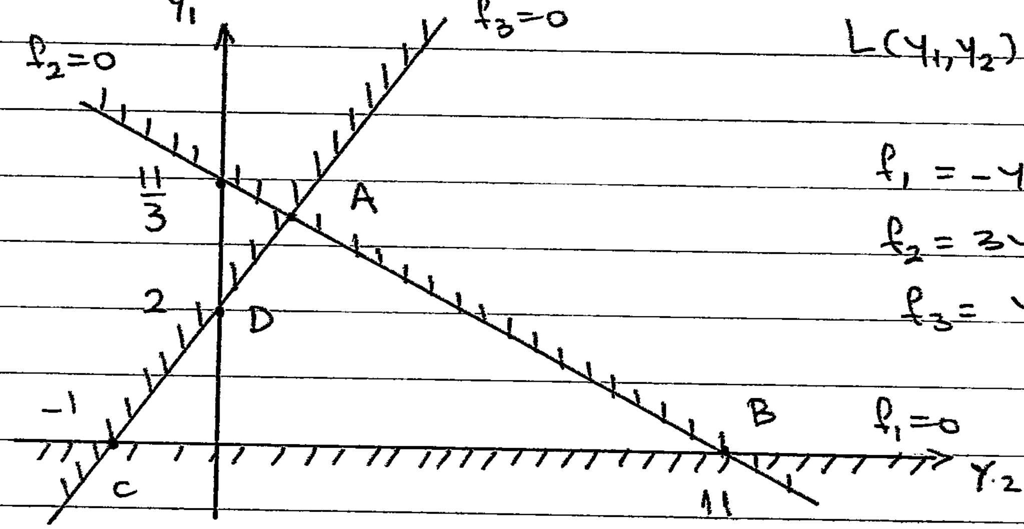


$L(y_1,y_2) = -5y_1 - y_2$

$f_1 = -y_1 \le 0$

$f_2 = 3y_1 + y_2 - 11 \le 0$

$f_3 = y_1 - 2y_2 - 2 \le 0$

The feasible region is sketched above. The candidate points for the minimum are A, B, C
The point that satisfies the Kuhn-Tucker conditions is the minimizing point.

Define $\quad H = -5y_1 - y_2 + \lambda_1(-y_1) + \lambda_2(3y_1 + y_2 - 11) + \lambda_3(y_1 - 2y_2 - 2)$

$$\frac{\partial H}{\partial y_1} = -5 - \lambda_1 + 3\lambda_2 + \lambda_3 = 0$$

$$\frac{\partial H}{\partial y_2} = -1 + \lambda_2 - 2\lambda_3 = 0$$

At the minimizer $\quad \lambda_i \ge 0 \quad i = 1,2,3$

At point A: $\quad \lambda_1 = 0 \quad$ so $\quad \left.\begin{array}{l} 3\lambda_2 + \lambda_3 = 5 \\ \lambda_2 = 2\lambda_3 + 1 \end{array}\right\} \Rightarrow \begin{array}{l} \lambda_3 = \frac{2}{7} \\ \lambda_2 = \frac{11}{7} \end{array}$

possibly a minimizer

at point B: $\lambda_3=0$ $\quad \left.\begin{array}{l}\lambda_2-1=0\\ -5-\lambda+3\lambda_2=0\end{array}\right\} \Rightarrow \quad \begin{array}{l}\lambda_2=1\\ \lambda_1=-2\end{array}$

not a minimizer

At point C: $\lambda_2=0$ so $\left.\begin{array}{l}-5-\lambda_1+\lambda_3=0\\ -1-2\lambda_3=0\end{array}\right\} \Rightarrow \begin{array}{l}\lambda_1=-\frac{11}{2}\\ \lambda_3=-\frac{1}{2}\end{array}$

not a minimizer

The minimizer is point A. At this point,

$$\left.\begin{array}{l}3y_1+y_2=11\\ y_1-2y_2=2\end{array}\right\} \Rightarrow y_1^*=\frac{24}{7}, \quad y_2^*=\frac{5}{7}$$

$$L_{min}=L(y_1^*,y_2^*)=-5\left(\frac{24}{7}\right)-\frac{5}{7}=-\frac{125}{7}$$

Note: The unconstrained minimum of L does not exist since it is a linear function wrt y_1, y_2.

$$\boxed{\begin{array}{lll} f(x_1,x_2) = x_1^2 - x_2 & & \\ x_1^2+x_2^2 \le 1 & x_2 \le 2 & x_1^3+x_2=1 \end{array}}$$

$$L = x_1^2 - x_2 + \lambda_1(x_1^2+x_2^2-1) + \lambda_2(x_2-2) + \lambda_3(x_1^3+x_2-1)$$

$$\frac{\partial L}{\partial x_1} = 2x_1 + 2\lambda_1 x_1 + 3\lambda_3 x_1^2 = 0$$

$$\frac{\partial L}{\partial x_2} = -1 + 2\lambda_1 x_2 + \lambda_2 + \lambda_3 = 0$$

Case 1: $\lambda_1 = \lambda_2 = 0$ (both I.C. inactive)

$$x_1^3 + x_2 = 1 \Rightarrow x_2 = 1 - x_1^3$$

$$f = x_1^2 - (1-x_1^3) = x_1^2 - 1 + x_1^3 \quad \text{no minimum}$$

Case 2: $\lambda_1 \neq 0 \quad \lambda_2 \neq 0$ (both i.c active)

$x_2 = 2$ cannot hold together with $x_1^2+x_2^2=1$

Case 3: $\lambda_2 \neq 0 \quad \lambda_1 = 0$ ($x_2=2$ is active)

$$2x_1 + 3\lambda_3 x_1^2 = 0$$
$$-1 + \lambda_2 + \lambda_3 = 0$$

$$\underline{x_2 = 2} \Rightarrow x_1^3 = 1 - 2 = -1 \Rightarrow \underline{x_1 = -1}$$

$$-2+3\lambda_3=0 \Rightarrow \lambda_3=\frac{2}{3}$$

$$\lambda_2=1-\lambda_3=1-\frac{2}{3}=\frac{1}{3}$$

<u>Case 4:</u> $\lambda_2=0 \quad \lambda_1\neq 0 \quad (x_1^2+x_2^2=1$ is active$)$

$$2x_1+2\lambda_1x_1+3\lambda_3x_1^2=0 \qquad (1)$$
$$-1+2\lambda_1x_2+\lambda_3=0$$
$$x_1^2+x_2^2=1$$
$$x_1^3+x_2=1$$

$$(1) \Rightarrow x_1(2+2\lambda_1+3\lambda_3x_1)=0 \begin{cases} \underline{x_1=0} \\ 2+2\lambda_1+3\lambda_3x_1=0 \end{cases}$$

If
$$2+2\lambda_1+3\lambda_3x_1=0$$
$$-1+2\lambda_1x_2+\lambda_3=0$$
$$\left.\begin{aligned} x_1^2+x_2^2&=1 \\ x_1^3+x_2&=1 \end{aligned}\right\} \quad \begin{aligned} x_1^2+(1-x_1^3)^2&=1 \\ x_1^2+1-2x_1^3+x_1^6&=1 \end{aligned}$$

or

$$x_1^2-2x_1^3+x_1^6=0 \Rightarrow x_1^2(1-2x_1+x_1^4)=0$$

Assume $1-2x_1+x_1^4=0 \Rightarrow \underline{x_1=1}$ or $\underline{x_1=0.5437}$

$$\min\,(x_1 + x_2^2 + x_2x_3 + 2x_3^2)$$

subject to $\quad \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) = 1$

Lagrangian $\quad L = x_1^2 + x_2^2 + x_2x_3 + 2x_3^2 + \frac{\lambda}{2}(x_1^2 + x_2^2 + x_3^2 - 2)$

First order necessary conditions

$$1 + \lambda x_1 = 0$$

$$2x_2 + x_3 + \lambda x_2 = 0$$

$$x_2 + 4x_3 + \lambda x_3 = 0$$

One solution is

$$x_0 = (-\sqrt{2}, 0, 0) \quad \text{and} \quad \lambda = \sqrt{2}/2$$

Second-order conditions at this point

$$L_{xx} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 2+\lambda & 1 \\ 0 & 1 & 4+\lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$g'(x) = [x_1 \; x_2 \; x_3] \Rightarrow g'(x_0) = [-1 \; 0 \; 0]$$

$$N(g'(x_0)) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$N^T L_{xx} N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$$

This matrix is positive definite

Hence L_{xx} is positive definite on $N(g'(x_0))$

It follows that $x_0 = (-\sqrt{2}, 0, 0)$ is a local minimizer for this problem

This problem is equivalent to the following problem in standard form.

$$\min \; -x_1$$

subject to

$$x_2 - (1-x_1)^3 \le 0$$
$$-x_2 \le 0$$

Lagrangian

$$L(x_1, x_2, \lambda_0, \lambda_1, \lambda_2) = -\lambda_0 x_1 + \lambda_1 (x_2 - (1-x_1)^3) + \lambda_2(-x_2)$$

$$L_x = \begin{bmatrix} -\lambda_0 + 3\lambda_1 (1-x_1)^2 \\ \lambda_1 - \lambda_2 \end{bmatrix} = 0$$

Let $\lambda_0^* = 0$. Then a solution is

$$\lambda_1^* = \lambda_2^* = \alpha > 0 \quad \text{and} \quad x_1^* = 1, \; x_2^* = 0$$

This solution satisfies all necessary conditions and at $(1,0)$ both constraints are active.

This is an <u>abnormal</u> extremal since $\lambda_0^* = 0$.

Let $\lambda_0^* = 1$ and assume constraint #1 is active and constraint #2 is inactive ($\lambda_2^* = 0$)

The necessary conditions give $\lambda_1^* = \lambda_2^* = 0$ and hence also $\lambda_0^* = 0$, which is not possible

Similarly, if constraint #2 is active and constraint #1 is inactive ($\lambda_1^* = 0$) we also get $\lambda_1^* = \lambda_2^* = 0$ and hence also $\lambda_0^* = 0$, which is not possible.

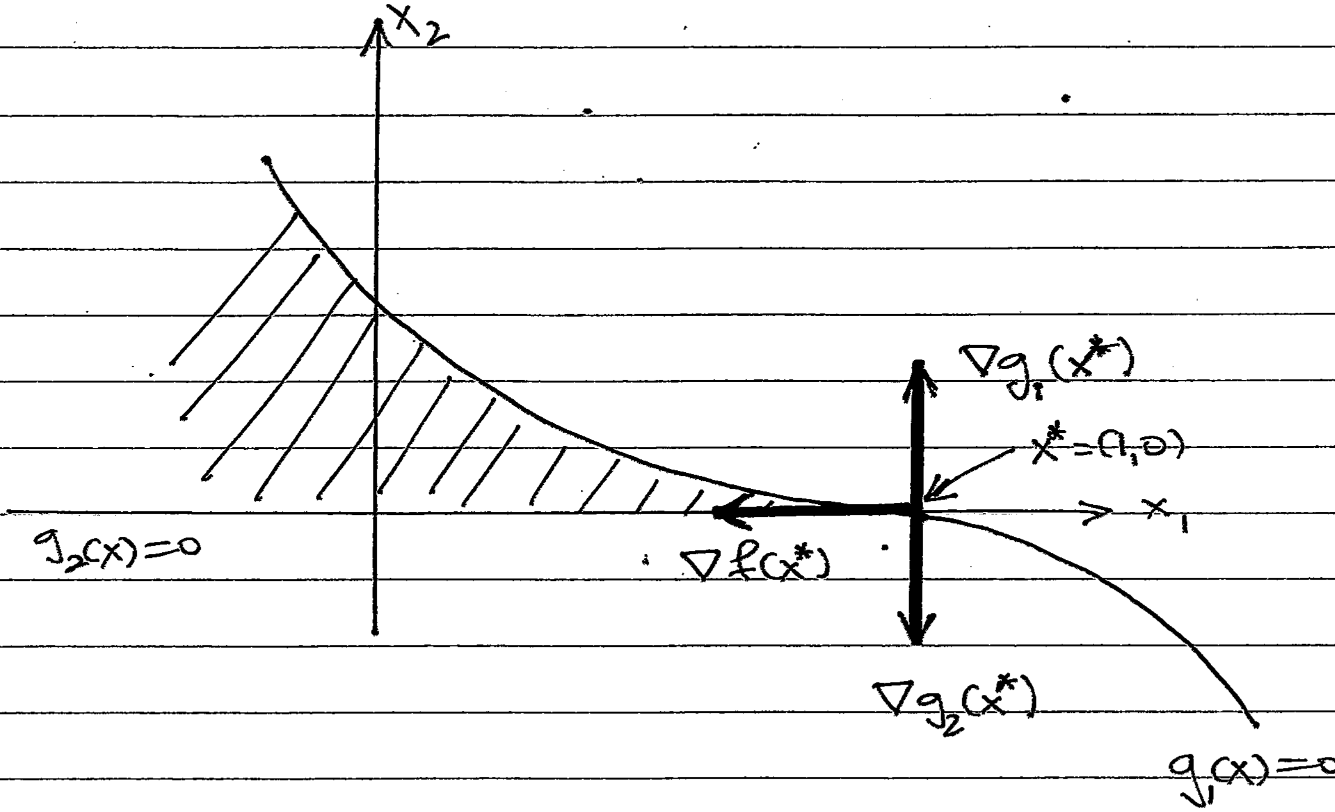
So if $\lambda_0^* = 1$ we need to have both constraints active. In this case we need to solve

$$\left.\begin{array}{l} x_2 - (1-x_1)^3 = 0 \\ x_2 = 0 \end{array}\right\} \Rightarrow x_1^* = 1 \quad x_2^* = 0$$

which is already the point we have computed.

For this problem, $x^* = (1,0)$ is the abnormal candidate local minimizer

The figure in the next page shows pictorialy the situation for this problem



x_2
$\nabla g_1(x^*)$
$x^* = (1,0)$
x_1
$g_2(x) = 0$
$\nabla f(x^*)$
$\nabla g_2(x^*)$
$g_1(x) = 0$

$$\min f(x) = x$$

subject to $\quad g(x,y) = y^2 + x^4 - x^3 = 0$

$$L = \lambda_0 x + \lambda (y^2 + x^4 - x^3)$$

$$\frac{\partial L}{\partial x} = \lambda_0 + 4\lambda x^3 - 3\lambda x^2 = 0 \qquad (1)$$

$$\frac{\partial L}{\partial y} = 2\lambda y = 0 \qquad (2)$$

$$\frac{\partial L}{\partial \lambda} = y^2 + x^4 - x^3 = 0 \qquad (3)$$

$(2) \Rightarrow \quad \lambda = 0 \quad$ or $\quad y = 0$

For $\underline{\lambda = 0}$ $\quad (1) \Rightarrow \lambda_0 = 0 \quad$ but $\quad (\lambda_0, \lambda) \neq 0$

Hence $\boxed{y = 0} \Rightarrow (3) \Rightarrow x^4 - x^3 = x^3(x-1) = 0 \begin{cases} x = 0 \\ x = 1 \end{cases}$

for $x = 0 \quad (1) \Rightarrow \lambda_0 = 0$

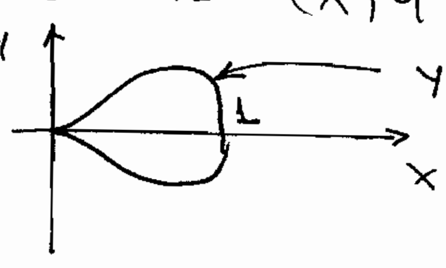
for $x = 1 \quad (1) \Rightarrow \lambda_0 + 4\lambda - 3\lambda = \lambda_0 + \lambda = 0$

$$-\lambda = \lambda_0 = 1$$

Hence

$x = 0$		$x = 1$
$y = 0$	or	$y = 0$
$\lambda_0 = 0$		$\lambda_0 = 1$
$\lambda \in \mathbb{R}$		$\lambda = -1$
(abnormal)		(regural)

Both satisfy the constraina $g(x,y) = 0$ but the minimum is $(x^*, y^*) = (0,0)$ abnormal



Based on the problem data we can create the following table

Seattle	$\boxed{400}$ x_1	$\boxed{550}$ x_2	$\boxed{600}$ $500-x_1-x_2$	500
Chicago	$\boxed{360}$ $200-x_1$	$\boxed{470}$ $360-x_2$	$\boxed{500}$ $-160+x_1+x_2$	400
	Denver	Miami	New York	
	200	360	340	

The inequality constraints for this problem are

$$x_1 \geq 0 \qquad x_2 \geq 0$$

$$500 - x_1 - x_2 \geq 0$$

$$200 - x_1 \geq 0$$

$$360 - x_2 \geq 0$$

$$-160 + x_1 + x_2 \geq 0$$

or in matrix form

$$Ax \leq b$$

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 500 \\ 200 \\ 360 \\ -160 \end{bmatrix}$$

The total transportation cost is

$$\text{Cost} = 400x_1 + 500x_2 + (500 - x_1 - x_2)\,600$$
$$+ 360(200 - x_1) + 470(360 - x_2) + 500(-160 + x_1 + x_2)$$

$$= -60x_1 - 70x_2 + 461{,}200$$

We want to minimize

$$\underbrace{\begin{bmatrix} -60 & -70 \end{bmatrix}}_{c} x$$

subject to $Ax \le b$

MATLAB gives $x_1 = 140$ $x_2 = 360$

Total transportation cost is $\underline{\$427{,}600}$