Examples – 3D Representations

Example 1:

Earth

Given : $\bar{r}_1 = 1.6772\,R_\oplus\,\hat{x} - 1.6772\,R_\oplus\,\hat{y} + 2.3719\,R_\oplus\,\hat{z}$
at t_0 $\bar{v}_1 = 3.1574\,\hat{x} + 2.4987\,\hat{y} + 0.4658\,\hat{z}$ km/s

UNITS!!

Find : $a, e, i, \Omega, \omega, \theta^*$

$\theta = \omega + \theta^*$ argument of latitude

$(t - t_p)$? What type of conic?

Analysis :

Shape → $r_1 = |\bar{r}_1| \qquad v_1 = |\bar{v}_1|$ magnitude

$$\underbrace{\varepsilon = -\frac{\mu}{2a}}_{} = \frac{v_1^2}{2} - \frac{\mu}{r_1} \quad \Rightarrow \quad \boxed{a = 3\,R_\oplus}$$

check/confirm orbit elliptical $v < \sqrt{2}\,v_c$

$$\bar{h} = \bar{r}_1 \times \bar{v}_1 \rightarrow h = |\bar{h}| = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$$
$$\rightarrow \boxed{e = 0.2}$$

check r_p and r_a → any collisions? impacts?

From conic equation →

$$\theta_1^* = \pm\cos^{-1}\left\{\frac{1}{e}\left(\frac{p}{r_1} - 1\right)\right\}$$

double-valued

quadrant check!

$\theta_1^* = \pm 135°$

$$h^2 = \mu a(1-e^2)$$
$$\frac{h^2}{\mu a} = 1 - e^2$$
$$e^2 = 1 - \frac{h^2}{\mu a}$$
$$e = \sqrt{1 - \frac{h^2}{\mu a}}$$

orbit orientations → $\bar{h} \Rightarrow \frac{\bar{h}}{|\bar{h}|} = \hat{h}$

$$\hat{h} = \underbrace{\frac{\bar{r}\times\bar{v}}{|\bar{r}\times\bar{v}|}}_{\text{careful!}} = -.5\,\hat{x} + .5\,\hat{y} + .7071\,\hat{z}$$

check! is it a unit vector?

careful! Generally $\bar{r}$ & $\bar{v}$ are not $\perp$

${}^I C^R$	$\hat{r}_1$	$\hat{\theta}_1$	$\boxed{\hat{h}_1}$
$\hat{x}$	....	....	$s_\Omega s_i$ ← -0.5
$\hat{y}$	....	....	$-c_\Omega s_i$ ← 0.5
$\hat{z}$	$s_i s_\theta$	$s_i c_\theta$	c_i ←

$0 \le \Omega \le 2\pi$

$c_i = .7071 \Rightarrow i = \pm 45^\circ$ (choose $i = 45^\circ$) $0 \le i \le \pi$

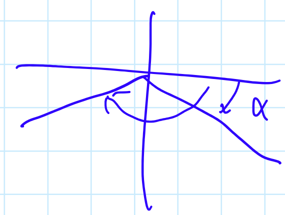
$s_\Omega s_i = -.5 \quad \{ \Omega = -45^\circ, 225^\circ$

$-c_\Omega s_i = +.5 \quad \{ \Omega = \pm 135^\circ$

$\Omega = 225^\circ$

$-(180+\alpha)$

$\alpha - 180$



Numerical values

Note that we can also obtain the remaining elements of the direction cosine matrix

$$\hat{r}_1 = \frac{\bar{r}_1}{|\bar{r}_1|} = .5\,\hat{x} - .5\,\hat{y} + .7071\,\hat{z}$$

$$\hat{\theta}_1 = \hat{h}\times\hat{r}_1 = .7071\,\hat{x} + .7071\,\hat{y}$$

$$\left.\begin{aligned} s_i s_{\theta_1} &= .7071 \\ s_i c_{\theta_1} &= 0 \end{aligned}\right\} \quad \boxed{\theta_1 = 90^\circ}$$

Back to θ_1^* ($\pm 135^\circ$) $\longrightarrow$ Recall

$$\bar{v}_1 = \underbrace{(\bar{v}_1 \bullet \hat{r}_1)}_{\dot{r}_1}\hat{r}_1 + \underbrace{(\bar{v}_1 \bullet \hat{\theta}_1)}_{r_1\dot{\theta}_1}\hat{\theta}_1 \qquad \bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$\dot{r}_1 = \bar{v}_1 \bullet \hat{r}_1 \rightarrow$ if $\dot{r}_1 > 0$ ascending in orbit
if $\dot{r}_1 < 0$ descending in orbit

(check γ)

in this case $\dot{r}_1 = +0.658 > 0 \Rightarrow$ ascending

$\rightarrow \boxed{\theta^* = +135^\circ} \longrightarrow \dot{r} > 0$

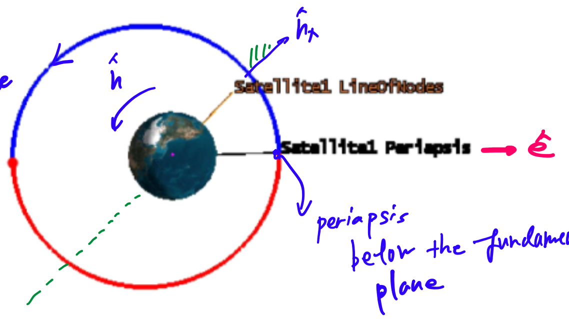
in this case $\dot{r}_1 = +0.659 > 0 \Rightarrow$ ascending

$$\Rightarrow \boxed{\theta_1^* = +135^\circ} \longrightarrow \gamma > 0$$

$$\omega = \theta_1 - \theta_1^* = -45^\circ$$

orbit plane

$\hat{h}$



periapsis below the fundamental plane

Example 2:

Given : $\bar{r}_1 = 14450.6\,\hat{x} - 1529.9\,\hat{y} - 6524.0\,\hat{z}$ km
$\bar{r}_2 = -6199.5\,\hat{x} + 14699.2\,\hat{y} + 8531.9\,\hat{z}$ km

UNITS!!

there is a method to get p from vectors

$p = 2.88\,R_\oplus$ ← one additional piece of information required. Typically, it is time between the position → assume we have is p

Find: $a, e, i, \Omega, \omega, \theta_1^*, \theta_2^*, \bar{v}_1, \bar{v}_2$

Analysis

Available $\quad r_1 = |\bar{r}_1| \qquad r_2 = |\bar{r}_2|$

$\bar{h} \neq \bar{r}_1 \times \bar{r}_2$
but $\hat{h} \perp (r_1 \wedge r_2)$

unit vector $\quad \hat{h} = \dfrac{\bar{r}_1 \times \bar{r}_2}{|\bar{r}_1 \times \bar{r}_2|}$ $\quad$ why? $\qquad$ Are $\hat{h}_1, \hat{h}_2$ both necessary?

$\hat{r}_1 = \dfrac{\bar{r}_1}{|\bar{r}_1|} \Longrightarrow$ unit vector in $\hat{x}, \hat{y}, \hat{z}$

$\hat{\theta} = \hat{h} \times \hat{r}_1$

${}^I C^R$	$\hat{r}_1$	$\hat{\theta}_1$	$\hat{h}$
$\hat{x}$	.9072	.2280	.3536
$\hat{y}$	-.0960	.9305	-.3536
$\hat{z}$	-.4096	.2868	.8660

Can I check ?

What conditions must be satisfied so there is a chance that this DC matrix is correct?

orthogonality
mag(row) = 1
mg (col) = 1
dot product rows/cols = $\cancel{0}$

$\hat{r}_2, \hat{\theta}_2, \hat{h}$ do I need $\underbrace{\hat{h}_1, \hat{h}_2}_{\text{same}}$

transformation matrix

$\left.\begin{matrix} \hat{r}_1 & \hat{\theta}_1 & \hat{h}_1 \\ \hat{r}_2 & \hat{\theta}_2 & \hat{h} \end{matrix}\right\}$ not same

$\bar{r}_1\ \theta_1\ h_1$ $\hat{r}_2\ \hat{\theta}_2\ \hat{h}$ } not same

Compare to

${}^{I}C^{R}$	$\hat{r}_1$	$\hat{\theta}_1$	$\hat{h}$
$\hat{x}$			$\sin\Omega\sin i$
$\hat{y}$			$-\cos\Omega\sin i$
$\hat{z}$	$\sin i\sin\theta_1$	$\sin i\cos\theta_1$	$\cos i$

$\cos i = .866$ → $i = 30°$

$\sin\Omega\sin i = .3536$ { $\Omega = 45°, 135°$

$-\cos\Omega\sin i = -.3536$ { $\Omega = 45°, -45°$ ⇒ $\Omega = 45°$

$\sin\theta_1 \sin i = -.4096$ { $\theta_1 = -55°, 235°$

$\cos\theta_1 \sin i = .2868$ { $\theta_1 = \pm 55$ ⇒ $\theta_1 = -55°$

velocities?

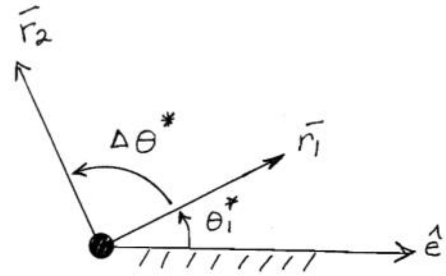
$\bar{r}_2 = f\,\bar{r}_1 + g\,\bar{v}_1$ ➡ $\bar{v}_1 = \dfrac{\bar{r}_2 - f\bar{r}_1}{g}$ ⇒ f? g?

Use f and g in terms of $\Delta\theta^*$!!

$f, g = f\left(r_1, r_2, \Delta\theta^*, p\right);\quad g\left(r_1, r_2, \Delta\theta^*, p\right)$

How to find $\Delta\theta^*$? $\theta_2^* - \theta_1^* = \Delta\theta^*$

Any assumptions required?



$\bar{r}_1 \bullet \bar{r}_2 = r_1 r_2 \cos\Delta\theta^*$

$\Rightarrow \boxed{\Delta\theta^* = \pm 125.6^\circ}$

Assume sign is positive

$\Downarrow$ from f & g func

$\bar{v}_1 = 0.6814\,\hat{x} + 5.0560\,\hat{y} + 1.7859\,\hat{z}$ km/s

θ_1^*? Is the vehicle ascending or descending?

Recall $\bar{v}_1 = \underbrace{(\bar{v}_1 \bullet \hat{r}_1)}_{\dot{r}_1}\hat{r}_1 + \underbrace{(\bar{v}_1 \bullet \hat{\theta}_1)}_{r_1\dot{\theta}_1}\hat{\theta}_1$

$\dot{r}_1 = \dfrac{\bar{v}_1 \bullet \bar{r}_1}{r_1} = -0.5987 < 0$ descending

Can γ_1 now be determined? How? Would it be + or - ? $\bar{r}_1, \bar{v}_1 \Rightarrow \bar{h}$

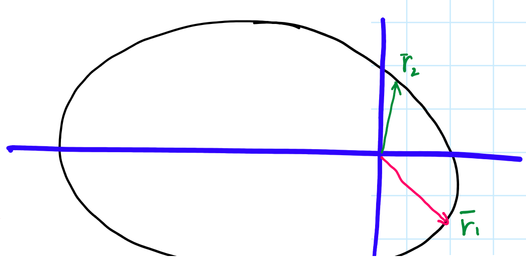
$h = rv\,c_\gamma \Rightarrow \gamma = neg$

Now $v_1 = |\bar{v}_1|$ $r_1 = |\bar{r}_1|$

$\varepsilon = \dfrac{v_1^2}{2} - \dfrac{\mu}{r_1} \Rightarrow \boxed{a = 3R_\oplus}$

$p = a(1-e^2) \Rightarrow \boxed{e = .2}$

conic equation $\theta_1^* = \pm\cos^{-1}\left\{\dfrac{1}{e}\left(\dfrac{p}{r_1} - 1\right)\right\} \Rightarrow \theta_1^* = \pm 40^\circ$ $\theta_1^* = 320^\circ$



$\theta_2^* = \theta_1^* + \Delta\theta^* \rightarrow \boxed{\theta_2^* = 85.6^\circ}$

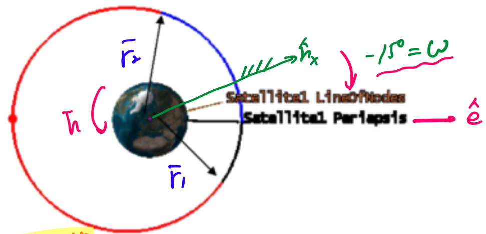
$\omega = \theta_1 - \theta_1^* \rightarrow \omega = -15^\circ \rightarrow \theta_2$

$\omega = \theta_1 - \theta_1^*$ $\longrightarrow$ $\omega = -15° \rightarrow \theta_2$

θ_2 ?

$\theta_2 = \omega + \theta_2^*$?

$$\boxed{\bar{v}_2 = \dot{f}\,\bar{r}_1 + \dot{g}\,\bar{v}_1}$$



natural behavior relative 2BP

orbit characteristics $\left\{ \begin{array}{l} \text{various forms} \\ \bar{r}, \bar{v} \\ \text{orbital elements} \\ \bar{h}, \bar{e} \\ \vdots \end{array} \right.$

next? for s/c → $\left. \begin{array}{l} \text{change} \\ \text{orbital} \\ \text{characteristics} \end{array} \right\}$ maneuver