Lecture: Introduction to Graph Theory

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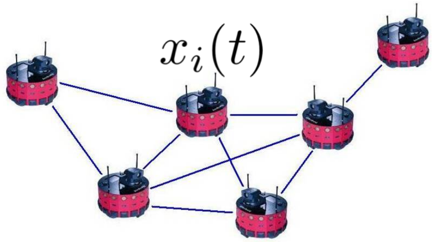
Review Distributed Consensus



agent's dynamics: $x_i(t+1) = u_i$

distributed control: $u_i = f_i(x_j(t), j \in \mathcal{N}_i)$

✓ **Objective:** $x_1(t) = x_2(t) = \cdots = x_m(t) = x^*$

✓ **Update:** $x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$

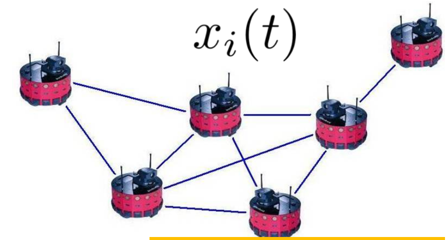
w_{ij} : the weight assigned by agent *i* to agent *j*

Consensus Goals	Choices of Weights
x^* is an unknown constant	$w_{ij} = \begin{cases} >0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \quad \sum_{j=1}^{m} w_{ij} = 1$
x^* is the global average $\frac{1}{m}\sum_{i=1}^{m} x_i(0)$	$w_{ij} = \begin{cases} \min\{\frac{1}{d_i}, \frac{1}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, \text{ otherwise.} \end{cases}$
x^* is a specific convex combination $\sum_{i=1}^{m} \gamma_i x_i(0)$	$w_{ij} = \begin{cases} \frac{1}{\gamma_i} \min\{\frac{\gamma_i}{d_i}, \frac{\gamma_j}{d_j}\} & j \in \mathcal{N}_i,\ j \neq i; \\ 1 - \sum_{j \in \mathcal{N}_i,\ j \neq i} w_{ij} & j = i \\ 0, \text{ otherwise.} \end{cases}$

Distributed Algorithm for Consensus

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$$

$$w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \qquad \sum_{j=1}^{m} w_{ij} = 1$$



Network connectivity

$$\boxed{x(t+1) = Ax(t)}$$

Graph Theory

?

A is row stochastic

If A is also Primitive

Gershgorin Circle Theorem

Perron - Frobenius Theorem

$$A\mathbf{1} = \mathbf{1}$$

1 is the largest eigenvalue in magnitude.

1 is a simple eigenvalue

all the other eigenvalues are with magnitude strictly less than 1

$$\lim_{t \to \infty} A^t = \mathbf{1}w'$$

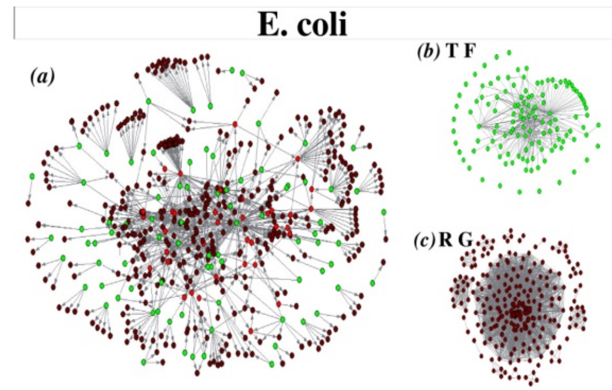
$$\lim_{t \to \infty} x(t) = \mathbf{1}w'x(0)$$

What is a **graph**? ***Graph:*** A set of **vertices**(nodes) connected by a group of **edges.**

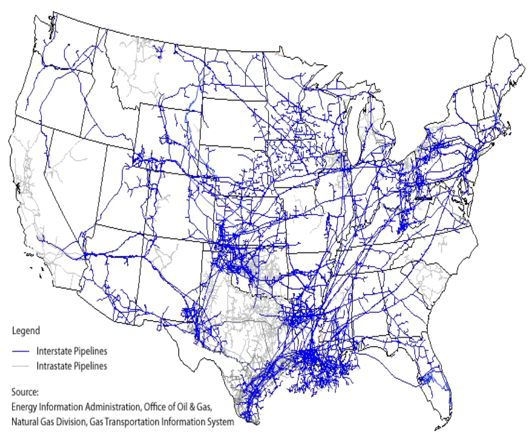
Why Graphs? Graphs can model many types of **relations** in many real-world systems.

Social Networks
(Nodes: People; Edges: Friendship)

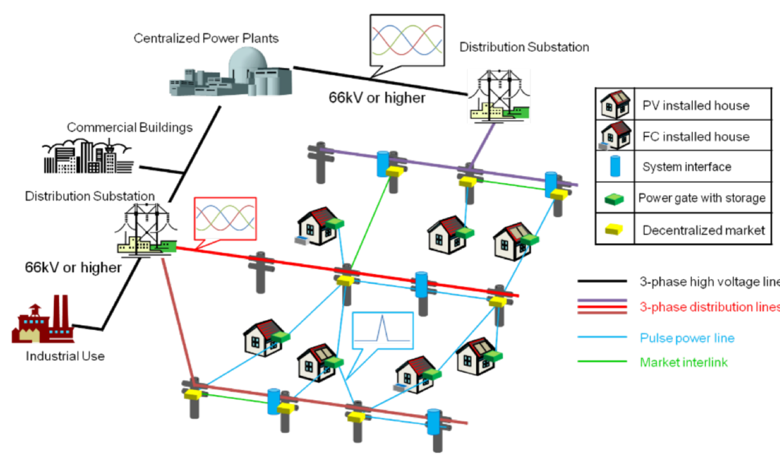
Biological Networks
(Nodes: genes; Edges: Regulation)



Transportation Networks
(Nodes: cities; Edges: transportation flow)



Power Networks
(Nodes: cites; Edges: power flow)



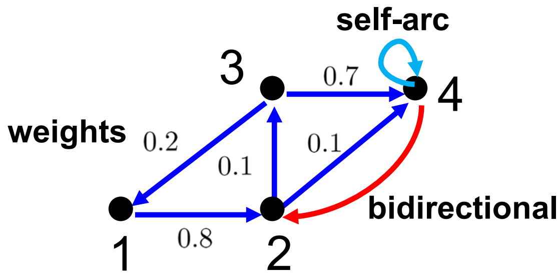
Graph and Its Elements

$\mathbb{G}$ $\mathcal{V}$ $\mathcal{E}$ **relations**

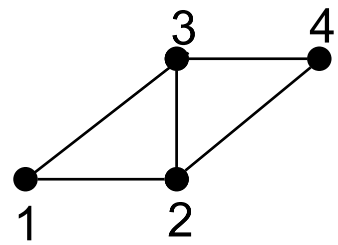
Graph: A set of **vertices**(nodes) connected by a group of **edges.**

$$\mathcal{V} = \{1, 2, 3, 4\}$$

$$\mathcal{E} = \{(1,2), (2,3), (3,1), (2,4), (3,4)\}$$



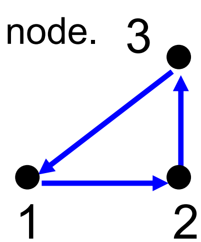
directed
(information flow; leader/follower,…)



undirected
(distance, bidirectional communications,…)

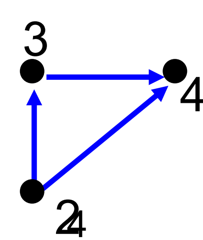
- **Path:** (1,2),(2,3),(3,4) of length 3.
- **Cycle:** (1,2),(2,3),(3,1) of length 3.
  (4,4) of length 1.

- **Distance between i and j:** the length of the shortest path connecting them. Distance between 1 and 4=2
- **Diameter:** the largest one among all distances.
- **In-neighbors/out-neighbors of vertex i:** For a directed edge from 1 to 2 1 is an in-neighbor of 2; 2 is an out-neighbor of 1.
- **In-degree/out-degree of vertex i:** the number of incoming/outgoing edges. (degree for undirected graphs. Maximum degree)
- A directed graph is **strongly connected** if there is a path from any node to any other node.



- A directed graph is **weakly connected** if its undirected version is connected.

  Connected for undirected graphs.



- A **globally reachable node** if this node can be reached from any other node by a path.

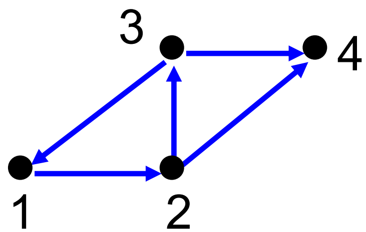
  Every node is globally reachable=strongly connected.

$\mathbb{A}_2$ $\mathbb{A}_1$

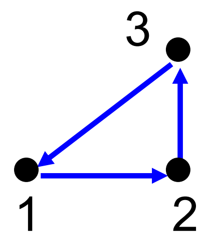
- A graph $\mathbb{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1\}$ is a **subgraph** of $\mathbb{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2\}$ if $\mathcal{V}_1 \subset \mathcal{V}_2, \mathcal{E}_1 \subset \mathcal{E}_2$

$\mathbb{A}_3$ $\mathbb{A}_1$

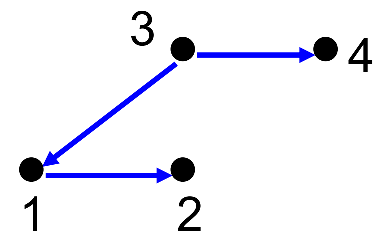
- A graph $\mathbb{G}_1 = \{\mathcal{V}_1, \mathcal{E}_1\}$ is a **spanning subgraph** of $\mathbb{G}_2 = \{\mathcal{V}_2, \mathcal{E}_2\}$ if $\mathcal{V}_1 = \mathcal{V}_2, \mathcal{E}_1 \subset \mathcal{E}_2$



$\mathbb{A}_1$



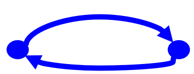
$\mathbb{A}_2$



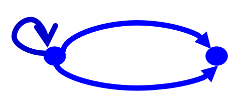
$\mathbb{A}_3$

- **Period** of graphs: For a strongly connected directed graph, let k denote the greatest common divisor of lengths of all its cycles.

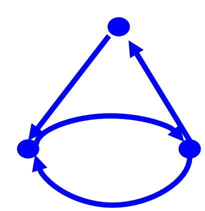
- A strongly connected directed graph is **periodic if k>1**; is **aperiodic,** otherwise.



periodic with period 2



aperiodic



aperiodic

- Any strongly connected directed graph with a self-loop is aperiodic.

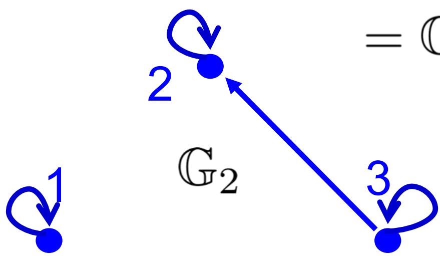
Graph Operations

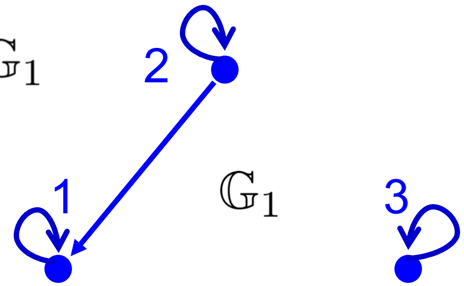
Given two graphs with the **same vertex set** $\mathbb{G}_1 = \{\mathcal{V}, \mathcal{E}_1\}$ $\mathbb{G}_2 = \{\mathcal{V}, \mathcal{E}_2\}$

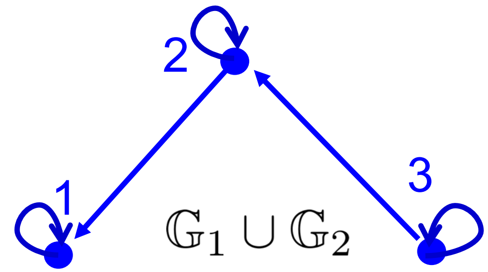
- Graph **Union:** $\mathbb{G}_1 \cup \mathbb{G}_2 = \{\mathcal{V}, \mathcal{E}\}$ $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$

$= \mathbb{G}_2 \cup \mathbb{G}_1$

Union: ***direct impacts***







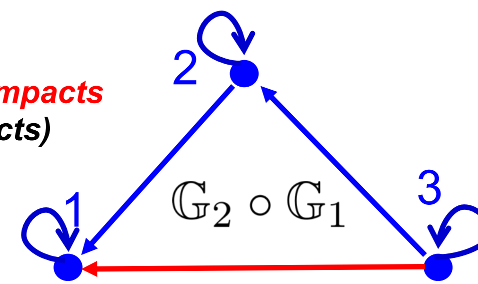
$\mathcal{E}_1 = \{(3,2),(1,1),(2,2),(3,3)\}$ $\mathcal{E}_2 = \{(2,1),(1,1),(2,2),(3,3)\}$ $\mathcal{E} = \{(3,2),(2,1),(1,1),(2,2),(3,3)\}$

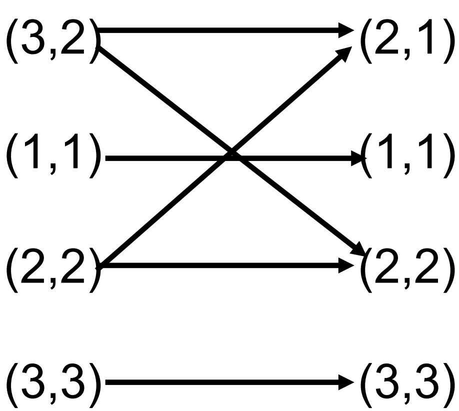
- Graph **Composition:** $\mathbb{G}_2 \circ \mathbb{G}_1 = \{\mathcal{V}, \mathcal{E}\}$

There exists *k* such that

$$(i,k) \in \mathcal{E}_2 \quad (k,j) \in \mathcal{E}_1 \Longleftrightarrow (i,j) \in \mathcal{E}$$

Composition: ***indirect impacts***
(transition of impacts)





(3,1)

(3,2)
(1,1)

(2,1)

(2,2)

(3,3)

- ❑ If all graphs are with **self-arcs** at every vertex, then

$$\mathbb{G}_1 \cup \mathbb{G}_2 \subset \mathbb{G}_2 \circ \mathbb{G}_1$$

Proof?

- ❑ **Composition for graphs with self-arcs include both direct and indirect impacts.**

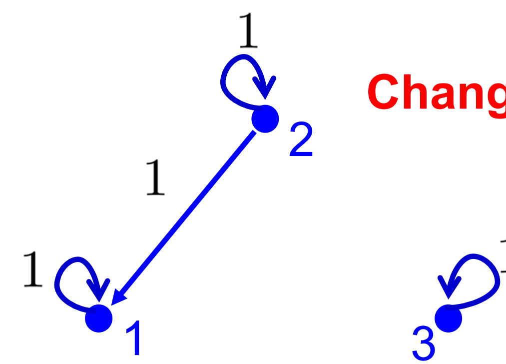
Graph Representation for Non-Negative Matrices

Given an non-negative matrix $A \in \mathbb{R}^{n \times n}$,

the ***graph of a matrix*** A *is* a directed graph of ***n*** nodes such that

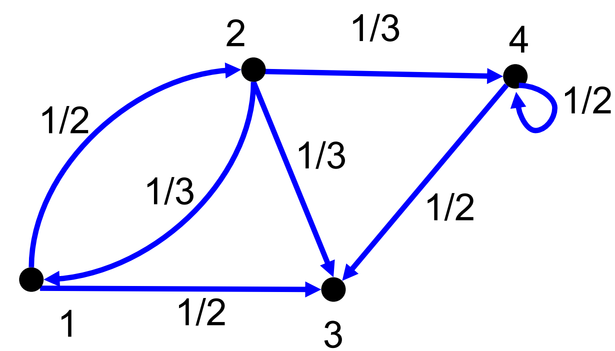
there exists a directed **edge** $i \to j$ with the weight A_{ij} if and only if $A_{ij} > 0$.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



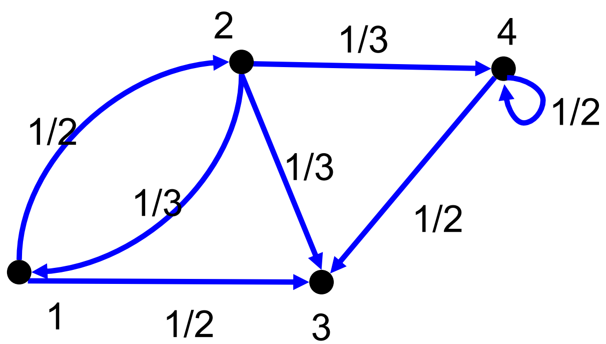
Change the definition, ji

$$A = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$



Correspondingly, the matrix *A* is called the *adjacency matrix of the graph*.

Adjacency Matrix of a Graph



Algebraic Graph Theory
applying algebraic methods in solving graph problems.

$$\mathbb{G} = \{\mathcal{V}, \mathcal{E}\} \qquad \mathcal{V} = \{1, 2, 3, 4\}$$

$$\mathcal{E} = \{(1,2), (1,3), (2,1), (2,3), (2,4), (4,3), (4,4)\}$$

➢ **Adjacency Matrix** $A = [a_{ij}]_{n\times n}$

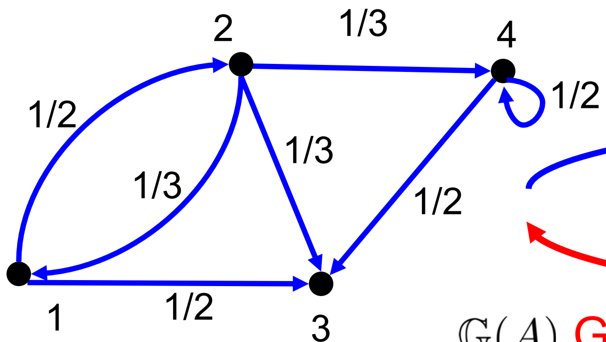
$$a_{ij} = \begin{cases} w_{ij}, & i \to j; \\ 0, & \text{otherwise} \end{cases} \qquad A = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Matrix of a graph.

Adjacency matrix of an unweighted graph $A = [a_{ij}]_{n\times n}$

$$a_{ij} = \begin{cases} 1, & i \to j; \\ 0, & \text{otherwise} \end{cases} \qquad A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Adjacency matrix of an undirected/unweighted graph? Symmetric



an *n*-node **graph** $\mathbb{G}$

Adjacency Matrix $A_{\mathbb{G}}$

$$i \xrightarrow{a_{ij}} j \qquad [A]_{ij} = a_{ij}$$

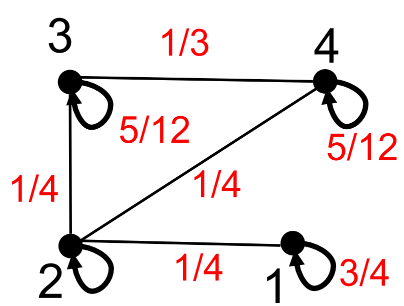
$\mathbb{G}(A)$ Graph of a non-negative matrix.

$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

an n×n **matrix A**

Example: Distributed Averaging using Metropolis Weights

$$w_{ij} = w_{ji} \qquad x(t+1) = Nx(t)$$

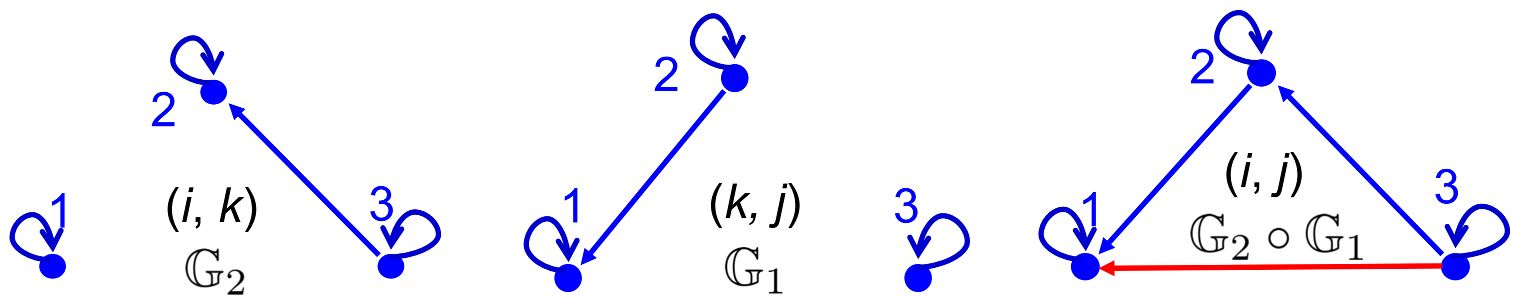


- Write out the weights for each edge
- Write out the adjacency matrix

$$N = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

Graph composition $\mathbb{G}_2 \circ \mathbb{G}_1$

of graphs of the same vertex set, with **self-arcs**.



$$M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad M_2 M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\boxed{\mathbb{G}(M_2) \circ \mathbb{G}(M_1) = \mathbb{G}(M_2 M_1)}$$

Graph Composition **Matrix Product**

Seminar 2 for Extra Credit

- **Title:** “Visual-Inertial State Estimation and Perception for Autonomous Vehicles”
- **Time:** 9:30am on Thursday (Feb. 20), 2020
- **Location:** ARMS B071
- **Speaker:** Guoquan Huang (Assistant Professor, University of Delaware)

❖ *Correspondingly, the class on Wednesday (Feb. 19) will be canceled and replaced by this seminar video.*

❖ *Please start to work on the problem formulation of your course projects. Feel free to email me or stop by my office during office hours for discussion.*