College of Engineering

School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW3

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Exercise 1

Determine whether or not the following functions are *lpd*.
(a)

$$V(x) = x_1^2 - x_1^4 + x_2^2$$

(b)

$$V(x) = x_1 + x_2^2$$

(c)

$$V(x) = 2x_1^2 - x_1^3 + x_1x_2 + x_2^2$$

Solution:
(a) Since this equation can be written as

$$V(x) = x_1^2(1 - x_1)(1 + x_1) + x_2^2$$

we can see that $V(x) = 0$ when

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now,

$$DV(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 4x_1^3 & 2x_2 \end{bmatrix}$$

and

$$D^2V(x) = \begin{bmatrix} \frac{\partial^2 V}{\partial {x_1}^2} & \frac{\partial^2 V}{\partial {x_2}^2} \end{bmatrix} = \begin{bmatrix} 2 - 12x_1^2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Next, when $x_e = [\pm 1, 0]^T$,

$$DV \neq 0$$

Thus, this equation is not *lpd* about $x_e = \pm 1$. Albeit, when $x_e = 0$,

$$\begin{aligned} V(0) &= 0 \\ DV(0) &= 0 \\ D^2V(0) &= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} > 0. \end{aligned}$$

This equation is *lpd* about $x_e = 0$.

(b) We can see that $V(x) = 0$ when

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now,

$$DV(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & 2x_2 \end{bmatrix}$$

and

$$D^2V(x) = \begin{bmatrix} \frac{\partial^2 V}{\partial {x_1}^2} & \frac{\partial^2 V}{\partial {x_2}^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

When $x_e = [0, 0]^T$

$$DV(0) \neq 0.$$

Thus, this system is not *lpd*.

(c) Since this equation can be written as

$$V(x) = x_1^2(1 - x_1)(1 + x_1) + x_2^2$$

we can see that $V(x) = 0$ when

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now,

$$DV(x) = \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 - 3x_1^2 + x_2 & x_1 + 2x_2 \end{bmatrix}$$

and

$$D^2V(x) = \begin{bmatrix} \frac{\partial^2 V}{\partial {x_1}^2} & \frac{\partial^2 V}{\partial {x_2}^2} \end{bmatrix} = \begin{bmatrix} 4 - 6x_1 & 2 \end{bmatrix}.$$

When $x_e = 0$,

$$\begin{aligned} V(0) &= 0 \\ DV(0) &= 0 \\ D^2V(0) &= \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} > 0. \end{aligned}$$

This equation is *lpd* about $x_e = 0$.

Exercise 2

By appropriate choice of Lyapunov function, show that the origin is a stable equilibrium state for

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= -x_1^3 \end{aligned}$$

Note that the linearization of this system about the origin is unstable.

Solution:
Choose the following Lyapunov function for the equilibrium state $x_e = [0, 0]^T$

$$V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2.$$

Since,

$$DV(x) = \begin{bmatrix} x_1^3 & x_2 \end{bmatrix} \qquad \textit{and} \qquad D^2V = \begin{bmatrix} 3x_1^2 & 0 \\ 0 & 1 \end{bmatrix}$$

we have

$$\begin{aligned} V(x_e) &= 0 \\ DV(x_e) &= 0 \\ D^2V(x_e) &> 0 \end{aligned}$$

Thus, V is *lpd* about x_e. Subsequently, we solve for

$$DV(x)f(x) = \begin{bmatrix} x_1^3 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1^3 \end{bmatrix} = x_1^3 x_2 - x_2 x_1^3 = 0.$$

Hence the origin is stable.

Exercise 3

By appropriate choice of Lyapunov function, show that the origin is a stable equilibrium state for

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_1 + x_1^3$$

Solution:
Choose the following Lyapunov function for the equilibrium states

$$x_e = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V(x) = -\frac{1}{4}x_1^4 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2.$$

Since,

$$DV(x) = \begin{bmatrix} -x_1^3 + x_1 & x_2 \end{bmatrix} \qquad and \qquad D^2V = \begin{bmatrix} -3x_1^2 + 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

When, $x_e = [-1, 0]^T$

$$\begin{aligned} V(x_e) &= 0 \\ DV(x_e) &= 0 \\ D^2V(x_e) &< 0 \end{aligned}$$

Thus, V is not *lpd* about $x_e = [-1, 0]^T$. When, $x_e = [1, 0]^T$

$$\begin{aligned} V(x_e) &= 0 \\ DV(x_e) &= 0 \\ D^2V(x_e) &< 0 \end{aligned}$$

Thus, V is not *lpd* about $x_e = [0, 0]^T$. When, $x_e = [0, 0]^T$

$$\begin{aligned} V(x_e) &= 0 \\ DV(x_e) &= 0 \\ D^2V(x_e) &> 0 \end{aligned}$$

Thus, V is *lpd* about $x_e = [0, 0]^T$. Subsequently, we solve for

$$DV(x)f(x) = \begin{bmatrix} -x_1^3 + x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 + x_1^3 \end{bmatrix} = -x_2x_1^3 + x_1x_2 - x_2x_1 + x_2x_1^3 = 0.$$

Hence the origin is stable.

Exercise 4

Show that the following system is stable about the zero state.

$$\dot{x_1} = x_2^3$$
$$\dot{x_2} = -x_2^2 x_1$$

Solution:
Choose a candidate Lyapunov function of

$$V(x) = x_1^2 + x_2^2$$

Since,

$$DV(x) = \begin{bmatrix} 2x_1 & 2x_2 \end{bmatrix} \qquad and \qquad D^2V = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

when, $x_e = [0, 0]^T$

$$\begin{aligned} V(x_e) &= 0 \\ DV(x_e) &= 0 \\ D^2V(x_e) &> 0 \end{aligned}$$

Thus, V is *lpd* about $x_e = [0, 0]^T$. Subsequently, we solve for

$$DV(x)f(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2^3 \\ -x_2^2 x_1 \end{bmatrix} = x_1 x_2^3 - x_2^3 x_1 = 0.$$

Hence this system is stable about the zero state.

Exercise 5

Show that the following system is GAS about the zero.

$$\dot{x} = -(2 + \cos x)x$$

Solution:
Choose a candidate Lyapunov function of

$$V(x) = x^2$$

Since,

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \qquad for \quad \forall x \neq 0 \\ \lim_{\|x\| \to \infty} V(x) &= \infty \end{aligned}$$

Thus, V is *pd* about $x_e = 0$. Subsequently, we solve for

$$DV(x)f(x) = -x(2 + \cos x)(2x) = -2(2 + \cos x)x^2 < 0 \qquad for \quad \forall x \neq 0$$
$$\because 1 < 2 + \cos x < 3$$

Hence this system is GAS about zero.

Exercise 6

Show that the following system is GAS about 1.

$$\dot{x} = -(2 + \cos x)(x - 1)$$

Solution:
Choose a candidate Lyapunov function of

$$V(x) = (x - 1)^2$$

Since,

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \qquad for \quad \forall x \neq 0 \\ \lim_{\|x\| \to \infty} V(x) &= \infty \end{aligned}$$

Thus, V is *pd* about $x_e = 1$. Subsequently, we solve for

$$DV(x)f(x) = -(x-1)(2+\cos x)2(x-1) = -2(2+\cos x)(x-1)^2 < 0 \qquad for \quad \forall x \neq 0$$
$$\because 1 < 2 + \cos x < 3$$

Hence this system is GAS about 1.