College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 10
Controllability of Control Systems

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Exercise 1

Determine (by hand) whether or not each of the following systems are controllable.

(a)

$$\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

(b)

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

(c)

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

(a)

The A matrix and B matrix are

$$\mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore rank(Q_c) = 2 \ .$$

Thus, this system is controllable.

(b)

The A matrix and B matrix are

$$\mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} .$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore rank(Q_c) = 1 \;.$$

Thus, this system is uncontrollable.

(c)

The A matrix and B matrix are

$$\mathrm{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Then the controllability matrix becomes

$$Q_c = (B \quad AB) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore rank(Q_c) = 1 \;.$$

Thus, this system is uncontrollable.

MATLAB Verification

```
function res = checkControllability(A, B)
    dim = size(A); n = dim(1);
    Qc = ctrb(A, B);
    res.check = rank(Qc) == n;
    res.Qc = Qc;
end
```

```
% Ex1
% (a)
A = [-1, 0; 0, 1];
B = [1; 1];
res = checkControllability(A, B);
res.check
res.Qc
% (b)
A = [-1, 0; 0, 1];
B = [0; 1];
res = checkControllability(A, B);
res.check
res.Qc
% (c)
A = [1, 0; 1, 0];
B = [1; 1];
res = checkControllability(A, B);
res.check
res.Qc
```

Exercise 2

(By hand) Determine whether or not the following system is controllable.

$$\begin{aligned} \dot{x}_1 &= 5x_1 + x_2 - x_3 + u_1 \\ \dot{x}_2 &= -x_1 + 3x_2 - x_3 + u_1 + u_2 \\ \dot{x}_3 &= -2x_1 - 2x_2 + 4x_3 + u_2 \end{aligned}$$

If the system is uncontrollable, compute the uncontrollable eigenvalues.

The A and B matrices of this system are

$$\mathrm{A} = \begin{pmatrix} 5 & 1 & -1 \\ -1 & 3 & -1 \\ -2 & -2 & 4 \end{pmatrix}, \qquad \mathrm{B} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

Then,

$$\mathrm{AB} = \begin{pmatrix} 5 & 1 & -1 \\ -1 & 3 & -1 \\ -2 & -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 2 & 2 \\ -4 & 2 \end{pmatrix} .$$

$$A^2\mathrm{B} = \begin{pmatrix} 5 & 1 & -1 \\ -1 & 3 & -1 \\ -2 & -2 & 4 \end{pmatrix}\begin{pmatrix} 5 & 1 & -1 \\ -1 & 3 & -1 \\ -2 & -2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 10 & -10 \\ -6 & 10 & -6 \\ -16 & -16 & 20 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 36 & 0 \\ 4 & 4 \\ -32 & 4 \end{pmatrix} .$$

Thus, the controllability matrix becomes

$$Q_c = (B \quad AB \quad A^2B) = \begin{pmatrix} 1 & 0 & 6 & 0 & 36 & 0 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ 0 & 1 & -4 & 2 & -32 & 4 \end{pmatrix} .$$

The reduced echelon form of this matrix is

$$\begin{pmatrix} 1 & 0 & 6 & 0 & 36 & 0 \\ 1 & 1 & 2 & 2 & 4 & 4 \\ 0 & 1 & -4 & 2 & -32 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 6 & 0 & 36 & 0 \\ 0 & 1 & -4 & 2 & -32 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

$$\therefore \ rank(Q_c) = 2 \ .$$

This system is uncontrollable.

Then we conduct the PBH test to find the uncontrollable eigenvalues.

$$A - \lambda I \ = \begin{pmatrix} 5-\lambda & 1 & -1 \\ -1 & 3-\lambda & -1 \\ -2 & -2 & 4-\lambda \end{pmatrix}$$

$$det(A - \lambda \mathrm{I}) = (5-\lambda)\begin{vmatrix} 3-\lambda & -1 \\ -2 & 4-\lambda \end{vmatrix} - \begin{vmatrix} -1 & -1 \\ -2 & 4-\lambda \end{vmatrix} - \begin{vmatrix} -1 & 3-\lambda \\ -2 & -2 \end{vmatrix}$$

$$= (5-\lambda)[(3-\lambda)(4-\lambda)-2] - (-4+\lambda-2) - (2+6-2\lambda)$$

$$= (5-\lambda)(\lambda^2 - 7\lambda + 10) - (-6+\lambda) - (8-2\lambda)$$

$$= (2-\lambda)(\lambda-4)(\lambda-6)$$

$$\therefore \lambda = 2, 4, 6 \ .$$

For $\lambda = 2$,

$$\mathrm{Z} = (A - \lambda I \quad B) = \begin{pmatrix} 3 & 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 & 1 \\ -2 & -2 & 2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 & -0.25 \\ 0 & 1 & -1 & 0 & -0.25 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$rank(Z) = 3$$

This eigenvalue is observable.

For $\lambda = 4$,

$$\mathrm{Z} = (A - \lambda I \quad B) = \begin{pmatrix} 1 & 1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 1 & 1 \\ -2 & -2 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & -0.5 \\ 0 & 0 & 1 & -1 & -0.5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$rank(Z) = 2 \neq 3$$

This eigenvalue of 4 is uncontrollable.

For $\lambda = 6$,

$$\mathrm{Z} = (A - \lambda I \quad B) = \begin{pmatrix} -1 & 1 & -1 & 1 & 0 \\ -1 & -3 & -1 & 1 & 1 \\ -2 & -2 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 & -0.25 \\ 0 & 1 & 0 & 0 & -0.25 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$rank(Z) = 3$$

This eigenvalue is observable.

Exercise 3

Carry out the following for linearizations L1, L3, L7 of the two pendulum cart system.

(a) Determine which linearizations are controllable?
(b) Compute the singular values of the controllability matrix.
(c) Determine the uncontrollable eigenvalues for the uncontrollable linearizations.

You may want to use MATLAB.

The system equation for the double pendulum cart system is

$$\begin{aligned}(m_0+m_1+m_2)\ddot{y} - m_1 l_1 cos\theta_1\ddot{\theta}_1 - m_2 l_2 cos\theta_2\ddot{\theta}_2 + m_1 l_1 sin\theta_1\dot{\theta}_1^2 + m_2 l_2 sin\theta_2\dot{\theta}_2^2 &= u\\ -m_1 l_1 cos\theta_1\ddot{y} + m_1 l_1^2\ddot{\theta}_1 \quad + \quad m_1 l_1 g sin\theta_1 &= 0\\ -m_2 l_2 cos\theta_2\ddot{y} + m_2 l_2^2\ddot{\theta}_2 \quad + \quad m_2 l_2 g sin\theta_2 &= 0\end{aligned}$$

Have the system be a single output of the displacement y.

$$E1: \quad (y^e, \theta_1^e, \theta_2^e) = (0,0,0)$$

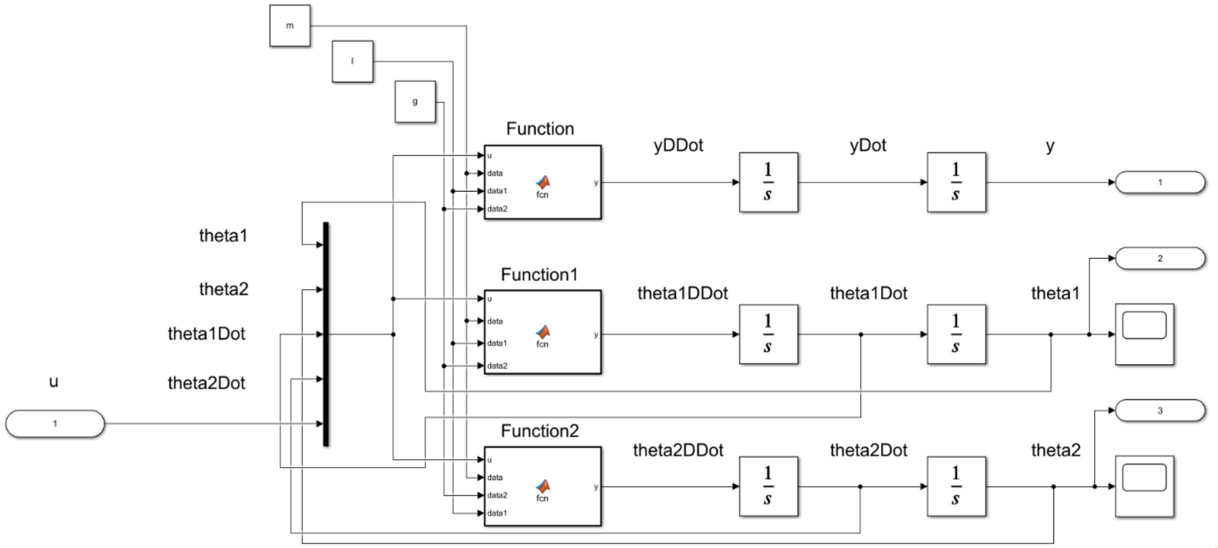
$$E2: \quad (y^e, \theta_1^e, \theta_2^e) = (0,\pi,\pi)$$

	m_0	m_1	m_2	l_1	l_2	g	u
P1	2	1	1	1	1	1	0
P2	2	1	1	1	0.99	1	0
P3	2	1	0.5	1	1	1	0
P4	2	1	1	1	0.5	1	0

L1	P1	E1
L2	P1	E2
L3	P2	E1
L4	P2	E2
L5	P3	E1
L6	P3	E2
L7	P4	E1
L8	P4	E2

(a)

The Simulink model used for this is shown below,



Embedded MATLAB Block – Function (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION
    %}

    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -m1*l1*sin(u(1))*u(3)*u(3) - m2*l2*sin(u(2))*u(4)*u(4)...
            - m1*g*sin(u(1))*cos(u(1)) - m2*g*sin(u(2))*cos(u(2))...
            + u(5);
    den = m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2;
    y = num / den;
end
```

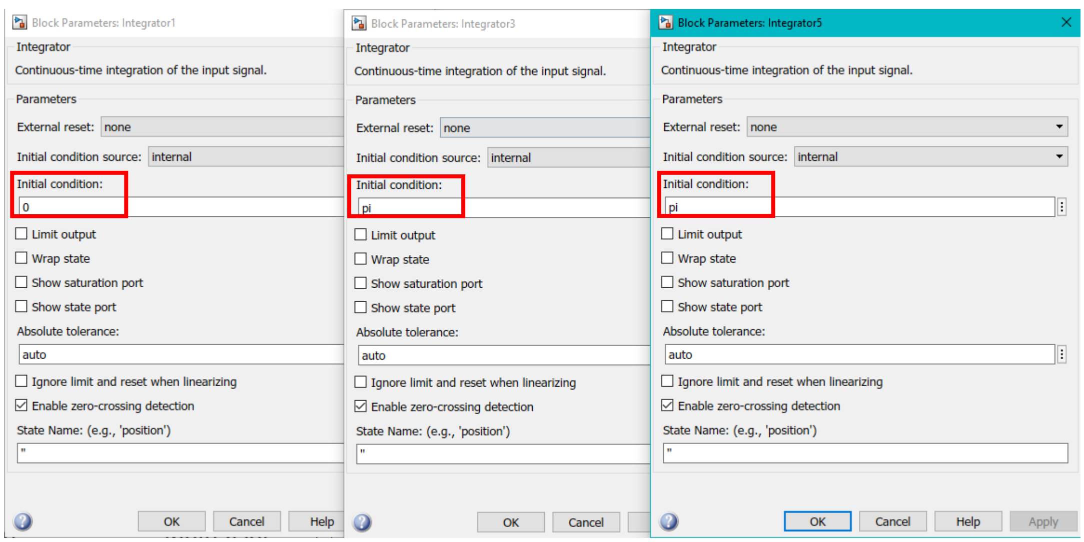
Embedded MATLAB Block – Function1 (code)

```
function y = fcn(u, data, data1, data2)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION1
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(1))*sin(u(2))*u(4)*u(4))...
            + m2*g*(sin(u(1))*cos(u(2))^2 - cos(u(1))*sin(u(2))*cos(u(2)))...
            - (m0 + m1 + m2)*g*sin(u(1)) + u(5)*cos(u(1));
    den = l1*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

Embedded MATLAB Block – Function2 (code)

```
function y = fcn(u, data, data2, data1)
    %{
        EMBEDDED MATLAB BLOCK FUNCTION2
    %}
    m0 = data(1); m1 = data(2); m2 = data(3); l1 = data1(1); l2 = data1(2);
    g = data2;

    num = -(m1*l1*cos(u(2,1))*sin(u(1))*u(3)*u(3) +
m2*l2*cos(u(2))*sin(u(2))*u(4)*u(4))...
            + m1*g*(sin(u(2))*cos(u(1))^2 - cos(u(2))*sin(u(1))*cos(u(1)))...
            - (m0 + m1 + m2)*g*sin(u(2)) + u(5)*cos(u(2));
    den = l2*(m0 + m1 + m2 - m1*cos(u(1))^2 - m2*cos(u(2))^2);
    y = num / den;
end
```

For the conditions E1 and E2, we set the initial conditions of the integrator block of y, θ_1, and θ_2 correspondingly to y^e, θ_1^e, θ_2^e; like in the following windows,



L1:

```
A = 6×6
        0         0         0    1.0000         0         0
        0         0         0         0    1.0000         0
        0         0         0         0         0    1.0000
        0   -0.5000   -0.5000         0         0         0
        0   -1.5000   -0.5000         0         0         0
        0   -0.5000   -1.5000         0         0         0

B = 6×1
        0
        0
        0
   0.5000
   0.5000
   0.5000

C = 1×6
     1     0     0     0     0     0

D = 0
```

The controllability matrix for this system is

```
Qc_L1 = 6×6
        0    0.5000         0   -0.5000         0    1.0000
        0    0.5000         0   -1.0000         0    2.0000
        0    0.5000         0   -1.0000         0    2.0000
   0.5000         0   -0.5000         0    1.0000         0
   0.5000         0   -1.0000         0    2.0000         0
   0.5000         0   -1.0000         0    2.0000         0
```

The reduced echelon form of this matrix is

```
Qc_L1_rref = 6×6
   1.0000         0         0         0         0         0
        0    1.0000         0         0         0         0
        0         0    1.0000         0   -2.0000         0
        0         0         0    1.0000         0   -2.0000
        0         0         0         0         0         0
        0         0         0         0         0         0
```

Thus,

$$rank(Q_c) = 4 < 6$$

This system linearized by L1 is uncontrollable.

L3:

```
A = 6×6
     0         0         0    1.0000         0         0
     0         0         0         0    1.0000         0
     0         0         0         0         0    1.0000
     0   -0.5000   -0.5000         0         0         0
     0   -1.5000   -0.5000         0         0         0
     0   -0.5051   -1.5152         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    0.5051

C = 1×6
     1     0     0     0     0     0

D = 0
```

The controllability matrix for this system is

```
Qc_L3 = 6×6
         0    0.5000         0   -0.5025         0    1.0101
         0    0.5000         0   -1.0025         0    2.0127
         0    0.5051         0   -1.0178         0    2.0484
    0.5000         0   -0.5025         0    1.0101         0
    0.5000         0   -1.0025         0    2.0127         0
    0.5051         0   -1.0178         0    2.0484         0
```

The reduced echelon form of this matrix is

```
Qc_L3_rref = 6×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

Thus,

$$rank(Q_c) = 6$$

This system linearized by L3 is controllable.

L7:

```
A = 6×6
         0         0         0    1.0000         0         0
         0         0         0         0    1.0000         0
         0         0         0         0         0    1.0000
         0   -0.5000   -0.5000         0         0         0
         0   -1.5000   -0.5000         0         0         0
         0   -1.0000   -3.0000         0         0         0

B = 6×1
         0
         0
         0
    0.5000
    0.5000
    1.0000

C = 1×6
     1     0     0     0     0     0

D = 0
```

The controllability matrix for this system is

```
Qc_L7 = 6×6
         0    0.5000         0   -0.7500         0    2.3750
         0    0.5000         0   -1.2500         0    3.6250
         0    1.0000         0   -3.5000         0   11.7500
    0.5000         0   -0.7500         0    2.3750         0
    0.5000         0   -1.2500         0    3.6250         0
    1.0000         0   -3.5000         0   11.7500         0
```

The reduced echelon form of this matrix is

```
Qc_L7_rref = 6×6
     1     0     0     0     0     0
     0     1     0     0     0     0
     0     0     1     0     0     0
     0     0     0     1     0     0
     0     0     0     0     1     0
     0     0     0     0     0     1
```

Thus,

$$rank(Q_c) = 6$$

This system linearized by L7 is controllable.

```
% (a)
global m l g ye theta1e theta2e
param_combo = ["L1","L3","L7"];
for i = 1:numel(param_combo)
    define_params(param_combo(i));
    [A, B, C, D] = linmod('db_pend_cart_lin');
    lin_sys(i).Amat = A;
    lin_sys(i).Bmat = B;
    lin_sys(i).Cmat = C;
    lin_sys(i).Dmat = D;
    sys_ss = ss(A, B, C, D);  % get the state space system
    CTR(i) = checkControllability(A, B);  % check the observability of the system
    eigCTR{i} = find_unctrb_eigVal(A, B);  % check the observability of the
eigenvalues
end

Qc_L1 = CTR(1).Qc
Qc_L1_rref = rref(Qc_L1)
Qc_L3 = CTR(2).Qc
Qc_L3_rref = rref(Qc_L3)
Qc_L7 = CTR(3).Qc
Qc_L7_rref = rref(Qc_L7)

function define_params(L)
    % Function to define parameters
    global m l g ye theta1e theta2e
    if L == "L1"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L2"
        m = [2,1,1]; l = [1,1]; g = 1; % P1
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L3"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L4"
        m = [2,1,1]; l = [1,0.99]; g = 1; % P2
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L5"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L6"
        m = [2,1,0.5]; l = [1,1]; g = 1; % P3
        ye = 0; theta1e = pi; theta2e = pi; % E2
    elseif L == "L7"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; theta1e = 0; theta2e = 0; % E1
    elseif L == "L8"
        m = [2,1,1]; l = [1,0.5]; g = 1; % P4
        ye = 0; theta1e = pi; theta2e = pi; % E2
    else
        print('error: did not match any')
    end
end
```

(b)

For this problem, we conduct single value decomposition using MATLAB.

```
% (b)
[U_L1, S_L1, V_L1] = svd(Qc_L1);
[U_L3, S_L3, V_L3] = svd(Qc_L3);
[U_L7, S_L7, V_L7] = svd(Qc_L7);
S_L1
S_L3
S_L7
```

L1:

```
S_L1 = 6×6
    3.4565         0         0         0         0         0
         0    3.4565         0         0         0         0
         0         0    0.2287         0         0         0
         0         0         0    0.2287         0         0
         0         0         0         0    0.0000         0
         0         0         0         0         0    0.0000
```

L3:

```
S_L3 = 6×6
    3.5019         0         0         0         0         0
         0    3.5019         0         0         0         0
         0         0    0.2290         0         0         0
         0         0         0    0.2290         0         0
         0         0         0         0    0.0016         0
         0         0         0         0         0    0.0016
```

L7:

```
S_L7 = 6×6
   13.1371         0         0         0         0         0
         0   13.1371         0         0         0         0
         0         0    0.3513         0         0         0
         0         0         0    0.3513         0         0
         0         0         0         0    0.1083         0
         0         0         0         0         0    0.1083
```

(c)

The uncontrollable system is only L1.

The eigenvalues for L1 are

```
eigVal = 6×1 complex
   0.0000 + 0.0000i
   0.0000 + 0.0000i
   0.0000 + 1.4142i
   0.0000 - 1.4142i
  -0.0000 + 1.0000i
  -0.0000 - 1.0000i
```

For $\lambda = 0$,

$$Z = (A - \lambda I \quad B)$$

```
Z = 6×7
         0         0         0    1.0000         0         0         0
         0         0         0         0    1.0000         0         0
         0         0         0         0         0    1.0000         0
         0   -0.5000   -0.5000         0         0         0    0.5000
         0   -1.5000   -0.5000         0         0         0    0.5000
         0   -0.5000   -1.5000         0         0         0    0.5000
```

The reduced echelon form of Z is

```
Zrref = 6×7
     0     1     0     0     0     0     0
     0     0     1     0     0     0     0
     0     0     0     1     0     0     0
     0     0     0     0     1     0     0
     0     0     0     0     0     1     0
     0     0     0     0     0     0     1
```

$$rank(Z) = 6$$

The eigenvalue 0 is controllable.

For $\lambda = 1.4142j$,

$$Z = (A - \lambda I \quad B)$$

```
Z = 6×7 complex
  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.4142i   0.5000 + 0.0000i
```

The reduced echelon form of Z is

```
Zrref = 6×7 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.3536i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.7071i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 0.7071i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i  -1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
```

$$rank(Z) = 6$$

The eigenvalue 1.4142j is controllable.

For $\lambda = -1.4142j$,

$$Z = \begin{pmatrix} A - \lambda I & B \end{pmatrix}$$

```
Z = 6×7 complex
  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.4142i   0.5000 + 0.0000i
```

The reduced echelon form of Z is

```
Zrref = 6×7 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.3536i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.7071i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.7071i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i  -0.5000 - 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i  -1.0000 - 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i
```

$$rank(Z) = 6$$

The eigenvalue -1.4142j is controllable.

For $\lambda = j$,

$$Z = (A - \lambda I \quad B)$$

```
Z = 6×7 complex
   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i   0.5000 + 0.0000i
```

The reduced echelon form of Z is

```
Zrref = 6×7 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 1.0000i  -1.0000 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 1.0000i  -0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 - 1.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

$$rank(Z) = 5$$

The eigenvalue j is uncontrollable.

For $\lambda = -j$,

$$Z = \begin{pmatrix} A - \lambda I & B \end{pmatrix}$$

```
Z = 6×7 complex
   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -1.5000 + 0.0000i  -0.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   0.0000 + 0.0000i   0.5000 + 0.0000i
   0.0000 + 0.0000i  -0.5000 + 0.0000i  -1.5000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i   0.5000 + 0.0000i
```

The reduced echelon form of Z is

```
Zrref = 6×7 complex
   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 1.0000i  -1.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 1.0000i  -0.0000 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   0.0000 + 0.0000i  -0.0000 - 0.0000i   0.0000 - 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   1.0000 + 0.0000i   1.0000 - 0.0000i   0.0000 + 1.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
```

$$rank(Z) = 5$$

The eigenvalue -j is uncontrollable.

Exercise 4

(BB in laundromat: external excitation) Obtain a state representation of the following system.

$$
\begin{aligned}
m\ddot{q}_1 - m\Omega^2 q_1 + k(q_1 - q_2) &= 0 \\
m\ddot{q}_2 - m\Omega^2 q_2 - k(q_1 - q_2) &= u
\end{aligned}
$$

Determine whether or not your state space representation system is controllable.

Manipulating the system, we obtain

$$
\begin{aligned}
\ddot{q}_1 &= \frac{m\Omega^2 - k}{m} q_1 + \frac{k}{m} q_2 \\
\ddot{q}_2 &= \frac{k}{m} q_1 + \frac{m\Omega^2 - k}{m} q_2 + \frac{u}{m}
\end{aligned}
$$

Let $x_1 \coloneqq q_1,\ x_2 \coloneqq q_2,\ x_3 \coloneqq \dot{q}_1,\ x_4 \coloneqq \dot{q}_2$, then the state representation of this system becomes

$$
\begin{pmatrix}
\dot{x}_1 & = & x_3 \\
\dot{x}_2 & = & x_4 \\
\dot{x}_3 & = & \frac{m\Omega^2 - k}{m} x_1 + \frac{k}{m} x_2 \\
\dot{x}_4 & = & \frac{k}{m} x_1 + \frac{m\Omega^2 - k}{m} x_2 + \frac{u}{m}
\end{pmatrix}
$$

Thus, the A and B matrices become

$$
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{m\Omega^2 - k}{m} & \frac{k}{m} & 0 & 0 \\
\frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & 0
\end{pmatrix}, \qquad
B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{pmatrix}
$$

Then the controllability matrix becomes

$$
Q_c = (B \quad AB \quad A^2B \quad A^3B)
$$

$$
AB = \begin{pmatrix} 0 \\ 1/m \\ 0 \\ 0 \end{pmatrix}
$$

$$A^2B = AAB = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{m\Omega^2 - k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1/m \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{k}{m^2} \\ \frac{m\Omega^2 - k}{m^2} \end{pmatrix}$$

$$A^3B = A(AAB) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{m\Omega^2 - k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{k}{m^2} \\ \frac{m\Omega^2 - k}{m^2} \end{pmatrix} = \begin{pmatrix} \frac{k}{m^2} \\ \frac{m\Omega^2 - k}{m^2} \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$$Q_c = \begin{pmatrix} 0 & 0 & 0 & \frac{k}{m^2} \\ 0 & 1 & 0 & \frac{m\Omega^2 - k}{m^2} \\ 0 & 0 & \frac{k}{m^2} & 0 \\ 1 & 0 & \frac{m\Omega^2 - k}{m^2} & 0 \end{pmatrix}.$$

The reduced echelon form of this matrix is

$$\sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore rank(Q_c) = 4 \ .$$

This system is controllable.

Exercise 5

(BB in Laundromat: self excited.) (By hand.) Obtain a state space representation of the following system.

$$\begin{aligned} m\ddot{\phi}_1 - m\Omega^2\phi_1 + k(\phi_1 - \phi_2) &= -u \\ m\ddot{\phi}_2 - m\Omega^2\phi_2 - k(\phi_1 - \phi_2) &= u \\ y &= \phi_1 \end{aligned}$$

(a) Determine the uncontrollable eigenvalues. Consider $\omega \coloneqq \sqrt{\frac{k}{2m}} > \Omega$.

(b) Obtain a basis for its controllable subspace

(c) Obtain a reduced order controllable system which has the same input-output behavior as the original system when initial conditions are zero.

(a)

Manipulating the system, we obtain

$$\begin{aligned} \ddot{\phi}_1 &= \frac{m\Omega^2 - k}{m}\phi_1 + \frac{k}{m}\phi_2 - \frac{u}{m} \\ \ddot{\phi}_2 &= \frac{k}{m}\phi_1 + \frac{m\Omega^2 - k}{m}\phi_2 + \frac{u}{m} \end{aligned}$$

Let $x_1 \coloneqq \phi_1,\ x_2 \coloneqq \phi_2,\ x_3 \coloneqq \dot{\phi}_1,\ x_4 \coloneqq \dot{\phi}_2$, then the state representation of this system becomes

$$\begin{pmatrix} \dot{x}_1 &=& x_3 \\ \dot{x}_2 &=& x_4 \\ \dot{x}_3 &=& \frac{m\Omega^2 - k}{m}x_1 + \frac{k}{m}x_2 - \frac{u}{m} \\ \dot{x}_4 &=& \frac{k}{m}x_1 + \frac{m\Omega^2 - k}{m}x_2 + \frac{u}{m} \end{pmatrix}$$

$$y = x_1$$

Then the A and B matrices become

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{m\Omega^2 - k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ -1/m \\ 1/m \end{pmatrix}.$$

Then the controllability matrix become

$$Q_c = (B \quad AB \quad A^2B \quad A^3B)$$

$$\mathrm{AB} = \begin{pmatrix} -1/m \\ 1/m \\ 0 \\ 0 \end{pmatrix}$$

$$A^2B = AAB = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & 0 & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1/m \\ 1/m \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{k}{m^2} - \dfrac{m\Omega^2 - k}{m^2} \\ -\dfrac{k}{m^2} + \dfrac{m\Omega^2 - k}{m^2} \end{pmatrix}$$

$$A^3B = A(AAB) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & 0 & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dfrac{k}{m^2} - \dfrac{m\Omega^2 - k}{m^2} \\ -\dfrac{k}{m^2} + \dfrac{m\Omega^2 - k}{m^2} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{k}{m^2} - \dfrac{m\Omega^2 - k}{m^2} \\ -\dfrac{k}{m^2} + \dfrac{m\Omega^2 - k}{m^2} \\ 0 \\ 0 \end{pmatrix}$$

Thus,

$$Q_c = \begin{pmatrix} 0 & -1 & 0 & \dfrac{k}{m^2} - \dfrac{m\Omega^2 - k}{m^2} \\ 0 & 1 & 0 & -\dfrac{k}{m^2} + \dfrac{m\Omega^2 - k}{m^2} \\ -1 & 0 & \dfrac{k}{m^2} - \dfrac{m\Omega^2 - k}{m^2} & 0 \\ 1 & 0 & -\dfrac{k}{m^2} + \dfrac{m\Omega^2 - k}{m^2} & 0 \end{pmatrix}.$$

The reduced echelon form of this matrix is

$$\sim \begin{pmatrix} 1 & 0 & \dfrac{m\Omega^2 - 2k}{m} & 0 \\ 0 & 1 & 0 & \dfrac{m\Omega^2 - 2k}{m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore rank(Q_c) = 2 \neq 4$$

This system in uncontrollable.

Now we find the uncontrollable eigenvalues.

The eigenvalues of this system are

$$\lambda = \pm\Omega, \quad \pm\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$$

When $\lambda = \Omega$,

$$Z = (A - \lambda I \quad B)$$

$$Z = \begin{pmatrix} -\Omega & 0 & 1 & 0 & 0 \\ 0 & -\Omega & 0 & 1 & 0 \\ \frac{m\Omega^2 - k}{m} & \frac{k}{m} & -\Omega & 0 & \frac{-1}{m} \\ \frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & -\Omega & \frac{1}{m} \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & -\frac{1}{\Omega} & \frac{1}{k} \\ 0 & 1 & 0 & -\frac{1}{\Omega} & 0 \\ 0 & 0 & 1 & -1 & \frac{\Omega}{k} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$\text{rank}(Z) = 3 \neq 4\,.$$

The eigenvalue of Ω is uncontrollable.

When $\lambda = -\Omega$,

$$Z = (A - \lambda I \quad B)$$

$$Z = \begin{pmatrix} \Omega & 0 & 1 & 0 & 0 \\ 0 & \Omega & 0 & 1 & 0 \\ \frac{m\Omega^2 - k}{m} & \frac{k}{m} & \Omega & 0 & \frac{-1}{m} \\ \frac{k}{m} & \frac{m\Omega^2 - k}{m} & 0 & \Omega & \frac{1}{m} \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{\Omega} & \frac{1}{k} \\ 0 & 1 & 0 & \frac{1}{\Omega} & 0 \\ 0 & 0 & 1 & -1 & -\frac{\Omega}{k} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$rank(Z) = 3 \neq 4$$

The eigenvalue of $-\Omega$ is uncontrollable.

When $\lambda = -\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$,

$$Z = (A - \lambda I \quad B)$$

$$Z = \begin{pmatrix} -\frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 & 1 & 0 & 0 \\ 0 & -\frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 & 1 & 0 \\ \frac{m\Omega^2-k}{m} & \frac{k}{m} & -\frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 & \frac{-1}{m} \\ \frac{k}{m} & \frac{m\Omega^2-k}{m} & 0 & -\frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & \frac{1}{m} \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & -\frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 \\ 0 & 1 & 0 & \frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$rank(Z) = 4$$

The eigenvalue of $-\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$ is controllable.

When $\lambda = \frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$,

$$Z = (A - \lambda I \quad B)$$

$$Z = \begin{pmatrix} \frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 & 1 & 0 & 0 \\ 0 & \frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 & 1 & 0 \\ \frac{m\Omega^2-k}{m} & \frac{k}{m} & \frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 & \frac{-1}{m} \\ \frac{k}{m} & \frac{m\Omega^2-k}{m} & 0 & \frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & \frac{1}{m} \end{pmatrix}$$

The reduced echelon form is

$$Z = \begin{pmatrix} 1 & 0 & 0 & \frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 \\ 0 & 1 & 0 & -\frac{\sqrt{-m\,(2\,k-\Omega^2\,m)}}{2\,k-\Omega^2\,m} & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$$rank(Z) = 4$$

The eigenvalue of $\frac{\sqrt{-m(2k-\Omega^2 m)}}{m}$ is controllable.

(b)

From part (a) we know that the reduced echelon form of the controllability matrix is

$$Q_c = \begin{pmatrix} 1 & 0 & \frac{m\Omega^2-2k}{m} & 0 \\ 0 & 1 & 0 & \frac{m\Omega^2-2k}{m} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then the basis of the controllable subspace becomes the column space of Q_c

$$c_1 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad c_1, c_2 \neq 0$$

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c)

The reduced order is $\dot{x}_c = A_{cc}x_c + B_c u,\ \ y = C_c x_c + Du$

From the basis

$$\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \ \ \textit{where}\ \ c_1, c_2 \neq 0$$

Then

$$\mathrm{T} = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\widetilde{\boldsymbol{A}} = \boldsymbol{T}^{-1}\mathrm{AT}$$

$$= \begin{pmatrix} -0.5 & 0.5 & 0 & 0 \\ 0 & 0 & -0.5 & 0.5 \\ 0.5 & 0.5 & 0 & 0 \\ 0 & 1 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{m\Omega^2 - k}{m} & \dfrac{k}{m} & 0 & 0 \\ \dfrac{k}{m} & \dfrac{m\Omega^2 - k}{m} & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{2\,k - \Omega^2\,m}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \Omega^2 & 0 \end{pmatrix}$$

$$\widetilde{\boldsymbol{B}} = \boldsymbol{T}^{-1}\boldsymbol{B} = \begin{pmatrix} 0 \\ 1/m \\ 0 \\ 0 \end{pmatrix}$$

$$\widetilde{\boldsymbol{C}} = \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix}$$

Thus,

$$\dot{\boldsymbol{x}}_c = \begin{pmatrix} 0 & 1 \\ -\dfrac{2\,k - \Omega^2\,m}{m} & 0 \end{pmatrix} \boldsymbol{x}_c + \begin{pmatrix} 0 \\ 1/m \end{pmatrix} \boldsymbol{u}$$

$$\mathrm{y} = \begin{pmatrix} -1 & 1 \end{pmatrix} \boldsymbol{x}_c$$

Exercise 6

(By hand.) Consider a system described by

$$
\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 + b_1 u \\
\dot{x}_2 &= \lambda_2 x_2 + b_2 u \\
&\vdots \\
\dot{x}_n &= \lambda_n x_n + b_n u
\end{aligned}
$$

where all quantities are scalar. Obtain conditions on the numbers $\lambda_1, \cdots, \lambda_n$ and $b_1, \cdots, b_n$ which are necessary and sufficient for the controllability of this system. (Hint: PBH time.)

The A matrix of this system is

$$
A = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}
$$

The C matrix is

$$
\mathrm{C} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}
$$

Then the PBH test for controllability is

$$
\begin{aligned}
(A - \lambda I \quad B) &= \left(\begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix} - \lambda_i \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \quad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \right) \\
&= \begin{pmatrix} \lambda_1 - \lambda_i & \cdots & 0 & b_1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \lambda_n - \lambda_i & b_n \end{pmatrix}
\end{aligned}
$$

The leading columns are surrounded by rounded rectangle, and these leading columns cannot be zero in order for the matrix to have a full rank. Also, the system must have an input that is non-zero. Thus, the condition for controllability becomes

$$
\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n
$$

and

$$
b_i \neq 0 \in [b \mid 1 \leq i \leq n]
$$

Exercise 7

Consider the system described by

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= 4x_1 + 2u \end{aligned}$$

Find (by hand) a non-zero w such for every input $u(\cdot)$, every solution $x_1(\cdot)$ of this system satisfies

$$w'x(t) = e^{-2t} w' x(0).$$

The A and B matrix of this system is as follows.

$$A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The controllability matrix of this system is

$$Q_c = (B \quad AB) = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow rank(Q_c) = 1 \neq 2$$

This system is uncontrollable.

The eigenvalues of this system is

$$det(A - \lambda I) = 0 \Rightarrow (\lambda - 2)(\lambda + 2) = 0 \ .$$

Choosing $\lambda = -2$, we check if this eigenvalue is uncontrollable or not.

$$Z = (A - \lambda I \quad B) = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0.5 & 0.5 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow rank(Z) = 1 \neq 2$$

Thus, we verified that this eigenvalue is uncontrollable.

Since,

$$w'AB = \lambda w'B$$

$$\Rightarrow (w_1 \quad w_2)\begin{pmatrix} 2 \\ 4 \end{pmatrix} = -2(w_1 \quad w_2)\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow w_1 + 2w_2 = 0$$

$$\begin{pmatrix} -2w_2 \\ w_2 \end{pmatrix} = w_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} \Rightarrow w' = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Exercise 8

Suppose that λ is an uncontrollable complex eigenvalue of the system

$$\dot{x} = Ax + Bu$$

where $x, A,$ and B are real. Show that there are real vectors u and v such that for every initial condition $x(0) = x_0$ and every $u(\cdot)$,

$$u'x(t) = e^{\alpha t}(u\cos\omega t - v\sin\omega t)'x_0$$

$$v'x(t) = e^{\alpha t}(u\sin\omega t + v\cos\omega t)'x_0$$

where $\lambda = \alpha + j\omega$.

Let $w = u + jv$. Then if the eigenvalue of λ is uncontrollable for the system we know that there exists a w that satisfies

$$w'x(t) = e^{\lambda t}w'x_0.$$

Where w is a non-zero vector such that for every input $u(\cdot)$, every solution $x_1(\cdot)$ of this system suffices the equation above.

This will then become

$$\Rightarrow \left(u' + jv'\right)x(t) = e^{(\alpha+j\omega)t}(u' + jv')x_0$$

$$\Rightarrow u'x(t) + jv'x(t) = e^{(\alpha+j\omega)t}u'x_0 + je^{(\alpha+j\omega)t}v'x_0$$

Since

$$e^{(\alpha+j\omega)t} = e^{\alpha t}(\cos\omega t + j\sin\omega t)$$

$$\Rightarrow u'x(t) + jv'x(t) = e^{\alpha t}(\cos\omega t + j\sin\omega t)u'x_0 + je^{\alpha t}(\cos\omega t + j\sin\omega t)v'x_0$$

$$\begin{aligned}\Rightarrow u'x(t) + jv'x(t) \\ = e^{\alpha t}\cos(\omega t)u'x_0 - e^{\alpha t}\sin(\omega t)v'x_0 + je^{\alpha t}\sin(\omega t)u'x_0 + je^{\alpha t}\cos(\omega t)v'x_0\end{aligned}$$

Which can be separated into the real and imaginary part

$$\begin{aligned} u'x(t) &= e^{\alpha t}(u'\cos\omega t - v'\sin\omega t)x_0 \\ v'x(t) &= e^{\alpha t}(u'\sin\omega t + v'\cos\omega t)x_0 \end{aligned}$$

Which proves to be

$$u'x(t) = e^{\alpha t}(u\cos\omega t - v\sin\omega t)'x_0$$

$$v'x(t) = e^{\alpha t}(u\sin\omega t + v\cos\omega t)'x_0$$

$q.e.d$