Non-orthogonal bases: Finite dimensions

A non-orthogonal basis is simply a basis where the elements are not orthogonal to one another, but are linearly independent.

In a finite, N-dimensional space $\mathcal{S}$, any collection of N linearly independent vectors is a basis. Given such a set of vectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$, any $\boldsymbol{x} \in \mathcal{S}$ can be written

$$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{v}_n, \tag{1}$$

for some set of coefficients $\boldsymbol{\alpha} \in \mathbb{R}^N$. We know already that the mapping from the vector $\boldsymbol{x} \in \mathcal{S}$ to the vector $\boldsymbol{\alpha} \in \mathbb{R}^N$ is a bijection — each vector in $\mathcal{S}$ has a different set of N expansion coefficients, and this set of coefficients is unique. What we want in addition to this is stability in this mapping. Small changes in the expansion coefficients should not lead to large changes in the re-synthesized function. We also want a concrete method for calculating the coefficients in a basis expansion.

With an inner product $\langle \cdot, \cdot \rangle$ defined on $\mathcal{S}$ (and induced norm $\|\cdot\|$), we can measure the stability of a basis using the Gram matrix. To see this, let $\boldsymbol{x} \in \mathcal{S}$ have an expansion as in (1). We can now write the induced norm of $\boldsymbol{x}$ as

$$\begin{aligned}
\|\boldsymbol{x}\|^2 &= \langle \sum_{n=1}^{N} \alpha_n \boldsymbol{v}_n, \sum_{m=1}^{N} \alpha_m \boldsymbol{v}_m \rangle \\
&= \sum_{n-1}^{N} \sum_{m=1}^{N} \alpha_n \alpha_m \langle \boldsymbol{v}_n, \boldsymbol{v}_m \rangle \\
&= \boldsymbol{\alpha}^{\mathrm{T}} \boldsymbol{G} \boldsymbol{\alpha},
\end{aligned}$$

where

$$\boldsymbol{G} = \begin{bmatrix} \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle & \langle \boldsymbol{v}_2, \boldsymbol{v}_1 \rangle & \cdots & \langle \boldsymbol{v}_N, \boldsymbol{v}_1 \rangle \\ \langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle & \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle & \cdots & \langle \boldsymbol{v}_N, \boldsymbol{v}_2 \rangle \\ \vdots & & \ddots & \\ \langle \boldsymbol{v}_1, \boldsymbol{v}_N \rangle & \cdots & & \langle \boldsymbol{v}_N, \boldsymbol{v}_N \rangle \end{bmatrix}.$$

As we will see in the next segment of this course, since $\boldsymbol{G}$ is symmetric, we have for any $\boldsymbol{\alpha} \in \mathbb{R}^N$,

$$A\|\boldsymbol{\alpha}\|_2^2 \;\leq\; \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\alpha} \;\leq\; B\|\boldsymbol{\alpha}\|_2^2,$$

where A is the smallest eigenvalue of $\boldsymbol{G}$ and B is the largest eigenvalue of $\boldsymbol{G}$ (we will also see that symmetric matrices always have real eigenvalues). So we have

$$A\left(\sum_{n=1}^{N}\alpha_n^2\right) \;\leq\; \underbrace{\left\|\sum_{n=1}^{N}\alpha_n\boldsymbol{v}_n\right\|^2}_{\|\boldsymbol{x}\|^2} \;\leq\; B\underbrace{\left(\sum_{n=1}^{N}\alpha_n^2\right)}_{\|\boldsymbol{\alpha}\|_2^2},$$

Note that we have the norm induced by the inner product (squared) on the inside above, and the standard ℓ_2 norm (squared) on the outside above. The inequality is tight on both sides, there are $\boldsymbol{x}$ (with associated $\boldsymbol{\alpha}$) that achieve equality for each of the upper and lower bounds. Also note that when $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ is an orthobasis, then $\boldsymbol{G} = \mathbf{I}$ and we have $A = B = 1$ and so the above reduces to the Parseval theorem. We can also reverse the inequalities above; for a given $\boldsymbol{\alpha} \in \mathbb{R}^N$,

$$\frac{1}{B}\left\|\sum_{n=1}^{N}\alpha_n\boldsymbol{v}_n\right\|^2 \;\leq\; \underbrace{\sum_{n=1}^{N}\alpha_n^2}_{\|\boldsymbol{\alpha}\|_2^2} \;\leq\; \frac{1}{A}\underbrace{\left\|\sum_{n=1}^{N}\alpha_n\boldsymbol{v}_n\right\|^2}_{\|\boldsymbol{x}\|^2},$$

where again $\boldsymbol{x} = \sum_n \alpha_n \boldsymbol{v}_n$.

What does this tell us about stability? Well, if we have two distinct vectors in $\boldsymbol{\mathcal{S}}$, they have distinct basis coefficients,

$$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{v}_n, \qquad \tilde{\boldsymbol{x}} = \sum_{n=1}^{N} \tilde{\alpha}_n \boldsymbol{v}_n,$$

then applying the above, we have

$$\frac{1}{B}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^2 \;\leq\; \sum_{n=1}^{N} (\alpha_n - \tilde{\alpha}_n)^2 \;\leq\; \frac{1}{A}\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|^2$$

So if A is small, even minor differences between $\boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\alpha}}$ (in ℓ_2 norm) can blow up into huge differences between $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ (in induced norm)[1]. Likewise, if B is large tiny differences between $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ (in induced norm) can amount to large differences between the coefficient expansions.

Example: Polynomials on $[0, 1]$. Consider the space of 5th order polynomials equipped with the standard L_2 inner product with basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_6\}$ where $v_n(t) = t^{n-1}$. The Gram matrix entries are

$$G_{n,m} = \int_0^1 t^{n-1} t^{m-1}\, dt = \frac{1}{n+m-1},$$

[1]We are making a point of keeping this discussion general by carefully distinguishing between the ℓ_2 norm in coefficient space, and the induced norm in vector (e.g. function) space. The qualitative lesson is the same, though, if the induced norm is something standard like L_2

so

$$\boldsymbol{G} = \begin{bmatrix} 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 1/2 & 1/3 & 1/4 & 1/5 & 1/6 \\ 1/3 & 1/4 & 1/5 & 1/6 & 1/7 \\ 1/4 & 1/5 & 1/6 & 1/7 & 1/8 \\ 1/5 & 1/6 & 1/7 & 1/8 & 1/9 \end{bmatrix}.$$

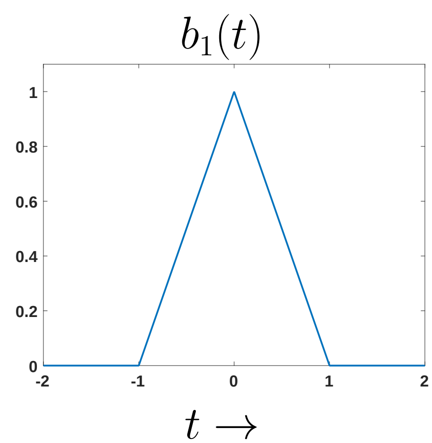
The maximum eigenvalue of $\boldsymbol{G}$ is $B = 1.6189$, while the minimum eigenvalue is $A = 1.082 \cdot 10^{-7}$. Thus the "standard" representation of polynomials is very unstable. (See also the discussion at the beginning of Notes 6.)

Example: Linear splines. Let $\mathcal{S}$ be the $N = 6$ dimensional space of piecewise linear functions $f(t)$ with the following properties

1. $f(t)$ is zero outside of the interval $-1 \leq t \leq 6$,
2. $f(t)$ is linear between the integers,
3. $f(t)$ is continuous at the integers,
4. $f(-1) = 0$ and $f(6) = 0$.

This space is spanned by $\{b_1(t-k),\ k = 0, \ldots, 5\}$, where

$$b_1(t) = \begin{cases} t+1, & -1 \leq t \leq 0, \\ 1-t, & 0 \leq t \leq 1, \\ 0, & \text{otherwise} \end{cases}$$



The entries in the Gram matrix $\boldsymbol{G}$ can be computed using

$$G_{m,n} = \int_{-1}^{6} b_1(t-m) b_1(t-n)\, dt,$$

and so

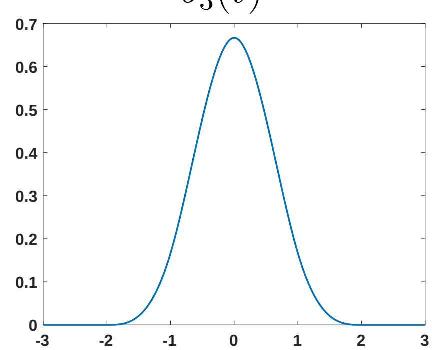
$$\boldsymbol{G} = \begin{bmatrix} 2/3 & 1/6 & 0 & 0 & 0 & 0 \\ 1/6 & 2/3 & 1/6 & 0 & 0 & 0 \\ 0 & 1/6 & 2/3 & 1/6 & 0 & 0 \\ 0 & 0 & 1/6 & 2/3 & 1/6 & 0 \\ 0 & 0 & 0 & 1/6 & 2/3 & 1/6 \\ 0 & 0 & 0 & 0 & 1/6 & 2/3 \end{bmatrix}$$

The smallest eigenvalues of $\boldsymbol{G}$ is $A = 0.366$ and the largest eigenvalue is $B = 0.967$. So the $\{b_1(t-k)\}$ provide a very stable way to represent functions in this space.

Example: Cubic splines. Let $b_3(t)$ be the cubic spline

$$b_3(t) = \begin{cases} \frac{t^3}{6} + t^2 + 2t + \frac{4}{3} & -2 \le t \le -1 \\ -\frac{t^3}{2} - t^2 + \frac{2}{3} & -1 \le t \le 0 \\ \frac{t^3}{2} - t^2 + \frac{2}{3} & 0 \le t \le 1 \\ -\frac{t^3}{6} + t^2 - 2t + \frac{4}{3} & 1 \le t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

$b_3(t)$



$t \rightarrow$

Consider $\mathcal{S} = \text{Span}(\{b_3(t-k),\ k = 0, \ldots, 5\}$. This is a $N = 6$ dimensional space of functions $f(t)$ with the following properties

1. $f(t)$ is zero outside of $-2 \le t \le 7$,
2. $f(t)$ is a cubic polynomial between the integers,

3. $f(t)$ is continuous and has two continuous derivatives at the integers,
4. $f'(-1) = 3f(-1)$, and $f'(6) = -3f(6)$,
5. $f'(-2) = 0, f''(2) = 0, f'(7) = 0, f''(7) = 0$.

The last two conditions come from non-obvious properties of $b_3(t)$, but they are straightforward to verify. In this case, the entries of the Gram matrix $\boldsymbol{G}$ can be computed using

$$G_{m,n} = \int_{-2}^{7} b_3(t-m)b_3(t-n)\, dt,$$

and so (after a little tedious calculation or numerical integration).

$$\boldsymbol{G} = \begin{bmatrix} 0.4794 & 0.2363 & 0.0238 & 0.0002 & 0 & 0 \\ 0.2363 & 0.4794 & 0.2363 & 0.0238 & 0.0002 & 0 \\ 0.0238 & 0.2363 & 0.4794 & 0.2363 & 0.0238 & 0.0002 \\ 0.0002 & 0.0238 & 0.2363 & 0.4794 & 0.2363 & 0.0238 \\ 0 & 0.0002 & 0.0238 & 0.2363 & 0.4794 & 0.2363 \\ 0 & 0 & 0.0002 & 0.0238 & 0.2363 & 0.4794 \end{bmatrix}$$

The smallest eigenvalue of $\boldsymbol{G}$ is $A = 0.086$ and the largest eigenvalue is $B = 0.938$. So the shifts of $b_3(t)$ are a reasonably stable representation of functions in this space.

Computing expansion coefficients

If we have an orthobasis for a space, not only is the mapping from vector space to coefficient space perfectly stable, there is also a concrete way to compute the expansion coefficients: you simply take the inner product of the vector you are representing with the corresponding basis function.

The coefficients for a nonorthogonal basis can be computed by taking the inner product with *dual basis* functions. Suppose we are given $\boldsymbol{x}$ in an inner product space $\mathcal{S}$ with basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N$. We know we can write $\boldsymbol{x}$ as

$$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{v}_n, \quad \text{where} \quad \boldsymbol{\alpha} = \boldsymbol{G}^{-1} \begin{bmatrix} \langle \boldsymbol{x}, \boldsymbol{v}_1 \rangle \\ \vdots \\ \langle \boldsymbol{x}, \boldsymbol{v}_N \rangle \end{bmatrix},$$

as of course $\boldsymbol{x}$ is the closest point to itself in $\mathcal{S}$. If we let $\boldsymbol{H} = \boldsymbol{G}^{-1}$ be the inverse of the Gram matrix, then we can write a particular α_n as

$$\alpha_n = \sum_{\ell=1}^{N} H_{n,\ell} \langle \boldsymbol{x}, \boldsymbol{v}_\ell \rangle = \left\langle \boldsymbol{x}, \sum_{\ell=1}^{N} H_{n,\ell} \boldsymbol{v}_\ell \right\rangle,$$

meaning

$$\alpha_n = \langle \boldsymbol{x}, \tilde{\boldsymbol{v}}_n \rangle, \quad \text{where} \quad \tilde{\boldsymbol{v}}_n = \sum_{\ell=1}^{N} H_{n,\ell} \boldsymbol{v}_\ell.$$

The collections of vectors $\{\tilde{\boldsymbol{v}}_1, \ldots, \tilde{\boldsymbol{v}}_N\}$ is called the **dual basis**, and since we already know $\boldsymbol{H}$ is invertible, we have

$$\operatorname{Span}(\{\tilde{\boldsymbol{v}}_1, \ldots, \tilde{\boldsymbol{v}}_N\}) = \operatorname{Span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}) = \mathcal{S}.$$

Example: Let $\mathcal{S}$ be the space of all second-order polynomials on $[0, 1]$. Set

$$v_1(t) = 1, \quad v_2(t) = t, \quad v_3(t) = t^2.$$

Then

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix},$$

and

$$\tilde{v}_1(t) = 30t^2 - 36t + 9, \quad \tilde{v}_2(t) = -180t^2 + 192t - 36, \quad \tilde{v}_3(t) = 180t^2 - 180t + 30$$

Non-orthogonal basis in infinite dimensions: Riesz Bases

Many of the concepts above translate directly to Hilbert spaces with infinite dimensions. The following definition extends the notion of a stable basis, where we can go back and forth between the vectors in the space and the coefficient sequence with confidence.

Definition. We say[2] $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ is a **Riesz basis** for Hilbert space $\mathcal{S}$ if

$$\operatorname{cl}\operatorname{Span}\left(\{\boldsymbol{v}_n\}_{n=1}^{\infty}\right) = \mathcal{S},$$

and there exists constants $A, B > 0$ such that[3]

$$A \sum_{n=1}^{\infty} \alpha_n^2 \;\leq\; \left\| \sum_{n=1}^{\infty} \alpha_n \boldsymbol{v}_n \right\|^2 \;\leq\; B \sum_{n=1}^{\infty} \alpha_n^2 \tag{2}$$

uniformly for all sequences $\{\alpha_n\}$ with $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

While in finite dimensions, the fact that the basis functions were linearly independent was enough to guarantee that the existence of $A > 0$ above, there is absolutely no guarantee that this is true in infinite dimensions; it is possible that we have a infinite set of vectors which are linearly independent and span $\mathcal{S}$ (after closure), but the representation is completely unstable (i.e. no such $A > 0$ exists

[2] This definition uses the natural numbers to index the set of basis functions, but of course it applies equally to any countably infinite sequence.

[3] If the scalars in the Hilbert space are complex, replace α_n^2 with $|\alpha_n|^2$.

above). Let's discuss this a little before looking at an example of when things go wrong.

Recall that a (possibly infinite) set of vectors is called linearly independent if no finite subset is linearly dependent. Trouble can come when larger and larger sets are coming closer and closer to being linearly dependent. That is, if $\{\boldsymbol{v}_n,\ 1 \leq n \leq \infty\}$ is a set of vectors, there might be no $\alpha_2, \ldots, \alpha_L$ such that

$$\boldsymbol{v}_1 = \sum_{\ell=2}^{L} \alpha_\ell \boldsymbol{v}_\ell,$$

exactly, no matter how large L is. But there could be an infinite sequence $\{\alpha_\ell\}$ such that

$$\lim_{L\to\infty} \|\boldsymbol{v}_1 - \sum_{\ell=2}^{L} \alpha_\ell \boldsymbol{v}_\ell\| = 0.$$

We will see an example of this below.

Example (of how things can go wrong): Multiscale Tent Functions

Consider this set of functions on $[0, 1]$.

$$\phi_0(t) = \sqrt{2}(1-t), \quad \phi_1(t) = \sqrt{2}t,$$

$$\psi_0(t) = \begin{cases} \sqrt{3}t, & 0 \leq t \leq 1/2, \\ \sqrt{3}(1/2 - t), & 1/2 \leq t \leq 1. \end{cases}$$

$$\psi_{j,n}(t) = 2^{j/2}\psi_0(2^j t - n), \quad j \geq 1, \quad n = 0, \ldots, 2^j - 1.$$

Sketch:

From your sketch above, it should be clear that

$$\mathrm{Span}(\{\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \boldsymbol{\psi}_0, \boldsymbol{\psi}_{j,n}, \quad 1 \leq j \leq J,\ n = 0, \ldots, 2^j - 1\})$$

is the set of all continuous piecewise-linear functions on dyadic intervals of length 2^{-J}. Since this set is dense in $L_2([0,1])$, we can write

$$x(t) = b_0\phi_0(t) + b_1\phi_1(t) + \sum_{j,n} c_{j,n}\boldsymbol{\psi}_{j,n}(t)$$

for some sequence of numbers $\{b_0, b_1, c_{j,n}\}$. The problem is that this sequence of numbers might not be well-behaved.

To see that this collection of functions cannot be a Riesz basis, notice that using the functions with $1 \leq j \leq J$, we can match the samples of any function on the grid with spacing 2^{-J}, with linear interpolation in between these samples. From this, we see that $\phi_0(t)$ can be matched exactly on the interval $[2^{-J-1}, 1]$ with a linear combination of $\boldsymbol{\psi}_{j,n}$, $0 \leq j \leq J$. This means that there is a sequence of numbers $\{\beta_{j,n}\}$ such that

$$\phi_0(t) = \sum_{j \geq 0} \sum_{n=0}^{2^j - 1} \beta_{j,n}\psi_{j,n}(t).$$

This means that the non-zero sequence of numbers $\{1, 0, \beta_{j,n},\ j \geq 0,\ n = 0, \ldots, 2^j - 1\}$ synthesizes the $\mathbf{0}$ signal, thus violating the condition that $A > 0$ uniformly.

Example: B-splines

Let $\mathcal{S}_{L,\mathbb{Z}}$ be the space of Lth-order *polynomial splines* with *knots* at the integers (for L odd) or half-integers (L even). This space contains functions $f(t)$ that

1. are Lth order polynomials between the (half) integers,
2. at the (half) integers, they are continuous (for $L \geq 1$) and have $L - 1$ continuous derivatives (for $L \geq 2$).

Clearly, this space is infinite dimensional.

We have already encountered a basis for this space: the set of shifted B-splines. Recall the definition of the B-spline functions:

$$b_0(t) = \begin{cases} 1, & -1/2 \leq t \leq 1/2, \\ 0, & \text{otherwise}, \end{cases}$$

$$b_1(t) = (b_0 * b_0)(t) = \int_{-\infty}^{\infty} b_0(s) b_0(t - s) ds,$$

$$b_2(t) = (b_1 * b_0)(t),$$

$$\vdots$$

$$b_L(t) = (b_{L-1} * b_0)(t).$$

Notice that

$$b_L(t) = (b_{L-1} * b_0)(t) = \int_{t-1/2}^{t+1/2} b_{L-1}(s)ds,$$

so we can construct these functions through iterative integration. Expressions for $b_0(t), b_1(t), b_2(t)$, and $b_3(t)$ all appear earlier in the notes; the higher order splines can be computed through recursion with the expressions above.

The set $\{b_L(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for $\mathcal{S}_{L,\mathbb{Z}}$. It is clear that any $f(t)$ of the form

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, b_L(t-n) \tag{3}$$

is a polynomial spline (in $\mathcal{S}_{L,\mathbb{Z}}$), so[4] $\operatorname{Span}(\{b_L(t-n)\}_{n\in\mathbb{Z}}) \subset \mathcal{S}_{L,\mathbb{Z}}$. It is also true that any polynomial spline in $\mathcal{S}_{L,\mathbb{Z}}$ can be written as a superposition of B-splines as in (3), so $\mathcal{S}_{L,\mathbb{Z}} \subset \operatorname{Span}(\{b_L(t-n)\}_{n\in\mathbb{Z}})$ ... establishing this second fact takes more work than we are willing to do at this point[5]. Combining these two points means

$$\operatorname{Span}(\{b_L(t-n)\}_{n\in\mathbb{Z}}) = \mathcal{S}_{L,\mathbb{Z}}.$$

Reasonably tight bounds on the values of A and B in (2) are also known; we can take

$$A = \left(\frac{2}{\pi}\right)^{2L+2}, \quad B = 1.$$

[4] Technically, this should be the closure of the span, since we are in infinite dimensional space here.

[5] For all of the gritty details about B-splines, see `https://goo.gl/T8Bnzk` and the references therein.

The B-spline basis is easy to modify to create richer spaces of polynomial splines that have knots that are more closely spaced. Let $S_{L,\mathbb{Z}/2}$ be the space of L-th order polynomial splines with knots at $\ldots,-1,-1/2,0,1/2,1,3/2,\ldots$. If we take

$$\psi_n(t) = \sqrt{2}\, b_L(2t-n),$$

then $\operatorname{Span}(\{\boldsymbol{\psi}_n\}_{n\in\mathbb{Z}})$ is a Riesz basis[6] for $S_{L,\mathbb{Z}/2}$ with the same A, B as above.

It should be clear that $S_{L,\mathbb{Z}} \subset S_{L,\mathbb{Z}/2}$, so this new space is "richer" (or "higher resolution" or "more expressive").

Of course this idea extends to any equally spaced "grid" of points on the real line. If we denote $\mathcal{S}_{L,a\mathbb{Z}}$ to be the space of polynomial splines whose knots are spaced a apart, then

$$\psi_n(t) = a^{-1/2}\, b_L(at-n)$$

will be a Riesz basis for this space. As $a \to 0$, this space becomes dense in $L_2(\mathbb{R})$.

Example: Non-harmonic sinusoids

Think of this example as Fourier series, but with sinusoids that are not harmonic and have unequally spaced frequencies.

Consider the set of (complex valued) functions on $[0,1]$

$$v_k(t) = e^{j2\pi\gamma_k t}, \quad k \in \mathbb{Z}$$

[6]Note that with this definition, $\psi_0(t)$ is centered at $t=0$, $\psi_1(t)$ is centered at $t=1/2$, $\psi_2(t)$ is centered at 1, etc. Also, the factor of $\sqrt{2}$ in front means that the basis functions have the same L_2 norm as in the base case.

where the frequencies γ_k are a sequence of numbers obeying

$$\gamma_k < \gamma_{k+1}, \quad \gamma_k \to -\infty \ \text{ as } \ k \to -\infty, \quad \gamma_k \to +\infty \ \text{ as } \ k \to +\infty.$$

Of course, if $\gamma_k = k$, this is the classical Fourier Series basis, and the $\{\boldsymbol{v}_k\}$ form an orthobasis. If the γ_k are no longer equally spaced by an integer multiple, these signals are not orthogonal. However, if they are not too far from being uniformly spaced, they still form a Riesz basis. "Kadec's 1/4-Theorem" is a result from harmonic analysis that says: If there exists a $\delta < 1/4$ such that

$$|\gamma_k - k| \leq \delta \quad \text{for all } k,$$

then $\{\boldsymbol{v}_k\}$ is a Riesz basis with

$$A = (\cos(\pi\delta) - \sin(\pi\delta))^2, \quad B = (2 - \cos(\pi\delta) + \sin(\pi\delta))^2.$$

Example:
Any orthobasis is a Riesz basis with $A = B = 1$.

Computing the expansion coefficients and the dual basis

We have not said anything yet about how to compute the expansion coefficients for a Riesz basis in infinite dimensions. This is because it is much less straightforward than in the finite dimensional case — instead matrix equations, we have to manipulate linear operators that act on sequences of numbers of infinite length.

But still, we can do this in certain cases[7]. Let's draw some parallels to the finite dimensional case to see what needs to be done. For a

[7] For B-splines, the set of notes `https://goo.gl/T8Bnzk` shows how to compute the dual vectors

finite N-dimensional space, we form the $N \times N$ Gram matrix $\boldsymbol{G}$ by filling in the entries $G_{n,\ell} = \langle \boldsymbol{v}_\ell, \boldsymbol{v}_n \rangle$, invert it to get another $N \times N$ matrix $\boldsymbol{H}$, and then set $\tilde{\boldsymbol{v}}_n = \sum_{\ell=1}^N H_{n,\ell} \boldsymbol{v}_\ell$.

We can follow the same procedure in infinite dimensions, but now the Gram "matrix" has an infinite number of rows and columns. The Grammian is a linear operator $\mathscr{G} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$; it maps infinite length sequences to infinite length sequences. Given an input $\boldsymbol{x} \in \ell_2(\mathbb{Z})$ to this operator, the output at the nth index is given by

$$[\mathscr{G}(\boldsymbol{x})](n) = \sum_{\ell=-\infty}^{\infty} \langle \boldsymbol{v}_\ell, \boldsymbol{v}_n \rangle x_\ell.$$

It turns out that the conditions for being a Riesz basis ensures that $\mathscr{G}$ is invertible, that is, that there is another linear operator $\mathscr{H} : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ such that

$$\mathscr{H}(\mathscr{G}(\boldsymbol{x})) = \boldsymbol{x}, \quad \text{for all } \ \boldsymbol{x} \in \ell_2(\mathbb{Z}).$$

In general, we need completely different methods to compute the inverse $\mathscr{H} = \mathscr{G}^{-1}$ than we do in the finite dimensional case. But in the end, the action of $\mathscr{H}$ will be specified by a two-dimensional array of numbers $\{H_{n,\ell},\ n, \ell \in \mathbb{Z}\}$; for any $\boldsymbol{w} \in \ell_2(\mathbb{Z})$,

$$[\mathscr{H}(\boldsymbol{w})](n) = \sum_{\ell=-\infty}^{\infty} H_{n,\ell}\, w_\ell.$$

With this array of numbers, we can, similar to the finite dimensional case, compute the expansion coefficients for a $\boldsymbol{x} \in \mathcal{S}$ using

$$\begin{aligned}
\alpha_n &= \sum_{\ell=-\infty}^{\infty} H_{n,\ell} \langle \boldsymbol{x}, \boldsymbol{v}_\ell \rangle \\
&= \left\langle \boldsymbol{x}, \sum_{\ell=-\infty}^{\infty} H_{n,\ell} \boldsymbol{v}_\ell \right\rangle \\
&= \langle \boldsymbol{x}, \tilde{\boldsymbol{v}}_n \rangle
\end{aligned}$$

where

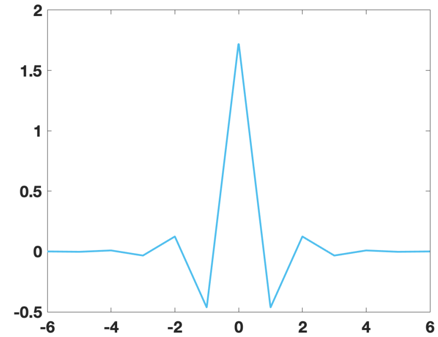
$$\tilde{\boldsymbol{v}}_n = \sum_{\ell=-\infty}^{\infty} H_{n,\ell} \boldsymbol{v}_\ell.$$

Example: Dual B-splines. We will not go into detail about how to compute the dual B-spline functions, we will just outline the results. We start by noting that the basis for the spline spaces was generated by shifting around a single function; we took $b_L(t)$, and generated a basis by collecting together all the integer shifts: $\{b_L(t-k),\ k \in \mathbb{Z}\}$. For all values of L, the dual basis will have the same structure; there is a single dual basis functions $\tilde{b}_L(t)$ such that $\{\tilde{b}_L(t-k)\ k \in \mathbb{Z}\}$ is the dual basis.

The basis $\{b_0(t-k),\ k \in \mathbb{Z}\}$ is self-dual, as it is an orthobasis (for the space of functions that are piecewise constant between the half integers. So $\tilde{b}_0(t) = b_0(t)$.

The basis $\{b_1(t-k),\ k \in \mathbb{Z}\}$ for the linear spline space has dual basis function shown below.

$\tilde{b}_1(t)$

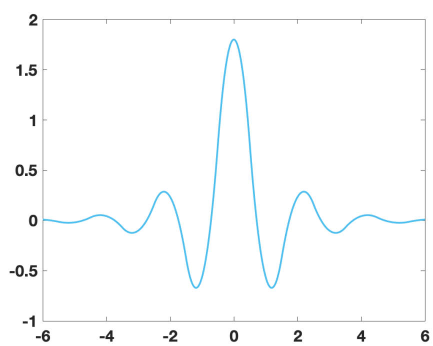


$t \rightarrow$

Note that while $b_1(t)$ is compactly supported on $[-1, 1]$, $\tilde{b}_1(t)$ has infinite support. This is reflected through the Grammian and its inverse: while there are only three values of ℓ for which $\boldsymbol{G}_{n,\ell}$ is non-zero for every fixed n (each "row" of $\mathscr{G}$ has only three non-zero terms), the inverse matrix is technically dense. However, there are only about 15 terms that are significant ... they die off pretty quickly as ℓ moves away from n. You can see above that the function they synthesize is effectively supported on $[-6, 6]$.

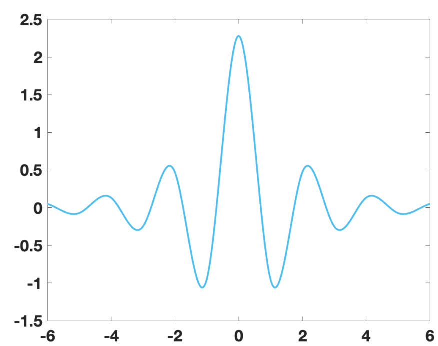
Here are the dual quadratic spline $\tilde{b}_2(t)$ and cubic spline $\tilde{b}_3(t)$:

$\tilde{b}_2(t)$



$t \rightarrow$

$\tilde{b}_3(t)$



$t \rightarrow$