P3: P

2020年3月13日 金曜日 午後0:56

Return to Poinsot Construction

Rolling of a spheroid σ on a plane π

Inertia Ellipsoid

$$\frac{x_1^2}{\left(k\, I_1^{-\frac{1}{2}}\right)^2} + \frac{x_2^2}{\left(k\, I_2^{-\frac{1}{2}}\right)^2} + \frac{x_3^2}{\left(k\, I_3^{-\frac{1}{2}}\right)^2} = 1$$

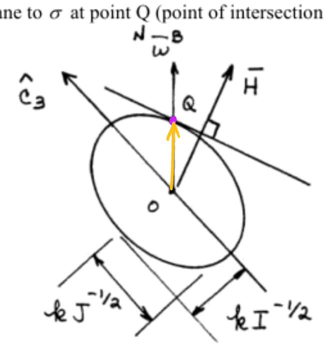
Energy Ellipsoid

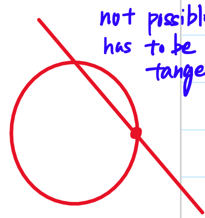
$$\frac{\omega_1^2}{\left(c\, I_1^{-\frac{1}{2}}\right)^2} + \frac{\omega_2^2}{\left(c\, I_2^{-\frac{1}{2}}\right)^2} + \frac{\omega_3^2}{\left(c\, I_3^{-\frac{1}{2}}\right)^2} = 1$$

σ is spheroid with center O; for axisymmetric body, principal semi-diameters equal to $k\, I^{-\frac{1}{2}}$, $k\, J^{-\frac{1}{2}}$

Symmetry axis of σ parallel to $\hat{c}_3$

π is tangent plane to σ at point Q (point of intersection of σ with ω)





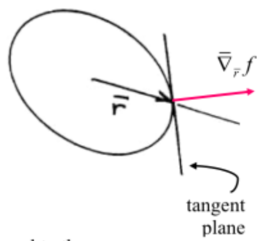
not possible
has to be
tangent!

How do we know that π is tangent to σ?

Determine the gradient of the ellipsoid at point Q

ellipsoid:

$$\frac{x_1^2}{\left(k\, I^{-\frac{1}{2}}\right)^2} + \frac{x_2^2}{\left(k\, I^{-\frac{1}{2}}\right)^2} + \frac{x_3^2}{\left(k\, J^{-\frac{1}{2}}\right)^2} = 1$$



OR

$$f(\bar{r}) = I\,x_1^2 + I\,x_2^2 + J\,x_3^2 - k^2$$

constant

Plane tangent to σ at any position $\bar{r}$ is normal to the gradient vector $\bar{\nabla}_{\bar{r}} f$

$$\bar{\nabla}_{\bar{r}} f = \frac{\partial f}{\partial x_1}\hat{c}_1 + \frac{\partial f}{\partial x_2}\hat{c}_2 + \frac{\partial f}{\partial x_3}\hat{c}_3$$

$$= 2I\,x_1\,\hat{c}_1 + 2I\,x_2\,\hat{c}_2 + 2J\,x_3\,\hat{c}_3$$

$$= 2\bar{\bar{I}} \cdot \bar{r}$$

Choose "position vector" at Q: $\bar{q} = \lambda\, {}^N\bar{\omega}^B$

$$\bar{\nabla}_{\bar{q}} f = 2\bar{\bar{I}} \cdot \bar{q}$$

$$= 2\lambda\, \bar{\bar{I}} \cdot {}^N\bar{\omega}^B$$

$$\bar{\nabla}_{\bar{q}} f = 2\lambda\, \bar{H} \Longrightarrow$$

π is normal to $\bar{H}$ by definition; plane tangent to ellipsoid is also normal to $\bar{H}$

Space Cones / Body Cones

Consistent with the decomposition of the motion into the sum of two rotations are some additional definitions and observations

Assume $I > J$ then $\frac{1}{\sqrt{I}} < \frac{1}{\sqrt{J}}$ (axisymmetric "rod" shape)

Assume initial ${}^N\bar{\omega}^B$ known $\rightarrow$ $\omega_1, \omega_2, \omega_3$ constants

$\bar{\omega}$ remains in same relative position with respect to the $\hat{c}$'s

angle constant

$$\bar{H} = I\omega_1\hat{c}_1 + I\omega_2\hat{c}_2 + J\omega_3\hat{c}_3$$

Each component of constant magnitude

angle constant

Note: All three vectors $\bar{H}, \bar{\omega}, \hat{c}_3$ will remain in the same relative position throughout the motion

$\bar{H}, \bar{\omega}, \hat{c}_3$ define a plane

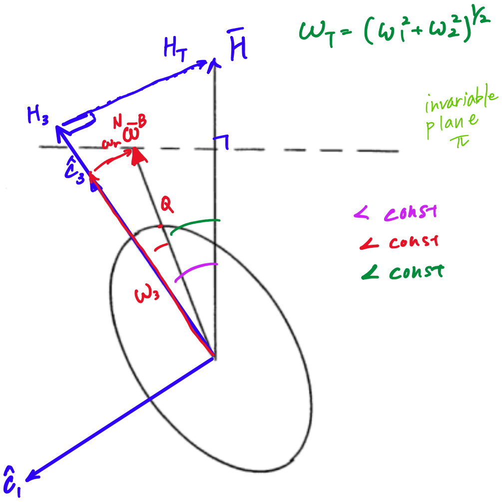
System in motion:

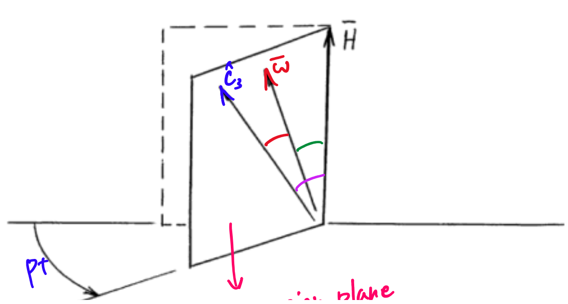
#1 $\bar{H}$ fixed; plane rotation about $\bar{H}$

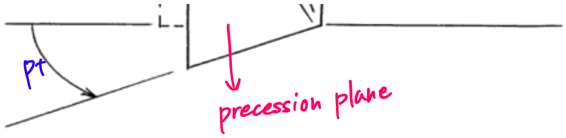
${}^N\bar{\omega}^C = p\,\hat{h}$ (precession)

$H_T = (H_1^2 + H_2^2)^{1/2}$

$\omega_T = (\omega_1^2 + \omega_2^2)^{1/2}$







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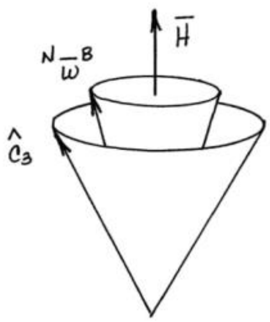
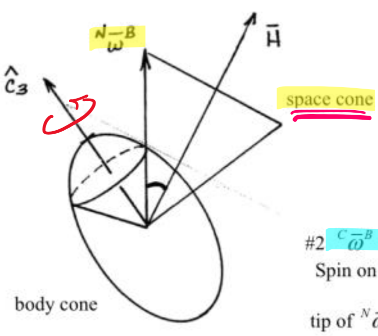
Note: $\hat{\omega}$, $\hat{c}_3$ each trace out a cone about $\bar{H}$



tip of $^N\bar{\omega}^B$ traces out curve on the invariable plane π

→ herpolhode curve

closed circle because it's axisymmetric



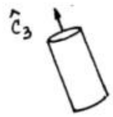
#2 $^C\bar{\omega}^B = s\,\hat{c}_3$

Spin on own axis → st

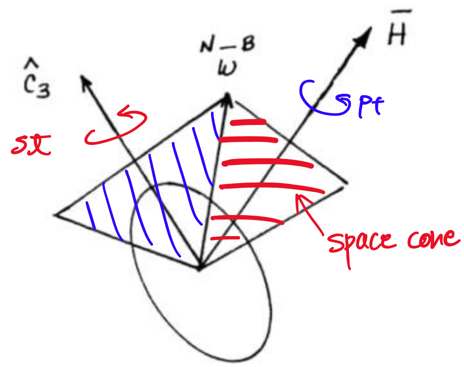
tip of $^N\bar{\omega}^B$ traces curve on σ

circle on ellipsoid → polhode curve

Example 1: σ prolate (elongated)



"rod" $I > J$

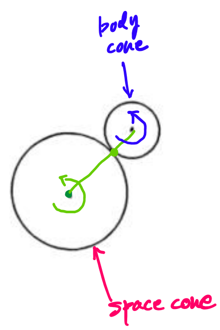


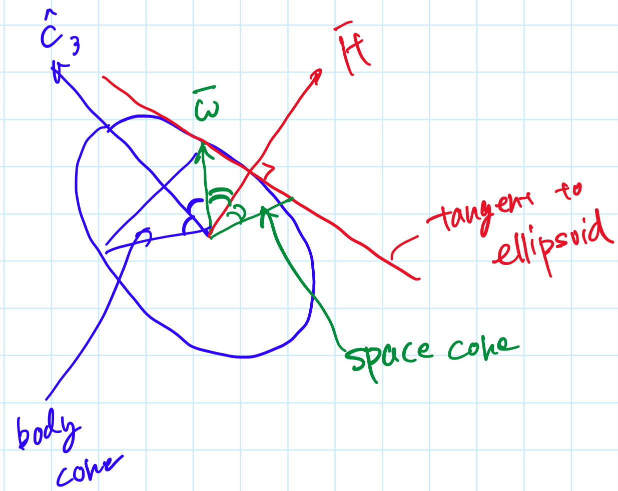
Motion of body: body cone rolls on space cone

const. $\left\{ \begin{array}{l} s = \dfrac{I-J}{I}\omega_3 \\ p = \dfrac{H}{I} \end{array} \right\}$ same sign

Direct Precession

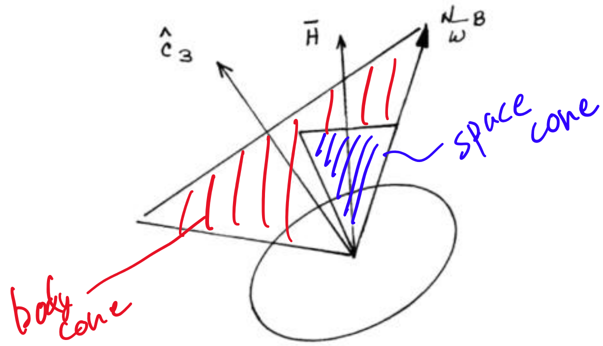
"rod-like" shape!





Example: σ oblate (squashed)

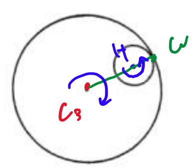
"disk" $I < J$



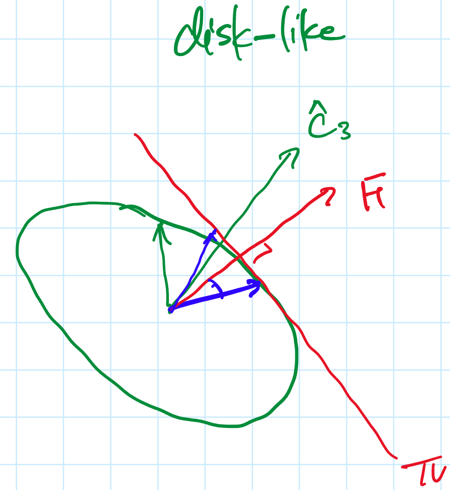
Motion of body: body cone rolls on space cone; space cone is <u>inside</u> body cone

$$s = \frac{I-J}{I}\omega_3 \qquad p = \frac{H}{I}$$

opposite signs



<u>Retrograde Precession</u>



What happens when $I = J$ (spherical inertia ellipsoid)?

$\bar{H} = I\,{}^N\bar{\omega}^B \longrightarrow \bar{H},\ {}^N\bar{\omega}^B$ parallel