College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 7
Input/Output Responses of LTI Systems

Author:
Tomoki Koike

Supervisor:
Martin Corless

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Purdue University
West Lafayette, Indiana

Exercise 1

Consider the differential equation

$$\dot{x} = Ax$$

where A is a square matrix. Show that if A has $j2\pi$ as an eigenvalue, then there is a nonzero initial state x_0 such that the equation has a solution x which satisfies $x(1) = x(0) = x_0$.

If the given equation has an eigenvalue of

$$\lambda = j2\pi$$

the complex conjugate is also an eigenvalue of the matrix A. Thus, we know that

$$\lambda = \pm j2\pi$$

From this we can deduce the solution of this equation to be

$$e^{\lambda t}v = \left(cos(2\pi t) + jsin(2\pi t)\right)(\sigma + j\omega) \quad \because v = \sigma + j\omega$$

For

$$x(1) = \left(cos(2\pi) + jsin(2\pi)\right)(\sigma + j\omega) = (\sigma + j\omega) = x_0$$

$$x(0) = \left(cos(0) + jsin(0)\right)(\sigma + j\omega) = (\sigma + j\omega) = x_0$$

$$\therefore x(1) = x(0) = x_0$$

$q.e.d$

Exercise 2

Compute e^{At} at $t = ln(2)$ for

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

using all four methods mentioned in the notes.

Method 1: MATLAB "expm()"

```
% Problem 2
% Method 1 expm()
A = [0, 1; 1, 0];
t = log(2);
eA1 = expm(A*t);
```

The result is

```
eA1 = 2×2
    1.2500    0.7500
    0.7500    1.2500
```

Method 2: Numerical Simulation

Using ode45 we accomplish this. Following is the function

```
function dx = numSim(t, x, A)
    dx = A*x;
end
```

and the following is the MATLAB script

```
tspan = 0:0.1:10;
A = [0, 1; 1, 0];
x0 = [1, 0];
res = ode45(@(t,x) numSim(t,x,A), tspan, x0)
E1 = deval(res, log(2))


x0 = [0, 1];
res = ode45(@(t,x) numSim(t,x,A), tspan, x0)
E2 = deval(res, log(2))
```

This gives us the following result

E1 = 2×1 1.2500 0.7500	E2 = 2×1 0.7500 1.2500

This agrees with the answer of method 1.

Method 3: Jordan form

The eigenvalue of this matrix is

$$(A-\lambda I)=\begin{pmatrix}-\lambda & 1\\ 1 & -\lambda\end{pmatrix}$$

$$det(A-\lambda I)=\lambda^2-1=(\lambda+1)(\lambda-1)$$

$$\lambda=\pm 1\ .$$

The eigenvectors become,

$$v_1=\begin{pmatrix}1\\1\end{pmatrix},\qquad \lambda_1=1$$

$$v_2=\begin{pmatrix}-1\\1\end{pmatrix},\qquad \lambda_2=-1$$

$$\therefore T=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

The transition matrix becomes

$$\Phi(t)=T\begin{pmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{pmatrix}T^{-1}$$

$$\Phi(t)=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}e^{t} & 0\\ 0 & e^{-t}\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}$$

In MATLAB, if $t=ln(2)$

```
% Method 2 numerical simulation
[v, d] = eig(A)
T = [-1, 1; 1, 1];
Phi = T*[exp(d(1,1)*t), 0; 0, exp(d(2,2)*t)]/T
```

```
Phi = 2×2
    1.2500    0.7500
    0.7500    1.2500
```

Which is equal to e^{At}.

Method 4: Laplace style

We already know the eigenvalues and eigenvectors, so we get the following relationship

$$(sI - A)^{-1} = \begin{pmatrix} \frac{s}{(s+1)(s-1)} & \frac{s-1}{(s+1)(s-1)} \\ \frac{s-1}{(s+1)(s-1)} & \frac{s}{(s+1)(s-1)} \end{pmatrix} = \begin{pmatrix} \frac{s}{(s+1)(s-1)} & \frac{1}{(s+1)} \\ \frac{1}{(s+1)} & \frac{s}{(s+1)(s-1)} \end{pmatrix}$$

Since,

$$\frac{s}{(s+1)(s-1)} = \frac{1}{2}\left(\frac{1}{s+1} + \frac{1}{s-1}\right)$$

$$e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$$

$$e^{At} = \begin{pmatrix} \frac{1}{2}(\mathrm{e}^{-\mathrm{t}} + e^{t}) & \mathrm{e}^{-\mathrm{t}} \\ \mathrm{e}^{-\mathrm{t}} & \frac{1}{2}(\mathrm{e}^{-\mathrm{t}} + e^{t}) \end{pmatrix}$$

Plug in $t = ln(2)$,

$$e^{At} = \begin{pmatrix} 1.25 & 0.75 \\ 0.75 & 1.25 \end{pmatrix} .$$

Exercise 3

Compute e^{At} for matrix

$$A = \begin{pmatrix} 3 & -5 \\ -5 & 3 \end{pmatrix}$$

(a) Using the eigenvalues and eigenvectors of A;
(b) Using the Laplace Transform

(a).

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} 3-\lambda & -5 \\ -5 & 3-\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (3 - \lambda)^2 - 25 = (8 - \lambda)(-2 - \lambda)$$

$$\lambda = -2, 8 \ .$$

The eigenvectors become,

$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda_1 = -2$$

$$v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = 8$$

$$\therefore T = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The transition matrix becomes

$$\mathrm{e}^{\mathrm{At}} = T \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} T^{-1}$$

$$\mathrm{e}^{\mathrm{At}} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{8t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 0.5e^{-2\,t} + 0.5e^{8\,t} & 0.5e^{-2\,t} - 0.5e^{8\,t} \\ 0.5e^{-2\,t} - 0.5e^{8\,t} & 0.5e^{-2\,t} + 0.5e^{8\,t} \end{pmatrix} .$$

(b).

Since we already know the eigenvalues and eigenvectors

$$(sI - A)^{-1} = \begin{pmatrix} \dfrac{s-3}{(s+2)(s-8)} & \dfrac{5}{(s+2)(s-8)} \\ \dfrac{5}{(s+2)(s-8)} & \dfrac{s-3}{(s+2)(s-8)} \end{pmatrix}$$

Since,

$$\frac{s-3}{(s+2)(s-8)} = \frac{0.5}{\mathrm{s}-8} + \frac{0.5}{\mathrm{s}+2}$$

$$\frac{5}{(s+2)(s-8)} = \frac{0.5}{s-8} - \frac{0.5}{s+2}$$

the answer becomes

$$e^{At} = \mathcal{L}^{-1}((sI-A)^{-1})$$

$$e^{At} = \begin{pmatrix} 0.5e^{-2\,t} + 0.5e^{8\,t} & 0.5e^{-2\,t} - 0.5e^{8\,t} \\ 0.5e^{-2\,t} - 0.5e^{8\,t} & 0.5e^{-2\,t} + 0.5e^{8\,t} \end{pmatrix}.$$

Exercise 4

Compute e^{At} at $t = ln(2)$ for

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Using MATLAB, we can compute the results

```
A = [0, 1; -1, 0];
t = log(2);
eA = expm(A*t)
```

```
eA = 2×2
    0.7692    0.6390
   -0.6390    0.7692
```

Exercise 5

Obtain (by hand) the state response $x(\cdot)$ of each of the following systems due to a unit impulse input and the zero initial conditions. For each case, determine whether the response contains all the system modes.

(a) .

$$\begin{aligned}\dot{x}_1 &= -5x_1 + 2x_2 + u \\ \dot{x}_2 &= -12x_1 + 5x_2 + u\end{aligned}$$

(b).

$$\begin{aligned}\dot{x}_1 &= -5x_1 + 2x_2 + u \\ \dot{x}_2 &= -12x_1 + 5x_2 + 2u\end{aligned}$$

(c).

$$\begin{aligned}\dot{x}_1 &= -5x_1 + 2x_2 + u \\ \dot{x}_2 &= -12x_1 + 5x_2 + 3u\end{aligned}$$

The A matrix for all three (a)~(c) are the same,

$$A = \begin{pmatrix} -5 & 2 \\ -12 & 5 \end{pmatrix}$$

But the B matrix is different. If the input is a unit impulse input $u(t) = \delta(t)$, and we know that function of $x(t)$ becomes

$$x(t) = e^{At}x_0 + e^{At}B \ .$$

Since we assume a zero initial condition

$$x(t) = e^{At}B \ .$$

We know can compute e^{At}.

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} -5-\lambda & 2 \\ -12 & 5-\lambda \end{pmatrix}$$

$$det(A - \lambda I) = (-5-\lambda)(5-\lambda) + 24 = (1-\lambda)(-1-\lambda)$$

$$\lambda = \pm 1 \ .$$

The eigenvectors become,

$$\begin{pmatrix} -6 & 2 \\ -12 & 4 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1/3 \\ 1 \end{pmatrix}, \qquad \lambda_1 = 1$$

And,

$$\begin{pmatrix} -4 & 2 \\ -12 & 6 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -1$$

$$\therefore T = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$$

The transition matrix becomes

$$\mathrm{e}^{\mathrm{At}} = T \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} T^{-1}$$

$$\mathrm{e}^{\mathrm{At}} = \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{8t} \end{pmatrix} (-1) \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 3e^{-t} - 2e^{t} & e^{t} - e^{-t} \\ 6e^{-t} - 6e^{t} & 3e^{t} - 2e^{-t} \end{pmatrix} .$$

(a).

For this given system

$$B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Hence,

$$x(t) = e^{At} B$$

$$x(t) = \begin{pmatrix} 3e^{-t} - 2e^{t} & e^{t} - e^{-t} \\ 6e^{-t} - 6e^{t} & 3e^{t} - 2e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} 2e^{-t} - e^{t} \\ 4e^{-t} - 3e^{t} \end{pmatrix} .$$

(b).

For this given system

$$B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Hence,

$$x(t) = e^{At} B$$

$$x(t) = \begin{pmatrix} 3e^{-t} - 2e^{t} & e^{t} - e^{-t} \\ 6e^{-t} - 6e^{t} & 3e^{t} - 2e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^{-t} \\ 2e^{-t} \end{pmatrix}$$

(c).

For this given system

$$B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Hence,

$$x(t) = e^{At}B$$

$$x(t) = \begin{pmatrix} 3e^{-t} - 2e^{t} & e^{t} - e^{-t} \\ 6e^{-t} - 6e^{t} & 3e^{t} - 2e^{-t} \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$x(t) = \begin{pmatrix} e^{t} \\ 3e^{t} \end{pmatrix}$$

We have a mode of continuously growing do not have the other two modes. So we cannot observe all modes with these responses.

Exercise 6

Consider the system with input u output y and state variables $x_1,\cdots,x_4$ described by

$$\begin{aligned} \dot{x_1} &= -x_1 \\ \dot{x_2} &= x_1 - 2x_2 \\ \dot{x_3} &= x_1 - 3x_3 + u \\ \dot{x_4} &= x_1 + x_2 + x_3 + x_4 \\ y &= x_3 \end{aligned}$$

Obtain (by hand) an expression for the impulse response of this system. Does it contain all the state space modes?

The A matrix is

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The state space

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

for this system is represented by the matrices

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}, \qquad D = 0$$

If the input is a unit impulse input $u(t) = \delta(t)$, we know that the output response is expressed as

$$G(t) = Ce^{At}B \ .$$

Assuming a zero initial condition for input response

$$x(t) = e^{At}B \ .$$

We know can compute e^{At}.

The eigenvalue of this matrix is

$$(A - \lambda I) = \begin{pmatrix} -1-\lambda & 0 & 0 & 0 \\ 1 & -2-\lambda & 0 & 0 \\ 1 & 0 & -3-\lambda & 0 \\ 1 & 1 & 1 & 1-\lambda \end{pmatrix}$$

$$det(A-\lambda I) = (\lambda-1)(\lambda+1)(\lambda+2)(\lambda+3)$$

$$\lambda = \pm 1, -2, -3 \ .$$

The eigenvectors become,

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 1 & 0 & -4 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_1 = 1$$

And,

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 3 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -2$$

And,

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 4 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 0 \\ -4 \\ 1 \end{pmatrix}, \qquad \lambda_3 = -3$$

And,

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 2 \end{pmatrix} v_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_4 = \begin{pmatrix} -0.8 \\ -0.8 \\ -0.4 \\ 1 \end{pmatrix}, \qquad \lambda_4 = -1$$

$$\therefore T = \begin{pmatrix} 0 & 0 & 0 & -0.8 \\ 0 & -3 & 0 & -0.8 \\ 0 & 0 & -4 & -0.4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The transition matrix becomes

$$\mathrm{e}^{\mathrm{At}} = T \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_3 t} & 0 \\ 0 & 0 & 0 & e^{\lambda_4 t} \end{pmatrix} T^{-1}$$

$$e^{At} = \begin{pmatrix} e^{-t} & 0 & 0 & 0 \\ e^{-t} - e^{-2t} & e^{-2t} & 0 & 0 \\ 0.5e^{-t} - 0.5e^{-3t} & 0 & e^{-3t} & 0 \\ 0.3333e^{-2t} - 1.25e^{-t} + 0.125e^{-3t} + 0.7917e^{t} & 0.3333e^{t} - 0.3333e^{-2t} & 0.25e^{t} - 0.25e^{-3t} & e^{t} \end{pmatrix}.$$

Thus,

$$x(t) = e^{At}B$$

$$x(t) = \begin{pmatrix} 0 \\ 0 \\ e^{-3t} \\ 0.25e^{t} - 0.25e^{-3t} \end{pmatrix}$$

$$G(t) = Ce^{At}B$$

$$G(t) = e^{-3t}$$

And, the output response is characterized as

$$y(t) = Ce^{(t-t_0)}x_0 + \int_{t_0}^{t} G(t-\tau)u(\tau)d\tau$$

$$x_0 = 0$$

Since the input is an impulse input, $y(t) = G(t)$

$$y(t) = e^{-3t}$$

We have only a continuously decaying but do not have the other two. So we cannot observe all modes with these responses.

Exercise 7

Consider an LTI system described by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2x_1 - 3x_2 + u \\ y &= 3x_1 - x_2 \end{aligned}$$

Is there a persistent input (does not go to zero) u for which the corresponding output always going to zero regardless of initial conditions? If answer is yes provided an example.

The A matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

The state space

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

for this system is represented by the matrices

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = (3 \quad -1), \qquad D = 0$$

We can compute the eigenvalues and eigenvectors as well as e^{At} using MATLAB

$$\lambda = -1, -2$$

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{pmatrix}$$

Consider the term

$$Ce^{A(t-t_0)}\left(\bar{x}(t_0) - \bar{x}_p(t_0)\right)$$

$$= CTe^{\Lambda(t-t_0)}T^{-1}\left(\bar{x}(t_0) - \bar{x}_p(t_0)\right)$$

$$= CT\begin{pmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{pmatrix}T^{-1}\left(\bar{x}(t_0) - \bar{x}_p(t_0)\right)$$

When $t \to \infty$, this goes to 0. Thus, A is stable.

The steady state part of the output response is defined as

$$y_{ss}(t) = \hat{G}(\lambda)e^{\lambda t}v.$$

If Λ is 0 of $\hat{G}(\Lambda)$, then the steady state response is also 0.

Next we find,

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{3 - s}{s^2 + 3s + 2}$$

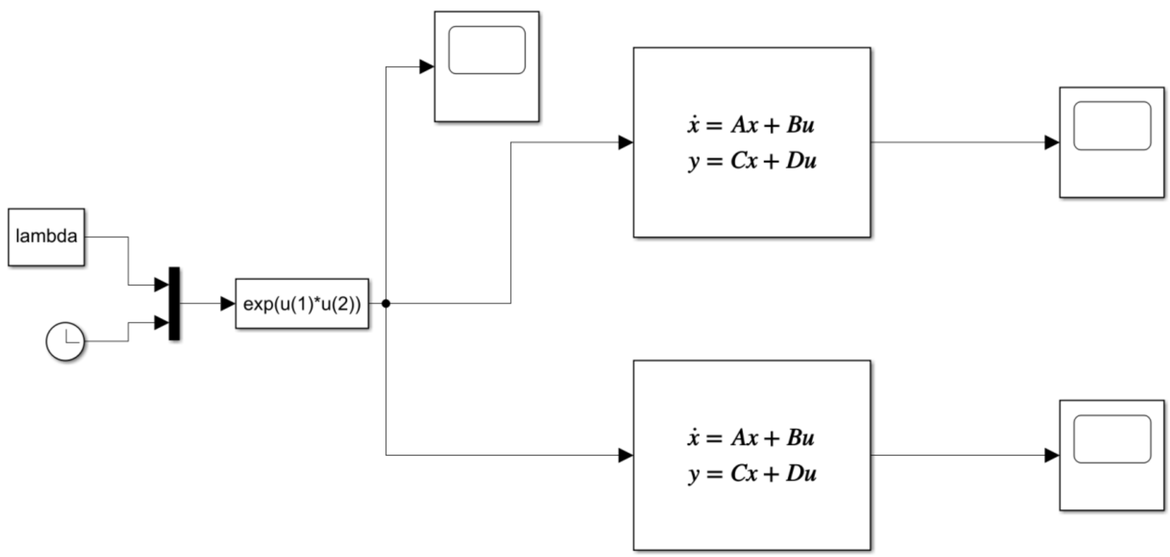
This goes to zero when $s = 3$. Thus, $\lambda = 3$

If the input of this system is

$$u(t) = e^{3t}v$$

With this input the steady state part of the output response becomes 0

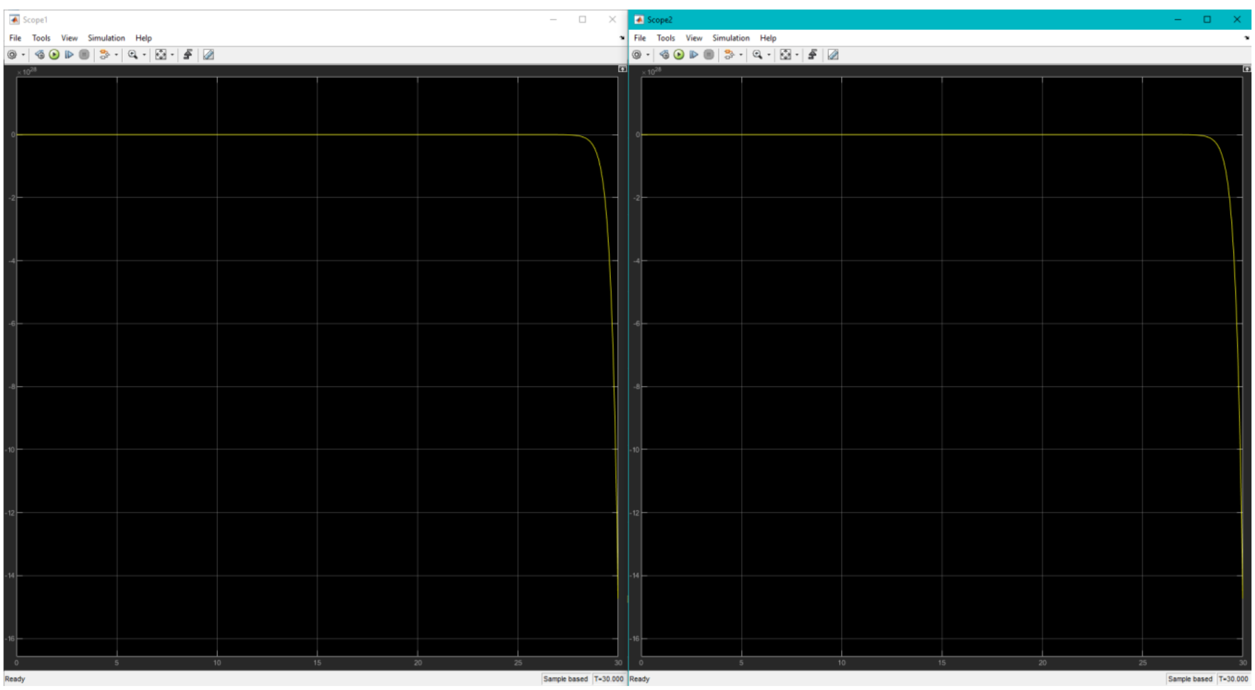
We verify this with Simulink




```
% Matrices
A = [0, 1; -2, -3];
B = [0; 1];
C = [3, -1];
D = 0;

lambda = 3;

% random numbers between 0 and 20
low = -20; high = 20;
x01 = low + (high - low) * rand(2,1)
x02 = low + (high - low) * rand(2,1)
sim('ex7_v2');
```

This is the exponential input

The responses were (for 2 random initial conditions)



Exercise 8

Consider an LTI system described by

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -2x_1 - 3x_2 + u \\ y &= -x_1 - 3x_2 + u \end{aligned}$$

Is there a persistent input (does not go to zero) u for which the corresponding output always going to zero regardless of initial conditions? If answer is yes provided an example.

The A matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$$

The state space

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

for this system is represented by the matrices

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & -3 \end{pmatrix}, \qquad D = 1$$

We can compute the eigenvalues and eigenvectors as well as e^{At} using MATLAB

$$\lambda = -1, -2$$

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{pmatrix}$$

Consider the term

$$Ce^{A(t-t_0)}\left(\bar{x}(t_0) - \bar{x}_p(t_0)\right)$$

$$= CTe^{\Lambda(t-t_0)}T^{-1}\left(\bar{x}(t_0) - \bar{x}_p(t_0)\right)$$

$$= CT\begin{pmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-2(t-t_0)} \end{pmatrix}T^{-1}\left(\bar{x}(t_0) - \bar{x}_p(t_0)\right)$$

When $t \to \infty$, this goes to 0. Thus, A is stable.

The steady state part of the output response is defined as

$$y_{ss}(t) = \hat{G}(\lambda)e^{\lambda t}v.$$

If Λ is 0 of $\hat{G}(\Lambda)$, then the steady state response is also 0.

Next we find,

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{s^2 + 1}{s^2 + 3s + 2}$$

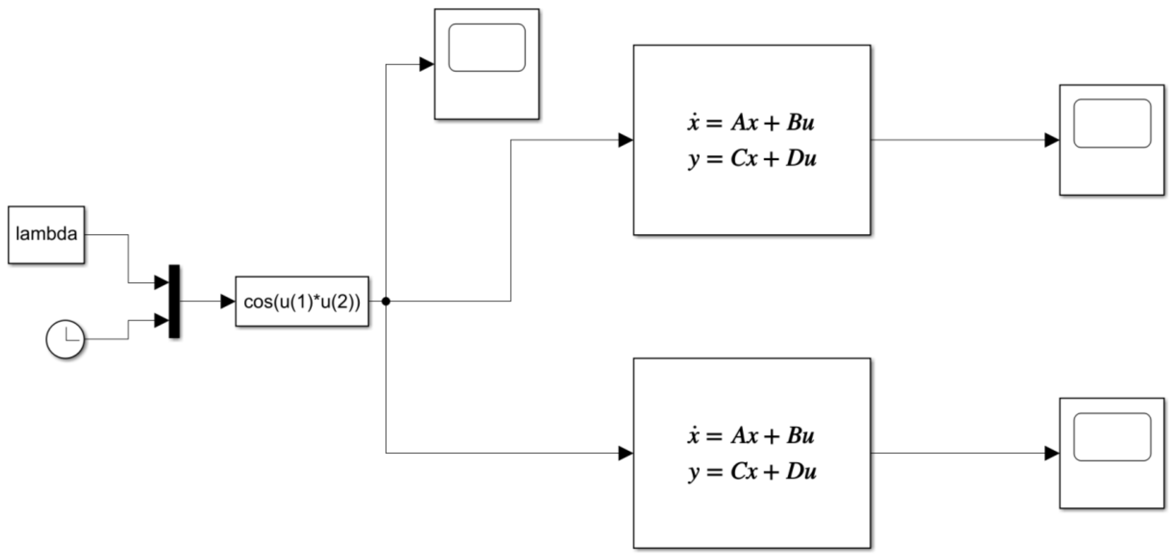
This goes to zero when $s = \pm j$. Thus, $\lambda = \pm j$

If the input of this system is

$$u(t) = cos(t)$$

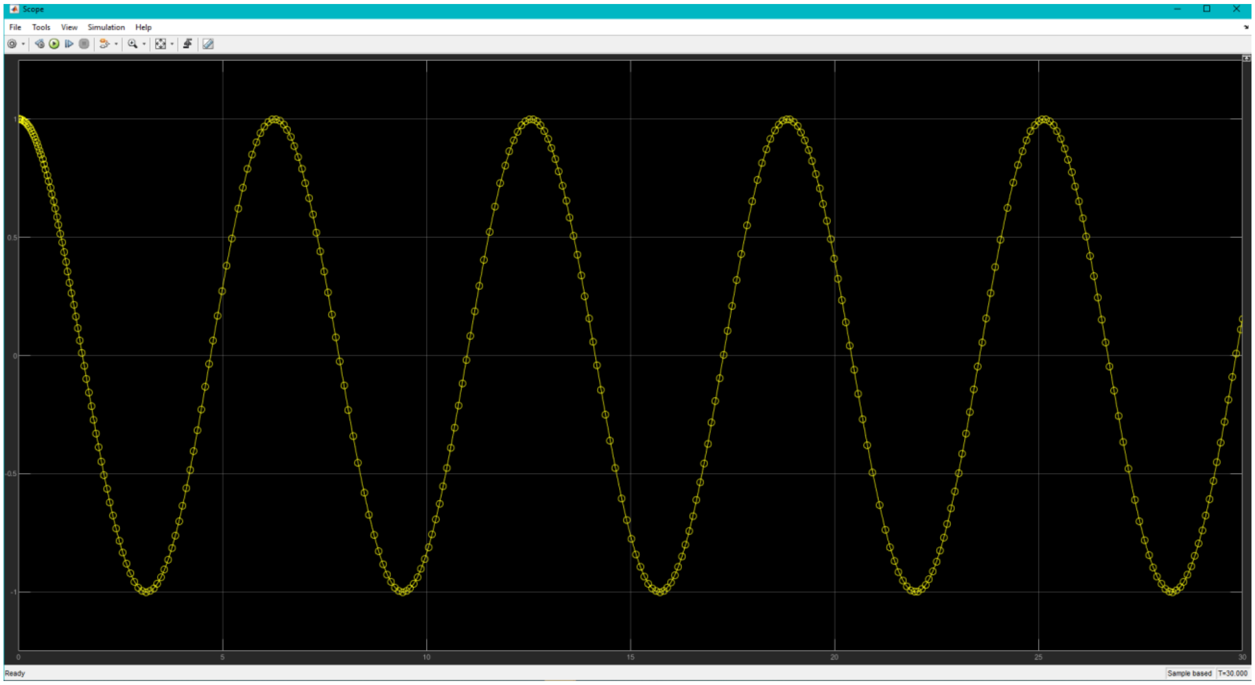
With this input the steady state part of the output response becomes 0

Verify this with Simulink

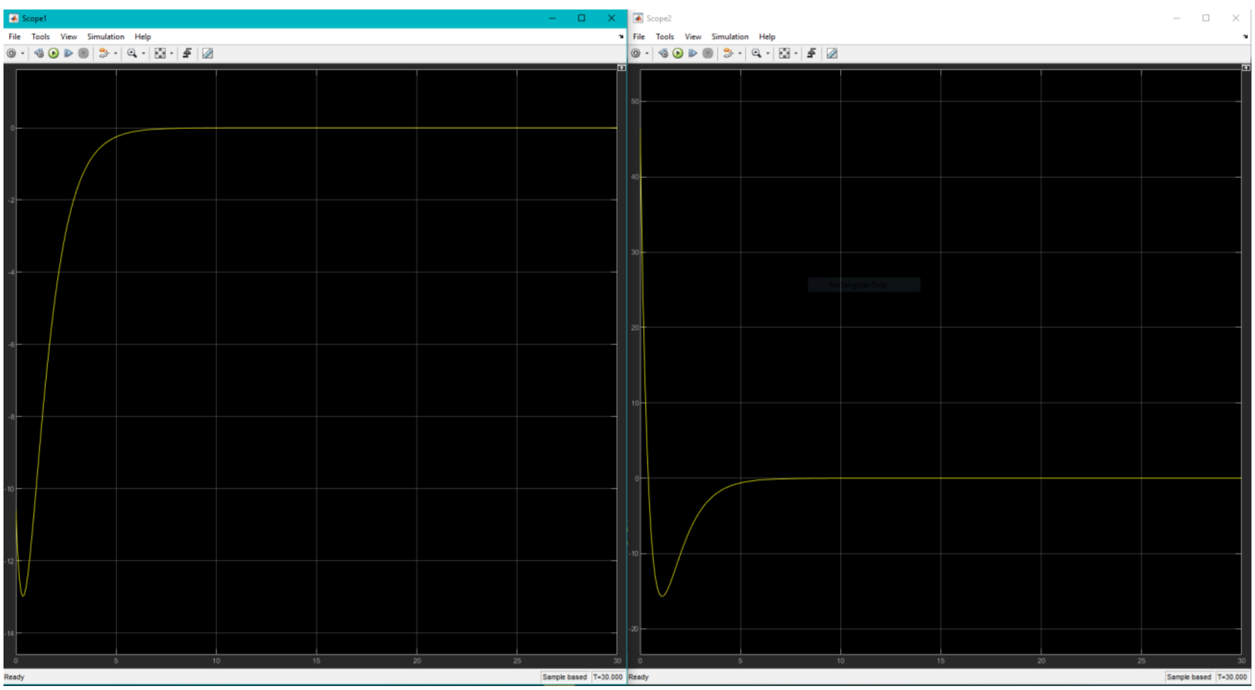



```
% Matrices
A = [0, 1; -2, -3];
B = [0; 1];
C = [-1, -3];
D = 1;

lambda = 1;

% random numbers between -20 and 20
low = -20; high = 20;
x01 = low + (high - low) * rand(2,1)
x02 = low + (high - low) * rand(2,1)
sim('ex7_v2');
```

This is the cosine input



The responses were this (for 2 random inputs)



Exercise 9

Disturbing the cart. Consider the pendulum cart system with parameter set P4.

We will subject it to passive stabilization and a disturbance input w, that is, we let

$$u = -ky - c\dot{y} + w$$

where $k > 0$ and $c > 0$. Regard the resulting system as an input-output system with input w and output y and answer the following questions.

(a) Using MATLAB, obtain the poles and zeros of the system linearized about $E1$.
(b) Consider a sinusoidal disturbance input of the form

$$w(t) = asin(\omega t)$$

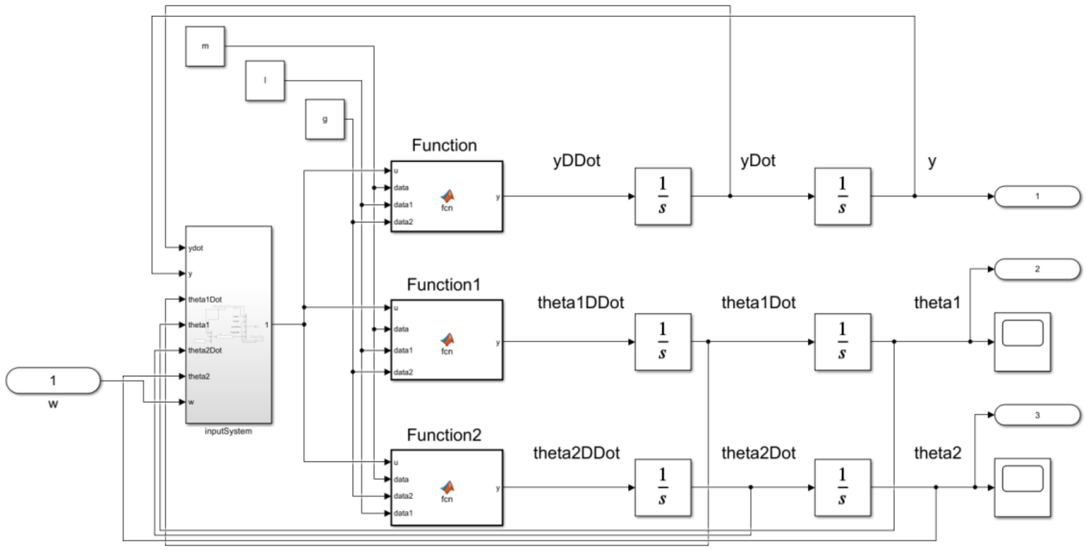
Choose ω so that the steady state response of the linearized system to this disturbance is zero. Simulate both the nonlinear system and the linearized system with zero initial conditions and this disturbance.

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

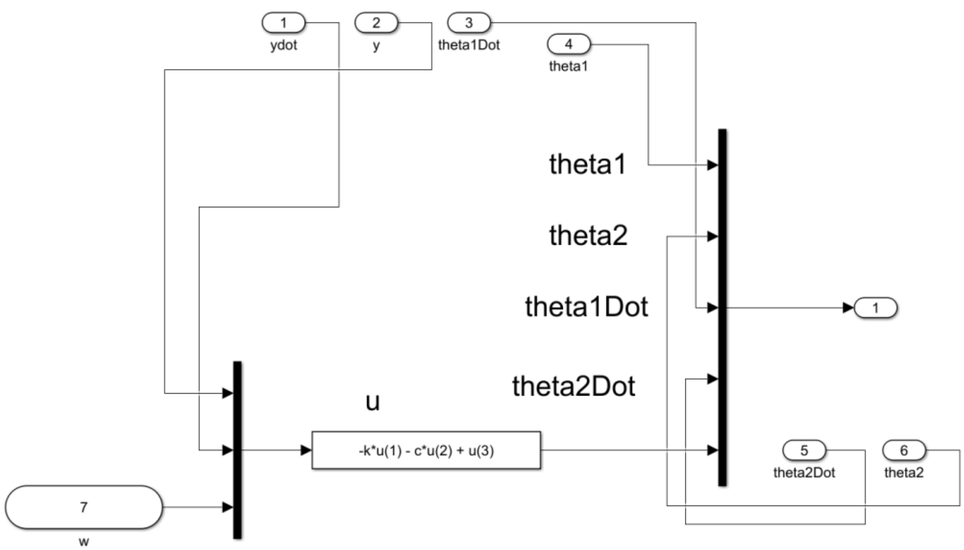
$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

(a).

The linear Simulink model is the following



The “inputSystem” subsystem is the following



The Embedded MATLAB Blocks are the following

Function:

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;
```

```
    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

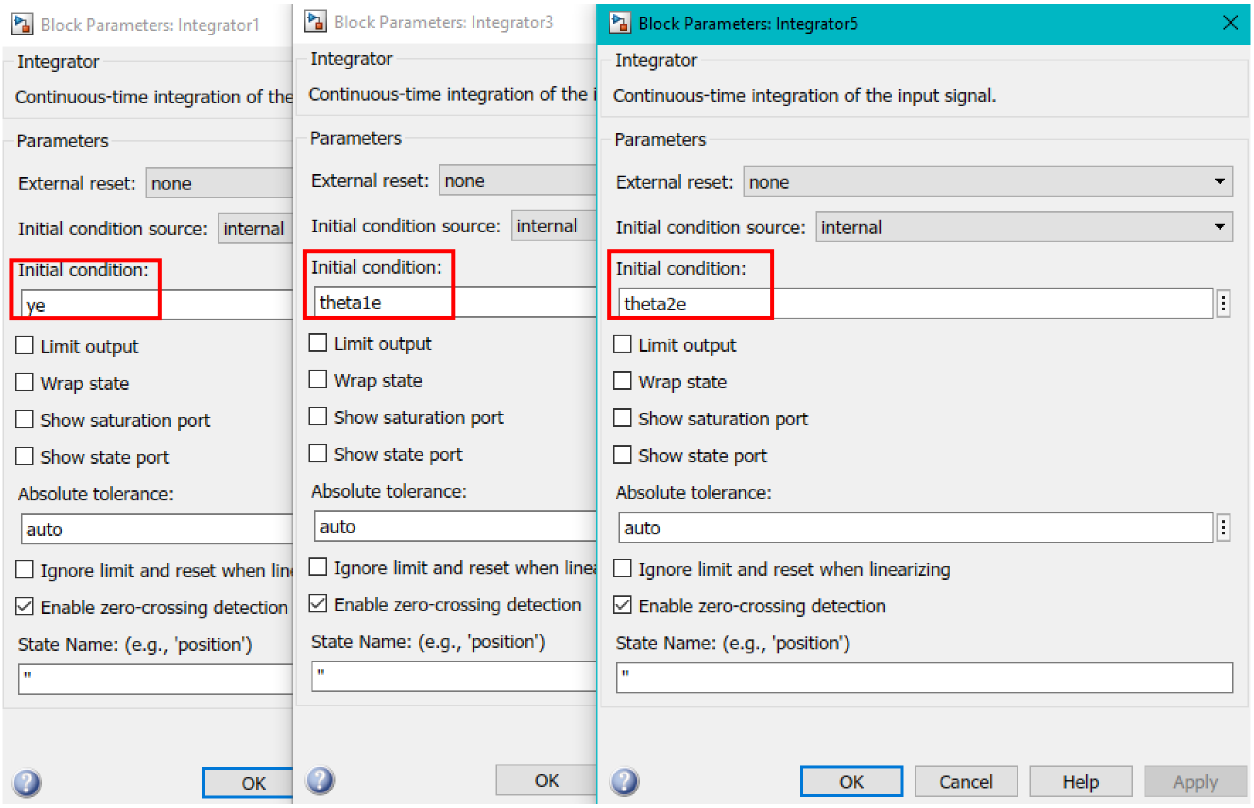
Function1:

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Function2:

```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the equilibriums conditions E1, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



And set arbitary values for the constants k and c (which presumably are the spring constant and damping coefficient)

k	c
1	1

Then we run the following MATLAB code to find the trim conditions and perform linearization on the system.

```
% Set parameters and equilibrium conditions
m = [2,1,1]; l = [1,0.5]; g = 1; % P4
ye = 0; theta1e = 0; theta2e = 0; % E1

k = 1;     % spring constant
c = 1;     % damping coefficient

xe = trim("db_pend_cart_lin_modInput")
[A, B, C, D] = linmod('db_pend_cart_lin_modInput',xe)
```

```
sys_ss = ss(A, B, C, D);  % get the state space system
sys_tf = tf(sys_ss)  % get the transfer function
p = pole(sys_tf)  % get the eigenvalues

% Define a transfer function for each one to obtain the zeros
tf1 = tf(sys_tf.Numerator(1), sys_tf.Denominator(1))
tf2 = tf(sys_tf.Numerator(2), sys_tf.Denominator(2))
tf3 = tf(sys_tf.Numerator(3), sys_tf.Denominator(3))
z1 = zero(tf1)
z2 = zero(tf2)
z3 = zero(tf3)
p1 = pole(tf1)
p2 = pole(tf2)
p3 = pole(tf3)
```

The results are the following

```
A = 6×6
        0         0         0    1.0000         0         0
        0         0         0         0    1.0000         0
        0         0         0         0         0    1.0000
  -0.5000   -0.5000   -0.5000   -0.5000         0         0
  -0.5000   -1.5000   -0.5000   -0.5000         0         0
  -1.0000   -1.0000   -3.0000   -1.0000         0         0

B = 6×1
        0
        0
        0
   0.5000
   0.5000
   1.0000

C = 3×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0

D = 3×1
     0
     0
     0
```

The transfer function for the first input “tf1” (for y):

```
tf1 =

  0.5 s^4 + 6.688e-17 s^3 + 1.5 s^2 + 6.688e-17 s + 1
  ---------------------------------------------------
   s^6 + 0.5 s^5 + 5 s^4 + 1.5 s^3 + 5.5 s^2 + s + 1

Continuous-time transfer function.
```

The corresponding poles and zeros

p1 = 6×1 complex -0.1259 + 1.8518i -0.1259 - 1.8518i -0.0205 + 1.1189i -0.0205 - 1.1189i -0.1036 + 0.4702i -0.1036 - 0.4702i	z1 = 4×1 complex -0.0000 + 1.4142i -0.0000 - 1.4142i 0.0000 + 1.0000i 0.0000 - 1.0000i

The transfer function for the first input “tf2” (for θ_1):

```
tf2 =

  0.5 s^4 + 8.668e-17 s^3 + s^2 + 1.742e-16 s + 4.697e-18
  -------------------------------------------------------
     s^6 + 0.5 s^5 + 5 s^4 + 1.5 s^3 + 5.5 s^2 + s + 1

Continuous-time transfer function.
```

The corresponding poles and zeros

p2 = 6×1 complex -0.1259 + 1.8518i -0.1259 - 1.8518i -0.0205 + 1.1189i -0.0205 - 1.1189i -0.1036 + 0.4702i -0.1036 - 0.4702i	z2 = 4×1 complex 0.0000 + 1.4142i 0.0000 - 1.4142i -0.0000 + 0.0000i -0.0000 - 0.0000i

The transfer function for the first input "tf3" (for θ_2):

```
tf3 =

  s^4 + 2.388e-16 s^3 + s^2 + 1.254e-16 s - 6.89e-18
  --------------------------------------------------
  s^6 + 0.5 s^5 + 5 s^4 + 1.5 s^3 + 5.5 s^2 + s + 1

Continuous-time transfer function.
```

The corresponding poles and zeros

```
p3 = 6×1 complex
  -0.1259 + 1.8518i
  -0.1259 - 1.8518i
  -0.0205 + 1.1189i
  -0.0205 - 1.1189i
  -0.1036 + 0.4702i
  -0.1036 - 0.4702i
```

```
z3 = 4×1 complex
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
  -0.0000 + 0.0000i
   0.0000 + 0.0000i
```

(b).

Since the code in part (a)

```
[xe, ue, ye, dxe] = trim("db_pend_cart_lin_modInput")
```

where the system was linearized for an equilibrium point of (0, 0, 0). The equilibrium condition for the input w was found to be 0 by the variable "ue" which is highlighted red above.

Thus, if $w = 0$

$$w(t) = asin(\omega t) = 0$$

and

$$\omega = \; 0 \; or \; \pi$$

For the initial condition,

$$\begin{aligned} y &= y^e + \delta y \\ \theta_1 &= \theta_1^e + \delta\theta_1 \\ \theta_2 &= \theta_2^e + \delta\theta_2 \end{aligned}$$

Thus,

$$\begin{aligned} y_i &= y^e + \delta y_i \\ \theta_{1i} &= \theta_1^e + \delta\theta_{1i} \\ \theta_{2i} &= \theta_2^e + \delta\theta_{2i} \end{aligned}$$

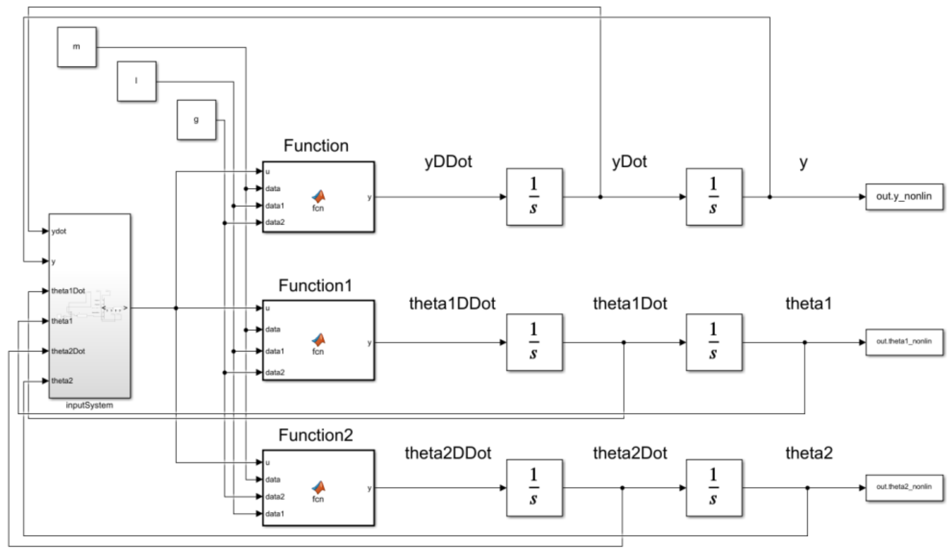
For E1, the initial condition $(y^e, \theta_1^e, \theta_2^e) = (0,0,0)$

$$(\delta y_i, \delta\theta_{1i}, \delta\theta_{2i}) = (0,0,0) = (0,0,0)$$

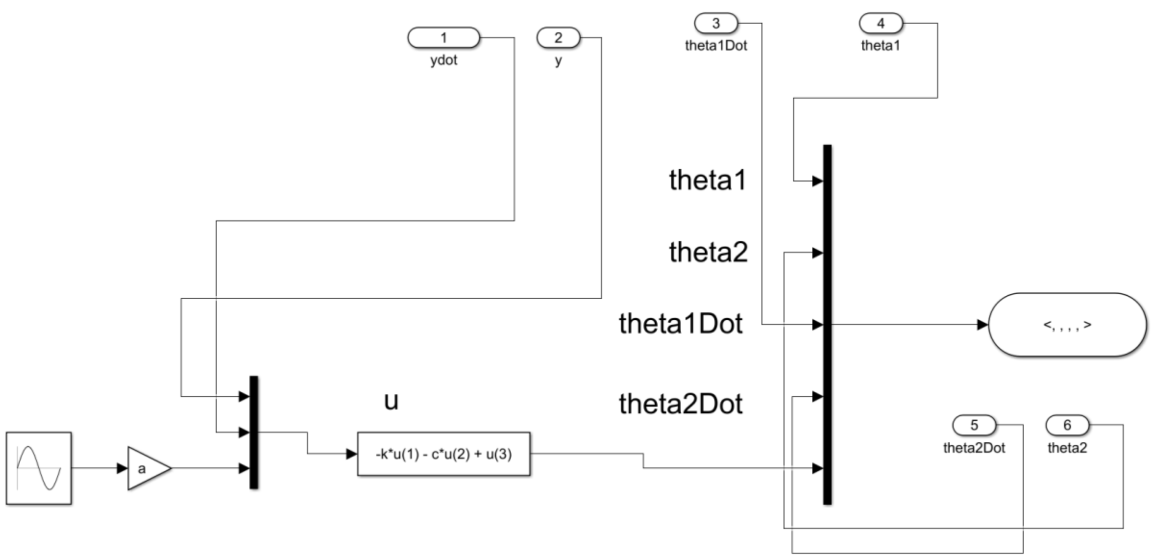
Then,

$$(y_i, \theta_{1i}\,, \theta_{2i}) \; = \; (0,0,0)$$

For the nonlinear system we use the following model

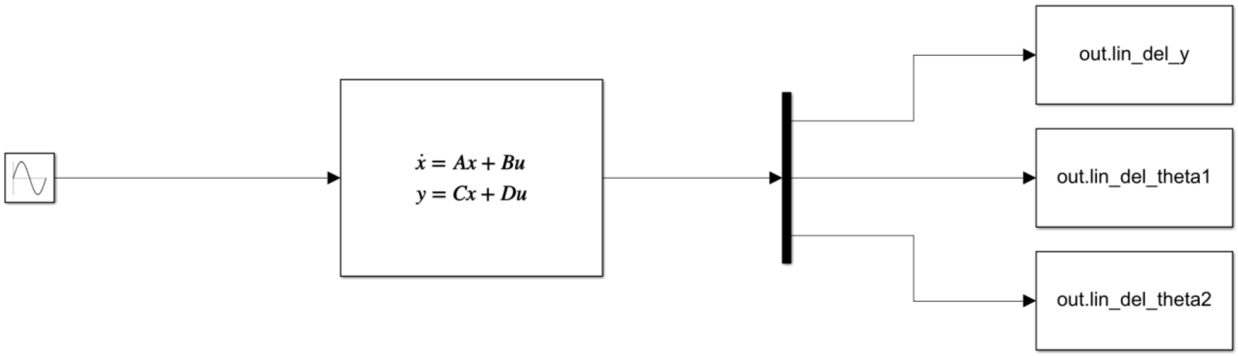


The subsystem "inputSystem" is the following



The Embedded MATLAB Blocks are all the same as the linearization model.

For the linearized one we use the following model



Using MATLAB, we can solve $\hat{G}(s)$ of this matrix A for the linearized model it becomes

$$\hat{G}(\mathrm{s}) = C(sI - A)^{-1}B + D$$

$$= \begin{pmatrix} \dfrac{1.6266s^4 + 4.8678s^2 + 3.2453}{3.2452s^6 + 1.6226s^5 + 1.6226s^4 + 4.8678s^3 + 17.849s^2 + 3.2452s + 3.2452} \\ \dfrac{1.6226s^4 + 3.2452s^2}{3.2452s^6 + 1.6226s^5 + 1.6226s^4 + 4.8678s^3 + 17.849s^2 + 3.2452s + 3.2452} \\ \dfrac{3.2452s^4 + 3.2452s^2}{3.2452s^6 + 1.6226s^5 + 1.6226s^4 + 4.8678s^3 + 17.849s^2 + 3.2452s + 3.2452} \end{pmatrix}$$

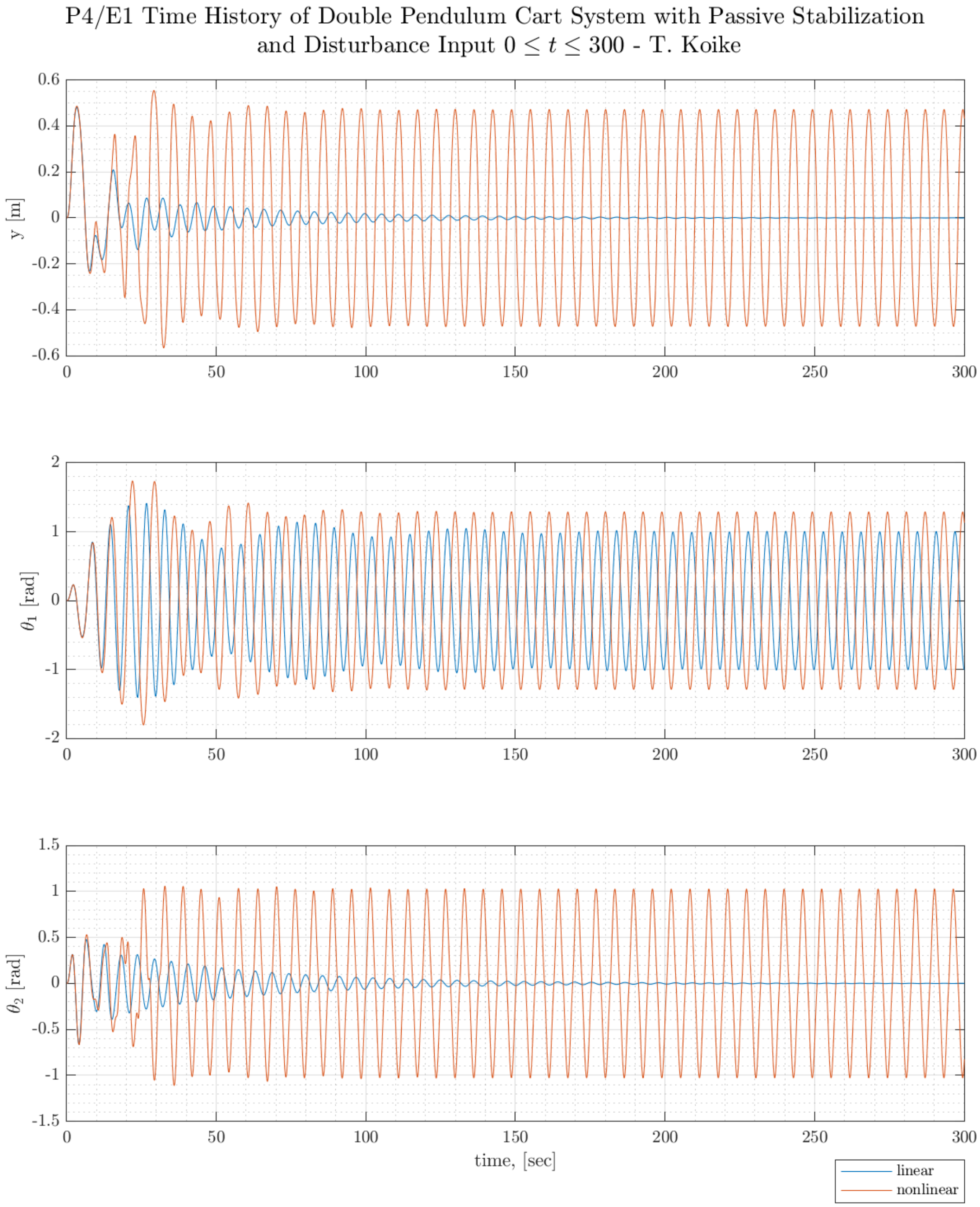
Solving each numerator to be zero, we find that the combination of (for the input of y) which corresponds to the first row

$$s = \begin{pmatrix} -1.4142\mathrm{j} \\ 1.4142\mathrm{j} \\ \mathrm{j} \\ -\mathrm{j} \end{pmatrix}$$

So, for when $s = j$ we can choose an omega of 1!

$$\omega = 1$$

Then we have the following plot of the simulation.



The y-displacement does decay to 0. So the simulation is congruent with our results.

This was done with the following code

```
% (b)
warning("off")
% Set parameters and equilibrium conditions
m = [2,1,1]; l = [1,0.5]; g = 1; % P4
ye = 0; theta1e = 0; theta2e = 0; % E1

k = 1;    % spring constant
c = 1;    % damping coefficient
a = 1;   % amplitude in disturbance
omega = 1;

xe = trim("db_pend_cart_lin_modInput")
[A, B, C, D] = linmod("db_pend_cart_lin_modInput",xe)

% % % Proof
% syms t s
% G = vpa(simplify(C*inv(s*eye(6)-A)*B+D))
% res1 = solve(G(1)==0, s)
% res2 = solve(G(2)==0, s)
% res3 = solve(G(3)==0, s)

% Initial conditions for the perturbation variables
del_yi = 0; del_theta1i = 0; del_theta2i = 0;
del_yi_dot = 0; del_theta1i_dot = 0; del_theta2i_dot = 0;

% Initial conditions for the variables
yi = ye + del_yi;
theta1i = theta1e + del_theta1i;
theta2i = theta2e + del_theta2i;
yi_dot = del_yi_dot;
theta1i_dot = del_theta1i_dot;
theta2i_dot = del_theta2i_dot;

% Linearized system
IC_lin = [del_yi, del_theta1i, del_theta2i, 0, 0, 0];
lin_res = sim('ss_lin_sys');

% Nonlinear system
nonlin_res = sim('db_pend_cart_nonlin_modInput')

% get values from results

tspan_lin  = lin_res.tout;
y_lin      = lin_res.lin_del_y.signals.values;
theta1_lin = lin_res.lin_del_theta1.signals.values;
theta2_lin = lin_res.lin_del_theta2.signals.values;

tspan_nonlin  = nonlin_res.tout;
y_nonlin      = nonlin_res.y_nonlin.signals.values;
theta1_nonlin = nonlin_res.theta1_nonlin.signals.values;
```

```
theta2_nonlin = nonlin_res.theta2_nonlin.signals.values;

% Plotting
fig1 = figure('Renderer',"painters", 'Position', [10 10 900 1000]);
subplot(3,1,1)
hold on; grid on; grid minor; box on;
plot(tspan_lin, y_lin)
plot(tspan_nonlin, y_nonlin)
ylabel('y [m]')
hold off
subplot(3,1,2)
hold on; grid on; grid minor; box on;
plot(tspan_lin, theta1_lin)
plot(tspan_nonlin, theta1_nonlin)
ylabel('$\theta_1$ [rad]')
hold off
subplot(3,1,3)
hold on; grid on; grid minor; box on;
plot(tspan_lin, theta2_lin)
plot(tspan_nonlin, theta2_nonlin)
ylabel('$\theta_2$ [rad]')
xlabel('time, [sec]')
h = legend('linear','nonlinear'); set(h, 'Position',  [0.8, 0.05, .1, .025]);
hold off
sgtitle({['P4/E1 Time History of Double Pendulum Cart System with Passive ' ...
    'Stabilization'], ' and Disturbance Input $0\leq t \leq 100$ - T. Koike'})
saveas(fig1, 'p9_b.png')
```