College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 11
Hermitian Matrices and Single Value Decomposition

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Exercise 1

Determine (by hand) whether each one of the following matrices is pd, psd, nd, nsd, or none of the above.

$$\begin{pmatrix} 1 & j & 0 \\ -j & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

Check you answers using MATLAB command `eig()`.

For the Hermitian matrix

$$\begin{pmatrix} 1 & j & 0 \\ -j & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

the eigenvalues become

$$\begin{vmatrix} 1-\lambda & j & 0 \\ -j & 2-\lambda & 1 \\ 0 & 1 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} - j\begin{vmatrix} -j & 1 \\ 0 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((2-\lambda)(4-\lambda)-1\big) - j(-j)(4-\lambda)$$

$$= (1-\lambda)(\lambda^2 - 6\lambda + 7) - (4-\lambda) = -\lambda^3 + 7\lambda^2 - 12\lambda + 3$$

$$\Rightarrow \lambda^3 - 7\lambda^2 + 12\lambda - 3 = 0$$

$$\therefore \lambda = \ 4.4605,\ \ 2.2391,\ \ 0.3004$$

All of the eigenvalues are positive, so this matrix is positive definite.

For the Hermitian matrix

$$\begin{pmatrix} 1 & j \\ -j & 1 \end{pmatrix}$$

the eigenvalues become

$$\begin{vmatrix} 1-\lambda & j \\ -j & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = (-\lambda)(2-\lambda) = 0$$

$$\therefore \lambda = \ 0, 2$$

All the eigenvalues are non-negative, so this matrix is positive semi-definite.

For the Hermitian matrix

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

The eigenvalues of this matrix are

$$\begin{vmatrix} -\lambda & 2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) = 0$$

$$\therefore \lambda = \pm 2$$

Since there are positive and negative eigenvalues this matrix is indefinite.

The Hermitian matrix

$$\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$$

has the eigenvalues

$$\begin{vmatrix} -1-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = (-1-\lambda)(-2-\lambda) - 1 = \lambda^2 + 3\lambda + 1 = 0$$

$$\therefore \lambda = -2.6180, -0.3820$$

All the eigenvalues are negative values, so this matrix is negative definite.

MATLAB Verification

```
% (1)
P = [1, 1j, 0; -1j, 2, 1; 0, 1, 4];
[v, d] = eig(P)

% (2)
P = [1, 1j; -1j, 1];
[v, d] = eig(P)

% (3)
P = [0, 2; 2, 0];
[v, d] = eig(P)

% (4)
P = [-1, 1; 1, -2];
[v, d] = eig(P)
```

The outputs are

(1)	(2)

<pre>d = 3×3
    0.3004   0   0
         0   2.2391   0
         0   0   4.4605</pre>
	<pre>d = 2×2
	     0 0
	     0 2</pre>

(3)	(4)

<pre>d = 2×2
    -2   0
     0   2</pre>
	<pre>d = 2×2
	   -2.6180 0
	         0 -0.3820</pre>

Exercise 2

Determine (by hand) the maximum singular value of the following matrices.

$$(1)\ A = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad (2)\ A = \begin{pmatrix} 3 & 4 \end{pmatrix} \qquad (3)\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(1)

$$M = AA' = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$$

The eigenvalues of M become

$$\begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = \lambda^2 - 25\lambda + 144 - 144 = \lambda(\lambda - 25) = 0$$

$$\therefore \lambda = 0, 25$$

Then the Σ of the singular value decomposition becomes

$$\Sigma = \begin{pmatrix} \sqrt{25} \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$$

Hence, the maximum singular value is 5.

(2)

$$M = A'A = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$$

The eigenvalues of M become

$$\begin{vmatrix} 9-\lambda & 12 \\ 12 & 16-\lambda \end{vmatrix} = \lambda^2 - 25\lambda + 144 - 144 = \lambda(\lambda - 25) = 0$$

$$\therefore \lambda = 0, 25$$

Then the Σ of the singular value decomposition becomes

$$\Sigma = \begin{pmatrix} \sqrt{25} & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 \end{pmatrix}$$

Hence, the maximum singular value is 5.

(3)

$$M = A'A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The eigenvalues of M become

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 = 0$$

$$\therefore \lambda = 1$$

Then the Σ of the singular value decomposition becomes

$$\Sigma = \begin{pmatrix} \sqrt{1} \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence, the maximum singular value is 1.

Exercise 3

Determine (by hand) the singular value decomposition of

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Check your answer in MATLAB.

We use the equations

$$A'A = V\Sigma'\Sigma V'$$

$$AV = U\Sigma$$

We then start with,

$$M = A'A = \begin{pmatrix} 3 & 1 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 10 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & 10 \end{pmatrix}.$$

$$det(M - \Lambda I) = \begin{vmatrix} 10-\Lambda & 0 & 6 \\ 0 & -\Lambda & 0 \\ 6 & 0 & 10-\Lambda \end{vmatrix}$$

$$(10-\Lambda)\begin{vmatrix} -\Lambda & 0 \\ 0 & 10-\Lambda \end{vmatrix} + 6\begin{vmatrix} 0 & -\Lambda \\ 6 & 0 \end{vmatrix} = (10-\Lambda)(-\Lambda)(10-\Lambda) + 36\Lambda = 0$$

$$\Rightarrow -\Lambda^3 + 20\Lambda^2 - 100\Lambda + 36\Lambda = 0$$

$$\Rightarrow -\Lambda^3 + 20\Lambda^2 - 64\Lambda = 0$$

$$\Rightarrow \Lambda(\Lambda - 16)(\Lambda - 4) = 0$$

$$\therefore \Lambda = 16, 4, 0$$

$$\therefore \lambda = 4, 2, 0$$

$$\sigma_1 = 4, \qquad \sigma_2 = 2$$

Then,

$$\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

Next, we find the eigenvectors for M.

If $\Lambda = 16$,

$$M - \Lambda I = \begin{pmatrix} -6 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_1 = c_1\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \; where \; c_1 \neq 0$$

If $\Lambda = 4$,

$$M - \Lambda I = \begin{pmatrix} 6 & 0 & 6 \\ 0 & -4 & 0 \\ 6 & 0 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_2 = c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \; where \; c_2 \neq 0$$

If $\Lambda = 0$,

$$M - \Lambda I = \begin{pmatrix} 10 & 0 & 6 \\ 0 & 0 & 0 \\ 6 & 0 & 10 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_3 = c_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; where \; c_3 \neq 0$$

Then normalize v_1, v_2, v_3

$$\hat{v}_1 = \begin{pmatrix} 0.7071 \\ 0 \\ 0.7071 \end{pmatrix}, \; \hat{v}_2 = \begin{pmatrix} -0.7071 \\ 0 \\ 0.7071 \end{pmatrix}, \; \hat{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$V = (\hat{v}_1 \quad \hat{v}_2 \quad \hat{v}_3) = \begin{pmatrix} 0.7071 & -0.7071 & 0 \\ 0 & 0 & 1 \\ 0.7071 & 0.7071 & 0 \end{pmatrix}$$

Then,

$$U = \begin{pmatrix} \frac{A\hat{v}_1}{\sigma_1} & \frac{A\hat{v}_2}{\sigma_2} \end{pmatrix}$$

$$U = \left(\frac{1}{4} \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0.7071 \\ 0 \\ 0.7071 \end{pmatrix} \quad \frac{1}{2} \begin{pmatrix} 3 & 0 & 1 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} -0.7071 \\ 0 \\ 0.7071 \end{pmatrix} \right)$$

$$\Rightarrow U = \begin{pmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{pmatrix}$$

Thus,

$$A = U \Sigma V'$$

$$A = \begin{pmatrix} 0.7071 & -0.7071 \\ 0.7071 & 0.7071 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0.7071 & 0 & 0.7071 \\ -0.7071 & 0 & 0.7071 \\ 0 & 1 & 0 \end{pmatrix}$$

MATLAB Verification

```
A = [3,0,1; 1,0,3];
[U, S, V] = svd(A)
```

U = 2×2 -0.7071 0.7071 -0.7071 -0.7071	S = 2×3 4.0000 0 0 0 2.0000 0
V = 3×3 -0.7071 0.7071 0 0 0 -1.0000 -0.7071 -0.7071 0	