College of Engineering
School of Aerospace Engineering

AE 6511: Optimal Guidance and Controls

HW3

Professor:
Panagiotis Tsiotras
Gtech AE Professor

Student:
Tomoki Koike
Gtech MS Student

October 14, 2021

Table of Contents

Problem 1

Consider the problem of minimizing

$$\mathcal{J}(x) = \int_0^1 \big(-a^2x^2(t) + \dot{x}^2(t) \big) dt$$

subject to $x(0) = x(1) = 0$. Show that if these boundary conditions are satisfied, then all solutions of Euler's equation are of the form

$$\begin{aligned} &x(t) = 0, & & \text{if } a \neq n\pi \\ &x(t) = A\sin(n\pi t) \quad \text{or} \quad x(t) = 0, & & \text{if } a = n\pi \end{aligned}$$

(a) Show that $\mathcal{J}(x(t)) = 0$ for all these solutions.

(b) Do all the solutions actually minimize $\mathcal{J}$? What does the Legendre condition give?

(c) Are there some values of a^2 such that $\mathcal{J}$ can be negative? To answer this, evaluate $\mathcal{J}(x)$ for a few choices of $x(t)$:

$$\begin{aligned} &x(t) = t(1-t) \\ &x(t) = t^m(1-t), \qquad m > 0 \\ &x(t) = \sin(\pi t) \end{aligned}$$

Solution:
(a) Regardless of $a \neq n\pi$ or $a = n\pi$, when $x(t) = 0$

$$\boxed{\mathcal{J}(0) = \int_0^1 (0)dt = 0 \qquad \text{if} \quad a \neq n\pi \quad \text{or} \quad a = n\pi.}$$

If $a = n\pi$ and $x(t) = A\sin(n\pi t)$

$$\begin{aligned} &\mathcal{J}\big(A\sin(n\pi t)\big) \\ &= \int_0^1 \Big(-a^2A^2\sin^2(n\pi t) + n^2\pi^2A^2\cos^2(n\pi t)dt \\ &= \int_0^1 \Big[-\frac{1}{2}a^2A^2\big(1 - \cos(2n\pi t)\big) + \frac{1}{2}n^2\pi^2A^2\big(1 + \cos(2n\pi t)\big)\Big] dt \\ &= \Big[-\frac{1}{2}a^2A^2\big(t - \frac{1}{2n\pi}\sin(2n\pi t)\big) + \frac{1}{2}n^2\pi^2A^2\big(t + \frac{1}{2n\pi}\sin(2n\pi t)\big)\Big]_0^1 \\ &= -\frac{1}{2}a^2A^2 + \frac{1}{2}n^2\pi^2A^2 = -\frac{1}{2}n^2\pi^2A^2 + \frac{1}{2}n^2\pi^2A^2 = 0 \end{aligned}$$

Hence,

$$\boxed{\mathcal{J}\big(A\sin(n\pi t)\big) = 0 \qquad \text{if} \quad a = n\pi.}$$

(b) Let

$$F(y,r,t) = -a^2y^2 + r^2$$

then

$$F_{rr} = \frac{\partial}{\partial r}\left(\frac{\partial F}{\partial r}\right) = 2 > 0.$$

Hence, the Legendre conditions is satisfied. Next, if we use the Jacobi equation

$$\frac{d}{dt}\left(F_{yr}\phi + F_{rr}\dot{\phi}\right) = F_{yr}\dot{\phi} + F_{yy}\phi$$
$$\ddot{\phi} + a^2\phi = 0$$

which gives

$$\phi = A\cos(at) + B\sin(at)$$

and since $x(0) = x(1) = 0$, $A = 0$ and

$$\phi = B\sin(at).$$

Now, with this sinusoidal function which oscillates we know that

$$\phi(0) = 0$$
$$\phi\left(\frac{\pi}{a}\right) = 0 \qquad \text{if} \quad \frac{\pi}{a} < 1 \quad \text{such as} \quad a = n\pi.$$

Thus, we know that there exists a conjugate point such that $t_2 = \pi/a < 1 = t_1$, and therefore, all solutions do not minimize $\mathcal{J}$.

(c) We evaluate $\mathcal{J}$ for the given choices, (calculations are done with MATLAB in Problem 1: MATLAB Code)
First choice:

$$\mathcal{J}\big(t(1-t)\big) = \frac{1}{3} - \frac{1}{30}a^2$$

For this choice $\mathcal{J} < 0$ when

$$\boxed{a < -\sqrt{10}, \qquad \sqrt{10} < a.}$$

Second choice:

$$\mathcal{J}\big(t^m(1-t)\big) = \begin{cases} -0.0833\,a^2 + \infty & \text{if } \ m = 0.5000 \\[2ex] \dfrac{a^2 - m\,(2\,a^2-3) + 5\,m^2 + 2\,m^3}{(2\,m-1)\,(2\,m+1)\,(2\,m+3)\,(m+1)} & \text{if } \ 0.5000 < m \\[2ex] -\dfrac{a^2 - \infty}{(2\,m+1)\,(2\,m+3)\,(m+1)} & \text{if } \ 0 < m < 0.5000 \end{cases}$$

For this choice (assuming that m is an integer value and $0.5000 < m$) $\mathcal{J} < 0$ when

$$\boxed{\begin{gathered} a^2 - m(2a^2-3) + 5m^2 + 2m^3 < 0 \\[2ex] a < -\sqrt{\frac{2m^3+5m^2+3m}{2m-1}}, \qquad \sqrt{\frac{2m^3+5m^2+3m}{2m-1}} < a. \end{gathered}}$$

Third choice:

$$\mathcal{J}\big(\sin(\pi t)\big) = \frac{\pi^2}{2} - \frac{a^2}{2}$$

For this choice $\mathcal{J} < 0$ when

$$\boxed{a < -\pi, \qquad \pi < a.}$$

Hence, there exists some values of a^2 such that $\mathcal{J}$ can be negative.

Problem 2

Consider a particle sliding along a ramp from point $(0, 0)$ to the point (a, b) under the force of gravity with zero initial velocity and assuming no friction.

(a) Show that the trip takes t_f seconds, where

$$t_f = \sqrt{\frac{2(a^2 + b^2)}{gb}}.$$

(b) Show that the brachistochrone solution is a cycloid given by

$$\begin{aligned} x &= \alpha + \beta(\psi + \sin(\psi)) \\ y &= \beta(1 + \cos(\psi)) \end{aligned}$$

The curve is parameterized by ψ, with constants α and β. If ψ_1 and ψ_2 are the values of the parameter ψ at the initial and final points, respectively, show that the time to transverse the cycloid is

$$t_f = \sqrt{\frac{\beta}{g}}(\psi_2 - \psi_1).$$

(c) Show that $\psi_2 = \theta + \pi$, where θ satisfies

$$(1 - \cos(\theta)) - \frac{b}{a}(\theta - \sin(\theta)) = 0$$

and solve for ψ_1, α, and β.

(d) How much faster than the ramp is the cycloid? Let $a = 4$ ft and $b = 2$ ft and compare the time difference. Where is the particle on the ramp when the particle on the cycloid finishes? Show that this distance is more pronounced for $a \gg b$. (Assume that the gravitational acceleration is $g = 32$ ft/sec^2.)

Solution:
(a) The Bernoulli's Brachistochrone Problem is visually explained in Figure 1. From Pythagorean Theorem we know that

$$\Delta s = \sqrt{\Delta x + \Delta y} = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\Delta x$$

where $\Delta x \neq 0$. Then from the conservation of energy law

$$\begin{aligned} \frac{1}{2}mv^2 - mgy &= 0 \\ v &= \sqrt{2gy}. \end{aligned}$$

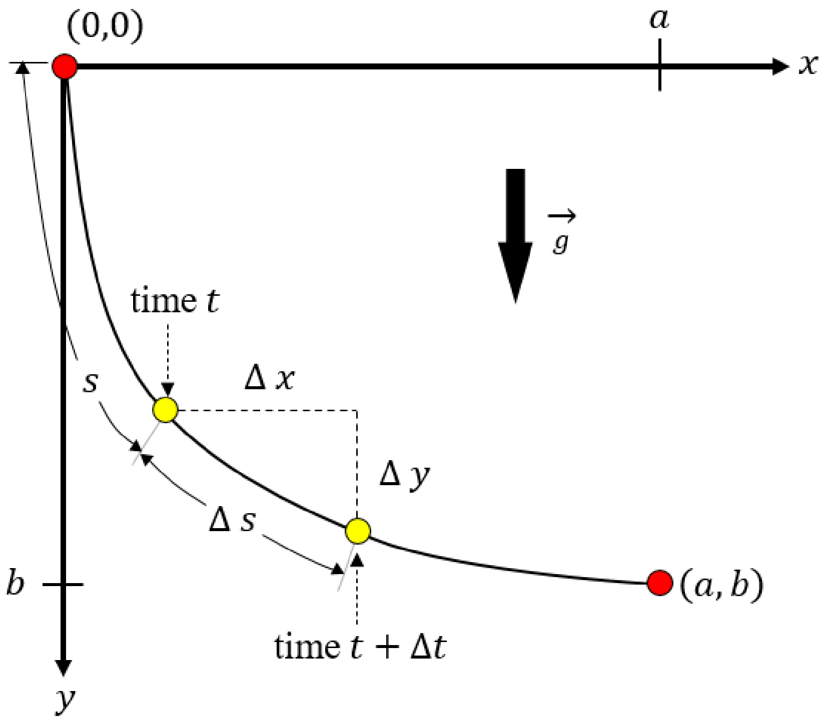

Figure 1: Bernoulli's Brachistochrone problem diagram

Next, we compute the velocity in another method

$$\begin{aligned} v = \frac{\Delta s}{\Delta t} &= \frac{\Delta s}{\Delta x}\frac{\Delta x}{\Delta t} \\ &= \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\frac{\Delta x}{\Delta t}. \end{aligned}$$

By equating the two velocity expressions we have

$$\sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2}\frac{\Delta x}{\Delta t} = \sqrt{2gy}$$

$$\Delta t = \sqrt{\frac{1+\left(\frac{\Delta y}{\Delta x}\right)^2}{2gy}}\Delta x$$

Now if we let y be a function of x, i.e. $y = y(x)$, we can rewrite the above equation into

$$\Delta t = \sqrt{\frac{1+(y')^2}{2gy}}\Delta x$$

and also let $\Delta t = dt$ and $\Delta x = dx$, which brings us to

$$dt = \sqrt{\frac{1+(y')^2}{2gy}}dx$$

Therefore,

$$T(x) = \int_0^x \sqrt{\frac{1+(y')^2}{2gy}}dx.$$

Now, at (a, b) let $m = b/a$. Then,

$$\begin{aligned} t_f = T(a) &= \int_0^a \sqrt{\frac{1+m^2}{2gmx}}dx = \sqrt{\frac{1+m^2}{2gm}}\left[2\sqrt{x}\right]_0^a \\ &= \sqrt{\frac{2(1+m^2)a}{gm}} = \sqrt{\frac{2(1+b^2/a^2)a}{g(b/a)}} \end{aligned}$$

Hence,

$$\boxed{t_f = \sqrt{\frac{2(a^2+b^2)}{gb}}.}$$

(b) For this question we use Euler-Lagrange. Let

$$F(y, y', x) = \sqrt{\frac{1+(y')^2}{2gy}}$$

then the minimum T requires that the Euler-Lagrange equation suffice

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0.$$

Then using Beltrami Identity, Euler-Lagrange equation reduces to

$$y'\frac{\partial F}{\partial y'} - F = C_1 = const.$$

which becomes

$$\begin{aligned} \frac{y'^2}{\sqrt{y(1+y'^2)}} - \sqrt{\frac{1+y'^2}{y}} = C_1 \frac{1}{y(1+y'^2)} &= C_1 \\ y(1+y'^2) &= \frac{1}{C_1^2} \\ y' = \sqrt{\frac{C-y}{y}} \qquad &\because C = \frac{1}{C_1^2}. \end{aligned}$$

This becomes

$$
\begin{aligned}
dx &= \sqrt{\frac{y}{C-y}}dy \\
x &= \int \sqrt{\frac{y}{C-y}}dy
\end{aligned}
$$

and if we use

$$
\begin{cases}
y = C\sin^2\phi \\
dy = 2C\sin\phi\cos\phi d\phi
\end{cases}
$$

we obtain

$$
\begin{aligned}
x &= \int \sqrt{\frac{C\sin^2\phi}{C\cos^2\phi}} 2C\sin\phi\cos\phi d\phi \\
&= 2C\int \sin^2\phi d\phi \\
&= C\int (1-\cos 2\phi)d\phi \\
&= C\left(\phi - \frac{1}{2}\sin 2\phi\right) + D
\end{aligned}
$$

Let $\theta = 2\phi$

$$
x = \frac{C}{2}(\theta - \sin\theta) + D
$$

then let $\psi = \theta + \pi$

$$
x = \frac{C}{2}(\psi + \sin\psi) + D - \frac{\pi C}{2}
$$

Now if we let

$$
\alpha = D - \frac{\pi C}{2}, \qquad \beta = \frac{C}{2}
$$

we obtain

$$
x = \alpha + \beta(\psi + \sin\psi).
$$

For y we have

$$
\begin{aligned}
y &= Cs\sin^2\phi \\
&= C(1-\cos^2\phi) \\
&= C(1 - \frac{1}{2}(1+\cos 2\phi)) \\
&= \frac{C}{2}(1-\cos 2\phi) \\
&= \frac{C}{2}(1-\cos\theta) \\
&= \frac{C}{2}(1+\cos\psi)
\end{aligned}
$$

Hence we have the solution of the brachistrochrone probblem to be (answer for (b))

$$\boxed{\begin{aligned} x &= \alpha + \beta(\psi + \sin(\psi)) \\ y &= \beta(1 + \cos(\psi)) \end{aligned}}$$

Next we compute

$$\begin{aligned} \frac{dy}{d\psi} &= -\beta \sin(\psi) \\ \frac{dx}{d\psi} &= \beta(1 + \cos(\psi)) \end{aligned}$$

and

$$\begin{aligned} \frac{1 + y'^2}{y} &= \frac{1 + \frac{\sin^2(\psi)}{(1+\cos(\psi))^2}}{\beta(1 + \cos(\psi))} \\ &= \frac{\beta(1 + 2\cos(\psi) + \cos^2(\psi) + \sin^2(\psi))}{1 + \cos(\psi)} \\ &= 2\beta. \end{aligned}$$

Thus, when ψ_1 and ψ_2 corresponds to points $x = 0$ and $x = a$ respectively

$$\begin{aligned} t_f = T(a) &= \int_{\psi_1}^{\psi_2} \frac{2\beta}{2g} dx \\ &= \sqrt{\frac{\beta}{g}} \int_{\psi_1}^{\psi_2} \frac{dx}{d\psi} d\psi \\ &= \sqrt{\frac{\beta}{g}} \Big[x \Big]_{\psi_1}^{\psi_2} \end{aligned}$$

Hence,

$$\boxed{t_f = \sqrt{\frac{\beta}{g}} \left(\psi_2 - \psi_1\right).}$$

(c) From the boundary conditions we have

$$\begin{aligned} a &= \alpha + \beta(\psi_2 + \sin(\psi_2)) \\ b &= \beta(1 + \cos(\psi_2)) \end{aligned}$$

which gives

$$\frac{b}{a} = \frac{\beta(1+\cos(\psi_2))}{\alpha+\beta(\psi_2+\sin(\psi_2))}.$$

We know from what we are given that

$$\frac{b}{a} = \frac{A(1-\cos\theta)}{A(\theta-\sin\theta)}$$

where this factor of A$\in \mathbb{R}$ is a common divider making the fraction reducible. Comparing these two we know that

$$\begin{aligned} 1-\cos\theta &= \beta(1+\cos\psi_2) \\ \theta-\sin\theta &= \alpha+\beta(\psi_2+\sin\psi_2) \end{aligned}$$

Thus, if we substitute $\psi_2 = \theta+\pi$, the right hand side of the equations become

$$\begin{aligned} \beta(1+\cos(\theta+\pi)) &= \beta(1-\cos\theta) \\ \alpha+\beta(\theta+\pi+\sin(\theta+\pi)) &= \alpha+\pi\beta+\beta(\theta-\sin\theta) \end{aligned}$$

Therefore, when $\beta = A$ and $\alpha = -\pi A$ we have $\psi_2 = \theta+\pi$.

$$\boxed{\alpha = -\pi A, \qquad \beta = A, \qquad \psi_1 = \pi \qquad \text{where} \quad A \in \mathbb{R}.}$$

(d) From Problem (c) we know that the time for the cycloid is

$$t_f = \sqrt{\frac{\beta}{g}}\theta$$

then the time difference for the ramp and the cycloid becomes

$$\Delta T = \sqrt{\frac{2(a^2+b^2)}{gb}} - \sqrt{\frac{\beta}{g}}\theta.$$

Using MATLAB we can compute this time difference as well as show the visualize the problem (refer to the code in). To numerically compute θ, we use the 15th order (for precision) Taylors series expansion and solve the given equation

$$(1-\cos(\theta)) - \frac{b}{a}(\theta-\sin(\theta)) = 0.$$

Then, since we know that the cycloid is

$$\begin{aligned} x &= A(\theta-\sin\theta) \\ y &= A(1-\cos\theta) \end{aligned}$$

and by substituting $x = a$ and $y = b$ we can find A. Then we can visualize the locations of the point at t_f. Thus, for $a = 4$ and $b = 2$ we get the time difference of

$$\boxed{\Delta T = 0.7906 - 0.6202 = 0.1704 \quad \text{s.}}$$

and the positions of each point is as follows.

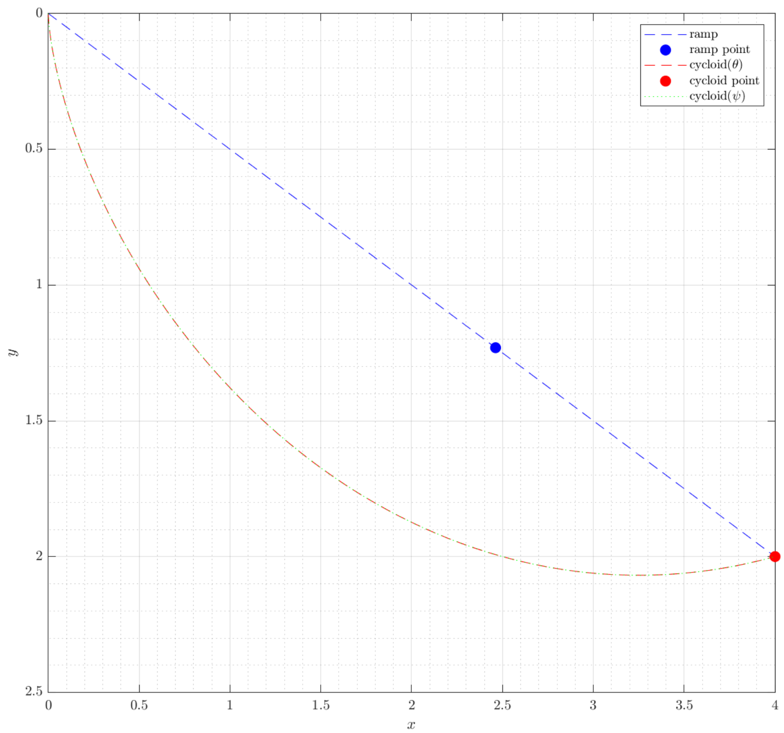

Figure 2: Position of points for ramp and brachistochrone solution ($a = 4$, $b = 2$)

Now if we choose an $a = 10$ and $b = 2$ that is more $a \gg b$ we get the following results

$$\Delta T = 0.9906 \quad \text{s}$$

and the plot looking as follows.

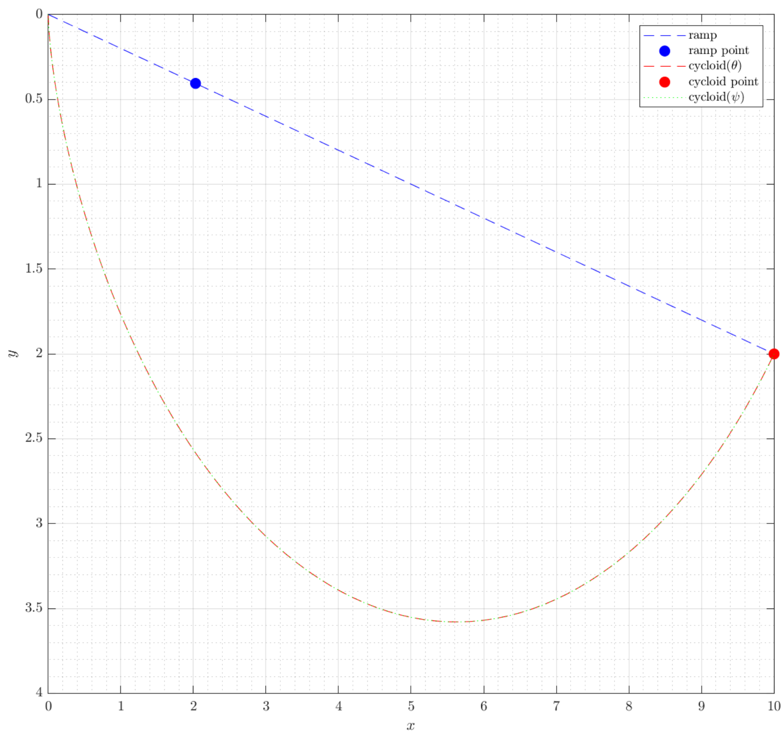

Figure 3: Position of points for ramp and brachistochrone solution ($a = 10$, $b = 2$)

Hence, we can observe that the distance difference between the ramp case and the cycloid case becomes larger for the condition of $a \gg b$.

Problem 3

Analyze the following problem

$$\min\left\{\int_0^1 \big(\dot{y}^2(t) + 12ty(t)\big)dt\right\}$$

subject to $y(0) = y(1) = 0$.

Solution:
Let

$$F(y, r, t) = r^2 + 12ty$$

then,

$$\begin{aligned} F_r &= 2r, \qquad F_{rr} &= 2 > 0 \\ F_y &= 12t, \qquad F_{yy} = 0, \qquad F_{yr} = 0 \end{aligned}$$

From the fact that $F_{rr} = 2 > 0$ we know that the Legendre condition is statisfied and that the problem is a regular problem. Next, with the Euler Lagrange equation we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) &= \frac{\partial F}{\partial y} \\ 2\ddot{y} &= 12t \\ \ddot{y} &= 6t \\ \therefore y^* &= t^2 + c_1 t + c_2 \end{aligned}$$

and with the boundary conditions of $y(0) = y(1) = 0$ we have

$$\begin{aligned} c_1 = -1, \qquad c_2 = 0 \\ \therefore y^* = t^3 - t. \end{aligned}$$

From the Weierstrass condition

$$\begin{aligned} E(t, y, r, q) &= q^2 + 12ty - r^2 - 12ty - (q - r)2r \\ &= q^2 - r^2 - (q - r)2r \\ &= (q - r)(q + r - 2r) \\ &= (q - r)^2 > 0. \end{aligned}$$

Therefore, the necessary condition for a strong local minimizer is sufficed. Lastly, we check to see if there are any conjugate points. From the Jacobi equation

$$\begin{aligned} \frac{d}{dt}\left(F_{yr}\phi + F_{rr}\dot{\phi}\right) &= F_{yr}\dot{\phi} + F_{yy}\phi \\ \frac{d}{dt}\left(0 + 2\dot{\phi}\right) &= 0 \\ 2\ddot{\phi} &= 0 \\ \phi(t) &= at + b \end{aligned}$$

and since $\phi(0) = 0$

$$\phi(t) = at$$

from this equation there is no repeating values in the range of $(0, 1]$, and therefore, there are no conjugate points for this solution. Hence, the minimizer for this problem becomes

$$\boxed{y^* = t^3 - t.}$$

With this solution the minimum value is

$$\min \mathcal{J} = \int_0^1 (3t^2 - 1)^2 + 12t(t^3 - t)dt$$

Hence,

$$\boxed{\min \mathcal{J} = -\frac{4}{5}.}$$

Problem 4

Find the extremals for the problem

$$\min\{\mathcal{J}\} = \int_{t_0}^{t_1} \left(3t^2x^2 + 2t^3x\dot{x}\right)dt$$

with boundary conditions $x(t_0) = x_0$ and $x(t_1) = x_1$. Calculate the optimal value of the cost $\mathcal{J}$.

Solution:
Let

$$F(x, r, t) = 3t^2x^2 + 2t^3xr$$

then,

$$\begin{aligned} F_r = 2t^3x, \qquad F_{rr} = 0, \qquad F_x = 6t^2x + 2t^3r \\ F_{xr} = 2t^3, \qquad F_{xx} \; = 6t^2. \end{aligned}$$

From the Euler-Lagrange equation we have

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} \\ &= \frac{d}{dt}\left(2t^3x\right) - 6t^2x - 2t^3\dot{x} \\ &= 0. \end{aligned}$$

From these results we know that this problem $J(x)$ is *singular*. Hence, from Corollary 4.3 in [1] page 23, we know that for when $F = M(t, x) + N(t, x)\dot{x}$ the Euler Lagrange equation becomes $N_t - M_x = 0$ for extremals. Therefore,

$$\begin{aligned} M = 3t^2x^2, \qquad N = 2t^3x \\ M_x = 6t^2x, \qquad N_t = 6t^2x \\ \therefore M_x \equiv N_t \end{aligned}$$

and $\mathcal{J}(x)$ is independent of the path $t \in [t_0, t_1]$. Thus, the problem with fixed end points has no relevance. Hence,

$$\boxed{x^* = 0.}$$

and

$$\boxed{\min \mathcal{J} = \int_{t_0}^{t_1} 0 dt = 0.}$$

Problem 5

Recall that the conjugate points for the problem

$$\min_{y(x)} \left\{ \int_a^b F(x, y, y')dx \right\}$$

are given by the solution $\phi(x)$ of the Euler-Lagrange equations of the accessory minimization problem

$$\min_{\phi(x)} \left\{ \int_a^b \left(F_{yy}\phi^2 + 2F_{yy'}\phi\phi' + F_{y'y'}(\phi')^2\right)dx \right\}$$

also known as the Jacobi equation.

(a) Show that the Jacobi equation can be written as follows

$$\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)\phi - \frac{d}{dx}\left(F_{y'y'}\frac{d\phi}{dx}\right) = 0$$

where F_{yy}, $F_{yy'}$, $F_{y'y'}$ are evaluated at the candidate weak local minimizer, say $y^*(x)$.

(b) Show that the ratio $\phi_1(x)/\phi_2(x)$ is constant for all conjugate points where $\phi_1(x)$ and $\phi_2(x)$ are two independent solutions of the Jacobi equation. (**Hint:** since the Jacobi equation is a second-order ordinary differential equation, its soluions are given by $\phi(x) = c_1\phi_1(x) + c_2\phi_2(x)$ where c_1 and c_2 are some constants. For $\phi(x) = 0$ for $x = a$, then we have $\phi_1(a)/\phi_2(a) = -c_2/c_1$.

Solution:
(a) We know that the Jacobi equation is

$$\omega_{\phi'\phi'}\phi'' + \omega_{\phi'\phi}\phi' + \omega_{\phi x} - \omega_\phi = 0.$$

With some computation we know that

$$\omega_\phi = F_{yy}\phi + Fyy'\phi' \qquad \text{and} \qquad \omega_{\phi'} = F_{yy'}\phi + F_{yy'}\phi'$$

and plugging this into the Euler-Lagrange equation we have

$$\frac{d}{dx}\left(F_{yy'}\phi + F_{yy'}\phi'\right) = F_{yy}\phi + Fyy'\phi'.$$

If we manipulate this a little bit, we have

$$\begin{aligned} F_{yy}\phi + F_{yy'}\phi' - \left(\frac{d}{dx}F_{yy'}\right)\phi - F_{yy'}\phi' - \frac{d}{dx}F_{y'y'}\phi' = 0 \\ F_{yy}\phi - \left(\frac{d}{dx}F_{yy'}\right)\phi - \frac{d}{dx}F_{y'y'}\phi' = 0 \end{aligned}$$

Now since,

$$\frac{d}{dx}F_{y'y'}\phi' = \frac{d}{dx}\left(F_{y'y'}\frac{d\phi}{dx}\right)$$

the equation becomes,

$$\boxed{\left(F_{yy} - \frac{d}{dx}F_{yy'}\right)\phi - \frac{d}{dx}\left(F_{y'y'}\frac{d\phi}{dx}\right) = 0.}$$

(b) Observing the Jacobi equation we know that it is a second-order ordinary differential equation of ϕ, and for such ODE with two independent solutions we can presume the solution to be in the shape of

$$\phi(x) = C_1\phi_1(x) + C_2\phi_2(x)$$

where C_1 and C_2 are constant values. Now if at some point $x = a$, if this problem is $\phi(a) = 0$, we have a conjugate point $x = b$ in which $\phi(b) = 0$ is also satisfied and we know that the following holds true.

$$\begin{gathered} 0 = C_1\phi_1(a) + C_2\phi_2(a) \\ \therefore \frac{\phi_1(a)}{\phi_2(a)} = -\frac{C_2}{C_1}. \end{gathered}$$

and

$$\begin{gathered} 0 = C_1\phi_1(b) + C_2\phi_2(b) \\ \therefore \frac{\phi_1(b)}{\phi_2(b)} = -\frac{C_2}{C_1}. \end{gathered}$$

which proves that the ratio $\phi_1(x)/\phi_2(x)$ is a constant for all conjugate points where $\phi_1(x)$ and $\phi_2(x)$ are two independent solutions of the Jacobi equation.

Problem 6

Consider the problem of minimizing

$$\mathcal{J}(y) = \int_{t_0}^{t_1} \left(\dot{y}^2(t) - 1\right)^2 dt$$

(a) Write down the Euler-Lagrange equations for this problem and show that the extremals for this problem are curves of constant slope (e.g., line segments).

(b) Using the Erdmann corner conditions, show that the only extremals with corners are those such that the slope is ± 1.

(c) Let $t_0 = 0$ and $t_1 = 3$, and assume that $y(0) = 1$ and $y(3) = 2$. Find the *global* minimizer for this case.

(d) What about the case when $t_0 = 0$ and $t_1 = 1$ and $y(0) = 0$ and $y(1) = 2$?

Solution:
(a) Let

$$F = (\dot{y}^2 - 1)^2$$

then, the Euler Lagrange equation becomes

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) &= \frac{\partial F}{\partial y}\\ \frac{d}{dt}\left(4(\dot{y}^2 - 1)\dot{y}\right) &= 0\\ 4\ddot{y}(3\dot{y}^2 - 1) &= 0.\end{aligned}$$

Thus, from

$$\begin{cases} \dot{y}^2 = \dfrac{1}{3} \\ \ddot{y} = 0 \end{cases}$$

the minimizer for this problem becomes a constant slope

$$\boxed{\begin{cases} y = \pm\dfrac{1}{\sqrt{3}}t + C \\ y = C_1 t + C_2 \end{cases}}$$

Hence, the extremals for this problem are curves of constant slope.

(b) Using Erdmann corner (weak and strong) conditions, when $F(y, r, t) = (r^2 - 1)^2$ we have

$$p(p^2 - 1) = q(q^2 - 1)$$

where p, q are the left and right derivatives. Furthermore, the strong case being

$$(p^2 - 1)(3p^2 - 1) = (q^2 - 1)(3q^2 + 1).$$

Looking at these equations, especially the first one for the weak condition, we can say that we have an infinite amount of trivial solutions for when $p = q$. However, for when $p \neq q$, we only have $(p, q) = (1, -1), (-1, 1)$ that satisfy both equations. Thus, the only extremals with corners are those such that the slope is ± 1.

(c) Applying the boundary conditions to

$$y = C_1 t + C_2 \qquad y(0) = 1, \quad y(3) = 2$$

we have

$$\begin{cases} C_2 = 1 \\ C_1 = \dfrac{1}{3}(2 - C_2) = \dfrac{1}{3}. \end{cases}$$

Hence, the global minimizer for this is

$$\boxed{y^* = \frac{1}{3}t + 1.}$$

where

$$\boxed{\min \mathcal{J} = \frac{64}{27} \approx 2.370.}$$

(d) Applying the boundary conditions to

$$y = C_1 t + C_2 \qquad y(0) = 0, \quad y(1) = 2$$

we have

$$\begin{cases} C_2 = 0 \\ C_1 = (2 - C_2) = 2. \end{cases}$$

Hence, the global minimizer for this is

$$\boxed{y^* = 2t.}$$

where

$$\boxed{\min \mathcal{J} = 9.}$$

References

[1] M. Bendersky. *The Calculus of Variations*. 2008.

Appendix

8.1 Problem 1: MATLAB Code

```
% AE6511 Hw3 Problem 1 MATLAB code
% Tomoki Koike
clear all; close all; clc;  % housekeeping commands
%%
syms a x(t)

F = -a^2 * x^2 + diff(x, t)^2
J = int(F, t, 0, 1)
%%
syms m
assume(m, {'real', 'positive'})
subs(J, x, t*(1-t))
subs(J, x, t^m * (1-t))
subs(J, x, sin(pi*t))
```

8.2 Problem 2: MATLAB Code

```
% AE6511 Hw3 Problem 2 MATLAB code
% Tomoki Koike
clear all; close all; clc;  % housekeeping commands
%%
syms psi_1 psi_2 theta
assume(0 <= psi_1 & psi_1 < 2*pi);
assume(0 <= psi_2 & psi_2 < 2*pi);
assume(0 <= theta & theta < 2*pi);
a = 4;
b = 2;
m = b/a;

alpha = -pi;
beta = 1;
g = 32;

% 10th order Taylor series of sine and cosine to make equation solvable
tsin = @(x) taylor(sin(x), 'Order', 25);
tcos = @(x) taylor(cos(x), 'Order', 25);
eqn0 = 1 - tcos(theta) - m * (theta - tsin(theta)) == 0;
%%
```

```matlab
% Brachistrochrone solution
sol = double(solve([eqn0], [theta]))
theta_sol = sol(sol > 0)
tf_brachi = sqrt(beta/g) * theta_sol

% Ramp solution
tf_ramp = sqrt(2 * (a^2 + b^2)/ g / b)

% Delta T
DT = tf_ramp - tf_brachi
%%
% Get the exact cycloid
syms A
x_cyc = @(x) A*(x - sin(x));
y_cyc = @(x) A*(1 - cos(x));

A1 = vpasolve(subs(x_cyc(theta), theta, theta_sol)==a, A)
A2 = vpasolve(subs(y_cyc(theta), theta, theta_sol)==b, A)
assert(round(A1 - A2, 3) == 0, 'Precision is not acceptable for A not matching.')
A_sol = A1;
%%
% Get the location of point on ramp
syms X
x_ramp_term = solve(sqrt(2*X*(1 + m^2)/g/m) == tf_brachi, X)
%%
% Plot

% Cycloid
th = linspace(0, theta_sol, 100);
xc = A_sol * (th - sin(th));
yc = A_sol * (1 - cos(th));

% second cycloid for verification
psi = linspace(pi, pi + theta_sol, 100);
% beta = A_sol and alpha = - pi*A_sol
xc2 = -pi*A_sol + A_sol*(psi + sin(psi));
yc2 = A_sol * (1 + cos(psi));


% Ramp
x_ramp = linspace(0, a, 100);
y_ramp = x_ramp * m;

set(groot, 'defaulttextinterpreter','latex');
```

```
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
fig = figure("Renderer","painters","Position",[60 60 900 800]);
    plot(x_ramp, y_ramp, '--b', 'DisplayName', 'ramp')
    hold on; grid on; grid minor; box on;
    plot(x_ramp_term, x_ramp_term*m, '.b', 'MarkerSize',25, 'DisplayName','ramp point')
    plot(xc, yc, '--r', 'DisplayName', 'cycloid($\theta$)')
    plot(xc(end), yc(end), '.r', 'MarkerSize',25, 'DisplayName','cycloid point')
    plot(xc2, yc2, ':g', 'DisplayName', 'cycloid($\psi$)')
    hold off
    xlabel('$x$')
    ylabel('$y$')
    legend
    set(gca, 'Ydir','reverse');
% saveas(fig, 'p2_brachi_2.png')
```