# P2: K

2020年2月28日 金曜日　　午後0:33

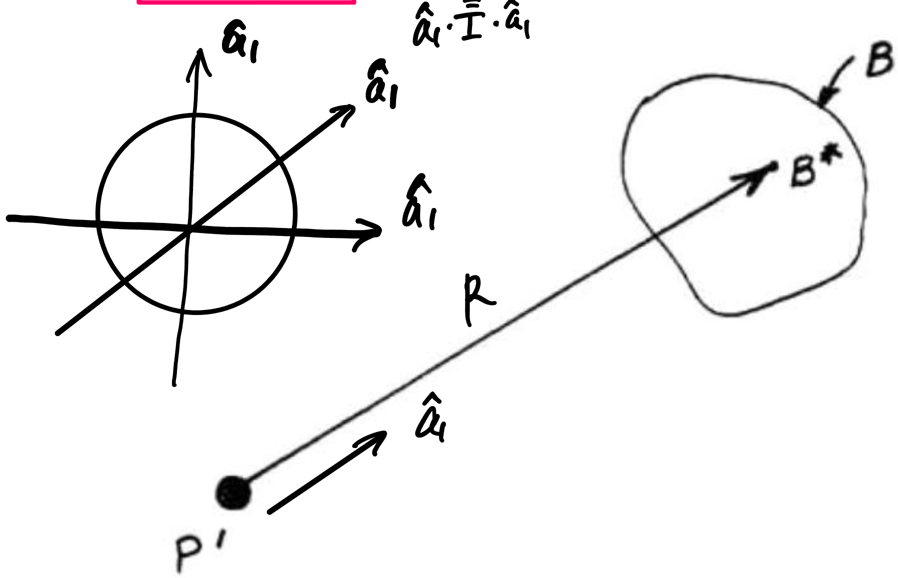
## Centrobaric Body

$\boxed{cm = cg}$

→ gravity torque is 0

Body B is centrobaric if its exact gravity force can be written

$\boxed{\bar{F} = -\frac{Gm'm}{R^2}\hat{a}_1}$ for all attracting bodies $P'$ (outside B)




* replace body w/ a particle

• Newton treated planets as centrobaric bodies

So body B has the same gravity as a particle inverse square law point mass

Body B **can** be replaced with a particle of mass $m$ at $B^*$

→ $\boxed{cg = cm}$

Examples: particle
sphere → sputnik
spherical shell → uniform mass density

exact

$$\bar{F} = -\int Gm'\, \bar{p}\left(p^2\right)^{-\frac{3}{2}} \upsilon\, d\tau \quad \text{for any body B}$$

cm = cg

$$\bar{F} = -\underbrace{\frac{Gm'm}{R^2}}\left[\hat{a}_1 + \overbrace{\sum_{i=2}^{\infty} \bar{f}^{(i)}}\right]$$

$\Sigma = 0$

$\bar{f}^{(2 \to \infty)} = 0$

Proving a body IS centrobaric means integrating to determine the form of the total gravity force

Proving a body IS NOT centrobaric means finding that $\boxed{\sum_{i=2}^{\infty} \bar{f}^{(i)} \neq 0}$

body is not centrobaric

Note:

$$\bar{f}^{(2)} \propto R^{-2}$$
$$\bar{f}^{(3)} \propto R^{-3}$$
$$\vdots$$

$$\therefore \bar{f}^{(i)} = 0 \qquad i = 1,2,3,\ldots,\infty$$

to make sum equal to zero, each $\bar{f}^{(i)}$ must vanish separately.

One consequence of this fact is associated with $\bar{f}^{(2)}$

➔ To be centrobaric, ONE requirement: $\bar{f}^{(2)} = 0$

If $\bar{f}^{(2)} = 0$, all components must be zero

$\hat{a}_1 \bullet \bar{f}^{(2)} = 0 \longrightarrow$ component along the orbit radial direction

$$\hat{a}_1 \bullet \underbrace{\frac{1}{mR^2}\left\{\frac{3}{2}\left[tr\left(\bar{\bar{I}}\right) - 5\hat{a}_1 \bullet \bar{\bar{I}} \bullet \hat{a}_1\right]\hat{a}_1 + 3\bar{\bar{I}} \bullet \hat{a}_1\right\}}_{\bar{f}^{(2)}} = 0$$

$$\frac{3}{2}\left[tr(\bar{\bar{I}}) - 5\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1\right] + 3\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1 = 0$$

$$\frac{3}{2} tr(\bar{\bar{I}}) - \frac{9}{2}\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1 = 0$$

↳ necessary condition for body to be centrobaric

OR

$$\boxed{\hat{a}_1 \cdot \bar{\bar{I}} \cdot \hat{a}_1 = \frac{1}{3} tr(\bar{\bar{I}}) = \text{const.}}$$

↳ $tr(\bar{\bar{I}})$ is invariant, no matter which unit vector $\bar{\bar{I}}$ is expressed $tr(\bar{\bar{I}})$ is always constant.

Trace is invariant (constant)

$\hat{a}_1 \bullet \bar{\bar{I}} \bullet \hat{a}_1$ must have the same value no matter how body is oriented

Centrobaric body has the same moment of inertia about <u>every</u> line passing through the mass center

→ Central inertial ellipsoid is a sphere

↳ definition of centrobaric

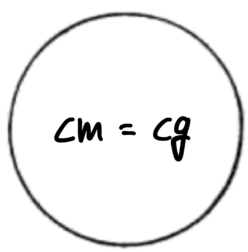
$$\boxed{\text{Central inertia ellipsoid is a sphere}}$$

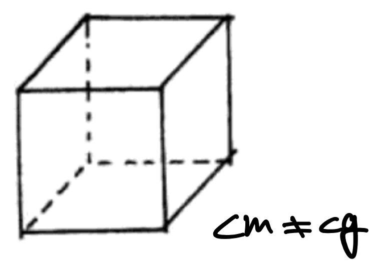
So, all centrobaric bodies have a spherical inertia ellipsoid

BUT

All bodies with a spherical inertia ellipsoid are not centrobaric

$\bar{f}^{(2)} = 0 \longrightarrow$ necessary condition




shell
sphere
collection of spherical shells

$\bar{f}^{(2)} = 0$

$\downarrow$

all $\bar{f}^{(i)} = 0$

is centrobaric

density const.

$\longrightarrow$ central inertial ellipsoid is a sphere

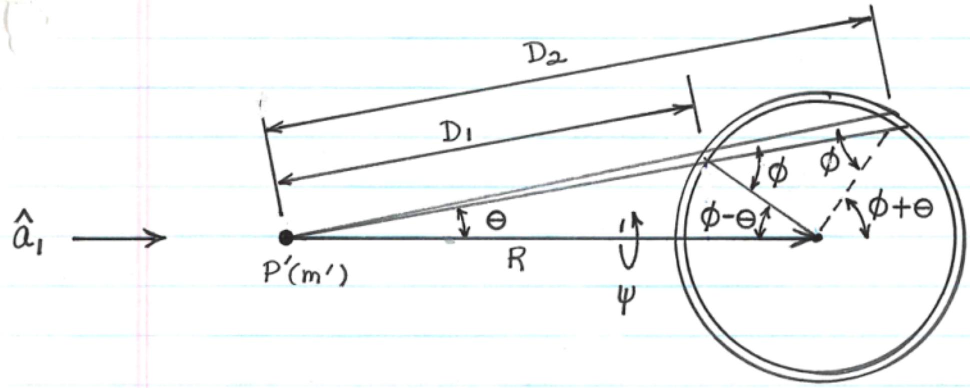
$\hat{a}_1 \cdot \bar{f}^{(2)} = 0$

not all $\bar{f}^{(i)} = 0$

* meets one of the cond. but not all.

not centrobaric

Example: Spherical Shell is centrobaric (radius r)




Resultant of all gravity forces

$$\bar{F} = -Gm' \int \bar{p}(p^2)^{-3/2} \rho \, d\tau$$

What follows is one of a number of ways in which the integral can be evaluated:

[Note: Components in $\hat{a}_2$ and $\hat{a}_3$ directions will cancel because of symmetry so it is necessary only to be concerned with the $\hat{a}_1$ direction]

Force on a differential element

$$\hat{a}_1 \cdot d\bar{F} = -\frac{Gm' \, dm \cos\theta}{D^2}$$

So

$$\hat{a}_1 \cdot \bar{F} = -Gm' \int \frac{\overbrace{\rho \, d\tau}^{dm}}{D^2} \cos\theta$$

Differential elements (rings)

$$d\tau_1 = \left[r\,d(\phi-\theta)\right]\left[r\sin(\phi-\theta)\,d\psi\right]$$

$$= \left[\frac{D_1\,d\theta}{\cos\phi}\right]\left[D_1\sin\theta\,d\psi\right]$$

$$= \sec\phi\,D_1^2\sin\theta\,d\theta\,d\psi$$

$$d\tau_2 = \left[r\,d(\phi+\theta)\right]\left[r\sin(\phi+\theta)\,d\psi\right]$$

$$= \left[\frac{D_2\,d\theta}{\cos\phi}\right]\left[D_2\sin\theta\,d\psi\right]$$

$$= \sec\phi\,D_2^2\sin\theta\,d\theta\,d\psi$$

$$\hat{a}_1\cdot\bar{F} = -Gm'\int_{\psi=0}^{2\pi}\int_{\theta=0}^{\sin^{-1}\frac{r}{R}}\left[\frac{\sec\phi\,\rho\,D_1^2\sin\theta\,d\theta\,d\psi}{D_1^2} + \frac{\sec\phi\,\rho\,D_2^2\sin\theta\,d\theta\,d\psi}{D_2^2}\right]\cos\theta$$

$$\hat{a}_1\cdot\bar{F} = -4\pi\,Gm'\rho\int_{\theta=0}^{\sin^{-1}\frac{r}{R}}\sec\phi\sin\theta\cos\theta\,d\theta$$

Change variable of integration to $\phi$

$$R s_\theta = r s_\phi$$

$$R c_\theta\,d\theta = r c_\phi\,d\phi$$

$$\hat{a}_1 \cdot \bar{F} = -4\pi G m' \rho \int_{\phi=0}^{\pi/2} \sec\phi \left(\frac{r}{R}\sin\phi\right)\frac{r}{R}\cos\phi \, d\phi$$

$$= -4\pi G m' \rho \int_{\phi=0}^{\pi/2} \left(\frac{r}{R}\right)^2 \sin\phi \, d\phi$$

$$= -4\pi G m' \rho \frac{r^2}{R^2} \left(-c_\phi\right)_0^{\pi/2}$$

$$= \frac{-Gm'}{R^2} \rho 4\pi r^2$$ but $4\pi r^2$ = area of a thin spherical shell

So $m = \rho 4\pi r^2$

$$\boxed{\bar{F} = -\frac{Gm'm\,\hat{a}_1}{R^2}} \Longleftarrow \text{centrobaric}$$

A sphere is a collection of thin spherical shells. It is also centrobaric.