College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW8

Professor:
Martin Corless
Purdue AAE Professor

Student:
Tomoki Koike
Purdue AAE Senior

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Exercise 1

What are the positive limit sets of the following solutions?

$$\text{(a)} \qquad x(t) = \sin(t^2)$$

$$\text{(b)} \qquad x(t) = e^t \sin(t)$$

Solution:

(a)
Since,

$$-1 \leq sin(t^2) \leq 1$$

we can say that the interval [-1, 1] is the positive limit set of $sin(t^2)$.

(b)
For the equation

$$x(t) = e^t \sin(t)$$

since e^t has an empty limit set and grows exponentially, it dominates the sinusoid which makes the positive limit set for $e^t \sin(t)$ be empty.

Exercise 2

Using LaSalle's Theorem, show that all solutions of the system

$$\dot{x}_1 = x_2^2$$
$$\dot{x}_2 = -x_1 x_2$$

must approach the x_1 axis.

Solution:
We choose a candidate Lyapunov function of

$$V(x) = \frac{1}{2}x_1^2 - x_1 + \frac{1}{2}x_2^2$$

This function is not positive definite but radially unbounded since the x_1^2 and x_2^2 terms dominate the function to increase infinitely. Taking the derivative of this, we have

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 - \dot{x}_1 + x_2\dot{x}_2 \\ &= x_1x_2^2 - x_2^2 - x_1x_2^2 \\ &= -x_2^2 \\ &\leq 0.\end{aligned}$$

From La Salle's Theorem, we can say

$$\dot{V}(x) \equiv 0 \quad \rightarrow \quad x_2 \equiv 0 \quad \rightarrow \quad \dot{x}_2 \equiv 0 \quad \rightarrow \quad \dot{x}_1 \equiv 0$$

and we know that

$$-x_1x_2 = 0$$

and since $x_2 \equiv 0$, we can say that any x_1 is possible, and therefore, all solutions of $x(\bullet)$ converges to the largest invariant set $\mathcal{M}$ which is defined as

$$\mathcal{M} \subseteq S := \Big\{ x \in \mathbb{R}^n \quad | \quad \forall x_1 \Big\}.$$

Thus, all solutions of $x(\bullet)$ approaches x_1.

Exercise 3

Considering the scalar nonlinear mechanical system

$$\ddot{q} + c\dot{q} + kq = 0$$

If the term $-c\dot{q}$ is due to the damping forces it is reasonable to assume that $c(0) = 0$ and

$$c\dot{q}\dot{q} > 0 \qquad \forall \quad \dot{q} \neq 0$$

Suppose the term $-kq$ is due to the conservative forces and define the potential energy by

$$P(q) = \int_0^q k(\eta) d\eta$$

Show that if $\lim_{q\to\infty} P(q) = \infty$, then all motions of this system must approach one of its equilibrium positions.

Solution:
The second assumption means that the $P(q)$ or the energy term

$$\frac{1}{2}kq^2$$

is radially unbounded. Now we redefine the system equation to be

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -kx_1 - cx_2\end{aligned}$$

and if we use a candidate Lyapunov function of

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$$

we get

$$\begin{aligned}\dot{V} &= kx_1\dot{x}_1 + x_2\dot{x}_2\\ &= kx_1x_2 + x_2(-kx_1 - cx_2)\\ &= -cx_2^2\end{aligned}$$

and from LaSalle's Theorem,

$$\dot{V} \equiv 0 \to x_2 = 0 \to \dot{x}_2 = 0$$

which means that

$$-kx_1 - cx_2 = 0$$

which is equivalent to the equilibrium solutions. Thus, we can say that all motions of this system approach one of its equilibrium positions.

q.e.d

Exercise 4

Consider a nonlinear mechanical system described by

$$m\ddot{q} + c\dot{q} + kq = 0$$

where q is scalar, m, $c > 0$ and k is a continuous function which satisfies

$$\begin{aligned} k(0) &= 0 \\ k(q)q &> 0 \end{aligned}$$

$$\lim_{q\to\infty} \int_0^q k(\eta)d\eta = \infty$$

(a) Obtain a state space description of this system.

(b) Prove that the state space model is GAS about the state corresponding to the equilibrium position $q = 0$.

(i) Use a La Salle type result.

(ii) Do not use a La Salle type result.

Solution:

(a) The state space representation of this system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

(b)(i) Using the results from Exercise 3 in this homework, we can see that all motions of this system approach one of its equilibrium condition. That is

$$\begin{aligned} 0 &= x_2 \\ 0 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{aligned}$$

which means that all motions approach the equilibrium condition of

$$q = 0 \qquad and \qquad \dot{q} = 0$$

Therefore, we have proved using La Salle's Theorem to say that the state space model is GAS about the state corresponding to the equilibrium position $q = 0$.

(b)(ii) We choose a candidate Lypunov function that is positive definite

$$V(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}x_2^2$$

and we get

$$\begin{aligned}\dot{V} &= kx_1\dot{x}_1 + x_2\dot{x}_2 \\ &= kx_1x_2 + x_2(-kx_1 - cx_2) \\ &= -cx_2^2 \\ &\leq 0\end{aligned}$$

Now, using **Theorem 23** on page 152 of the notes,

$$DV(x(t))f(x(t)) \equiv 0$$

only when

$$\dot{q} = x_2 \equiv 0$$

which implies that

$$q = x_1 \equiv 0$$

Therefore, we have proved without La Salle's Theorem to say that the state space model is GAS about the state corresponding to the equilibrium position $q = 0$.

Exercise 5

Consider an inverted pendulum $\mathcal{B}$ (or one link manipulator) subject to a control torque u. This system can be described by

$$\ddot{q} - a \sin q = bu$$

where q is the angle between the pendulum and a vertical line, $a = mgl/I$, $b = 1/I$, m is the mass of $\mathcal{B}$, I is the moment of inertia of $\mathcal{B}$ about its axis of rotation through $\mathcal{O}$, l is the distance between $\mathcal{O}$ and the mass center of $\mathcal{B}$, and g is the gravitational acceleration constant of YFHB. We wish to stabilize this system about the position corresponding to $q = 0$ by a linear feedback controller of the form

$$u = -k_p q - k_d \dot{q}$$

Using the results of the last problem, obtain the least restrictive conditions on the controller gains k_p, k_d which assure that the closed loop system is GAS about the state corresponding to $q(t) \equiv 0$. Illustrate your results with numerical simulations.

Solution:
Reorganizing the given equation we have

$$\ddot{q} = a \sin q + bu$$

and therefore, if $x_1 := q$ and $x_2 := \dot{q}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ a \sin x_1 + bu \end{bmatrix}$$

which also can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ a \sin x_1 - bk_p x_1 - bk_d x_2 \end{bmatrix}$$

Let a candidate Lyapunov function be

$$V(x) = \frac{1}{2} x_2^2 - a + a \cos x_1 + \frac{bk_p}{2} x_1^2.$$

Next we show that this is positive definite.

$$\begin{aligned} V(0) &= -a + a = 0 \\ DV(0) &= \begin{bmatrix} -a \sin x_1 + bk_p x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \\ D^2V(x) &= \begin{bmatrix} -a \cos x_1 + bk_p & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Since, $a > 0$ and $-a \leq -a\cos x_1 \leq 1$, for $D^2V(x) > 0$ to be true k_p must satisfy

$$k_p > \frac{a}{b}.$$

When this is true, the candidate Lyapunov function $V(x)$ is positive definite. Then,

$$\begin{aligned}\dot{V}(x) &= x_2\dot{x}_2 - a\dot{x}_1\sin x_1 + bk_p x_1\dot{x}_1 \\ &= ax_2\sin x_1 - bk_p x_1 x_2 - bk_d x_2^2 - ax_2\sin x_1 + bk_p x_1 x_2 \\ &= -bk_d x_2^2 \\ &\leq 0.\end{aligned}$$

This is true when k_d satisfies

$$k_d > 0.$$

From La Salle's Theorem

$$\dot{V}(x) \equiv 0 \quad \rightarrow x_2 \equiv 0 \quad \rightarrow \dot{x}_2 \equiv 0.$$

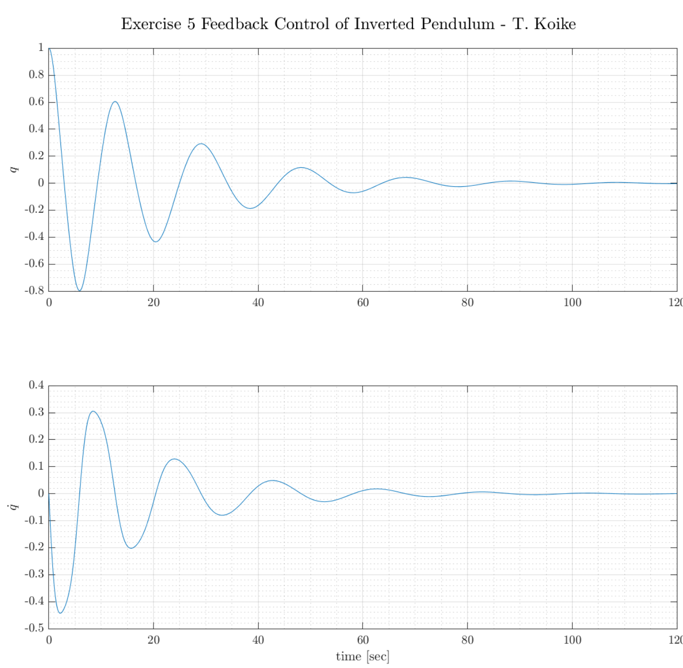
Then the origin is GAS. Now if we let the constants be

$$g = 9.8, \quad l = 2, \quad m = 1, \quad I = 10$$

and

$$k_p = \frac{a}{b} + 1 \quad \text{and} \quad k_d = 1$$

we can simulate the following results

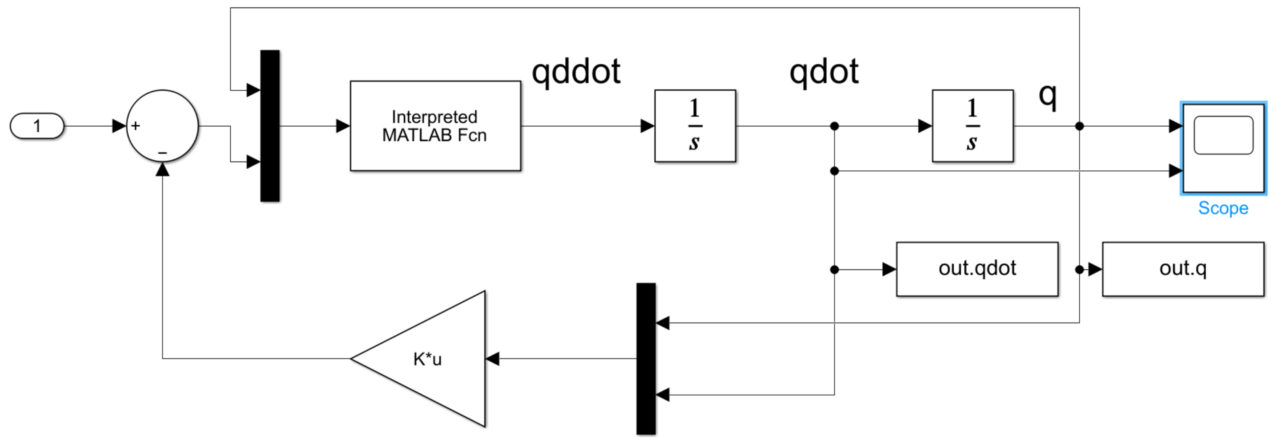


MATLAB Code:

```
% AAE 666 HW8 Exercise 5
% Tomoki Koike
close all; clear all; clc;
%%
% Define arbitrary constants
g = 9.8;  % [m/s2]
l = 2;  % [m]
m = 1;  % [kg]
I = 10;  % [kg—m2]

% Define parameters
a = m*g*l/I;
b = 1/I;

% Uncertain parameter
kp = a/b + 1;
kd = 1;
K = [kp kd];
%%
% Simulate and plot
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');

res = sim('ex5');
%%
t = res.tout;
q = res.q.signals.values;
qdot = res.qdot.signals.values;
%%
% — Plot
fig = figure("Renderer","painters","Position",[60 60 900 800]);
    subplot(2,1,1)
    plot(t, q)
    grid on; grid minor; box on;
    ylabel('$q$')
    subplot(2,1,2)
    plot(t, qdot)
    grid on; grid minor; box on;
    ylabel('$\dot{q}$')
    xlabel('time [sec]')
    title_string = 'Exercise 5 Feedback Control of Inverted Pendulum — T. Koike';
```

```
    sgtitle(title_string)
saveas(fig, 'ex5_invPend.png')
```

Simulink Model:



Exercise 6

Consider the system described by

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - x_2 + \theta \sin x_1 + u\end{aligned}$$

with the control input u where θ is an unknown constant parameter. Obtain an adaptive feedback controller which guarantees that, for any initial conditions, $\lim_{t\to\infty} x(t) = 0$ and $u(\bullet)$ is bounded. (Hint: As a candidate Lyapunov function for the closed loop system, consider something of the form $V(x)+U(\hat{\theta}-\theta)$ where V is a Lyapunov function for the nominal uncontrolled linear system.) Illustrate the effectiveness of your controller with simulations.

Solution:
Let a feedback controller be defined as

$$u = -\hat{\theta}\sin x_1$$

and we also define

$$\Delta\theta \triangleq \hat{\theta} - \theta$$

and therefore,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 - x_2 + \theta \sin x_1 - \hat{\theta}\sin x_1\\ &= -x_1 - x_2 + \theta \sin x_1 - (\Delta\theta + \theta)\sin x_1\\ &= -x_1 - x_2 - \Delta\theta \sin x_1\end{aligned}$$

Now since $\Delta\dot{\theta} = \dot{\hat{\theta}}$, and if we define a positive definite candidate Lyapunov function to be

$$V(x) = x_1^2 + x_2^2 + \gamma\Delta\theta^2 \qquad\qquad \gamma > 0$$

we can use then take the derivative of this to see what $\dot{\hat{\theta}}$ value would be optimal for the adapative controller.

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 + 2\gamma\Delta\theta\Delta\dot{\theta}\\ &= 2x_1x_2 + 2x_2(-x_1 - x_2 - \Delta\theta\sin x_1) + 2\gamma\Delta\theta\Delta\dot{\theta}\\ &= 2x_1x_2 - 2x_1x_2 - 2x_2^2 - 2\Delta_2\sin x_1 + 2\gamma\Delta\theta\dot{\hat{\theta}}\\ &= -2x_2^2 \underbrace{-2\Delta\theta x_2\sin x_1 + 2\gamma\Delta\theta\dot{\hat{\theta}}}_{0}\end{aligned}$$

Like how it is shown above we want the second and third term to become 0 so that $\dot{V} \leq 0$ which allows the system to be GAS about the origin. So we solve

$$2\gamma\Delta\theta\dot{\hat{\theta}} = 2\Delta\theta x_2 \sin x_1$$
$$\dot{\hat{\theta}} = \frac{1}{\gamma} x_2 \sin x_1$$

when this is true, we have

$$\dot{V}(x) = -2x_2^2$$

and from La Salle's Theorem, we have $x_2 \equiv 0$ and $\dot{x}_2 \equiv 0$ when $\dot{V} \equiv 0$, which also means that the following are bounded

$$\begin{cases} x \text{ is bounded} \\ \Delta\theta \text{ is bounded} \\ \hat{\theta} \text{ is bounded} \end{cases}$$

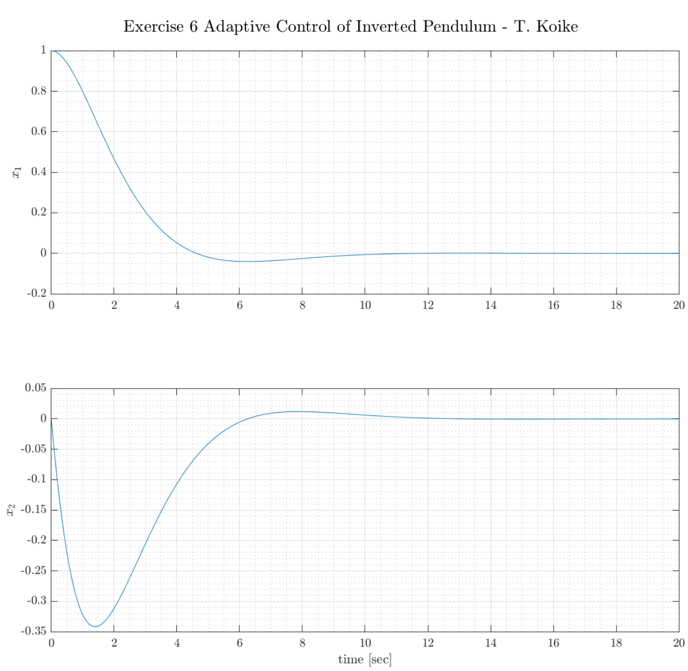
and therefore, $u(\bullet)$ is bounded. Because we know what $\dot{\hat{\theta}}$ we can integrate out $\hat{\theta}$ can create a adaptive controller of

$$u = -\hat{\theta} \sin x_1$$

Now we will simulate by setting the following constants

$$\theta = 0.5, \quad \gamma = 10$$

and the initial condition as $x_0 = [1, 0]^T$. The simulation result is as follows.



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