AAE364: Control Systems Analysis

HW4: Block Diagrams & Transfer Functions

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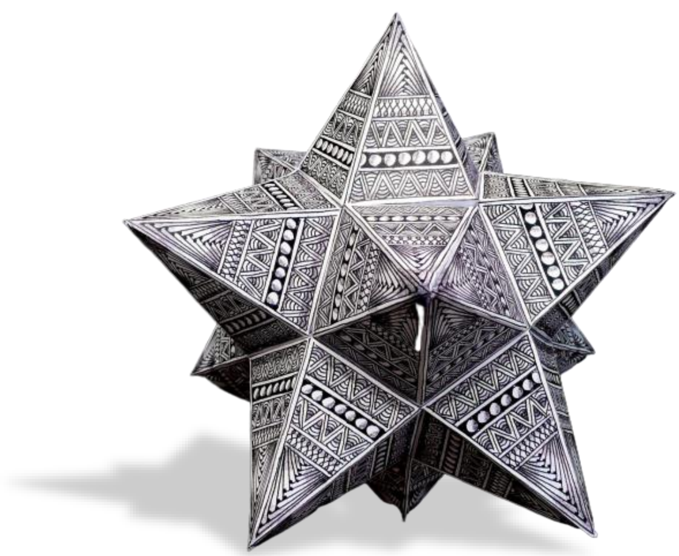
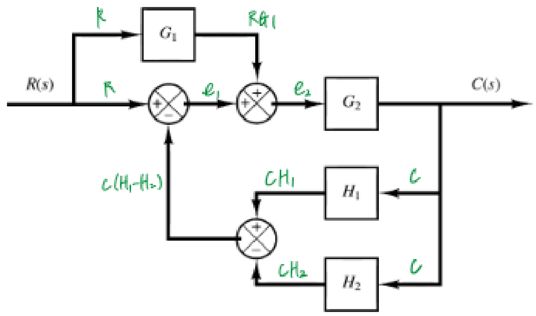


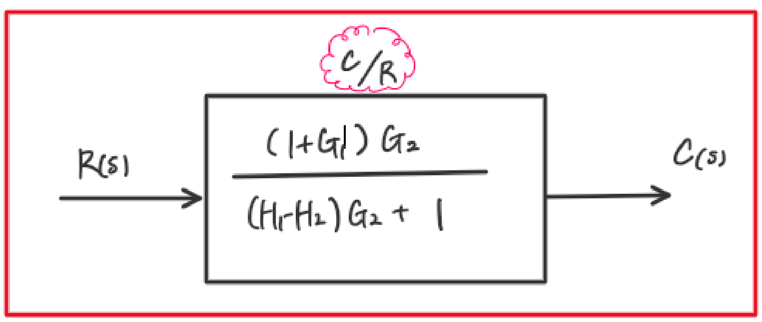
B-2-2) Simplify and find TF $C(s)/R(s)$



$$\begin{cases} e_2 G_2 = C & \cdots ① \\ e_1 + RG_1 = e_2 & \cdots ② \\ R - C(H_1 - H_2) = e_1 & \cdots ③ \end{cases} \Rightarrow \begin{array}{ll} e_2 = \dfrac{C}{G_2} & \cdots ③ \\ e_1 = \dfrac{C}{G_2} - RG_1 & \cdots ③ \rightarrow ② = ④ \\ R - C(H_1 - H_2) = \dfrac{C}{G_2} - RG_1 & \cdots ④ \rightarrow ③ \end{array}$$

$$\frac{C(H_1 - H_2)G_2 + C}{G_2} = R(1 + G_1)$$

$$\frac{C\left[(H_1 - H_2)G_2 + 1\right]}{G_2} = R(1 + G_1)$$



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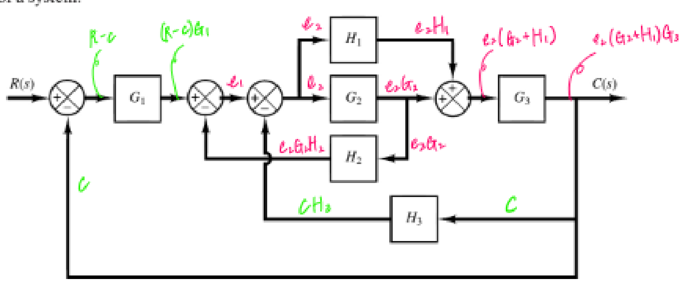
B-2-3) Simplify block diagram and find TF C/R

Block diagram of a system.



from the diagram and notes above

$$
\begin{cases}
e_1 = (R-C)G_1 - e_2 G_2 H_2 & ① \\
C = e_2 (G_2 + H_1) G_3 & ② \\
e_2 = e_1 - CH_3 & ③
\end{cases}
\rightarrow \quad e_2 = \frac{C}{(G_2+H_1)G_3} \quad ④
$$

① & ④ → ③

$$\frac{C}{(G_2+H_1)G_3} = (R-C)G_1 - \frac{C}{(G_2+H_1)G_3} G_2 H_2 - CH_3$$

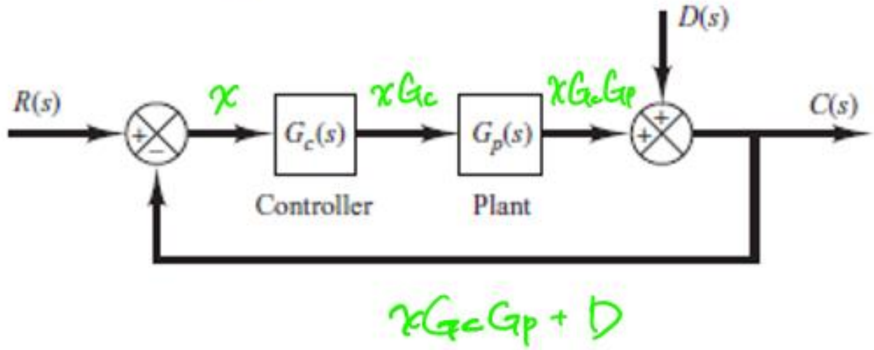
$$\frac{C}{(G_2+H_1)G_3} = RG_1 - CG_1 - \frac{G_2H_2}{(G_2+H_1)G_3} C - CH_3$$

$$\frac{C}{(G_2+H_1)G_3} + \frac{CG_1(G_2+H_1)G_3}{(G_2+H_1)G_3} + \frac{CG_2H_2}{(G_2+H_1)G_3} + \frac{(G_2+H_1)G_3H_3}{(G_2+H_1)G_3} C = RG_1$$

$$C\,\frac{1 + G_1(G_2+H_1)G_3 + G_2H_2 + (G_2+H_1)G_3H_3}{(G_2+H_1)G_3G_1} = R$$

$$\boxed{\frac{C}{R} = \frac{G_1G_3(G_2+H_1)}{1 + G_2H_2 + (G_1G_2G_3 + G_1G_3H_1 + G_2G_3H_3 + G_3H_1H_3)}}$$

B–2–5. Figure 2–32 shows a closed-loop system with a reference input and disturbance input. Obtain the expression for the output $C(s)$ when both the reference input and disturbance input are present.



$$x = R - xG_cG_p - D$$

$$C = xG_cG_p + D$$

$$x = \frac{C-D}{G_cG_p}$$

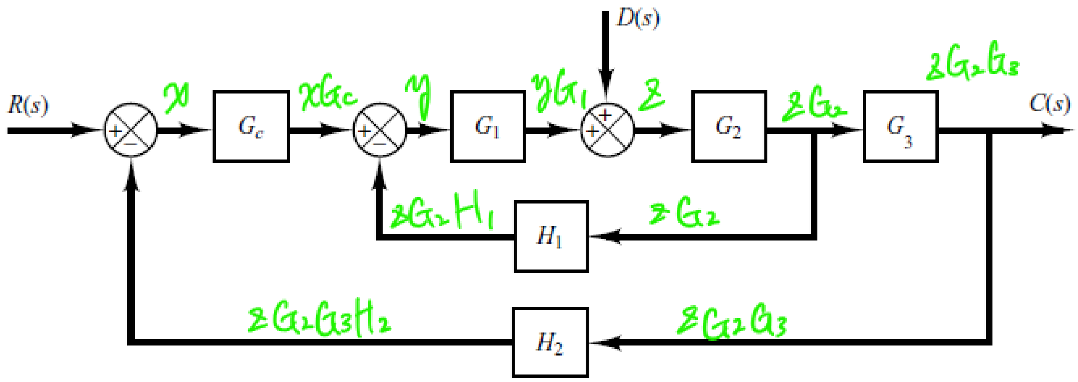
$$\Rightarrow \quad \frac{C-D}{G_cG_p} = R - D - (C-D)$$

$$C - D = (R - C)G_cG_p$$

$$C(1 + G_cG_p) = RG_cG_p + D$$

$$\boxed{C = \frac{RG_cG_p + D}{1 + G_cG_p}}$$

B–2–7. Obtain the transfer functions $C(s)/R(s)$ and $C(s)/D(s)$ of the system shown in Figure 2–34.



$$\begin{cases} C = zG_2G_3 \\ z = yG_1 + D \\ y = xG_c - zG_2H_1 \\ x = R - zG_2G_3H_2 \end{cases}$$

$$y = (R - zG_2G_3H_2)G_c - zG_2H_1$$

$$y = RG_c - zG_cG_2G_3H_2 - zG_2H_1$$

$$y = RG_c - zG_2(G_cG_3H_2 + H_1)$$

$$z = \left[R G_c - z G_2 (G_c G_3 H_2 + H_1) \right] G_1 + D$$

$$z = R G_c G_1 - z G_1 G_2 (G_c G_3 H_2 + H_1) + D$$

$$\left[1 + G_1 G_2 (G_c G_3 H_2 + H_1) \right] z = R G_c G_1 + D$$

$$z = \frac{R G_c G_1 + D}{1 + G_1 G_2 (G_c G_3 H_2 + H_1)}$$

$$\therefore C = \frac{R G_c G_1 G_2 G_3 + D G_2 G_3}{1 + G_1 G_2 (G_c G_3 H_2 + H_1)}$$

to find

$$\frac{C}{R} \qquad D \rightarrow 0$$

$$\therefore \quad \boxed{\frac{C}{R} = \frac{G_c G_1 G_2 G_3}{1 + G_1 G_2 (G_c G_3 H_2 + H_1)}}$$

to find

$$\frac{C}{D} \quad R \longrightarrow 0$$

$$\boxed{\frac{C}{D} = \frac{G_2 G_3}{1 + G_1 G_2 (G_c G_3 H_2 + H_1)}}$$

B–3–6. Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$ of the mechanical system shown in Figure 3–35.

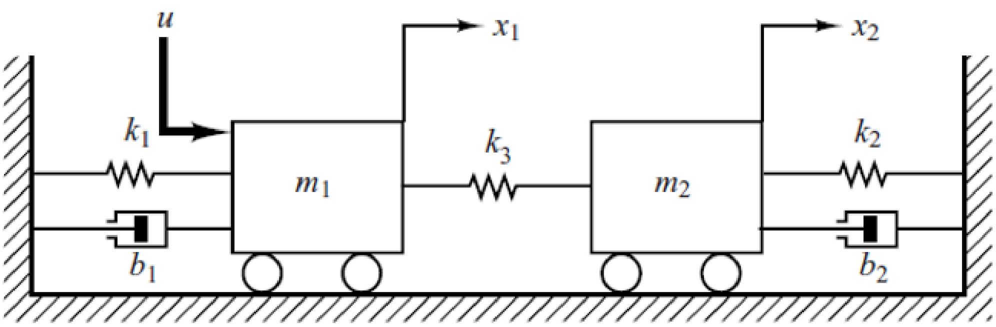


Figure 3–35 Mechanical system.

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + k_1x_1 + k_3(x_1 - x_2) = u$$

$$m_2\ddot{x}_2 + b_2\dot{x}_2 + k_2x_2 + k_3(x_2 - x_1) = 0$$

initial conditions are zero

take laplace transform

$$[m_1s^2 + b_1s + (k_1+k_3)]X_1(s) - k_3X_2(s) = U(s) \quad \cdots ①$$

$$[m_2s^2 + b_2s + (k_2+k_3)]X_2(s) - k_3X_1(s) = 0 \quad \cdots ②$$

$$\Rightarrow X_1 = \frac{m_2 s^2 + b_2 s + (k_2+k_3)}{k_3} X_2 \quad \cdots ③$$

$$\Rightarrow X_2 = \frac{k_3}{m_2 s^2 + b_2 s + (k_2+k_3)} X_1 \quad \cdots ④$$

④ → ①

$$[m_1 s^2 + b_1 s + (k_1+k_3)] X_1 - \frac{k_3^2}{m_2 s^2 + b_2 s + (k_2+k_3)} X_1 = U$$

$$\therefore \frac{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}{m_2 s^2 + b_2 s + k_2 + k_3} X_1 = U$$

$$\therefore \boxed{\frac{X_1}{U} = \frac{m_2 s^2 + b_2 s + k_2 + k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}}$$

thus,

$$\boxed{\frac{X_2}{U} = \frac{k_3}{(m_1 s^2 + b_1 s + k_1 + k_3)(m_2 s^2 + b_2 s + k_2 + k_3) - k_3^2}}$$

Problem 2: Aircraft Control

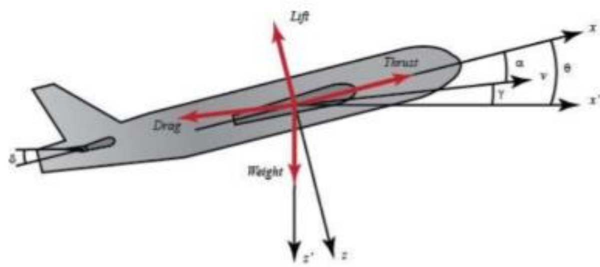


Figure 1: Forces acting on an aircraft in the Longitudinal plane.

Figure 1 shows the coordinate axes and forces acting on the aircraft in the longitudinal plane of motion. Assuming that the aircraft is cruising at constant velocity and altitude along with other simplifying assumptions, the equations of motion describing the longitudinal motion of the aircraft are given below.

$$\dot{\alpha} = \mu\Omega\sigma[-(C_L + C_D)\alpha + \frac{1}{(\mu - C_L)}q - (C_W sin\gamma)\theta + C_L] \quad (1)$$

$$\dot{q} = \frac{\mu\Omega}{2i_{yy}}[[C_M - \eta(C_L + C_D)]\alpha + [C_M + \sigma C_M(1 - \mu C_L)]q + (\eta C_W sin\gamma)\delta] \quad (2)$$

$$\dot{\theta} = \Omega q \quad (3)$$

Nomenclature

α	Angle of attack	C_D	Coefficient of drag
q	Pitch rate	C_L	Coefficient of lift
θ	Pitch angle	C_W	Coefficient of weight
δ	Elevator deflection angle	C_M	Coefficient of pitching moment
ρ	Density of air	γ	Flight path angle
S	Wing planform area	i_{yy}	Normalized moment of inertia
c	Average chord length	μ	$\frac{\rho Sc}{4m}$ (constant)
m	Aircraft mass	η	$\mu\sigma C_M$ (constant)
U	Equilibrium flight speed	Ω	$= \frac{2U}{c}$
C_T	Coefficient of thrust		

Once we have the above equations of motion, we can substitute values for the various aircraft characteristics and aerodynamic coefficients for a particular aircraft to obtain the longitudinal equations of motion for that aircraft. Taking these values for a representative commercial aircraft, we obtain the simplified form of the equations of motion shown below:

$$\dot{\alpha} = -0.312\alpha + 53.4q + 0.232\delta \quad (4)$$

$$\dot{q} = -0.0125\alpha - 0.426q + 0.0207\delta \quad (5)$$

$$\dot{\theta} = 53.4q \quad (6)$$

Using the equations of motion in (4)-(6), derive the transfer function describing the aircraft pitch angle response output θ to the elevator deflection input δ.

Take the Laplace transforms (4) ~ (6)
w/ all initial conditions as $\emptyset$

$$(4) \rightarrow \quad sA = -0.312A + 53.4q + 0.232\Delta \quad \cdots (i)$$

$$(5) \rightarrow \quad sq = -0.0125A - 0.426q + 0.0207\Delta \quad \cdots (ii)$$

$$(6) \rightarrow \quad sH = 53.4q \quad \cdots (iii)$$

from (i)

$$A = \frac{53.4q + 0.232\Delta}{s + 0.312} \quad \cdots (iv)$$

$$A = \frac{-(s + 0.426)q + 0.0207\Delta}{0.0125} \quad \cdots (v)$$

then (iv) = (v)

$$0.0125(53.4q + 0.232\Delta) = (s + 0.312)\left[-(s + 0.426)q + 0.0207\Delta\right]$$

$$0.6675\theta + 2.9\times10^{-3}\Delta = -(s^2 + 0.738s + 0.132912)\theta + (0.0207s + 1.4584\times10^{-3})\Delta$$

$$\theta(s^2 + 0.738s + 0.800412) = (0.0207s + 3.5584\times10^{-3})\Delta$$

$$\theta = \frac{(0.0207s + 3.5584\times10^{-3})\Delta}{s^2 + 0.738s + 0.800412}$$

$$s\,H = \frac{53.4(0.0207s + 3.5584\times10^{-3})\Delta}{s^2 + 0.738s + 0.800412}$$

$$\therefore \boxed{\frac{H}{\Delta} = \frac{1.10538s + 0.1900156}{s(s^2 + 0.738s + 0.800412)}}$$