AAE 564, Fall 2019

$\boxed{\text{Homework 2 - Solution}}$

Exercise 1 To obtain the matrices A, B, C, D, we first need to obtain a state space representation.

(a)

$$
\begin{aligned}
u &= a_0 q + a_1 \dot{q} + \ldots + a_{n-1} q^{(n-1)} + q^{(n)} \\
y &= \beta_0 q + \beta_1 \dot{q} + \ldots + \beta_{n-1} q^{(n-1)} + \gamma u
\end{aligned}
$$

Defining the state variables

$$
x_1 = q \qquad \text{and} \qquad x_i = q^{(i-1)} \quad , \qquad i = 2, \ldots, n
$$

the following state space representation is obtained.

$$
\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ u - a_{n-1} x_n - \ldots - a_1 x_2 - a_0 x_1 \end{pmatrix}
$$

$$
y = \beta_0 x_1 + \beta_1 x_2 + \ldots + \beta_{n-1} x_n + \gamma u
$$

Defining the state space vector $x = (x_1\, x_2\, \ldots\, x_n)^T$ these equations can be written as:

$$
\boxed{x = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} x + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u}
$$

$$
\boxed{y = \underbrace{\begin{bmatrix} \beta_0 & \beta_1 & \ldots & \beta_{n-1} \end{bmatrix}}_{\mathbf{C}} x + \underbrace{\begin{bmatrix} \gamma \end{bmatrix}}_{\mathbf{D}} u}
$$

(b) In this exercise we are asked to find the matrices for a discrete time system.

$$
\begin{aligned}
u(k) &= a_0 q(k) + a_1 q(k+1) + \ldots + a_{n-1} q(k+n-1) + q(k+n) \\
y(k) &= \beta_0 q(k) + \beta_1 q(k+1) + \ldots + \beta_{n-1} q(k+n-1) + \gamma u(k)
\end{aligned}
$$

We introduce the following state variables:

$$x_i(k) = q(k+i) \quad , \qquad i = 0, \ldots, n-1$$

which leads to the state space representation

$$\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{pmatrix} = \begin{pmatrix} x_2(k) \\ x_3(k) \\ \vdots \\ x_n(k) \\ -a_{n-1}x_n(k) - \ldots - a_1 x_2(k) - a_0 x_1(k) + u(k) \end{pmatrix}$$

$$y(k) = \beta_0 x_1(k) + \beta_1 x_2(k) + \ldots + \beta_{n-1} x_n(k) + \gamma u(k)$$

Defining the vector $x(k) = (x_1(k)\, x_2(k)\, \ldots\, x_n(k))^T$ the state space representation can be expressed as

$$\boxed{\bar{x}(k+1) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix}}_{\mathbf{A}} \bar{x}(k) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}} u(k)}$$

$$\boxed{y(k) = \underbrace{\begin{bmatrix} \beta_0 & \beta_1 & \ldots & \beta_{n-1} \end{bmatrix}}_{\mathbf{C}} \bar{x}(k) + \underbrace{\begin{bmatrix} \gamma \end{bmatrix}}_{\mathbf{D}} u(k)}$$

(c) The equations for the simple structure can be found inon page 9 of the lecture notes. Rearranging them to solve for the highest order derivatives gives:

$$\begin{aligned} \ddot{q}_1 &= \frac{1}{m_1}\left[-(c_1+c_2)\dot{q}_1 - (k_1+k_2)q_1 + c_2\dot{q}_2 + k_2 q_2 + u_1\right] \\ \ddot{q}_2 &= \frac{1}{m_2}\left[c_2\dot{q}_1 + k_2 q_1 - c_2\dot{q}_2 - k_2 q_2 + u_2\right] \end{aligned}$$

Now the state variables can be defined as:

$$x_1 = q_1, \quad x_2 = \dot{q}_1, \quad x_3 = q_2, \quad x_4 = \dot{q}_2, \quad \bar{x} = (x_1, x_2, x_3, x_4)^T, \quad \bar{u} = (u_1, u_2)^T$$

which leads to the state space representation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ \frac{1}{m_1}\left[-(c_1+c_2)x_2 - (k_1+k_2)x_1 + c_2 x_4 + k_2 x_3 + u_1\right] \\ x_4 \\ \frac{1}{m_2}\left[c_2 x_2 + k_2 x_1 - c_2 x_4 - k_2 x_3 + u_2\right] \end{pmatrix}$$

With $y = (q_1\, q_2)^T$ as output, we can put this into a matrix-form

$$\boxed{\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{k_2}{m_1} & \frac{c_2}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{k_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix}}_{\mathbf{A}} x + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{\mathbf{B}} u}$$

$$\boxed{y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\mathbf{C}} x + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\mathbf{D}} u}$$

Exercise 2 (a)

$$\dot{x} = x^3 - x$$

The equilibrium states are given by

$$(x^e)^3 - x^e = 0$$

Hence

$$x^e = 0, \quad x^e = 1, \quad \text{or} \quad x^e = -1$$

Linearization about $x^e = 0$ yields

$$\boxed{\delta\dot{x} = -\delta x}$$

Linearization about $x^e = 1$ yields

$$\boxed{\delta\dot{x} = 2\delta x}$$

Linearization about $x^e = -1$ yields

$$\boxed{\delta\dot{x} = 2\delta x}$$

(b) The graph of $f(x) = \sqrt{|x|}$ is shown in figure 1. The only equilibrium state is $x^e = 0$. The derivative of f does not exist at zero. Therefore the $\boxed{\text{linearization is not possible}}$.

(c) The 'attitude dynamics' system is described by

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\omega_2\omega_3 \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\omega_3\omega_1 \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\omega_1\omega_2 \end{aligned}$$

We consider the non-symmetric case ($I_1 \neq I_2 \neq I_3$) in this exercise. Equilibrium conditions are

$$\begin{aligned} (I_2 - I_3)\omega_2\omega_3 &= 0 \\ (I_3 - I_1)\omega_3\omega_1 &= 0 \\ (I_1 - I_2)\omega_1\omega_2 &= 0 \end{aligned}$$

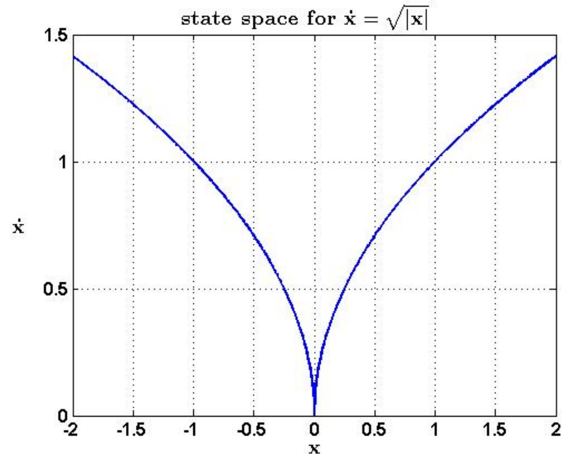

Figure 1: Graph of $f(x) = \sqrt{|x|}$

Since $I_1 \neq I_2 \neq I_3$ these conditions are equivalent to

$$\begin{aligned}\omega_2\omega_3 &= 0\\ \omega_3\omega_1 &= 0\\ \omega_1\omega_2 &= 0\end{aligned}$$

Hence at least two of the angular velocities have to be 0 and all the equilibrium angular velocity vectors are given by

$$\omega^e = \begin{pmatrix} c\\ 0\\ 0\end{pmatrix}, \quad \omega^e = \begin{pmatrix} 0\\ c\\ 0\end{pmatrix}, \quad \omega^e = \begin{pmatrix} 0\\ 0\\ c\end{pmatrix}$$

where $\omega = (\omega_1\,\omega_2\,\omega_3)^T$ and c is arbitrary.

The state variables can be defined as:

$$(x_1\,x_2\,x_3)^T = (\omega_1\,\omega_2\,\omega_3)^T$$

Linearization about an equilibrium start results in

$$\boxed{\begin{array}{lllll} \delta\dot{x}_1 &=& \frac{(I_2-I_3)}{I_1}x_3^e\delta x_2 &+& \frac{(I_2-I_3)}{I_1}x_2^e\delta x_3\\ \delta\dot{x}_2 &=& \frac{(I_3-I_1)}{I_2}x_3^e\delta x_1 &+& \frac{(I_3-I_1)}{I_2}x_1^e\delta x_3\\ \delta\dot{x}_3 &=& \frac{(I_1-I_2)}{I_3}x_2^e\delta x_1 &+& \frac{(I_1-I_2)}{I_3}x_1^e\delta x_2 \end{array}}$$

where $x^e = \omega^e$. Now we can obtain the system matrices (see also previous exercise) for the three different cases by plugging in the values for x^e.

$$\boxed{\mathbf{A}_{e1} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & \frac{I_3-I_1}{I_2}c\\ 0 & \frac{I_1-I_2}{I_3}c & 0 \end{pmatrix}}$$

$$\boxed{\mathbf{A}_{e2} = \begin{pmatrix} 0 & 0 & \frac{I_2-I_3}{I_1}c \\ 0 & 0 & 0 \\ \frac{I_1-I_2}{I_3}c & 0 & 0 \end{pmatrix}}$$

$$\boxed{\mathbf{A}_{e3} = \begin{pmatrix} 0 & \frac{I_2-I_3}{I_1}c & 0 \\ \frac{I_3-I_1}{I_2}c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

The linearized system is now described as:

$$\boxed{\delta\dot{x} = \mathbf{A}\delta x}$$

Note that for the special case of $c = 0$ or $x^e = 0$ the linearized system matrix $\mathbf{A}$ will be $\mathbf{0}$.

Exercise 3 The system dynamics are described by:

$$\begin{aligned} \dot{x}_1 &= 2x_2(1-x_1) - x_1 \\ \dot{x}_2 &= 3x_1(1-x_2) - x_2 \end{aligned}$$

(a) The equilibrium points can be found by solving the equation for $\dot{x}_1^e = \dot{x}_2^e = 0$.

$$\begin{aligned} x_1^e &= 2x_2^e(1-x_1^e) \quad (I) \\ x_2^e &= 3x_1^e(1-x_2^e) \quad (II) \end{aligned}$$

Assuming an equilibrium with $x_1^e = 0$ than $x_2^e = 0$ follows. Hence: $\boxed{\bar{x}_1^e = (0,0)^T}$ is an equilibrium. It also follows that either both or non of the states is 0 at an equilibrium point.
To solve for other than zero equilibrium points, we solve equation (I) for x_1^e and plug it into (II).

$$x_1^e = 2x_2^e(1-x_1^e) \quad \Leftrightarrow \quad x_1^e = 2x_2^e - 2x_2^e x_1^e \quad \Leftrightarrow \quad (1+2x_2^e)x_1^e = 2x_2^e$$

Now we observe that $(1+2x_2^e)$ is 0 for $2x_2^e = -1$. However, this would cause a contradiction in the equation, which would say $0 = -1$. Therefore we can divide by $(1+2x_2^e)$ and get:

$$x_1^e = \frac{2x_2^e}{1+2x_2^e}$$

Now x_1 is plugged into (II):

$$x_2^e = \frac{6x_2^e}{1+2x_2^e}(1-x_2^e)$$

We already know the solution for $x_2^e = 0$. Hence, we can divide by x_2^e and multiply by $(1+2x_2^e)$.

$$1 + 2x_2^e = 6 - 6x_2^e \quad \Leftrightarrow \quad 8x_2^e = 5 \quad \Leftrightarrow \quad x_2^e = \frac{5}{8} \quad (\neq -\frac{1}{2})$$

The second equilibrium can be obtained as: $\boxed{\bar{x}_2^e = \left(\frac{5}{9}, \frac{5}{8}\right)^T}$.

(b) If we linearize the system dynamics, we get:

$$\begin{aligned}\delta\dot{x}_1 &= -2x_2^e\delta x_1 - \delta x_1 + 2(1-x_1^e)\delta x_2\\ \delta\dot{x}_2 &= 3(1-x_2^e)\delta x_1 - 3x_1^e\delta x_2 - \delta x_2\end{aligned}$$

Using the equilibrium point $\bar{x}_1^e = (0,0)^T$ this reduces to:

$$\boxed{\begin{aligned}\delta\dot{x}_1 &= -\delta x_1 + 2\delta x_2\\ \delta\dot{x}_2 &= \ \ 3\delta x_1 - \delta x_2\end{aligned} \qquad \Leftrightarrow \qquad \begin{pmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{pmatrix} = \begin{bmatrix}-1 & 2\\ 3 & -1\end{bmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix}}$$

Exercise 4 Before the equations are linearized, we search for the equilibrium point. Afterwards the linearization is performed by using the values from the equilibrium. Therefore some terms might cancel out (see also previous exercise)

(a)

$$\ddot{y} + (y^2-1)\dot{y} + y = 0\,.$$

An equilibrium requires: $\ddot{y}^e = \dot{y}^e = 0$. Hence $y^e = 0$ is the only equilibrium solution. Linearizing the equation gives:

$$\delta\ddot{y} + 2y^e\dot{y}^e\delta y + (0-1)\delta\dot{y} + \delta y = \boxed{\delta\ddot{y} - \delta\dot{y} + \delta y = 0} \quad \Leftrightarrow \quad \delta\ddot{y} = -\delta y + \delta\dot{y}$$

Defining the state space variables as:

$$x_1 = \delta y, \quad x_2 = \delta\dot{y}$$

the system with the system matrix $\mathbf{A}$ would be:

$$\boxed{\dot{\bar{x}} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix}}_{\mathbf{A}}\bar{x}}$$

(b)

$$\ddot{y} + \dot{y} + y - y^3 = 0 \quad \Leftrightarrow \quad \ddot{y} = -y + y^3 - \dot{y}$$

As before: $\ddot{y}^e = \dot{y}^e = 0$, which leaves

$$y^e - (y^e)^3 = 0 \quad \Leftrightarrow \quad y^e(1-(y^e)^2) = 0$$

Hence the solutions would be: $y_1^e = -1$, $y_2^e = 0$ and $y_3^e = 1$.
The linearized system will be:

$$\delta\ddot{y} = -\delta y + 3(y^e)^2\delta y - \delta\dot{y}$$

Using the notation for the states as in (a), we get:

$$\boxed{\mathbf{A}_{e1} = \mathbf{A}_{e3} = \begin{bmatrix}0 & 1\\ 2 & -1\end{bmatrix} \quad \text{and} \quad \mathbf{A}_{e2} = \begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}} \quad \text{where} \quad \delta\dot{\bar{x}} = \mathbf{A}\delta\bar{x}$$

(c)

$$\begin{aligned}(M+m)\ddot{y}+ml\,\ddot{\theta}\cos\theta-ml\,\dot{\theta}^2\sin\theta+ky &= 0\\ ml\ddot{y}\cos\theta+ml^2\,\ddot{\theta}+mgl\,\sin\theta &= 0\end{aligned}$$

Again, we require $\ddot{y}^e=\ddot{\theta}^e=\dot{\theta}^e=0$ for the equilibrium, which leaves us with:

$$\begin{aligned}ky &= 0\\ mgl\,\sin\theta &= 0\end{aligned}$$

Hence we get: $y^e=0$ and $\theta^e=n\cdot\pi,\quad n\in\mathbb{Z}$.
Knowing that M,m,l,k and g are constants, the equations can be linearized:

$$\begin{aligned}(M+m)\delta\ddot{y}+ml\,\left(\cos\theta^e\delta\ddot{\theta}-0\right)-ml\,\left(\sin\theta^e 2\dot{\theta}^e\delta\dot{\theta}+0\right)+k\delta y &= 0\\ ml\,(\cos\theta^e\delta\ddot{y}+0)+ml^2\,\delta\ddot{\theta}+mgl\,\cos\theta^e\delta\theta &= 0\end{aligned}$$

Note: 0s in the equation symbolize a multiplication by a derivative like $\ddot{\theta}^e$, which is 0 and therefore the whole term gets 0 and doesn't have to be calculated. $\dot{\theta}^e=0$ can be used to obtain:

$$\begin{aligned}(M+m)\delta\ddot{y}+ml\,\cos\theta^e\delta\ddot{\theta}+k\delta y &= 0\\ ml\cos\theta^e\delta\ddot{y}+ml^2\,\delta\ddot{\theta}+mgl\,\cos\theta^e\delta\theta &= 0\end{aligned}$$

Those equations can be written in vectorform as:

$$\begin{bmatrix}(M+m) & ml\,\cos\theta^e\\ ml\cos\theta^e & ml^2\end{bmatrix}\begin{pmatrix}\delta\ddot{y}\\ \delta\ddot{\theta}\end{pmatrix}=\begin{bmatrix}-k & 0\\ 0 & -mgl\,\cos\theta^e\end{bmatrix}\begin{pmatrix}\delta y\\ \delta\theta\end{pmatrix}\quad\Leftrightarrow\quad\begin{pmatrix}\delta\ddot{y}\\ \delta\ddot{\theta}\end{pmatrix}=\mathbf{M}\begin{pmatrix}\delta y\\ \delta\theta\end{pmatrix}$$

To get a direct expression of the second derivatives, we need to invert the 2×2-matrix using standardized techniques:

$$\begin{aligned}\begin{bmatrix}(M+m) & ml\,\cos\theta^e\\ ml\cos\theta^e & ml^2\end{bmatrix}^{-1} &= \frac{1}{(M+m)ml^2-m^2l^2\cos^2\theta^e}\begin{bmatrix}ml^2 & -ml\,\cos\theta^e\\ -ml\cos\theta^e & (M+m)\end{bmatrix}\\ &= \frac{1}{Ml+ml-ml\cos^2\theta^e}\begin{bmatrix}l & -\cos\theta^e\\ -\cos\theta^e & \frac{M+m}{ml}\end{bmatrix}\end{aligned}$$

Hence the matrix $\mathbf{M}$ for would be:

$$\mathbf{M}=\frac{1}{Ml+ml-ml\cos^2\theta^e}\begin{bmatrix}-kl & mgl\cos^2\theta^e\\ -k\cos\theta^e & -(M+m)g\cos\theta^e\end{bmatrix}$$

If $\sin^2\theta^e=0$, $\cos^2\theta^e=1$:

$$\mathbf{M}=\frac{1}{Ml}\begin{bmatrix}-kl & mgl\\ k\cos\theta^e & -(M+m)g\cos\theta^e\end{bmatrix}$$

The system describes a pendulum. Therefore only values of θ inside a 2π interval make sense. However, we observe inside the equations, that we only have to differ between odd or even numbers of n ($\cos\theta^e = \pm 1$), which comes back to the argument with the 2π interval.

$$\mathbf{M}_{\text{even}} = \begin{bmatrix} -\frac{k}{M} & \frac{mg}{M} \\ \frac{k}{Ml} & -\frac{(M+m)g}{Ml} \end{bmatrix} \quad \text{and} \quad \mathbf{M}_{\text{odd}} = \begin{bmatrix} -\frac{k}{M} & \frac{mg}{M} \\ -\frac{k}{Ml} & \frac{(M+m)g}{Ml} \end{bmatrix}$$

Using a state description like:

$$x_1 = \delta y, \quad x_2 = \delta\dot{y}, \quad x_3 = \delta\theta, \quad x_4 = \delta\dot{\theta}$$

the system **A**-matrix can be obtained out of **M**:

$$\boxed{\mathbf{A}_{\text{even}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M} & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{Ml} & 0 & -\frac{(M+m)g}{Ml} & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{A}_{\text{odd}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{M} & 0 & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{Ml} & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix}}$$

(d)

$$\ddot{y} + 0.5\dot{y}|\dot{y}| + y = 0 \quad \Leftrightarrow \quad \ddot{y} = -y - 0.5\dot{y}|\dot{y}|$$

using $\ddot{y}^e = \dot{y}^e = 0$ the only equilibrium is at $y^e = 0$.
To linearize the equation, we consider the derivative of a function $z|z|$:

$$z|z| = \begin{cases} z^2, & z > 0 \\ 0, & z = 0 \\ -z^2, & z < 0 \end{cases} \quad \Rightarrow \quad f(z) = \frac{d}{dz} z|z| = \begin{cases} 2z, & z > 0 \\ 0, & z = 0 \\ -2z, & z < 0 \end{cases}$$

Note that $f(0^-) = f(0^+) = 0$ and therefore the equation can be linearized at $\dot{y} = 0$:

$$\delta\ddot{y} = -\delta y$$

with:

$$x_1 = \delta y, \quad x_2 = \delta\dot{y}$$

the system **A**-matrix gets:

$$\boxed{\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

Exercise 5 To obtain the transfer function matrix $\hat{\mathbf{G}}(s)$ the equation will be Laplace-transformed (assuming 0 initial states) and afterwards rearranged

$$\begin{aligned} \ddot{y}_1 + \ddot{y}_2 + y_1 + y_2 = u_1 + \dot{u}_2 \quad & \overset{\mathcal{L}}{\Rightarrow} s^2 Y_1(s) + s^2 Y_2(s) + Y_1(s) + Y_2(s) = U_1(s) + sU_2(s) \\ 2\ddot{y}_1 + 3\ddot{y}_2 + y_1 - y_2 = 0 \quad & \overset{\mathcal{L}}{\Rightarrow} 2s^2 Y_1(s) + 3s^2 Y_2(s) + Y_1(s) - Y_2(s) = 0 \end{aligned}$$

Hence the system can be written as:

$$\begin{bmatrix} s^2+1 & s^2+1 \\ 2s^2+1 & 3s^2-1 \end{bmatrix} \begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \begin{bmatrix} 1 & s \\ 0 & 0 \end{bmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

Applying the inverse-rule for a 2×2-matrix:

$$\begin{pmatrix} Y_1(s) \\ Y_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} 3s^2-1 & -s^2-1 \\ -2s^2-1 & s^2+1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 0 \end{bmatrix} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix}$$

with:

$$\Delta = (3s^2-1)(s^2+1) - (2s^2+1)(s^2+1) = 3s^4 + 2s^2 - 1 - (2s^4 + 3s^2 + 1) = s^4 - s^2 - 2$$

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \underbrace{\frac{1}{\Delta} \begin{bmatrix} 3s^2-1 & 3s^3-s \\ -2s^2-1 & -2s^3-s \end{bmatrix}}_{\hat{\mathbf{G}}(\mathbf{s})} \begin{pmatrix} U_1(s) \\ U_2(s) \end{pmatrix} \quad \Rightarrow \quad \boxed{\hat{\mathbf{G}}(\mathbf{s}) = \begin{bmatrix} \frac{3s^2-1}{s^4-s^2-2} & \frac{3s^3-s}{s^4-s^2-2} \\ -\frac{2s^2+1}{s^4-s^2-2} & -\frac{2s^3+s}{s^4-s^2-2} \end{bmatrix}}$$

Exercise 6 Again, we need to apply the Laplace-transform (assuming 0 initial states)

$$\mathcal{L}\left\{\begin{array}{rcl} \ddot{q}_1 + 3\dot{q}_2 + \dot{q}_1 + 2q_2 & = & \dot{u} + 4u \\ \ddot{q}_1 + 4\dot{q}_2 + 3q_2 & = & u \\ y & = & q_1 + q_2 \end{array}\right\} \quad \Rightarrow \quad \begin{array}{rcl} s^2Q_1(s) + 3sQ_2(s) + sQ_1(s) + 2Q_2(s) & = & sU(s) + 4U(s) \\ s^2Q_1(s) + 4sQ_2(s) + 3Q_2(s) & = & U(s) \\ Y(s) & = & Q_1(s) + Q_2(s) \end{array}$$

The first two equations can be written as

$$\begin{bmatrix} s^2+s & 3s+2 \\ s^2 & 4s+3 \end{bmatrix} \begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \begin{bmatrix} s+4 \\ 1 \end{bmatrix} U(s)$$

Inverting:

$$\begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \frac{1}{\Delta} \begin{bmatrix} 4s+3 & -3s-2 \\ -s^2 & s^2+s \end{bmatrix} \begin{bmatrix} s+4 \\ 1 \end{bmatrix} U(s)$$

$$\Delta = (4s^3 + 7s^2 + 3s) - (3s^3 + 2s^2) = s \cdot (s^2 + 5s + 3)$$

$$\begin{pmatrix} Q_1(s) \\ Q_2(s) \end{pmatrix} = \frac{1}{s \cdot (s^2+5s+3)} \begin{bmatrix} 4s^2 + 16s + 10 \\ -s^3 - 3s^2 + s \end{bmatrix} U(s)$$

Now we can plug in the relation for $Q_1(s)$ and $Q_2(s)$ into $Y(s)$:

$$Y(s) = \underbrace{\frac{1}{\Delta}(-s^3 + s^2 + 17s + 10)}_{\hat{G}(s)} U(s) \quad \Rightarrow \quad \boxed{\hat{G}(s) = \frac{-s^3 + s^2 + 17s + 10}{s(s^2+5s+3)}}$$

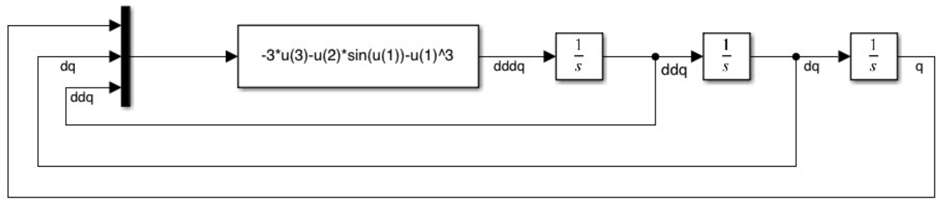

Figure 2: SIMULINK model for Exercise 7.

Exercise 7 An exemplary SIMULINK model for

$$\frac{d^3q}{dt^3} + 3\ddot{q} + \dot{q}\sin q + q^3 = 0$$

is shown in Fig. 2.

Exercise 8 (a) An exemplary SIMULINK model for

$$\begin{aligned} \ddot{q}_1 + 2\dot{q}_2 + q_1 &= 0 \\ \ddot{q}_2 + \dot{q}_1 + 6q_2 &= 0 \end{aligned}$$

is shown in Fig. 3.

(b) An exemplary SIMULINK model for

$$\begin{aligned} \ddot{q}_1 + \ddot{q}_2 + 6q_2 &= 0 \\ \ddot{q}_1 - \ddot{q}_2 + 4q_1 &= 0 \end{aligned}$$

is shown in Fig. 4. (Hint: first write the highest orders in terms of lower orders)

(c) An exemplary SIMULINK model for

$$\begin{aligned} \ddot{q}_1 + \dot{q}_1 + q_1q_2 &= 0 \\ (1 + q_2^2)\ddot{q}_2 + 2\dot{q}_1q_2 &= 0 \end{aligned}$$

is shown in Fig. 5. (Hint: first write the highest orders in terms of lower orders)

Note that: other correct answers are also accepted.

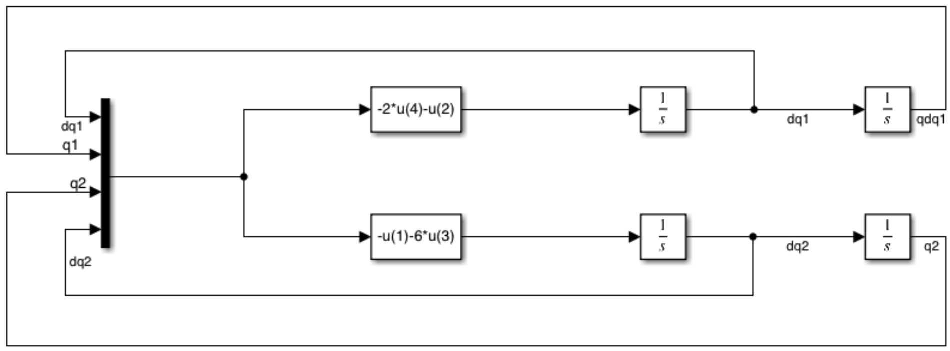

Figure 3: SIMULINK model for Exercise 8(a).

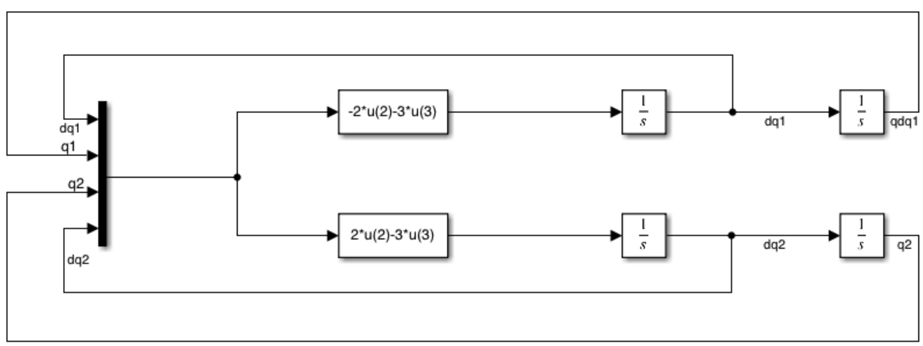

Figure 4: SIMULINK model for Exercise 8(b).

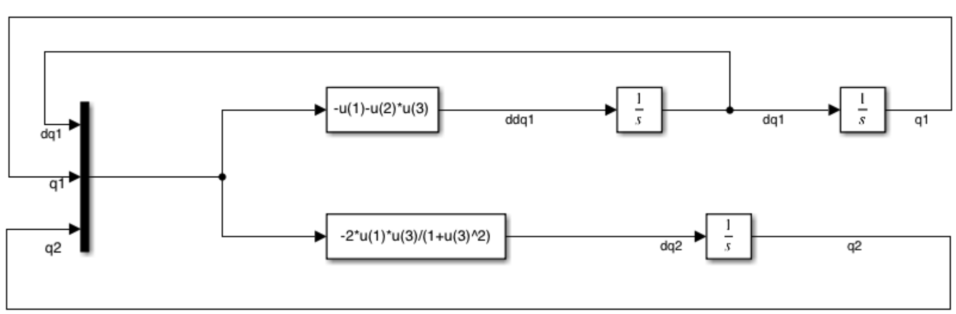

Figure 5: SIMULINK model for Exercise 8(c).