Lecture: Rigidity Matrix

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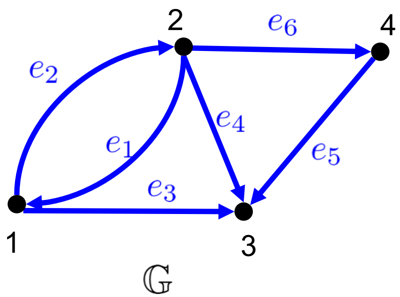
Review

- Incidence Matrix of an *n*-node-*m*-edge directed graph

$$H = [h_{ik}]_{n\times m} = \begin{cases} 1, & \text{node } i \text{ is the } \textbf{head} \text{ of edge } k; \\ -1, & \text{node } i \text{ is the } \textbf{tail} \text{ of edge } k; \\ 0, & \text{otherwise} \end{cases}$$

self-arcs are excluded

Label all edges



$$\begin{array}{c} \text{Edges:} \\ H = \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \left(\begin{array}{cccccc} 1 & -1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right)_{4\times 6} \end{array} \begin{array}{c} \text{Nodes} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

- The *k*th column of **H** corresponds to the *k*th edge $i \rightarrow j$ with the *i*th entry -1 and *j*th entry 1.

- At each column of H, there is one entry equal to 1, one entry equal to -1, and all other entries are 0s.

$$\boxed{\begin{array}{c} (H'x)_k = x_j - x_i \\ \mathbf{1}'H = 0 \end{array}}$$

- For $M \in \mathbb{R}^{n\times m}$, its **pseudo-inverse** $M^\dagger$ is the unique $m \times n$ matrix such that

$$\begin{cases} MM^\dagger M = M \\ M^\dagger M M^\dagger = M^\dagger \\ MM^\dagger, \quad M^\dagger M \text{ are both symmetric} \end{cases}$$

- ***For an undirected connected graph, what is the pseudo-inverse of its Laplacian?***

L is symmetric $L = U\text{diag}\{0, \lambda_2, ..., \lambda_n\}U'$

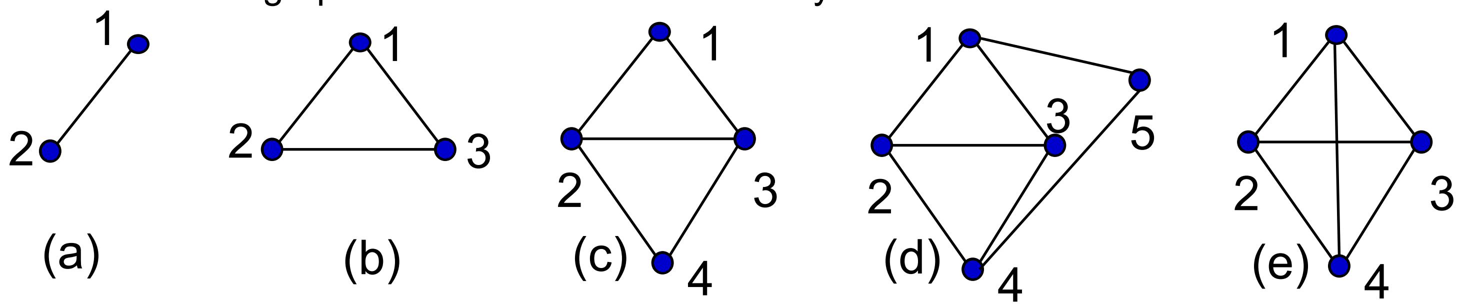
Columns of U are **orthonormal eigenvectors** of L.

$$U'U = I$$

$$L^\dagger = U\text{diag}\{0, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}\}U'$$

Rigid Graph

A graph that can not be deformed by continuous motions.



- A **minimally rigid** graph is a rigid graph and deletion of any edge will violate the rigidity.

a,b,c,d are minimally rigid; e is not.

- A **rigid** graph is graph which contains a minimally rigid graph as a *spanning subgraph*.

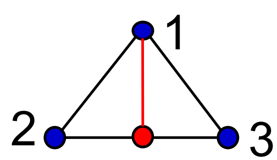
(Same vertex set;Subset of edges)

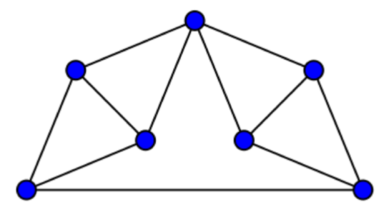
c is a spanning subgraph of e

❖ How to produce a minimally rigid graph in 2D?

- Vertex Addition: Add a new vertex by connecting it to two other vertices by two new edges. a,b,c,d

Henneberg Operations





The Moser spindle

- Edge Splitting: Insert a new vertex into one edge to split it into two and also connect it to another node.

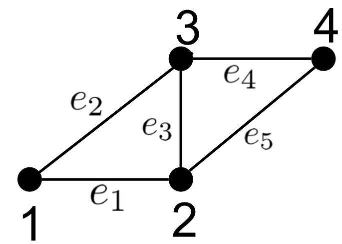
How many edges are there for a minimally rigid graph in 2D? $2n-3$

Henneberg Operations provide a **geometric way** to determine whether a graph is rigid.

Is there any algebraic way? Since computers usually do not understand geometric shapes but matrices.

Rigidity Matrix

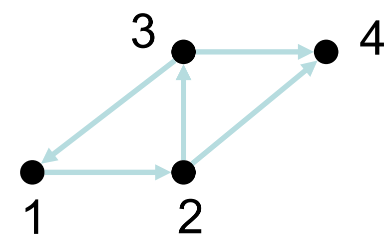
$x_i \in \mathbb{R}^2$



$$R(x) = \begin{array}{c} e_1 \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} x_1' - x_2' & x_2' - x_1' & 0 & 0 \\ x_1' - x_3' & 0 & x_3' - x_1' & 0 \\ 0 & x_2' - x_3' & x_3' - x_2' & 0 \\ 0 & 0 & x_3' - x_4' & x_4' - x_3' \\ 0 & x_2' - x_4' & 0 & x_4' - x_2' \end{bmatrix}_{m \times 2n}$$

(infinitesimally) rigid $\quad \mathrm{rank}\ R = 2n - 3$

(infinitesimally) minimally rigid: full row rank



- Connection to incidence matrix
- Assign an arbitrary direction fro each edge;

$$H = [h_{ik}]_{n \times m}$$

$$= \begin{cases} 1, & \text{node } i \text{ is the } \textbf{head} \text{ of edge } k; \\ -1, & \text{node } i \text{ is the } \textbf{tail} \text{ of edge } k; \\ 0, & \text{otherwise} \end{cases}$$

$$H = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 \end{bmatrix}_{4 \times 5}$$

For the kth edge from i to j, one define $\quad z_k = x_j - x_i \qquad Z = \mathrm{diag}\{z_1, z_2, ..., z_m\}$

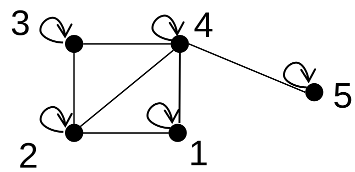
$$R = Z'_{2m \times m}(H'_{n \times m} \otimes I_2)$$

- Kronecker Product $\otimes$

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{bmatrix},$$

$$H \otimes I_2 = \begin{bmatrix} I_2 & -I_2 & 0 & 0 & 0 \\ -I_2 & 0 & I_2 & 0 & I_2 \\ 0 & I_2 & -I_2 & I_2 & 0 \\ 0 & 0 & 0 & -I_2 & -I_2 \end{bmatrix}_{8 \times 10}$$

Let $\mathbb{G}$ denote a five-node undirected graph with self-arcs as shown in the following Figure:



1. Is the network strongly connected or not? (1pt); Is the network periodic or aperiodic? (1pt); Draw one of its spanning subgraphs which is a tree. (1pt)

2. Write out the adjacency matrix A_d for the network $\mathbb{G}$ (1pt). Is A_d irreducible, primitive, or positive? (1pt).

3. Write out one incidence matrix H for the network $\mathbb{G}$. (1pt); What is the rank of H? (1pt)

4. Write out the Laplacian matrix L for the network $\mathbb{G}$. (1pt); What is the smallest eigenvalue of L? (1pt); What is the rank of L? (1pt)

5. Let $x(t) \in \mathbb{R}^5$ denote a state vector with the ith element $x_i(t) \in \mathbb{R}$ as the state of each node i, $i = 1, 2, ..., 5$. Write out a distributed update for reaching consensus for each $x_i(t)$. (1pt); Let $x(t+1) = Mx(t)$ denote the resulted state update for consensus. Write out the matrix M. (1pt). What is the spectral radius for M? (1pt); Write out one right eigenvector corresponding to the largest eigenvalue of M (1pt). Is M primitive? (1pt); Employ the Perron-Frobenius Theorem to prove that
$$\lim_{t\to\infty} M^t = \mathbf{1}v' \tag{1}$$
where v is one left eigenvector of M corresponding to the largest eigenvalue of M (3pt).

6. Write out the distributed update for each node to achieve the global average $\frac{1}{5}\sum_{i=1}^{t} x_i(0)$. (2pt)

Solution

Q1

Yes, it is strongly connected.
The network is aperiodic.
Spanning tree: ①-②-③-④-⑤.

Q2

$$A_d = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

A_d is irreducible and primitive.

Q3

Each column of H corresponds an edge of $\mathbb{G}$.

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Rank of H is 4.

Q4

$$L = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$\min \lambda(L) = 0$. Rank of L is 4.

Q5

$$x_i(t+1) = \frac{1}{d_i} \sum_{j \in \mathcal{N}_i} (x_j(t)).$$

where d_i is the number of neighbors of agent i, $\mathcal{N}_i$ is the neighbor set of agent i. Here, $i \in \mathcal{N}_i$.

$$x(t+1) = Mx(t), \quad M = D^{-1}A_d$$

where $D = \text{diag}\,\{d_1, d_2, d_3, d_4, d_5\}$, A_d is in Q2.

For $\max \lambda(M) = 1, v = [1, 1, 1, 1, 1]^\top$. M is primitive.

Since M is primitive, by PF Theorem, it has a simple larges eigenvalue. Thus, M can be written as

$$M = T \begin{bmatrix} 1 & 0_{1\times 4} \\ 0_{4\times 1} & S \end{bmatrix} T^{-1}$$

where $S \in \mathbb{R}^{4\times 4}$ and $0 \leq |\lambda(S)| < 1$. The first column of T equals $\mathbf{1}$, which is the right eigenvector of $\lambda(M) = 1$. The first row of T^{-1} equals v', which is the left eigenvector of $\lambda(M) = 1$. Thus,

$$\begin{aligned}\lim_{t\to\infty} M^t &= \lim_{t\to\infty} \left(T \begin{bmatrix} 1 & 0_{1\times 4} \\ 0_{4\times 1} & S \end{bmatrix} T^{-1} \right)^t = \lim_{t\to\infty} T \begin{bmatrix} 1 & 0_{1\times 4} \\ 0_{4\times 1} & S^t \end{bmatrix} T^{-1} \\ &= T \begin{bmatrix} 1 & 0_{1\times 4} \\ 0_{4\times 1} & 0_{4\times 4} \end{bmatrix} T^{-1} = \mathbf{1}v'\end{aligned}$$

Q6

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t), \quad w_{ij} = \begin{cases} \min\{\frac{1}{d_i}, \frac{1}{d_j}\} & j \in \mathcal{N}_i, j \neq i \\ 1 - \sum_{j \in \mathcal{N}_i, j\neq i} w_{ij} & j = i \\ 0 & \text{otherwise} \end{cases}$$