AAE364: Controls System Analysis

HW5: Stability and Error Analysis

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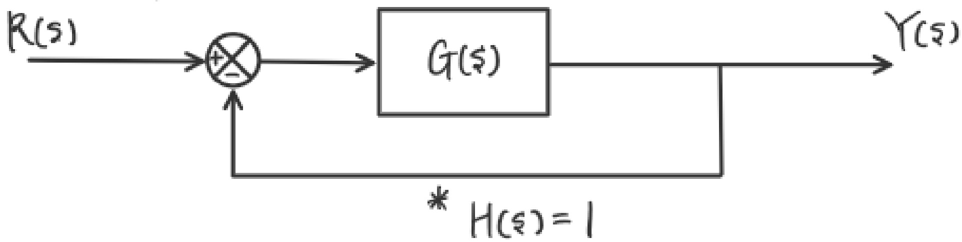
B–5–5. Obtain the unit-impulse response and the unit-step response of a unity-feedback system whose open-loop transfer function is

$$G(s) = \frac{2s+1}{s^2}$$

Block diagram



For this response system,

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{2s+1}{s^2}}{1+\frac{2s+1}{s^2}} = \frac{\frac{2s+1}{s^2}}{\frac{s^2+2s+1}{s^2}}$$

$$\frac{Y(s)}{R(s)} = \frac{2s+1}{(s+1)^2}$$

<i> unit-impulse response $\quad r(t) = \delta(t) \Rightarrow R(s) = 1$

then $\quad Y(s) = \frac{2s+1}{(s+1)^2} = -\frac{1}{(s+1)^2} + \frac{2}{s+1}$

and $\quad y(t) = \mathcal{L}^{-1}[Y(s)] = -te^{-t} + 2e^{-t}$

$$\boxed{y(t) = (2-t)e^{-t}}$$

<ii> unit-step response $\quad r(t) = 1 \Rightarrow R(s) = \frac{1}{s}$

then $\quad Y(s) = \frac{2s+1}{s(s+1)^2} = \frac{A}{s} + \frac{B}{(s+1)^2} + \frac{C}{s+1}$

$$A = \lim_{s\to 0} sY(s) = 1$$

$$B = \lim_{s\to -1} (s+1)^2 Y(s) = 1$$

$$X(s) = Y(s) - \frac{B}{(s+1)^2} = \frac{2s+1}{s(s+1)^2} - \frac{1}{(s+1)^2} = \frac{s+1}{s(s+1)^2}$$

$$X(s) = \frac{1}{s(s+1)}$$

$$C = \lim_{s\to -1} (s+1)X(s) = -1$$

$$\Rightarrow \quad Y(s) = \frac{1}{s} + \frac{1}{(s+1)^2} - \frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = 1 + te^{-t} - e^{-t}$$

$$\boxed{y(t) = 1 + (t-1)e^{-t}}$$

B–5–9. Consider the system shown in Figure 5–76. Determine the value of k such that the damping ratio ζ is 0.5. Then obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s in the unit-step response.

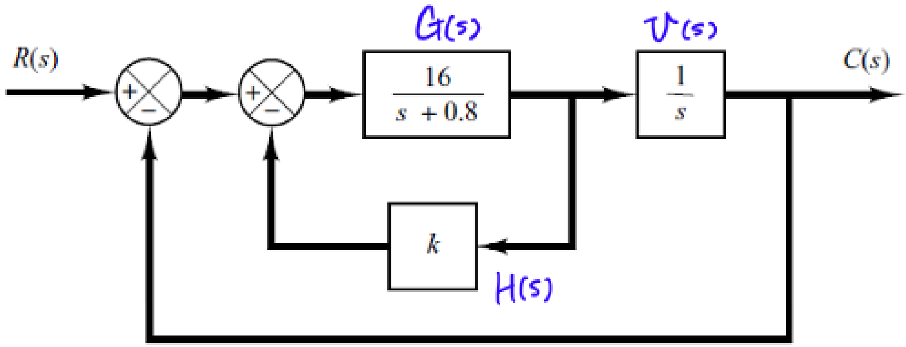
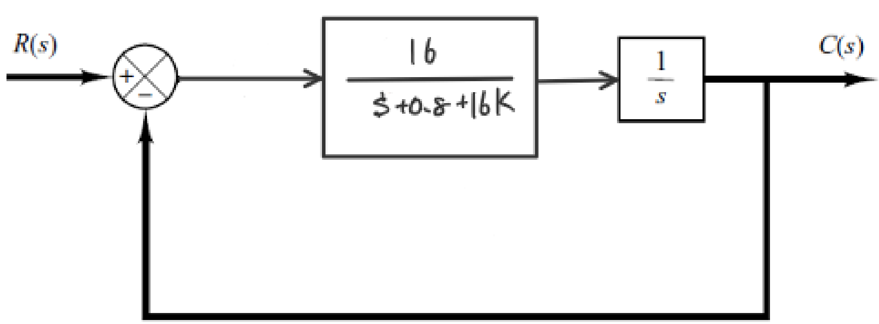


Figure 5–76
Block diagram of a system.

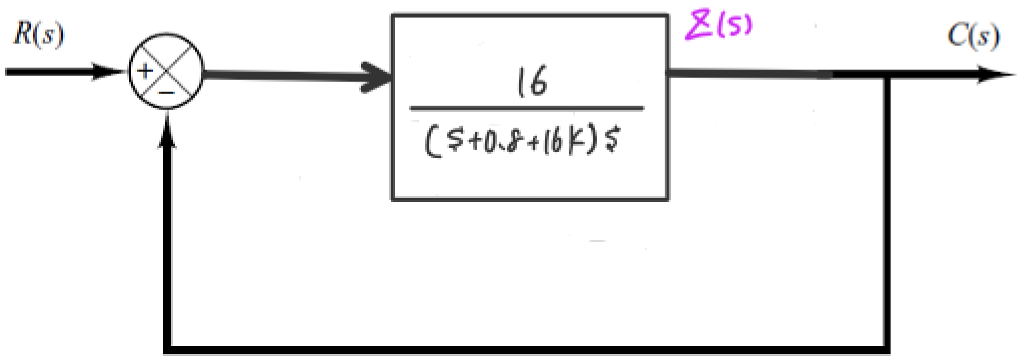
<i> The inner CL has a TF of

$$\frac{G(s)}{1+G(s)H(s)} = \frac{\frac{16}{s+0.8}}{1+\frac{16}{s+0.8}k} = \frac{\frac{16}{s+0.8}}{\frac{s+0.8+16k}{s+0.8}}$$

$$= \frac{16}{s+0.8+16k}$$



$\Downarrow$



This CL TF becomes

$$\frac{C(s)}{R(s)} = \frac{Z(s)}{1 + Z(s)\cdot 1} = \frac{\frac{16}{(s+0.8+16K)s}}{\frac{s^2+(0.8+16K)s+16}{(s+0.8+16K)s}}$$

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+(0.8+16K)s+16}$$

since $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$$\omega_n = \sqrt{16} = 4 \quad (>0)$$

then if $\xi = 0.5$

$$2\xi\omega_n = 2(0.5)(4) = 0.8 + 16K$$

$$16K = 4 - 0.8$$

$$\boxed{K = \frac{3.2}{16} = 0.2}$$

<ii> Find t_r, t_p, M_p, t_s (* for unit-step response)

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 4\sqrt{1-0.5^2} \cong 3.4641$$

$$\frac{C(s)}{R(s)} = \frac{16}{s^2+4s+16} = Z(s)$$

for $0 < \xi < 1$ underdamped case

$\sigma = \xi\omega_n$ and

$$t_r = \frac{1}{\omega_d}\arctan\left(\frac{\omega_d}{-\sigma}\right)$$

$$t_r = \frac{1}{3.4641}\arctan\left(\frac{3.4641}{0.5\times 4}\right)$$

$$\boxed{t_r \cong 0.3023\ s}$$

and $$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{3.4641} = 0.9069\ s$$

$$\boxed{t_p = 0.9069\ s}$$

next

$$M_p = \exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right)\cdot 100$$

$$M_p = \exp\left(\frac{-0.5\pi}{\sqrt{1-0.5^2}}\right)\cdot 100$$

$$\boxed{M_p = 16.30\ \%}$$

finally,

$$\boxed{\begin{array}{ll} 2\% & t_s = \dfrac{4}{\xi\omega_n} = 2\ s \\ 5\% & t_s = \dfrac{3}{\xi\omega_n} = 1.5\ s \end{array}}$$

B–5–10. Using MATLAB, obtain the unit-step response, unit-ramp response, and unit-impulse response of the following system:

$$\frac{C(s)}{R(s)} = \frac{10}{s^2 + 2s + 10}$$

where $R(s)$ and $C(s)$ are Laplace transforms of the input $r(t)$ and output $c(t)$, respectively.

from MATLAB we obtain the following results

<i> unit-impulse

$$\boxed{y(t) = \frac{10}{3} e^{-t} \sin 3t}$$

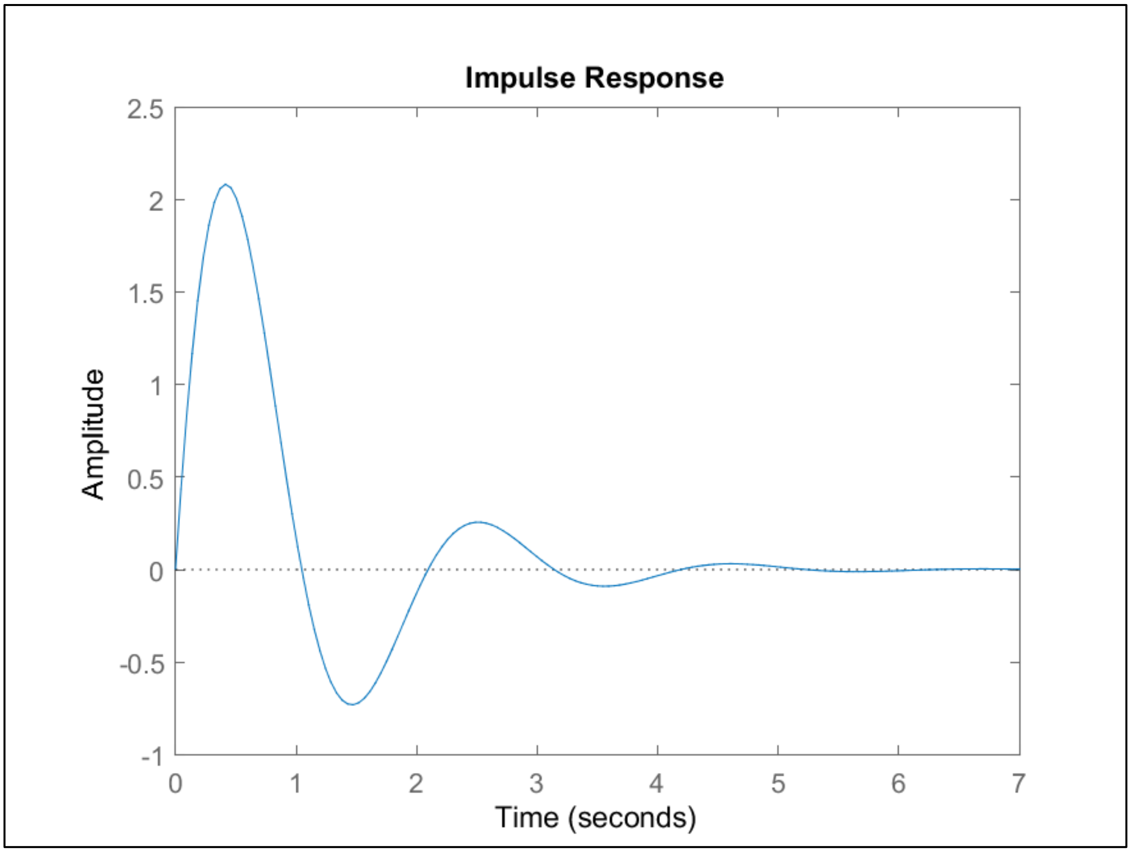
input $R(s) = 1$

thus,

$$C(s) = \frac{10}{s^2+2s+10}$$

take $\mathcal{L}^{-1}[C(s)] = y(t)$

and then plot


Impulse Response
Amplitude
Time (seconds)
2.5
2
1.5
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7

(ii) unit-step

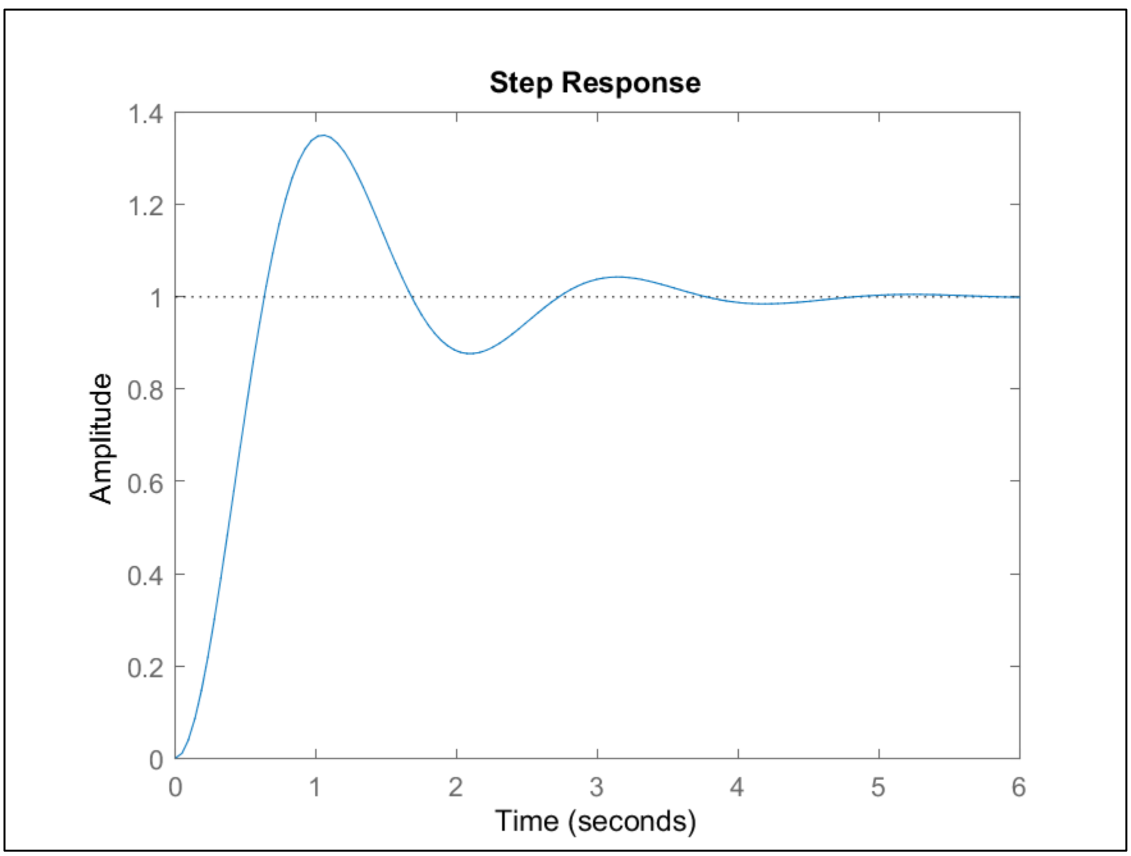
$$\boxed{y(t) = 1 + e^{-t}\left(\cos 3t - \frac{1}{3}\sin 3t\right)}$$

input $R(s) = \frac{1}{s}$

thus, $C(s) = \frac{10}{s^3+2s^2+10s}$

take $\mathcal{L}^{-1}[C(s)] = y(t)$

and then plot



<iii> unit-ramp

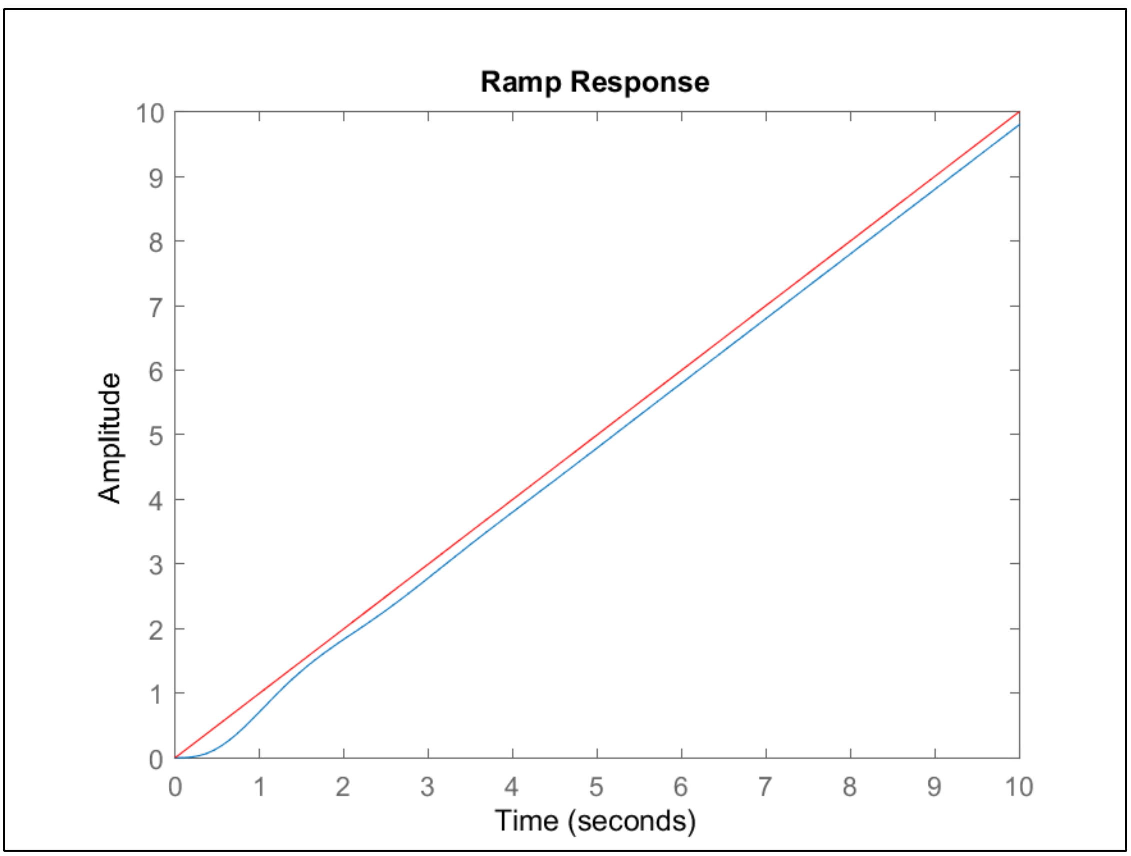
$$\boxed{y(t) = t - \frac{1}{5} + e^{-t}\left(\frac{1}{5}\cos 3t - \frac{4}{15}\sin 3t\right)}$$

input $R(s) = \frac{1}{s^2}$

thus $C(s) = \frac{10}{s^4 + 2s^3 + 10s^2}$

take $\mathcal{L}^{-1}[C(s)] = y(t)$

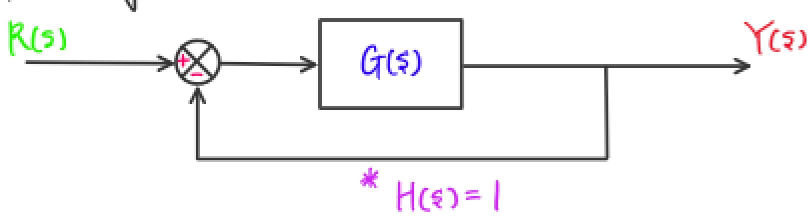
then plot



B–5–20. Determine the range of K for stability of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}$$

Block diagram



$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$= \frac{\frac{K}{s(s+1)(s+2)}}{1 + \frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}$$

factor out the denominator of $\frac{Y(s)}{R(s)}$

$$s^3 + 3s^2 + 2s + K$$

now do RSC (Routh's Stability Criterion)

s^4	0	3	K
s^3	1	2	0
s^2	3	K	
s^1	$\frac{6-K}{3}$	0	
s^0	K		

thus, $\quad k \geqq 0 \quad$ and $\quad \frac{6-k}{3} \geqq 0$

$$6 \geqq k$$

$$\therefore \quad \boxed{k \in [0, 6]}$$

B–5–22. Consider the closed-loop system shown in Figure 5–79. Determine the range of K for stability. Assume that $K > 0$.

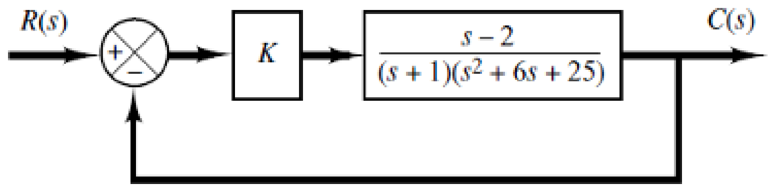


Figure 5–79 Closed-loop system.

$$\frac{C(s)}{R(s)} = \frac{\dfrac{(s-2)K}{(s+1)(s^2+6s+25)}}{1 + \dfrac{(s-2)K}{(s+1)(s^2+6s+25)}}$$

$$\frac{C(s)}{R(s)} = \frac{(s-2)K}{(s+1)(s^2+6s+25)+(s-2)K}$$

factor out the denominator

$$den = s^3+6s^2+25s+s^2+6s+25+Ks-2K$$

$$= s^3+7s^2+(31+K)s+(25-2K)$$

conduct RSC

s^4	0	7	$(25-2K)$
s^3	1	$(31+K)$	0
s^2	7	$(25-2K)$	
s^1	$\frac{192+9K}{7}$	0	
s^0	$25-2K$		

$$\frac{7(31+K)-(25-2K)}{7}$$

thus for stability

$$\frac{192+9K}{7} \geqq 0 \qquad 25-2K \geqq 0$$

$$192 \geqq -9K \quad \text{and} \quad 25 \geqq 2K$$

$$-\frac{64}{3} \leqq K \qquad \frac{25}{2} \geqq K$$

However, since $K > 0$

$$\therefore \boxed{K \in \left(0, \frac{25}{2}\right]}$$

2. Consider the closed-loop system given by

$$\frac{y(s)}{r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Determine the values of ζ and ω_n so that the system responds to a step input with approximately 15% overshoot and with a settling time of 3 *sec*. (Use the 2% criterion.)

using the equation of Max overshoot (%)

$$M_p = \exp\left(\frac{-\xi\pi}{\sqrt{1-\xi^2}}\right) \times 100$$

we can obtain the equation for ξ, damping ratio

$$\xi = \frac{-\ln(M_p/100)}{\sqrt{\pi^2 + [\ln(M_p/100)]^2}} = \boxed{0.5169}$$

calculation w/ MATLAB (code in appendix)

then, since $t_d = \frac{4}{\xi\omega_n}$

$$\therefore \omega_n = \frac{4}{t_d\,\xi}$$

$$\omega_n = \frac{4}{(0.5169)(3)}$$

$$\boxed{\omega_n = 2.5795 \text{ rad/s}}$$

3. Figure 1 is a block diagram of a spacecraft attitude control system. Assuming the time constant T of the controller to be $3\,sec$ and the ratio K/J to be $\frac{2}{9}$ rad^2/sec^2, find the damping ratio of the closed-loop system.

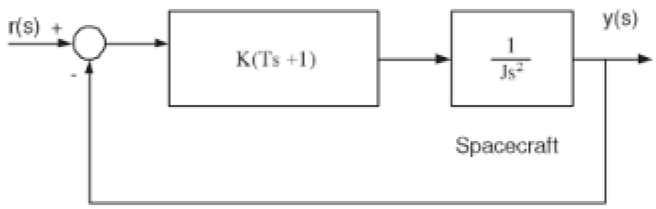


Figure 1: Spacecraft attitude control system

$$\frac{Y(s)}{R(s)} = \frac{\frac{K(Ts+1)}{Js^2}}{1 + \frac{K(Ts+1)}{Js^2}} = \frac{K(Ts+1)}{Js^2 + KTs + K}$$

$$= \frac{\frac{KT}{J}s + \frac{K}{J}}{s^2 + \frac{KT}{J}s + \frac{K}{J}} = \frac{\frac{2}{9}s + \frac{2}{9}}{s^2 + \left(\frac{2}{9}\right)3 \cdot s + \frac{2}{9}}$$

$$\frac{K}{J} = \frac{2}{9}\ rad^2/sec^2$$

$$\omega_n = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3}\ rad/s \quad (>0)$$

and

$$2\xi\omega_n = \left(\frac{2}{9}\ rad/s\right)(3\,s)$$

$$\xi = \frac{1}{3\omega_n}$$

$$\xi = \frac{1}{\sqrt{2}}$$

$$\boxed{\xi = 0.7071}$$

APPENDIX

AAE364 HW5 MATLAB CODE

problem 1 >> B-5-10

```
s = tf('s');
G = 10/(s^2+2*s+10);


fdir = 'C:\Users\Tomo\Desktop\studies\2020-Spring\AAE364\matlab\matlab_output';


% Impulse reponse
impF = 1;
[num_imp, den_imp] = tfdata(G*impF);
output_impulse = return_inverseLaplace_expression(num_imp, den_imp)
fig1 = figure("Renderer","painters");
impulse(G);
saveas(fig1, fullfile(fdir,'HW5_b_5_10_impulse.png'));


% Step Response
stepF = 1/s;
[num_step, den_step] = tfdata(G*stepF);
output_step = return_inverseLaplace_expression(num_step, den_step)
fig2 = figure("Renderer","painters");
step(G);
saveas(fig2, fullfile(fdir,'HW5_b_5_10_step.png'));
```

```


% Ramp Response
rampF = 1/s^2;
[num_ramp, den_ramp] = tfdata(G*rampF);
output_ramp = return_inverseLaplace_expression(num_ramp, den_ramp)
fig3 = figure("Renderer","painters");
step(G / s);
ylim([0, 10])
xlim([0, 10])
hold on
plot(linspace(0,10,20), linspace(0,10,20), '-r')
hold off
title('Ramp Response')
saveas(fig3, fullfile(fdir,'HW5_b_5_10_ramp.png'));
```

problem 2

```
MOS = 15;  % percent
zeta = calc_zetaFromMOS_or_MOSFromzeta(MOS, "zeta")
```

```
function inverted_expr = return_inverseLaplace_expression(num, den)
    %{
      inputs:  1) num: numerator of the transfer function times input
                  function G(s)*R(s)
               2) den: denominator of the transfer function times input
                  function G(s)*R(s)
      outputs: 1) inverted_expr: returns the expression for the inverse
                  laplace equation of the output laplace equation
    %}
    syms s t                                                    % Invoke Symbolic
Math Toolbox
    snum = poly2sym(num, s);                                    % Symbolic
Numerator Polynomial
    sden = poly2sym(den, s);                                    % Symbolic
Denominator Polynomial
    G_time_domain = ilaplace(snum/sden);                        % Inverse Laplace
Transform
    G_time_domain = simplify(G_time_domain, 'Steps',10);        % Simplify To Get
Nice Result
    inverted_expr = collect(G_time_domain, exp(-t));            % Optional
Further Factorization
end


function output = calc_zetaFromMOS_or_MOSFromzeta(MOS_or_zeta, type)
    %{
      inputs:  1) MOS_or_zeta: maximum overshoot or zeta (damping ratio) input
                  the one of the two will be chosen depending on the second
                  input "type"
               2) type: string "MOS" or "zeta" indicates what output the
                  user requires
      outputs: 1) output: returns either the MOS or zeta
    %}
    if type == "MOS"
        zeta = MOS_or_zeta;
        output = exp(-zeta*pi/sqrt(1-zeta^2))*100;
    elseif type == "zeta"
        MOS = MOS_or_zeta;
        output = -log(MOS/100)/sqrt(pi^2 + (log(MOS/100))^2);
    end
end
```