Problem Set 3 Solution

AAE 532: Orbital Mechanics
MWF: 11:30-12:30
Professor Howell

September 18, 2020

Contents

Problem 1

A mission concept was propsed recently to deliver a spacecraft to one or more of the Trojan asteroids, e.g., 8317 Eurysaces. Of course, as seen in figure, there are many, many asteroids. As they move along their orbital path, the 'green' Trojans remain in the same relative locations with respect to the Sun and Jupiter. To develop a trajectory for the mission, it is also necessary to understand the path of the asteroid.

(a) Return to the small body database and check the orbit of the asteroid Eurysaces. Use a view to "Look at Sun" and "Look from Above". Is Eurysaces in the Greek camp or the Trojan camp? The database lists the orbital period of Eurysaces inyears. Compare it to the period of Jupiter in its orbit relative to the Sun. Take 3 images as Jupiter moves through its orbit relative to the Sun. Use a start date of 9/18/20 and select 3 other dates along Jupiter's orbit. From the images, measure approximately the angle between the lines Sun-Jupiter and Sun-Eurysaces. How much does the angle change over your three dates?

(b) For a prelimiary assessment, the positions of a sample Trojan asteroid (e.g., Eurysaces), the Sun, and Jupiter can be modeled as located at the vertices of an equilateral triangle as envisioned below. Assume that the distance between the Sun and Jupiter is equal to the semi-major axis of Jupiter's orbit. Let the mass of the asteroid be assumed as $\mu = 75km^3/sec^2$. Consider the net acceleration on the asteroid.

(i) Write the expression for the acceleration of the asteroid relative to the Sun where Jupiter is a perturbing body. [Note that the definition of a set of unit vectors is necessary.] Write this expression in the form $\ddot{r}_{\oplus \rightarrow asteroid}$. Label each of the following terms in this expression: dominant term, direct perturbing terms, indirect perturbing terms. Determine the magnitude and direction of each of the terms in the expression, as well as the net perturbing accelerations for each body and the total net acceleration.

(ii) Re-formulate the probelm and write the expression for the acceleration of the asteroid relative Jupiter. Again, determine the magnitude direction of each of the terms in the expression, as well as the net perturbing accelerations for each body and the total net acceleration.

(iii) Which term is the largest in each formulation? How does the magnitude of the dominant term in each formulation compare? Determine the net perturbing acceleration in each case? Which has the largest impact in each formulation, Sun or Jupiter? Is the net total acceleration on the asteroid the same in each formulation? Should it be the same? Why or why not? Which formulation is correct? Why? From the results here, is it reasonable to model the motion of the asteroid as a two-body problem, i.e., Sun-asteroid or Jupiter-asteroid? Why?

Problem 1 Solution

(a)

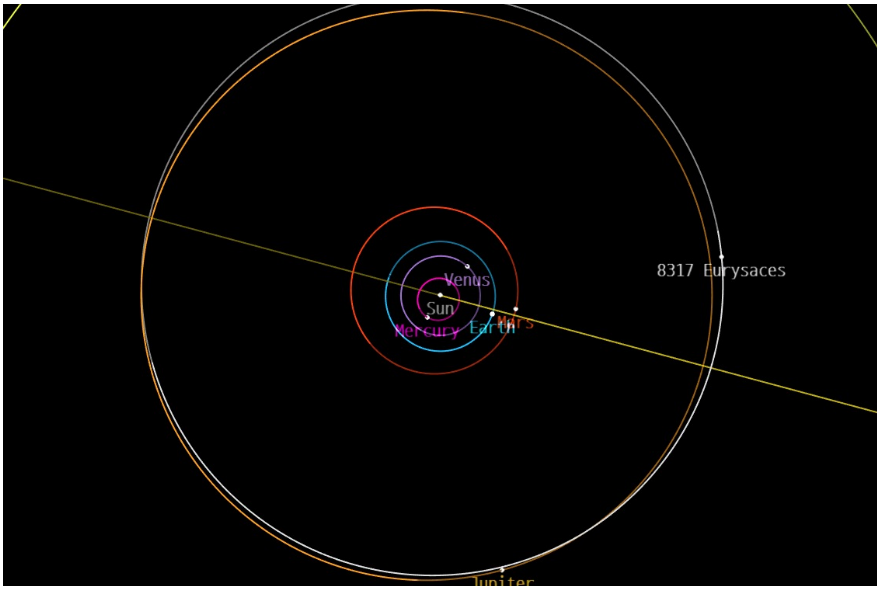


Figure 1: Configuration at 09/18/2020, 85°

In the figures, the asteroid appears to be ahead of Jupiter's orbit, which means that it is a $\boxed{\text{greek}}$.

$$\mathbb{P}_{♃} = 11.87\text{years}$$
$$\mathbb{P}_{Eurysaces} = 12.21\text{years}$$
$$\boxed{\frac{\mathbb{P}_{Eurysaces}}{\mathbb{P}_{♃}}} = \frac{12.21}{11.87} = \boxed{102.86\%}$$

So the period of Eurysaces is slightly longer than that of Jupiter. Along the four configurations of Eurysaces and Jupiter, and we can measure the angle between Sun-Jupiter and Sun-Eurysaces. Over the three periods that we measured, the angle appears to be decreasing from 85° to 75°. For the shorter period of time (days or months), this angle would appear to change slowly.

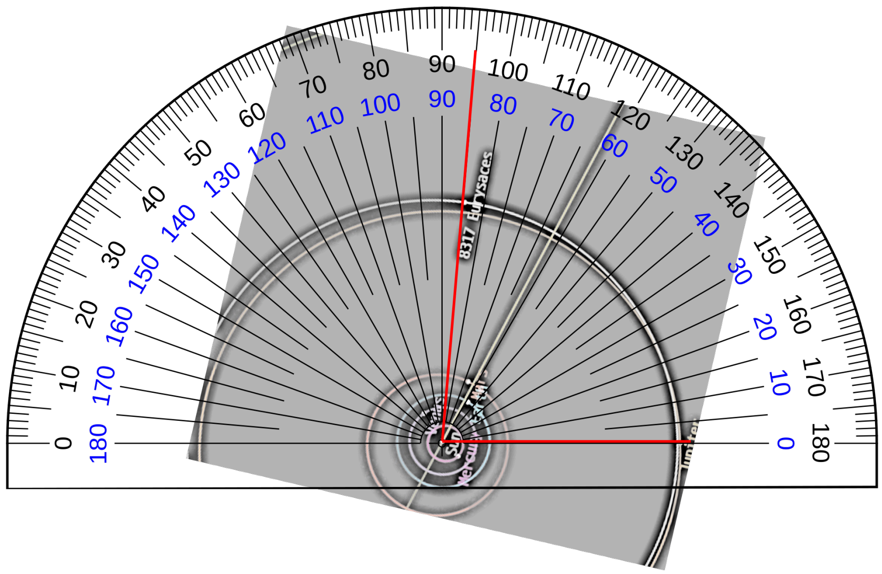


Figure 2: Approximate angle measurement

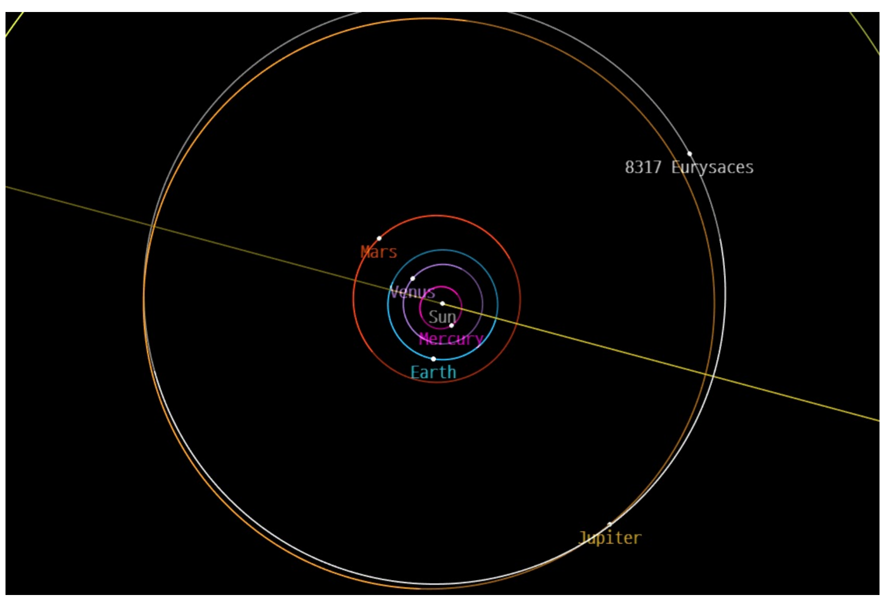


Figure 3: Configuration at 07/31/2021, 83°

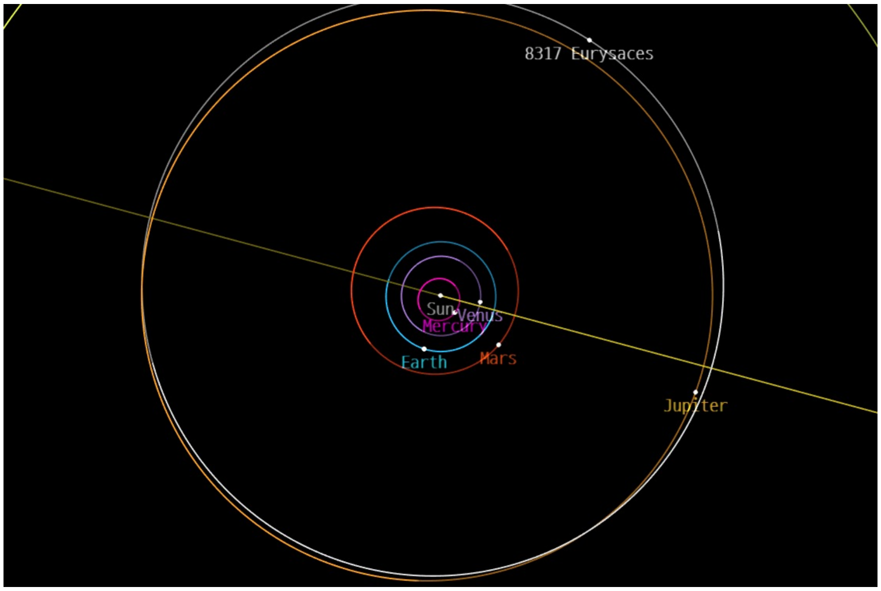


Figure 4: Configuration at 06/19/2022, 81°

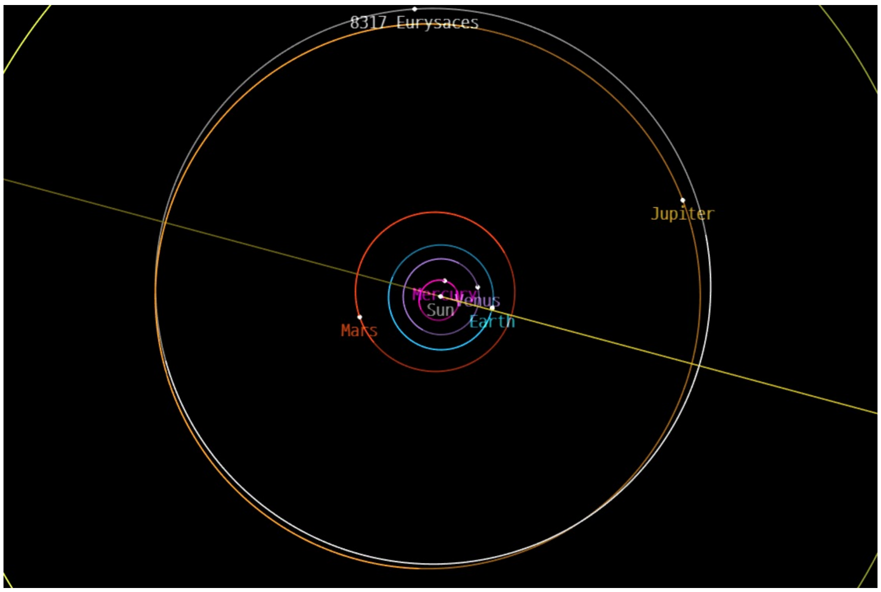


Figure 5: Configuration at 09/26/2023, 75°

(b) - (i)

First, let's make clear what our assumptions are:

1. It is a three-body system. We have Sun, Jupiter and the asteroid, nothing else.
2. All bodies are approximated as point masses.
3. Three bodies form an equilateral triangle at the moment, length being the semi-major axis of Jupiter.

And then we define the unit vectors. In Figure 6, note that S, J, E represent Sun, Jupiter and Eurysaces respectively. Although we can define the inertial directions arbitrarily, the most straightforward way to define from the given configuration is assigning the first unit vector, $\hat{\imath}$, parallel to the vector from the Sun to Jupiter with the same direction. And the third unit vector $\hat{k}$ is assumed to be coming out of the page. Also note that these unit vectors do not require a specific origin, they represent the inertial directions only. For example, we could attach these unit vectors to the center of mass, but also to one of the bodies.

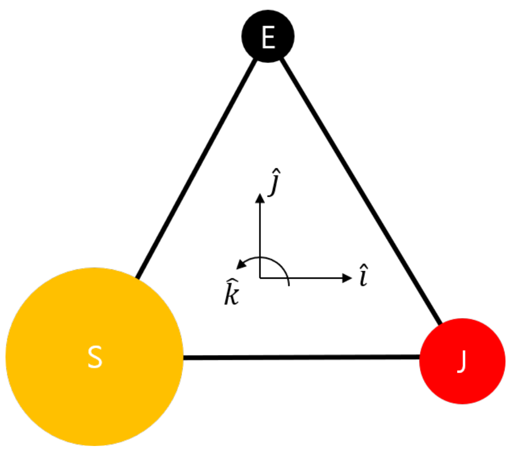


Figure 6: Inertial unit vectors

Now, moving on to the relative acceleration of the asteroid with respect to the Sun:

$$\ddot{\bar{r}}_{\odot\rightarrow asteroid} = \underbrace{-\frac{\mu_{\odot} + \mu_{asteroid}}{r^3_{\odot\rightarrow asteroid}}\bar{r}_{\odot\rightarrow asteroid}}_{Dominant} + (\underbrace{\frac{\mu_{♃}}{r_{asteroid\rightarrow ♃}}\bar{r}_{asteroid\rightarrow ♃}}_{direct\ pert.} - \underbrace{\frac{\mu_{♃}}{r_{\odot\rightarrow ♃}}\bar{r}_{\odot\rightarrow ♃}}_{indirect\ pert.})$$

Note that the indirect term itself does not have negative sign, but when we compute the net perturbing force from Jupiter, we *subtract* the indirect term from the direct term, which is why we have the negative sign. Since we have the equation, we can compute values and vectors needed. First from the table of constants:

$$\mu_{\odot} = 132712440017.99 km^3/sec^2 = 1.3271 \times 10^{11} km^3/sec^2$$
$$\mu_{♃} = 126712767.8578 km^3/sec^2 = 1.2671 \times 10^{8} km^3/sec^2$$
$$\mu_{asteroid} = 75 km^3/sec^2$$
$$a_{♃} = 778279959 km = 7.7828 \times 10^{4} km$$

And the relative position vectors from the Figure 6:

$$\bar{r}_{\odot \to asteroid} = a_{♃} \cos 60^\circ \hat{i} + a_{♃} \sin 60^\circ \hat{j} = 3.8914 \times 10^8\ km\ \hat{i} + 6.7401 \times 10^4\ km\ \hat{j}$$
$$\bar{r}_{asteroid \to ♃} = a_{♃} \cos 60^\circ \hat{i} - a_{♃} \sin 60^\circ \hat{j} = 3.8914 \times 10^8\ km\ \hat{i} - 6.7401 \times 10^4\ km\ \hat{j}$$
$$\bar{r}_{\odot \to ♃} = a_{♃} \hat{i} = 7.7828 \times 10^8\ km\ \hat{i}$$
$$r_{\odot \to asteroid} = r_{asteroid \to ♃} = r_{\odot \to ♃} = a_{♃} = 7.7828 \times 10^8\ km$$

And plug in the numbers for each of the acceleration terms to get:

$$\boxed{\begin{aligned}
&-\frac{\mu_\odot + \mu_{asteroid}}{r^3_{\odot \to asteroid}} \bar{r}_{\odot \to asteroid} = -1.0955 \times 10^{-7}\ km/s^2\ \hat{i} - 1.8974 \times 10^{-7}\ km/s^2 \hat{j} \\
&\frac{\mu_{♃}}{r_{asteroid \to ♃}} \bar{r}_{asteroid \to ♃} = 1.0460 \times 10^{-10}\ km/s^2\ \hat{i} - 1.8117 \times 10^{-10}\ km/s^2\ \hat{j} \\
&\frac{\mu_{♃}}{r_{\odot \to ♃}} \bar{r}_{\odot \to ♃} = 2.0919 \times 10^{-10}\ km/s^2\ \hat{i} \\
&\text{Net perturbing acceleration} = \frac{\mu_{♃}}{r_{asteroid \to ♃}} \bar{r}_{asteroid \to ♃} - \frac{\mu_{♃}}{r_{\odot \to ♃}} \bar{r}_{\odot \to ♃} \\
&= -1.0460 \times 10^{-10}\ km/s^2\ \hat{i} - 1.8117 \times 10^{-10}\ km/s^2\ \hat{j} \\
&\text{Net acceleration} = -\frac{\mu_\odot + \mu_{asteroid}}{r^3_{\odot \to asteroid}} \bar{r}_{\odot \to asteroid} + \text{Net perturbing acceleration} \\
&= -1.0965 \times 10^{-7}\ km/s^2\ \hat{i} - 1.8993 \times 10^{-7}\ km/s^2\ \hat{j}
\end{aligned}}$$

One good way to check our answer is correct, is to see if the directions of the accelerations are consistent with our intuition. For example, the dominant term is the Sun pulling the asteroid towards the Sun. So it should have negative $\hat{i}$ direction and also negative $\hat{j}$ direction. And indeed it does, which aligns with our intuition. We can do similar analysis for the perturbing accelerations too.

(b) - (ii)

Re-formulating the problem with respect to Jupiter, and considering the Sun as the perturbing body, we get:

$$\ddot{\bar{r}}_{♃ \to asteroid} = \underbrace{-\frac{\mu_{♃} + \mu_{asteroid}}{r^3_{♃ \to asteroid}} \bar{r}_{♃ \to asteroid}}_{Dominant} + (\underbrace{\frac{\mu_\odot}{r_{asteroid \to \odot}} \bar{r}_{asteroid \to \odot}}_{direct\ pert.} - \underbrace{\frac{\mu_\odot}{r_{♃ \to \odot}} \bar{r}_{♃ \to \odot}}_{indirect\ pert.})$$

The relative position vectors change to:

$$\bar{r}_{♃ \to asteroid} = -a_{♃} \cos 60^\circ \hat{i} + a_{♃} \sin 60^\circ \hat{j} = -3.8914 \times 10^8\ km\ \hat{i} + 6.7401 \times 10^4\ km\ \hat{j}$$
$$\bar{r}_{asteroid \to ☼} = -a_{♃} \cos 60^\circ \hat{i} - a_{♃} \sin 60^\circ \hat{j} = -3.8914 \times 10^8\ km\ \hat{i} - 6.7401 \times 10^4\ km\ \hat{j}$$
$$\bar{r}_{♃ \to \odot} = -a_{♃} \hat{i} = -7.7828 \times 10^4\ km\ \hat{i}$$
$$r_{♃ \to asteroid} = r_{asteroid \to ☼} = r_{♃ \to \odot} = a_{♃} = 7.7828 \times 10^4\ km$$

And plug the numbers in:

$$-\frac{\mu_{\jupiter} + \mu_{asteroid}}{r^3_{\jupiter \to asteroid}} \bar{r}_{\jupiter \to asteroid} = 1.0460 \times 10^{-10}\ km/s^2\ \hat{\imath} - 1.8117 \times 10^{-10}\ km/s^2\ \hat{\jmath}$$

$$\frac{\mu_{\odot}}{r_{asteroid \to \odot}} \bar{r}_{asteroid \to \odot} = -1.0955 \times 10^{-7}\ km/s^2\ \hat{\imath} - 1.8974 \times 10^{-7}\ km/s^2\ \hat{\jmath}$$

$$\frac{\mu_{\odot}}{r_{\jupiter \to \odot}} \bar{r}_{\jupiter \to \odot} = -2.1910 \times 10^{-7}\ km/s^2\ \hat{\imath}$$

$$\text{Net perturbing acceleration} = \frac{\mu_{\odot}}{r_{asteroid \to \odot}} \bar{r}_{asteroid \to \odot} - \frac{\mu_{\odot}}{r_{\jupiter \to \odot}} \bar{r}_{\jupiter \to \odot}$$

$$= 1.0955 \times 10^{-7}\ km/s^2\ \hat{\imath} - 1.8975 \times 10^{-7}\ km/s^2\ \hat{\jmath}$$

$$\text{Net acceleration} = -\frac{\mu_{\jupiter} + \mu_{asteroid}}{r^3_{\jupiter \to asteroid}} \bar{r}_{\jupiter \to asteroid} + \text{Net perturbing acceleration}$$

$$= 1.0965 \times 10^{-7}\ km/s^2\ \hat{\imath} - 1.8993 \times 10^{-7}\ km/s^2\ \hat{\jmath}$$

(b) - (iii)

Formulation		$\ddot{\bar{r}}_{\odot \to asteroid}$	$\ddot{\bar{r}}_{\jupiter \to asteroid}$
Dominant	Vector	$-1.0955 \times 10^{-7}\hat{\imath} - 1.8974 \times 10^{-7}\hat{\jmath}$	$1.0460 \times 10^{-10}\hat{\imath} - 1.8117 \times 10^{-10}\hat{\jmath}$
	Magnitude	2.1910×10^{-7}	2.0919×10^{-10}
Net perturbing	Vector	$-1.0460 \times 10^{-10}\hat{\imath} - 1.8117 \times 10^{-10}\hat{\jmath}$	$1.0955 \times 10^{-7}\hat{\imath} - 1.8975 \times 10^{-7}\hat{\jmath}$
	Magnitude	2.0919×10^{-10}	2.1909×10^{-7}
Net total	Vector	$-1.0965 \times 10^{-7}\hat{\imath} - 1.8993 \times 10^{-7}\hat{\jmath}$	$1.0965 \times 10^{-7}\hat{\imath} - 1.8993 \times 10^{-7}\hat{\jmath}$
	Magnitude	2.1931×10^{-7}	2.1931×10^{-7}

Table 1: Summary of accelerations from two formulations (Unit: km/s^2)

Discussions regarding Table 1:

- Dominant term is the largest magnitude in the first formulation (w.r.t. the Sun), and the perturbing forces are the largest magnitudes in the second formulation. Note that the direct/indirect perturbing forces have the same magnitude since they are in the equilateral configuration as in Figure 6.

- The dominant term in the first formulation is on the order of 1e-7, whereas in the second formulation it is on the order of 1e-10. We have about 1000 times larger dominant term in the first formulation.

- In both formulations, the Sun has larger impact on the asteroid, whether is a dominant acceleration (first formulation) or is a perturbing acceleration (second formulation)

- The net accelerations are different in each formulation. But we notice that the magnitude is the same, coming from the equilateral configuration. They should not be the same in general. Depending on the reference body that we are looking at, the acceleration of the asteroid with respect to the body is going to be different.

- Both formulations are mathematically correct! And we gain different information from each formulation.
- If we have to use the two-body problem, it should be Sun-asteroid. This is obvious from the fact that the perturbing acceleration from Jupiter in the first formulation is not large compared to the dominant acceleration. And thus the net total acceleration does not differ much from the dominant acceleration. So we could approximate it as Sun-asteroid two-body problem. This of course depends on the accuracy we want in our model. If we are looking for a higher-fidelity motion of the asteroid, then we should take the Jupiter's effect into account too. So, it depends on what we want to do!

Problem 2

An Introductory Manual (Intro_Manual_F20 GMATR2020a with OpenFrames) for the General Mission Analysis Tool (GMAT) software is posted under GMAT on Brightspace. GMAT is open source and is easily downladed. To obtain some practice using GMAT, step through the manual carefully. Complete all the steps and view the final orbits.

(a) Now use an Epoch of 18 Sept 2020. Produce a satellite orbit with a 'semi-major axis' of 60,000 km. 'eccentricity' of 0.7, and an 'inclination' of 45°. Note that you are using an Earth point mass model. (In the Resource tree, for the 'LowEarthProp' propagator, replace Gravity Model with 'JGM-2' but set degree and order to zero to render a point mass model. Under 'DefaultOrbitView' it will be a cleaner image if you do MOT enable the constellations.) Plot image from GMAT. Also print the summary of the orbit detatils from the "Report" option. Under the Resources tree, right click on 'Output'. You are offered the opportunity to add a report file with numerical data from the simulation. The report will appear in the Output tree. From the output data, determine:

  (i) radius at closest approach or Rad. Peri.
  (ii) radius at farthest excursion or Rad. Apo.
  (iii) energy
  (iv) semi-major axis
  (v) semi-latus rectum
  (vi) angular momentum
  (vii) the Cartesian components of position and velocity at the initial time. With what referecne frame are these associated?

(b) Given the orbit in the scenario in part (a), set the inclination to zero. Add a second spacecraft with an orbit of a different color. With the same eccentricity and zero inclination, try a different semi-major axis, i.e., 40,000km. Add a third satellite with $a = 75,000km$. Repeat the exercise for $a = 60,000km$ and three eccentricities, i.e., $e = 0.2, 0.65, 0.88$ in each case, hold the inclination fixed at 45°. Repeat the exercise for $e = 0.65$ and three semi-major axes, i.e., $a = 20,000km, 35,000km, 75,000km$

You should have three plots from GMAT; use a view that is looking down on the orbit plane for variations in semi-major axis and eccentricity.

Problem 2 Solution

(a)

First, let's summarize the values that we are using for the initial state. Note that we are using the values from the tutorial unless stated otherwise.

Data	Value
Epoch	18 Sep 2020 00:00:00
Semi-major axis	60000km
Eccentricity	0.7
Inclination	45°
RAAN	292.8362°
AOP	218.9805°
TA	180°

Table 2: Summary of the initial state

We should be able to see the report file like below:

Sat1.Earth.RadPer	Sat1.Earth.RadApo	Sat1.Earth.Energy	Sat1.Earth.SMA	Sat1.Earth.SemilatusRectum	Sat1.Earth.HMAG	Sat1.EarthMJ2000Eq.HX	Sat1.EarthMJ2000Eq.HY	Sat1.EarthMJ2000Eq.HZ
18000	102000	-3.321670345833334	60000	30600.00000000001	110440.8145112123	-0	0	110440.8145112123
18000	102000	-3.321670345833334	60000	30600.00000000001	110440.8145112123	1.189246902080999e-15	2.222577679142185e-15	110440.8145112123
18000	102000	-3.321670345833334	60000	30600	110440.8145112123	1.11196232065747e-16	1.990554296913954e-16	110440.8145112123
18000	102000	-3.321670345833334	60000	30600	110440.8145112123	3.657015669199321e-15	6.997204992454691e-15	110440.8145112123
18000	102000	-3.321670345833335	59999.99999999998	30599.99999999999	110440.8145112123	2.756165168112097e-14	5.367773337676224e-14	110440.8145112123
18000	102000	-3.321670345833334	60000	30599.99999999999	110440.8145112123	3.822889453611538e-14	7.641978479875835e-14	110440.8145112123
17999.99999999997	102000	-3.321670345833335	59999.99999999997	30599.99999999997	110440.8145112122	3.672067006138519e-14	7.371180909823331e-14	110440.8145112122
17999.99999999999	102000	-3.321670345833335	59999.99999999998	30599.99999999998	110440.8145112123	6.167542982048272e-14	1.331419477154316e-13	110440.8145112123
17999.99999999999	102000	-3.321670345833335	59999.99999999997	30599.99999999999	110440.8145112123	9.72566491126128e-14	2.21582284563698e-13	110440.8145112123
17999.99999999998	102000	-3.321670345833335	59999.99999999997	30599.99999999999	110440.8145112123	1.045821657787461e-13	2.428923053641197e-13	110440.8145112123
18000	102000	-3.321670345833334	59999.99999999999	30600.00000000002	110440.8145112123	8.72472118672373 4e-14	1.898601083138428e-13	110440.8145112123
17999.99999999999	102000	-3.321670345833335	59999.99999999998	30600.00000000001	110440.8145112123	9.574154355941904e-14	2.197884080460248e-13	110440.8145112123
18000.00000000001	102000	-3.321670345833334	60000	30600.00000000001	110440.8145112123	9.696782698459911e-14	2.263863663021108e-13	110440.8145112123
18000	102000	-3.321670345833334	60000	30600	110440.8145112123	9.3850777920074e-14	2.150852485574384e-13	110440.8145112123

Sat1.EarthMJ2000Eq.X	Sat1.EarthMJ2000Eq.Y	Sat1.EarthMJ2000Eq.Z	Sat1.Earth.RMAG	Sat1.EarthMJ2000Eq.VX	Sat1.EarthMJ2000Eq.VY	Sat1.EarthMJ2000Eq.VZ	Sat1.EarthMJ2000Eq.VMAG
89906.997197181	-48173.97487218684	0	102000	0.5113776454370423	0.9543831219448243	0	1.082753083443258
89937.6190630279	-48116.67932743464	-1.356145707072567e-19	101999.9517265841	0.5093511006281947	0.9554681548439528	-2.471320430459043e-20	1.082754791553499
90028.23996996845	-47945.38334508126	-4.228619489673713e-18	101999.2341941694	0.5033018800716655	0.9586971039494646	-2.23467327471654e-21	1.082780180648299
90294.55796974309	-47426.1274974413	1.486518155790475e-17	101991.88579664	0.4850492559300543	0.9683507912651149	-7.741324399132724e-20	1.083040181905581
90998.3867825897	-45920.37798236495	-3.908340801284868e-16	101928.3449835082	0.4327784357646545	0.9952645135859765	-5.917344234601974e-19	1.085287347418384
92041.26676502806	-43171.27486633325	-1.987456777903457e-15	101662.9419272114	0.3394262672250343	1.040700038130963	-8.376066795230804e-19	1.094653717048505
92829.53231805719	-40302.76277880149	-3.965681293279568e-15	101200.9622384766	0.2441216518189359	1.083728798116711	-8.044844770979143e-19	1.11088410230514
93357.4641805561 8	-37321.24950173235	-7.142630049797487e-15	100540.995532143	0.1465274822135104	1.12441170837333	-1.437362727135414e-18	1.133918865250447
93618.38507244232	-34233.02855596053	-1.375920450461429e-14	99681.00254153396	0.04626999217981616	1.162772167704053	-2.373667551267553e-18	1.163692410460558
93604.55003179822	-31044.38391344813	-2.036316557132898e-14	98618.2820739693	-0.05706943910170498	1.198793221593301	-2.58246197267666e-18	1.200150869273288
93329.09456641444	-27930.41262228216	-2.571361080471087e-14	97418.82693729004	-0.158447362290093	1.230766410544637	-1.99065313308782e-18	1.240923657579956

Figure 7: Report file example

If we put the required data into a table:

Data	Value
Rad.Peri.	$18000 = 1.8 \times 10^4$km
Rad.Apo.	$102000 = 1.02 \times 10^5$km
Energy	-3.3217km^2/s^2
SMA(semi-major axis)	$60000 = 6 \times 10^4$ km
Semi-latus rectum	$30600 = 3.06 \times 10^4$km
Angular momentum	$-71972\hat{i} - 30308\hat{j} + 78093\hat{k}(km^2/s)$
Angular momentum magnitude (HMAG)	$110440.8145 = 1.1044 \times 10^5(km^2/s)$
Initial position	$72586\hat{i} - 55468\hat{j} + 45371\hat{k}(km) = (7.2586\hat{i} - 5.5468\hat{j} + 4.5371\hat{k}) \times 10^4(km)$
Initial distance from the Earth (RMAG)	$102000(km) = 1.0200 \times 10^5(km)$
Initial velocity	$0.28418\hat{i} + 0.85871\hat{j} + 0.59516\hat{k}(km/s)$
Initial velocity magnitude (VMAG)	$1.0827(km/s)$

Table 3: Output data for problem 2(a)

Note that since we used Earth-point mass model and ignored other bodies and perturbations, some of the data will be constant over the simulation. Perigee, apogee, energy, SMA, semi-latus rectum

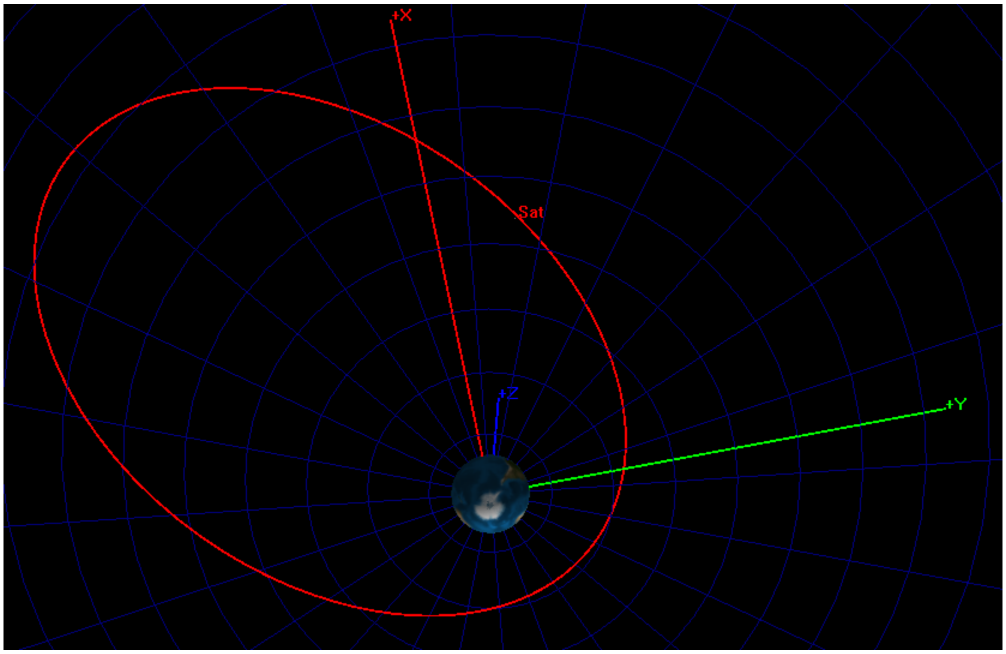


Figure 8: Orbit in EarthJ2000Eq inertial view

and the angular momentum will remain constant. We can see numerical errors building up by watching the variations of these constants. But the positions and velocities evolve over time.

(b)

1. Orbit 1: $a = 60000km$, $e = 0.7$, $i = 0°$
2. Orbit 2: $a = 40000km$, $e = 0.7$, $i = 0°$
3. Orbit 3: $a = 75000km$, $e = 0.7$, $i = 0°$

Since the inclinations are all zero, the viewing direction is parallel to the inertial $\hat{z}$ direction in the figure. This is not the case with the rest of the plots with non-zero inclinations.

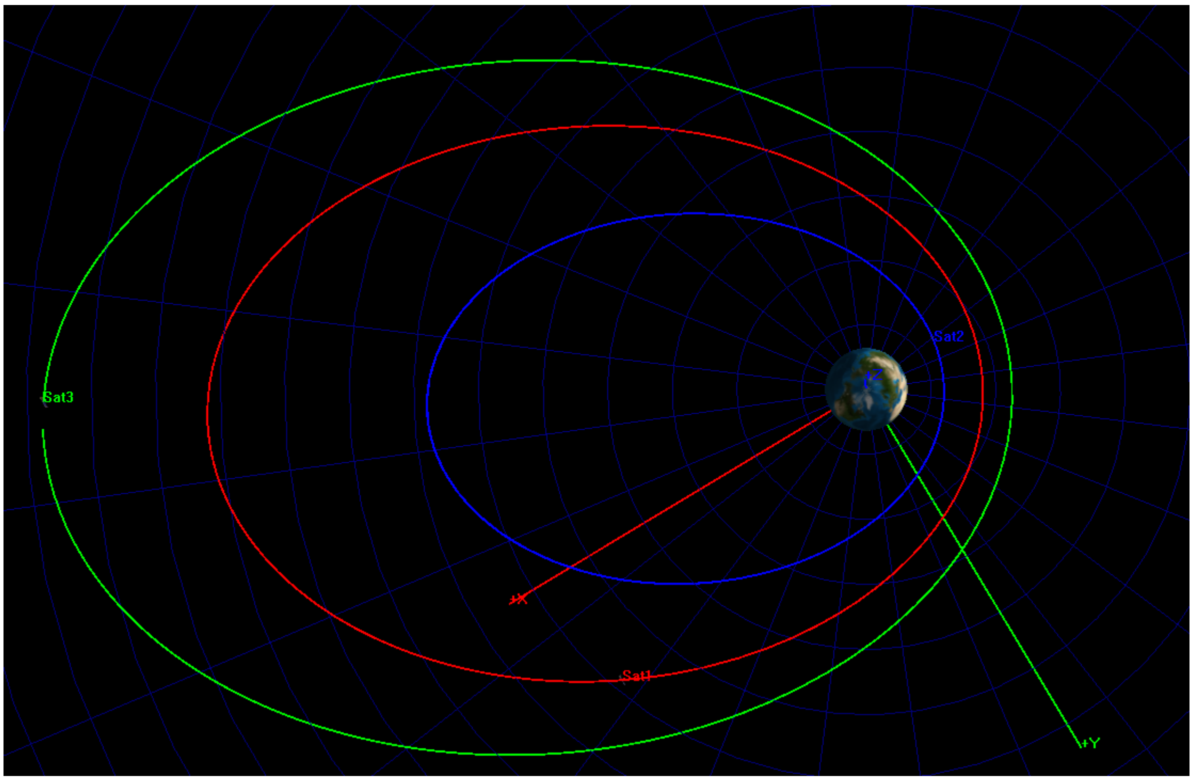


Figure 9: First exercise: three orbits in EarthJ2000Eq inertial view

1. Orbit 1: $a = 60000km$, $e = 0.2$, $i = 45°$
2. Orbit 2: $a = 60000km$, $e = 0.65$, $i = 45°$
3. Orbit 3: $a = 60000km$, $e = 0.88$, $i = 45°$

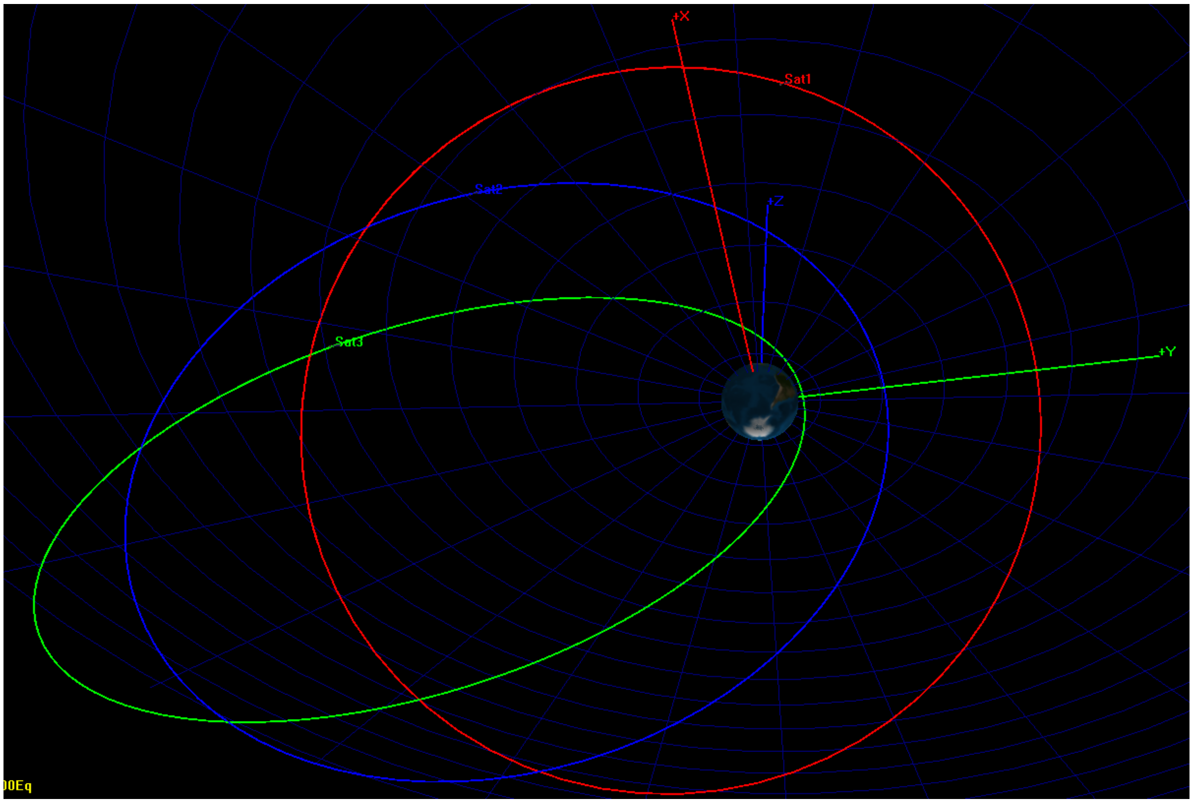


Figure 10: Second exercise: three orbits in EarthJ2000Eq inertial view

1. Orbit 1: $a = 20000km$, $e = 0.65$, $i = 45°$
2. Orbit 2: $a = 35000km$, $e = 0.65$, $i = 45°$
3. Orbit 3: $a = 75000km$, $e = 0.65$, $i = 45°$

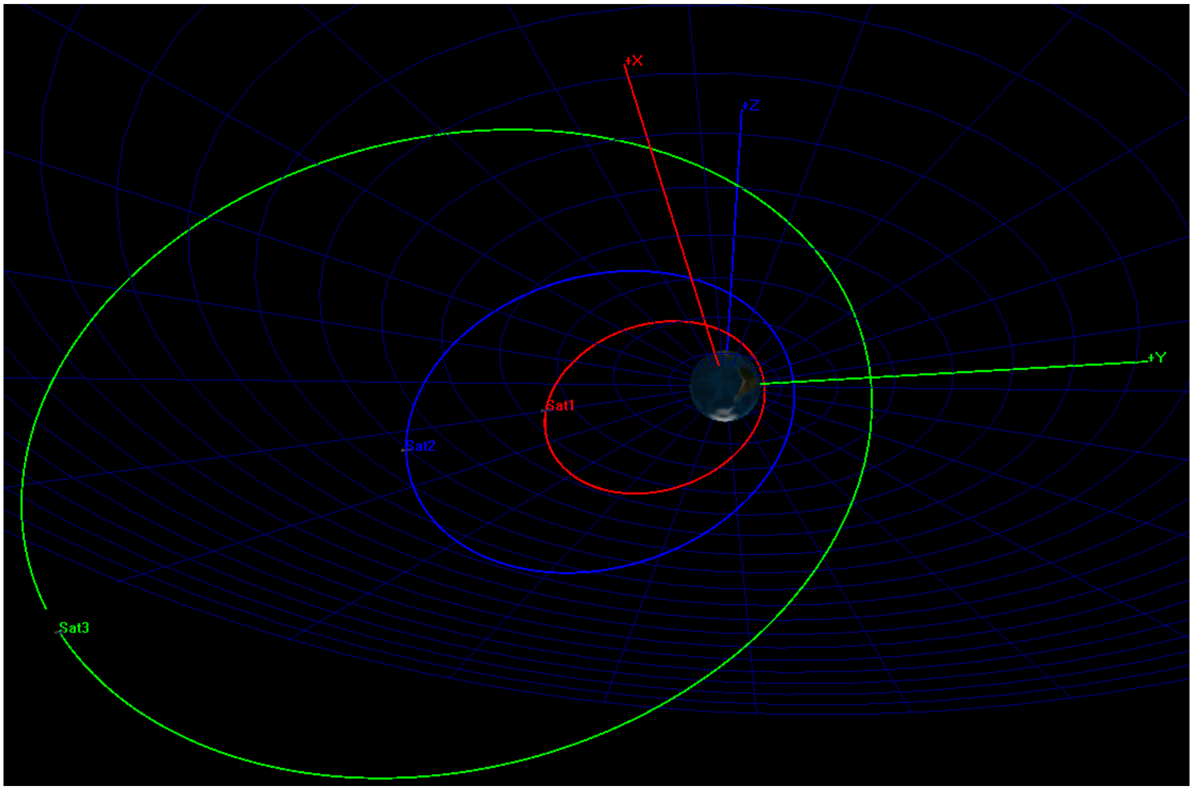


Figure 11: Third exercise: three orbits in EarthJ2000Eq inertial view

Problem 3

Consider only the relative two-body problem (Earth and spacecraft). An Earth-orbiting vehicle is tracked from ground stations; the spacecraft mass is $600kg$. At a certain instant (t_0), the following position and velocity information is obtained relative to an inertial observer:

Altitude = 8560km

Radial component of relative velocity = 2.11km/s

Transverse component of relative velocity = 4.89 km/s

(Our tracking system is perfect so these states are completely without error!!)

(a) Compute the total system angular momentum $\bar{C}_3$, specific angular momentum, total kinetic energy for the system, total energy C_4, specific energy, areal velocity.

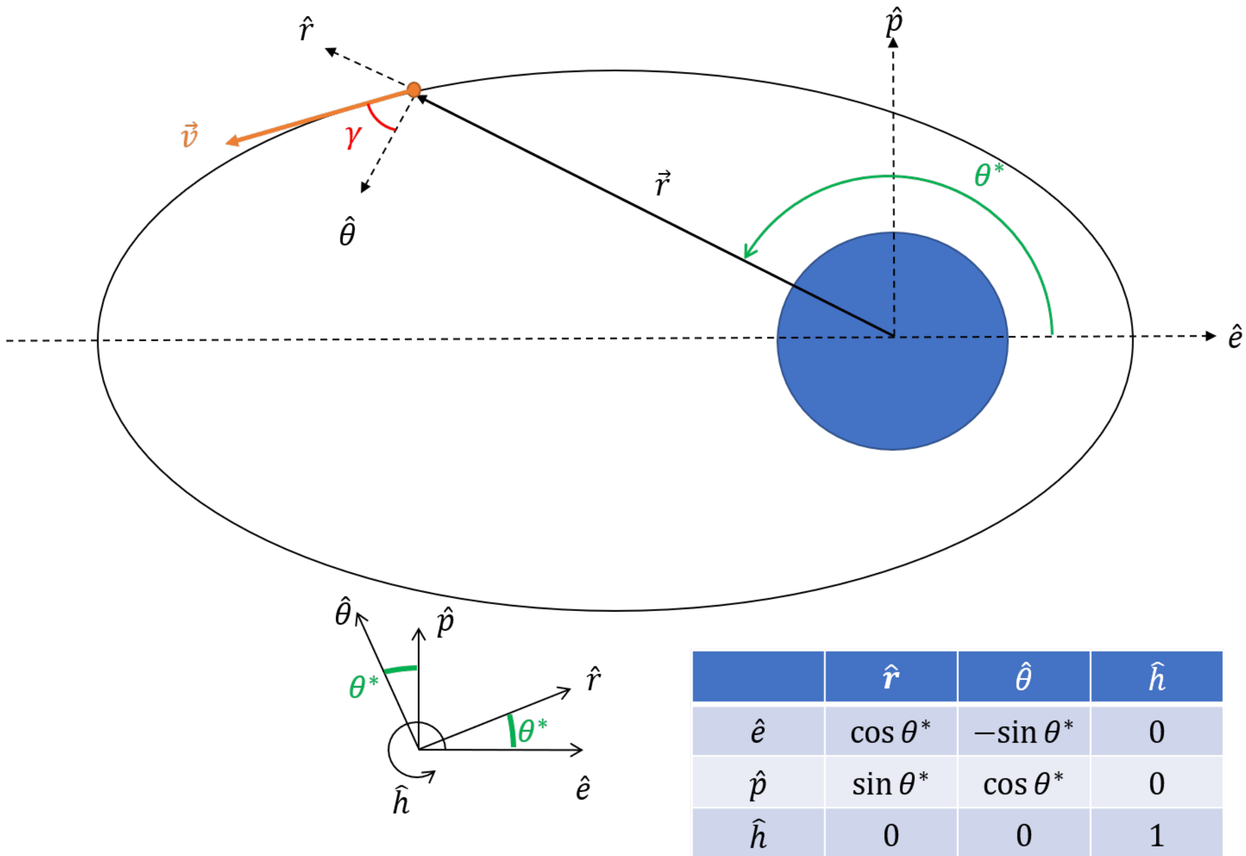
(b) What is the value of the coefficient by which to multiply C_4 to obtain specific energy?

(c) Within the context of the relative two-body problem, determine the following orbital characteristics: $p, e, a, \text{period}, \gamma, \theta^*$. Write the position vector in terms of the inertial unit vectors $\hat{e}, \hat{p}$

(d) Compare this relative velocity to the circular relative velocity at this altitude.

Problem 3 Solution

First, we begin with a sketch of the problem and denote it with the proper unit vectors:



	$\hat{r}$	$\hat{\theta}$	$\hat{h}$
$\hat{e}$	$\cos\theta^*$	$-\sin\theta^*$	0
$\hat{p}$	$\sin\theta^*$	$\cos\theta^*$	0
$\hat{h}$	0	0	1

Figure 12: Orbit in MJ2000 Frame, two sets of unit vectors

(a)

Now, we shall compute the system angular momentum $\vec{C}_3$. We can employ the following expression:

$$\vec{C}_3 = \frac{m_\oplus m_{s/c}}{m_\oplus + m_{s/c}}(\vec{r} \times \dot{\vec{r}}) \tag{1}$$

where we note that $\vec{r}, \dot{\vec{r}}$ are the *relative* position and velocity vectors. To solve for our velocity vector, we can decompose the magnitude of the velocity into its radial and transverse components. Using the bottom left drawing in Figure 12, we note that

$$\vec{r} = (8560\ km + R_\oplus)\hat{r} = (8560 + 6378.1363)\ \hat{r}\ \ \text{km} = 1.4938 \times 10^4 \hat{r}\ km \tag{2}$$
$$\dot{\vec{r}} = \vec{v} = \sin\gamma\ \hat{r} + v\cos\gamma\ \hat{\theta} \tag{3}$$
$$\dot{\vec{r}} = 2.11\ \hat{r} + 4.89\ \hat{\theta}\ \text{km/s} \tag{4}$$

Therefore, we can compute $\vec{C}_3$ as

$$\vec{C}_3 = \frac{m_\oplus m_{s/c}}{m_\oplus + m_{s/c}}(\vec{r} \times \dot{\vec{r}}) = \boxed{4.3828 \times 10^7 \ \hat{h} \ \text{ kg*km}^2/\text{s}} \tag{5}$$

Note that the out-of-plane direction is:

$$\hat{r} \times \hat{\theta} = \hat{e} \times \hat{p} = \hat{h}$$

The specific angular momentum is same expression as $\vec{C}_3$, however mass is not included in the units. The resulting expression then becomes:

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \boxed{7.3047 \times 10^4 \ \hat{h} \ \text{ km}^2/\text{s}} \tag{6}$$

The total kinetic energy T for the system can be expressed in relative quantities as:

$$T = \frac{1}{2}\frac{m_\oplus m_{s/c}}{m_\oplus + m_{s/c}}(\dot{\vec{r}} \cdot \dot{\vec{r}}) = \boxed{8.5093 \times 10^3 \ \text{ kg*km}^2/\text{s}^2} \tag{7}$$

Next, we can evaluate gravitational potential to compute the total energy C_4 of the system:

$$U = \frac{G m_\oplus m_{s/c}}{r} = 1.6010 \times 10^4 \ \text{ kg*km}^2/\text{s}^2 \tag{8}$$

$$C_4 = T - U = \boxed{-7.5008 \times 10^3 \ \text{ kg*km}^2/\text{s}^2} \tag{9}$$

To obtain specific energy, we simply multiply C_4 by a mass-based quantity as follows:

$$\mathscr{E} = C_4 \frac{m_\oplus + m_{s/c}}{m_\oplus m_{s/c}} = \frac{v^2}{2} - \frac{\mu_\oplus + \mu_{s.c}}{r} = \boxed{-12.501 \ \text{ km}^2/\text{s}^2} \tag{10}$$

Finally, we compute areal velocity as

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{h}{2} = \boxed{3.6079 \times 10^4 \ \text{ km}^2/\text{s}} \tag{11}$$

(b) It follows that the value of the coefficient by which C_4 is multiplied to obtian specific energy $\mathscr{E}$ is

$$\mathscr{E} = C_4 \boxed{\frac{m_\oplus + m_{s/c}}{m_\oplus m_{s/c}}} \tag{12}$$

which stems from the definition of center of mass. Since we are expressing information pertaining to the *relative vector*, we are not surprised that this expression is used to obtain specific energy which does not contain mass units.

(c) Now, we can solve for specific parameters beginning with semilatus rectum

To solve for semilatus rectum, we express

$$\boxed{p = \frac{h^2}{\mu_\oplus} = 1.3387 \times 10^4 \text{ km}} \tag{13}$$

Next, we solve for semi-major axis using specific energy

$$\boxed{a = -\frac{\mu_\oplus}{2\mathscr{E}} = 1.5942 \times 10^4 \text{ km}} \tag{14}$$

now, we proceed to solve for eccentricity as follows:

$$\boxed{e = \sqrt{1 + \frac{2\mathscr{E}h^2}{\mu_\oplus^2}} = 0.40038} \tag{15}$$

To solve for period, we can write

$$\boxed{\mathbb{P} = 2\pi\sqrt{\frac{a^3}{\mu_\oplus}} = 2.0033 \times 10^4 \text{ sec or } 5.5646 \text{ hrs}} \tag{16}$$

For the flight path angle:

$$\boxed{\gamma = \arctan(\frac{v_r}{v_\theta}) = 23.340°}$$

Next, we can solve inversely for true anomaly using the polar equation

$$\boxed{\theta^* = \cos^{-1}\left(\frac{1}{e}\Big(\frac{p}{r} - 1\Big)\right) = \pm 105.03° \ \rightarrow \ 105.03°} \tag{17}$$

We note that the true anomaly is positive to be consistent with the positive flight path angle given in the initial specifications of the problem. This indicates that the spacecraft is on the ascending leg of its orbit. In the polar frame,

$$\vec{r} = \boxed{1.4756 \times 10^4 \ \hat{r} \ \text{ km}} \tag{18}$$

$$\dot{\vec{r}} = \boxed{\vec{v} = v\sin\gamma \ \hat{r} + v\cos\gamma \ \hat{\theta} = 2.11 \ \hat{r} + 4.89 \ \hat{\theta} \text{ km/s}} \tag{19}$$

Using the DCM formulated in figure 6, we can obtain the position and velocity vectors in the inertial $\hat{e} - \hat{p} - \hat{h}$ coordinate system

$$\underset{\hat{e},\hat{p},\hat{h}}{\vec{r}} = DCM * \underset{\hat{r},\hat{\theta},\hat{h}}{\vec{r}} = \boxed{-0.3875 \times 10^4 \ \hat{e} + 1.4427 \times 10^4 \ \hat{p} \ \text{ km}} \tag{20}$$

$$\underset{\hat{e},\hat{p},\hat{h}}{\dot{\vec{r}}} = DCM * \underset{\hat{r},\hat{\theta},\hat{h}}{\dot{\vec{r}}} = \boxed{-5.2699 \ \hat{e} + 0.7693 \ \hat{p} \ \text{ km/s}} \tag{21}$$

(d) If we compare the current velocity with the circular velocity at that altitude we find that

$$\boxed{v = 5.3258 \text{ km/s} \quad > \quad v_c = \sqrt{\frac{\mu_\oplus}{r}} = 5.1656 \text{ km/s}} \tag{22}$$

the current velocity at this altitude is *larger than* the circular velocity at this altitude.