplot(3,2,2) % cdot over t
            plot(t,cdot); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\dot{c}$');
        subplot(3,2,3) % h over t
            plot(t,h); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$h$');
        subplot(3,2,4) % hdot over t
            plot(t,hdot); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\dot{h}$');
        subplot(3,2,5) % theta over t
            plot(t,theta); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\theta$');
        subplot(3,2,6) % thetadot over t
            plot(t,thetadot); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\dot{\theta}$');

    elseif flag == "gc"
        p = x(:,1);
        pdot = x(:,2);
        theta = x(:,3);
        thetadot = x(:,4);
        h = x(:,5);
        hdot = x(:,6);

        % Compute sigmas
        sigma21 = sin(theta).*cos(phi) - cos(theta).*sin(phi);
        sigma22 = sin(theta).*sin(phi) + cos(theta).*cos(phi);

        % convert h(t) to q(t)
        q = h - gamma*R*sigma22;
        qdot = hdot + gamma*R*thetadot.*sigma21;

        % Trajectory
```

```
        fig1 = figure(Renderer="painters");
            plot(p,q); grid on; grid minor; box on;
            xlabel('$p$'); ylabel('$q$');
        % states over t
        fig2 = figure(Renderer="painters", Position=[90 90 650 800]);
        subplot(3,2,1) % p over t
            plot(t,p); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$p$');
        subplot(3,2,2) % pdot over t
            plot(t,pdot); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\dot{p}$');
        subplot(3,2,3) % q over t
            plot(t,q); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$q$');
        subplot(3,2,4) % qdot over t
            plot(t,qdot); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\dot{q}$');
        subplot(3,2,5) % theta over t
            plot(t,theta); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\theta$');
        subplot(3,2,6) % thetadot over t
            plot(t,thetadot); grid on; grid minor; box on;
            xlabel('$t$'); ylabel('$\dot{\theta}$');

    else
        error("Wrong flag statement. Only accept 'cog' and 'gc'.");
    end

    % Save figures
    saveas(fig1,im_filenames(1));
    saveas(fig2,im_filenames(2));
end
```