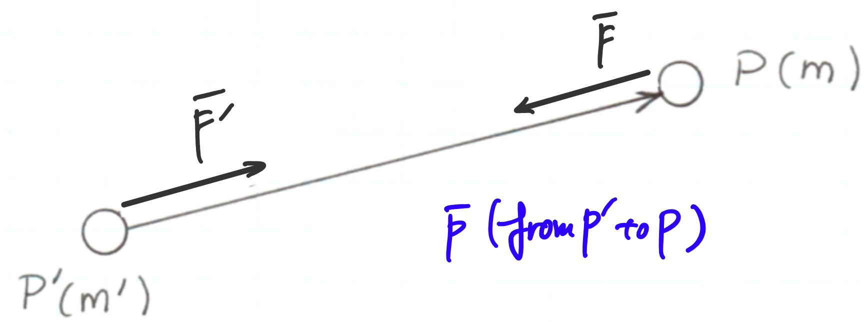
P2: IBP

2020年2月14日 金曜日　　午後0:30

Gravity

Gravitational interaction of two particles



Particle P will experience a force $\bar{F}$ due to the existence of P'

$$\bar{F} = -G\,m'\,m\,\bar{p}\left(p^2\right)^{-\frac{3}{2}}$$

directed along $-\bar{p}$

universal gravitational constant $\cong 6.67 \times 10^{-11}\ \dfrac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$

P' experiences equal and opposite force $\bar{F}'$

$$\bar{F}' = -\bar{F} \quad \text{directed in } +\bar{p}$$

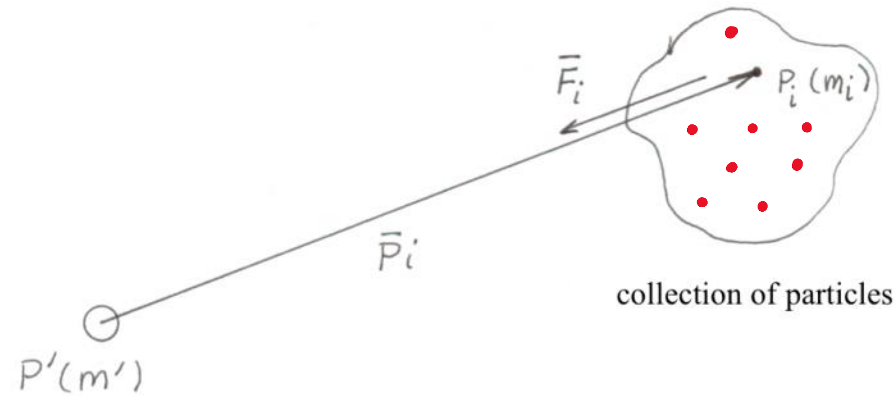
Assume Earth $(\oplus)$ and Moon $(☾)$ are particles

$P'(\oplus)$ $\quad m' \cong 5.976 \times 10^{24}\,\text{kg} \quad \rightarrow \quad |\bar{p}| = 3.844 \times 10^{8}\,\text{meters}$

$P(☾)$ $\quad m \cong 7.34 \times 10^{22}\,\text{kg} \quad \rightarrow \quad |\bar{F}| = 1.98 \times 10^{20}\,\dfrac{\text{kg}\cdot\text{m}}{\text{s}^2}$

may be large

Replace particle P with a set of particles $P_i \; (i = 1, \dots N)$

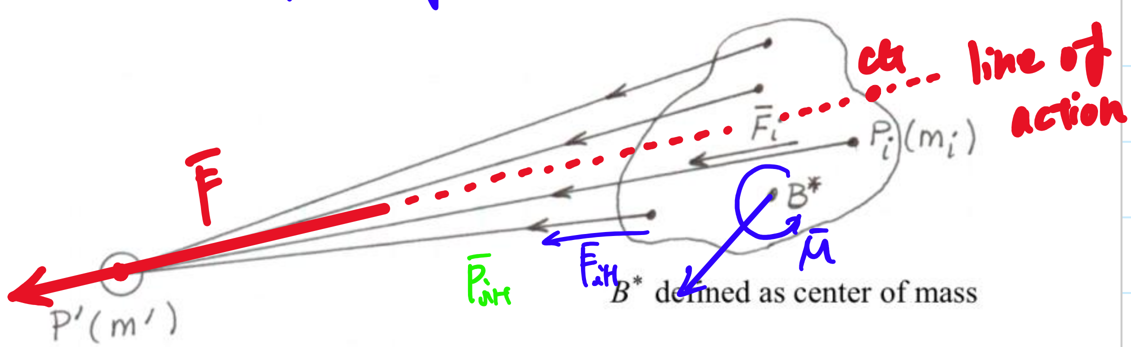


Resultant force exerted on collection of particles by P' ?

Force exerted by P' on each particle of the collection:

$$\bar{F}_i = -G\, m' m\, \bar{p}_i \left(p_i^2\right)^{-\frac{3}{2}}$$

line of action for each $\bar{F}_i \Rightarrow -\bar{p}_i$ thru P'



Resultant Gravity Force $\bar{F}$

magnitude direction

System of forces exerted on $P_1, P_2, \ldots, P_N$ is

equivalent to the force resultant $\bar{F}$

$$\bar{F} = \sum_{i=1}^{N} \bar{F}_i = -Gm' \sum_{i=1}^{N} m_i (p_i^2)^{-3/2} \bar{p}_i$$

Note: "force" is always associated with a point of application (line of action)

This force resultant $\bar{F}$ → line of action?
point of application?

Force Models Equivalent force / Moment model

(a) identifies effect of a particular system of forces on motion of a body $\Rightarrow \bar{F}$

(b) effect can be represented different ways → impact on body's motion must be the same

force resultant @ A(P')

force resultant @ point B + moment resultant

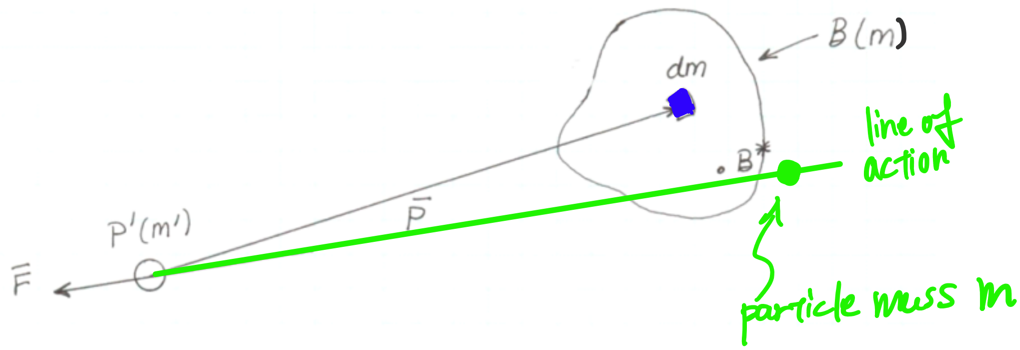
Note that force resultant acts through P'

Gravity forces not parallel

Consider moving force resultant to act at B^*

line of action not pass thru P'

Increase sophistication of model $\quad$ B now continuous distribution of matter density υ



$\bar{p}$ is the vector to any generic point P, i.e., to a differential element of mass dm

Force resultant / gravitational force on B due to P'

exact $$\bar{F} = -Gm'\int \bar{p}\left(p^2\right)^{-\frac{3}{2}} \underbrace{\overset{\text{density}}{\upsilon}\, d\tau}_{dm}$$ differential volume

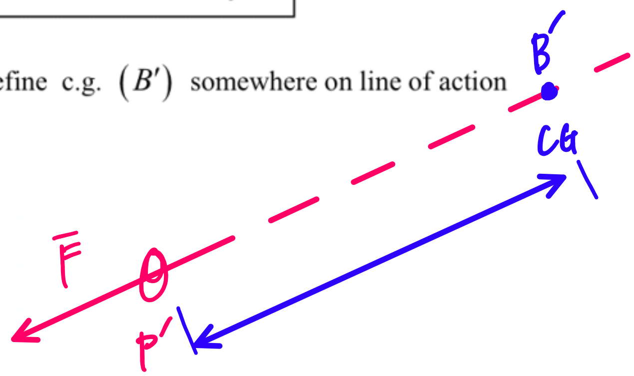
Assuming this could be successfully integrated analytically, it is the exact gravity force for this model ➡ NOT an approximation

Again, resultant gravity force (true line of action) at P' →

Important because it is the basis for actual definition of **center of gravity**

Center of Gravity

Define c.g. (B') somewhere on line of action



R^{cg} location where a particle with mass equal to mass of body $B\ (m)$ would be placed to yield the same resultant gravity force as the body

cm property of body

cg not property of body; determined by gravity field.

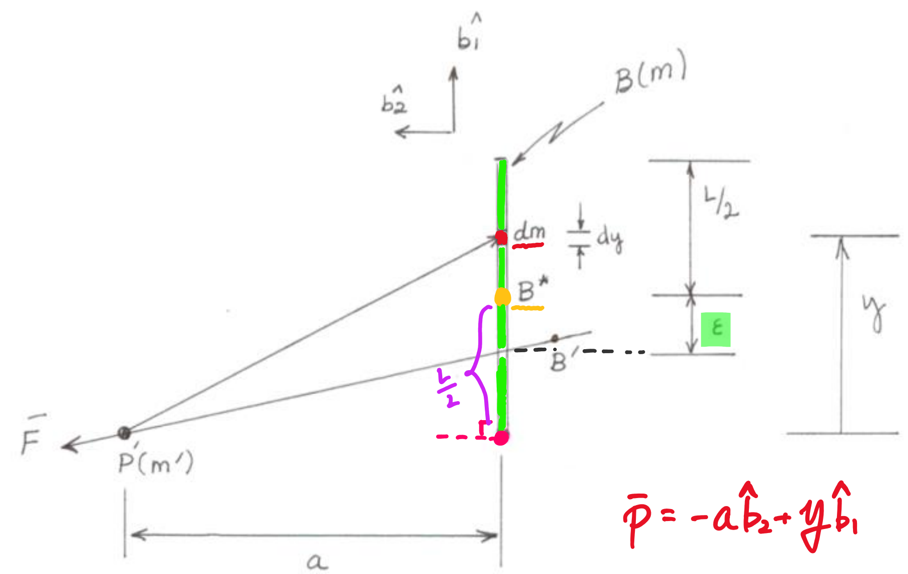
$cm \neq cg$ in general; $\underbrace{\text{may coincide rarely}}$

Examples: "point mass" model
spherical mass with uniform density

$$R^{cg} = \left[\frac{G m' m}{|\bar{F}|} \right]^{\frac{1}{2}}$$

$\begin{cases} m' : \text{mass of } P' \\ m : \text{total mass of } B \end{cases}$

<u>Example</u> – vehicle modeled as a thin homogeneous rod



$$\bar p = -a\hat b_2 + y\hat b_1$$

Exact

$$\bar F = -Gm'\int \bar p\left(p^2\right)^{-\frac{3}{2}} \overbrace{\upsilon\, d\tau}^{dm}$$

differential element along L

$$= -Gm'\int_0^L\left(-a\hat b_2 + y\hat b_1\right)\left(a^2+y^2\right)^{-\frac{3}{2}} \underbrace{\frac{m}{L}}_{} dy$$

$\frac{\text{density}}{\text{unit length}} = \text{constant}$

$$\boxed{\begin{aligned} dm &= \upsilon\, d\tau \\ &= \frac{m}{L}\, dy \end{aligned}}$$

$$= \frac{Gm'm}{L}\int_0^L a\left(a^2+y^2\right)^{-\frac{3}{2}} dy\, \hat b_2 - \frac{Gm'm}{L}\int_0^L y\left(a^2+y^2\right)^{-\frac{3}{2}} dy\, \hat b_1$$

$$\bar F = +\left\{\frac{Gm'm}{a\left(a^2+L^2\right)^{1/2}}\right\}\hat b_2 + \left\{-\frac{G\, m'm}{a\left(a^2+L^2\right)^{1/2}}\left[\left(1+\frac{a^2}{L^2}\right)^{\frac{1}{2}} - \frac{a}{L}\right]\right\}\hat b_1$$

generally we cannot analytically integrate

for actual system – in motion – extremely unlikely that can location can be analytically integrated

To locate cg requires magnitude of resultant force and line of action

$$|\bar{F}| = G\,m'm\left[\left\{\frac{1}{a\left(a^2+L^2\right)^{\frac{1}{2}}}\right\}^2 + \left\{\frac{1}{a\left(a^2+L^2\right)^{\frac{1}{2}}}\left[\left(1+\frac{a^2}{L^2}\right)^{\frac{1}{2}} - \frac{a}{L}\right]\right\}^2\right]^{\frac{1}{2}}$$

Reduces to

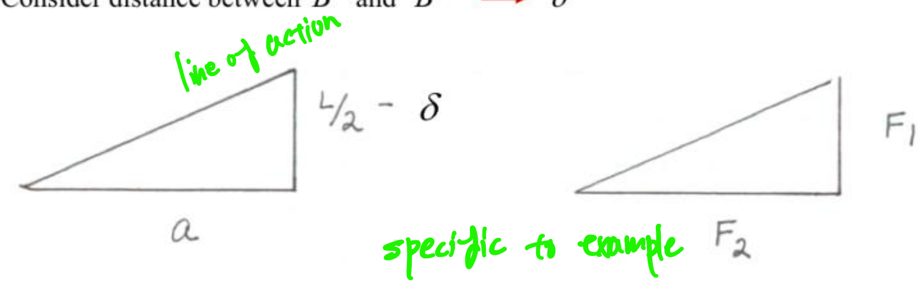
$$|\bar{F}| = \frac{G\,m'm}{a\,L}\left[2 - \frac{2a}{L}\left(1+\frac{a^2}{L^2}\right)^{-\frac{1}{2}}\right]^{\frac{1}{2}}$$

Magnitude of force resultant (exact)

$$\boxed{R^{cg} = \left[\frac{G m' (m_R)}{|F|}\right]^{\frac{1}{2}}}$$

function of a

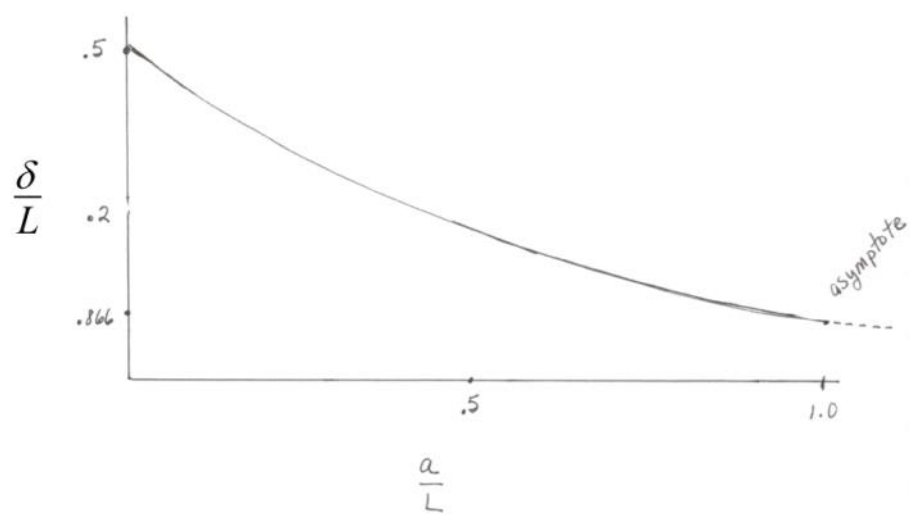
Consider distance between B^* and B^{cg} $\longrightarrow$ δ



$$\frac{\frac{L}{2}-\delta}{a}=\frac{F_1}{F_2}$$

$$\frac{\delta}{L}=\frac{1}{2}-\frac{a}{L}\frac{F_1}{F_2}$$

$$\frac{\delta}{L}=\frac{1}{2}-\frac{a}{L}\left[\left(1+\frac{a^2}{L^2}\right)^{\frac{1}{2}}-\frac{a}{L}\right]$$

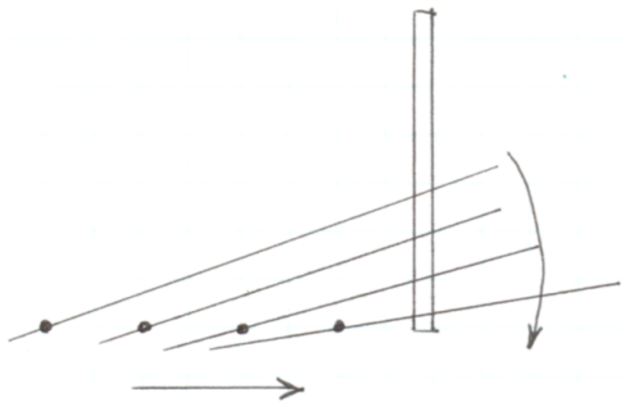


Observations

(a) Location of *cg* is a function of distance a to P'

(b) $\frac{a}{L} = 0$ is excluded since two points cannot occupy same point in space

(c) when $\frac{a}{L}$ small $\left(P' \text{ very close}\right)$, $\bar{F}$ is "flatter"



$F \propto \frac{1}{r^2}$

r small then F large

(d) for any finite value of $\frac{a}{L}$ line of action cannot pass through P' and B^*

→ B^* and B^{cg} will not coincide;

when $\frac{a}{L}$ large, difference between *cm* and *cg* may be very small but NEVER coincide

$\delta \to$ small / aproach asymptote

but the fact that they are close when $\frac{a}{L}$ large may be useful

How can difference be reasonably approximated? yes.