AAE 421 Homework 1: Solutions
Fall 2020

Problem 1 (10pts)

Show that $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} + \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) = 0$

Solution

We have vector triple product:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a}.\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}.\boldsymbol{b})\boldsymbol{c}$$

Expanding the given expression using the above formula:

$$\begin{aligned}
\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) + \boldsymbol{v} \times (\boldsymbol{w} + \boldsymbol{u}) + \boldsymbol{w} \times (\boldsymbol{u} \times \boldsymbol{v}) &= [(\boldsymbol{u}.\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u}.\boldsymbol{v})\boldsymbol{w}] + [(\boldsymbol{v}.\boldsymbol{u})\boldsymbol{w} - (\boldsymbol{v}.\boldsymbol{w})\boldsymbol{u}] + [(\boldsymbol{w}.\boldsymbol{v})\boldsymbol{u} - (\boldsymbol{w}.\boldsymbol{u})\boldsymbol{v}] \\
&= [(\boldsymbol{u}.\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{w}.\boldsymbol{u})\boldsymbol{v}] + [-(\boldsymbol{u}.\boldsymbol{v})\boldsymbol{w} + (\boldsymbol{v}.\boldsymbol{u})\boldsymbol{w}] + [-(\boldsymbol{v}.\boldsymbol{w})\boldsymbol{u} + (\boldsymbol{w}.\boldsymbol{v})\boldsymbol{u}] \\
&= 0 \\
&\quad q.e.d
\end{aligned}$$

Problem 2 (20pts)

Two particles moving with constant velocity are described by the position vectors:

$$\boldsymbol{p} = \boldsymbol{p_0} + \boldsymbol{v}t, \quad \boldsymbol{s} = \boldsymbol{s_0} + \boldsymbol{w}t$$

1. Show that the shortest distance between their trajectories is given by:

$$d = \frac{(|\boldsymbol{s_0} - \boldsymbol{p_0}).(\boldsymbol{w} \times \boldsymbol{v})|}{|\boldsymbol{w} \times \boldsymbol{v}|}$$

2. Find the shortest distance between the particles themselves.

Solution

(a)

Shortest distance between the two trajectories.

Let $\boldsymbol{d}$ be a vector representing the shortest disatnce between the trajectories, and defined by:

$$\boldsymbol{p} = \boldsymbol{p_0} + \boldsymbol{v}t_1 + \boldsymbol{d} = \boldsymbol{s_0} + \boldsymbol{w}t_2 \tag{1}$$

Where, t_1, t_2 are the times of closest approach to the respective particles to the other trajectory. The trajectories are rectilinear, with orientations defined by the velocity vectors $\boldsymbol{v}, \boldsymbol{w}$. For shortest distance, vector $\boldsymbol{d}$ must be orthogonal to both $\boldsymbol{w}, \boldsymbol{v}$ and can be written as:

$$\boldsymbol{d} = |\boldsymbol{d}| \frac{\boldsymbol{w} \times \boldsymbol{v}}{|\boldsymbol{w} \times \boldsymbol{v}|} \tag{2}$$

Also from (1),

$$(\boldsymbol{s_0} - \boldsymbol{p_0}) + \boldsymbol{w}t_2 - \boldsymbol{v}t_1 = \boldsymbol{d} \tag{3}$$

Taking dot product of $\boldsymbol{w} \times \boldsymbol{v}$ and eqn.(3), and making use of (2):

$$(\boldsymbol{s_0} - \boldsymbol{p_0}).(\boldsymbol{w} \times \boldsymbol{v}) = |\boldsymbol{d}||\boldsymbol{w} \times \boldsymbol{v}| \tag{4}$$

Therefore, the shortest distance between the trajectories is given by:

$$|\boldsymbol{d}| = \frac{(\boldsymbol{s_0} - \boldsymbol{p_0}).(\boldsymbol{w} \times \boldsymbol{v})}{|\boldsymbol{w} \times \boldsymbol{v}|} \tag{5}$$

$$q.e.d$$

(b)

Shortest distance between particles
At any time, t, the vector distance between the particles is:

$$(\boldsymbol{s_0} - \boldsymbol{p_0}) + (\boldsymbol{w} - \boldsymbol{v})t = \boldsymbol{a} + \boldsymbol{b}t$$

whose magnitude is,

$$(\boldsymbol{a} + \boldsymbol{b}t).(\boldsymbol{a} + \boldsymbol{b}t) = |a|^2 + 2(\boldsymbol{a.b}) + |b|^2 t$$

Setting derivative to zero yields the conditions for minimum distance:

$$t = -\frac{\boldsymbol{a.b}}{|\boldsymbol{b}|^2}$$

The negative sign confirms that, if the relative velocity $\boldsymbol{b} = (\boldsymbol{w} - \boldsymbol{v}$ makes an angle θ with initial relative position $\boldsymbol{a} = (\boldsymbol{s}_0 - \boldsymbol{p}_0)$, the trajectories will close if $\theta > \pi/2$. Substituting for t yields:

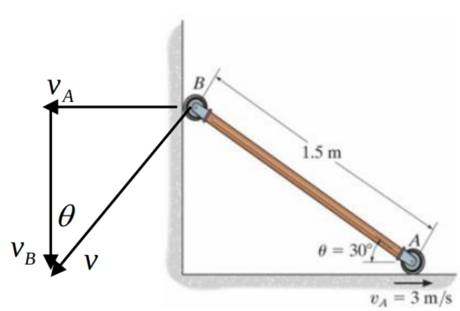
$$(min\ distance)^2 = |\boldsymbol{a}|^2 - \frac{(\boldsymbol{a.b})^2}{|\boldsymbol{b}|} = |\boldsymbol{a}|^2(1 - \cos^2\theta) = |\boldsymbol{a}|^2 \sin^2\theta$$

$$\therefore min.\ distance = |\boldsymbol{s}_0 - \boldsymbol{p}_0| \sin\theta$$

note: This solution could also have been derived very simply by keeping $\boldsymbol{p}_0$ stationary, applying the relative velocity at $\boldsymbol{s}_0$, and solving the right-triangle for the closest approach to $\boldsymbol{p}_0$.

Problem 3 (15pts)

If a roller A moves to the right with a constant velocity of $v_A = 3m/s$, determine the angular velocity of the link and the velocity of the roller B when $\theta = 30^0$.

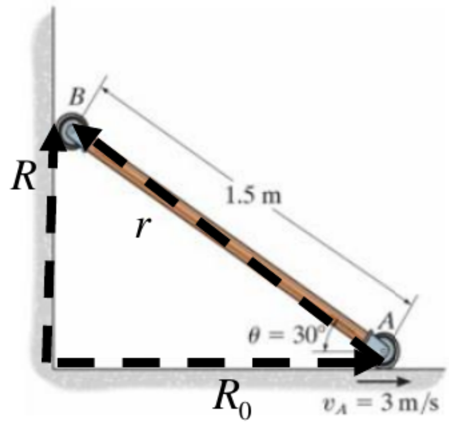


Solution Method:1

The velocity diagram of the point B is relation to reference point A is shown in the figure. From this diagram we obtain:

1. $\sin\theta = \frac{v_A}{v} \implies v = l\dot{\theta} = v_A \sin\theta \implies \dot{\theta} = \frac{v_A}{l\sin\theta} = \frac{3}{(1.5)(0.5)} = 4\, rad/sec$
2. $\tan\theta = \frac{v_A}{v_B} \implies v_B = \frac{V_A}{\tan\theta} = \frac{3}{0.5774} = 5.196\, m/s$

Solution Method:2



$$\bar{R} = \bar{R}_0 + \bar{r} \implies \frac{d\bar{R}}{dt} = \frac{d\bar{R}_0}{dt} + \frac{d\bar{r}}{dt} \tag{6}$$

$$\bar{V} = \bar{V}_0 + \frac{\partial \bar{r}}{\partial t} + \omega \times \bar{r} \tag{7}$$

where,

$$V_0 = 3\hat{\imath} + 0\hat{\jmath} + 0\hat{k} \tag{8}$$

$$\frac{\partial \bar{r}}{\partial t} = 0 \tag{9}$$

$$\bar{V}_R = 0\hat{\imath} + V_B\hat{\jmath} + 0\hat{k} \tag{10}$$

and

$$\bar{\omega} \times \bar{r} = (0\hat{\imath} + 0\hat{\jmath} + \omega\hat{k}) \times (-1.5\cos 30^0 \hat{\imath} + -1.5\sin 30^0 \hat{\jmath} + 0\hat{k}) = \left\| \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \omega \\ -1.3 & -0.75 & 0 \end{bmatrix} \right\| = 0.75\omega\hat{\imath} + 1.3\hat{\jmath} + 0\hat{k} \tag{11}$$

Substituting above results in (2):

$$0\hat{\imath} + v_b\hat{\jmath} + 0\hat{k} = (3\hat{\imath} + 0\hat{\jmath} + 0\hat{k}) + (0.75\hat{\imath} + 1.3\hat{\jmath} + 0\hat{k}) = ((3 + 0.75\omega)\hat{\imath} + 1.3\omega\hat{\jmath} + 0\hat{k} \tag{12}$$

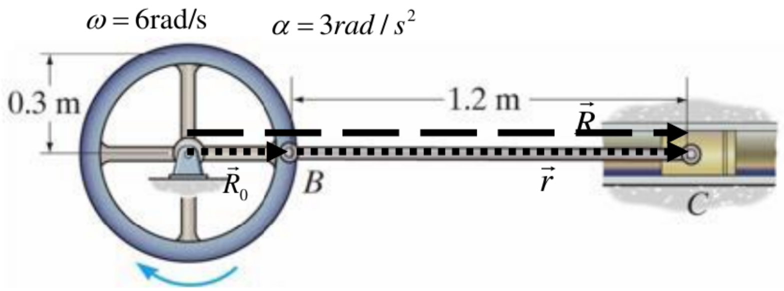
Hence,

$$3 + 0.75\omega = 0 \implies \omega = -3/0.75 = -4\, rad/sec \tag{13}$$

$$v_B = 1.3\omega = 1.3 \times -4 = -5.2\, m/s \tag{14}$$

Problem 4 (15pts)

Determine the angular acceleration of link BC and the acceleration of the piston C at the instant shown below where the wheel has angular velocity of $\omega = 6\ rad/s$ and angular acceleration $\alpha = 3\ rad/s^2$.



Solution:

Given:

$$\bar{R}_0 = R_0\hat{i} + 0\hat{j} + 0\hat{k}$$
$$\bar{\omega}_0 = 0\hat{i} + 0\hat{j} + \omega_0\hat{k}$$
$$\bar{r} = r\hat{i} + 0\hat{j} + 0\hat{k}$$
$$\bar{\omega}_{BC} = 0\hat{i} + 0\hat{j} + \omega_{BC}\hat{k}$$
$$\bar{R} = R\hat{i} + 0\hat{j} + 0\hat{k}$$

Velocities:

$$\bar{V}_0 = \bar{\omega}_0 \times \bar{R}_0 = (0\hat{i} + 0\hat{j} + \omega_0\hat{k}) \times (R_0\hat{i} + 0\hat{j} + 0\hat{k}) = 0\hat{i} + R_0\omega_0\hat{j} + 0\hat{k}$$
$$\bar{v}_r = \bar{\omega}_{BC} \times \bar{r} = (0\hat{i} + 0\hat{j} + \omega_{BC}\hat{k}) \times (r\hat{i} + 0\hat{j} + 0\hat{k}) = 0\hat{i} + r\omega_{BC}\hat{j} + 0\hat{k}$$
$$\bar{V} = \bar{V}_0 + \bar{v}_r = 0\hat{i} + (R_0\omega_0 + r\omega_{BC})\hat{j} + 0\hat{k}$$

Hence,

$$\omega_{BC} = -(R_0/r)\omega_0$$

Accelerations:

$$\bar{a}_C = \frac{d\bar{V}_0}{dt} + \left(\frac{\partial \bar{\omega}_{BC}}{\partial t}\right) \times \bar{r} + \omega_{BC} \times \bar{v}_r$$
$$\frac{d\bar{V}_0}{dt} = \frac{\partial V_0}{\partial t} + \bar{\omega}_0 \times \bar{V}_0$$
$$= (0\hat{i} + R_0\alpha_0\hat{j} + 0\hat{k}) + (0\hat{i} + 0\hat{j} + \omega_0\hat{k}) \times (0\hat{i} + R_0\omega_0\hat{j} + 0\hat{k})$$
$$= -R_0\omega_0^2\hat{i} + R_0\alpha_0\hat{j} + 0\hat{k}$$
$$\left(\frac{\partial \bar{\omega}_{BC}}{\partial t}\right) \times \hat{r} = (0\hat{i} + 0\hat{j} + \alpha_{BC}\hat{k}) \times (r_x\hat{i} + 0\hat{j} + 0\hat{k}) = 0\hat{i} + r_x\alpha_{BC}\hat{j} + 0\hat{k}$$
$$\bar{\omega}_{BC} \times \bar{v}_r = (0\hat{i} + 0\hat{j} + \omega_{BC}\hat{k}) \times (0\hat{i} + r_x\omega_{BC}\hat{j} + 0\hat{k}) = -r_x\omega_{BC}^2\hat{i} + 0\hat{j} + 0\hat{k}$$

hence,

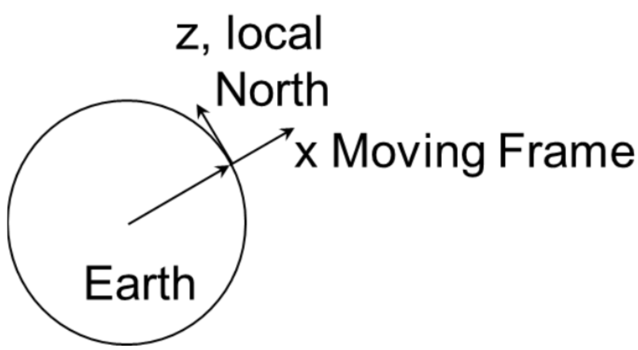
$$\bar{a}_c = (-R_{0_x}\omega^2\hat{\imath} + R_{0_x}\alpha_0\hat{\jmath} + 0\hat{k}) + (0\hat{\imath} + r_x\alpha_{BC}\hat{\jmath} + 0\hat{k}) + (-r_x\omega_{BC}^2\hat{\imath} + 0\hat{\jmath} + 0\hat{k})$$
$$= -(R_{0_x}\omega_0^2 + r_x\omega_{BC}^2)\hat{\imath} + (R_{0_x}\alpha_0 + r_x\alpha_{BC})\hat{\jmath} + 0\hat{k} = a_c\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

And so,

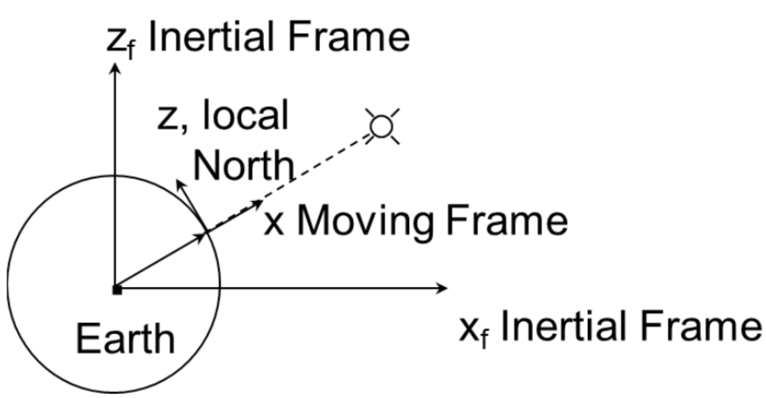
$$a_C = -(R_{0_x}\omega_0^2 + r_x\omega_{BC}^2) = -\left[R_{0_x}\omega_0^2 + r\left(\frac{R}{r}\right)^2\omega_0^2\right] = -R_0^2\omega_0^2\left(1 + \frac{R_0}{r}\right) = -13.5\ m/s^2$$
$$\alpha_{BC} = -\frac{R_0}{r}\alpha_0 = 0.75\ rad/s^2\ ccw$$

Problem 5. (20pt)

An observer positioned at 30 degrees North latitude on the surface of the earth tracks a satellite.



You may assume that the inertial frame of reference for the problem is attached to the center of the earth, but doesn't rotate with the earth. The zf axis passes through the North Pole of the earth. The xf axis passes through the equator at a point nearest the observer. You may assume that the radius of the earth is 4000 miles.



At a particular instant, the satellite appears to be 1000 miles directly above him and, by his observations, appears to be traveling 5000 miles per hour due North (local North). What is the absolute velocity of the satellite in terms of the unit vectors of the moving coordinate system within which the observer resides.

Solution:

$$\begin{aligned}
\bar{V}_p &= \bar{V}_0 + \frac{\partial \bar{r}}{\partial t} + \bar{\omega} \times \bar{r} \\
\bar{\omega} &= \frac{2\pi}{24}\hat{k}\ rad/hour = \frac{2\pi}{24}(\sin(30^o)\hat{\imath} + \cos(30^o)\hat{k})\ rad/hour \\
\bar{V}_0 &= \frac{d\bar{R}}{dt} = \bar{\omega} \times \bar{R} = \frac{2\pi}{24}(\sin(30^o)\hat{\imath} + \cos(30^o)\hat{k}) \times (4000\hat{\imath}) = 906.90\hat{\jmath}\ mph \\
\frac{\partial \bar{r}}{\partial t} &= 5000\hat{k}\ mph \\
\bar{\omega} \times \bar{r} &= \frac{2\pi}{24}\hat{k}\ rad/hour = \frac{2\pi}{24}(\sin(30^o)\hat{\imath} + \cos(30^o)\hat{k}) \times 1000\hat{\imath} = 226.72\hat{\jmath}\ mph \\
\bar{V}_p &= 906.9\hat{\jmath} + 5000\hat{k} + 226.72\hat{\jmath} = 1133.6\hat{\jmath} + 5000\hat{k}\ mph
\end{aligned}$$

Problem 6 (20 pts)

Expand the equations of motions for translation $m(\dot{V}|_{body} + \omega \times V) = F$ and rotation $I\dot{\omega} + \omega \times (I\omega) = M$ into components in the body-fixed reference frame where $V = u\hat{\imath} + v\hat{\jmath} + w\hat{k}$, $\omega = p\hat{\imath} + q\hat{\jmath} + r\hat{k}$, $F = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$, $M = L\hat{\imath} + M\hat{\jmath} + B\hat{k}$ and I is the moment of inertia matrix given by $I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$.

Solution:

After expansion, we obtain:

$$m(\dot{u} - rv + qw) = F_x$$

$$m(\dot{v} - pw + ru) = F_y$$

$$m(\dot{w} - qu + pv) = F_z$$

and

$$\begin{aligned}
I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{yz}(q^2 - r^2) - I_{zx}(\dot{r} + pq) - I_{xy}(\dot{q} - rp) &= L \\
I_{yy}\dot{q} - (I_{zz} - I_{xx})rp - I_{zx}(r^2 - p^2) - I_{xy}(\dot{p} + qr) - I_{yz}(\dot{r} - pq) &= M \\
I_{zz}\dot{r} - (I_{xx} - I_{yy})pq - I_{xy}(p^2 - q^2) - I_{yz}(\dot{q} + rp) - I_{zx}(\dot{p} - qr) &= N
\end{aligned}$$