August 31, 2020

AAE 564

Homework 1 - Solution

Exercise 1 To transform the linear system into a state space description, we need to relabel the variables accoring to the following rule:

$$x_i = q^{(i-1)} \quad , \qquad i = 1..n$$

Hence x_i refers to the (i-1)-th derivative of q, where $x_1 = q$. Following this definition, we mention that:

$$\dot{x}_i = q^i = x_{i+1} \qquad \forall i$$

Finally we need to find an explicit expression for $q^{(n)}$ by rearranging the given system description:

$$q^{(n)} = u - a_{n-1}q^{(n-1)} - \ldots - a_1\dot{q} - a_0 q$$

Hence, the state space representation would be:

$$\boxed{\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ u - a_{n-1}x_n - \ldots - a_1 x_2 - a_0 x_1 \end{pmatrix}}$$

Exercise 2 procedure to find a state space representation:

- rearrange equations to an explicit expression of the highest derivative for every variable
- make sure your equations are not coupled in the highest derivatives of the variables
- relabel the variables and formulate the state space

for this and the next exercise, the following relabeling of the variables is used (x_4 is only used in exercise 2i):

$$x_1 = q_1, \quad x_2 = \dot{q}_1, \quad x_3 = q_2, \quad x_4 = \dot{q}_2$$

(i)

$$\begin{aligned} 2\ddot{q}_1 + \ddot{q}_2 + \sin q_1 &= 0 \qquad (I) \\ \ddot{q}_1 + 2\ddot{q}_2 + \sin q_2 &= 0 \qquad (II) \end{aligned}$$

decouple variables and formulate an explicit expression for the highest derivatives ($\ddot{q}_1$ and $\ddot{q}_2$):

$$\begin{aligned}
2(I)-(II): &\qquad 3\ddot{q}_1 + 2\sin q_1 - \sin q_2 = 0 &&\Leftrightarrow\quad \ddot{q}_1 = -\frac{2}{3}\sin q_1 + \frac{1}{3}\sin q_2\\
2(II)-(I): &\qquad 3\ddot{q}_2 - \sin q_1 + 2\sin q_2 = 0 &&\Leftrightarrow\quad \ddot{q}_2 = \frac{1}{3}\sin q_1 - \frac{2}{3}\sin q_2
\end{aligned}$$

this can be transformed into state space as:

$$\boxed{\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{pmatrix} = \begin{pmatrix} x_2\\ -\frac{2}{3}\sin x_1 + \frac{1}{3}\sin x_3\\ x_4\\ \frac{1}{3}\sin x_1 - \frac{2}{3}\sin x_3\end{pmatrix}}$$

(ii)

$$\begin{aligned}
\ddot{q}_1 + \dot{q}_2 + q_1^3 &= 0 \qquad (I)\\
\dot{q}_1 + \dot{q}_2 + q_2^3 &= 0 \qquad (II)
\end{aligned}$$

There is no instance of $\ddot{q}_2$. Hence, we solve for $\ddot{q}_1$ and $\dot{q}_2$:

$$\begin{aligned}
(I)-(II): &\qquad \ddot{q}_1 - \dot{q}_1 + q_1^3 - q_2^3 = 0 &&\Leftrightarrow\quad \ddot{q}_1 = \dot{q}_1 - q_1^3 + q_2^3\\
(II): &\qquad \dot{q}_1 + \dot{q}_2 + q_2^3 = 0 &&\Leftrightarrow\quad \dot{q}_2 = -\dot{q}_1 - q_2^3
\end{aligned}$$

Now we can obtain a state space representation:

$$\boxed{\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix} x_2\\ -x_1^3 + x_2 + x_3^3\\ -x_2 - x_3^3\end{pmatrix}}$$

Exercise 3 As in (2ii), the equations have to be solved for $\ddot{q}_1$ and $\dot{q}_2$

$$\begin{aligned}
\ddot{q}_1 + \dot{q}_2 + \sin q_1 &= u \qquad (I)\\
\dot{q}_2 + q_1 + q_2 &= 0 \qquad (II)\\
y &= q_1 + q_2
\end{aligned}$$

$$\begin{aligned}
(I)-(II): &\qquad \ddot{q}_1 - q_1 + \sin q_1 - q_2 = u &&\Leftrightarrow\quad \ddot{q}_1 = q_1 - \sin q_1 + q_2 + u\\
(II): &\qquad \dot{q}_2 + q_1 + q_2 = 0 &&\Leftrightarrow\quad \dot{q}_2 = -q_1 - q_2
\end{aligned}$$

By relabeling according to the definitions in exercise 2, the state space representation can be written as:

$$\boxed{\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{pmatrix} = \begin{pmatrix} x_2\\ x_1 - \sin x_1 + x_3\\ -x_1 - x_3\end{pmatrix} + \begin{pmatrix}0\\ u\\ 0\end{pmatrix}}$$

$$\boxed{y = x_1 + x_3}$$

Exercise 4 As we relabeled the variables for the continuous systems in exercise 1-3, we can do this as well with the discrete time system

$$q(k+3) + 7q(k+2) + q(k+1) + 6q(k) + 7u(k) = 0$$

The variables will be changed to:

$$x_1(k) = q(k), \quad x_2(k) = q(k+1), \quad x_3(k) = q(k+2) \qquad \textit{note: } x_1(k+1) = q(k+1) = x_2(k)$$

and rewrite the system as:

$$q(k+3) = -7q(k+2) - q(k+1) - 6q(k) - 7u(k)$$

The state space representation turns out to be:

$$\boxed{\begin{aligned} x_1(k+1) &= \; x_2(k) \\ x_2(k+1) &= \; x_3(k) \\ x_3(k+1) &= -6x_1(k) - x_2(k) - 7x_3(k) - 7u(k) \end{aligned}}$$

Exercise 5 Compared to exercise 1, we now need to obtain a state space representation for a discrete time system by defining the variables according to:

$$x_i(k) = q(k+i-1) \quad , \qquad i = 1..n \qquad \text{with } x_i(k+1) = q(k+i) = x_{i+1}(k)$$

and rewrite the system as:

$$q(k+n) = -a_{n-1}q(k+n-1) - \ldots - a_1 q(k+1) - a_0 q(k)$$

to obtain the state space description:

$$\boxed{\begin{pmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{pmatrix} = \begin{pmatrix} x_2(k) \\ x_3(k) \\ \vdots \\ x_n(k) \\ -a_{n-1}x_n(k) - \ldots - a_1 x_2(k) - a_0 x_1(k) \end{pmatrix}}$$

Exercise 6 As we solved a continiuous system for the highest order derivative in each variable, the give discrete time system has to be solved for $q_1(k+2)$ and $q_2(k+1)$.

$$\begin{aligned} &q_1(k+2) + q_2(k+1) + q_1(k) = u(k) && (I) \\ &q_1(k+2) - q_2(k+1) + q_2(k) = 0 && (II) \\ &y(k) = q_1(k+1) + q_2(k) \end{aligned}$$

The system is rearranged as:

$$(I) + (II): \qquad 2q_1(k+2) + q_1(k) + q_2(k) = u(k) \qquad \Leftrightarrow \quad q_1(k+2) = \frac{1}{2}\left[u(k) - q_1(k) - q_2(k)\right]$$

$$(I) - (II): \qquad 2q_2(k+1) + q_1(k) - q_2(k) = u(k) \qquad \Leftrightarrow \quad q_2(k+1) = \frac{1}{2}\left[u(k) - q_1(k) + q_2(k)\right]$$

by defining the variables as:

$$x_1(k) = q_1(k), \quad x_2(k) = q_1(k+1), \quad x_3(k) = q_2(k)$$

the state space representation is:

$$\boxed{\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= \tfrac{1}{2}\left[-x_1(k) - x_3(k) + u(k)\right] \\ x_3(k+1) &= \tfrac{1}{2}\left[-x_1(k) + x_3(k) + u(k)\right] \end{aligned}}$$

$$\boxed{y(k) = x_2(k) + x_3(k)}$$

Exercise 7 The equations of motion for the double pendulum cart as well as a figure can be found on page 14 in the notes.
By defining $\bar{q} = [y, \theta_1, \theta_2]^T$ the equations can be rearranged to match a form, like $\mathbf{M}(\bar{q}) \cdot \ddot{\bar{q}} = \bar{F}(\bar{q}, \dot{\bar{q}}, u)$

$$\underbrace{\begin{bmatrix} m_0 + m_1 + m_2 & -m_1 l_1 \cos\theta_1 & -m_2 l_2 \cos\theta_2 \\ -m_1 l_1 \cos\theta_1 & m_1 l_1^2 & 0 \\ -m_2 l_2 \cos\theta_2 & 0 & m_2 l_2^2 \end{bmatrix}}_{\mathbf{M}(\bar{q})} \cdot \ddot{\bar{q}} = \underbrace{\begin{bmatrix} u - m_1 l_1 \sin\theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 \\ -m_1 l_1 g \sin\theta_1 \\ -m_2 l_2 g \sin\theta_2 \end{bmatrix}}_{\bar{F}(\bar{q}, \dot{\bar{q}}, u)}$$

Using the different parameter sets and initial conditions in Matlab/Simulink, the second order derivatives can be expressed by:

$$\ddot{\bar{q}} = \mathbf{M}^{-1}(\bar{q}) \cdot \bar{F}(\bar{q}, \dot{\bar{q}}, u)$$

Now, the only thing left is to set up a system model with two integrators (see figure 1) as Matlab can do the matrix inverse computation for you (see listing 1).

By going through the different parameter and initial condition sets (see figures 2 to 9), we especially notice the following two things:

- the simulation has a truncation-error in the computation (e.g. see the figure 2)
- a deviation from the intial conditions/parameter set causes a chaotic looking behaviour (e.g. see figure 5)

Listing 1: exercise 7, m-file getQDDot

```matlab
function qDDot = getQDDot(m, l, u, q, qDot)
g = 1;
% define multiple used matrix entries for M: -m*l*cos theta
mLCosT1 = -m(2)*l(1)*cos(q(2));
mLCosT2 = -m(3)*l(2)*cos(q(3));

M = [(m(1)+m(2)+m(3)), mLCosT1, mLCosT2;...
     mLCosT1, m(2)*l(1)*l(1), 0;...
     mLCosT2, 0, m(3)*l(2)*l(2)];

% define multiple used entries for F: -m*l*sin theta
mLSinT1 = -m(2)*l(1)*sin(q(2));
mLSinT2 = -m(3)*l(2)*sin(q(3));

F = [(u+mLSinT1*qDot(2)*qDot(2) + mLSinT2*qDot(3)*qDot(3));...
     mLSinT1*g; mLSinT2*g];

qDDot = M\F;
```

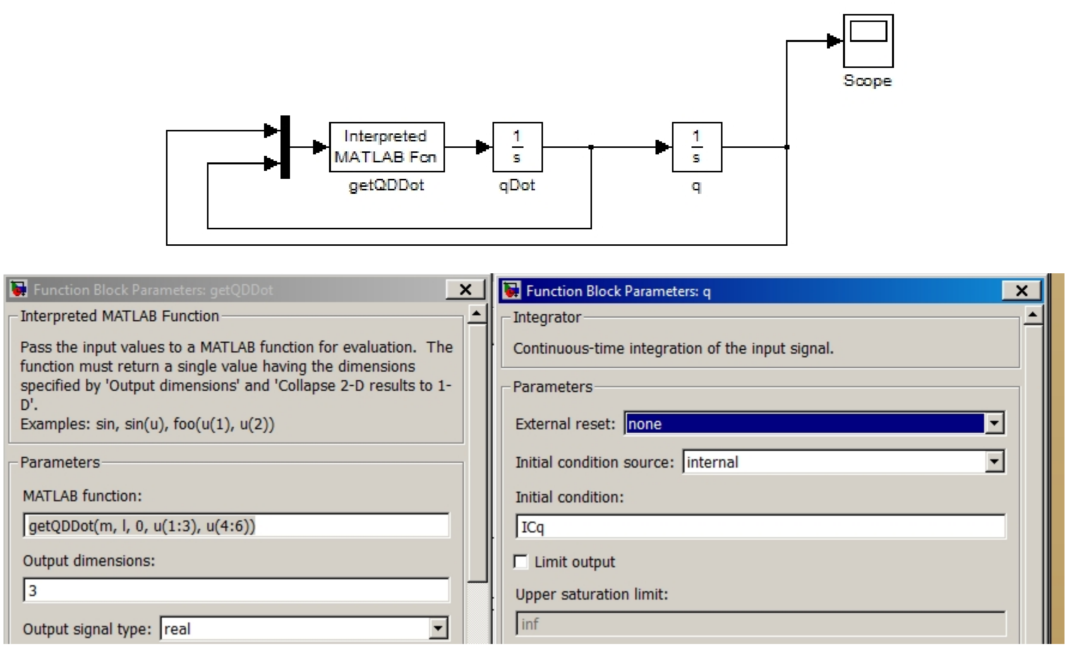

Figure 1: Simulinkmodel

Listing 2: exercise 7, Matlab script

```
%m = [2; 1; 1]; l = [1; 1]; %P1
m = [2; 1; 1]; l = [1; 0.5]; %P4

ICqDot = [0; 0; 0];
%ICq = [0; -10; 10]; %IC1
%ICq = [0; 10; 10]; %IC2
%ICq = [0; -90; 90]; %IC3
ICq = [0; -90.01; 90]; %IC4
%ICq = [0; 179.99; 0]; %IC7

% convert angles to rad
piFac = pi/180;
ICq(2) = ICq(2)*piFac; ICq(3) = ICq(3)*piFac;

sim('Ex7');
```

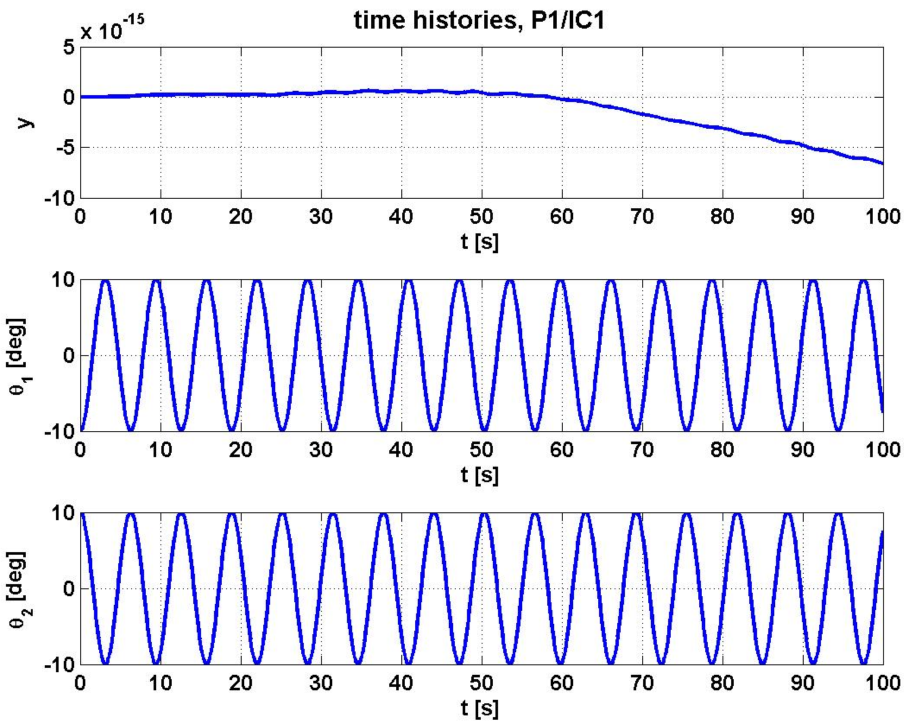

Figure 2: time histories P1/IC1

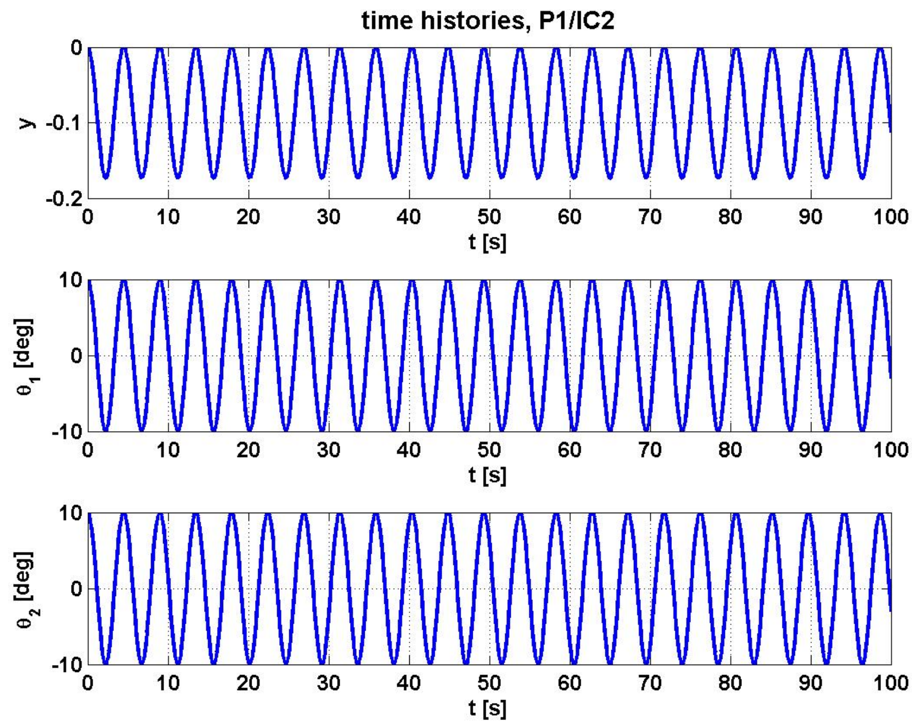

Figure 3: time histories P1/IC2

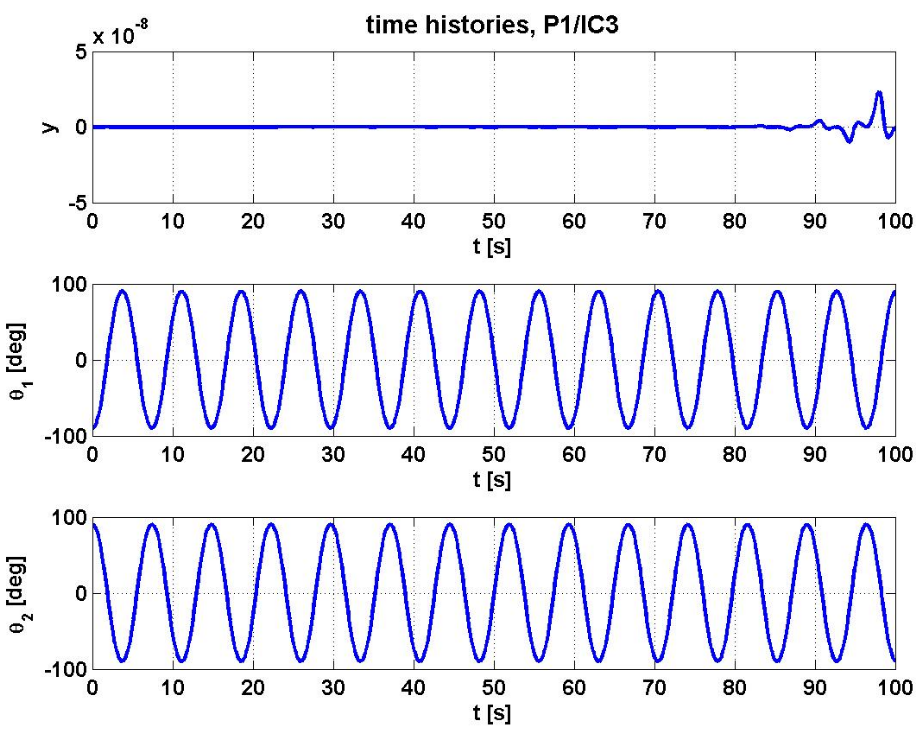

Figure 4: time histories P1/IC3

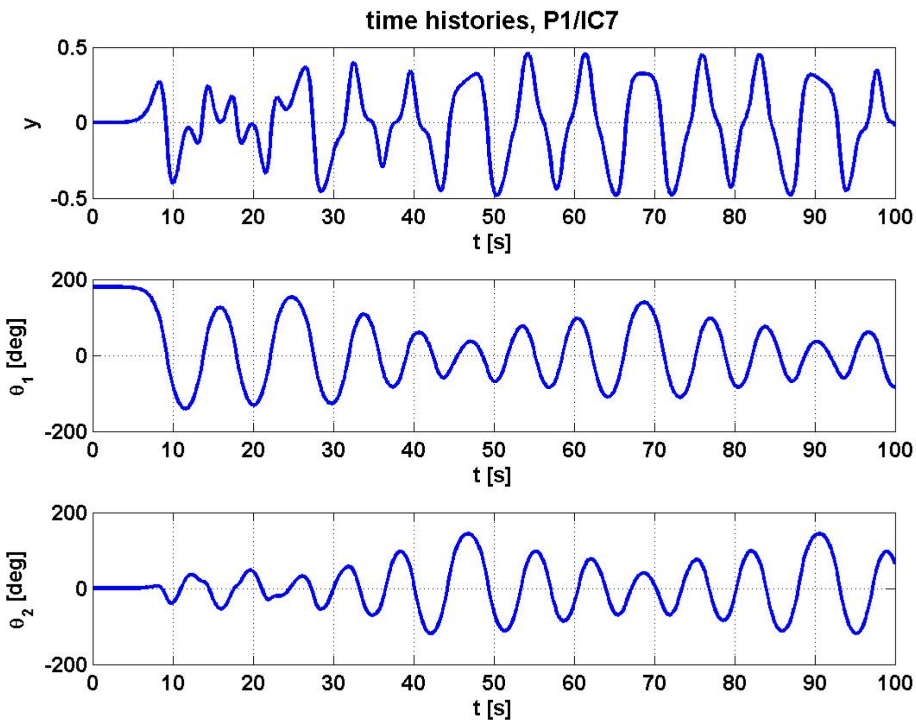

Figure 5: time histories P1/IC7

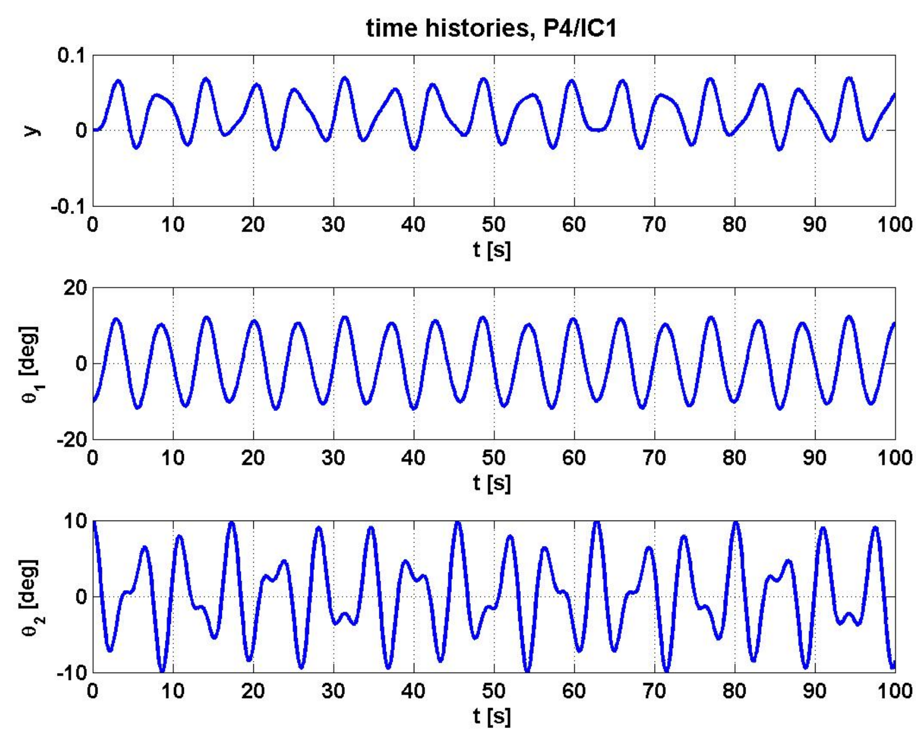

Figure 6: time histories P4/IC1

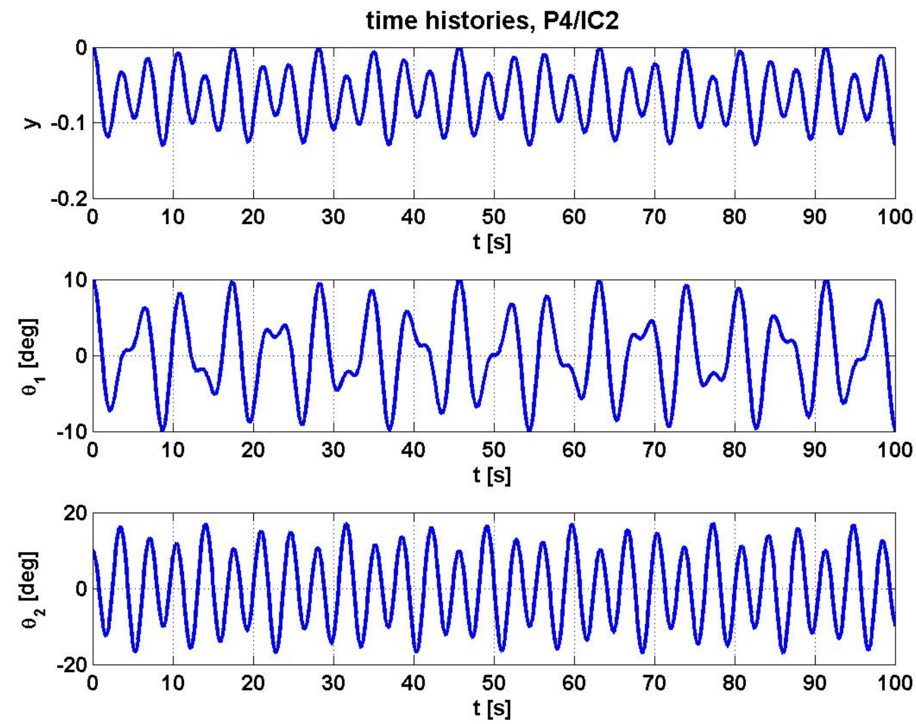

Figure 7: time histories P4/IC2

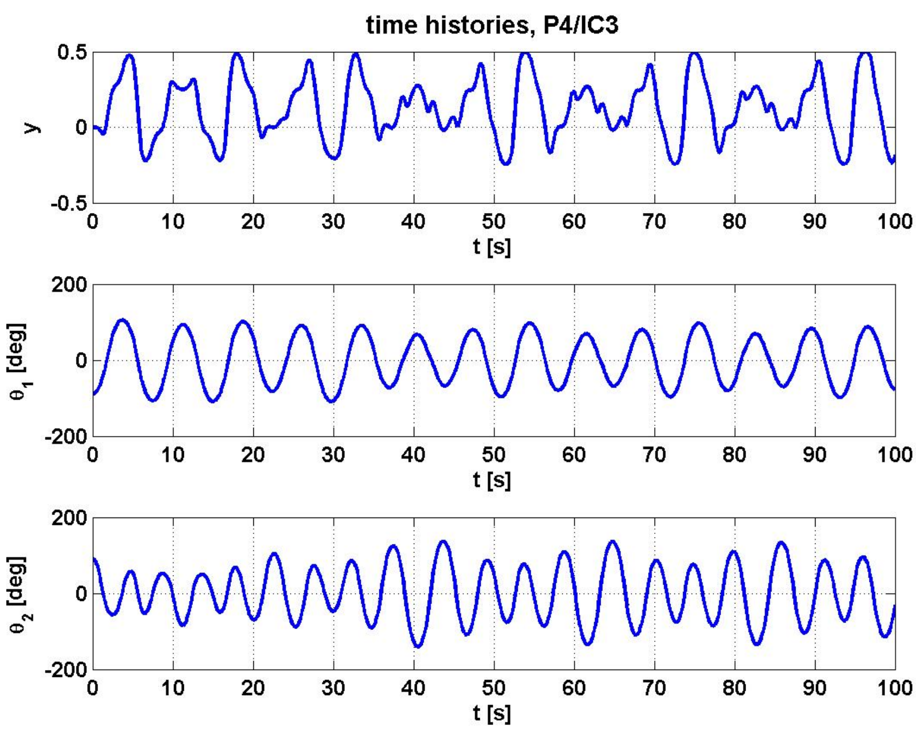

Figure 8: time histories P4/IC3

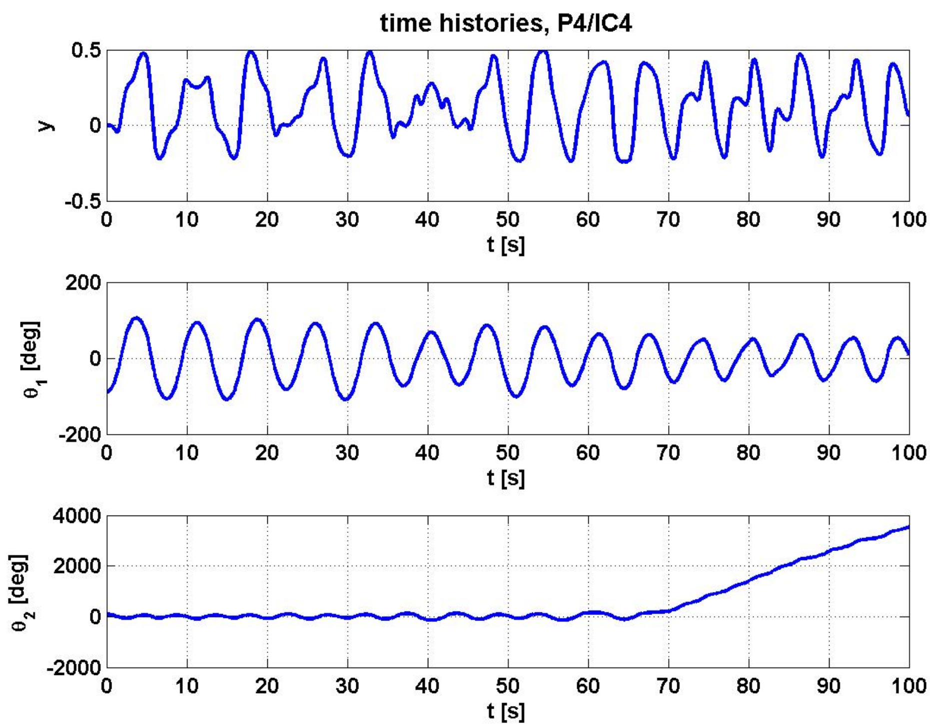

Figure 9: time histories P4/IC4