College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 8
Stability of LTI Systems

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Exercise 1

Determine (by hand) whether each of the following systems is asymptotically stable, stable, or unstable.

(a)

$$\begin{aligned} \dot{x_1} &= -x_1 + 2001x_2 \\ \dot{x_2} &= -x_1 \end{aligned}$$

(b)

$$\begin{aligned} \dot{x_1} &= -x_1 \\ \dot{x_2} &= x_2 \end{aligned}$$

(c)

$$\begin{aligned} \dot{x_1} &= jx_1 + x_2 \\ \dot{x_2} &= jx_2 \end{aligned}$$

(d)

$$\begin{aligned} \mathrm{x}_1(\mathrm{k}+1) &= -2x_1(k) \\ x_2(k+1) &= 0.5x_2(k) \end{aligned}$$

(a)

$$A = \begin{pmatrix} -1 & 2001 \\ -1 & 0 \end{pmatrix}$$

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} -1-\lambda & 2001 \\ -1 & -\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (-1-\lambda)(-\lambda) + 2001 = \lambda^2 + \lambda + 2001$$

$$\lambda = -0.5 \pm j44.7297 \ .$$

The eigenvectors become,

$$v_1 = \begin{pmatrix} 0.5 - j44.7297 \\ 1 \end{pmatrix}, \qquad \lambda_1 = -0.5 + j44.7297$$

$$v_2 = \begin{pmatrix} 0.5 + j44.7297 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -0.5 - j44.7297$$

Thus, the A matrix is nondefective and has a complex eigenvalue with a negative real part. Hence this system is asymptotically stable and bounded.

(b)

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} -1 - \lambda & 0 \\ 0 & 1 - \lambda \end{pmatrix}$$

$$det(A - \lambda I) = (-1 - \lambda)(1 - \lambda)$$

$$\lambda = 1, -1.$$

The geometric and algebraic multiplicity is equal, so this matrix is nondefective. The eigenvalues of this A matrix consists of one negative real value and another value that is a positive real value. Thus, this system is unstable and unbounded.

(c)

$$A = \begin{pmatrix} j & 1 \\ 0 & j \end{pmatrix}$$

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} j - \lambda & 1 \\ 0 & j - \lambda \end{pmatrix}$$

$$det(A - \lambda I) = (j - \lambda)^2$$

$$\lambda = j.$$

This is a defective matrix A, and has a repeating eigenvalue on the imaginary axis of the complex plane. Thus, this system is unstable and unbounded.

(d)

$$A = \begin{pmatrix} -2 & 0 \\ 0 & 0.5 \end{pmatrix}$$

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} -2 - \lambda & 0 \\ 0 & 0.5 - \lambda \end{pmatrix}$$

$$det(A - \lambda I) = (-2 - \lambda)(0.5 - \lambda)$$

$$\lambda = 0.5, -2.$$

Since this system is nondefective and has a negative eigenvalue that is smaller than -1, the system will grow with increasing age and is unstable and unbounded.

Exercise 2

Determine (by hand) the stability properties of a linear continuous-time system with

$$A = \begin{pmatrix} \frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 1 \\ 0 & \frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix}$$

Using the eig() command on MATLAB, what would you stability conclusion be?

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} \frac{1}{2} - \lambda & 1 & -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} - \lambda & 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} - \lambda & 1 \\ 0 & \frac{1}{2} & -1 & -\frac{1}{2} - \lambda \end{pmatrix}$$

$$\therefore det(A - \lambda I) = \lambda^4 + 2\lambda^2 + 1$$

$$\lambda = \pm j\,.$$

This matrix A is defective because the geometric multiplicity and the algebraic multiplicity are not equal. Thus, it will have solutions such as

$$e^{jt}, \qquad e^{-jt}, \qquad te^{jt}, \qquad te^{-jt}$$

Since there is a repeated eigenvalue on the imaginary axis of the complex plane, we know that this A matrix is unstable and unbounded.

The eig() command on MATLAB, gives the following result

$$e^{At} = \begin{pmatrix} (0.5t+1)cos(t) & (0.5t+1)sin(t) & -0.5tcos(t) & -0.5tsin(t) \\ -(0.5t+1)sin(t) & (0.5t+1)cos(t) & 0.5tsin(t) & -0.5tcos(t) \\ 0.5tcos(t) & 0.5tsin(t) & -0.5(t-2)cos(t) & -0.5(t-2)sin(t) \\ -0.5tsin(t) & 0.5tcos(t) & 0.5t-1 & -0.5(t-2)cos(t) \end{pmatrix}$$

As you can see all of them keep growing with respect to time. So, it is unstable and unbounded.

Exercise 3

Using linearization determine (if possible) the stability properties of each of the following systems about their corresponding specified equilibrium solution q^e. If not possible, provide a reason.

(a)

$$\ddot{q} + (\dot{q} - 1)|\dot{q} - 1| + 2sinq = 0$$

and $q^e = \pi/6$

(b)

$$\begin{aligned} \dot{q}_1 &= e^{q_1}q_2 - q_1^3 \\ \dot{q}_2 &= -q_1 cosq_2 \end{aligned}$$

and $q^e = [0,\ 0]^T$.

(c)

$$\begin{aligned} \ddot{q}_1 &= q_2 \\ \ddot{q}_2 &= sinq_1 \end{aligned}$$

and $q^e = [0,\ 0]^T$.

(a)

Linearize the system

$$\ddot{q} + (\dot{q}-1)|\dot{q}-1| + 2\sin q = 0$$

$$\rightarrow \begin{cases} \ddot{q} + (\dot{q}-1)^2 + 2\sin q = 0 & \text{if } \dot{q} > 1 \\ \ddot{q} - (\dot{q}-1)^2 + 2\sin q = 0 & \text{if } \dot{q} < 1 \end{cases}$$

$$q = q^e + \delta q \rightarrow \dot{q} = \delta\dot{q} \rightarrow \ddot{q} = \delta\ddot{q}$$

when $\dot{q} > 1$,

$$\delta\ddot{q} + (\delta\dot{q} - 1)^2 + 2\sin(q^e + \delta q) = 0$$

$$\because \sin(q^e+\delta q) = \sin q^e \cos\delta q + \sin\delta q \cos q^e$$
$$\approx \sin q^e + \delta q \cos q^e$$

$$\delta\ddot{q} + \cancelto{0}{\delta\dot{q}^2} - 2\delta\dot{q} + 1 + 2\sin q^e + 2\delta q\cos q^e = 0$$

$$\delta\ddot{q} - 2\delta\dot{q} + 2\delta q\cos q^e + 2\sin q^e + 1 = 0$$

When $\dot{q} < 1$,

$$\delta\ddot{q} - (\delta\dot{q} - 1)^2 + 2\sin(q^e+\delta q) = 0$$
$$\because \sin\delta q \approx \sin q^e + \delta q\cos q^e$$
$$\delta\ddot{q} - \cancelto{0}{\delta\dot{q}^2} + 2\delta\dot{q} - 1 + 2\sin q^e + 2\delta q\cos q^e = 0$$
$$\delta\ddot{q} + 2\delta\dot{q} + 2\delta q\cos q^e + 2\sin q^e - 1 = 0$$

Plug in $q^e = \frac{\pi}{6}$

When $\dot{q} > 1$

$$\delta\ddot{q} - 2\delta\dot{q} + \sqrt{3}\,\delta q + 3 = 0$$

When $\dot{q} < 1$

$$\delta\ddot{q} + 2\delta\dot{q} + \sqrt{3}\,\delta q = 0$$

When $\dot{q} > 1$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\sqrt{3} & 2 \end{pmatrix} x + \begin{pmatrix} 0 \\ -3 \end{pmatrix} u$$

The eigenvalues of the A matrix is

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 1 \\ -\sqrt{3} & 2-\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (2 - \lambda)(-\lambda) + \sqrt{3} = \lambda^2 - 2\lambda + \sqrt{3}$$

$$\lambda = 1 \pm j0.8556.$$

Since the real part is a positive this is unstable and unbounded.

When $\dot{q} < 1$

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\sqrt{3} & -2 \end{pmatrix} x$$

The eigenvalues of the A matrix is

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 1 \\ -\sqrt{3} & -2 - \lambda \end{pmatrix}$$

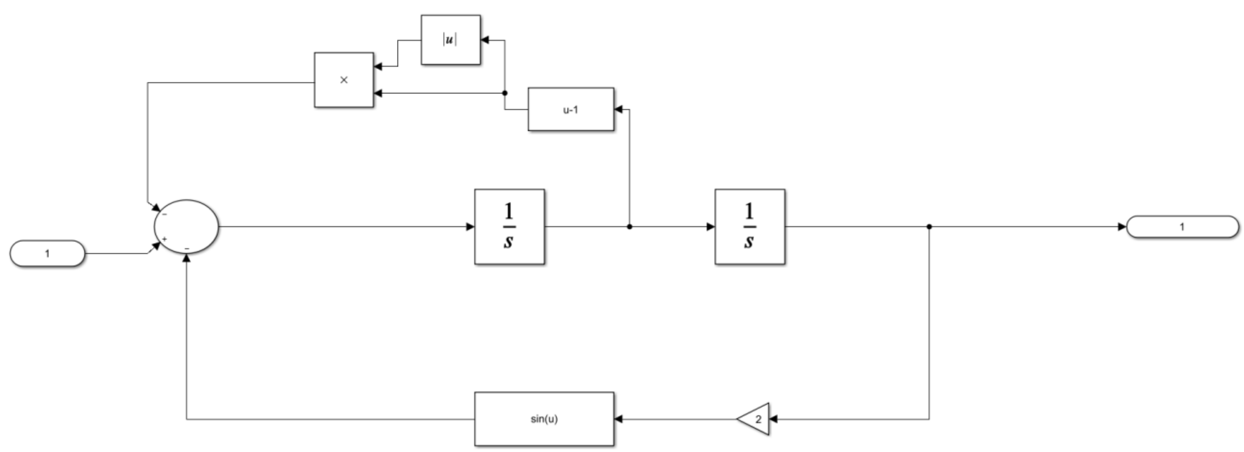
$$det(A - \lambda I) = (-2 - \lambda)(-\lambda) + \sqrt{3} = \lambda^2 + 2\lambda + \sqrt{3}$$

$$\lambda = -1 \pm j0.8556.$$

Since the real part is a negative this is GAS and bounded.

Thus, if $\dot{q} > 1$, the nonlinear system is unstable. Whereas if $\dot{q} < 1$, the nonlinear system is stable.

Verify this with Simulink



Then,

```
% simulink
qe = pi/6; qde = 0;
```

```
xe = trim("ex3_a")
[A, B, C, D] = linmod("ex3_a",xe)
```

A = 2×2 0 1.0000 -1.5238 -2.0000	B = 2×1 0 1
C = 1×2 1 0	D = 0

This agrees with our results done by hand.

(b)

Linearize the system

$$\delta \dot{q}_1 = \left(q_1^e e^{q_1^e} q_2^e - 3q_1^{e^2}\right)\delta q_1 + e^{q_1^e}\delta q_2$$

$$\delta \dot{q}_2 = \left(-\cos q_2^e\right)\delta q_1 + \left(q_1^e \sin q_2^e\right)\delta q_2$$

$$q_1^e = q_2^e = 0$$

$$\Longrightarrow \quad \delta \dot{q}_1 = \delta q_2$$

$$\delta \dot{q}_2 = -\delta q_1$$

Then the matrix A becomes

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

The eigenvalues of the A matrix is

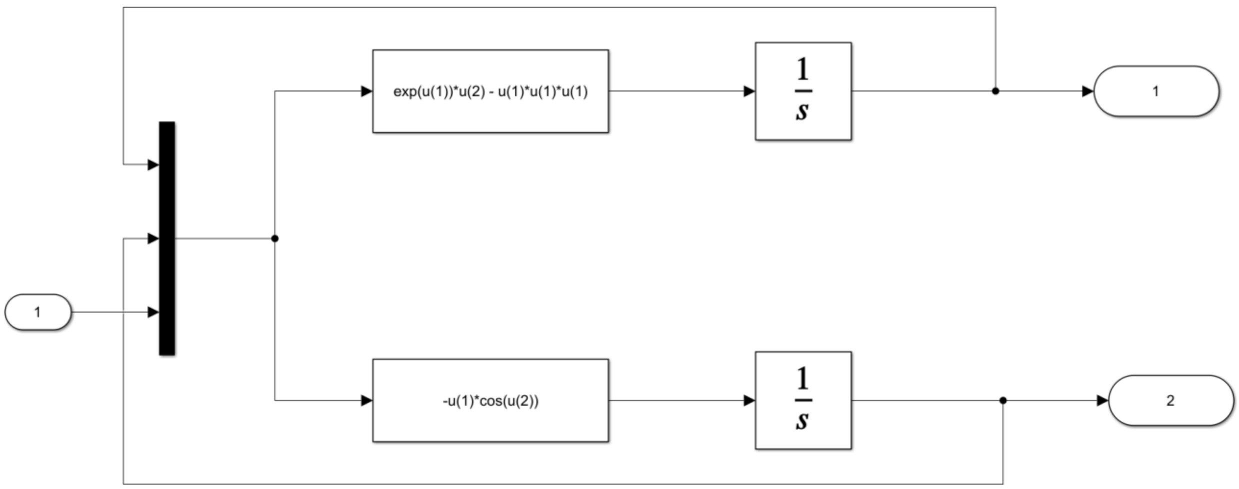
$$(A - \lambda I) = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$$

$$det(A - \lambda I) = \lambda^2 + 1$$

$$\lambda = \pm j.$$

Since the linearized system has an eigenvalue on the imaginary axis, it is unstable and unbounded. Thus, the nonlinear system is undetermined for an eigenvalue on the imaginary axis.

Verify this with Simulink




```
% simulink
xe = trim("ex3_b")
[A, B, C, D] = linmod("ex3_b",xe)
```

A = 2×2 -0.0000 1.0000 -1.0000 0	B = 2×1 0 0
C = 2×2 1 0 0 1	D = 0

This agrees with our results done by hand.

(c)

Linearize the system

$$\delta \ddot{q}_1 = q_2^e + \delta q_2$$
$$\delta \ddot{q}_2 = \cos q_1^e \, \delta q_1$$

$$q_1^e = q_2^e = 0$$

$$\delta \ddot{q}_1 = \delta q_2$$
$$\delta \ddot{q}_2 = \delta q_1$$

The A matrix becomes

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues of the A matrix is

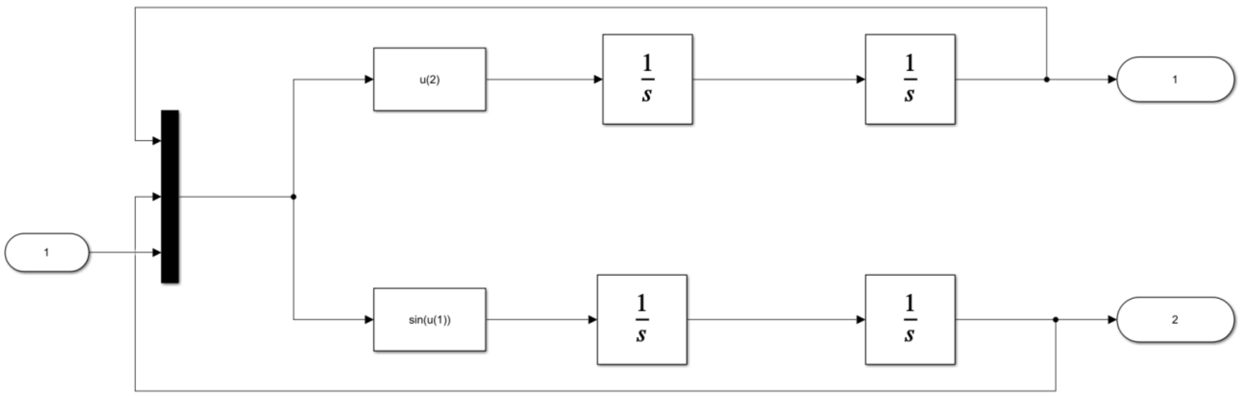
$$(A - \lambda I) = \begin{pmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{pmatrix}$$

$$det(A - \lambda I) = \lambda^4 - 1$$

$$\lambda = \pm 1, \pm \mathrm{j}.$$

Since the eigenvalue includes a positive real value the linearized system will grow exponentially and is unstable and unbounded. Thus, the nonlinear system is also unstable.

Verify with Simulink




```
% simulink
xe = trim("ex3_c")
[A, B, C, D] = linmod("ex3_c",xe)
```

A = 4×4 0 0 1.0000 0 0 0 0 1.0000 0 1.0000 0 0 1.0000 0 0 0	B = 4×1 0 0 0 0
C = 2×4 1 0 0 0 0 1 0 0	D = 2×1 0 0

This agrees with our results done by hand.

Exercise 4

Determine the stability properties of the following system about the zero solution.

$$x_1(k+1) = x_1(k)^3 + sin\big(x_2(k)\big)$$

$$x_2(k+1) = -\frac{1}{2}cos\big(x_2(k)\big)x_1(k) + x_2(k)^3$$

Linearize the discrete time system

$$\delta x_1(k+1) = 3x_1^e(k)^2\,\delta x_1(k) + cos(x_2^e(k))\,\delta x_2(k)$$

$$\delta x_2(k+1) = -\frac{1}{2}cos(x_2^e(k))\,\delta x_1(k) + \left[\frac{1}{2}sin(x_2^e(k))x_1(k) + 3x_2^e(k)\right]\delta x_2(k)$$

$$\because\ x_1^e(k) = x_2^e(k) \doteq 0$$

$$\Longrightarrow \begin{aligned} \delta x_1(k+1) &= \delta x_2(k) \\ \delta x_2(k+1) &= -\frac{1}{2}\,\delta x_1(k) \end{aligned}$$

The nondefective A matrix becomes

$$A = \begin{pmatrix} 0 & 1 \\ -0.5 & 0 \end{pmatrix}$$

The eigenvalues of the A matrix is

$$(A - \lambda I) = \begin{pmatrix} -\lambda & 1 \\ -0.5 & -\lambda \end{pmatrix}$$

$$det(A - \lambda I) = \lambda^2 + 0.5$$

$$\lambda = \pm j0.7071\ .$$

Since the eigenvalues are on the imaginary axis the linearized system is unstable but the nonlinear system is undetermined.

Exercise 5

Stability properties of the two pendulum cart system. Using MATLAB, determine the stability properties of the linearizations L1 and L2. What can you say about the stability properties of the nonlinear system about the corresponding equilibrium states?

Given parameters and initial and equilibrium conditions

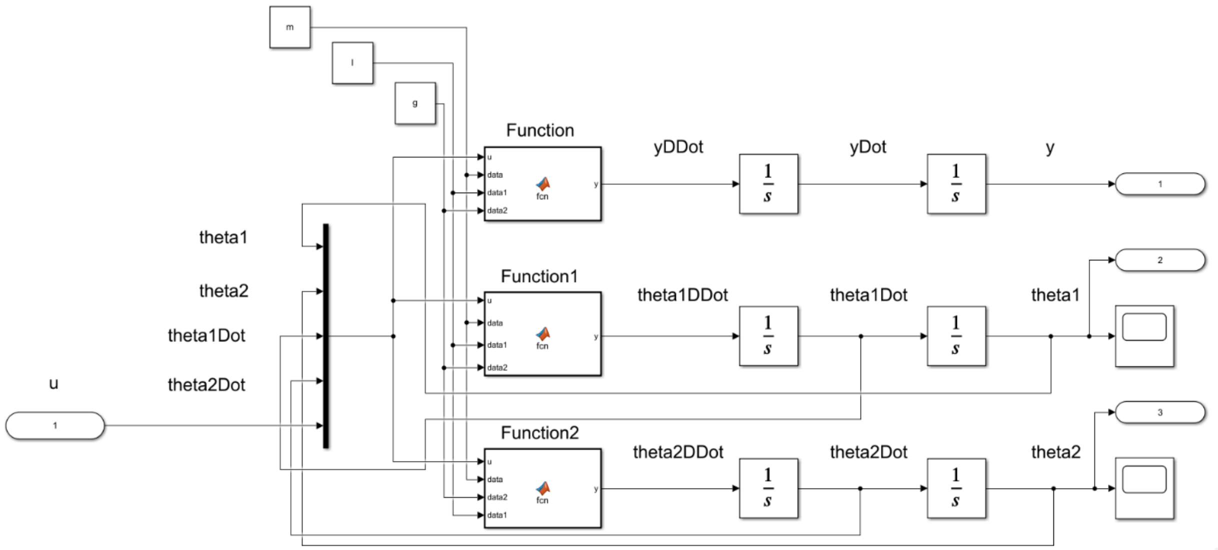
$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

$$E2: \quad (y^e, \theta_1^e, \theta_2^e) = (0,\pi,\pi)$$

	m_0	m_1	m_2	l_1	l_2	g	u
$P1$	2	1	1	1	1	1	0
$P2$	2	1	1	1	0.99	1	0
$P3$	2	1	0.5	1	1	1	0
$P4$	2	1	1	1	0.5	1	0

L1	P1	E1
L2	P1	E2
L3	P2	E1
L4	P2	E2
L5	P3	E1
L6	P3	E2
L7	P4	E1
L8	P4	E2

The Simulink model used for this is shown below,



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

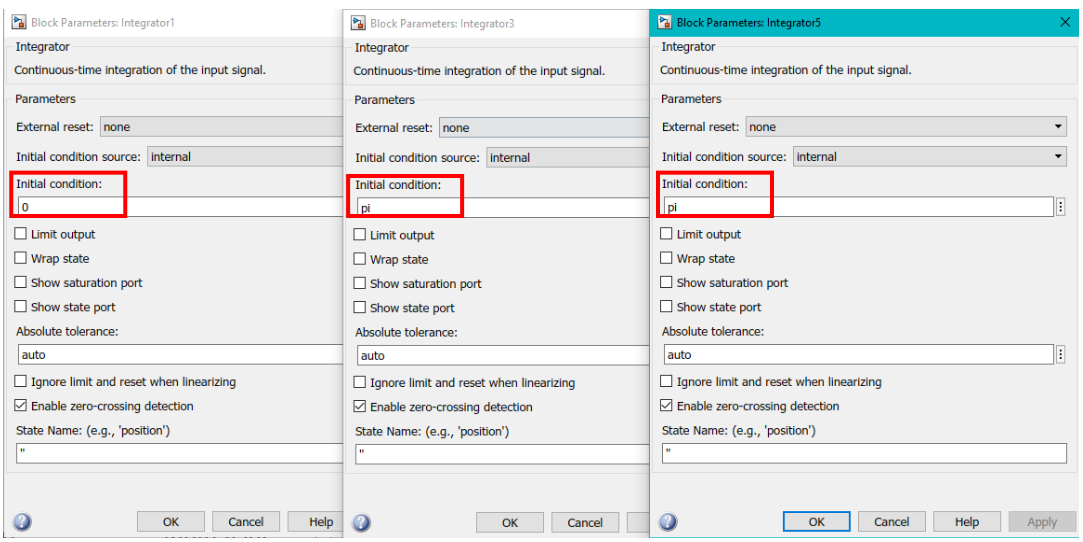
Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

```matlab
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



The code to run the linearization and eigenvalue computation is the following

```
% (a)
global m l g ye t1e t2e
param_combo = ["L1","L2"];
for i = 1:numel(param_combo)
    define_params(param_combo(i));
    [A, B, C, D] = linmod('db_pend_cart_lin');
    lin_sys(i).Amat = A;
    lin_sys(i).Bmat = B;
    lin_sys(i).Cmat = C;
    lin_sys(i).Dmat = D;
    sys_ss = ss(A, B, C, D);  % get the state space system
    sys_tf = tf(sys_ss);  % get the transfer function
    lin_sys(i).eigVal = pole(sys_tf);  % get the eigenvalues
end
```

```
function define_params(L)
    % Function to define parameters
    global m l g ye t1e t2e
    if L == "L1"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L2"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; t1e = pi; t2e = pi; % E2
    elseif L == "L3"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L4"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; t1e = pi; t2e = pi; % E2
    elseif L == "L5"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L6"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; t1e = pi; t2e = pi; % E2
    elseif L == "L7"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; t1e = 0; t2e = 0; % E1
    elseif L == "L8"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; t1e = pi; t2e = pi; % E2
    else
        print('error: did not match any')
    end
end
```

The eigenvalues for the configurations L1 and L2 are

L1:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -0.5000   -1.5000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

L2:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0    1.5000    0.5000         0         0         0
         0    0.5000    1.5000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
   -0.5000
   -0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

```
eigVal = 6×1
         0
         0
   -1.4142
   -1.0000
    1.4142
    1.0000
```

From the linearized models we can see that L1 has eigenvalues on the imaginary axis. Thus, the linearized model is unstable and unbounded. However, for the nonlinear model the stability is undetermined.

Whereas for L2 there are positive real values that blow up the linearized system. This means that the linearized system is unstable and unbounded, and the nonlinear model is also unstable and unbounded.

	Linear	*Nonlinear*
L1	unstable	undetermined
L2	unstable	unstable