AE 6230 – HW4: Mode Shapes and Responses of 1-D Continuous Systems

Out: November 15, 2022; **Due:** November 23, 2022 by 11:59 PM ET in Canvas

Guidelines

- Read each question carefully before doing any work;
- If you find yourself doing pages of math, pause and consider if there is an easier approach;
- You can consult any relevant materials;
- You can discuss solution approaches with others, but your submission must be your own work;
- If you have doubts, please ask questions in class, during office hours, and/or Piazza (no questions via email);
- The solution to each question should concisely and clearly show the steps;
- Simplify your results as much as possible;
- Box the final answer for each question;
- Submit any code with the solution (but remember to also submit all relevant plots).

Problem 1 – 30 points

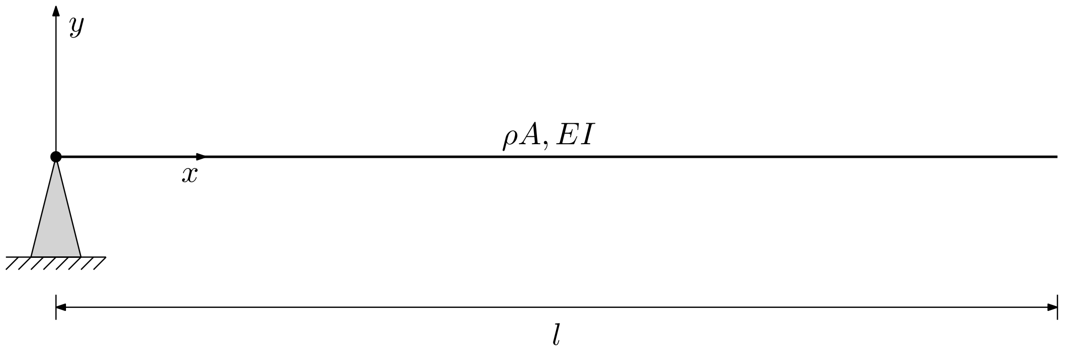


Figure 1: Schematic of a pinned-free uniform beam undergoing out-of-plane bending.

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Length	l	1 m
Bending stiffness	EI	5 Nm^2
Mass per unit length	ρA	0.5 kg/m

Figure 1 shows a simplified model for the anti-symmetric out-of-plane bending vibrations of an aircraft. Using the anti-symmetry condition, the model considers one half wing represented as a pinned-free uniform beam. Considering the parameters in Table 1, answer the following questions:

1. Verify that the characteristic equation is given by

$$\tanh \alpha l - \tan \alpha l = 0 \tag{1}$$

2. Evaluate the first four eigenvalues α_i;
3. Evaluate the natural frequencies ω_i;
4. Determine the analytical expressions of the eigenfunctions $X_i(x)$;
5. Plot the mode shapes $\phi_i(x)$ obtained by normalizing the eigenfunctions to have unit maximum displacement;
6. Verify (mathematically) that there is one rigid-body eigenfunction.

Problem 2 – 15 points

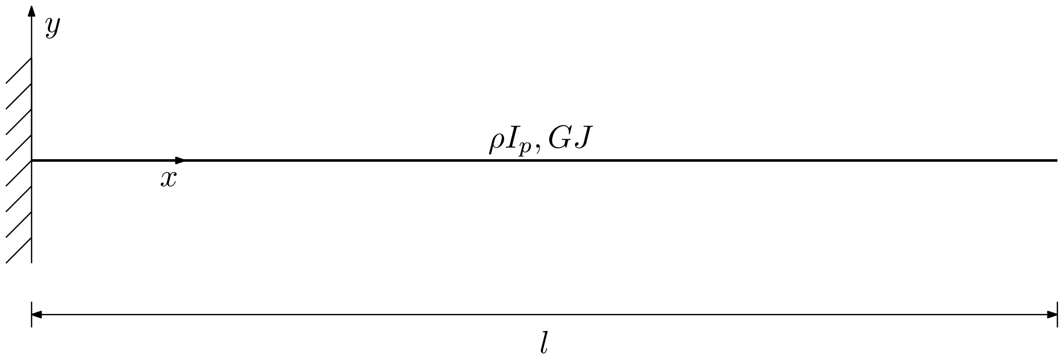


Figure 2: Schematic of a clamped-free uniform beam undergoing torsion.

Figure 2 shows a clamped-free uniform beam undergoing torsion. The beam is subject to the initial conditions

$$\theta(x,0) = \theta_0(x) = \frac{\bar{\theta}x}{l} \qquad \dot{\theta}(x,0) = \dot{\theta}_0(x) = 0 \tag{2}$$

where $\bar{\theta}$ is the tip twist angle at the initial time. Answer the following questions:

1. Determine the analytical expressions of the modal initial conditions $\eta_{0_i}, \dot{\eta}_{0_i}$;
2. Write the undamped free response in the form

$$\theta(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t) \tag{3}$$

   showing the appropriate expressions of the mode shapes, natural frequencies, and modal coordinates;
3. Verify that the response from Question 2 satisfies the initial conditions.

Problem 3 – 30 points

Table 2: Parameter values for Problem 3.

Parameter	Symbol	Value
Length	l	1.0 m
Torsional stiffness	GJ	5.0 Nm^2
Moment of inertia per unit length	ρI_p	0.005 kg·m
Modal viscous damping factor	ζ_i	0.02
Excitation amplitude	r_0	1 N
Excitation frequency	ω_0	125 rad/s

Consider the beam of Problem 2 but now subject to the distributed moment

$$r(x,t) = r_0 \sin \omega_0 t \tag{4}$$

Considering the parameters in Table 2, answer the following question:

1. Determine the analytical expressions of the modal forces $N_i(t)$;
2. Write the damped steady-state response in the form

$$\theta(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t) \tag{5}$$

   showing the appropriate expressions of the modal coordinates;
3. Tabulate the quantities needed to evaluate Eq. (5) considering the first four modes[1];
4. Plot the tip twist angle for $0 \le t \le 0.2$ s considering an increasing number of modes $N = 1, 2, 3, 4$;
5. Explain the trend in the results for increasing N;
6. How should the distributed moment $r(x,t)$ be shaped to avoid exciting the first torsional vibration mode?

[1]MATLAB prinouts are acceptable.

Problem 4 – 25 points

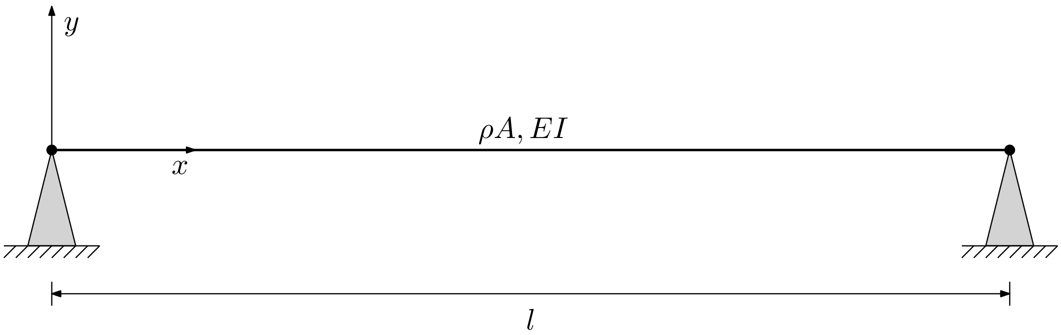


Figure 3: Schematic of a pinned-pinned uniform beam undergoing out-of-plane bending.

Table 3: Parameter values for Problem 4.

Parameter	Symbol	Value
Length	l	1 m
Bending stiffness	EI	50 Nm^2
Mass per unit length	ρA	0.25 kg/m
Excitation amplitude	F_0	1 N
Excitation application point	x_0	$l/2$

Figure 3 shows a pinned-pinned uniform beam in bending. The beam is at rest when it experiences the excitation

$$F(t) = F_0\,\delta(t) \quad \text{at} \quad x = x_0 \tag{6}$$

where F_0 is the amplitude, x_0 the application point, and $\delta(t)$ the unit impulse function. The eigenvalues and mode shapes (normalized to have unit maximum displacement) are the same as for a uniform string

$$\alpha_i = \frac{i\pi}{l} \qquad \phi_i(x) = \sin\left(\frac{i\pi x}{l}\right) \tag{7}$$

Considering the parameters in Table 3, answer the following questions:

1. Determine the analytical expressions of the modal forces $N_i(t)$;
2. Write the undamped forced response in the form

$$v(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t) \tag{8}$$

   showing the appropriate expressions of the natural frequencies and modal coordinates;
3. Tabulate the quantities needed to evaluate Eq. (7) considering the first eight modes[2];
4. Plot the midpoint displacement for $0 \le t \le 0.1$ s considering an increasing number of modes $N = 1, \ldots, 8$;
5. Explain the trend in the results for increasing N.

[2]MATLAB prinouts are acceptable.

Problem 1 Solution – 30 points

Question 1 – 5 points

The displacement field for a uniform beam undergoing out-of-plane bending must satisfy

$$EI\,\frac{\partial^2 v(0,t)}{\partial x^2}\,\delta\left[\frac{\partial v(0,t)}{\partial x}\right] = EI\,\frac{\partial^3 v(0,t)}{\partial x^3}\,\delta v(0,t) = 0 \tag{9}$$

$$EI\,\frac{\partial^2 v(l,t)}{\partial x^2}\,\delta\left[\frac{\partial v(l,t)}{\partial x}\right] = EI\,\frac{\partial^3 v(l,t)}{\partial x^3}\,\delta v(l,t) \tag{10}$$

For a pinned-free beam, the bending displacement is restrained and the bending rotation is unrestrained at $x = 0$, while the displacement and rotation are both unrestrained at $x = l$. In this case, Eqs. (9) and (10) give

$$v(0,t) = \frac{\partial^2 v(0,t)}{\partial x^2} = 0 \tag{11}$$

$$\frac{\partial^2 v(l,t)}{\partial x^2} = \frac{\partial^3 v(l,t)}{\partial x^3} = 0 \tag{12}$$

which are the boundary conditions for this problem. Setting $v(x,t) = X(x)Y(t)$, we obtain

$$X(0) = X''(0) = 0 \tag{13}$$

$$X''(l) = X'''(l) = 0 \tag{14}$$

These conditions must be applied to

$$X(x) = A_1 \sin\alpha x + B_1 \cos\alpha x + A_2 \sinh\alpha x + B_2 \cosh\alpha x \tag{15}$$

which is the general solution for $X(x)$ for $\alpha \neq 0$. Enforcing Eq. (13) gives

$$\begin{aligned} X(0) &= B_1 + B_2 = 0 \\ X''(0) &= \alpha^2\left(-B_1 + B_2\right) = 0 \end{aligned} \tag{16}$$

For $\alpha \neq 0$, we obtain $B_1 = B_2 = 0$. Using this result, Eq. (15) reduces to

$$X(x) = A_1 \sin\alpha x + A_2 \sinh\alpha x \tag{17}$$

Enforcing that Eq. (17) satisfies Eq. (14) gives

$$\begin{aligned} X''(l) &= \alpha^2\left(-A_1 \sin\alpha l + A_2 \sinh\alpha l\right) = 0 \\ X'''(l) &= \alpha^3\left(-A_1 \cos\alpha l + A_2 \cosh\alpha l\right) = 0 \end{aligned} \tag{18}$$

For $\alpha \neq 0$, Eq. (18) gives the 2×2 system

$$\begin{bmatrix} -\sin\alpha l & \sinh\alpha l \\ -\cos\alpha l & \cosh\alpha l \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

which admits a non-trivial solutions only if the coefficient matrix is singular, that is, if

$$-\sin\alpha l \cosh\alpha l + \sinh\alpha l \cos\alpha l = 0 \tag{20}$$

Diving by $\cosh\alpha l \cos\alpha l$ gives the expected characteristic equation

$$\tanh\alpha l - \tan\alpha l = 0 \tag{21}$$

Note that $\alpha = 0$ is not a valid root of Eq. (21) because the process used to derive the characteristic equation assumes $\alpha \neq 0$. The correct process to deal with the case $\alpha = 0$ is explained in Question 6.

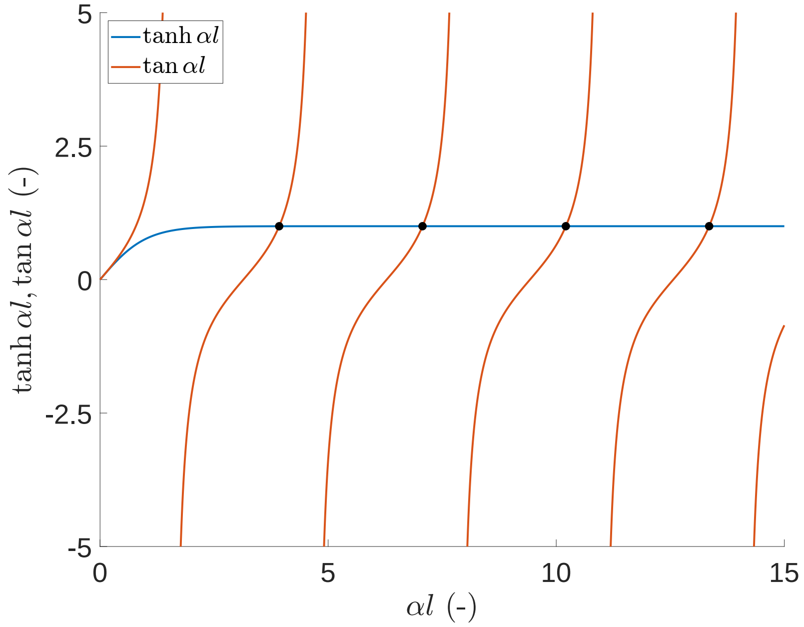


Figure 4: Characteristic equation in Eq. (21) for Problem 1 Question 2.

Question 2 – 5 points

Solving the characteristic equation Eq. (21) gives the *elastic* eigenvalues $\alpha_i \neq 0$, as the case $\alpha = 0$ must be treated separately (see Question 6). To solve Eq. (21), it is convenient to write it in the form

$$\tanh \alpha l = \tan \alpha l \tag{22}$$

Figure 4 plots the sides of Eq. (22) for $\alpha l \in [0, 15]$. We see that $\tanh \alpha l \to 1$ before its first non-zero intersection with $\tan \alpha l$ for $\alpha l > \pi$. Thus, we set the first four first guesses as

$$\alpha_i l \approx \frac{(4i+1)\pi}{4} \qquad (i = 1, 2, 3, 4) \tag{23}$$

These values practically coincide with the roots of the characteristic equation. For $l = 1$ m, we obtain

$$\begin{aligned} \alpha_1 &= 3.9266 \text{ m}^{-1} \\ \alpha_2 &= 7.0686 \text{ m}^{-1} \\ \alpha_3 &= 10.2102 \text{ m}^{-1} \\ \alpha_4 &= 13.3518 \text{ m}^{-1} \end{aligned} \tag{24}$$

While $\alpha = 0$ satisfies Eq. (21), this root must be discarded at this stage[3] because the process followed to derive the equation assumes $\alpha \neq 0$. The case $\alpha = 0$ is treated in Question 6.

Question 3 – 5 points

The natural frequencies are related to the eigenvalues by

$$\omega_i = \alpha_i^2 \sqrt{\frac{EI}{\rho A}} \tag{25}$$

which gives

$$\begin{aligned} \omega_1 &= 48.7566 \text{ rad/s} \\ \omega_2 &= 158.0028 \text{ rad/s} \\ \omega_3 &= 329.6602 \text{ rad/s} \\ \omega_4 &= 563.7384 \text{ rad/s} \end{aligned} \tag{26}$$

[3] Submissions that have reported $\alpha = 0$ as a solution for Question 2 are considered correct only if they discuss that $\alpha = 0$ is obtained following the process in Question 6 and, consequently, its associated eigenfunction is not $X(x) = 0$.

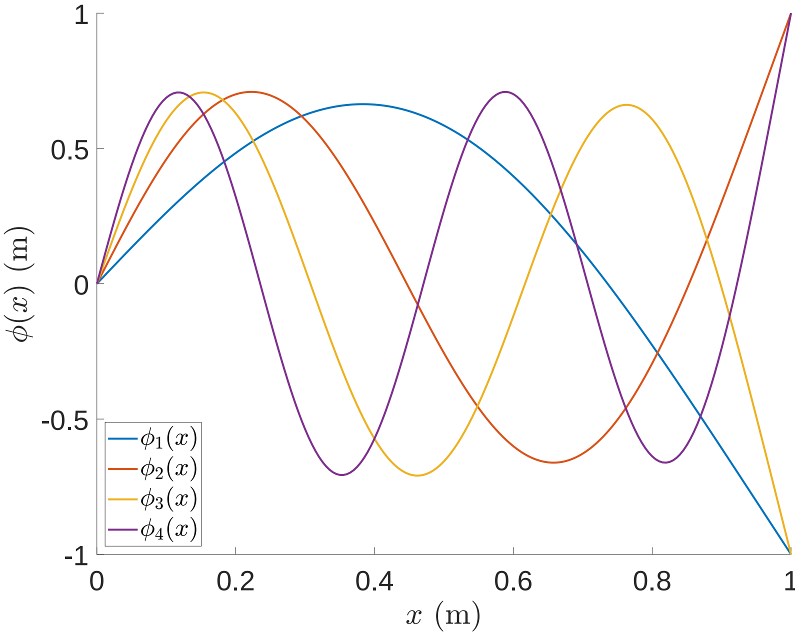


Figure 5: Mode shapes for Problem 1 Question 5.

Question 4 – 5 points

To determine the *elastic* eigenfunctions, we plug the eigenvalues from Question 2 back into Eq. (19). Because the two equations are not linearly independent for $\alpha = \alpha_i$ $(i = 1, \ldots, \infty)$ (by definition of eigenvalue), we must assign an arbitrary value to either A_1 or A_2. Setting $A_1 = 1$, we obtain

$$A_2 = \frac{\sin \alpha_i l}{\sinh \alpha_i l} \qquad \text{or} \qquad A_2 = \frac{\cos \alpha_i l}{\cosh \alpha_i l} \tag{27}$$

which must give the same result for $\alpha = \alpha_i$ $(i = 1, \ldots, \infty)$. Using the first relation in Eq. (27), we obtain

$$X_i(x) = \sin \alpha_i x + \frac{\sin \alpha_i l}{\sinh \alpha_i l} \sinh \alpha_i x \qquad (i = 1, \ldots, \infty) \tag{28}$$

Alternatively, we can set $A_2 = 1$ and determine A_1 from either equation in the system.

Question 5 – 5 points

The mode shapes $\phi_i(x)$ $(i = 1, \ldots, \infty)$ are obtained by evaluating Eq. (28) normalized by its maximum value along the beam length. Figure 5 shows the results for $i = 1, 2, 3, 4$. It is good practice to check that the mode shapes meet the boundary conditions. All the curves start from zero at $x = 0$ (restrained displacement) with a non-zero slope (unrestrained rotation) and zero curvature (zero bending moment). Additionally, all the curves have non-zero slope and zero curvature at $x = l$ (again, because of the unrestrained rotation). The curvature rate is also zero at $x = l$ (unrestrained displacement requiring zero shear force), though this is hardly appreciable graphically. Finally, the number of "waves" along the length increases with the mode order, leading to an increasing number of nodes.

Because the first *elastic* mode shape has one node, we can expect the problem to have one more mode shape that has not been considered yet. This is a rigid-body mode shape representing a constant rotation along the pinned point, which is determined in the following question.

Question 6 – 5 points

To verify that there is one rigid-body eigenfunction, we set $\alpha = 0$, which gives

$$X''''(x) = 0 \tag{29}$$

satisfied by the general solution

$$X(x) = Ax^3 + Bx^2 + Cx + D \tag{30}$$

Enforcing Eq. (13) gives

$$\begin{aligned} X(0) &= D = 0 \\ X''(0) &= 2B = 0 \end{aligned} \tag{31}$$

Using $B = D = 0$, makes Eq. (30) reduce to

$$X(x) = Ax^3 + Cx \tag{32}$$

Enforcing the second condition in Eq. (14) gives

$$X'''(l) = 6A = 0 \tag{33}$$

This gives $A = 0$ and

$$X(x) = Cx \tag{34}$$

Equation (34) satisfies the first condition in Eq. (14), leaving the coefficient C undetermined. Thus, the pinned-free boundary conditions admit a non-trivial eigenfunction for $\alpha = 0$

$$X_0(x) = Cx \tag{35}$$

which represents a rigid-body rotation about $x = 0$ (expected for this problem). Note that $X(x) = 0$ is not a rigid-body eigenfunction but a trivial solution (no motion).

Problem 2 Solution – 15 points

For this problem (and Problem 3), we use the eigenvalues, natural frequencies, and mode shapes for a clamped-free uniform beam undergoing torsion

$$
\begin{aligned}
\alpha_i &= \frac{(2i-1)\pi}{2l} \\
\omega_i &= \frac{(2i-1)\pi}{2l}\sqrt{\frac{GJ}{\rho I_p}} \\
\phi_i(x) &= \sin\left[\frac{(2i-1)\pi x}{2l}\right]
\end{aligned}
\tag{36}
$$

with $i = 1, \dots, \infty$. The mode shapes give unit maximum rotation and non-unit modal masses

$$
m_i = \rho I_p \int_0^l \phi_i^2(x)dx = \rho I_p \frac{l}{2} \tag{37}
$$

Question 1 – 5 points

Using the mode shapes in Eq. (36), the initial conditions for the ith modal coordinate are given by

$$
\begin{aligned}
\eta_{0_i} &= \frac{2}{l}\int_0^l \phi_i(x)\theta_0(x)dx = \frac{2\bar{\theta}}{l^2}\int_0^l \sin\left[\frac{(2i-1)\pi x}{2l}\right] x dx = \frac{8\bar{\theta}}{(2i-1)^2\pi^2}\sin\left[\frac{(2i-1)\pi}{2}\right] = \frac{8\bar{\theta}\,(-1)^{i+1}}{(2i-1)^2\pi^2} \\
\dot{\eta}_{0_i} &= \frac{2}{l}\int_0^l \phi_i(x)\dot{\theta}_0(x)dx = 0
\end{aligned}
\tag{38}
$$

Question 2 – 5 points

Using the modal initial conditions determined in Question 1, the undamped free response is given by

$$
\theta(x,t) = \sum_{i=1}^{\infty}\phi_i(x)\eta_i(t) = \sum_{i=1}^{\infty}\phi_i(x)\,\eta_{0_i}\cos\omega_i t = \frac{8\bar{\theta}}{\pi^2}\sum_{i=1}^{\infty}\sin\left[\frac{(2i-1)\pi x}{2l}\right]\frac{(-1)^{i+1}}{(2i-1)^2}\cos\omega_i t \tag{39}
$$

where the ith modal coordinate is given by

$$
\eta_i(t) = \eta_{0_i}\cos\omega_i t = \frac{8\bar{\theta}\,(-1)^{i+1}}{(2i-1)^2\pi^2}\cos\omega_i t \tag{40}
$$

and the natural frequencies ω_i ($i = 1, \dots, \infty$) are given by Eq. (2). We expect higher-order modes to contribute less and less to the response as their amplitudes are inversely proportional to i^2. The term involving $\sin\omega_i t$ is omitted because it vanishes due to the zero initial twist rate.

Question 3 – 5 points

The solution in Eq. (39) satisfies zero initial twist rate because it only contains a cosine function of time. One approach to verifying that Eq. (39) converges to the assigned initial twist angle $\theta_0(x) = \bar{\theta}x/l$ is to evaluate it at the initial time

$$
\theta(x,0) = \frac{8\bar{\theta}}{\pi^2}\sum_{i=1}^{\infty}\sin\left[\frac{(2i-1)\pi x}{2l}\right]\frac{(-1)^{i+1}}{(2i-1)^2} \tag{41}
$$

and plot it for an increasing number of modes.

Figure 6 shows $\theta(x,0)/\bar{\theta}$ as a function of x/l, showing it tends to the assigned initial twist angle as the number of modes increases. The convergence is slow at the tip because the assigned triangular shape is difficult to capture by means of sinusoidal functions (as known from experience with the Fourier series).

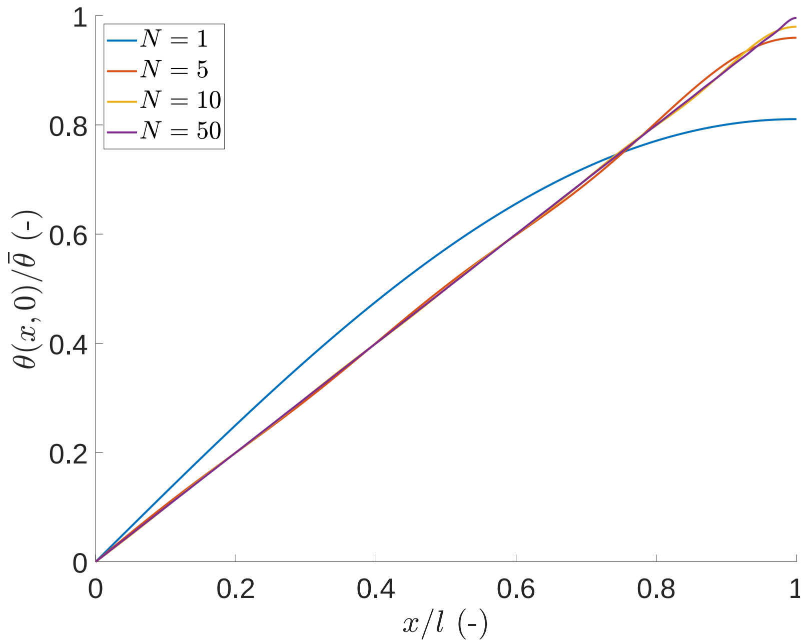


Figure 6: Solution verification at $t = 0$ for Problem 2 Question 3.

Problem 3 Solution – 30 points

The eigenvalues, natural frequencies, mode shapes, and modal masses are given by Eqs. (36) and (37).

Question 1 – 5 points

The ith modal force is given by

$$N_i(t) = \int_0^l \phi_i(x) r(x,t) dx = r_0 \sin \omega_0 t \int_0^l \sin\left[\frac{(2i-1)\pi x}{2l}\right] dx = \frac{2lr_0}{(2i-1)\pi} \sin \omega_0 t = N_{0_i} \sin \omega_0 t \tag{42}$$

where the amplitude is given by

$$N_{0_i} = \frac{2lr_0}{(2i-1)\pi} \tag{43}$$

The amplitudes of the modal forces are inversely proportional to i. However, the excitation frequency also plays a role in determining which how different modes contribute to the response.

Question 2 – 5 points

The damped steady-state response is given by

$$\begin{aligned} \theta(x,t) &= \sum_{i=1}^{\infty} \phi_i(x) \eta_i(t) \\ &= \sum_{i=1}^{\infty} \sin\left[\frac{(2i-1)\pi x}{2l}\right] N_{0_i} |H_i(\omega_i)| \sin\left[\omega_0 t - \varphi_i(\omega_i)\right] \\ &= \frac{2lr_0}{\pi} \sum_{i=1}^{\infty} \sin\left[\frac{(2i-1)\pi x}{2l}\right] \frac{|H_i(\omega_i)|}{(2i-1)} \sin\left[\omega_0 t - \varphi_i(\omega_i)\right] \end{aligned} \tag{44}$$

where the ith modal coordinate is given by

$$\eta_i(t) = N_{0_i} |H_i(\omega_i)| \sin\left[\omega_0 t - \varphi_i(\omega_i)\right] \tag{45}$$

The magnitude of the ith modal frequency response function is given by

$$|H_i(\omega_i)| = \frac{1}{m_i \sqrt{\left(\omega_i^2 - \omega_0^2\right)^2 + (2\zeta_i \omega_i \omega_0)^2}} \tag{46}$$

and the corresponding phase delay is given by

$$\varphi_i(\omega_i) = \tan^{-1}\left(\frac{2\zeta_i \omega_i \omega_0}{\omega_i^2 - \omega_0^2}\right) \tag{47}$$

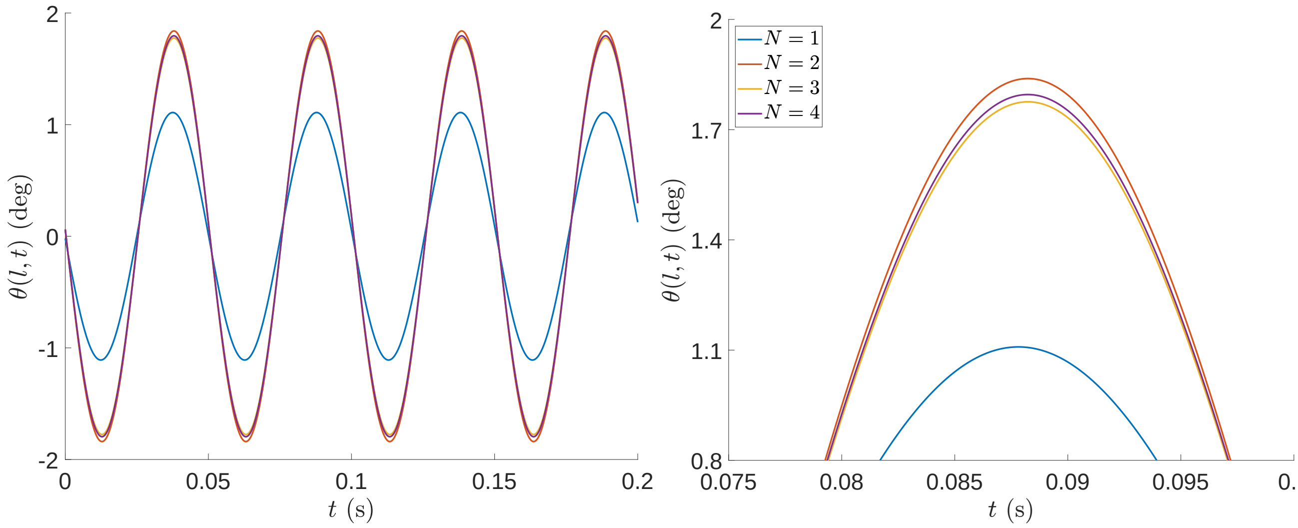
The modal frequency response function in Eq. (46) accounts for the non-unit modal mass in Eq. (37). Alternatively, we could have used the formula for unit modal masses by normalizing the mode shapes consistently. Remember that the phase delay must be computed using the inverse tangent function that can deal with complex numbers, and that the steady-state response has the same type of time dependency (sine or cosine function of time) as the excitation.

Question 3 – 5 points

Table 4 lists the quantities needed to evaluate Eq. (46) for the first four modes. The modal masses are constant for all modes and equal to $m_i = 0.0025$ kg m^2.

Table 4: Quantities needed to evaluate the solution for Problem 3 Question 2

i	α_i (1/m)	ω_i (rad/s)	N_{0_i} (Nm)	$\lvert H_i(\omega_i)\rvert$ (1/Nm)	$N_{0_i}\lvert H_i(\omega_i)\rvert$ (-)	$\varphi_i(\omega_i)$ (rad)	$\varphi_i(\omega_i)$ (deg)
1	1.5708	49.6729	0.6366	0.0304	0.0193	3.1227	178.9186
2	4.7124	149.0188	0.2122	0.0604	0.0128	0.1127	6.4589
3	7.8540	248.3647	0.1273	0.0087	0.0011	0.0270	1.5444
4	10.9956	347.7106	0.0909	0.0038	0.0003	0.0165	0.9461



(a) Complete amplitude range

(b) Zoom

Figure 7: Tip twist angle for increasing number of modes for Problem 3 Question 4.

Question 4 – 5 points

The tip twist angle is obtained by evaluating Eq. (44) at $x = l$, which gives

$$\theta(l,t) = \sum_{i=1}^{\infty} (-1)^{i+1} |H_i(\omega_i)|(2i-1) \sin\left[\omega_0 t - \varphi_i(\omega_i)\right] \tag{48}$$

Figure 7 shows the results obtained by truncating Eq. (48) to $N = 1, 2, 3, 4$ modes. Remember that this is a *steady-state* response, so even if it is plotted from $t = 0$ s, it must interpreted as the response after a long enough time that the transient phase has ended. Each curve in Fig. 7 (except the first one) is not the contribution from an individual mode, but the response obtained considering the first $N = 1, 2, 3, 4$ modes in Eq. (48).

Question 5 – 5 points

The convergence trend in Fig. 7 is explained considering that the excitation frequency $\omega_0 = 125$ rad/s is between the first two natural frequencies, $\omega_1 < \omega_0 < \omega_2$, and it is closer to ω_2 (see Table 4). This causes the solution for $N = 1$ (one-mode approximation) to miss the amplitude of the steady-state response. However, it captures the phase with relative accuracy because this is mainly influenced by the modal frequency response functions with $\omega_i < \omega_0$ (only the first one). The first and second modal coordinates contribute with amplitudes of the same order of magnitude. The third and fourth modal coordinates have amplitudes one and two orders of magnitude smaller, respectively. While the overall response amplitude changes moving from $N = 2$ (two-mode approximation) to $N = 3, 4$ (three- and four-mode approximations), the changes are slight. The convergence trend is not monotonic because the mode shapes alternate between ± 1 at $x = l$, such that the exact solution for $N \to \infty$ is between the two approximations obtained for N and $N + 1$ modes.

Note that these convergence trends are specific to this problem and not general rules. The convergence trend in a particular problem depends on the spatial distribution and frequency of the harmonic excitation, and on the modal characteristics of the structure.

Question 6 – 5 points

To avoid exciting the first mode, we must apply a load

$$r(x,t) = f(x)\sin\omega_0 t \tag{49}$$

with a spatial distribution $f(x)$ such that

$$N_1(t) = \int_0^l \phi_1(x) r(x,t) dx = 0 \tag{50}$$

Because the modal equations are decoupled, Eq. (50) guarantees that the *steady-state* response of the first modal coordinate is zero (for the overall response of the first modal coordinate to be zero, we must also have no initial conditions on that coordinate). Plugging Eq. (49) into Eq. (50), the spatial distribution $f(x)$ must satisfy

$$\int_0^l \phi_1(x) f(x) dx = 0 \tag{51}$$

One possible class of solutions is

$$f(x) = f_0\,\phi_i(x) \qquad i \neq 1 \tag{52}$$

This corresponds to a load distribution proportional to any other mode shape in Eq. (36) than the first one. This class of solutions builds on the orthogonality conditions for a *uniform* beam in torsion, which can be written as

$$\rho I_p \int_0^l \phi_i(x)\phi_j(x) dx = \delta_{ij} m_i \qquad (i, j = 1\ldots,\infty) \tag{53}$$

If the beam had non-uniform inertia properties, the orthogonality conditions become

$$\int_0^l [\rho I_p(x)]\phi_i(x)\phi_j(x) dx = \delta_{ij} m_i \qquad (i, j = 1\ldots,\infty) \tag{54}$$

where $\rho I_p(x)$ is the polar moment of inertia per unit length. In this situation, we should weight the load distribution with the inertia properties to satisfy Eq. (51) using orthogonality, which would require changing Eq. (52) to

$$f(x) = f_0[\rho I_p(x)]\phi_i(x) \qquad i \neq 1 \tag{55}$$

Additionally, note that the mode shapes are no longer given by Eq. (36) in the case of non-uniform properties.

Problem 4 Solution – 25 points

For this problem, we use the eigenvalues, natural frequencies, and mode shapes for a pinned-pinned uniform beam undergoing out-of-plane bending

$$\alpha_i = \frac{i\pi}{l}$$

$$\omega_i = \frac{(i\pi)^2}{l^2}\sqrt{\frac{EI}{\rho A}} \tag{56}$$

$$\phi_i(x) = \sin\left(\frac{i\pi x}{l}\right)$$

with $i = 1, \ldots, \infty$. The modal masses are given by

$$m_i = \rho A \int_0^l \phi_i^2(x)dx = \rho A \frac{l}{2} \tag{57}$$

Question 1 – 5 points

The applied concentrated load can be written as a distributed load using a Dirac's delta function in space

$$f(x,t) = F(t)\,\delta(x - x_0) = F_0\,\delta(t)\,\delta(x - x_0) \tag{58}$$

not to be confused with the Dirac's delta function *in time* $\delta(t)$ that describes the impulsive nature of the excitation. The ith modal force is given by

$$N_i(t) = F_0\,\delta(t)\int_0^l \phi_i(x)\,\delta(x - x_0)\,dx = F_0\,\phi_i(x_0)\,\delta(t) = N_{0_i}\delta(t) \tag{59}$$

where the amplitude is given by

$$N_{0_i} = F_0 \sin\left(\frac{i\pi x_0}{l}\right) \tag{60}$$

Because $x_0 = l/2$ in this problem, we have

$$N_{0_i} = F_0 \sin\left(\frac{i\pi}{2}\right) \tag{61}$$

Based on this result, the even modes ($i = 2, 4, \ldots, \infty$) do not contribute to the response. This is expected because the even modes are anti-symmetric with respect to the midpoint (force application point).

Question 2 – 5 points

Because all the modal forces are impulse-type excitations, the undamped forced response is given by

$$v(x,t) = \sum_{i=1}^{\infty}\phi_i(x)\eta_i(t) = \sum_{i=1,3,\ldots}^{\infty}\sin\left(\frac{i\pi x}{l}\right)N_{0_i}h_i(t) \tag{62}$$

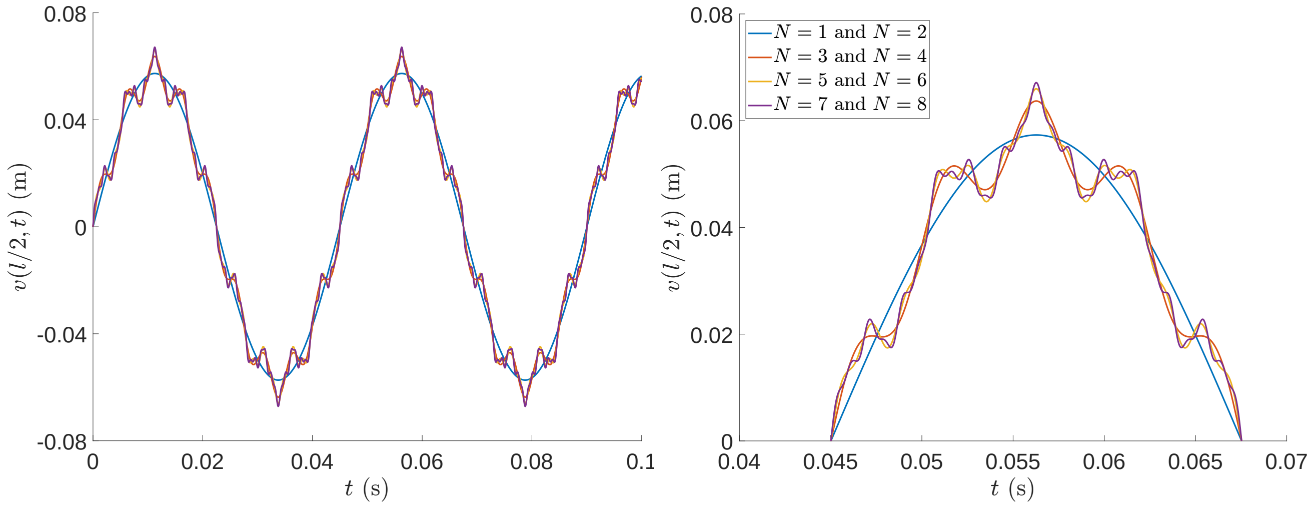
where the ith modal coordinate is given by

$$\eta_i(t) = N_{0_i}h_i(t) \tag{63}$$

and

$$h_i(t) = \frac{\sin\omega_i t}{m_i\omega_i} \tag{64}$$

is the ith modal unit impulse response function, with modal masses and natural frequencies given by Eq. (56). Equation (62) only includes the contributions from the odd modes because the other ones are zero. Note that there is no real need to use the convolution integral here because the impulse response function is, by its definition, the response to a unit impulse excitation. In fact, the convolution integral can be interpreted as an application of the properties of linear and time-invariant system to write their forced responses (with zero initial conditions) as a superposition of scaled and shifted unit impulse responses.



(a) Complete amplitude range

(b) Zoom

Figure 8: Midpoint vertical displacement for increasing number of modes for Problem 4 Question 4.

Question 3 – 5 points

Table 5 lists the quantities needed to evaluate Eq. (46) for the first eight modes. The terms associated with the even modes are omitted because these modes do not contribute to the response. The modal masses are constant for all modes and equal to $m_i = 0.1250$ kg m^2.[4]

Table 5: Quantities needed to evaluate the solution for Problem 3 Question 2

i	α_i (1/m)	ω_i (rad/s)	N_{0_i} (Nm)	$N_{0_i}/(m_i\omega_i)$ (-)
1	3.1416	139.5773	1.0000	0.0573
3	9.4248	1256.1956	−1.0000	−0.0064
5	15.7080	3489.4321	1.0000	0.0023
7	21.9911	6839.2869	−1.0000	−0.0012

Question 4 – 5 points

The vertical displacement at the midpoint is obtained by evaluating Eq. (44) at $x = l/2$, which gives

$$v(l/2,t) = \sum_{i=1}^{\infty} \phi_i(l/2)\eta_i(t) = \sum_{i=1,3,\ldots}^{\infty} \sin\left(\frac{i\pi}{2}\right) N_{0_i} h_i(t) \tag{65}$$

Figure 8 shows the results obtained by truncating Eq. (48) to $N = 1, 3, 5, 7$ modes (the results for $N = 2, 4, 6, 8$ are the same). Note that each curve in Fig. 8 except the first one is not the contribution from a mode, but the response resulting from considering the first N modes in Eq. (48).

Question 5 – 5 points

The convergence trend in Fig. 8 can be explained considering the impulsive nature of the excitation, which makes all modes be active in the response. This makes the response exhibit higher frequency content as the number of retained modes increases. Because the modal masses are the same for all modes and the modal forces also have the same magnitude for all modes, the contributions from higher-order odd mode shapes decrease like $1/\omega_i$ for increasing i. The even mode shapes do not contribute to the response because $x_0 = l/2$ is a node for these mode shapes.

[4] Units of the modal masses are determined assuming to keep the units of the displacement field with the mode shapes, which requires the modal coordinates non-dimensional. Alternatively, we could keep the units with modal coordinate, which requires the mode shapes to be non-dimensional and changes the units of the other quantities accordingly. In the end, the displacement field must have units of meters.

Note that, in theory, an impulsive excitation activates all modes. However, in the practice, even the fastest excitation would be applied for a finite amount of time, resulting in a cutoff for the modes involved in the response. Additionally, in the presence of damping, the contributions from higher-order modes would likely decay quickly due to their higher frequencies (assuming a constant modal viscous damping factor for all modes, which is usually a reasonable approximation for many structural dynamics problems).