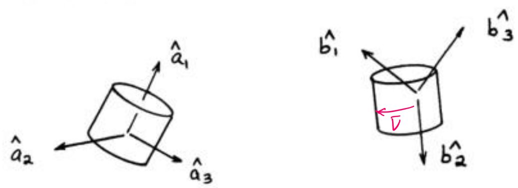
Kinematical Differential Equations

Numerous ways to describe orientation at any instant of time. Now consider rates of change of orientation.

Introduce familiar concept of angular velocity

Basic Definitions



Let A and B be two spacecraft rotating with respect to each other

$$\boxed{{}^A\bar{\omega}^B = \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3} \qquad \boxed{{}^A\bar{\omega}^B = -{}^B\bar{\omega}^A}$$

Ang vel of. B wrt A

Scalar measure #s have meaning in terms of body frame

Express in terms of any vector basis

Trans

pos (different variable sets)

x, y, z

r, θ

velocity

» diff. observers

» diff. var. s

Rot

orientation (diff. variable sets)

$\theta_1, \theta_2, \theta_3, \varepsilon$

angular velocity

» diff. observers

» diff. vars

$\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \dot{\varepsilon}$

G2

$${}^A\bar{\omega}^B = \omega_a \hat{a}_1 + \omega_b \hat{a}_2 + \omega_c \hat{a}_3$$

Angular velocity is particularly important in the BKE ⟵ relating the change of a vector as observed from different frames

If $\bar{v}$ is any vector

$$\boxed{{}^A\frac{d\bar{v}}{dt} = {}^B\frac{d\bar{v}}{dt} + {}^A\bar{\omega}^B \times \bar{v}}$$

If $\bar{v}$ is fixed in B → ${}^B\frac{d\bar{v}}{dt} = 0$

Then ${}^A\frac{d\bar{v}}{dt} = {}^A\bar{\omega}^B \times \bar{v}$

$${}^A\frac{d\hat{b}_2}{dt} = {}^A\bar{\omega}^B \times \hat{b}_2 = \boxed{\omega_1\hat{b}_3 - \omega_3\hat{b}_1}$$

change in $\hat{b}_2$ as observed by A

$${}^A\frac{d\hat{b}_1}{dt} = -\omega_2\hat{b}_3 + \omega_3\hat{b}_2$$

$${}^A\frac{d\hat{b}_3}{dt} = -\omega_1\hat{b}_2 + \omega_2\hat{b}_1$$

Recall ${}^A\bar{\omega}^B = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$

More general form

$${}^A\bar{\omega}^B = \underbrace{\hat{b}_1 \frac{{}^A d\hat{b}_2}{dt}}_{\text{dyad}} \bullet \hat{b}_3 + \hat{b}_2 \frac{{}^A d\hat{b}_3}{dt} \bullet \hat{b}_1 + \hat{b}_3 \frac{{}^A d\hat{b}_1}{dt} \bullet \hat{b}_2$$

G3

Let ${}^A\frac{d\hat{b}_i}{dt} = {}^A\dot{\hat{b}}_i = \dot{\hat{b}}_i$

$$\boxed{{}^A\bar{\omega}^B = \hat{b}_1 \dot{\hat{b}}_2 \bullet \hat{b}_3 + \hat{b}_2 \dot{\hat{b}}_3 \bullet \hat{b}_1 + \hat{b}_3 \dot{\hat{b}}_1 \bullet \hat{b}_2}$$

$${}^A\bar{\alpha}^B = \frac{{}^A d\, {}^A\bar{\omega}^B}{dt}$$

For a simple rotation ($\hat{\lambda}$ constant throughout rotation)

$$\boxed{{}^A\bar{\omega}^B = \dot{\theta}\hat{\lambda}}$$

$${}^A\hat{\lambda}^B = \lambda_1 \hat{b}_1 + \lambda_2 \hat{b}_2 + \lambda_3 \hat{b}_3$$

$${}^A\bar{\omega}^B = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3$$

$$\begin{cases} \omega_1 = \dot{\theta}_1 \lambda_1 \\ \omega_2 = \dot{\theta}_2 \lambda_2 \\ \omega_3 = \dot{\theta}_3 \lambda_3 \end{cases}$$

Like $\hat{\lambda}, \theta$ have used C, ε as variables to describe orientation at a particular instant of time; angular velocity can be used to tell how the orientation is changing

$${}^{A}C^{B} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

1. Direction Cosines

Define $$\dot{C} = \begin{bmatrix} \dot{C}_{11} & \dot{C}_{12} & \dot{C}_{13} \\ \dot{C}_{21} & \dot{C}_{22} & \dot{C}_{23} \\ \dot{C}_{31} & \dot{C}_{32} & \dot{C}_{33} \end{bmatrix}$$

derivatives of scalars – no need to indicate frame

Example using relationships from page C3 and concept of a simple rotation

$$C_{11} = c_\theta + \lambda_1^2 (1 - c_\theta)$$

$$\dot{C}_{11} = -\dot{\theta} s_\theta + \lambda_1^2 (\dot{\theta} s_\theta)$$

$$= (-1 + \lambda_1^2)\dot{\theta} s_\theta$$

Recall $|\hat{\lambda}| = 1$

$$\dot{C}_{11} = -\left(\lambda_2^2 + \lambda_3^2\right)\dot{\theta} s_\theta$$

$$\dot{C}_{11} = \left\{-\lambda_2^2 s_\theta - \lambda_1\lambda_2\lambda_3(1 - c_\theta) - \lambda_3^2 s_\theta + \lambda_1\lambda_2\lambda_3(1 - c_\theta)\right\}\dot{\theta}$$

ω_2

$$= [-\lambda_2 s_\theta - \lambda_1\lambda_3(1 - C_\theta)]\lambda_2\dot{\theta}$$

ω_3

$$+ [-\lambda_3 s_\theta + \lambda_1\lambda_2(1 - C_\theta)]\lambda_3\dot{\theta}$$

$$\dot{C}_{11} = -C_{13}\,\omega_2 + C_{12}\,\omega_3 \implies$$

9 scalar
1st order
DE

⋮

$$\boxed{\dot{C} = C\,\tilde{\omega}}$$

Poisson's Kinematical Equations for change in elements of the direction cosine matrix

$$\tilde{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Also $$\boxed{\tilde{\omega} = C^T \dot{C}}$$

can be viewed as the definition of angular velocity

Above, use ω_i to write expressions for changes in elements of the direction cosine matrix

2. Euler Parameters

$$\tilde{\omega} = C^T \dot{C}$$

Example using relationship between C and ε

$$\boxed{\omega_1 = C_{13}\dot{C}_{12} + C_{23}\dot{C}_{22} + C_{33}\dot{C}_{32}}$$

But $C_{13} = 2(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)$

$C_{12} = 2(\varepsilon_1\varepsilon_2 - \varepsilon_3\varepsilon_4)$

$\dot{C}_{12} = 2(\dot{\varepsilon}_1\varepsilon_2 + \varepsilon_1\dot{\varepsilon}_2 - \dot{\varepsilon}_3\varepsilon_4 - \varepsilon_3\dot{\varepsilon}_4)$

$$\boxed{\begin{aligned}\omega_1 &= 4(\varepsilon_3\varepsilon_1 + \varepsilon_2\varepsilon_4)(\dot{\varepsilon}_1\varepsilon_2 + \varepsilon_1\dot{\varepsilon}_2 - \dot{\varepsilon}_3\varepsilon_4 - \varepsilon_3\dot{\varepsilon}_4)\\ &+ 4(\varepsilon_2\varepsilon_3 - \varepsilon_1\varepsilon_4)(\varepsilon_2\dot{\varepsilon}_2 - \varepsilon_3\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_1 + \varepsilon_4\dot{\varepsilon}_4)\\ &+ 2(1 - 2\varepsilon_1^2 - 2\varepsilon_2^2)(\dot{\varepsilon}_2\varepsilon_3 + \varepsilon_2\dot{\varepsilon}_3 + \dot{\varepsilon}_1\varepsilon_4 + \varepsilon_1\dot{\varepsilon}_4)\end{aligned}}$$

$$\begin{aligned}\omega_1 &= (4\varepsilon_1\varepsilon_2\varepsilon_3 + 4\varepsilon_2^2\varepsilon_4 - 4\varepsilon_1\varepsilon_2\varepsilon_3 + 4\varepsilon_1^2\varepsilon_4 + 2\varepsilon_4 - 4\varepsilon_1^2\varepsilon_4 - 4\varepsilon_2^2\varepsilon_4)\dot{\varepsilon}_1\\ &+ (\quad\cdots\quad)\dot{\varepsilon}_2\\ &+ (\quad\cdots\quad)\dot{\varepsilon}_3\\ &+ (\quad\cdots\quad)\dot{\varepsilon}_4\end{aligned}$$

$$\boxed{\omega_1 = 2\varepsilon_4\dot{\varepsilon}_1 + 2\varepsilon_3\dot{\varepsilon}_2 - 2\varepsilon_2\dot{\varepsilon}_3 - 2\varepsilon_1\dot{\varepsilon}_4}$$

$$\vdots$$

$$\boxed{\omega_2 = 2\varepsilon_4\dot{\varepsilon}_2 + 2\varepsilon_1\dot{\varepsilon}_3 - 2\varepsilon_3\dot{\varepsilon}_1 - 2\varepsilon_2\dot{\varepsilon}_4}$$

$$\boxed{\omega_3 = 2\varepsilon_4\dot{\varepsilon}_3 + 2\varepsilon_2\dot{\varepsilon}_1 - 2\varepsilon_1\dot{\varepsilon}_2 - 2\varepsilon_3\dot{\varepsilon}_4}$$

more convenient form?

constraint

Matrix format

derivative of constraint

Define 4-element vectors $\omega = [\omega_1 \ \omega_2 \ \omega_3 \ 0]$

$\varepsilon = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_3 \ \varepsilon_4]$

Define $$E = \begin{bmatrix} \varepsilon_4 & -\varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_4 & -\varepsilon_1 & \varepsilon_2 \\ -\varepsilon_2 & \varepsilon_1 & \varepsilon_4 & \varepsilon_3 \\ -\varepsilon_1 & -\varepsilon_2 & -\varepsilon_3 & \varepsilon_4 \end{bmatrix}$$

$\boxed{\omega = 2\,\dot{\varepsilon}\,E}$ ← $\boxed{\dot{\varepsilon} = \frac{1}{2}\omega E^T}$

1st order kinematic DE for rate of change of orientation expressed in $\dot{\varepsilon}$

Note: three equations above but add constraint equation

$$\frac{d}{dt}\left(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \varepsilon_4^2 = 1\right)$$

$$2\varepsilon_1\dot{\varepsilon}_1 + 2\varepsilon_2\dot{\varepsilon}_2 + 2\varepsilon_3\dot{\varepsilon}_3 + 2\varepsilon_4\dot{\varepsilon}_4 = 0$$

Assumptions associated with the ω equation?

$$\boxed{{}^A\omega^B = 2\ {}^A\dot{\varepsilon}^B\ {}^AE^B}$$

components in vector $\hat{b}$ body fixed

components in either $\hat{a}$ or $\hat{b}$

Invert $\boxed{\dot{\varepsilon} = \frac{1}{2}\omega E^T}$ differential equations for change in ε in terms of ω

Can also develop a vector version

in body-fixed

$$
\begin{aligned}
{}^{A}\bar{\omega}^{B} = & \, 2\left(\varepsilon_4\dot{\varepsilon}_1 + \varepsilon_3\dot{\varepsilon}_2 - \varepsilon_2\dot{\varepsilon}_3 - \varepsilon_1\dot{\varepsilon}_4\right)\hat{b}_1 \\
& + 2\left(\varepsilon_4\dot{\varepsilon}_2 + \varepsilon_1\dot{\varepsilon}_3 - \varepsilon_3\dot{\varepsilon}_1 - \varepsilon_2\dot{\varepsilon}_4\right)\hat{b}_2 \\
& + 2\left(\varepsilon_4\dot{\varepsilon}_3 + \varepsilon_2\dot{\varepsilon}_1 - \varepsilon_1\dot{\varepsilon}_2 - \varepsilon_3\dot{\varepsilon}_4\right)\hat{b}_3
\end{aligned}
$$

cross product

$$
\begin{aligned}
{}^{A}\bar{\omega}^{B} = 2\{ & \varepsilon_4\left(\dot{\varepsilon}_1\hat{b}_1 + \dot{\varepsilon}_2\hat{b}_2 + \dot{\varepsilon}_3\hat{b}_3\right) \\
& + \boxed{\left(\varepsilon_3\dot{\varepsilon}_2 - \varepsilon_2\dot{\varepsilon}_3\right)\hat{b}_1 + \left(\varepsilon_1\dot{\varepsilon}_3 - \varepsilon_3\dot{\varepsilon}_1\right)\hat{b}_2 + \left(\varepsilon_2\dot{\varepsilon}_1 - \varepsilon_1\dot{\varepsilon}_2\right)\hat{b}_3} \\
& - \dot{\varepsilon}_4\left(\varepsilon_1\hat{b}_1 + \varepsilon_2\hat{b}_2 + \varepsilon_3\hat{b}_3\right)\}
\end{aligned}
$$

$$
{}^{A}\bar{\omega}^{B} = 2\left\{ \varepsilon_4 \frac{{}^{B}d\bar{\varepsilon}}{dt} - \bar{\varepsilon} \times \frac{{}^{B}d\bar{\varepsilon}}{dt} - \dot{\varepsilon}_4\bar{\varepsilon} \right\}
$$

Check $\dfrac{{}^{B}d\bar{\varepsilon}}{dt} = \dot{\varepsilon}_1\hat{b}_1 + \dot{\varepsilon}_2\hat{b}_2 + \dot{\varepsilon}_3\hat{b}_3$

$$
\frac{{}^{A}d\bar{\varepsilon}}{dt} = \frac{{}^{B}d\bar{\varepsilon}}{dt} + {}^{A}\bar{\omega}^{B} \times \bar{\varepsilon} \neq \frac{{}^{B}d\bar{\varepsilon}}{dt}
$$

Must carry frame information

Invert

$$
\boxed{
\begin{aligned}
\frac{{}^{B}d\bar{\varepsilon}}{dt} &= \frac{1}{2}\left(\varepsilon_4\,\bar{\omega} + \bar{\varepsilon} \times \bar{\omega}\right) \\
\dot{\varepsilon}_4 &= -\frac{1}{2}\bar{\omega} \bullet \bar{\varepsilon}
\end{aligned}
}
$$

vector format for G7 red box

3. Orientation Angles

Need a relationship for changes in the orientation angles as a function of $\bar{\omega}$ and the angles themselves

Actually, already familiar with relationship between angular velocity and SOME sets of angles → start there……

Chain rule for angular velocity

$$ {}^{A}\bar{\omega}^{B} = {}^{A}\bar{\omega}^{B'} + {}^{B'}\bar{\omega}^{B''} + {}^{B''}\bar{\omega}^{B'''} + \cdots + {}^{B^{n}}\bar{\omega}^{B} $$

where $B', B'', B''', B^{iv}, \ldots$ are auxiliary or intermediate frames

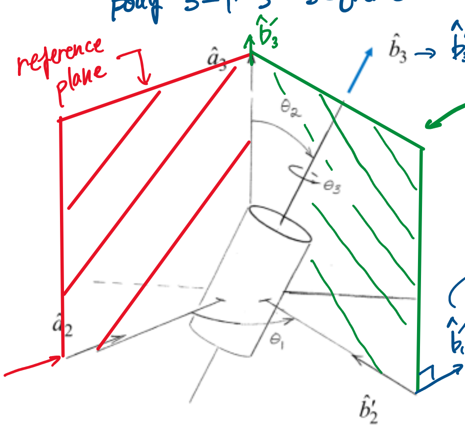
chain rule is extremely useful but ONLY applies to BODY sequence

Example: Classical Euler Angles
Body-two 3-1-3 (popular notation)
(3-2-3)

} notable for axissymmetric vehicles

[Note: not exactly the set used by Euler
Euler's original published work in 1760 used a left- and right-handed combination which is unacceptable today]

Body 3-1-3 sequence



Classical "top" problem in dynamics

Orbits:
$3-1-3$
$\Omega - i - \theta$

Rotations: 3-1-3

θ_1 $\hat{a}_3 = \hat{b}_3'$ precession

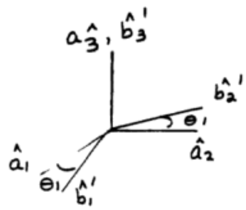
θ_2 $\hat{b}_1' = \hat{b}_1''$ nutation

θ_3 $\hat{b}_3'' = \hat{b}_3$ spin

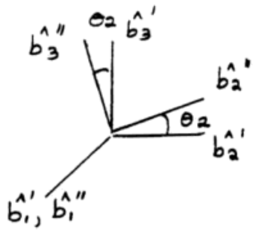
Angular velocity $\quad {}^{A}\bar{\omega}^{B} = {}^{A}\bar{\omega}^{B'} + {}^{B'}\bar{\omega}^{B''} + {}^{B''}\bar{\omega}^{B}$ chain rule

$$ {}^{A}\bar{\omega}^{B} = \dot{\theta}_1 \hat{a}_3 + \dot{\theta}_2 \hat{b}_1' + \dot{\theta}_3 \hat{b}_3'' $$

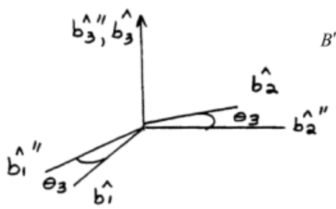
$$ {}^{A}\bar{\omega}^{B} = \underbrace{\dot{\theta}_1}_{\text{prec rate}} \hat{b}_3' + \underbrace{\dot{\theta}_2}_{\text{nut rate}} \hat{b}_1'' + \underbrace{\dot{\theta}_3}_{\text{spin rate}} \hat{b}_3 $$



$$ {}^{A}C^{B'} = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$



$$ {}^{B'}C^{B''} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{bmatrix} $$



$$ {}^{B''}C^{B} = \begin{bmatrix} c_3 & -s_3 & 0 \\ s_3 & c_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

G14

$$ {}^{A}\bar{\omega}^{B} = \dot\theta_1\left(s_2\underbrace{\hat b_2''}_{s_3\hat b_1 + c_3\hat b_2} + c_2\overbrace{\hat b_3''}^{\hat b_3}\right) + \dot\theta_2\left(c_3\hat b_1 - s_3\hat b_2\right) + \dot\theta_3\hat b_3 $$

substitute unit vectors

$$ {}^{A}\bar{\omega}^{B} = \left(\dot\theta_1 s_2 s_3 + \dot\theta_2 c_3\right)\hat b_1 + \left(\dot\theta_1 s_2 c_3 - \dot\theta_2 s_3\right)\hat b_2 + \left(\dot\theta_1 c_2 + \dot\theta_3\right)\hat b_3 $$

$$ {}^{A}\bar{\omega}^{B} = \omega_1\hat b_1 + \omega_2\hat b_2 + \omega_3\hat b_3 $$

$$ \boxed{\begin{aligned} \omega_1 &= \dot\theta_1 s_2 s_3 + \dot\theta_2 c_3 \\ \omega_2 &= \dot\theta_1 s_2 c_3 - \dot\theta_2 s_3 \\ \omega_3 &= \dot\theta_1 c_2 + \dot\theta_3 \end{aligned}} $$

Invert

$$ \dot\theta_1 = (\omega_1 s_3 + \omega_2 c_3)/s_2 \qquad \text{prec. rate} $$

$$ \dot\theta_2 = \omega_1 c_3 - \omega_2 s_3 \qquad \text{nut rate} $$

$$ \dot\theta_3 = -(\omega_1 s_3 + \omega_2 c_3)\, c_2/s_2 + \omega_3 \qquad \text{spin rate} $$