College of Engineering
School of Aeronautics and Astronautics

AAE 421
Flight Dynamics and Controls

HW 1
Basic Kinematics and Rigid Body Motion

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September 8th , 2020
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Problem 1. (10pts) Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$.

Since $\bar{a} \times (\bar{b} \times \bar{c}) = \bar{b}(\bar{a} \cdot \bar{c}) - \bar{c}(\bar{a} \cdot \bar{b})$

$$\bar{u} \times (\bar{v} \times \bar{w}) + \bar{v} \times (\bar{w} \times \bar{u}) + \bar{w} \times (\bar{u} \times \bar{v})$$

$$= \bar{v}(\bar{u} \cdot \bar{w}) - \bar{w}(\bar{u} \cdot \bar{v}) + \bar{w}(\bar{v} \cdot \bar{u}) - \bar{u}(\bar{v} \cdot \bar{w}) + \bar{u}(\bar{w} \cdot \bar{v}) - \bar{v}(\bar{w} \cdot \bar{u})$$

$$= \boxed{0} \quad (\because \ \bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a})$$

Problem 2. (20pts) Two particles moving with constant velocity are described by the position vectors:

$$\mathbf{p} = \mathbf{p_0} + v t, \quad \mathbf{s} = \mathbf{s_0} + w t$$

a) Show that the shortest distance between their trajectories is given by

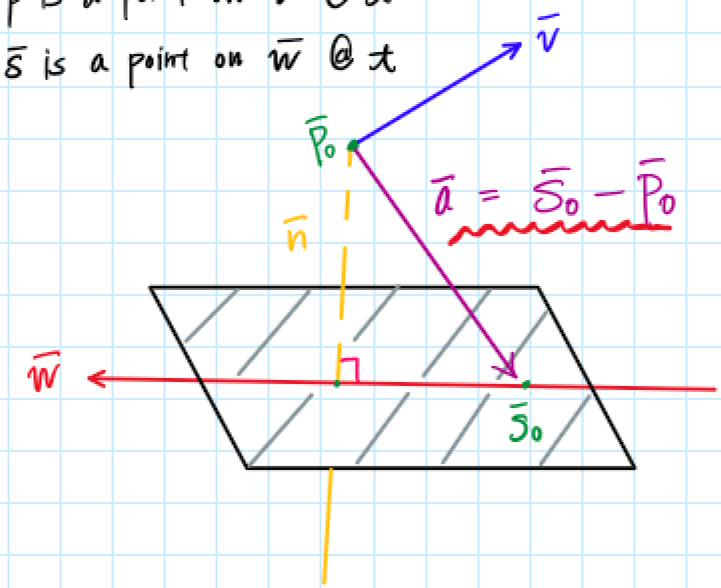
$$d = \frac{|(\mathbf{s_0} - \mathbf{p_0}) \cdot (\mathbf{w} \times \mathbf{v})|}{|\mathbf{w} \times \mathbf{v}|}$$

b) Find the shortest distance between the particles themselves.

(a)

$\bar{p}$ is a point on $\bar{v}$ @ t

$\bar{s}$ is a point on $\bar{w}$ @ t



normal vector $\bar{n} = \bar{w} \times \bar{v}$

now $\bar{d}$ which is the vector w/ magnitude of d is the projection of $\bar{a}$ onto $\bar{n}$

$$\bar{d} = \text{proj}_{\bar{n}} \bar{a} = \frac{\bar{a} \cdot \bar{n}}{|\bar{n}|}$$

$$= \frac{(\bar{s}_0 - \bar{p}_0) \cdot (\bar{w} \times \bar{v})}{|\bar{w} \times \bar{v}|}$$

$$\boxed{\therefore \quad d = |\bar{d}| = \frac{|(\bar{s}_0 - \bar{p}_0) \cdot (\bar{w} \times \bar{v})|}{|\bar{w} \times \bar{v}|}}$$

(b) say

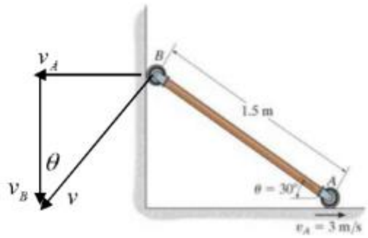
$$\bar{r} = \bar{s} - \bar{p}$$

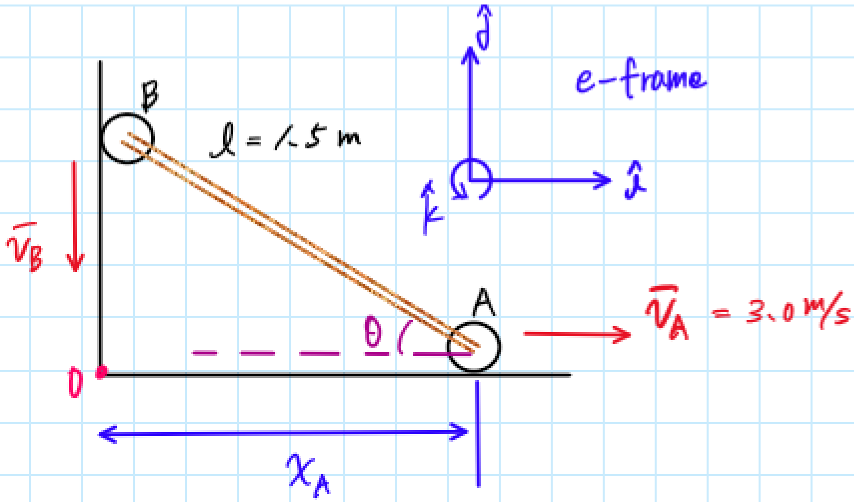
this the vector between the two position vectors. Thus, the shortest distance between the particles is

$$|\bar{r}| = |\bar{s} - \bar{p}| = |\bar{s}_0 + \bar{w}t - \bar{p}_0 - \bar{v}t|$$

$$\boxed{|\bar{r}| = |(\bar{s}_0 - \bar{p}_0) + (\bar{w} - \bar{v})t|}$$

Problem 3. (15pts) If a roller A moves to the right with a constant velocity of v_A
angular velocity of the link and the velocity of the roller B when θ=30 deg.





We know that

$$\bar{r}_{OA} = x_A \hat{i} \quad , \quad \bar{r}_{OB} = l \sin\theta \, \hat{j}$$

$$\bar{v}_A = (3 \ \hat{i}) \ m/s = \text{const.}$$

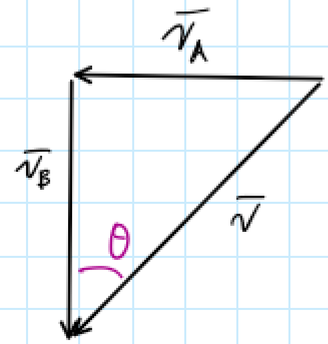
Assess the Kinematics

$$\bar{v}_{OA} = \dot{\bar{r}}_{OA} = \frac{{}^e d \bar{r}_{OA}}{dt} = \dot{x}_A \hat{i} = \bar{v}_A$$

$$\bar{v}_{OB} = \dot{\bar{r}}_{OB} = \frac{{}^e d \bar{r}_{OB}}{dt} = \frac{{}^e d}{dt} (l \sin\theta) \, \hat{j}$$

$$= l \dot{\theta} \cos\theta \, \hat{j}$$

Now, since



$$\bar{v}_{OB} = \bar{v}_B = -\frac{v_A}{\tan\theta}\hat{j} \quad \text{and} \quad {}^{O}\omega^{B} = \dot{\theta}$$

Thus,

$$l\,{}^{O}\omega^{B}\cos\theta = -\frac{v_A}{\tan\theta} = -\frac{v_A}{\sin\theta}\cos\theta$$

$$ {}^{O}\omega^{B} = -\frac{v_A}{l\sin\theta}$$

$$ {}^{O}\bar{\omega}^{B} = -\frac{v_A}{l\sin\theta}\hat{k}$$

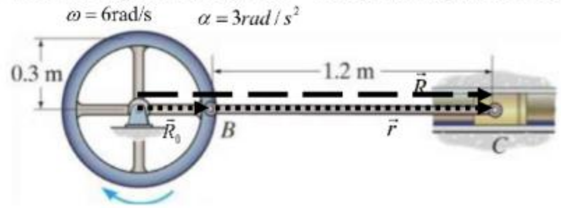
@ $\theta = 30°$

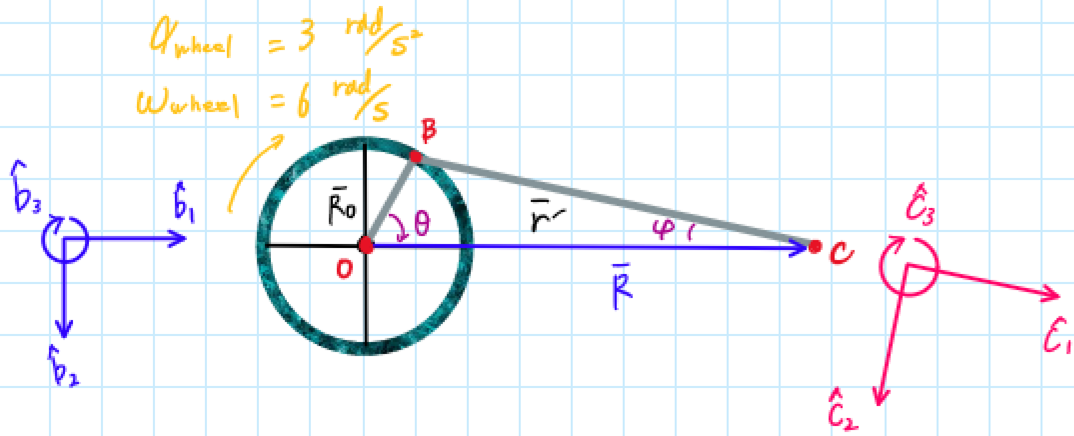
$$\boxed{{}^{O}\bar{\omega}^{B} = -\frac{3.0\ m/s}{(1.5m)(\frac{1}{2})}\hat{k} = (-4\hat{k})\frac{rad}{s}}$$

$$v_{OB} = l\,{}^{O}\omega^{B}\cos\theta$$

$$\boxed{v_{OB} = (1.5\,m)(4\ rad/s)\left(\frac{\sqrt{3}}{2}\right) \cong 5.20\ m/s}$$

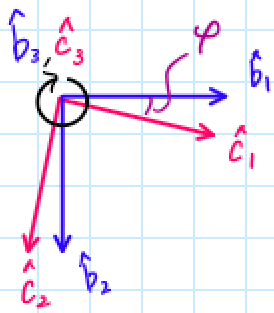
Problem 4. (15pts) Determine the angular acceleration of link BC and the acceleration of the piston C at the instant shown below where the wheel has angular velocity of $\omega = 6\text{rad/s}$ and angular acceleration $\alpha = 3rad/s^2$.





$|\bar{R}_0| = R_0 = 0.3\text{ m}$, $|\bar{r}'| = r' = 1.2\text{m}$

$\bar{r}_{OB} = \bar{r}' = 1.2\text{m}$



Analyze the kinematics

$$\bar{r}_{OB} = R_0 \cos\theta\, \hat{b}_1 + R_0 \sin\theta\, \hat{b}_2$$

$$\bar{v}_{OB} = {}^{b}\frac{d}{dt}(\bar{r}_{OB}) = -R_0\dot{\theta}\sin\theta\, \hat{b}_1 + R_0\dot{\theta}\cos\theta\, \hat{b}_2$$

$$\bar{a}_{OB} = {}^b\frac{d}{dt}(\bar{v}_{OB})$$
$$= (-\rho_0\ddot{\theta}\sin\theta - \rho_0\dot{\theta}^2\cos\theta)\hat{b}_1 + (\rho_0\ddot{\theta}\cos\theta - \rho_0\dot{\theta}^2\sin\theta)\hat{b}_2$$

$$\bar{r}_{OC} = \bar{r}_{OB} + \bar{r}_{BC}$$
$$= \bar{r}_{OB} + r'\hat{c}_1$$

$$\bar{v}_{OC} = {}^b\frac{d}{dt}(\bar{r}_{OC})$$
$$= {}^b\frac{d}{dt}(\bar{r}_{OB}) + {}^b\frac{d}{dt}(r'\hat{c}_1)$$
$$= \bar{v}_{OB} + \cancelto{0}{{}^c\frac{d}{dt}(r'\hat{c}_1)} + {}^b\bar{\omega}^c \times r'\hat{c}_1$$
$$= \bar{v}_{OB} + \omega_{wheel}\,\hat{c}_3 \times r'\hat{c}_1$$
$$= \bar{v}_{OB} + \omega_{wheel}\,r'\hat{c}_2$$

$$\bar{a}_{OC} = {}^b\frac{d}{dt}(\bar{v}_{OC})$$
$$= \bar{a}_{OB} + {}^c\frac{d}{dt}(\omega_{wheel}\,r'\hat{c}_2) + \omega_{wheel}\,\hat{c}_3 \times (\omega_{wheel}\,r'\hat{c}_2)$$
$$= \bar{a}_{OB} + \alpha_{wheel}\,r'\hat{c}_2 - \omega_{wheel}^2\,r'\hat{c}_1$$

Since, Link BC & the wheel is attached together they should have the same angular acceleration

$$\boxed{\bar{\alpha}_{Link} = \bar{\alpha}_{wheel} = 3\,\hat{b}_3 \ \text{rad}/s^2}$$

The figure shows the instance @ $\theta = 0°$

$$\bar{a}_{OB}\big|_{\theta=0°}$$

$$= \left(-R_0 \cancel{\alpha_{wheel} \sin 0°} - R_0 \omega_{wheel}^2 \cos 0°\right)\hat{b}_1 + \left(R_0 \alpha_{wheel} \cos 0° - R_0 \cancel{\omega_{wheel}^2 \sin 0°}\right)\hat{b}_2$$

$$= -(0.3\,m)\left(6\,\tfrac{rad}{s}\right)^2 \hat{b}_1 + (0.3\,m)\left(3\,\tfrac{rad}{s^2}\right)\hat{b}_2$$

$$= \left(-10.8\,\hat{b}_1 + 0.9\,\hat{b}_2\right)\,m/s^2$$

$$\because\ \hat{c}_1 = \cos\varphi\,\hat{b}_1 + \sin\varphi\,\hat{b}_2$$
$$\hat{c}_2 = -\sin\varphi\,\hat{b}_1 + \cos\varphi\,\hat{b}_2$$

$$\theta = 0° \longrightarrow \varphi = 0°$$

$$\therefore\ \hat{c}_1 = \hat{b}_1\ ,\quad \hat{c}_2 = \hat{b}_2$$

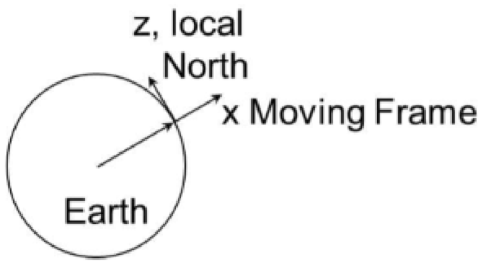
$$\bar{a}_{OC} = \bar{a}_{OB} + \alpha_{wheel}\,r'\,\hat{b}_2 - \omega_{wheel}^2\,r'\,\hat{b}_1$$

$$= \left(-10.8\,\hat{b}_1 + 0.9\,\hat{b}_2\right)\,m/s^2 + \left(3\,\tfrac{rad}{s^2}\right)(1.2\,m)\,\hat{b}_2 - \left(6\,\tfrac{rad}{s}\right)^2(1.2\,m)\,\hat{b}_1$$

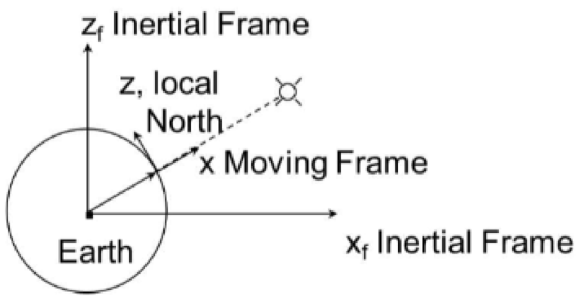
$$= \left(-54\,\hat{b}_1 + 4.5\,\hat{b}_2\right)\,m/s^2$$

$$\boxed{\bar{a}_{OC} = \left(-54\,\hat{b}_1 + 4.5\,\hat{b}_2\right)\,m/s^2}$$

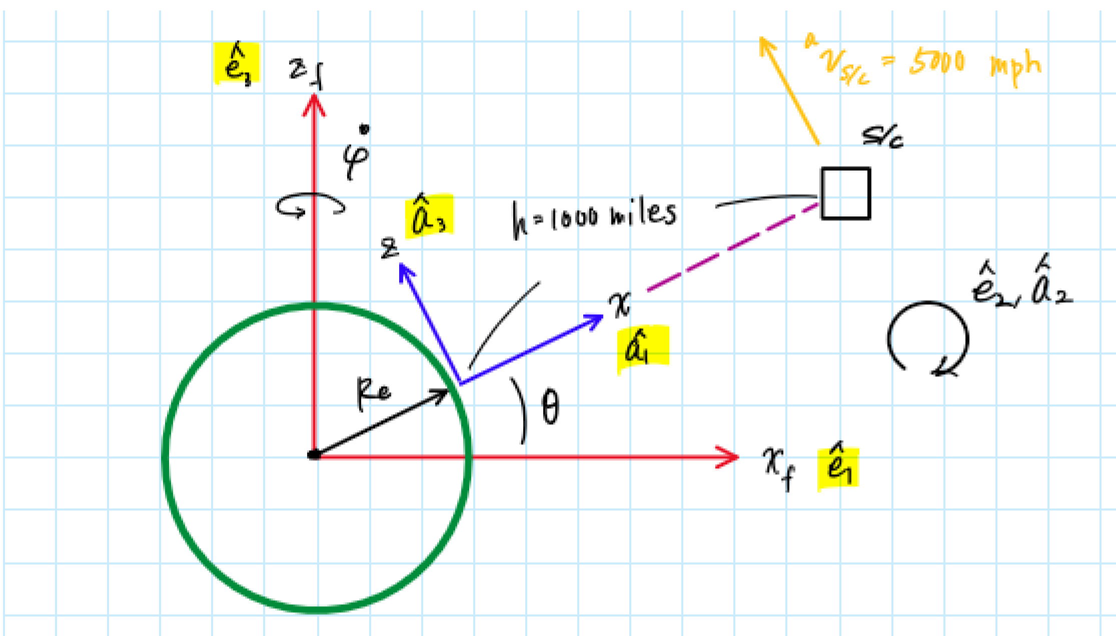
Problem 5. (20pt) An observer positioned at 30 degrees North latitude on the surface of the earth tracks a satellite.



You may assume that the inertial frame of reference for the problem is attached to the center of the earth, but doesn't rotate with the earth. The z_f axis passes through the North Pole of the earth. The x_f axis passes through the equator at a point nearest the observer. You may assume that the radius of the earth is 4000 miles.



At a particular instant, the satellite appears to be 1000 miles directly above him and, by his observations, appears to be traveling 5000 miles per hour due North (local North). What is the absolute velocity of the satellite in terms of the unit vectors of the moving coordinate system within which the observer resides.



$$\bar{r}_{S/c} = (R_e + h)\,\hat{a}_1$$

$$
\begin{aligned}
{}^{e}\bar{v}_{s/c} &= \frac{{}^{e}d}{dt}(\bar{r}_{s/c}) \\
&= \frac{{}^{e}d}{dt}(R_e+h) + \dot{\bar{\varphi}} \times (R_e+h)\,\hat{a}_1 \\
&= \dot{\varphi}\,\hat{e}_3 \times (R_e+h)\,\hat{a}_1 \\
&= \dot{\varphi}(R_e+h)\,|\hat{e}_3||\hat{a}_1|\sin(90^\circ-\theta)\,(\hat{e}_2) \\
&= \dot{\varphi}(R_e+h)\sin(90^\circ-\theta)\,\hat{e}_2
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\bar{v}_{s/c}^{ABS} &= {}^{e}\bar{v}_{s/c} + {}^{a}\bar{v}_{s/c} \\
&= \dot{\varphi}(R_e+h)\cos\theta\,\hat{e}_2 + {}^{a}v_{s/c}\,\hat{a}_3 \\
&= \dot{\varphi}(R_e+h)\cos\theta\,\hat{e}_2 \\
&\qquad + {}^{a}v_{s/c}(-\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_3) \\
&= -\,{}^{a}v_{s/c}\sin\theta\,\hat{e}_1 + {}^{a}v_{s/c}\cos\theta\,\hat{e}_3 \\
&\qquad + \dot{\varphi}(R_e+h)\cos\theta\,\hat{e}_2 \\
&= -\left(5000\,\frac{\text{miles}}{\text{hr}}\right)\left(\frac{1}{2}\right)\hat{e}_1 + \left(5000\,\frac{\text{miles}}{\text{hr}}\right)\left(\frac{\sqrt{3}}{2}\right)\hat{e}_3 \\
&\qquad + \left(\frac{2\pi}{24\,\text{hr}}\right)(4000+1000\text{ miles})\frac{\sqrt{3}}{2}\,\hat{e}_2 \\
&= -2500\text{ mph}\,\hat{e}_1 + 4330\text{ mph}\,\hat{e}_3 \\
&\qquad + 1134\,\hat{e}_2
\end{aligned}
$$

$$
\boxed{{}^{e}\bar{v}_{s/c}^{ABS} = (-2500\,\hat{e}_1 + 1134\,\hat{e}_2 + 4330\,\hat{e}_3)\text{ mph}}
$$

Problem 6. (20pt) Expand the equations of motion for translation $m(\dot{\boldsymbol{V}}|_{body} + \boldsymbol{\omega} \times \boldsymbol{V}) = \boldsymbol{F}$ and rotation $I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) = \boldsymbol{M}$ into components in the body-fixed reference frame where $\boldsymbol{V} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\widehat{\boldsymbol{k}}$, $\boldsymbol{\omega} = p\hat{\boldsymbol{\imath}} + q\hat{\boldsymbol{\jmath}} + r\widehat{\boldsymbol{k}}$, $\boldsymbol{F} = F_x\hat{\boldsymbol{\imath}} + F_y\hat{\boldsymbol{\jmath}} + F_z\widehat{\boldsymbol{k}}$, $\boldsymbol{M} = L\hat{\boldsymbol{\imath}} + M\hat{\boldsymbol{\jmath}} + N\widehat{\boldsymbol{k}}$ and I is the moment of inertia matrix given by $I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yz} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$.

$$\bar{V} = u\hat{i} + v\hat{j} + w\hat{k}, \quad \bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

$$\bar{\omega} = p\hat{i} + q\hat{j} + r\hat{k}, \quad \bar{M} = L\hat{i} + M\hat{j} + N\hat{k}$$

$$\bar{F} = m(\dot{\bar{V}}_{body} + \bar{\omega} \times \bar{V})$$

$$= m\left[(\dot{u}\hat{i} + \dot{v}\hat{j} + \dot{w}\hat{k}) + (p\hat{i} + q\hat{j} + r\hat{k}) \times (u\hat{i} + v\hat{j} + w\hat{k})\right]$$

$$= m(\dot{u}\hat{i} + \dot{v}\hat{j} + \dot{w}\hat{k} + pv\hat{k} - pw\hat{j} - qu\hat{k} + qw\hat{i} + ru\hat{j} - rv\hat{i})$$

$$\boxed{\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = m\begin{bmatrix} \dot{u} + qw - rv \\ \dot{v} + ru - pw \\ \dot{w} + pv - qu \end{bmatrix}}$$

$$\bar{M} = \bar{\bar{I}}\cdot\dot{\bar{\omega}} + \bar{\omega}\times\bar{\bar{I}}\cdot\bar{\omega}$$

$$\begin{aligned}
I\cdot\dot{\omega} = (&I_{xx}\hat{i}\hat{i} - I_{xy}\hat{i}\hat{j} - I_{xz}\hat{i}\hat{k} \\
&- I_{yz}\hat{j}\hat{i} + I_{yy}\hat{j}\hat{j} - I_{yz}\hat{j}\hat{k} \\
&- I_{zx}\hat{k}\hat{i} - I_{zy}\hat{k}\hat{j} + I_{zz}\hat{k}\hat{k})\cdot(\dot{p}\hat{i} + \dot{q}\hat{j} + \dot{r}\hat{k}) \\
= \ &I_{xx}\dot{p}\hat{i} - I_{yz}\dot{p}\hat{j} - I_{zx}\dot{p}\hat{k} \\
&- I_{xy}\dot{q}\hat{i} + I_{yy}\dot{q}\hat{j} - I_{zy}\dot{q}\hat{k} \\
&- I_{xz}\dot{r}\hat{i} - I_{yz}\dot{r}\hat{j} + I_{zz}\dot{r}\hat{k} \\
= \ &(I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r})\hat{i} \\
&+ (-I_{yz}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r})\hat{j} \\
&+ (-I_{zx}\dot{p} - I_{zy}\dot{q} + I_{zz}\dot{r})\hat{k}
\end{aligned}$$

$$\bar{\omega}\times\bar{\bar{I}}\cdot\bar{\omega}$$

$$\begin{aligned}
= (p\hat{i} + q\hat{j} + r\hat{k}) \times (&I_{xx}\hat{i}\hat{i} - I_{xy}\hat{i}\hat{j} - I_{xz}\hat{i}\hat{k} \\
&- I_{yz}\hat{j}\hat{i} + I_{yy}\hat{j}\hat{j} - I_{yz}\hat{j}\hat{k} \\
&- I_{zx}\hat{k}\hat{i} - I_{zy}\hat{k}\hat{j} + I_{zz}\hat{k}\hat{k})\cdot(p\hat{i} + q\hat{j} + r\hat{k})
\end{aligned}$$

$$\begin{aligned}
= (p\hat{i} + q\hat{j} + r\hat{k}) \times \ &(I_{xx}p - I_{xy}q - I_{xz}r)\hat{i} \\
&+ (-I_{yz}p + I_{yy}q - I_{yz}r)\hat{j} \\
&+ (-I_{zx}p - I_{zy}q + I_{zz}r)\hat{k}
\end{aligned}$$

the calculation of $\bar{\bar{I}}\cdot\bar{\omega}$ is same as $\bar{\bar{I}}\cdot\dot{\bar{\omega}}$

$$\begin{aligned}
= \ &p(-I_{yz}p + I_{yy}q - I_{yz}r)\hat{k} - p(-I_{zx}p - I_{zy}q + I_{zz}r)\hat{j} \\
&- q(I_{xx}p - I_{xy}q - I_{xz}r)\hat{k} + q(-I_{zx}p - I_{zy}q + I_{zz}r)\hat{i} \\
&+ r(I_{xx}p - I_{xy}q - I_{xz}r)\hat{j} - r(-I_{yz}p + I_{yy}q - I_{yz}r)\hat{i}
\end{aligned}$$

Thus,

$$
\boxed{
\begin{bmatrix} L \\ M \\ N \end{bmatrix}
=
\begin{bmatrix}
(I_{xx}\dot{p} - I_{xy}\dot{q} - I_{xz}\dot{r}) + q(-I_{zx}p - I_{zy}q + I_{zz}r) - r(-I_{yz}p + I_{yy}q - I_{yz}r) \\
(-I_{yz}\dot{p} + I_{yy}\dot{q} - I_{yz}\dot{r}) + r(I_{xx}p - I_{xy}q - I_{xz}r) - p(-I_{zx}p - I_{zy}q + I_{zz}r) \\
(-I_{zx}\dot{p} - I_{zy}\dot{q} + I_{zz}\dot{r}) + p(-I_{yz}p + I_{yy}q - I_{yz}r) - q(I_{xx}p - I_{xy}q - I_{xz}r)
\end{bmatrix}
}
$$