Linear algebra has become as basic and as applicable as calculus, and fortunately it is easier.

– Gilbert Strang

Linear vector spaces

A *vector space* is simply a collection of things that obeys certain abstract but familiar algebraic properties:

- A vector space $\mathcal{S}$ is composed of a set of elements, called *vectors*, and members of a field[1] $\mathbb{F}$ called *scalars*.
- The space also has rules for adding vectors and multiplying them by scalars
  - *vector addition*, which we will write as '+' combines two vectors to get a third
  - *scalar multiplication*, combines a scalar and a vector to get another vector
- The '+' operation must obey the following four rules for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$:
  1. $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (commutative)
  2. $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ (associative)
  3. There is a unique *zero vector* $\mathbf{0}$ such that

$$\boldsymbol{x} + \mathbf{0} = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathcal{S}$$

  4. For each vector $\boldsymbol{x} \in \mathcal{S}$, there is a unique vector (called

[1] A field is simply a set of numbers for which multiplication and addition are defined, and distribute/associate in the same manner as the reals.

$-\boldsymbol{x}$) such that

$$\boldsymbol{x} + (-\boldsymbol{x}) = \mathbf{0}$$

- Scalar multiplication must obey the following four rules for all $a, b \in \mathbb{F}$ and $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S}$:

  1. $a(\boldsymbol{x} + \boldsymbol{y}) = a\boldsymbol{x} + a\boldsymbol{y}$
     $(a + b)\boldsymbol{x} = a\boldsymbol{x} + b\boldsymbol{x}$ (distributive)
  2. $(ab)\boldsymbol{x} = a(b\boldsymbol{x})$ (associative)
  3. For the multiplicative identity of $\mathbb{F}$, which we write as 1, we have
     $$1\boldsymbol{x} = \boldsymbol{x} \quad \forall \boldsymbol{x} \in \mathcal{S}$$
  4. For the additive identity of $\mathbb{F}$, which we write as 0, we have
     $$0\boldsymbol{x} = \mathbf{0}$$
     (that's the scalar zero on the left, and the vector zero on the right).

- $\mathcal{S}$ is closed under scalar multiplication and vector addition:

$$\boldsymbol{x}, \boldsymbol{y} \in \mathcal{S} \;\Rightarrow\; a\boldsymbol{x} + b\boldsymbol{y} \in \mathcal{S}, \quad \forall a, b \in \mathbb{F}.$$

This last point is really the most salient piece of algebraic structure. In light of it, we will often use the more descriptive terminology **linear vector space**.

Examples of vector spaces

1. $\mathbb{R}^N$
$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \quad \text{where the } x_i \text{ are real}$$
and we use the standard rules for vector addition and scalar multiplication

2. $\mathbb{C}^N$, same as above, except the x_i are complex

3. Bounded, continuous functions $f(x)$ on the interval $[a, b]$ that are real valued.
Vector addition = adding functions pointwise,
scalar multiplication = multiplying by $a \in \mathbb{R}$ pointwise,
it should be easy to see that adding two bounded, continuous functions gives you another bounded and continuous function.

4. $GF(2)^N$
Here, the scalar field is $\{0, 1\}$, and so vectors are lists of N bits.
Addition for the field is modulo 2, so
$$\begin{aligned} 0 + 0 &= 0 \\ 0 + 1 &= 1 + 0 = 1 \\ 1 + 1 &= 0 \end{aligned}$$
For example,
$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$
This space is super useful in information/coding theory

Here is an example of something which is not a vector space:

5. Continuous functions $f(x)$ on $[a, b]$ such that

$$|f(x)| \leq 2.$$

Why is this not a linear vector space?

Linear subspaces

A (non-empty) subset $\mathcal{T}$ of $\mathcal{S}$ is call a **linear subspace** of $\mathcal{S}$ if

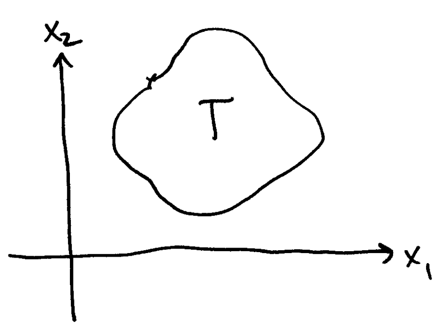
$$\forall\ a, b \in \mathbb{F}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathcal{T} \ \Rightarrow\ a\boldsymbol{x} + b\boldsymbol{y} \in \mathcal{T}$$

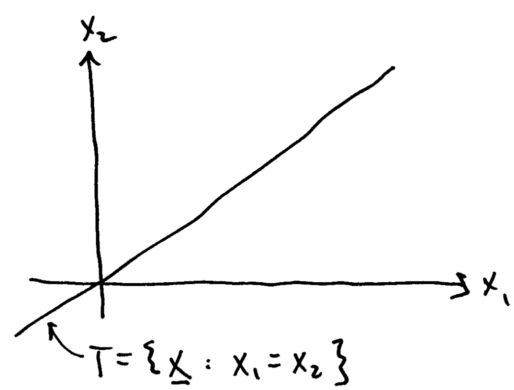
Note that is has to be true that

$$\mathbf{0} \in \mathcal{T}.$$

It is easy to show that $\mathcal{T}$ can be treated as a linear vector space by itself.

Easy examples: Are these subspaces of $\mathcal{S} = \mathbb{R}^2$?





Which of these are subspaces?

1. $\mathcal{S} = \mathbb{R}^5$
   $\mathcal{T} = \{\boldsymbol{x} \ : \ x_4 = 0, \ x_5 = 0\}$

2. $\mathcal{S} = \mathbb{R}^5$
   $\mathcal{T} = \{\boldsymbol{x} \ : \ x_4 = 1, \ x_5 = 1\}$

3. $\mathcal{S} = \mathcal{C}([0,1])$ (bounded, continuous functions on $[0,1]$)
   $\mathcal{T} = \{$polynomials of degree at most $p\}$

4. $\mathcal{S} =$ continuous functions on the real line
   $\mathcal{T} = \{f(x) \ : \ f$ is bandlimited to $\Omega\}$

5. $\mathcal{S} = \mathbb{R}^N$
   $\mathcal{T} = \{\boldsymbol{x} \ : \ \boldsymbol{x}$ has no more than 5 non-zero components$\}$

6. $\mathcal{S} = \mathbb{R}^N$
   $\mathcal{T} = \{\boldsymbol{x} \ : \ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = 3\}$, where $\boldsymbol{c} \in \mathbb{R}^N$ is a fixed vector
   (Recall the standard dot product $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \sum_{n=1}^{N} c[n]x[n]$)

7. $\mathcal{S} = \mathcal{C}([0,1])$
   $\mathcal{T} = \{f(x) \ : \ f(x) = a\cos(2\pi t) + b\sin(2\pi t)$ for some $a, b \in \mathbb{R}\}$

Linear combinations and spans

Let $\mathcal{V} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ be a set of vectors in a linear space $\mathcal{S}$.

Definition: A **linear combination** of vectors in $\mathcal{V}$ is a sum of the form

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_N\boldsymbol{v}_N$$

for some $a_1, \ldots, a_N \in \mathbb{F}$.

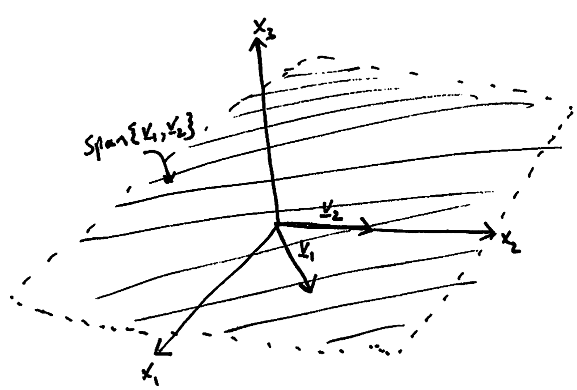
Definition: The **span** of $\mathcal{M}$ is the set of all linear combinations of $\mathcal{M}$. For $\mathcal{V} = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$, we write this as

$$\operatorname{Span}(\mathcal{V}) \;=\; \{a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + \cdots + a_N\boldsymbol{v}_N \;:\; a_1, \ldots, a_N \in \mathbb{F}\}$$

It should be clear that $\operatorname{Span}(\mathcal{V})$ is always a subspace, no matter what $\mathcal{V}$ contains.

Example:

$$\mathcal{S} = \mathbb{R}^3, \qquad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$



$\operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\}) = (x_1, x_2)$ plane

i.e. for any x_1, x_2 we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for some $a, b \in \mathbb{R}$

Question: What is the span of $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ for

$$\boldsymbol{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \qquad ?$$

What about if

$$\boldsymbol{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix} \qquad \boldsymbol{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad \boldsymbol{v}_3 = \begin{bmatrix}0\\-1\\1\end{bmatrix} \qquad ?$$

Example:

$$\mathcal{M} = \{b_0'(x-k),\ k = 0,1,2,3\},$$

where $b_0'(x)$ is (a slightly shifted version of) the zeroth order B-spline (see the last set of notes).

$$b_0'(x) = \begin{cases} 1, & 0 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned}\text{Span}(\mathcal{M}) = {} & \text{piecewise constant functions between the integers} \\ & \text{that are non-zero only on } [0,4].\end{aligned}$$

Example:

$$\mathcal{M} = \{b_0'(x-k),\ k \in \mathbb{Z}\},$$

Then

$$\text{Span}(\mathcal{M}) = \text{piecewise constant functions between the integers}$$

Example:

$$\mathcal{M} = \{b_1(x-k),\ k = 0,1,2,3\},$$

where $b_1(t)$ is the first order B-spline (see last set of notes). Then

$$\begin{gathered}\text{Span}(\mathcal{M}) = \text{piecewise linear functions on } [-1,4] \\ \text{with } f(-1) = f(4) = 0\end{gathered}$$

Linear independence

A set of vectors $\{\boldsymbol{v}_j\}_{j=1}^N$ is said to be **linearly dependent** if there exists scalars $a_1, \ldots, a_N$, not all $= 0$, such that

$$\sum_{n=1}^N a_n \boldsymbol{v}_n = \mathbf{0}$$

Likewise, if $\sum_n a_n \boldsymbol{v}_n = \mathbf{0}$ only when all the $a_j = 0$, then $\{\boldsymbol{v}_n\}_{n=1}^N$ is said to be **linearly independent**.

Example 1:

$$\mathcal{S} = \mathbb{R}^3, \quad \boldsymbol{v}_1 = \begin{bmatrix}2\\1\\0\end{bmatrix} \quad \boldsymbol{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \boldsymbol{v}_3 = \begin{bmatrix}1\\2\\0\end{bmatrix}$$

Find a_1, a_2, a_3 such that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = \mathbf{0}$$

Note that any two of the vectors above are linearly independent:

$$\operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}) \;=\; \operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_2\}) \;=\; \operatorname{span}(\{\boldsymbol{v}_1, \boldsymbol{v}_3\}) \;=\; \operatorname{span}(\{\boldsymbol{v}_2, \boldsymbol{v}_3\})$$

Example 2:

$$\begin{aligned} \mathcal{S} &= \mathcal{C}([0,1]) \\ \boldsymbol{v}_1 &= \cos(2\pi t) \\ \boldsymbol{v}_2 &= \sin(2\pi t) \\ \boldsymbol{v}_3 &= 2\cos(2\pi t + \pi/3) \end{aligned}$$

Find a_1, a_2, a_3 such that

$$a_1\boldsymbol{v}_1 + a_2\boldsymbol{v}_2 + a_3\boldsymbol{v}_3 = \mathbf{0}$$

Repeat for

$$\boldsymbol{v}_3 = A\cos(2\pi t + \phi) \qquad \text{for some } A > 0, \quad \phi \in [0, 2\pi).$$

Bases

Definition: A **basis** of a subspace $\mathcal{T}$ of a linear vector space $\mathcal{S}$ is a (countable) set of vectors $\mathcal{B}$ such that[2]

1. $\text{span}(\mathcal{B}) = \mathcal{T}$
2. $\mathcal{B}$ is linearly independent

We will often speak of bases for the entire space $\mathcal{S}$; these just follow the definition above with $\mathcal{T} = \mathcal{S}$.

In $\mathbb{R}^N$, we have seen (in the linear algebra review notes) that every subspace has a basis, and every basis for a subspace has the same number of elements. This result extends to all of the vector spaces that we will encounter in this course. The number of elements in a basis for $\mathcal{T}$ (which can be infinite) is called the **dimension** of $\mathcal{T}$.

Having a basis in place for $\mathcal{T}$ allows us to translate elements of $\mathcal{T}$ into lists of numbers. Suppose, for example, that $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_N\}$ is a basis for an N-dimensional subspace $\mathcal{T}$. Then since the $\{\boldsymbol{v}_n\}$ span $\mathcal{T}$, we know that for any $\boldsymbol{x} \in \mathcal{T}$ there exists $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ such that

$$\boldsymbol{x} = \sum_{n=1}^{N} \alpha_n \boldsymbol{v}_n.$$

Moreover, every vector has a *unique* set of expansion coefficients. Write down the argument for this here:

[2]In infinite dimensions, we really need to be more careful with this definition than what is being said here. We will revisit this issue later in the notes.

The same is true for infinite dimensional subspaces, but the argument is slightly more involved since we need some notion of convergence for infinite sums of vectors. More on this later in the notes.

Examples:

1. $\mathbb{R}^N$ with

$$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_N\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

   This is the **standard basis** for $\mathbb{R}^N$.

2. $\mathbb{R}^N$ with any set of N linearly independent vectors.

3. $\mathcal{S} = \{\text{polynomials of degree at most } p\}$.
   A basis for $\mathcal{S}$ is $\mathcal{B} = \{1, x, x^2, \ldots, x^p\}$.
   The dimension of $\mathcal{S}$ is $p+1$.

4. $\mathcal{S} = \{f(x) \ : \ f(x) \text{ is periodic with period } 2\pi\}$
   A basis for $\mathcal{S}$ is $\mathcal{B} = \{e^{jkx}\}_{k=-\infty}^{\infty}$ (Fourier Series)
   $\mathcal{S}$ is infinite dimensional.

5. $\mathcal{S} = GF(2)^3$ (length 3 bit vectors with mod 2 arithmetic).
   A basis for $\mathcal{S}$ is

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

How would you write

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{\hspace{2em}}\, \boldsymbol{v}_1 + \underline{\hspace{2em}}\, \boldsymbol{v}_2 + \underline{\hspace{2em}}\, \boldsymbol{v}_3 \qquad ?$$

Example: Second-order polynomial splines

Let $\mathcal{P}_2$ be the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that are second-order polynomial splines between the half-integers. This means that

$$f(x) = a_k x^2 + b_k x + c_k, \quad k - 1/2 \leq t \leq k + 1/2, \quad k \in \mathcal{Z},$$

and that $f(x)$ is both continuous and differentiable at the half integers $x = k + 1/2$.

1. Is is a fact that any $f \in \mathcal{P}_2$ is uniquely determined by its samples at the integers, $\ldots, f(-1), f(0), f(1), f(2), \ldots$. Why is this true?

2. Let $\mathcal{B} = \{b_2(t-k),\ k \in \mathcal{Z}\}$, where $b_2(t)$ is the second-order B-spline. Argue that $\mathrm{Span}(\mathcal{B}) \subset \mathcal{P}_2$. That is, every function of the form
$$f(x) = \sum_{k=-\infty}^{\infty} \alpha_k \, b_2(x-k)$$
has the properties described above.