P3: Oa

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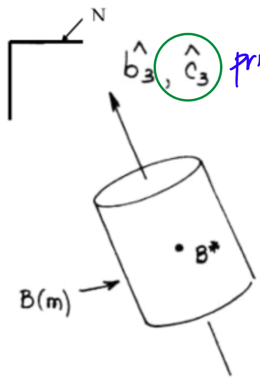
Axisymmetric RB; Torque-free



$\hat{b}_3, \hat{c}_3$ principal

$$\bar{M}^{B^*} = \frac{{}^{i}d\,{}^{i}\bar{H}^{B/B^*}}{dt}$$

$\bar{M}^{B^*}$ ↓ model of gravity moment

$\frac{{}^{i}d\,{}^{i}\bar{H}^{B/B^*}}{dt}$ ↓ kinematics

Model

Mass properties:
- I transverse moment of inertia
- J axial moment of inertia

⇓

$\hat{b}_i$ central, principal

$$\bar{\bar{I}}^{B/B^*} = I\hat{b}_1\hat{b}_1 + I\hat{b}_2\hat{b}_2 + J\hat{b}_3\hat{b}_3$$

$$\bar{\bar{I}}^{B/B^*} = I\hat{c}_1\hat{c}_1 + I\hat{c}_2\hat{c}_2 + J\hat{c}_3\hat{c}_3$$

$$\bar{M}^{B^*} = \frac{{}^{N}d\,{}^{N}\bar{H}^{B/B^*}}{dt} = \bar{0} \quad \Longrightarrow \quad \bar{H}^{B/B^*} \text{ constant } \begin{cases} \text{mag} \\ \text{direc} \end{cases}$$

$\bar{\bar{I}}^{B/B^*} \Longleftarrow$

has to be wrt. inertial frame $\rightarrow$ ${}^{N}\bar{\omega}^{B} \Longleftarrow$

$\Big\{\; {}^{N}\bar{H}^{B/B^*} = H\hat{h}$

Rotational motion of this body under these circumstances is well-established. Solution is analytical; can be derived or represented a number of ways

complete soln. is known:

Motion of B in N can be expressed as sum of 2 rotations

Introduce a nonphysical (fictitious) frame $\hat{c}_i$

1. $\hat{c}_3 = \hat{b}_3$ always so $\quad \bar{\bar{I}}^{B/B^*} = I\hat{c}_1\hat{c}_1 + I\hat{c}_2\hat{c}_2 + J\hat{c}_3\hat{c}_3$
2. $\hat{c}$ moves in N and $\hat{b}$ moves wrt $\hat{c}$ $\quad {}^C\bar{\omega}^B = s\hat{c}_3$

Then ${}^N\bar{\omega}^B = {}^N\bar{\omega}^C + {}^C\bar{\omega}^B$

Note: if $s = 0 \rightarrow \hat{c} = \hat{b}$

Choose to write EOM utilizing $\hat{c}$ unit vectors as the working frame

Define ${}^N\bar{\omega}^B$ for vector basis $\hat{c}$

$${}^N\bar{\omega}^B = \omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3$$

$$\Downarrow$$

$$\begin{aligned}
{}^N\bar{H}^{B/B^*} &= \bar{\bar{I}}^{B/B^*} \bullet {}^N\bar{\omega}^B \\
&= \left(I\hat{c}_1\hat{c}_1 + I\hat{c}_2\hat{c}_2 + J\hat{c}_3\hat{c}_3\right) \bullet \left(\omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3\right) \\
&= I\omega_1\hat{c}_1 + I\omega_2\hat{c}_2 + J\omega_3\hat{c}_3
\end{aligned}$$

$$\frac{{}^N d\, {}^N\bar{H}}{dt} = \frac{{}^C d\, {}^N\bar{H}}{dt} + {}^N\bar{\omega}^C \times \bar{H} = \bar{0}$$

$$\frac{{}^C d\bar{H}}{dt} = I\dot{\omega}_1\hat{c}_1 + I\dot{\omega}_2\hat{c}_2 + J\dot{\omega}_3\hat{c}_3$$

$${}^N\bar{\omega}^C \times \bar{H} = \Big(\underbrace{{}^N\bar{\omega}^B}_{\omega_i c_i} - \underbrace{{}^C\bar{\omega}^B}_{s\hat{c}_3}\Big) \times \bar{H}$$

$$= \left[(J-I)\omega_2\omega_3\hat{c}_1 + (I-J)\omega_1\omega_3\hat{c}_2 - I\omega_1 s\hat{c}_2 + I\omega_2 s\hat{c}_1\right]$$

$$\frac{{}^N d\, {}^N\bar{H}}{dt} = \bar{0} \qquad \boxed{\begin{aligned} &I\dot{\omega}_1 + (J-I)\omega_2\omega_3 + I\omega_2 s = 0 \\ &I\dot{\omega}_2 - (J-I)\omega_1\omega_3 - I\omega_1 s = 0 \\ &J\dot{\omega}_3 = 0 \end{aligned}}$$

Not Euler's equations (unless s = 0)

Solution: $\boxed{\omega_3 = \text{constant}}$

s can be defined arbitrarily

Define

$$\boxed{s = \frac{I-J}{I}\omega_3} \quad \text{constant}$$

$$\Downarrow$$

$\omega_1 = \text{const.}$

$\omega_2 = \text{const.}$

O4

$s\hat{c}_3$ constant $\quad s = \frac{I-J}{I}\omega_3$

$s = (1 - \frac{J}{I})\omega_3$

Describe motion

$$ {}^N\bar{\omega}^B = {}^N\bar{\omega}^C + {}^C\bar{\omega}^B $$

How are $\hat{c}$'s moving in N?

$$ H\hat{h} = I\omega_1\hat{c}_1 + I\omega_2\hat{c}_2 + J\omega_3\hat{c}_3 $$

$$ \frac{H}{I}\hat{h} = \omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \underbrace{\frac{J}{I}\omega_3}_{\omega_3 - s}\hat{c}_3 $$

$$ \frac{H}{I}\hat{h} = \underbrace{\omega_1\hat{c}_1 + \omega_2\hat{c}_2 + \omega_3\hat{c}_3}_{{}^N\bar{\omega}^B} - \underbrace{s\hat{c}_3}_{{}^C\bar{\omega}^B} = {}^N\bar{\omega}^C $$

$$ \frac{H}{I}\hat{h} = \boxed{{}^N\bar{\omega}^C = p\,\hat{h}} \Longrightarrow {}^N\bar{\omega}^C \text{ const. rate, const. direction} $$

constant

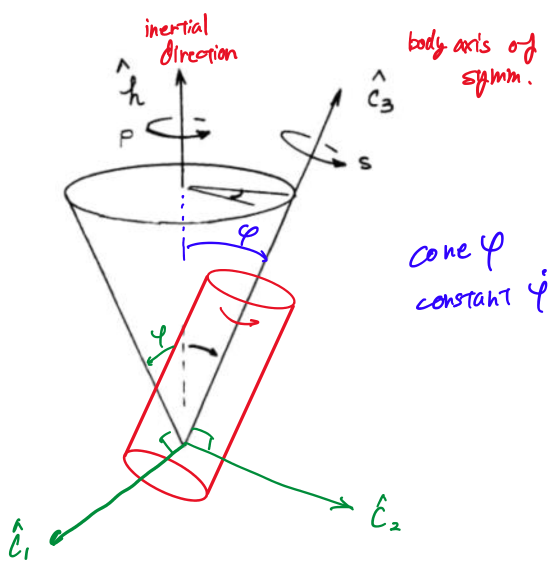
O5

Summarize: Rotational motion of B in N can be described as a sum of two simple rotations

${}^N\bar{\omega}^C = p\,\hat{h}$ constant mag; constant direction

${}^C\bar{\omega}^B = s\,\hat{c}_3$ constant mag

if $s = \left(\frac{I-J}{I}\right)\omega_3; \quad p = \frac{H}{I}$



cone φ

constant $\dot{\varphi}$

Recall $\quad {}^N\bar{H}^{B/B^*} = I\omega_1\hat{c}_1 + I\omega_2\hat{c}_2 + J\omega_3\hat{c}_3$

$\bar{H}\bullet\hat{c}_3 = J\omega_3$

But, it is also true $\quad \bar{H}\bullet\hat{c}_3 = H\,\hat{h}\bullet\hat{c}_3 = H\cos\phi$

$\Longrightarrow \quad \bar{H}\cos\varphi = J\omega_3$

ang. vel

$$\boxed{\begin{aligned} \phi &= \cos^{-1}\left(\frac{J}{H}\omega_3\right) \\ p &= \frac{H}{I} \\ s &= \left(\frac{I-J}{I}\right)\omega_3 \end{aligned}}$$

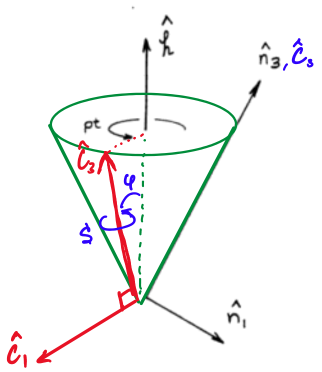
constant

$$\Longrightarrow \quad \boxed{\varphi = \arccos\left[\frac{Js}{(I-J)p}\right]}$$

$s \Rightarrow$ spin

$p \Rightarrow$ precession

Describe this same motion (orientation on B in N) in terms of Euler parameters ε_i



${}^N\bar{\varepsilon}^B \quad {}^N\varepsilon_4^B$

$\uparrow$

${}^N\bar{\varepsilon}^C \quad {}^N\varepsilon_4^C$

$+$

${}^C\bar{\varepsilon}^B \quad + \quad {}^C\varepsilon_4^B$

Two successive rotations

Rot #1 $\quad {}^N\bar{\varepsilon}^C = \hat{h}\,\sin\left(\frac{pt}{2}\right) \qquad {}^N\varepsilon_4^C = \cos\left(\frac{pt}{2}\right)$

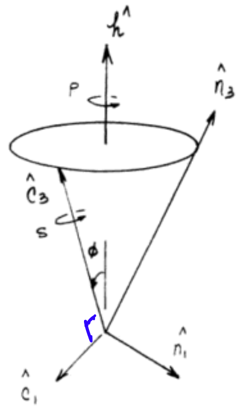
Rot #2 $\quad {}^C\bar{\varepsilon}^B = \hat{c}_3\,\sin\left(\frac{st}{2}\right) \qquad {}^C\varepsilon_4^B = \cos\left(\frac{st}{2}\right)$

Euler rule of successive rotations

$${}^N\bar{\varepsilon}^B = {}^N\bar{\varepsilon}^C\,{}^C\varepsilon_4^B + {}^C\bar{\varepsilon}^B\,{}^N\varepsilon_4^C + {}^C\bar{\varepsilon}^B \times {}^N\bar{\varepsilon}^C$$

O8

${}^N\bar{\varepsilon}^B$? ${}^N\varepsilon_4^B$



$${}^N\bar{\varepsilon}^B = {}^N\bar{\varepsilon}^C\,{}^C\varepsilon_4^B + {}^C\bar{\varepsilon}^B\,{}^N\varepsilon_4^C + {}^C\bar{\varepsilon}^B \times {}^N\bar{\varepsilon}^C$$

$${}^N\bar{\varepsilon}^B = \sin\left(\frac{pt}{2}\right)\hat{h}\cos\left(\frac{st}{2}\right) + \sin\left(\frac{st}{2}\right)\hat{c}_3\cos\left(\frac{pt}{2}\right) + \left(\hat{c}_3 \times \hat{h}\right)\sin\left(\frac{st}{2}\right)\sin\left(\frac{pt}{2}\right)$$

$-s_\phi\hat{c}_2$

$c_\phi\hat{c}_3 - s_\phi\hat{c}_1$

O9

Components of ${}^N\bar{\varepsilon}^B$ expressed in $\boxed{\hat{c}_i}$

$$\boxed{\begin{aligned} {}^{N}\varepsilon_1^{B} &= -\sin\phi\sin\left(\frac{pt}{2}\right)\cos\left(\frac{st}{2}\right) \\ \varepsilon_2 &= -\sin\phi\sin\left(\frac{st}{2}\right)\sin\left(\frac{pt}{2}\right) \\ \varepsilon_3 &= \cos\phi\sin\left(\frac{pt}{2}\right)\cos\left(\frac{st}{2}\right) + \cos\left(\frac{pt}{2}\right)\sin\left(\frac{st}{2}\right) \end{aligned}}$$

$${}^N\varepsilon_4^B = {}^N\varepsilon_4^C\,{}^C\varepsilon_4^B - {}^N\bar{\varepsilon}^C \cdot {}^C\bar{\varepsilon}^B$$

$$= \cos\left(\frac{pt}{2}\right)\cos\left(\frac{st}{2}\right) - \sin\left(\frac{pt}{2}\right)\sin\left(\frac{st}{2}\right)\hat{h}\cdot\hat{c}_3$$

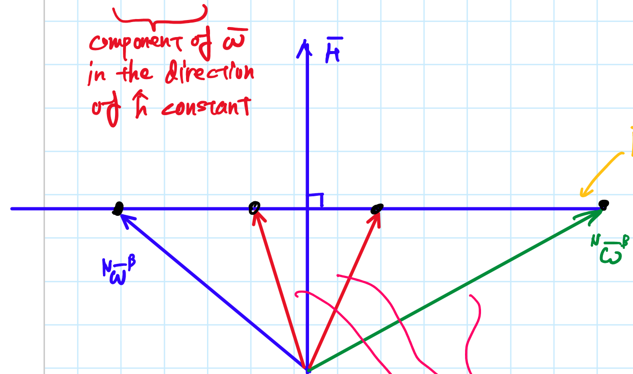
$$\boxed{{}^N\varepsilon_4^B = \cos\left(\frac{pt}{2}\right)\cos\left(\frac{st}{2}\right) - \cos\phi\sin\left(\frac{pt}{2}\right)\sin\left(\frac{st}{2}\right)}$$

$$\frac{1}{2}\,{}^N\bar{\omega}^B \cdot {}^N\bar{H}^{B/B*}$$

K.E.

$$T_{rot} = \frac{1}{2}\,{}^N\bar{\omega}^B \cdot \bar{\bar{I}}^{B/B*} \cdot {}^N\bar{\omega}^B$$

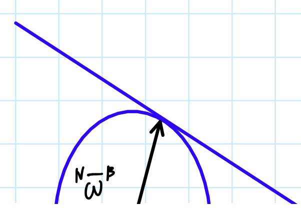
$$\underbrace{{}^N\bar{\omega}^B \cdot {}^N\bar{H}^{B/B*}}_{\text{component of }\bar{\omega}\text{ in the direction of }\hat{h}\text{ constant}} = \text{constant} = 2T_{rot}$$

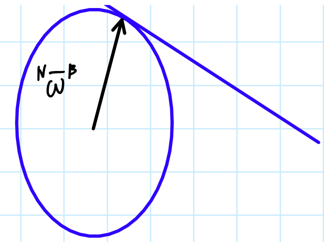


$$\underbrace{{}^N\bar{\omega}^B \cdot {}^N\bar{H}^{B/B*}}_{I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2} = 2T_{rot} = C^2$$

Eqn. of ellipsoid

Called "Energy" ellipsoid because it is conserved by the rotational KE




${}^{N}\bar{\omega}^{B}$