College of Engineering

School of Aeronautical and Astronautical Engineering

AAE 567: Introduction To Applied Stochastic Processes

HW1

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Problem 1

[Problem 7 from p.27 of the notes.] Solve the following optimization problem.

$$\begin{aligned} d^2 &= \inf\Big\{\int_0^\infty \left\|e^{-3t} - ae^{-t} - be^{-2t}\right\|^2 dt : \quad a \in \mathbb{C} \text{ and } b \in \mathbb{C}\Big\} \\ &= \int_0^\infty \left\|e^{-3t} - \alpha e^{-t} - \beta e^{-2t}\right\|^2 dt \end{aligned}$$

In other words, find α, β, and d^2.

Solution:
We know that from the inner product rule

$$\langle f, g\rangle = \int_0^\infty f(t)\overline{g(t)}dt$$
$$\mathcal{L}^2(0,\infty) = \Big\{f(t) : \int_0^\infty \|f(t)\|^2 dt < \infty\Big\}$$

and

$$e^{-3t} \notin \mathcal{L}^2(0,\infty)$$
$$\int_0^\infty \left\|e^{t}\right\|^2 dt = \infty$$

Now, since we know that

$$e^{-t}, \quad e^{-2t}, \qquad \text{are } l.i.$$
$$\alpha e^{-t} + \beta e^{-2t} = P_{\mathcal{H}} e^{-3t}$$
$$\mathcal{H} = span\{e^{-t}, e^{-2t}\}$$
$$\text{where} \qquad \hat{f} = \alpha e^{-t} + \beta e^{-2t} \in \mathcal{H}$$

This means that

$$e^{-3t} - \hat{f} \perp e^{-t}$$
$$e^{-3t} - \hat{f} \perp e^{-2t}$$

So we calculate,

$$\langle e^{-3t} - \alpha e^{-t} - \beta e^{-2t}, e^{-t}\rangle = 0$$
$$\langle e^{-3t}, e^{-t}\rangle - \alpha\langle e^{-t}, e^{-t}\rangle - \beta\langle e^{-2t}, e^{-3t}\rangle = 0$$
$$\langle e^{-3t}, e^{-t}\rangle = \alpha\langle e^{-t}, e^{-t}\rangle + \beta\langle e^{-2t}, e^{-t}\rangle$$

and

$$\begin{aligned} \langle\, e^{-3t} - \alpha e^{-t} - \beta e^{-2t}, e^{-2t} \rangle &= 0 \\ \langle\, e^{-3t}, e^{-2t} \rangle - \alpha \langle\, e^{-t}, e^{-2t} \rangle - \beta \langle\, e^{-2t}, e^{-2t} \rangle &= 0 \\ \langle\, e^{-3t}, e^{-2t} \rangle &= \alpha \langle\, e^{-t}, e^{-2t} \rangle + \beta \langle\, e^{-2t}, e^{-2t} \rangle \end{aligned}$$

In matrix form we can express this as

$$\begin{bmatrix} \langle\, e^{-3t}, e^{-t} \rangle \\ \langle\, e^{-3t}, e^{-2t} \rangle \end{bmatrix} = \begin{bmatrix} \langle\, e^{-t}, e^{-t} \rangle & \langle\, e^{-2t}, e^{-t} \rangle \\ \langle\, e^{-t}, e^{-2t} \rangle & \langle\, e^{-2t}, e^{-2t} \rangle \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Thus, the Gram Matrix, $\mathcal{G}$ becomes

$$\mathcal{G} = \begin{bmatrix} \langle\, e^{-t}, e^{-t} \rangle & \langle\, e^{-2t}, e^{-t} \rangle \\ \langle\, e^{-t}, e^{-2t} \rangle & \langle\, e^{-2t}, e^{-2t} \rangle \end{bmatrix}$$

Since we know that

$$\begin{aligned} \langle\, e^{-jt}, e^{-kt} \rangle &= \int_0^\infty e^{-jt}\overline{e^{-kt}} dt \\ &= \int_0^\infty e^{-(j+k)t} dt \\ &= -\frac{1}{j+k}\left[e^{-(j+k)t} \right]_0^\infty \\ &= -\frac{1}{j+k}(0-1) \\ &= \frac{1}{j+k} \end{aligned}$$

The matrix equation becomes

$$\begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Thus,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \end{bmatrix} = \frac{1}{(\frac{1}{2})(\frac{1}{4}) - (\frac{1}{3})(\frac{1}{3})} \begin{bmatrix} \frac{1}{4} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{1}{5} \end{bmatrix} = 72 \begin{bmatrix} -\frac{1}{250} \\ \frac{1}{60} \end{bmatrix} = \begin{bmatrix} -0.288 \\ 1.2 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -0.288 \\ 1.2 \end{bmatrix}$$

Then,

$$
\begin{aligned}
d^2 &= \int_0^\infty \left\| e^{-3t} - \alpha e^{-t} - \beta e^{-2t} \right\|^2 dt \\
&= \int_0^\infty \left\{ e^{-6t} + \alpha^2 e^{-2t} + \beta^2 e^{-4t} - 2e^{-3t}(\alpha e^{-t}) - 2e^{-3t}(\beta e^{-2t}) + 2(\alpha e^{-t})(\beta e^{-2t}) \right\} dt \\
&= \int_0^\infty \left\{ e^{-6t} + \alpha^2 e^{-2t} + \beta^2 e^{-4t} - 2\alpha e^{-4t} - 2\beta e^{-5t} + 2\alpha\beta e^{-3t} \right\} dt \\
&= \int_0^\infty \left\{ \alpha^2 e^{-2t} + 2\alpha\beta e^{-3t} + (-2\alpha + \beta^2) e^{-4t} - 2\beta e^{-5t} + e^{-6t} \right\} dt
\end{aligned}
$$

Since,

$$
\begin{aligned}
\int_0^\infty e^{-jt} dt &= -\frac{1}{j} \left[e^{-jt} \right]_0^\infty \\
&= \frac{1}{j}
\end{aligned}
$$

$$
\begin{aligned}
d^2 &= \frac{\alpha^2}{2} + \frac{2\alpha\beta}{3} + \frac{-2\alpha + \beta^2}{4} + \frac{-2\beta}{5} + \frac{1}{6} \\
&= 0.0017
\end{aligned}
$$