AAE 334: Aerodynamics

HW12: Supersonic, Subsonic, Transonic Linear Theories

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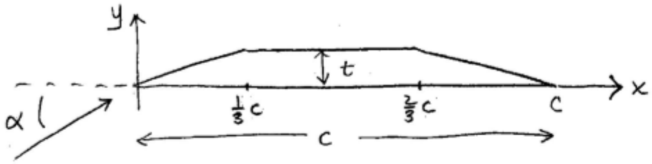
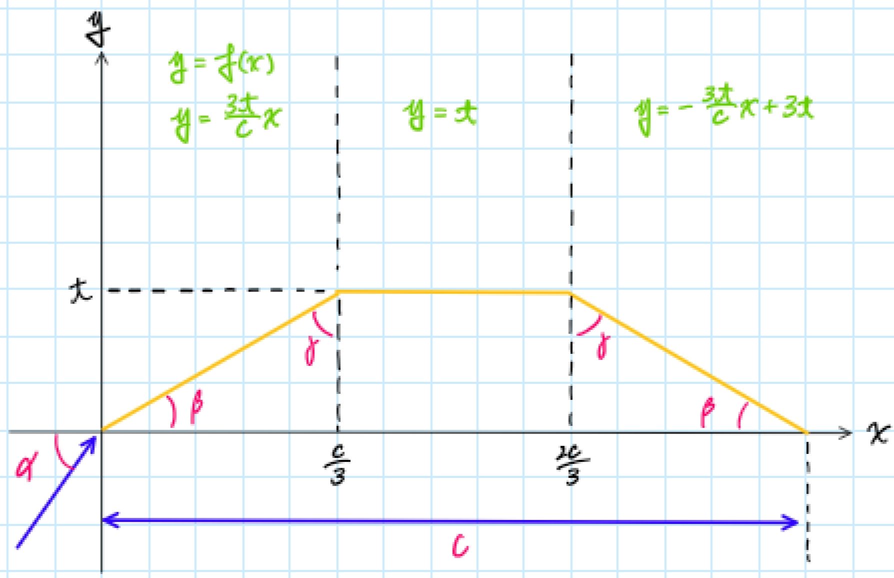
1.) [50 pts] Consider an airfoil made with angled leading and trailing edges as shown below. The thickness to chord ratio is $t/c = 0.05$. The lower surface of the airfoil is flat. The transition on the upper surface from the angled leading edge portion to the flat center section occurs at $x/c = 1/3$. Likewise the transition from the center section to the angled trailing edge portion occurs at $x/c = 2/3$. The freestream Mach number is $M_\infty = 2.0$. Use supersonic linear theory to analyze the airfoil.



(a) Determine the lift coefficient, c_ℓ, as a function of angle of attack, α.



Organize what we know from this figure

$$y_u = \begin{cases} \frac{3t}{c}x & 0 \le x \le \frac{c}{3} \\ t & \frac{c}{3} \le x \le \frac{2}{3}c \\ -\frac{3t}{c}x + 3t & \frac{2}{3}c \le x \le c \end{cases}$$

Since $y_\ell = 0$,

$$y_c = \frac{1}{2}(y_u + y_\ell) = \begin{cases} \frac{3t}{2c}x & 0 \le x \le \frac{c}{3} \\ \frac{t}{2} & \frac{c}{3} \le x \le \frac{2}{3}c \\ -\frac{3t}{2c}x + \frac{3t}{2} & \frac{2}{3}c \le x \le c \end{cases}$$

Let $\hat{x} = \frac{x}{c}$ & $\hat{y}_c = \frac{y}{c}$

Then

$$\frac{d\hat{y}_c}{d\hat{x}} = \begin{cases} \frac{3\tau}{2c} & 0 \leq \hat{x} \leq \frac{1}{3} \\ 0 & \frac{1}{3} \leq \hat{x} \leq \frac{2}{3} \\ -\frac{3\tau}{2c} & \frac{2}{3} \leq \hat{x} \leq 1 \end{cases}$$

Now, using the equation in our notes

$$\Delta\alpha = -\int_0^1 \frac{d\hat{y}_c}{d\hat{x}} d\hat{x}$$

$$= -\int_0^{1/3} \frac{3\tau}{2c} d\hat{x} - \cancelto{0}{\int_{1/3}^{2/3} 0\, d\hat{x}} - \int_{2/3}^{1} -\frac{3\tau}{2c} d\hat{x}$$

$$= -\left[\frac{3\tau}{2c}\hat{x}\right]_0^{1/3} + \left[\frac{3\tau}{2c}x\right]_{2/3}^{1}$$

$$= -\frac{\tau}{2c} + \frac{3\tau}{2c} - \frac{\tau}{c} = 0$$

Thus, from supersonic linear theory

$$c_l = \frac{4(\alpha + \Delta\alpha)}{\sqrt{M_\infty^2 - 1}}$$

Since $\Delta\alpha = 0$, $M_\infty = 2.0$

$$\boxed{c_l = \frac{4}{\sqrt{3}}\alpha}$$

(b) Determine the lift curve slope, $dc_\ell/d\alpha$ (where α is measured in radians). How does this compare to the incompressible result for thin airfoils, $dc_\ell/d\alpha = 2\pi$?

Supersonic linear Theory

$$\frac{dc_\ell}{d\alpha} = \frac{4}{\sqrt{3}} \cong 2.3094$$

Incompressible airflow

$$\frac{dc_\ell}{d\alpha} = 2\pi \cong 6.2832$$

Using supersonic linear theory we get a lift curve slope that is considerably smaller than an incompressible airflow.

(c) Determine the drag coefficient, c_d, due to wave drag, as a function of angle of attack.

$$K_1 = \int_0^1 \frac{d\hat{y}_c}{d\hat{x}} \hat{x}\, d\hat{x}$$

$$= \int_0^{1/3} \frac{3\tau}{2c}\hat{x}\, d\hat{x} + \int_{1/3}^{2/3} \cancel{0}\, \hat{x}\, d\hat{x} + \int_{2/3}^{1} -\frac{3\tau}{2c}\hat{x}\, d\hat{x}$$

$$= \left[\frac{3\tau}{4c}\hat{x}^2\right]_0^{1/3} - \left[\frac{3\tau}{4c}\hat{x}^2\right]_{2/3}^{1}$$

$$= \frac{\tau}{12c} - \frac{3\tau}{4c} + \frac{\tau}{3c} = \frac{\tau}{12c} - \frac{9\tau}{12c} + \frac{4\tau}{12c}$$

$$= -\frac{\tau}{3c}$$

$$K_2 = \int_0^1 \left(\frac{d\hat{y}_c}{d\hat{x}}\right)^2 d\hat{x}$$

$$= \int_0^{1/3} \frac{9\tau^2}{4c^2} d\hat{x} + \int_{1/3}^{2/3} 0\, d\hat{x} + \int_{2/3}^{1} \frac{9\tau^2}{4c^2} d\hat{x}$$

$$= \left[\frac{9\tau^2}{4c^2}\hat{x}\right]_0^{1/3} + \left[\frac{9\tau^2}{4c^2}\hat{x}\right]_{2/3}^{1}$$

$$= \frac{3\tau^2}{4c^2} + \frac{9\tau^2}{4c^2} - \frac{3\tau^2}{2c^2} = \frac{3\tau^2}{c^2} - \frac{3\tau^2}{2c^2} = \frac{3\tau^2}{2c^2}$$

Since, $y_t = y_c$

$$K_3 = K_2 = \frac{3\tau^2}{2c^2}$$

Then,

$$c_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\left[(\alpha + \cancel{\alpha_0})^2 + K_2 + K_3\right]$$

$$c_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + \frac{3\tau^2}{2c^2} + \frac{3\tau^2}{2c^2}\right)$$

$$c_d = \frac{4}{\sqrt{M_\infty^2 - 1}}\left(\alpha^2 + 3\frac{\tau^2}{c^2}\right)$$

Since $\frac{\tau}{c} = 0.05$, $M_\infty = 2$

$$\boxed{c_d = \frac{4}{\sqrt{3}}\left(\alpha^2 + 0.0075\right)}$$

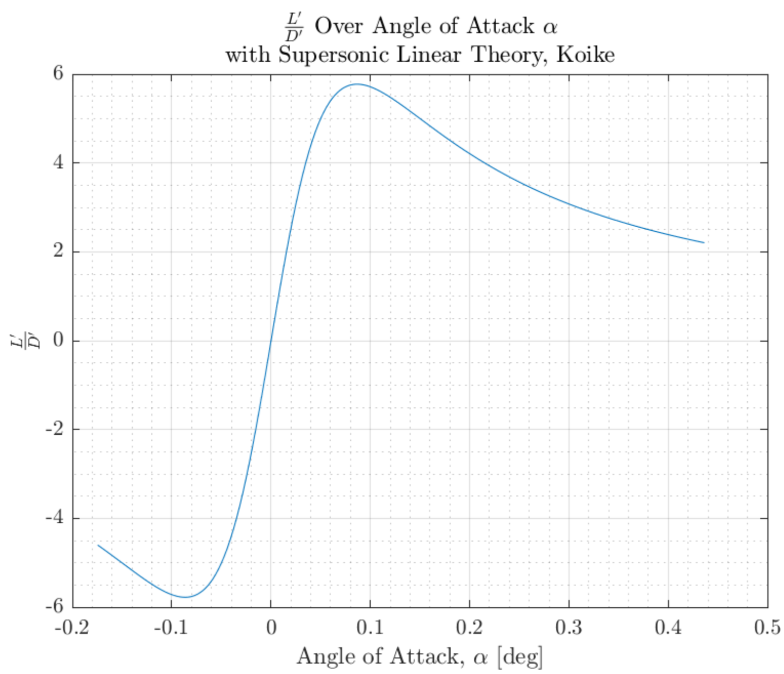
(d) Determine and plot the lift to drag ratio, L'/D' , as a function of angle of attack. Determine the angle of attack that gives the maximum lift to drag ratio and the corresponding maximum value of the lift to drag ratio.

From (a) & (c)

$$\frac{L'}{D'} = \frac{c_l}{c_d} = \frac{4}{\sqrt{3}}\alpha \cdot \frac{\sqrt{3}}{4}\left(\alpha^2 + 0.0075\right)^{-1}$$

$$\frac{L'}{D'} = \frac{\alpha}{\alpha^2 + 0.0075}$$

Plotting (using MATLAB)



Maximum value

$$\frac{d}{d\alpha}\left(\frac{L'}{D'}\right) = \frac{\alpha^2 + 0.0075 - \alpha(2\alpha)}{(\alpha^2 + 0.0075)^2}$$

$$= \frac{0.0075 - \alpha^2}{(\alpha + 0.0075)^2}$$

Maximum $\frac{L'}{D'}$ is when $0.0075 - \alpha^2 = 0$

and $\alpha > 0$ $\Longrightarrow$ $\boxed{\alpha_{max} = \sqrt{0.0075} = 0.08660}$

$$\therefore \boxed{\left(\frac{L'}{D'}\right)_{max} = \frac{0.08660}{2 \times 0.0075} = 5.7735}$$

(e) Determine the moment coefficient about the aerodynamic center, c_{mac}.

From the equations in the notes

$$C_{mac} = \frac{4(K_1 + \Delta a/2)}{\sqrt{M_\infty^2 - 1}}$$

$\because$ part (c) $K_1 = -\frac{t}{3c} = -\frac{1}{3} \times 0.05 = -\frac{0.05}{3}$

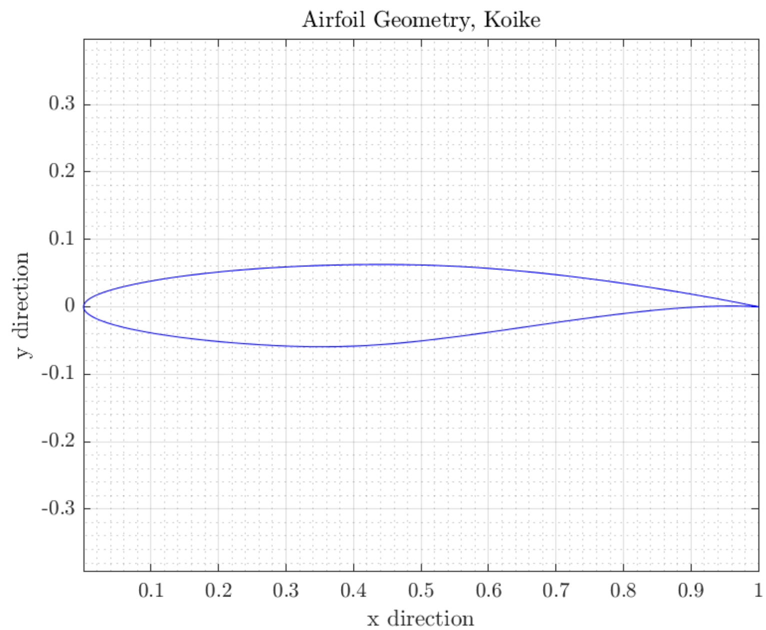
part (a) $\Delta a = 0$

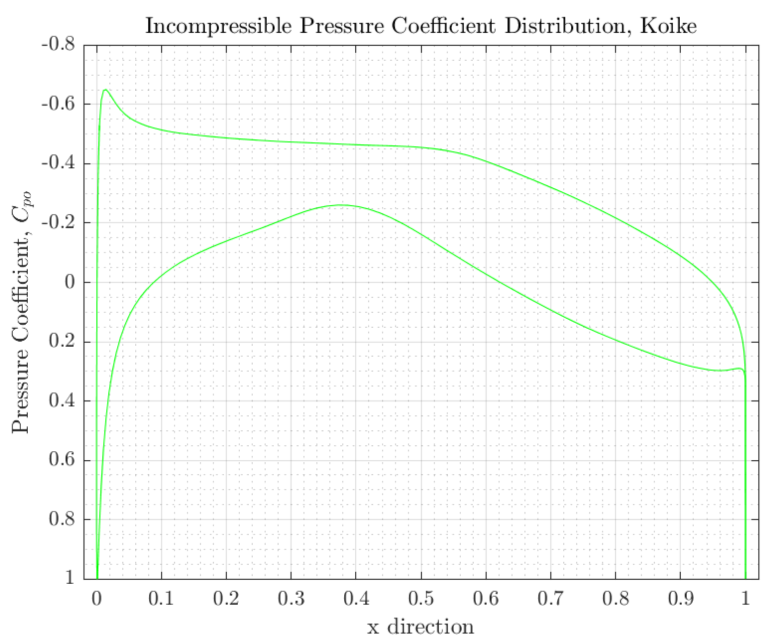
$$\boxed{C_{mac} = \frac{-\frac{0.2}{3}}{\sqrt{3}} = -0.03849}$$

2.) [50 pts] In the Homework/PS12 folder on Blackboard are the files **airfoil_x.txt**, **airfoil_y.txt**, **airfoil_cp.txt** that give the x and y coordinates for the RAE 2822 airfoil and the C_p distribution for incompressible conditions. (For parts (a)-(c), write a Matlab program to do the required tasks. **Turn in a listing of your program. This is necessary to receive credit for the problem.**)

(a) Plot the geometry of the airfoil from the x and y data files. (Make the scaling of the x and y axes the same so that the airfoil has its true shape. This can be done within Matlab by using the **axis('equal')** command.) Also, plot the incompressible pressure coefficient distribution, $C_{p,0}$. What is the value of C_p^{min}?

Plots



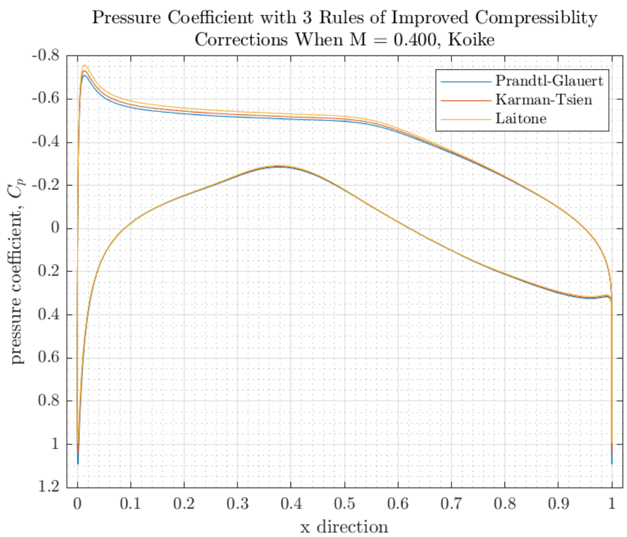


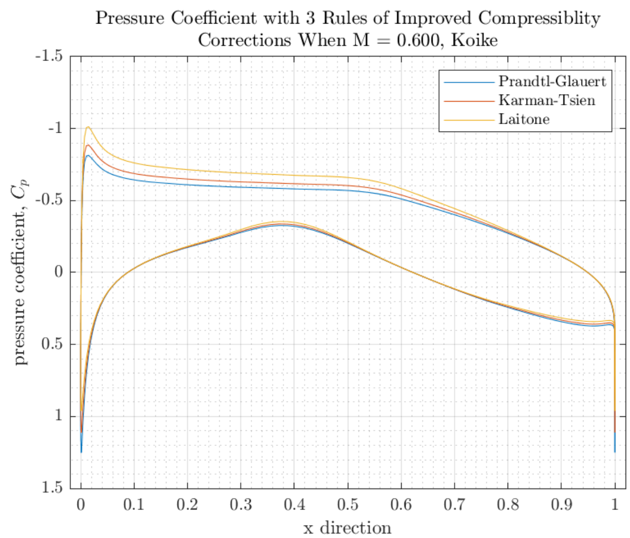
Minimum C_{po}
Using MATLAB, we can find the minimum incompressible pressure coefficient.

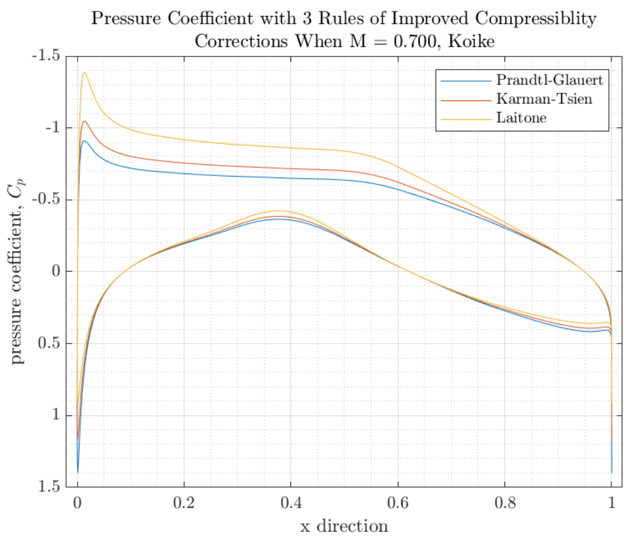
$$\boxed{C_{po,min} = -0.6507}$$

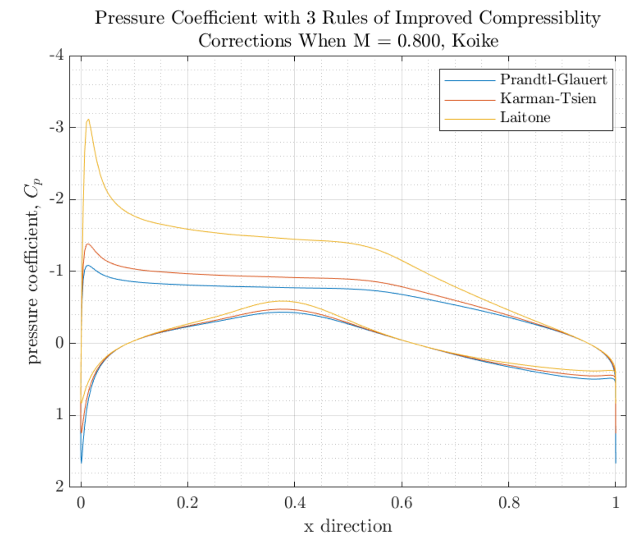
(b) Compute $C_p(x)$ for $M_\infty = 0.40, 0.60, 0.70$ and 0.80 using three methods: (i) the Prandtl-Glauert rule, (ii) the Karman-Tsien rule, and (iii) the Laitone rule. Plot the results with curves for the three methods together on the same plot, one plot for each Mach number. Tabulate the minimum pressure coefficient values, C_p^{min}, for the three methods for each of the Mach numbers.

Plots









Minimum Pressure Coefficients

Mach Number	Prandtl-Glauert	Karman-Tsien	Laitone
0.40	-0.709980307	-0.731664128	-0.758489723
0.60	-0.813384625	-0.88540185	-1.011909502
0.70	-0.911173033	-1.047605489	-1.387353085
0.80	-1.084512833	-1.38490153	-3.120387693

(c) Compute the critical pressure coefficient, C_{pcr}, for each of the Mach numbers in part (b). Compare the values of C_{pcr} you obtain to the minimum pressure coefficient values, C_p^{min}, tabulated in part (b) and determine the values of the Mach number used in part (b) that are just below and just above the critical Mach number for each of the three methods.

$C_{p,cr}$			
M = 0.40	M = 0.60	M = 0.70	M = 0.80
-3.66201724484731	-1.29434359045528	-0.779065964559632	-0.434640479155229

Analysis

- The critical pressure coefficient becomes larger as the Mach number increase (the magnitude decreases).
- At M = 0.40, the critical pressure coefficient is smaller than all 3 of the minimum pressure coefficients computed with the 3 rules
- At M = 0.60, the critical pressure coefficient is smaller than all 3 methods
- At, M = 0.70, the critical pressure coefficient is now larger than all 3 methods
- At, M = 0.80, the critical pressure coefficient is larger than all 3 methods like at M = 0.60

Critical Mach Numbers

The following is the tabulated data of the absolute difference between the minimum pressure coefficient for all 3 rules at each Mach number and the critical pressure coefficient for each Mach number. For example,

$$\text{Prandtl-Glauert: } \left|C_p - C_{p,cr}\right|_{M=0.40} = 2.952036938$$

Mach Number	Prandtl-Glauert	Karman-Tsien	Laitone
0.40	2.952036938	2.930353117	2.903527522
0.60	0.480958965	0.40894174	0.282434088
0.70	0.132107068	0.268539524	0.608287121
0.80	0.649872354	0.950261051	2.685747214

The light-blue highlight represents the smallest absolute difference, meaning that the minimum pressure coefficient is the closest to the critical pressure coefficient for the corresponding Mach number. Thus, the rough prediction of the critical Mach number for the 3 different methods can be deduced as the following.

Rough Prediction of the Critical Mach Number		
Prandtl-Glauert	Karman-Tsien	Laitone
0.70	0.70	0.60

(d) Determine the critical Mach number, M_{cr}, using the three different compressibility rules (see lecture35 slide 3 etc., and Sections 11.4-6 of the textbook). Do this by slolving the nonlinear equation obtained by setting the right hand side (RHS) of equation (11.59) equal to the right hand side of equation (11.51), for the Prandtl-Glauert rule; or equation (11.54), for the Karman-Tsien rule; or equation (11.55), for the Laitone rule. (Please note that equation (11.55) as it is written in the textbook is ambiguous. See the discussion in the errata sheet available on Blackboard. The class notes have this fixed.) To solve the nonlinear equation define a function of the Mach number to be the RHS of equation (11.59) minus the RHS of one of the subsonic scaling laws, e.g. for the Prandtl-Glauert rule,

$$G(M_\infty) = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+[(\gamma-1)/2]M_\infty^2}{1+(\gamma-1)/2}\right)^{\gamma/(\gamma-1)} - 1\right] - \frac{C_{p,0}}{\sqrt{1-M_\infty^2}}$$

Then solve for M_∞ such that $G(M_\infty) = 0$. The freestream Mach number at that condition is the critical Mach number, $M_{cr} = M_\infty$. You may solve the nonlinear equation in one of the following ways: (i) use Matlab's **fsolve** function (add a section to your program that you used to solve parts (a)-(c), making sure this section is included in the code listing you turn in), or (ii) guess values for M_{cr} and use a calculator (or Matlab as a calculator) to compute the value of $G(M_\infty)$ until you have found M_{cr} accurate to within three significant figures (you must show a table with each of the guesses you make for M_{cr} and the resulting values of G). If using **fsolve** in Matlab is proving difficult, do not waste your time; just solve it using the "guess check and revise" method. In that case it will be useful to program Matlab or a calculator to compute $G(M_\infty)$ based on your inputted values of M_∞.

Using the command patternsearch() which finds the global minimum of the absolute difference of the equations of Prandtl-Glauert, Karman-Tsien, or Laitone and the critical pressure coefficient equation. The following are the tabulated results.

Actual Critical Mach Number		
Prandtl-Glauert	Karman-Tsien	Laitone
0.676035594940186	0.658168983459473	0.632974100112915

(e) Consider a wing made with the RAE 2822 airfoil operating at an effective angle of attack corresponding to that used in the analaysis done in parts (a)-(d). Using the estimate of the critical Mach number from the Karman-Tsien rule, determine the sweep angle needed to obtain a critical Mach number of 0.85.

To find the sweep angle we use the following equation.

$$\sigma = \arccos\left(\frac{M_{n\infty}}{M_{\infty}}\right), \quad \sigma = \text{sweep angle}$$

Where $M_{n\infty}$ = Mach number normal to the wing, M_{∞} = Freestream Mach number

Since, we want the swept wing to resist a freestream Mach number of 0.85 while the Mach number of the airflow normal to the wing is at the critical Mach number for the case of the Karman-Tsien rule which we computed in the previous problem.

Thus, if

$$M_{n\infty} = 0.658168983459473 \text{ and } M_{\infty} = 0.85$$

Using MATLAB, we can compute the sweep angle to be

$$\boxed{\sigma \cong 39.257^{\circ}}$$

Appendix

MATLAB CODE

AAE334 HW12 P1

```
clear all; close all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Spring\AAE334\matlab\outputs\HW12';
set(groot, 'defaulttextinterpreter',"latex");
set(groot, 'defaultAxesTickLabelInterpreter',"latex");
set(groot, 'defaultLegendInterpreter',"latex");
```

```
% Plotting
a = deg2rad(-10:0.0001:25);
Lp_Dp = a./(a.^2 + 0.0075);

fig = figure("Renderer","painters");
    plot(a,Lp_Dp);
    title({'$\frac{L^{\prime}}{D^{\prime}}$ Over Angle of Attack $\alpha$', ['' ...
        'with Supersonic Linear Theory, Koike']});
    xlabel('Angle of Attack, $\alpha$ [deg]')
    ylabel('$\frac{L^{\prime}}{D^{\prime}}$')
    grid on; grid minor; box on;
saveas(fig,fullfile(fdir,'lift2dragRatio.png'));

% Maximum value
Lp_Dp_max = max(Lp_Dp)
```

AAE334 HW12 P2

```
clear all; close all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-Spring\AAE334\matlab\outputs\HW12';
set(groot, 'defaulttextinterpreter',"latex");
set(groot, 'defaultAxesTickLabelInterpreter',"latex");
set(groot, 'defaultLegendInterpreter',"latex");

% Global constants
gamma = 1.4;
```

a)

```
% Import data
xdata = read_txt_file("inputs\hw12\airfoil_x.txt");
ydata = read_txt_file("inputs\hw12\airfoil_y.txt");
Cpdata = read_txt_file("inputs\hw12\airfoil_cp.txt");
```

```
% Plot the airfoil geometry
fig = figure("Renderer","painters");
    plot(xdata,ydata,'-b');
    title({'Airfoil Geometry, Koike'});
    xlabel('x direction');
    ylabel('y direction');
    axis equal; grid on; grid minor; box on;
saveas(fig,fullfile(fdir,"airfoil.png"));
```

```
% Plot the incompressible pressure coefficient distribution
fig = figure("Renderer","painters");
    plot(xdata,Cpdata,'-g');
    title("Incompressible Pressure Coefficient Distribution, Koike");
    xlabel('x direction')
    ylabel("Pressure Coefficient, $C_{po}$")
    grid on; grid minor; box on; xlim([-0.02,1.02]);
     set(gca,'ydir','reverse');
saveas(fig,fullfile(fdir,"pressure_coeff.png"));
```

```
% Find the minimum Cpo
Cpo_min = min(Cpdata);
```

b)

```
% Computing the correct pressure coefficients based on subsonic condtions using
% Prandtl-Glauert, Karman-Tsien, and Latione formulas

% Mach numbers to investigate
M_b = [0.40 0.60 0.70 0.80];

% Preallocating an array to store all results
Cp_PG = zeros([numel(Cpdata),numel(M_b)]);
Cp_KT = zeros([numel(Cpdata),numel(M_b)]);
Cp_La = zeros([numel(Cpdata),numel(M_b)]);

% Loop to conduct calculations
for i = 1:length(M_b)
    M_i = M_b(i);
    Cp_PG(:,i) = Prandtl_Glauert(Cpdata,M_i);
    Cp_KT(:,i) = Karman_Tsien(Cpdata,M_i);
    Cp_La(:,i) = Laitone(Cpdata,M_i,gamma);

    % Plotting
    fig = figure(i);
        plot(xdata,Cp_PG(:,i))
        txt1 = "Pressure Coefficient with 3 Rules of Improved Compressiblity";
        txt2 = sprintf("Corrections When M = %.3f, Koike",M_i);
        title({txt1, txt2})
        xlabel('x direction')
        ylabel('pressure coefficient, $C_{p}$')
        hold on
        plot(xdata,Cp_KT(:,i))
        plot(xdata,Cp_La(:,i))
        hold off; grid on; grid minor; box on; xlim([-0.02,1.02]);
        legend('Prandtl-Glauert','Karman-Tsien',"Laitone","Location",'southeast')
         set(gca,'ydir','reverse');
    file_txt = sprintf("3rulesPlot_M%.3f.png",M_i);
    saveas(fig,fullfile(fdir,file_txt));
end
```

```
% Minimum pressure coefficients
min_Cp_PG = min(Cp_PG);
min_Cp_KT = min(Cp_KT);
min_Cp_La = min(Cp_La);
arr = [min_Cp_PG.' min_Cp_KT.' min_Cp_La.'];
```

```
T = array2table(arr,"VariableNames",{'Prandtl-Glauert','Karman-Tsien','Laitone'});
writetable(T,fullfile(fdir,'3rules_min_Cp.xlsx'),"WriteMode","overwritesheet");
```

c)

```
Cpcr = critical_Cp(M_b,gamma);
arr_p = abs(T{:,:} - Cpcr.');
T_p = array2table(arr_p,"VariableNames",{'Prandtl-Glauert','Karman-
Tsien','Laitone'});
writetable(T_p,fullfile(fdir,'3rules_diff_Cp.xlsx'),"WriteMode","overwritesheet");
```

d)

```
% Compute the Critical Mach Number using Optimization - PatternSearch

% Prandtl-Glauert
M0 = 0.2;
lb = 0.01; ub = 0.95;
A = []; b = [];
Aeq = []; beq = [];
objfunc = @(M) opt_Prandtl_Glauert_Mcr(M,Cpo_min,gamma);
[Mcr_PG, fval_PG] = patternsearch(objfunc,M0,A,b,Aeq,beq,lb,ub);

% Karman-Tsien
objfunc = @(M) opt_Karman_Tsien_Mcr(M,Cpo_min,gamma);
[Mcr_KT, fval_KT] = patternsearch(objfunc,M0,A,b,Aeq,beq,lb,ub);

% Laitone
objfunc = @(M) opt_Laitone_Mcr(M,Cpo_min,gamma);
[Mcr_La, fval_La] = patternsearch(objfunc,M0,A,b,Aeq,beq,lb,ub);
```

d)

```
Mn = 0.85;
sigma = acosd(Mcr_KT/Mn);
```

FUNCTIONS

```
function data = read_txt_file(file_str)
    % This function reads a txt.file and imports the data as an array
    afile = fopen(file_str,'r');
    formatSpec = '%f';
    data = fscanf(afile,formatSpec);
    fclose(afile);
end

function Cp = Prandtl_Glauert(Cpo,M)
    % Function that calculates the subsonic linear theory using
    % Prandtl-Glauert formula
    Cp = Cpo./sqrt(1 - M.^2);
end

function Cp = Karman_Tsien(Cpo,M)
    % Function that calculates the subsonic linear theory using
    % Karman-Tsien formula
    a1 = sqrt(1 - M.^2);
    a2 = M^2./(1 + sqrt(1 - M.^2));
    Cp = Cpo./(a1 + a2*Cpo/2);
end
```

```
function Cp = Laitone(Cpo,M,gamma)
    % Function that calculates the subsonic linear theory using Latione
    % formula
    a1 = sqrt(1 - M.^2);
    a2 = (1 + (gamma - 1)/2.*M.^2);
    a3 = M^2.*a2/2./sqrt(1 - M.^2);
    Cp = Cpo./(a1 + a3.*Cpo);
end

function Cpcr = critical_Cp(M,gamma)
    % This function computes the critical pressure coefficient for a given
    % Mach number
    a1 = 2/gamma./M.^2;
    a2 = 1 + (gamma - 1)/2.*M.^2;
    a3 = (gamma + 1)/2;
    a4 = gamma/(gamma - 1);
    Cpcr = a1.*((a2./a3).^a4 - 1);
end

function G_M = opt_Prandtl_Glauert_Mcr(M,Cpo_min,gamma)
    % Expression of critical pressure coefficient function
    A1 = 2/gamma/M^2;
    A2 = 1 + (gamma - 1)/2*M^2;
    A3 = (gamma + 1)/2;
    A4 = gamma/(gamma - 1);
    Cpcr = A1*((A2/A3)^A4 - 1);

    % Prandtl-Glauert
    PG = Cpo_min/sqrt(1 - M^2);

    % Difference between the two equations
    G_M = abs(Cpcr - PG);
end

function G_M = opt_Karman_Tsien_Mcr(M,Cpo_min,gamma)
    % Expression of critical pressure coefficient function
    A1 = 2/gamma/M^2;
    A2 = 1 + (gamma - 1)/2*M^2;
    A3 = (gamma + 1)/2;
    A4 = gamma/(gamma - 1);
    Cpcr = A1*((A2/A3)^A4 - 1);

    % Karman-Tsien
    a1 = sqrt(1 - M.^2);
    a2 = M^2./(1 + sqrt(1 - M.^2));
    KT = Cpo_min./(a1 + a2*Cpo_min/2);

    % Difference between the two equations
    G_M = abs(Cpcr - KT);
end

function G_M = opt_Laitone_Mcr(M,Cpo_min,gamma)
    % Expression of critical pressure coefficient function
```

```
    A1 = 2/gamma/M^2;
    A2 = 1 + (gamma - 1)/2*M^2;
    A3 = (gamma + 1)/2;
    A4 = gamma/(gamma - 1);
    Cpcr = A1*((A2/A3)^A4 - 1);

    % Laitone
    a1 = sqrt(1 - M.^2);
    a2 = (1 + (gamma - 1)/2.*M.^2);
    a3 = M^2.*a2/2./sqrt(1 - M.^2);
    La = Cpo_min./(a1 + a3.*Cpo_min);

    % Difference between the two equations
    G_M = abs(Cpcr - La);
end
```