College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW2

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Exercise 1

Determine the nature (node, focus, etc) of each equilibrium state of the damped duffing system.

$$\ddot{y} + 0.1\dot{y} - y + y^3 = 0$$

Numerically obtain the phase portrait.

Solution:
The equilibrium states are found by setting

$$\begin{cases} \ddot{y}, \dot{y} = 0 \\ y := y_e \end{cases}$$

Then

$$\begin{aligned} y_e^3 - y_e &= 0 \\ y_e(y_e+1)(y_e-1) &= 0 \\ y_e &= -1, 0, 1 \end{aligned}$$

Next, we linearize the equation.

$$\begin{aligned} \delta\ddot{y} + 0.1\delta\dot{y} - (\delta y + y_e) + (\delta y + y_e)^3 = 0 \\ \delta\ddot{y} + 0.1\delta\dot{y} - (\delta y + y_e) + \cancelto{0}{\delta y^3} + \cancelto{0}{3\delta y^2 y_e} + 3\delta y y_e^2 + y_e^3 = 0 \\ \delta\ddot{y} + 0.1\delta\dot{y} - (\delta y + y_e) + 3\delta y y_e^2 + y_e^3 = 0 \\ \delta\ddot{y} + 0.1\delta\dot{y} + (3y_e^2 - 1)\delta y + y_e^3 - y_e = 0 \end{aligned}$$

If $x_1 := \delta y$ and $x_2 := \delta\dot{y}$ the system becomes

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -(3y_e^2 - 1)x_1 - 0.1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -y_e^3 + y_e \end{bmatrix}$$

Now, if $y_e = \pm 1$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 0.1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

or

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 0.1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

They have the same A matrix so can be evaluated equally. The eigenvalues and corresponding eigenvectors for this linearized system are

$$\lambda_1 = -0.0500 + 1.4133j \qquad v_1 = \begin{bmatrix} -0.0204 - 0.5770j \\ 0.8165 \end{bmatrix}$$

$$\lambda_2 = -0.0500 - 1.4133j \qquad v_2 = \begin{bmatrix} -0.0204 + 0.5770j \\ 0.8165 \end{bmatrix}$$

The eigenvalues are complex values in the left-hand plane of the complex plane meaning that the linearized system is exponentially stable and stable focus. Thus, the equilibrium points of $y_e = \pm 1$ are asymptotically stable for the original nonlinear system.

Now, if $y_e = 0$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 - 0.1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

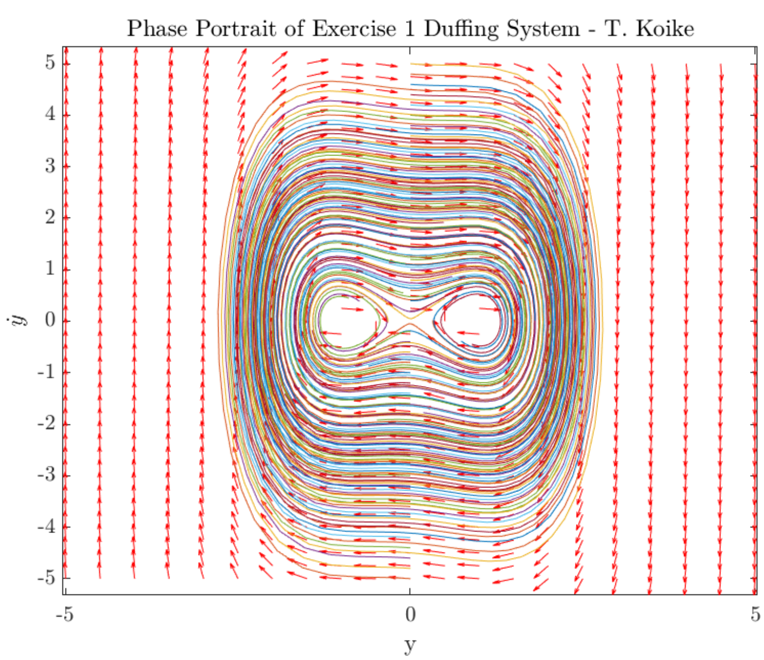
The eigenvalues and corresponding eigenvectors for this linearized system are

$$\lambda_1 = -1.0512 \qquad v_1 = \begin{bmatrix} -0.6892 \\ 0.7245 \end{bmatrix}$$

$$\lambda_2 = 0.9512 \qquad v_2 = \begin{bmatrix} -0.7245 \\ -0.6892 \end{bmatrix}$$

There is a real eigenvalue in both the right- and left-hand plane of the complex plane. This means that the the equilibrium point of $y_e = 0$ is saddle point, and is undetermined for the original nonlinear system.

The phase portrait of this Duffing system becomes



The MATLAB Code for this exercise is as follows.

```
Amat = @(ye) [0, 1; -(3*ye^2 - 1), -0.1];

% ye = -1
ye = -1;
A = Amat(ye);
[v, lambda] = eig(A);

% ye = 0
y = 0;
A = Amat(ye);
[v, lambda] = eig(A);

% ye = 1
ye = 1;
A = Amat(ye);
[v, lambda] = eig(A);
%%
% Phase Portrait
f = @(t,x) [x(2);-0.1*x(2)+x(1)-x(1)^3];
fig = vectfield(f,-5:.5:5,-5:.25:5);
hold on
for y20=-5:0.2:5
  [ts,ys] = ode45(f,[0,10],[0;y20]);
  plot(ys(:,1),ys(:,2))
end
hold off
title('Phase Portrait of Exercise 1 Duffing System - T. Koike')
xlabel('y')
ylabel('$\dot{y}$')
saveas(fig, fullfile(fdir, "ex1-phase-portrait.png"));
%%
function fig = vectfield(func,y1val,y2val,t)
    if nargin==3
      t=0;
    end
    n1=length(y1val);
    n2=length(y2val);
    yp1=zeros(n2,n1);
    yp2=zeros(n2,n1);
    for i=1:n1
      for j=1:n2
        ypv = feval(func,t,[y1val(i);y2val(j)]);
```

```
        yp1(j,i) = ypv(1);
        yp2(j,i) = ypv(2);
      end
    end
    len=sqrt(yp1.^2+yp2.^2);
    fig = quiver(y1val,y2val,yp1./len,yp2./len,.6,'r');
    axis tight;
end
```

Exercise 2

Determine the nature (if possible) of each equilibrium state of the simple pendulum system.

$$\ddot{y} + \sin y = 0$$

Numerically obtain the phase portrait.

Solution:
The equilibrium states are found by setting

$$\begin{cases} \ddot{y} = 0 \\ y := y_e \end{cases}$$

Then

$$\begin{aligned} \sin y_e &= 0 \\ y_e &= \pm n\pi \qquad \text{where } n = 0, 1, 2.. \end{aligned}$$

Next, we linearize the equation.

$$\begin{aligned} \delta\ddot{y} + \sin(\delta y + y_e) &= 0 \\ \delta\ddot{y} + \sin\delta y \cos y_e + \sin y_e \cos\delta y &= 0 \\ \delta\ddot{y} + \cancelto{\delta y}{\sin\delta y} \cos y_e + \sin y_e \cancelto{1}{\cos\delta y} &= 0 \\ \delta\ddot{y} + \delta y \cos y_e + \sin y_e &= 0 \end{aligned}$$

If $x_1 := \delta y$ and $x_2 := \delta\dot{y}$ the system becomes

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \cos y_e \end{bmatrix} + \begin{bmatrix} 0 \\ -\sin y_e \end{bmatrix}$$

Now, if $y_e = \pm 2m\pi$ where m=0,1,2...

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}$$

The eigenvalues and corresponding eigenvectors for this linearized system are

$$\lambda_1 = j \qquad\qquad v_1 = \begin{bmatrix} -0.7071 \\ 0.7071j \end{bmatrix}$$

$$\lambda_2 = -j \qquad\qquad v_2 = \begin{bmatrix} 0.7071 \\ -0.7071j \end{bmatrix}$$

The eigenvalues are on the imaginary axis of the complex plane and are non-defective which means that the linearized system is marginally stable and a center. Thus, the equilibrium point of $y_e = \pm 2m\pi$ where (m=0,1,2...) is stable for the original nonlinear system.

Now, if $y_e = \pm(2m+1)\pi$ where m=0,1,2...

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

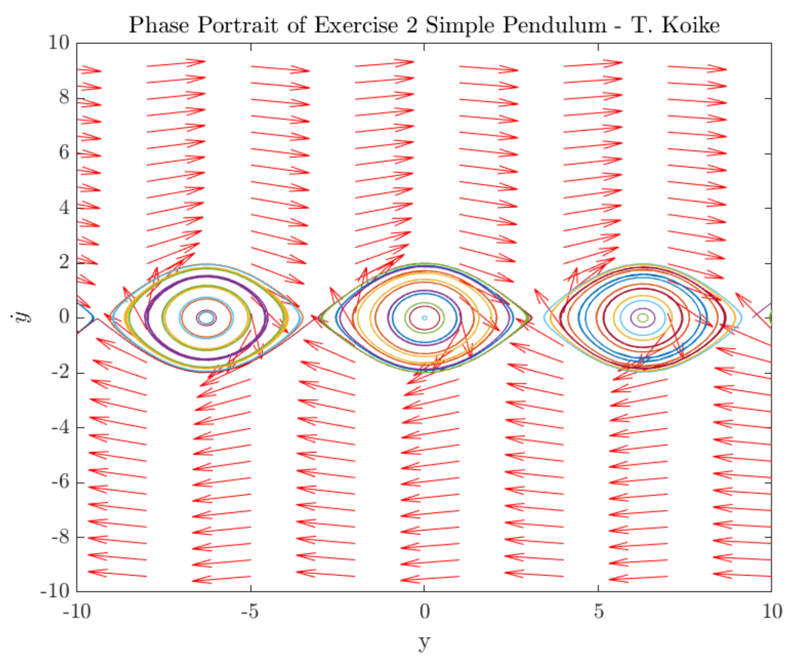
The eigenvalues and corresponding eigenvectors for this linearized system are

$$\lambda_1 = -1 \qquad\qquad v_1 = \begin{bmatrix} -0.7071 \\ 0.7071 \end{bmatrix}$$

$$\lambda_2 = 1 \qquad\qquad v_2 = \begin{bmatrix} 0.7071 \\ 0.7071 \end{bmatrix}$$

There is a real eigenvalue in both the right- and left-hand plane of the complex plane. This means that the the equilibrium point of $y_e = \pm(2m+1)\pi$ where (m=0,1,2...) is saddle point, and is undetermined for the original nonlinear system.

The phase portrait of this Duffing system becomes



The MATLAB Code for this exercise is as follows.

```
Amat = @(ye) [0, 1; -cos(ye), 0];

% ye = (2m)pi
ye = 0;
A = Amat(ye);
[v, lambda] = eig(A);
```

```matlab

% ye = (2m+1)pi
ye = pi;
A = Amat(ye);
[v, lambda] = eig(A);
%%
% Phase Portrait
f = @(t,x) [x(2);-sin(x(1))];
fig = vectfield(f,-20:3:20,-3*pi:0.6:3*pi);
hold on
for yi=-4*pi:0.5:4*pi
  [ts,ys] = ode45(f,[0,100],[yi;0]);
  plot(ys(:,1),ys(:,2))
end
hold off
title('Phase Portrait of Exercise 2 Simple Pendulum - T. Koike')
xlabel('y')
xlim([-10, 10])
ylabel('$\dot{y}$')
saveas(fig, fullfile(fdir, "ex2-phase-portrait.png"));
%%
function fig = vectfield(func,y1val,y2val,t)
    if nargin==3
      t=0;
    end
    n1=length(y1val);
    n2=length(y2val);
    yp1=zeros(n2,n1);
    yp2=zeros(n2,n1);
    for i=1:n1
      for j=1:n2
        ypv = feval(func,t,[y1val(i);y2val(j)]);
        yp1(j,i) = ypv(1);
        yp2(j,i) = ypv(2);
      end
    end
    len=sqrt(yp1.^2+yp2.^2);
    fig = quiver(y1val,y2val,yp1./len,yp2./len,.6,'r');
end
```

Exercise 3

For each of the following systems, determine (from the state portrait) the stability properties of each equilibrium state. For AS equilibrium states, give the region of attraction (RoA).
(a)

$$\dot{x} = -x - x^3$$

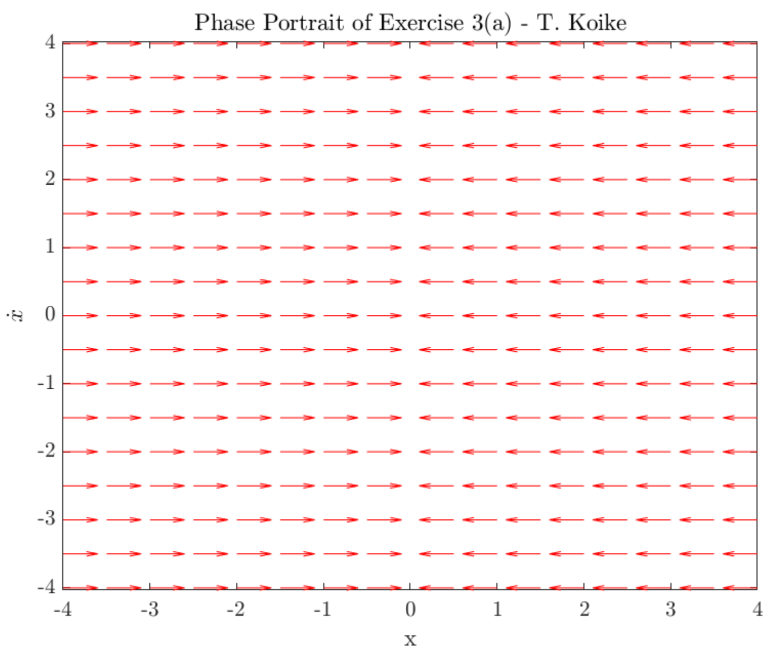
(b)

$$\dot{x} = -x + x^3$$

(c)

$$\dot{x} = x - 2x^2 + x^3$$

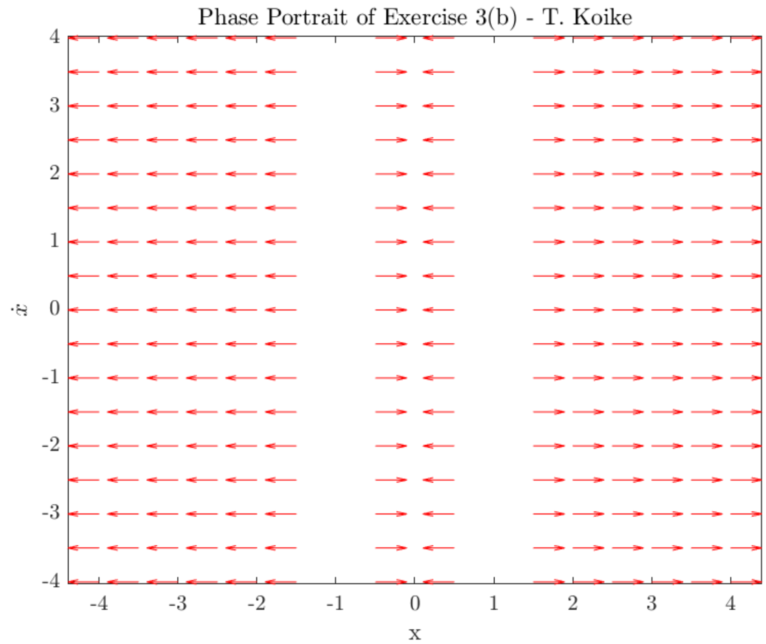
Solution:
(a)
The phase portrait of this system is



Observing the phase portrait, we can say that for each equilibrium states

$$x_e = \begin{cases} \pm j \rightarrow \text{ saddle, and therefore, undetermined} \\ 0 \rightarrow \text{ asymptotically stable node with RoA of } (-\infty, 0] \text{ and } [0, \infty) \end{cases}$$

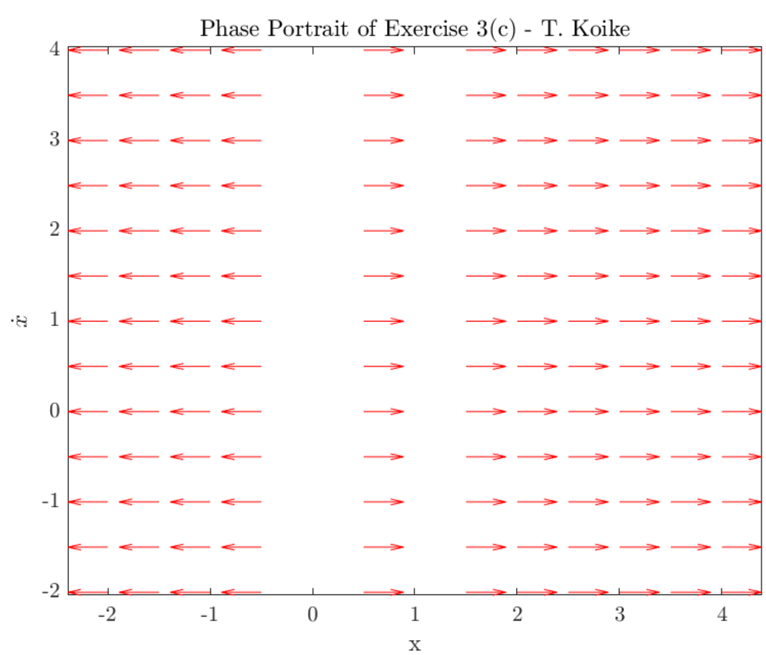
(b)
The phase portrait of this system is



Observing the phase portrait, we can say that for each equilibrium states

$$x_e = \begin{cases} \pm 1 \rightarrow \text{ unstable node} \\ 0 \rightarrow \text{ asymptotically stable node with RoA of } [-1, 0] \text{ and } [0, 1] \end{cases}$$

(c)
The phase portrait of this system is



Observing the phase portrait, we can say that for each equilibrium states

$$x_e = \begin{cases} 1 \rightarrow \text{ saddle, and therefore, undetermined} \\ 0 \rightarrow \text{ unstable node} \end{cases}$$

The MATLAB code for this exercise is as follows.

```
% (a)
f = @(t,x) [-x(1)-x(1)^3;0];
fig = vectfield(f,-4:0.5:4,-4:0.5:4);
hold on
for y20=-4:0.2:4
  [ts,ys] = ode45(f,[0,10],[0;y20]);
  plot(ys(:,1),ys(:,2))
end
hold off
title('Phase Portrait of Exercise 3(a) - T. Koike')
xlabel('x')
ylabel('$\dot{x}$')
saveas(fig, fullfile(fdir, "ex3a-phase-portrait.png"));
%%
% (b)
f = @(t,x) [-x(1)+x(1)^3;0];
fig = vectfield(f,-4:0.5:4,-4:0.5:4);
hold on
for y20=-4:0.2:4
  [ts,ys] = ode45(f,[0,10],[0;y20]);
  plot(ys(:,1),ys(:,2))
end
hold off
title('Phase Portrait of Exercise 3(b) - T. Koike')
xlabel('x')
ylabel('$\dot{x}$')
saveas(fig, fullfile(fdir, "ex3b-phase-portrait.png"));
%%
% (c)
f = @(t,x) [x(1)-2*x(1)^2+x(1)^3;0];
fig = vectfield(f,-2:0.5:4,-2:0.5:4);
hold on
for y20=-2:0.2:4
  [ts,ys] = ode45(f,[0,10],[0;y20]);
  plot(ys(:,1),ys(:,2))
end
hold off
```

```
title('Phase Portrait of Exercise 3(c) — T. Koike')
xlabel('x')
ylabel('$\dot{x}$')
saveas(fig, fullfile(fdir, "ex3c—phase—portrait.png"));
%%
function fig = vectfield(func,y1val,y2val,t)
    if nargin==3
      t=0;
    end
    n1=length(y1val);
    n2=length(y2val);
    yp1=zeros(n2,n1);
    yp2=zeros(n2,n1);
    for i=1:n1
      for j=1:n2
        ypv = feval(func,t,[y1val(i);y2val(j)]);
        yp1(j,i) = ypv(1);
        yp2(j,i) = ypv(2);
      end
    end
    len=sqrt(yp1.^2+yp2.^2);
    fig = quiver(y1val,y2val,yp1./len,yp2./len,.6,'r');
    axis tight;
end
```

Exercise 4

Show that all non-zero solutions of the following differential blow up in a finite time. Compute the "blow-up" time as a function of initial state.

$$\dot{x} = x^3$$

Solution:
Solve this differential equation analytically

$$\frac{dx}{dt} = x^3$$

$$x^{-3}dx = dt$$

$$\int_{x_0}^{x} \chi^{-3} d\chi = \int_0^t d\tau$$

$$-\frac{1}{2x^2} + \frac{1}{2x_0^2} = t$$

$$\frac{1}{x^2} = \frac{1}{x_0^2} - 2t$$

$$\frac{1}{x^2} = \frac{1 - 2x_0^2 t}{x_0^2}$$

Thus, the explicit solution for this differential equation is

$$x(t) = \pm \frac{x_0}{\sqrt{1 - 2x_0^2 t}} \qquad \text{where} \qquad x_0 = x(0)$$

This analytical solution blows up when the denominator $\sqrt{1 - 2x_0^2 t} = 0$. Thus, if we express the "blow-up" time as a function of initial state becomes

$$t(x_0) = \frac{1}{2x_0^2}$$

Exercise 5

Prove that no solution of the following differential equation can "blow up" in a finite time.

$$\dot{x} = \frac{x}{1+x^2} + \sin(x)$$

Solution:
The following condition guarantees that solutions can be extended indefinitely. There are constants α and $beta$ such that

$$\|f(x)\| = \alpha \|x\| + \beta$$

for all x. Thus, if we find a pair of constants α and $beta$ that satisfy this condition we can prove that no solution for the differential equation "blows up".

$$\left\| \frac{x}{1+x^2} + \sin x \right\| \leq \left\| \frac{x}{1+x^2} \right\| + \|\sin x\|$$

Since,

$$0 < \frac{x}{1+x^2} < 1$$
$$0 < \left\| \frac{x}{1+x^2} \right\| < 1$$

and

$$-1 \leq \sin x \leq 1$$
$$0 \leq \|\sin x\| \leq 1$$

Then,

$$0 \leq \left\| \frac{x}{1+x^2} \right\| + \|\sin x\| < 2$$

Thus, we find a pair of constants α and $beta$ that satisfy

$$2 \leq \alpha \|x\| + \beta$$

Since, $\|x\| > 0$, a possible pair that satisfies this is $\alpha = 1$ and $\beta = 2$. Hence,

$$\left\| \frac{x}{1+x^2} + \sin x \right\| \leq \|x\| + 2$$

and this proves that no solution of the given differential equation can "blow up" in a finite time.

q.e.d

Exercise 6

What initial states x_0 can you guarantee that the following equation has a unique solutions with $x(0) = x_0$. Justify your answer.

$$\dot{x} = -\sqrt{(1-x)^2}$$

Solution:
Since differentiability of $\dot{x} = f(x)$ guarantees uniqueness, we have to prove that the given equation $f(x)$ is differentiable. If we rewrite the equation we have

$$\dot{x} = -\|1-x\|$$

This means that this equation is not differentiable only at $x = 1$. Thus, this equation guarantees a unique solution whenever the initial condition $x(0) = x_0$ is not equal to 1.