College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 4
Linear Algebra

Author:
Tomoki Koike

Supervisor:
Martin Corless

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Purdue University
West Lafayette, Indiana

Exercise 1 Determine (by hand) whether or not the following system of linear equations has a solution. If a solution exists, determine whether or not it is unique, and if not unique, obtain an expression for all solutions.

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & x_4 & = & 5 \\ x_1 & - & x_2 & + & x_3 & & & = & 3 \\ -2x_1 & + & 2x_2 & & & + & 2x_4 & = & -2 \\ 2x_1 & - & 2x_2 & - & x_3 & - & 3x_4 & = & 0 \end{array}$$

Do some Gaussian Elimination

$$\begin{pmatrix} 1 & -1 & 2 & 1 & 5 \\ 1 & -1 & 1 & 0 & 3 \\ -2 & 2 & 0 & 2 & -2 \\ 2 & -2 & -1 & -3 & 0 \end{pmatrix}$$

Swap rows: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} -2 & 2 & 0 & 2 & -2 \\ 1 & -1 & 1 & 0 & 3 \\ 1 & -1 & 2 & 1 & 5 \\ 2 & -2 & -1 & -3 & 0 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \to R_2 + \frac{1}{2}R_1$

$$\sim \begin{pmatrix} -2 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 1 & -1 & 2 & 1 & 5 \\ 2 & -2 & -1 & -3 & 0 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \to R_3 + \frac{1}{2}R_1$

$$\sim \begin{pmatrix} -2 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 4 \\ 2 & -2 & -1 & -3 & 0 \end{pmatrix}$$

Cancel leading column in row 4: $R_4 \to R_4 + R_1$

$$\sim \begin{pmatrix} -2 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & -1 & -1 & -2 \end{pmatrix}$$

Cancel leading column in row 4: $R_4 \to R_4 + R_2$

$$\sim \begin{pmatrix} -2 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel R_3: $R_3 \rightarrow R_3 - 2R_2$

$$\sim \begin{pmatrix} -2 & 2 & 0 & 2 & -2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Divide common factors for columns in R_1: $R_1 \rightarrow R_1/(-2)$

$$\sim \begin{pmatrix} 1 & -1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For this reduced echelon form, we can tell that the linear system equations have an infinite number of solutions and not unique.

Hence, the expression of all solutions become the following

$$\boxed{\begin{pmatrix} x_1 - x_2 - x_4 & = 1 \\ x_3 + x_4 & = 2 \end{pmatrix}}$$

Now, let $x_2 = s$ and $x_4 = t$

$$\begin{pmatrix} x_1 & = & s + t + 1 \\ x_3 & = & 2 - t \end{pmatrix}$$

Thus,

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}}$$

Exercise 2 We consider here systems of linear equations of the form

$$Ax = b$$

$$\text{with} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For each of the following cases, we present the reduced row echelon form of the augmented matrix $[A\,b]$. Determine (by hand) whether or not the corresponding system of linear equations has a solution. If a solution exists, determine whether or not it is unique; if not unique, obtain an expression for all solutions and give two solutions.

$$\text{(a)} \begin{pmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{(b)} \begin{pmatrix} 1 & -2 & 0 & 6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(c)} \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(d)} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) This matrix does not have a solution because the last row has a leading column in the last column of the matrix and zero cannot be equal to one.

(b) The second matrix has a solution but has an infinite number of solutions and not unique.

$$\begin{pmatrix} x_1 - 2x_2 & = 6 \\ x_3 & = 4 \end{pmatrix}$$

Let, $x_2 = s$. Then,

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 6 \\ 0 \\ 4 \end{pmatrix}}$$

Here, x_1 and x_2 may be anything as long as they satisfy the first equation above.

(c) For this matrix, there is no solution because the third row has a leading column at the last column and 0 = 1 is impossible.

(d) The fourth matrix also has no solution since it the last row has a leading column at the last column. And 0 = 1 is impossible.

Exercise 3 (By hand) Consider the three vectors

$$v_1 = \begin{pmatrix}1\\2\\3\end{pmatrix} \quad v_2 = \begin{pmatrix}2\\3\\4\end{pmatrix} \quad v_3 = \begin{pmatrix}3\\4\\5\end{pmatrix}$$

(a) Express the vector

$$v = \begin{pmatrix}4\\5\\6\end{pmatrix}$$

as a linear combination of $\boldsymbol{v_1}$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$.

(b) Can every vector of the form

$$v = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

be expressed as a linear combination of $\boldsymbol{v_1}$, $\boldsymbol{v}_2$, and $\boldsymbol{v}_3$?

(c) Are the vectors v_1, v_2, and v_3 linearly independent?

(a) .

This is the same as

$$\begin{pmatrix}1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\5\\6\end{pmatrix}$$

Thus, conduct Gaussian Elimination on the following augmented matrix

$$\begin{pmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 5\\3 & 4 & 5 & 6\end{pmatrix}$$

Swap rows: $R_1 \leftrightarrow R_3$

$$\sim\begin{pmatrix}3 & 4 & 5 & 6\\2 & 3 & 4 & 5\\1 & 2 & 3 & 4\end{pmatrix}$$

Cancel leading column in row 2: $R_2 \rightarrow R_2 - \frac{2}{3} \cdot R_1$

$$\sim\begin{pmatrix}3 & 4 & 5 & 6\\0 & 1/3 & 2/3 & 1\\1 & 2 & 3 & 4\end{pmatrix}$$

Cancel leading column in row 3: $R_3 \to R_3 - \frac{1}{3} \cdot R_1$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 2/3 & 4/3 & 2 \end{pmatrix}$$

Cancel row 3: $R_3 \to R_3 - 2 \cdot R_2$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Divide row 1 with 3 and multiply row 2 by 3:

$$\sim \begin{pmatrix} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel the second column of row 1: $R_1 \to R_1 - \frac{4}{3} R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This indicates that there is an infinite number of solutions for x, y, and z.

$$\begin{aligned} x + -z &= -2 \\ y + 2z &= 3 \end{aligned}$$

So, we will choose,

$$\begin{aligned} z &= 1 \\ y &= 3 - 2 \cdot 1 = 1 \\ x &= -2 + 1 = -1 \end{aligned}$$

Hence,

$$\boxed{v = -v_1 + v_2 + v_3}$$

(b) .

To figure out what the span of these three vectors are we conduct a Gaussian Elimination on the following matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

We have already done this in part (a). We simply end up with the same kind of matrix but disregarding the 4th column in our calculations of part (a). Thus, the after Gaussian Elimination, we end up with the following

$$\sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We see that there are 2 non-zero rows in this final matrix (or 2 pivots). Thus, we can say that the three vectors v_1, v_2, and v_3 span a plane in R^3. Thus, any arbitrary vector, v in R^3 cannot be expressed in terms of v_1, v_2, and v_3.

(c).

Take the determinant of the matrix

$$det\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$= (3 \cdot 5 - 4 \cdot 4) - 2(2 \cdot 5 - 3 \cdot 4) + 3(2 \cdot 4 - 3 \cdot 3) \;=\; -1 + 4 - 3 \;=\; 0$$

Also, from the calculations in part (b), we know that

$$v_3 = -v_1 + 2v_2$$

Since the determinant is zero and one of the three vectors can be expressed by the other two vectors we know that the 3 vectors are NOT linearly independent.

Exercise 4 (By hand)

(a) Find a basis for the **null space** of the matrix,

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

(b) What is the nullity of A?

Check your answers using MATLAB.

(a).

First, conduct Gaussian Elimination on the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

Swap rows: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \rightarrow R_2 - \frac{2}{3} \cdot R_1$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \rightarrow R_3 - \frac{1}{3} \cdot R_1$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 2/3 & 4/3 & 2 \end{pmatrix}$$

Cancel row 3: $R_3 \rightarrow R_3 - 2 \cdot R_2$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Divide row 1 with 3 and multiply row 2 by 3:

$$\sim \begin{pmatrix} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel the second column of row 1: $R_1 \rightarrow R_1 - \frac{4}{3}R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider the solutions for $Ax = 0$.

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

Since, the leading columns do not show up in the 3rd and 4th column we know that x_3 and x_4 are free variables. Thus, let $x_3 = s$ and $x_4 = t$. Then x can be expressed as

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix}$$

Thus, the basis for the null space of A is

$$\boxed{\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}}$$

(b).

The nullity, or in other words,

$$\boxed{dim(\mathcal{N}(A)) = 2}$$

MATLAB verification

```
A = [1,2,3,4;2,3,4,5;3,4,5,6]
E_A = rref(A)
N_A = null(A,'r')
```

```
A = 3×4
     1     2     3     4
     2     3     4     5
     3     4     5     6
```

```
E_A = 3×4
     1     0    -1    -2
     0     1     2     3
     0     0     0     0
```

```
N_A = 4×2
     1     2
    -2    -3
     1     0
     0     1
```

Exercise 5 (By hand)

(a) Find a basis for the **range** of the matrix,

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}.$$

(b) What is the rank of A?

Check your answers using MATLAB.

(a).

First, conduct Gaussian Elimination on the following matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$$

Swap rows: $R_1 \leftrightarrow R_3$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Cancel leading column in row 2: $R_2 \to R_2 - \frac{2}{3} \cdot R_1$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Cancel leading column in row 3: $R_3 \to R_3 - \frac{1}{3} \cdot R_1$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 2/3 & 4/3 & 2 \end{pmatrix}$$

Cancel row 3: $R_3 \to R_3 - 2 \cdot R_2$

$$\sim \begin{pmatrix} 3 & 4 & 5 & 6 \\ 0 & 1/3 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Divide row 1 with 3 and multiply row 2 by 3:

$$\sim \begin{pmatrix} 1 & 4/3 & 5/3 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Cancel the second column of row 1: $R_1 \to R_1 - \frac{4}{3}R_2$

$$\sim \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The first and second column has a leading column of 1. Thus, the basis of range of A are the first and second column of matrix A,

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \right\}.}$$

(b).

The rank of the matrix A is

$$\boxed{dim\big(\mathcal{R}(A)\big) = 2}$$

MATLAB verification

```
R_A = rank(A)
```

```
R_A = 2
```

Exercise 6 Consider a 5 x 5 matrix

$$A = vw^T$$

where v and w are 5 x 1 matrices.

(a) What is the rank of A?
(b) What is the nullity of A?

(a).

Let matrices v and w be

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix}.$$

Then, $A = vw^T$ becomes

```
A =
```

$$\begin{pmatrix} v_1 w_1 & v_1 w_2 & v_1 w_3 & v_1 w_4 & v_1 w_5 \\ v_2 w_1 & v_2 w_2 & v_2 w_3 & v_2 w_4 & v_2 w_5 \\ v_3 w_1 & v_3 w_2 & v_3 w_3 & v_3 w_4 & v_3 w_5 \\ v_4 w_1 & v_4 w_2 & v_4 w_3 & v_4 w_4 & v_4 w_5 \\ v_5 w_1 & v_5 w_2 & v_5 w_3 & v_5 w_4 & v_5 w_5 \end{pmatrix}$$

Now, performing the Gaussian Elimination, we obtain the following outcome

```
E_A =
```

$$\begin{pmatrix} 1 & \frac{w_2}{w_1} & \frac{w_3}{w_1} & \frac{w_4}{w_1} & \frac{w_5}{w_1} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We know that there is only one column with a leading column of 1 in the row reduced echelon form of A. Hence,

$$\mathcal{R}(A) = \begin{pmatrix} v_1 w_1 \\ v_2 w_1 \\ v_3 w_1 \\ v_4 w_1 \\ v_5 w_1 \end{pmatrix}.$$

Therefore,

$$\boxed{dim\big(\mathcal{R}(A)\big) = 1}$$

(b).

Since,

$$dim\big(\mathcal{R}(A)\big) + dim\big(\mathcal{N}(A)\big) = number\ of\ columns$$

$$dim\big(\mathcal{R}(A)\big) + dim\big(\mathcal{N}(A)\big) = 5$$

$$\boxed{\therefore dim\big(\mathcal{N}(A)\big) = 5 - 1 = 4}$$

MATLAB verification

```
vv = sym('v',[5,1]); ww = sym('w',[5,1]);
assume(ww,'real')
A = vv*ww'
E_A = rref(A)
R_A = rank(A)
N_A = null(A)
```

Exercise 7 Suppose $\mathcal{U}$ and $\mathcal{W}$ are subspaces of a vector space $\mathcal{V}$. Prove or disprove (by counterexample) the following statements.

(a) $\mathcal{U} \cap \mathcal{W}$ is a subspace of $\mathcal{V}$.
(b) $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$.

(a).

Theorem. To say that B is a subspace of A.

1. B is a subset of A.
2. $0 \in B$,
3. For all $x, y \in B$ it holds that $f(x,y) \in B$, and
4. For all $c \in \mathcal{R}$ and $x \in B$ it holds that $g(c,x) \in B$.

We follow the theorem above.

1. We know that $\mathcal{U}$ and $\mathcal{W}$ are subspaces of a vector space $\mathcal{V}$
2. Since $\mathcal{U}$ and $\mathcal{W}$ are subspaces of $\mathcal{V}$ we know that $0 \in \mathcal{U}$ and $0 \in \mathcal{W}$ and therefore $0 \in \mathcal{U} \cap \mathcal{W}$
3. Let, $u, v \in \mathcal{U} \cap \mathcal{W}$. Then we have that $u, v \in \mathcal{U}$ and $u, v \in \mathcal{W}$ individually. This shows that $u + v \in \mathcal{U}$ and $u + v \in \mathcal{W}$. Therefore, we have $u + v \in \mathcal{U} \cap \mathcal{W}$
4. Let $u \in \mathcal{U} \cap \mathcal{W}$ and c be a scalar value. If we have $u \in \mathcal{U}$ and $u \in \mathcal{W}$ it indicates that $cu \in \mathcal{U}$ and $cu \in \mathcal{W}$, and therefore, $cu \in \mathcal{U} \cap \mathcal{W}$. This is true because $\mathcal{U}$ and $\mathcal{W}$ are subspaces of a vector space $\mathcal{V}$.

Since, all 4 of the criteria in the theorem above is satisfied, the statement " $\mathcal{U} \cap \mathcal{W}$ is a subspace of $\mathcal{V}$" is true.

$q.e.d$

(b).

Say $\mathcal{U} \not\subset \mathcal{W}$ and $\mathcal{W} \not\subset \mathcal{U}$. Also, there is a vector u that suffices $u \in \mathcal{U}$ but $u \notin \mathcal{W}$. There is another vector w that suffices $w \in \mathcal{W}$ but $w \notin \mathcal{U}$.

Seeking for a contradiction, let us assume that the union, $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$. The vectors u, w lie in the vector space $\mathcal{U} \cup \mathcal{W}$. Thus, there sum $u + w$ is also in $\mathcal{U} \cup \mathcal{W}$. This implies that we have either

$$u + w \in \mathcal{U} \quad or \quad u + w \in \mathcal{W}$$

If $u + w \in \mathcal{U}$, then there exists a vector $u' \in \mathcal{U}$ such that

$$u + w = u'.$$

Since $u \in \mathcal{U}$ and $u' \in \mathcal{U}$, then the difference $u' - u$ is also in $\mathcal{U}$. Thus,

$$w = u' - u \in \mathcal{U}\,.$$

However, this contradicts with our assumption $w \notin \mathcal{U}$.

Thus, we must have $u + w \in \mathcal{W}$.

In this case, there exists a $w' \in \mathcal{W}$ such that

$$u + w = w'\,.$$

Since both w, w' are vectors of $\mathcal{W}$ it should follow

$$u = w' - w \in \mathcal{W}$$

which happens to contradict with $u \notin \mathcal{W}$.

Hence, we can say that "$\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$" is false when $\mathcal{U} \not\subset \mathcal{W}$ and $\mathcal{W} \not\subset \mathcal{U}$.

$q.e.d$