G BP

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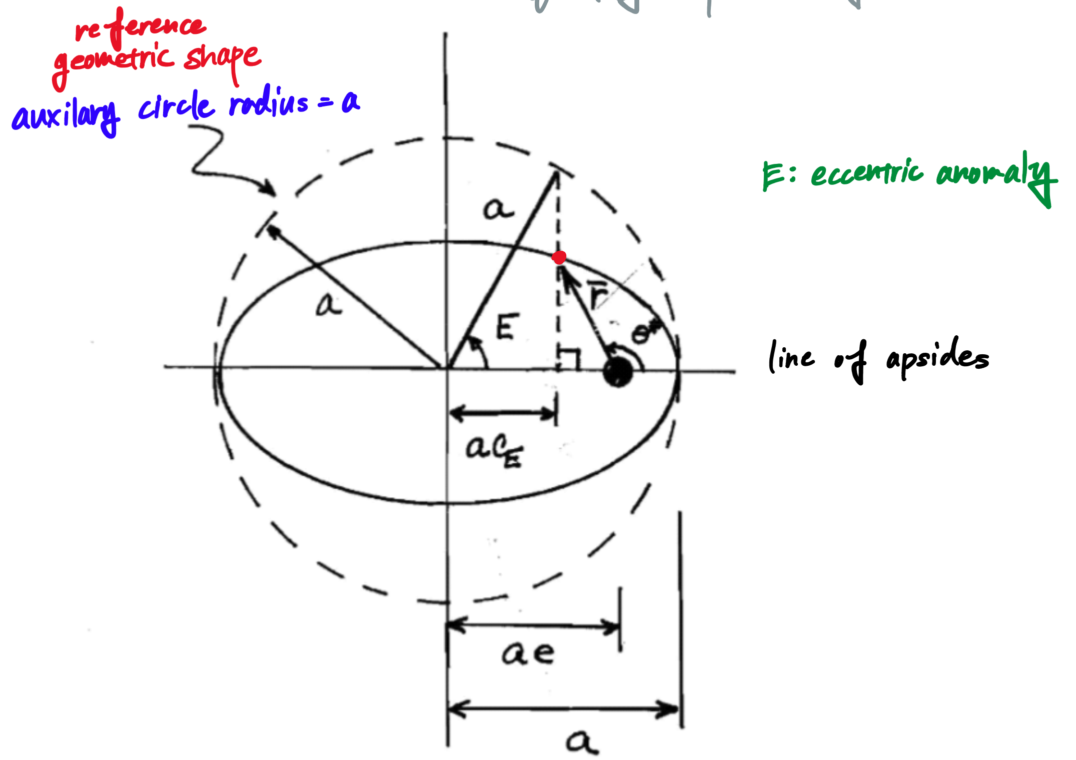
Eccentric Anomaly E

we still don't know TIME !

each conic section → introduce new quantity to produce info on time.

Additional variable defined for ellipse



$$a\cos E = ae - r\cos(180^\circ - \theta^*)$$
$$= ae + r\cos\theta^*$$

use polar eqn

$$r = \frac{p}{1+e\cos\theta^*} \rightarrow \cos\theta^* = \frac{p}{re} - \frac{1}{e}$$

$$\cos E = \frac{ae + r\cos\theta^*}{a}$$
$$= e + \frac{r}{a}\cos\theta^*$$

$$\boxed{\cos E = \frac{a-r}{ae}}$$

OR

$$\boxed{r = a(1 - e\cos E)}$$

$$r = \frac{p}{1+e\cos\theta^*}$$

E obviously related to θ^*

Previously $r\cos\theta^* = a\cos E - ae$

Identity $\cos 2\alpha = 2\cos^2\alpha - 1$

$$r\left(2\cos^2\left(\frac{\theta^*}{2}\right) - 1\right) = a\cos E - ae$$

$$2r\cos^2\left(\frac{\theta^*}{2}\right) = a\cos E - ae + a(1 - e\cos E)$$
$$= (a - ae)\cos E + (a - ae)$$

$$2r\cos^2\left(\frac{\theta^*}{2}\right) = a(1-e)(1+\cos E)$$

Identity $\cos 2\alpha = 1 - 2\sin^2\alpha$

$$r\left(1 - 2\sin^2\left(\frac{\theta^*}{2}\right)\right) = a\cos E - ae$$

$$-2r\sin^2\left(\frac{\theta^*}{2}\right) = a\cos E - ae - a(1 - e\cos E)$$
$$= (a + ae)\cos E - (a + ae)$$

$$-2r\sin^2\left(\frac{\theta^*}{2}\right) = a(1+e)(\cos E - 1)$$

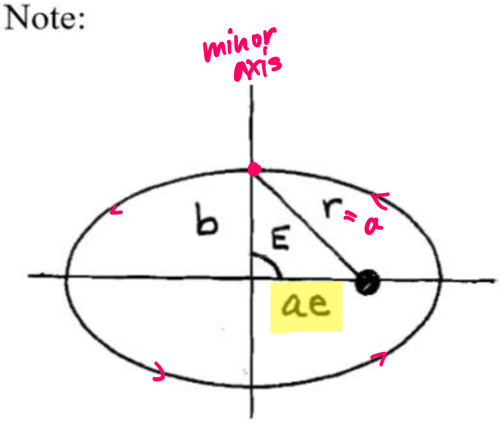
$$\frac{-2r\sin^2\left(\theta^*/2\right)}{2r\cos^2\left(\theta^*/2\right)} = \frac{a(1+e)(\cos E - 1)}{a(1-e)(1+\cos E)}$$

Identity: $\tan^2\frac{E}{2} = \frac{1-\cos E}{1+\cos E}$

$$\boxed{\tan\frac{\theta^*}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{E}{2}}$$

size of orbit ('a') doesn't matter

Note:



At $E = 90^\circ$ $\quad r = a(1 - e c_E)$

$$r = a(1 - e\cos 90^\circ) = a$$

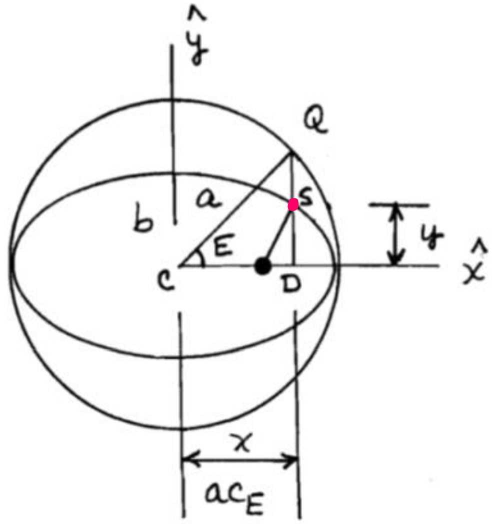
$$b^2 = r^2 - a^2e^2$$

$$b^2 = a^2(1-e^2)$$

$$\boxed{b = a\sqrt{1-e^2}}$$

$$a \quad \cancel{r} = \frac{p}{1+e\cos\theta^*} \rightarrow \cos\theta^* = \overbrace{\frac{p}{re}}^{r=a} - \frac{1}{e} = \frac{a(1-e^2)}{ae} - \frac{1}{e} = -e$$

$$\Longrightarrow \quad \boxed{\cos\theta^* = -e}$$



$\overset{x}{\boxed{CD = a\cos E}}$

Let x, y be point S on the ellipse

→ measured from center C

SD? → y

Equation of an ellipse ⟹ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

location of ellipse is measured from center x, y

$$\frac{(CD)^2}{a^2}+\frac{(SD)^2}{b^2}=1$$

$$\frac{a^2\cos^2 E}{a^2}+\frac{(SD)^2}{b^2}=1$$

$$(SD)^2 = b^2\left[1-\frac{a^2\cos^2 E}{a^2}\right] \Rightarrow \boxed{SD = b\sin E}$$

All of these fundamental relationships are useful but one of the most important reasons to introduce E is to obtain a relation between position and time

Kepler's Equation

How? Various approaches – consider one

Kepler's Equation (Relation between position and time)

Begin with some relationships that are already known

$$p = \frac{h^2}{\mu}$$

$$\boxed{r = \frac{p}{1 + e\cos\theta^*}}$$

$$h = r^2\dot{\theta} = r^2\frac{d\theta}{dt}$$

time

Combine to eliminate r and h

$$h = r^2\frac{d\theta}{dt}$$

$$\sqrt{\mu p} = \frac{p^2}{\left(1 + e\cos\theta^*\right)^2}\frac{d\theta}{dt}$$

Rearrange

$$\boxed{\sqrt{\frac{\mu}{p^3}}dt = \frac{d\theta}{\left(1 + e\cos\theta^*\right)^2}}$$

Need to integrate to get a useful relationship for time as a function of θ^* integration is nontrivial which is why t was always eliminated previously. However, it is possible to use eccentric anomaly E and the relationship → easier/convenient

$$r = a\left(1 - e\cos E\right)$$

Note: introducing E implies that the result will only apply to elliptical orbits.

To use E to relate position and time:

1) Relate equation for r

$$\cos E = \frac{a-r}{ae} \qquad \text{I}$$

2) Differentiate I and rearrange

$$\dot{r} = ae\dot{E}\sin E \qquad \text{II}$$

3) Given

$$\varepsilon = \frac{\dot{r}^2 + r^2\dot{\theta}^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

Multiply by $\dfrac{2r^2a}{\mu}$

$$\frac{ar^2\dot{r}^2}{\mu} = \frac{-ar^4\dot{\theta}^2}{\mu} + 2ra - r^2$$

But $$r^4\dot{\theta}^2 = h^2 = \mu p = \mu a\left(1-e^2\right)$$

$$\therefore \quad \frac{ar^2\dot{r}^2}{\mu} = a^2e^2 - (a-r)^2 \qquad \text{III}$$

4) Square I

$$a^2e^2\cos^2 E = (a-r)^2$$

Combine with II, III

$$\frac{ar^2}{\mu}\left[a^2e^2\dot{E}^2\sin^2 E\right] = a^2e^2 - a^2e^2\cos^2 E$$

$$\Rightarrow \qquad r\dot{E} = \sqrt{\frac{\mu}{a}}$$

5) Rewrite

$$r\,dE = \sqrt{\frac{\mu}{a}}\,dt$$

$$r = a(1 - e\cos E)$$

EASY

$$\sqrt{\frac{\mu}{a}}\,dt = a(1 - e\cos E)\,dE$$

6) Integrate (now easy)

$$\boxed{\sqrt{\frac{\mu}{a^3}}\,(t-t_p) = E - e\sin E}$$

Kepler's Equation (ellipse only)

time since we pass periapsis

integration const.

transcendental equation

$E_p = 0°$

Corresponding relationship for a hyperbolic orbit

$$\boxed{\sqrt{\frac{\mu}{|a|^3}}\,(t-t_p) = e\sinh H - H}$$

Kepler's Eqn. (hyperbolic)

Can you derive this?

$H :=$ hyperbolic anomaly

Define $\mathbf{n} = \sqrt{\frac{\mu}{a^3}}$ mean motion (const.)

$M = n(t - t_p)$ Mean anomaly

$$\boxed{M = E - e\sin E}$$

Kepler's Eqn for ellipces

Short equation but transcendental

Given time (M), cannot solve for E in closed form

Solution usually obtained iteratively →

Solution of Kepler's Equation

Because it is frequently required, the solution of Kepler's equation is of great interest. Consider the equation as written

$$M = E - e\sin E$$

By differentiation

$$dM = \left(1 - e\cos E\right)dE$$

Integration between limits 0 and t

$$\int_0^{E_t} dE = \int_0^{M_t} \frac{dM}{1 - e\cos E}$$

An expansion using Fourier series and noting that the period of the function is 2π has coefficients

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} dE = 1$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} \cos\left\{m\left(E - e\sin E\right)\right\} dE$$

$$= 2\, J_m(me)$$

J_m is a Bessel function of the first kind of order m. For calculation,

$$J_m(me) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{me}{2}\right)^{2n+m}}{n!\left(n+m\right)!}$$

Friedrich Bessel (1784–1846) ~1820s develop fcns.

So that an explicit formula for eccentric anomaly is given by

$$\boxed{E = M + 2\sum_{m=1}^{\infty} \frac{1}{m} J_m(me)\sin(mM)}$$

today initial guess

Numerically, at times a few terms can be used to start the solution and then with the first approximation, E_n, continue by a Newton procedure,

$n(t-t_p)$

$$E_{n+1} = E_n - \frac{E_n - e\sin E_n - M}{1 - e\cos E_n}, \qquad n = 1, 2, \ldots, p$$

Until E_p no longer varies significantly.

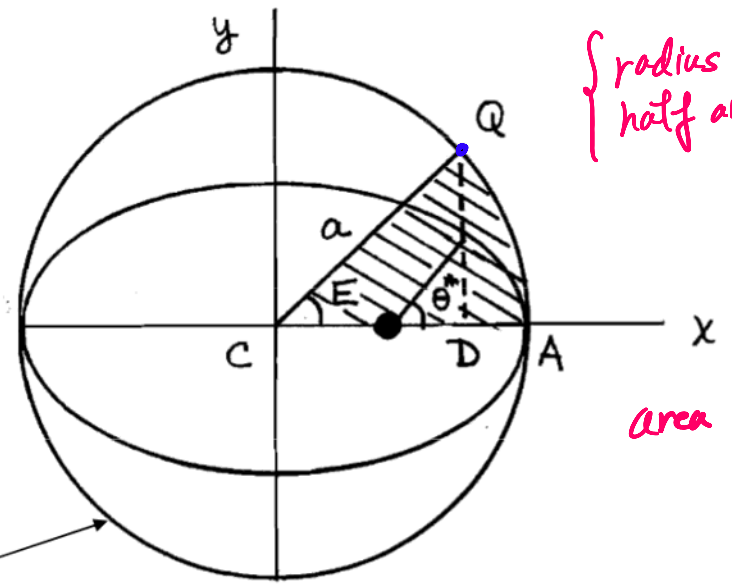
Hyperbolic Anomaly H

cannot sketch a pic

Need analog to E for hyperbola ← NOT an angle

To accomplish, note that E actually represents an area

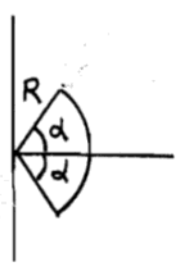
Recall → the definition for E was based on concept of auxiliary circle
→ now define an area as a sector of that circle



radius a
half angle $\frac{E}{2}$

area QCA

Reference geometric shape is circle



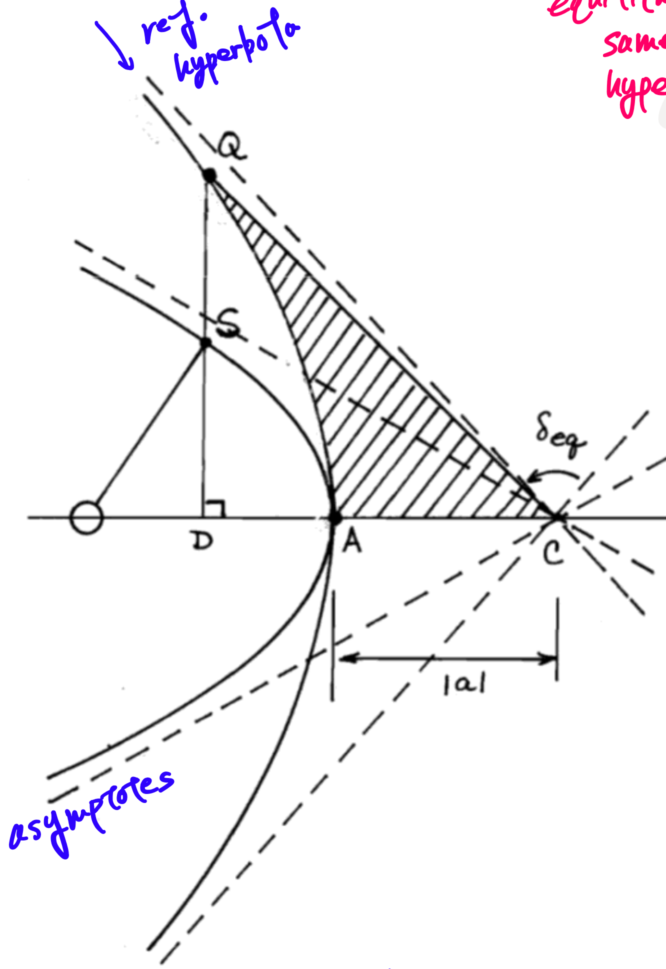
Area of a sector of a circle $= R^2 \alpha$

Area QCA $= R^2 \left(\frac{E}{2}\right) = a^2 \frac{E}{2}$

Corresponding situation for hyperbola:
Based on reference geometric shape –

equilateral hyperbola
same semimajor axis as
hyperbola of interest (focus differs)

consid
eration
(focus
differs)



same |a|
interest ;

$$H = \frac{2(\text{area } CQA)}{a^2}$$

Same $|a|$ as orbit of interest; same pericenter; actually different focus

H is a nondimensional real #

Define hyperbolic anomaly $\boldsymbol{H}$ in terms of relationship to area $\boldsymbol{CQA}$

$$\frac{|a|+r}{e} = |a|\cosh H$$

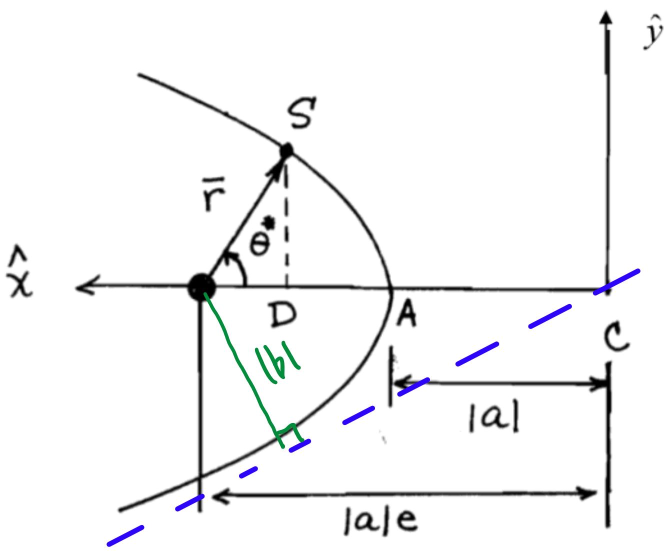
$$\longrightarrow \quad r = |a|(e\cosh(H) - 1)$$

$$\Downarrow$$

$$\tan\frac{\theta^*}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\frac{H}{2}$$

$$\sqrt{\frac{\mu}{|a|^3}}\,(t - t_p) = e\sinh(H) - H$$

Useful relationship in terms of ***H***



u = 2 (area ***ASC***)

Point S:

$$\left.\begin{aligned} \boldsymbol{x} &= \cosh \boldsymbol{u} \\ \boldsymbol{y} &= \sinh \boldsymbol{u} \end{aligned}\right\} \text{ for unit } |a|$$

Equation for point on hyperbola measured from the <u>center</u>

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{(CD)^2}{a^2} - \frac{(SD)^2}{b^2} = 1$$

$$CD = |a| \cosh H$$

$$SD = |b| \sinh H$$

Useful to obtain other relationships

Note:

$$|a|\, e - r \cos\theta^* = CD$$

$$|a|\, e - r \cos\theta^* = |a| \cosh H$$

$$|a|\,e - r\left(\frac{p}{re} - \frac{1}{e}\right) = |a|\cosh H$$

$$|a|\,e - \frac{r|a|\left(e^2-1\right)}{r\,e} + \frac{r}{e} = |a|\cosh H$$

$$\frac{|a|+r}{e} = |a|\cosh H$$

$$\Longrightarrow \boxed{r = |a|\left(e\cosh H - 1\right)}$$

$$\boxed{\begin{aligned} \tan\frac{\theta^*}{2} &= \left(\frac{e+1}{e-1}\right)^{1/2}\tanh\frac{H}{2} \\ \sqrt{\frac{\mu}{|a|^3}}\,(t-t_p) &= e\sinh H - H \end{aligned}}$$

Kepler's Equ of Hyperbolic orbit

$$N = \sqrt{\frac{\mu}{|a|^3}}\,(t-t_p)$$

Again, solution iterative!

Parabolic Orbits and Barker's Equation

$e=1$

leverage trig identies

$$r=\frac{p}{1+\cos\theta^*}=\frac{p}{2}\left(1+\tan^2\frac{\theta^*}{2}\right)$$

$$h=r^2\dot{\theta}=r^2\frac{d\theta}{dt}=\sqrt{\mu p}$$

$$\sqrt{\frac{\mu}{p^3}}\,dt=\frac{d\theta}{\left(1+\cos\theta^*\right)^2}=\frac{1}{4}\left(1+\tan^2\frac{\theta^*}{2}\right)^2 d\theta$$

$$4\sqrt{\frac{\mu}{p^3}}\,dt=\left(1+\tan^2\frac{\theta^*}{2}\right)^2 d\theta$$

$$4\sqrt{\frac{\mu}{p^3}}\,dt=\sec^4\frac{\theta^*}{2}\,d\theta$$

$\Downarrow$

Integrate

$$\boxed{6\sqrt{\frac{\mu}{p^3}}\,(t-t_p)=\tan^3\frac{\theta^*}{2}+3\tan\frac{\theta^*}{2}}$$

$\leftarrow$ Barker's Eqn

(Barker prepared extensive tables of solutions in the 18th century.)

Define $B = 3\sqrt{\frac{\mu}{p^3}}\left(t - t_p\right)$

$$\Downarrow$$

$$\tan\frac{\theta^*}{2} = \left(B + \sqrt{1+B^2}\right)^{\frac{1}{3}} - \left(B + \sqrt{1+B^2}\right)^{-\frac{1}{3}}$$

(from Jerome Cardan method of solving cubic equations 1545 AD)

Now better methods!

E, H, θ^* with definitions

$$\downarrow$$

other useful quantities