College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 567: Introduction To Applied Stochastic Processes

HW4

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Table of Contents

Problem 1

Consider the state space system

$$x(n+1) = 2x(n) + \frac{1}{\sqrt{5}}u(n) \quad and \quad y(n) = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} x(n) + \begin{bmatrix} v_1(n) \\ v_2(n) \\ v_3(n) \\ v_4(n) \end{bmatrix}.$$

Moreover, $u(n)$ and $v_j(n)_1^4$ are all independent Gaussian white noise process with variance one, which are also independent to the initial condition $x(0)$. Find the steady state Kalman filter and the steady state error covariance P. Hint: $A(I + BA)^{-1} = (I + AB)^{-1}A$.

Solution:
From the given system we know that the system matrices are

$$A = 2, \quad B = \frac{1}{\sqrt{5}}, \quad C = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From the information filter we know that

$$Q_{n+1} = A\Big(Q_n^{-1} + C^*(DD^*)^{-1}C\Big)^{-1} A^* + BB^*$$
$$Q_{n+1} = A\Big(Q_n^{-1} + C^*C\Big)^{-1} A^* + BB^*.$$

The steady state form of this is

$$\lim_{n\to\infty} Q_n = P.$$

Then we have

$$P = A(P^{-1} + C^*C)^{-1}A^* + BB^*$$
$$P = \frac{4}{\frac{1}{p} + 4} + \frac{1}{5}$$
$$0 = 20P^2 - 19P - 1$$
$$0 = (20P + 1)(P - 1).$$

$$P = \begin{bmatrix} 1 \\ -0.05 \end{bmatrix}.$$

Since $P > 0$, the answer becomes

$$P = 1.$$

The steady state becomes

$$K_p = APC^* \Big(CPC^* + CC^* \Big)^{-1}$$
$$= \begin{bmatrix} 5 & -8 & 8 & -8 \end{bmatrix}.$$

Thus, the steady state Kalman filter becomes

$$\hat{x}(n+1) = (A - K_p C)\hat{x}(n) + K_p y(n)$$

$$\hat{x}(n+1) = -27\hat{x}(n) + \begin{bmatrix} 5 & -8 & 8 & -8 \end{bmatrix} y(n).$$

Problem 2

Consider the state space system

$$x(n+1) = Ax(n) + u(n) \qquad and \qquad y(n) = Cx(n) + v(n)$$

where $A = A(n)$ are matrices on a state space $\mathcal{X}$ and $C = C(n)$ are matrices mapping $\mathcal{X}$ into $\mathcal{Y}$. Moreover, $u(n)$ and $v(n)$ are mean zero Gaussian random process which are independent to the initial condition $x(0)$ which is a Gaussian random vector. Furthermore, assume that

$$E\begin{bmatrix} u(n) \\ v(n) \end{bmatrix} \begin{bmatrix} u(m)^* & v(m)^* \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \delta_{n,m}$$

where $R_{ij} = R_{ij}(n)$ can be a function of n. (As expected, $\delta_{n,m}$ is the Kronecker delta, that is, $\delta_{n,m} = 1$ if $n = m$. and zero otherwise.) Let $\mathcal{M}_n = span\{1, y(k)\}_0^n$ and $\hat{x}(n) = P_{\mathcal{M}_{n-1}} x(n)$ denote the optimal state estimate. Let $\tilde{x}(n) = x(n) - \hat{x}(n)$ and Q_n be the error covariance matrix defined by

$$Q_n = E\tilde{x}(n)\tilde{x}(n)^* = E(x(n) - \hat{x}(n))(x(n) - \hat{x}(n))^*.$$

Find the Kalman filter for the state space system above. To be precise, find a recursive estimate for the optimal state $\hat{x}(n)$ and a recursive formula for the error covariance Q_n

(i) Show that the optimal state is given by

$$\begin{aligned} \hat{x}(n+1) &= A\hat{x}(n) + L_n(y(n) - C\hat{x}(n)) \\ &= (A - L_n C)\hat{x}(n) + Ly(n) \\ L_n &= (AQ_nC^* + R_{12})(CQ_nC^* + R_{22})^{-1}. \end{aligned}$$

The initial conditions is $\hat{x}(0) = \mu_0$.

(ii) Show that the error covariance Q_n is given by the solution to the Riccati difference equation

$$\begin{aligned} Q_{n+1} = AQ_nA^* &+ R_{11} \\ &- (AQ_nC^* + R_{12})(CQ_nC^* + R_{22})^{-1}(AQ_nC^* + R_{12})^*. \end{aligned}$$

The initial condition $Q_0 = E(x(0) - \mu_0)(x(0) - \mu_0)^*$. Another form for the Ricatti difference equation is given by

$$Q_{n+1} = (A_{Ln}C)Q_n(A - L_nC)^* + \begin{bmatrix} I & -L_n \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} I \\ -L_n^* \end{bmatrix}.$$

(iii) Show that $\hat{x}(n|n) = P_{\mathcal{M}_n} x(n)$ is determined by

$$\hat{x}(n|n) = \hat{x}(n) + Q_nC^*(CQ_nC^* + R_{22})^{-1}(y(n) - C\hat{x}(n)).$$

Solution:
For this problem we do not assume that the noises $u(n)$ and $v(n)$ are independent.

$$\mathcal{M}_n = \mathcal{M}_{n-1} \bigvee y(n)$$
$$\hat{x}(n) = P_{\mathcal{M}_{n-1}} x(n)$$

and

$$\begin{aligned}
\phi(n) = y(n) - P_{\mathcal{M}_{n-1}} y(n) &= y(n) - P_{\mathcal{M}_{n-1}} \Big(Cx(n) + v(n) \Big) \\
&= y(n) - CP_{\mathcal{M}_{n-1}} x(n) - \cancelto{0}{P_{\mathcal{M}_{n-1}} v(n)} \\
&= y(n) - C\hat{x}(n) \\
&= C\tilde{x}(n) + v(n).
\end{aligned}$$

Then we compute

$$\begin{aligned}
\hat{x}(n+1) &= P_{\mathcal{M}_n} x(n+1) = P_{\mathcal{M}_n} \Big(Ax(n) + u(n) \Big) \\
&= AP_{\mathcal{M}_n} x(n) + P_{\mathcal{M}_n} u(n) \\
&= AP_{\mathcal{M}_{n-1}} x(n) + AR_{x(n)\phi(n)} R_{\phi(n)}^{-1} \phi(n) + \cancelto{0}{P_{\mathcal{M}_{n-1}} u(n)} + R_{u(n)\phi(n)} R_{\phi(n)}^{-1} \phi(n) \\
&= A\hat{x}(n) + AR_{x(n)\phi(n)} R_{\phi(n)}^{-1} \phi(n) + +R_{u(n)\phi(n)} R_{\phi(n)}^{-1} \phi(n).
\end{aligned}$$

Here we must compute

$$\begin{aligned}
R_{x(n)\phi(n)} &= E(x(n)\phi(n)^*) = Ex(n) \Big(C\tilde{x}(n) + v(n) \Big)^* \\
&= Ex(n)\tilde{x}(n)^* C^* + \cancelto{0}{Ex(n)v(n)^*} \\
&= E\Big(\tilde{x}(n) + \hat{x}(n) \Big) \tilde{x}(n) C^* \\
&= E\tilde{x}(n)\tilde{x}(n)^* C^* \\
&= Q_n C^*
\end{aligned}$$

and

$$\begin{aligned}
R_{\phi(n)} &= E\phi(n)\phi(n)^* \\
&= E\Big(C\tilde{x}(n) + v(n) \Big) \Big(C\tilde{x}(n) + v(n) \Big)^* \\
&= E\tilde{x}(n)\tilde{x}(n)^* C^* + Ev(n)v(n)^* \\
&= CQ_n C^* + R_{22}
\end{aligned}$$

and

$$\begin{aligned}
R_{u(n)\phi(n)} &= Eu(n) \Big(C\tilde{x}(n) + v(n) \Big) \\
&= \cancelto{0}{Eu(n)\tilde{x}(n) C^*} + Eu(n)v(n)^* \\
&= R_{12}.
\end{aligned}$$

Therefore,

$$\hat{x}(n+1) = A\hat{x}(n) + AQ_nC^*\Big(CQ_nC^* + R_{22}\Big)^{-1}\phi(n) + R_{12}\Big(CQ_nC^* + R_{22}\Big)^{-1}\phi(n)$$

$$\hat{x}(n+1) = A\hat{x}(n) + \Big(AQ_nC^* + R_{12}\Big)\Big(CQ_nC^* + R_{22}\Big)^{-1}\Big(y(n) - C\hat{x}\Big)$$

Now if we let

$$L_n = \Big(AQ_nC^* + R_{12}\Big)\Big(CQ_nC^* + R_{22}\Big)^{-1}$$

Then we have

$$\hat{x}(n+1) = A\hat{x}(n) + L_n(y(n) - C\hat{x}(n))$$

or

$$\hat{x}(n+1) = (A - L_nC)\hat{x}(n) + Ly(n).$$

(ii) We start from

$$\begin{aligned} Q_{n+1} &= E\tilde{x}(n+1)\tilde{x}(n+1)^* \\ &= E\Big\{A\Big(x(n) - P_{\mathcal{M}_n}x(n)\Big) + \Big(u(n) - P_{\mathcal{M}_n}u(n)\Big)\Big\}\Big\{A\Big(x(n) - P_{\mathcal{M}_n}x(n)\Big) + \Big(u(n) - P_{\mathcal{M}_n}u(n)\Big)\Big\}^* \end{aligned}$$

the cross terms are zero. Then

$$\begin{aligned} &E\Big\{A\Big(x(n) - P_{\mathcal{M}_n}x(n)\Big)\Big\}\Big\{A\Big(x(n) - P_{\mathcal{M}_n}x(n)\Big)\Big\}^* \\ &= E\tilde{x}(n)\tilde{x}(n)^* - R_{x(n)\phi(n)}R_{\phi(n)}^{-1}R_{x(n)\phi(n)}^* \\ &= AQ_n - AQ_nC^*\Big(CQ_nC^* + R_{22}^*\Big)^{-1}CQ_nA^* \end{aligned}$$

and

$$\begin{aligned} &E\Big\{\Big(u(n) - P_{\mathcal{M}_n}u(n)\Big)\Big\}\Big\{\Big(u(n) - P_{\mathcal{M}_n}u(n)\Big)\Big\}^* \\ &= Eu(n)u(n)^* - R_{x(n)\phi(n)}R_{\phi(n)}^{-1}R_{x(n)\phi(n)}^* \\ &= R_{11} - R_{12}\Big(CQ_nC^* + R_{22}^*\Big)^{-1}R_{12}^*. \end{aligned}$$

Thus,

$$Q_{n+1} = AQ_n - AQ_nC^*\Big(CQ_nC^* + R_{22}^*\Big)^{-1}CQ_nA^* + R_{11} - R_{12}\Big(CQ_nC^* + R_{22}^*\Big)^{-1}R_{12}^*$$

$$Q_{n+1} = AQ_nA^* + R_{11} - (AQ_nC^* + R_{12})(CQ_nC^* + R_{22})^{-1}(AQ_nC^* + R_{12})^*.$$

(iii)

$$x(\hat{n|}n) = P_{\mathcal{M}_n}x(n) = P_{\mathcal{M}_{n-1}}x(n) + R_{x(n)\phi(n)}R_{\phi(n)}^{-1}\phi(n)$$

and thus

$$\hat{x}(n|n) = \hat{x}(n) + Q_nC^*(CQ_nC^* + R_{22})^{-1}(y(n) - C\hat{x}(n)).$$