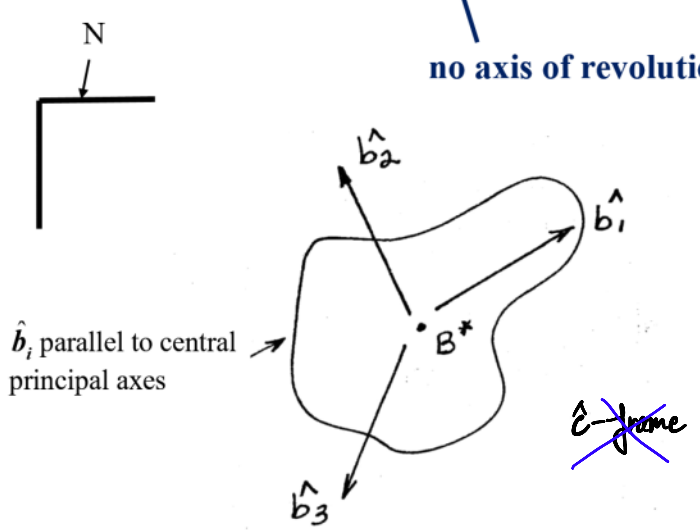
V1

Torque-Free Motion Unsymmetric Rigid Body

no axis of revolution



Derive EOM $\left(\bar{M}^{B^*} = \dfrac{{}^N d\, {}^N \bar{H}^{B^*}}{dt} = \bar{0} \right)$

$${}^N \bar{H}^{B^*} = \bar{\bar{I}}^{B/B^*} \bullet {}^N \bar{\omega}^B \longleftarrow \quad {}^N\bar{\omega}^B = \omega_i \hat{b}_i$$

$$\bar{H} = I_1 \omega_1 \hat{b}_1 + I_2 \omega_2 \hat{b}_2 + I_3 \omega_3 \hat{b}_3$$

$$\frac{{}^N d\, {}^N \bar{H}^{B^*}}{dt} = I_1 \dot{\omega}_1 \hat{b}_1 + I_2 \dot{\omega}_2 \hat{b}_2 + I_3 \dot{\omega}_3 \hat{b}_3 + {}^N \bar{\omega}^B \times \bar{H}$$

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$$\frac{{}^N d\, {}^N \bar{H}^{B^*}}{dt} = 0 = \left[I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 \right] \hat{b}_1 + \left[I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1 \right] \hat{b}_2 + \left[I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 \right] \hat{b}_3$$

→ "Euler Eqns"

torque-free

$$\boxed{\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3)\, {}^N\omega_2 \omega_3 &= 0 \\ I_2 \dot{\omega}_2 - (I_3 - I_1)\, \omega_3 \omega_1 &= 0 \\ I_3 \dot{\omega}_3 - (I_1 - I_2)\, \omega_2 \omega_1 &= 0 \end{aligned}}$$

Note: working in $\hat{b}$; no advantage now to use $\hat{c}$

Seek a solution to these equations, analytical if possible
Equations are nonlinear and coupled. However, this "standard" set of nonlinear equations also have a general solution

→ in terms of elliptic funcs

(As in the axisymmetric case, a problem must be put in proper form so solution is apparent!)

Review:

$s = 0$

$$I_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\omega_3$$
$$I_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\omega_1$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2$$

Axisymmetric $\quad I_1 = I_2 = I; \quad I_3 = J$

Let $\quad K = \dfrac{(I-J)}{I}$

$$\dot{\omega}_1 = K\omega_2\omega_3$$
$$\dot{\omega}_2 = -K\omega_3\omega_1$$
$$\dot{\omega}_3 = 0 \longrightarrow \boxed{\omega_3 = \omega_{3_o} = \text{constant}}$$

$$\ddot{\omega}_1 = K\omega_{3_o}\dot{\omega}_2 = K\omega_{3_o}\left(-K\omega_{3_o}\right)\omega_1$$
$$\ddot{\omega}_1 + K^2\omega_{3_o}^2\omega_1 = 0 \longrightarrow \begin{aligned}\omega_1 &= A\sin\left(K\omega_{3_o}t\right) + B\cos\left(K\omega_{3_o}t\right)\\ \omega_2 &= C\sin\left(K\omega_{3_o}t\right) + D\cos\left(K\omega_{3_o}t\right)\end{aligned}$$

➔ In terms of $\hat{b}$'s,
➔ In terms of $\hat{c}$'s,

➔ Not axisymmetric $\longrightarrow$ elliptic fcns

$$\left.\begin{matrix}\operatorname{sn}(x,k)\\ \operatorname{cn}(x,k)\\ \operatorname{dn}(x,k)\end{matrix}\right\} \quad \text{axisymmetric } k = 0 \ (k \text{ is elliptic modulus})$$

Elliptic Functions*

A function y can be defined in terms of x by direct calculation, such as

$y = x^2$ (equation that defines y in terms of x)

A function can also be defined using a differential equation and a set of initial conditions. The trigonometric function '$\sin x$' is defined by

$$\boxed{\begin{array}{c}\text{definition}\\ \text{of } \sin(x)\end{array}} \left\{ \begin{array}{l} \left(\frac{dy}{dx}\right)^2 = 1 - y^2 \\ \\ \text{I.C.} \quad y = 0, \quad \frac{dy}{dx} > 0 \ @ \ x = 0 \end{array} \right. \qquad \text{I}$$

since $y = \sin x$ then satisfies the differential equation.

Now consider the differential equation

$$\boxed{\begin{array}{c}\text{definition}\\ \text{of } \mathrm{sn}(x,k)\end{array}} \left\{ \begin{array}{l} \left(\frac{dy}{dx}\right)^2 = (1 - y^2)(1 - k^2y^2) \\ \\ \text{I.C.} \quad y = 0, \quad \frac{dy}{dx} > 0 \ @ \ x = 0 \end{array} \right. \qquad \text{II}$$

The solution for this differential equation with the given initial conditions is the **elliptic function** $y = \mathrm{sn}(x,k)$ and, thus, the function is defined by II. It is noted that the function is periodic since
$\mathrm{sn}(x + 4K, k) = \mathrm{sn}(x,k)$

Jacobi Elliptic Functions

$\longrightarrow$ MATLAB ellipj( )

*This discussion is a summary of that contained in Synge, J.L., and Griffith, B.A., **Principles of Mechanics**, McGraw-Hill Book Co., New York, 1959.

The period is defined through the complete elliptic integral K, i.e.,

$$K = \int_0^1 \left\{ \frac{dy}{\left[\left(1-y^2\right)\left(1-k^2y^2\right)\right]^{\frac{1}{2}}} \right\}$$

Other elliptic functions are defined

$$cn^2 x = 1 - sn^2 x$$
$$dn^2 x = 1 - k^2 sn^2 x$$

(The functions and derivatives are continuous.) In comparing I and II, note that when $k = 0$

$$\left.\begin{aligned} \operatorname{sn} x &= \sin x \\ \operatorname{cn} x &= \cos x \\ \operatorname{dn} x &= 1 \\ 4K &= 2\pi \end{aligned}\right\} \quad k = 0$$

Curves of these functions appear on the next pages

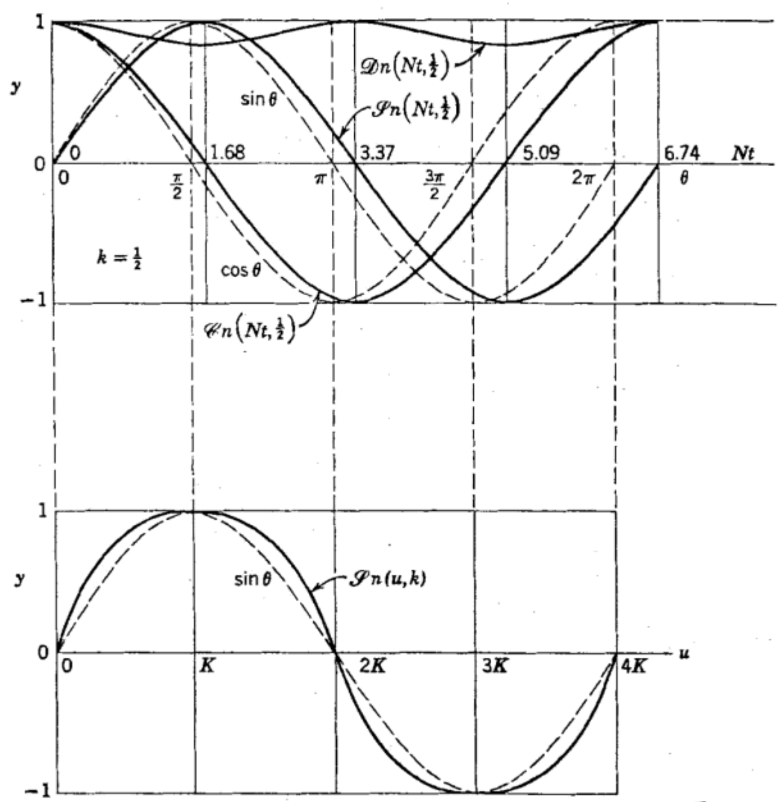

Fig. 5.11–1. Plot of elliptic functions. $k = .5$

Fig. 5.11–2. Plot of elliptic functions.

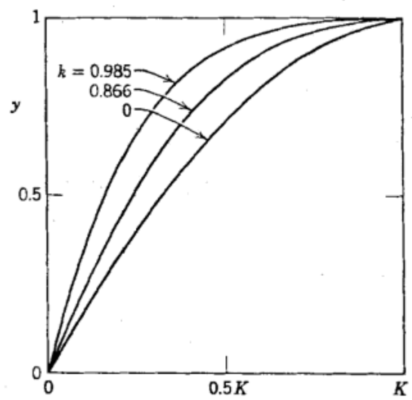

Table of $y = \sin\phi = \mathscr{S}n(u, k)$ Taken from Peirce's Table of Integrals
3rd Revised Ed. p. 122

ϕ	Ordinate, $\mathscr{S}n(u,k) = \sin\phi$	Abscissa $= u = Nt = \int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}$				
		$k = 0$	$k = 0.50$	$k = 0.707$	$k = 0.866$	$k = 0.9848$
0°	0	0	0	0	0	0
10°	0.1736	0.111	0.1037	0.0943	0.0814	0.0556
20°	0.3420	0.222	0.2080	0.190	0.1646	0.113
30°	0.500	0.3333	0.3140	0.289	0.2515	0.174
45°	0.707	0.500	0.477	0.445	0.3945	0.278
60°	0.866	0.666	0.645	0.616	0.563	0.413
75°	0.9563	0.834	0.821	0.802	0.765	0.617
90°	1.000	1.000	1.000	1.000	1.000	1.000
		K	= 1.686	= 1.854	= 2.156	= 3.153

Fig. 5.11-3. Plot of elliptic function $\mathscr{S}n(u, k)$.

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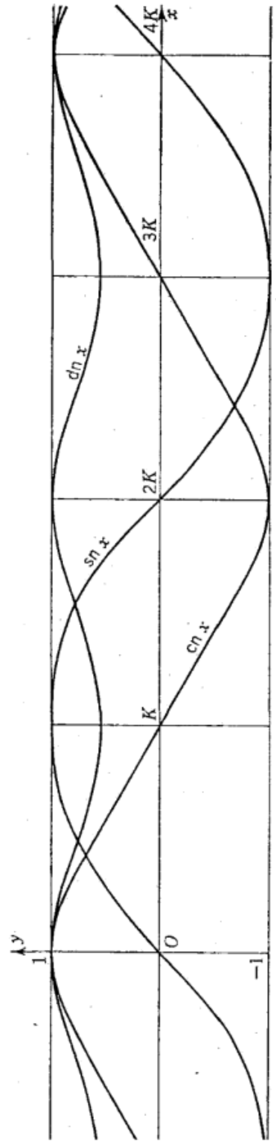


FIG. 133. Graphs of sn x, cn x, dn x ($k^2 = 0.7$).

Analytical solutions for ω_i !!!

Even with these analytical solutions, it doesn't seem to be obvious information about orientation here. (This same observation was also true in the axisymmetric case at this stage.)

The most that we can speculate: since ω_i's are elliptic functions, is there a cyclic or periodic nature to the motion of an unsymmetric rigid body?

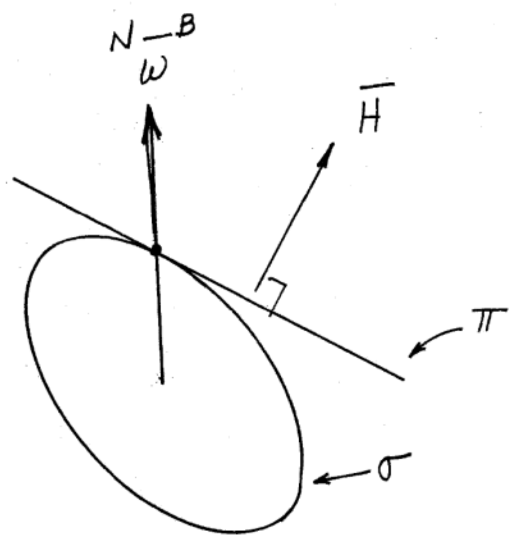
Number of options to discover more about orientation variables:

1. Choose kinematic variables
   Integrate DE (analytically if possible; numerically if necessary)

2. Consider the geometric solution
   $\Rightarrow$ Poinsot construction

Poinsot Construction

body principal axes

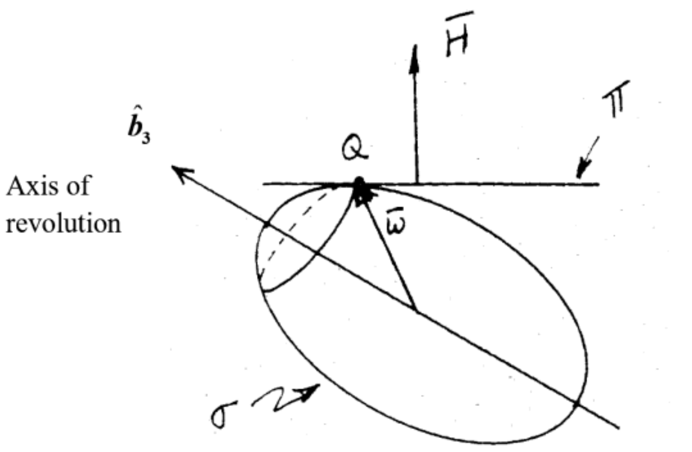


Still true

1. $2T_{ROT} = \bar{\omega} \bullet \bar{H}$ constant
2. π still tangent at point Q
3. σ still "rolls" on the invariable plane

But if σ NOT axisymmetric
How do angles change?
ω_i's now elliptic functions not constants; what is impact on angles?

Recall axisymmetric RB



$\omega_1, \omega_2, \omega_3 \quad (\hat{c})$

angle $\left(\hat{b}_3, \bar{\omega}\right)$

angle $\left(\bar{H}, \bar{\omega}\right)$

Unsymmetric RB

$\bar{H}$ constant BUT

changing nutation angle

Polhodes are closed but not circular

herpolhodes not even closed curves
— not very useful

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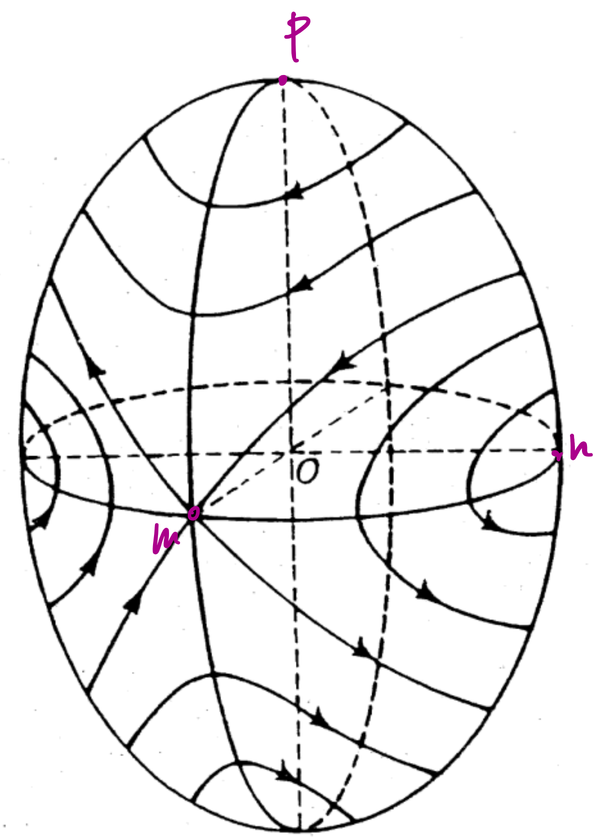

Polhodes do give much insight into the motion:

Axes NOT arbitrary
Label principal axes (same for body and σ)

o-p longest axis → smallest moment of inertia
o-n smallest axis → largest "
o-m intermediate axis → intermediate I

Geometric observations:

Polhodes are closed
Movement confined to one curve-depends on ICs that start near p or n tend to encircle that point

Interpretation:
Particular solution – maintain rotation about one principal axis
Consider rotating near axis of minimum or maximum moment of inertia $(o-n)$

→ will remain close to those axes
→ marginally stable by our definition

(closer you begin to axis of min/max inertia smaller changes in angle per cycle.

Try starting near axis of intermediate inertia
→ will not remain close
→ no initial angle is sufficiently small to keep it close

Can we get this result using linearization and eigenvalues?

Torque-free / unsymmetric rigid body
Stability investigation using linear analysis

1. EOM (NL)
   Also need kinematic equations?
2. Particular solution to NLDE
3. Linearize about particular solution
4. Put variational equations in 1st-order form
   $\dot{z} = zA$ (is matrix A constant?)
5. Check eigenvalues

NLDE

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2$$

rearrange of Euler's Eqns

Particular solution: Rotation about principle axis

Arbitrarily choose $\hat{b}_2$

$$\omega_1 = 0 + \tilde{\omega}_1$$

$$\omega_2 = \omega_{2_0} + \tilde{\omega}_2$$

$$\omega_1 = 0 + \tilde{\omega}_1$$
$$\omega_2 = \omega_{2_o} + \tilde{\omega}_2$$
$$\omega_3 = 0 + \tilde{\omega}_3$$

$$\dot{\tilde{\omega}}_1 = \frac{I_2 - I_3}{I_1}\left(\omega_{2_o} + \tilde{\omega}_2\right)\tilde{\omega}_3 = K_1 \omega_{20} \tilde{\omega}_3$$

$$\dot{\tilde{\omega}}_2 = \frac{I_3 - I_1}{I_2}\tilde{\omega}_3\tilde{\omega}_1 = 0$$

linear, constant coefficients?

$$\dot{\tilde{\omega}}_3 = \frac{I_1 - I_2}{I_3}\tilde{\omega}_1\left(\omega_{2_o} + \tilde{\omega}_2\right) = K_3 \omega_{20} \tilde{\omega}_1$$

$$\dot{\tilde{\omega}}_2 = 0$$

$$\begin{bmatrix} \dot{\tilde{\omega}}_1 & \dot{\tilde{\omega}}_3 \end{bmatrix} = \begin{bmatrix} \tilde{\omega}_1 & \tilde{\omega}_3 \end{bmatrix} \begin{bmatrix} 0 & K_3\omega_{2_o} \\ K_1\omega_{2_o} & 0 \end{bmatrix}$$

Characteristic Equation

$$\boxed{\lambda^2 - K_1 K_3 \omega_{20}^2 = 0}$$

$$\lambda_{2,3} = \pm \omega_{20}\sqrt{K_1 K_3}$$

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3 eigenvalues:

$$\lambda = 0\ ,\ \pm \omega_{20}\sqrt{\frac{(I_2 - I_3)(I_1 - I_2)}{I_1 I_3}}$$

For NL system, particular solution unstable

→ If **<u>any</u>** eigenvalue has a positive real part

For NL system, particular solution unstable

→ If **any** eigenvalue has a positive real part

True if $I_1 > I_2 > I_3$

or

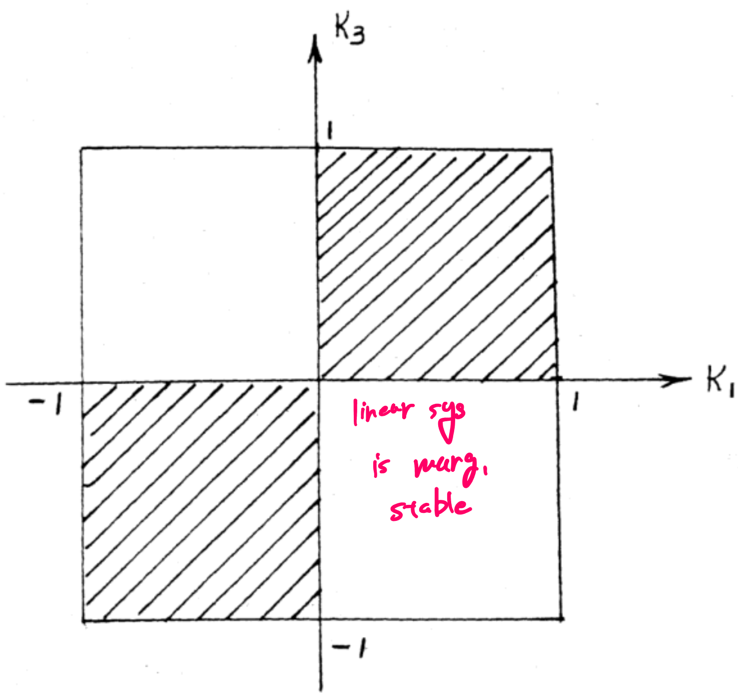
$I_1 < I_2 < I_3$

→ true if I_2 is the intermediate inertia

Note: This analysis gives NO conclusions for NL system if I_2 is min/max inertia because the linear analysis would predict only marginal stability for the linear system. We can say that **only** if axis if rotation corresponds to the axis of max or min inertia is there any chance of stability

(We know from Poinsot construction that rotation about axis of max/min inertia is indeed marginally stable for NL system.)

Common to represents results of linear analysis in the form of a stability chart



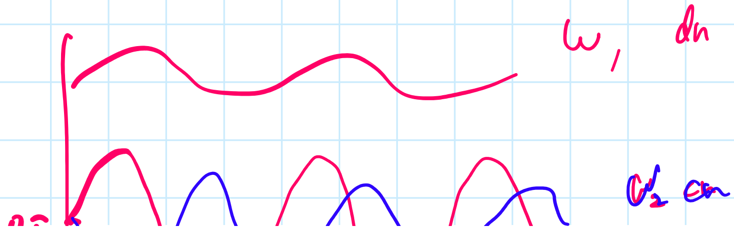
Will this change if a gravity torque is introduced?

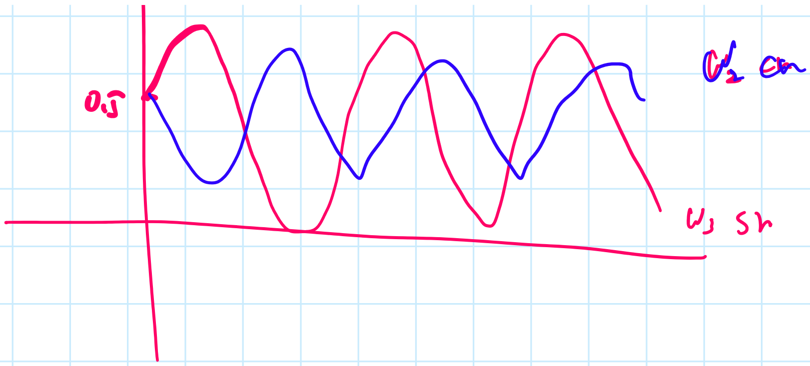
I = 10 , 5 , 1

ω_0 = 1 0.5 0.5

400 100 600

0.2 1 0.2





0.5
ω2 cn
ω3 sn