College of Engineering

School of Aeronautical and Astronautical Engineering

AAE 567: Introduction To Applied Stochastic Processes

HW4

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Problem 1

Let f be a random vector with values in $\mathbb{C}^k$. Let g be a random vector with values in $\mathbb{C}^n$ and $\mathcal{M}$ the subspace spanned by g. Assume that R_g is invertible. According to Theorem 2.2.1 in Chapter 2, we have

$$P_{\mathcal{M}} f = R_{fg} R_g^{-1} g$$
$$E(f - P_{\mathcal{M}} f)(f - P_{\mathcal{M}} f)^* = R_f - R_{fg} R_g^{-1} R_{gf}.$$

Now let y be a random vector with values in $\mathbb{C}^m$, and set $\phi = y - P_{\mathcal{M}} y$. Theorem 2.2.1 in Chapter 2 shows that

$$P_{\mathcal{M}} y = R_{yg} R_g^{-1} g$$
$$E(y - P_{\mathcal{M}} y)(y - P_{\mathcal{M}} y)^* = R_\phi = R_y - R_{yg} R_g^{-1} R_{gy}.$$

As in Lemma 3.3.1, consider the space $\mathcal{H}$ spanned by g and y, that is, $\mathcal{H} = g \bigvee y$. Let h be the random vector defined by $h = [g \quad y]^{tr}$ where tr denotes the transpose. Notice that $\mathcal{H}$ equals the span of h. Furthermore, R_h and R_{fh} are given by

$$R_h = \begin{bmatrix} R_g & R_{gy} \\ R_{yg} & R_y \end{bmatrix} \quad \textit{and} \quad R_{fh} = \begin{bmatrix} R_{fg} & R_{fy} \end{bmatrix}.$$

Observe that the Schur complement for R_h is given by $\Delta = R_y - R_{yg} R_g^{-1} R_{gy} = R_\phi$. In particular, R_h is invertible if and only if R_ϕ is invertible; see Lemma 2.4.1 in Chapter 2.

Now assume that R_ϕ is invertible. Then R_h is invertible and Theorem 2.2.1 in Chapter 2 implies that

$$P_{\mathcal{H}} f = R_{fh} R_h^{-1} h$$
$$E(f - P_{\mathcal{H}} f)(f - P_{\mathcal{H}} f)^* = R_f - R_{fh} R_h^{-1} R_{hf}.$$

Using the matrix inversion Lemma 2.4.1 in Chapter 2, give another proof of equations (3.1) and (3.2) in Lemma 3.3.1, that is, show that

$$P_{\mathcal{H}} f = P_{\mathcal{M}} f + R_{f\phi} R_\phi^{-1} \phi$$
$$E(f - P_{\mathcal{H}} f)(f - P_{\mathcal{H}} f)^* = E(f - P_{\mathcal{M}} f)(f - P_{\mathcal{M}} f)^* - R_{f\phi} R_\phi^{-1} R_{\phi f}$$

Solution:
From what we are given we can and Lemma 2.4.1 we can compute

$$R_h^{-1} = \begin{bmatrix} R_g^{-1} + R_g^{-1} R_{gy} R_\phi^{-1} R_{yg} R_g^{-1} & -R_g^{-1} R_{gy} R_\phi^{-1} \\ -R_\phi^{-1} R_{yg} R_g^{-1} & R_\phi^{-1} \end{bmatrix}.$$

Then,

$$
\begin{aligned}
R_{fh}R_h^{-1}h &= \begin{bmatrix} R_{fg} & R_{fy} \end{bmatrix} \begin{bmatrix} R_g^{-1} + R_g^{-1}R_{gy}R_\phi^{-1}R_{yg}R_g^{-1} & -R_g^{-1}R_{gy}R_\phi^{-1} \\ -R_\phi^{-1}R_{yg}R_g^{-1} & R_\phi^{-1} \end{bmatrix} \begin{bmatrix} g \\ y \end{bmatrix} \\
&= R_{fg}R_g^{-1}g + R_{fg}R_g^{-1}R_{gy}R_\phi^{-1}R_{yg}R_g^{-1}g - R_{fg}R_\phi^{-1}R_{yg}R_g^{-1}g \\
&\qquad - R_{fg}R_g^{-1}R_{gy}R_\phi^{-1}y + R_{fy}R_\phi^{-1}y.
\end{aligned}
$$

Since,

$$
\begin{aligned}
P_{\mathcal{M}}f &= R_{fg}R_g^{-1}g \\
\phi &= y - P_{\mathcal{M}}y = y - R_{yg}R_g^{-1}g.
\end{aligned}
$$

Then the equation above can be rewritten as

$$
\begin{aligned}
R_{fh}R_h^{-1}h &= P_{\mathcal{M}}f + R_{fg}R_g^{-1}R_{gy}R_\phi^{-1}(y - R_{yg}R_g^{-1}g) + R_{fy}R_\phi^{-1}(y - R_{yg}R_g^{-1}g) \\
&= P_{\mathcal{M}}f + (R_{fy} - R_{fg}R_g^{-1}R_{gy})R_\phi^{-1}\phi
\end{aligned}
$$

and because,

$$
\begin{aligned}
\phi &= span y \\
R_{gy} &= 0 \qquad \because g \perp y
\end{aligned}
$$

we have

$$
R_{fh}R_h^{-1}h = P_{\mathcal{M}}f + R_{f\phi}R_\phi^{-1}\phi
$$

$$
\therefore P_{\mathcal{H}} = P_{\mathcal{M}}f + R_{f\phi}R_\phi^{-1}\phi
$$

Secondly,

$$
\begin{aligned}
E(f - P_{\mathcal{H}}f)(f - P_{\mathcal{H}}f)^* &= E(f - P_{\mathcal{M}}f - R_{f\phi}R_\phi^{-1}\phi)(f - P_{\mathcal{M}}f - R_{f\phi}R_\phi^{-1}\phi)^* \\
&= E(f - P_{\mathcal{M}}f)(f - P_{\mathcal{M}}f)^* - R_{f\phi}R_\phi^{-1}\underbrace{E\phi\phi^*}_{R_\phi}R_\phi^{-1}R_{\phi f} \\
&= E(f - P_{\mathcal{M}}f)(f - P_{\mathcal{M}}f)^* - R_{f\phi}R_\phi^{-1}R_{\phi f}
\end{aligned}
$$

Thus,

$$
E(f - P_{\mathcal{H}}f)(f - P_{\mathcal{H}}f)^* = E(f - P_{\mathcal{M}}f)(f - P_{\mathcal{M}}f)^* - R_{f\phi}R_\phi^{-1}R_{\phi f}
$$

q.e.d

Problem 2

As in the Kalman filtering Theorem 3.4.1, consider the state space system

$$x(n+1) = Ax(n) + Bu(n)$$
$$y(n) = Cx(n) + Dv(n)$$

where $u(n)$ and $v(n)$ are independent white noise random processes, which are independent to the initial conditions $x(0)$. Recall that $\mathcal{M}_n$ equals the linear span of $\{y(k(0)\}_0^n$ and the optimal state estimate in the Kalman filter is given by $\hat{x}(n) = P_{\mathcal{M}_{n-1}}x(n)$. Find the state estimate $P_{\mathcal{M}n}x(n)$ for $x(n)$ in terms of $\hat{x}(n)$ and $y(n)$. Hint, according to Lemma 3.3.1, we have

$$P_{\mathcal{M}_n}f = P_{\mathcal{M}_{n-1}}f + R_{f\phi(n)}R_{\phi(n)}^{-1}\phi(n),$$

where f is any random vector, and $\phi(n) = y(n) - P_{\mathcal{M}_{n-1}}y(n)$.

Solution:

Lemma $\mathcal{H} = \mathcal{M} \vee y$, $\varphi = y - P_{\mathcal{M}} y$, $\mathcal{E} = \text{span}\{\varphi\}$

$\mathcal{H} = \mathcal{M} \oplus \mathcal{E} \Longrightarrow P_{\mathcal{H}} = P_{\mathcal{M}} + P_{\mathcal{E}}$

$P_{\mathcal{H}} f = P_{\mathcal{M}} f + P_{\mathcal{E}} f = P_{\mathcal{M}} f + R_{f\varphi} R_{\varphi}^{-1} \varphi$

$E(f - P_{\mathcal{H}} f)(f - P_{\mathcal{H}} f)^*$

$= E(f - P_{\mathcal{M}} f)(f - P_{\mathcal{M}} f)^* - R_{f\varphi} R_{\varphi}^{-1} R_{\varphi f}$

$\mathcal{M}_n \subseteq \text{span}\{1, x(0), u(0), \cdots, u(n-1), v(0), \cdots, v(n)\}$

$\mathcal{M}_n = \mathcal{M}_{n-1} \vee y(n)$

$\hat{x}(n) = P_{\mathcal{M}_{n-1}} x(n)$

$\varphi(n) = y(n) - P_{\mathcal{M}_{n-1}} y(n) = y(n) - P_{\mathcal{M}_{n-1}} (Cx(n) + Dv(n))$

$= y(n) - C P_{\mathcal{M}_{n-1}} x(n) - D \cancel{P_{\mathcal{M}_{n-1}} v(n)}^{\,0}$ $\quad \because v(n) \perp \mathcal{M}_{n-1}$

$\varphi(n) = y(n) - C\hat{x}(n)$

also $\varphi(n) = Cx(n) + DV(n) - C\hat{x}(n)$

$$\varphi(n) = C\tilde{x}(n) + DV(n)$$

$$x(n+1) = Ax(n) + Bu(n)$$

$$\hat{x}(n+1) = P_{\mathcal{M}_n} x(n+1) = P_{\mathcal{M}_n}(Ax(n) + Bu(n))$$

$$= AP_{\mathcal{M}_n} x(n) + B\cancel{P_{\mathcal{M}_n} u(n)}^{\,0} \qquad \because u(n) \perp \mathcal{M}_n$$

$$= AP_{\mathcal{M}_{n-1}} x(n) + AR_{x(n)\varphi(n)} R_{\varphi(n)}^{-1} \varphi(n) \qquad \because \mathcal{M}_n = \mathcal{M}_{n-1} \oplus \text{span}$$

$$\hat{x}(n+1) = A\hat{x}(n) + AR_{x(n)\varphi(n)} R_{\varphi(n)}^{-1} (y(n) - C\hat{x}(n))$$

$$= \underbrace{A\hat{x}(n)}_{\text{old estimate}} + \underbrace{\Delta_n (y(n) - Cx(n))}_{\text{orthogonal composite}}$$

$$Q_n = E(\tilde{x}(n)\tilde{x}(n)^*)$$

$$R_{x(n)\varphi(n)} = Ex(n)\varphi(n)^* = Ex(n)(C\tilde{x}(n) + DV(n))^*$$

$$= Ex(n)\tilde{x}(n)C^* + E\cancel{x(n)V(n)^*}^{\,0} D^*$$

$$x(n) = \Phi x(0) + \sum \Phi_j u(j)$$

$$x(0) \perp V(n), \quad u(j) \perp V(n)$$

$$x(j) \perp V(n)$$

$$\text{"} = E(\tilde{x}(n) + \hat{x}(n))\tilde{x}(n) C^*$$

$$= E(\tilde{x}(n)\tilde{x}(n)^*)C^* = Q_n C^*$$

$R_{x(n)\varphi(n)} = Q_n C^*$

$\tilde{x} = x(n) - \hat{x}(n)$

$= x(n) - P_{\mathcal{M}_{n-1}} x(n)$

$x(n) \perp v(n)$

$v(n) \perp \mathcal{M}_{n-1}$

$\tilde{x}(n) \perp v(n)$

$$R_{\varphi(n)} = E\,\varphi(n)\varphi(n)^*$$
$$= E(C\tilde{x}(n) + Dv(n))(C\tilde{x}(n) + Dv(n))^*$$
$$= CE\,\tilde{x}(n)\tilde{x}(n)^* C^* + D\underbrace{v(n)v(n)^*}_{I}D^*$$

cross terms go away

$$R_{\varphi(n)} = CQ_nC^* + DD^*$$

$$\boxed{\hat{x}(n+1) = A\hat{x}(n) + AQ_nC^*(CQC^* + DD^*)^{-1}(y(n) - C\hat{x}(n))}$$

$$\Delta_n = AQ_nC^*(CQ_nC^* + DD^*)^{-1}$$
$$= A R_{x(n)\varphi(n)} R_{\varphi(n)}^{-1}$$

$$\hat{x}(n+1) = A\hat{x}(n) + \Delta_n(y(n) - C\hat{x}(n))$$

$$\Delta_n = AQ_nC^*(CQ_nC^* + DD^*)^{-1}$$

$$Q_n = E(\tilde{x}(n)\tilde{x}(n)^*)$$

$$Q_{n+1} = AQ_nA^* + BB^* - AQ_nC^*(CQ_nC^* + DD^*)^{-1}CQ_nA^*$$

$$\tilde{x}(n+1) = x(n+1) - \hat{x}(n+1)$$
$$= x(n+1) - P_{\mathcal{M}_n} x(n+1)$$
$$= Ax(n) + Bu(n) - P_{\mathcal{M}_n}(Ax(n) + Bu(n)) \qquad \because \mathcal{M}_n \perp u(n)$$
$$= A(x(n) - P_{\mathcal{M}_n} x(n)) + Bu(n)$$

$$Q_{n+1} = E(\tilde{x}(n+1)\tilde{x}(n+1)^*)$$
$$= E[A(x(n) - P_{\mathcal{M}_n} x(n)) + Bu(n)][A(x(n) - P_{\mathcal{M}_{n-1}} x(n)) + B(u)]^*$$

$$= A E \underbrace{(x(n) - P_{\mathfrak{M}_n} x(n))(x(n) - P_{\mathfrak{M}_n} x(n))^*}_{\text{use Lemma}} A^* + B E \underbrace{u(n) u(n)^*}_{I} B^*$$

$$E(f - P_{\mathcal{H}} f)(f - P_{\mathcal{H}} f)^*$$

$$= E(f - P_{\mathfrak{M}} f)(f - P_{\mathfrak{M}} f)^* - R_{f\varphi} R_{\varphi}^{-1} R_{\varphi f}$$

$$\Longrightarrow A E \underbrace{(x(n) - P_{\mathfrak{M}_{n-1}} x(n))}_{\tilde{x}(n)} \underbrace{(x(n) - P_{\mathfrak{M}_{n-1}} x(n))^*}_{\tilde{x}(n)} A^* + B B^* \longrightarrow E\tilde{x}(n)\tilde{x}(n) = Q_n$$

$$- A R_{x(n)\varphi(n)} R_{\varphi(n)}^{-1} R_{x(n)\varphi(n)}^*$$

$$Q_{n+1} = A Q_n A^* + BB^* - A Q_n C^* (C Q_n C^* + DD^*)^{-1} C Q_n A^*$$

$$Q_n = E(\tilde{x}(n) \tilde{x}(n)^*) \geq 0$$

Problem 3

Consider the system

$$x(n+1) = ax(n) + u(n)$$
$$y(n) = x(n) + v(n)$$

where a is a scalar, $u(0), v(0), v(1), x(0)$ are all independent mean zero and variance one Gaussian random variables. Let $\mathcal{M}_0 = span\{y(0)\}$ and $\mathcal{M}_1 = span\{y(0), y(1)\}$. Find

(i) $\hat{x}(0) = P_{\mathcal{M}_1} x(0)$.

(ii) $E|x(0) - \hat{x}(0)|^2$.

(iii) Find α and β such that $\phi(1) = y(1) - P_{\mathcal{M}_0} y(1) = \alpha y(1) + \beta y(0)$.

Kalman filtering is not needed to solve this problem.

Solution:
(i) From what we are given we know that

$$\begin{aligned} x(1) &= ax(0) + u(0) \\ y(0) &= x(0) + v(0) \\ y(1) &= x(1) + v(1) = ax(0) + u(0) + v(1) \end{aligned}$$

and if we define

$$g = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}$$

since

$$\mathcal{M}_1 = span\Big\{y(0), \quad y(1)\Big\}.$$

Now,

$$\begin{aligned} \hat{x}(0) &= P_{\mathcal{M}_1} x(0) \\ &= R_{x(0)g} R_g^{-1} g. \end{aligned}$$

We can break down the calculations by components

$$\begin{aligned} R_{x(0)g} &= Ex(0)g^* = Ex(0)\begin{bmatrix} y(0) & y(1) \end{bmatrix} \\ &= E\begin{bmatrix} x(0)y(0) & x(0)y(1) \end{bmatrix} \\ &= E\begin{bmatrix} x(0)(x(0) + v(0)) & x(0)(ax(0) + u(0) + v(1)) \end{bmatrix} \\ &= \begin{bmatrix} 1 & a \end{bmatrix} \end{aligned}$$

and

$$
\begin{aligned}
R_g &= Egg^* \\
&= \begin{bmatrix} Ey(0)^2 & Ey(0)y(1) \\ Ey(1)y(0) & Ey(1)^2 \end{bmatrix} \\
&= \begin{bmatrix} E(x(0)+v(0))^2 & E(x(0)+v(0))(ax(0)+u(0)+v(1)) \\ E(x(0)+v(0))(ax(0)+u(0)+v(1)) & E(ax(0)+u(0)+v(1))^2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & a \\ a & a^2+2 \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\hat{x}(0) &= \begin{bmatrix} 1 & a \end{bmatrix} \begin{bmatrix} 2 & a \\ a & a^2+2 \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} \\
&= \frac{1}{a^2+4} \begin{bmatrix} 1 & a \end{bmatrix} \begin{bmatrix} a^2+2 & -a \\ -a & 2 \end{bmatrix}^{-1} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} \\
&= \frac{1}{a^2+4} \begin{bmatrix} 2 & a \end{bmatrix} \begin{bmatrix} y(0) \\ y(1) \end{bmatrix} \\
&= \frac{2y(0)+ay(1)}{a^2+4} \\
&= \frac{(a^2+2)x(0)+au(0)+2v(0)+av(1)}{a^2+4}.
\end{aligned}
$$

Thus,

$$
\hat{x}(0) = \frac{(a^2+2)x(0)+au(0)+2v(0)+av(1)}{a^2+4}.
$$

(ii) The error is calculated by

$$
\begin{aligned}
E|x(0)-\hat{x}(0)|^2 &= Ex(0)^2 - R_{x(0)g}R_g^{-1}R_{x(0)g}^* \\
&= 1 - \begin{bmatrix} 1 & a \end{bmatrix} \begin{bmatrix} 2 & a \\ a & a^2+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a \end{bmatrix} \\
&= 1 - \frac{1}{a^2+4} \begin{bmatrix} 1 & a \end{bmatrix} \begin{bmatrix} a^2+2 & -a \\ -a & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ a \end{bmatrix} \\
&= 1 - \frac{1}{a^2+4} \begin{bmatrix} 2 & a \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} \\
&= 1 - \frac{a^2+2}{a^2+4} \\
&= \frac{2}{a^2+4}.
\end{aligned}
$$

Thus,

$$E|x(0) - \hat{x}(0)|^2 = \frac{2}{a^2 + 4}$$

(iii) If

$$g = y(0)$$

we calculate

$$\begin{aligned}
\phi(1) &= y(1) - P_{\mathcal{M}_0} y(1) \\
&= ax(0) + u(0) + v(1) - R_{y(1)g} R_g^{-1} g \\
&= ax(0) + u(0) + v(1) - \frac{Ey(1)y(0)}{Ey(0)^2} y(0) \\
&= ax(0) + u(0) + v(1) - \frac{a}{2}(x(0) + v(0)) \\
&= \frac{a}{2} x(0) + u(0) - \frac{a}{2} v(0) + v(1) \\
&= y(1) - \frac{a}{2} y(0).
\end{aligned}$$

Thus,

$$\alpha = 1, \qquad \beta = -\frac{a}{2}.$$