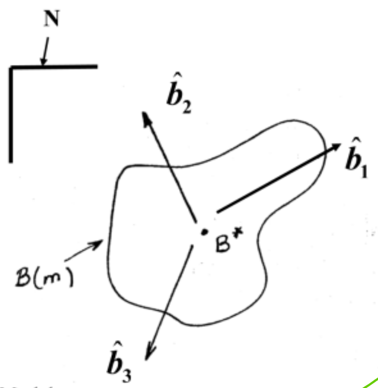
Effect of Gravity Moment on Unsymmetric Rigid Body



$\hat{a}_i$ orbit fixed

$\hat{b}_i$ parallel to central principal axes

$$\bar{\bar{I}}^{B/B^*} = I_1 \hat{b}_1 \hat{b}_1 + I_2 \hat{b}_2 \hat{b}_2 + I_3 \hat{b}_3 \hat{b}_3$$

$$^N\bar{\omega}^B = \omega_i \hat{b}_i$$

Model:

$$\bar{F} = -\frac{\mu m}{R^2}\hat{a}_1$$

$$\bar{M} = \frac{3\mu}{R^3}\hat{a}_1 \times \bar{\bar{I}}^{B/B^*} \cdot \hat{a}_1$$

Same as previous; assume orbit is independent of orientation (translational motion not affected by rotational motion); orientation IS influenced by orbit (rotational motion DOES depend on the translational motion)

$\bar{M}^{B*} = 3\Omega^2 \{ -k_1 I_1 {}^{AB}C_{12} C_{13} \hat{b}_1 - k_2 I_2 C_{11} C_{13} \hat{b}_2 - k_3 I_3 C_{11} C_{12} \hat{b}_3 \}$

Select kinematic Variables? ${}^{A}C^{B}$

Dynamic DE

$$\boxed{\begin{aligned} \dot{\omega}_1 &= k_1(\omega_2\omega_3 - 3\Omega^2 C_{12} C_{13}) \\ \dot{\omega}_2 &= k_2(\omega_3\omega_1 - 3\Omega^2 C_{13} C_{11}) \\ \dot{\omega}_3 &= k_3(\omega_1\omega_2 - 3\Omega^2 C_{11} C_{12}) \end{aligned}}$$

$$\boxed{\begin{aligned} K_1 &= \frac{I_2 - I_3}{I_1} \\ K_2 &= \frac{I_3 - I_1}{I_2} \\ K_3 &= \frac{I_1 - I_2}{I_3} \end{aligned}}$$

3 first order equations – independent variable: time

dependent variables: ω_1, ${}^{N}\omega_2^{B}$, ω_3 C_{11} C_{12} C_{13}

Need solution for C's or more DE

Poisson's equations for $\dot{C}$

$$\dot{C} = \overset{A\ B}{C}\,\tilde{\omega} \qquad \begin{bmatrix} 0 & -\overset{A\ B}{\omega_3} & \omega_2 \\ {}^A\omega_3^B & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

$${}^N\bar{\omega}^B = {}^N\bar{\omega}^A + {}^A\bar{\omega}^B$$

$${}^A\bar{\omega}^B = \underbrace{{}^N\bar{\omega}^B}_{\omega_i \hat{b}_i} + \underbrace{{}^N\bar{\omega}^A}_{-\Omega \hat{a}_3} \quad \Bigg\} \text{ need } {}^A C^B$$

ang vel in poisson

$${}^A\bar{\omega}^B = \underbrace{\left({}^N\omega_1^B - \Omega\,{}^A C_{31}^B\right)}_{{}^A\omega_1^B}\hat{b}_1 + \underbrace{\left({}^N\omega_2^B - \Omega\,{}^A C_{32}^B\right)}_{{}^A\omega_2^B}\hat{b}_2 + \underbrace{\left({}^N\omega_3^B - \Omega\,{}^A C_{33}^B\right)}_{{}^A\omega_3^B}\hat{b}_3$$

W

Kinematic DE

$$\begin{aligned}
\dot{C}_{11} &= \overset{A\ B}{C_{12}}\omega_3 - \overset{N\ B}{C_{13}\omega_2} + \Omega\left(C_{13}C_{32} - C_{12}C_{33}\right) \\
\dot{C}_{12} &= C_{13}\omega_1 - C_{11}\omega_3 + \Omega\left(C_{11}C_{33} - C_{13}C_{31}\right) \\
\dot{C}_{13} &= C_{11}\omega_2 - C_{12}\omega_1 + \Omega\left(C_{12}C_{31} - C_{11}C_{32}\right) \\
\dot{C}_{31} &= C_{32}\omega_3 - C_{33}\omega_2 \\
\dot{C}_{32} &= C_{33}\omega_1 - C_{31}\omega_3 \\
\dot{C}_{33} &= C_{31}\omega_2 - C_{32}\omega_1
\end{aligned}$$

$$\begin{aligned}
\dot{\omega}_1 &= K_1\left(\omega_2\omega_3 - 3\Omega^2 C_{12}C_{13}\right) \\
\dot{\omega}_2 &= K_2\left(\omega_3\omega_1 - 3\Omega^2 C_{13}C_{11}\right) \\
\dot{\omega}_3 &= K_3\left(\omega_1\omega_2 - 3\Omega^2 C_{11}C_{12}\right)
\end{aligned}$$

Non linear coupled 1st order DEs

9 nonlinear, coupled DE
Solve simultaneously (numerically)

↓

Numerical Integration

Investigate impact of gravity torque
(unsymmetric rigid body – circular orbit):

Consider what info we have

Torque-free

1. EOM
2. Analytical Solution } in terms of elliptic functions
3. Solution is complete (not always easy to work with; sometimes output numerically)

Gravity Torque put s/c in orbit to create torque

1. EOM
2. NOT solvable analytically in general ✓ no longer help from S'
   May be a particular solution that is of interest
   Once again, base investigation on the particular solution
3. Plan
   (i) Determine particular solution
   (ii) Investigate stability of particular solution
   [(iii) Analytical approximation for motion relative to particular solution]
   (iv) Numerical Simulations

Particular Solution of the EOM
(Motion of Interest)

MUST satisfy nonlinear DE; MUST know time history of all dependent variables
First consider possibility that a constant solution exists (easiest)

Consider the following motion: B fixed in A ← principal directions aligned with orbit

$$t=0 \qquad {}^{A}C^{B}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \qquad {}^{N}\bar{\omega}^{B}=\Omega\hat{a}_3$$

} satisfy the NL DE (const soln)

Will this solution satisfy the nonlinear DE?

$\omega_3 = \Omega$

$\omega_1 = \omega_2 = 0$

$$\dot{C}_{11}=C_{12}\omega_3-C_{13}\omega_2+\Omega\left(C_{13}C_{32}-C_{12}C_{33}\right)$$
$$\dot{C}_{12}=C_{13}\omega_1-C_{11}\omega_3+\Omega\left(C_{11}C_{33}-C_{13}C_{31}\right)$$
$$\dot{C}_{13}=C_{11}\omega_2-C_{12}\omega_1+\Omega\left(C_{12}C_{31}-C_{11}C_{32}\right)$$
$$\dot{C}_{31}=C_{32}\omega_3-C_{33}\omega_2$$
$$\dot{C}_{32}=C_{33}\omega_1-C_{31}\omega_3$$
$$\dot{C}_{33}=C_{31}\omega_2-C_{32}\omega_1$$
$$\dot{\omega}_1=K_1\left(\omega_2\omega_3-3\Omega^2C_{12}C_{13}\right)$$
$$\dot{\omega}_2=K_2\left(\omega_3\omega_1-3\Omega^2C_{13}C_{11}\right)$$
$$\dot{\omega}_3=K_3\left(\omega_1\omega_2-3\Omega^2C_{11}C_{12}\right)$$

Now we can numerically investigate this particular solution; test this solution for stability

First → a linear stability analysis
(perturb nominal motion, linearize, characteristic equation, eigenvalues)

1. Perturb nominal motion

$$C_{11}=1+\tilde{C}_{11}$$
$$C_{12}=0+\tilde{C}_{12}$$
$$C_{13}=0+\tilde{C}_{13}$$
$$C_{31}=0+\tilde{C}_{31}$$
$$C_{32}=0+\tilde{C}_{32}$$
$$C_{33}=1+\tilde{C}_{33}$$

reference solution

$$\omega_1=0+\tilde{\omega}_1$$
$$\omega_2=0+\tilde{\omega}_2$$
$$\omega_3=\Omega+\tilde{\omega}_3$$

2, Linearize

$$\dot{\tilde{C}}_{31}=\Omega\tilde{C}_{32}-\tilde{\omega}_2$$
$$\dot{\tilde{C}}_{32}=\tilde{\omega}_1-\Omega\tilde{C}_{31}$$
$$\dot{\tilde{C}}_{33}=0$$
$$\dot{\tilde{C}}_{11}=0$$

1) $\lambda_{1,2}=0$ ($9^{th}-7^{th}$)

7^{th} ↓ 6^{th}

$$\dot{\tilde{C}}_{12}=-\tilde{\omega}_3+\Omega\tilde{C}_{33}$$
$$\dot{\tilde{C}}_{13}=\tilde{\omega}_2-\Omega\tilde{C}_{32}$$

$$\dot{\tilde{C}}_{12} = \tilde{\omega}_2 - \Omega\tilde{C}_{32}$$

$$\dot{\tilde{\omega}}_1 = K_1 \Omega \tilde{\omega}_2$$

$$\dot{\tilde{\omega}}_2 = K_2 \Omega \tilde{\omega}_1 - 3\Omega^2 K_2 \tilde{C}_{13}$$

$$\dot{\tilde{\omega}}_3 = -3K_3 \Omega^2 \tilde{C}_{12}$$

Note: C_{13} and C_{31} equations
Why are they similar?

Use a 'constant' to reduce the system by one equation

$$C_{11}C_{21} + C_{12}C_{32} + C_{13}C_{33} = 0$$

$$(1+\tilde{C}_{11})\tilde{C}_{31} + \tilde{C}_{12}\tilde{C}_{32} + \tilde{C}_{13}(1+\tilde{C}_{33}) = 0$$

$$\tilde{C}_{31} + \tilde{C}_{13} = 0$$

Use just one equation in the model and reduce order of system

There exists non linear terms that we neglect which explain why MS result in LS → Does not apply to NL sys

marginal stability

Also note $\dot{\tilde{C}}_{12} = -\tilde{\omega}_3 + \Omega\tilde{C}_{33}$
OR

$$\tilde{\omega}_3 = -\dot{\tilde{C}}_{12} + \Omega\tilde{C}_{33}$$

But $\dot{\tilde{\omega}}_3 = -3K_3\Omega^2\tilde{C}_{12}$

$$-\ddot{\tilde{C}}_{12} + \Omega\cancel{\dot{\tilde{C}}_{33}}^{\text{const}} = -3K_3\Omega^2\tilde{C}_{12}$$

$$\ddot{\tilde{C}}_{12} + -3K_3\Omega^2\tilde{C}_{12} = 0$$

characteristic equ.

$$\lambda^2 - 3K_3\Omega^2 = 0$$

$$\lambda_{3,4} = \pm\sqrt{3K_3\Omega^2}$$

$\lambda^2 - 3K_3\Omega^2 = 0$

$\lambda_{3,4} = \pm\sqrt{3K_3\Omega^2}$

What is left?

$$\dot{\tilde{C}}_{31} = \Omega\tilde{C}_{32} - \tilde{\omega}_2$$
$$\dot{\tilde{C}}_{32} = \tilde{\omega}_1 - \Omega\tilde{C}_{31}$$
$$\dot{\tilde{\omega}}_1 = K_1\Omega\tilde{\omega}_2$$
$$\dot{\tilde{\omega}}_2 = K_2\Omega\tilde{\omega}_1 - 3\Omega^2K_2\tilde{C}_{13}$$

$K_3 = \dfrac{I_1 - I_2}{I_3}$

$$\left[\dot{\tilde{C}}_{31}\ \dot{\tilde{C}}_{32}\ \dot{\tilde{\omega}}_1\ \dot{\tilde{\omega}}_2\right] = \left[\tilde{C}_{31}\ \tilde{C}_{32}\ \tilde{\omega}_1\ \tilde{\omega}_2\right]\begin{bmatrix} 0 & -\Omega & 0 & 3\Omega^2K_2 \\ \Omega & 0 & 0 & 0 \\ 0 & 1 & 0 & K_2\Omega \\ -1 & 0 & K_1 & 0 \end{bmatrix}$$

$$\boxed{\begin{gathered}\lambda^4 + (1 - K_1K_2 + 3K_2)\Omega^2\lambda^2 - 4K_1K_2\Omega^4 = 0 \\ \lambda^2 - 3K_3\Omega^2 = 0 \\ \lambda_{1,2} = 0\end{gathered}}$$

$\lambda = \pm\left[3K_3\Omega^2\right]^{\frac{1}{2}} \longleftarrow$ unstable if $\boxed{K_3 > 0}$

$$\mu^4 + \underbrace{\left[(1 - K_1K_2 + 3K_2)\Omega^2\right]}_{2b}\mu^2 + \underbrace{\left(-4K_1K_2\Omega^4\right)}_{c} = 0$$

$\mu = \pm\left[-b \pm (b^2 - c)^{1/2}\right]^{1/2}$

$\searrow$ 4 eigen values!

Root with a positive real part will appear if:

$$\left.\boxed{\begin{aligned} &c \to \quad 1) \quad K_1K_2 > 0 \\ &b \to \quad 2) \quad 1 - K_1K_2 + 3K_2 < 0 \end{aligned}}\right\}$$ unstable results

c → 1) $k_1 k_2 > 0$

b → 2) $1 - k_1 k_2 + 3k_2 < 0$

$b^2 - c$ → 3) $(1 - k_1 k_2 + 3k_2)^2 - 16 k_1 k_2 < 0$

unstable results

4) $k_3 > 0$

$(x^2 + (y-1)^2 = r^2$

$r\cos^2\theta + (y \sin\theta - 1)^2 = r^2$

$(x+1)^2 + (y-1)^2$

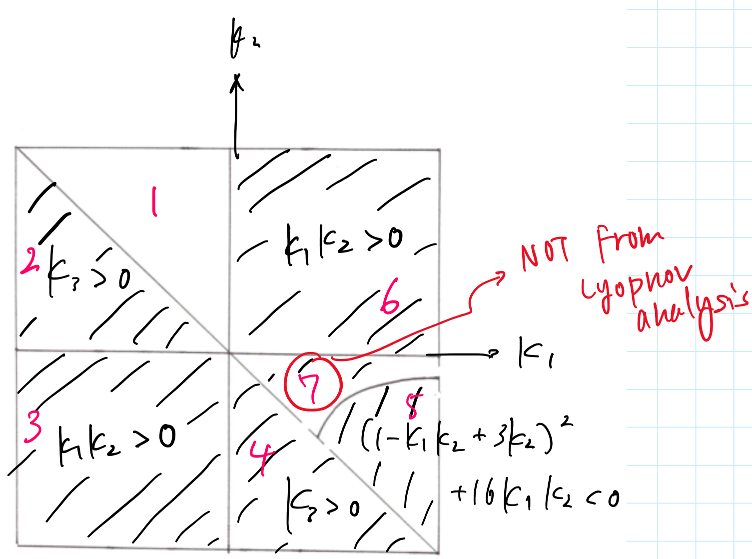
$y - 1 = r\sin\theta$

$y = r\sin\theta + 1$

$x = r\cos\theta - 1$

$y = r\sin\theta + 1$

W12



$k_1 = \frac{I_2 - I_3}{I_1}$, $k_2 = \frac{I_3 - I_1}{I_2}$ $k_2 = \frac{I_1 - I_2}{I_3}$

region 1

gravity gradient stabilization