1.

Added pre-stress

a. $\sigma_{11} = E_1 \epsilon_{11} + E_2 \epsilon_{11}^2 + \frac{T_0}{A}$

We can use U or δW^{INT} for this part since conservative

Let's use U:

$$u = \int \sigma_{11} \, d\epsilon_{11} = \int \left(E_1 \epsilon_{11} + E_2 \epsilon_{11}^2 + \frac{T_0}{A} \right) d\epsilon_{11}$$

$\epsilon_{11} = u_{,x} + \frac{1}{2}(u_{,x}^2 + v_{,x}^2 + w_{,x}^2)$

$$= \frac{1}{2} E_1 \epsilon_{11}^2 + \frac{1}{3} E_2 \epsilon_{11}^3 + \frac{T_0}{A} \epsilon_{11}$$

$$U = \int u \, dV = \int_0^{\ell} \left(\frac{1}{2} E_1 \epsilon_{11}^2 + \frac{1}{3} E_2 \epsilon_{11}^3 + \frac{T_0}{A} \epsilon_{11} \right) A \, dx$$

Take first variation

$$\delta U = \int_0^{\ell} \left(E_1 \epsilon_{11} + E_2 \epsilon_{11}^2 + \frac{T_0}{A} \right) \delta \epsilon_{11} A \, dx$$

Very little change in derivation from this point forward; e.g., δT unchanged, procedure unchanged.

H.P.:

$\epsilon = \epsilon_{11}$

$$\int_{t_1}^{t_2} \int_0^{\ell} \left[\rho A (\dot{u} \delta \dot{u} + \dot{v} \delta \dot{v} + \dot{w} \delta \dot{w}) - \left(E_1 \epsilon + E_2 \epsilon^2 + \frac{T_0}{A} \right) \cdot \right.$$
$$\left. \left((1 + u_{,x}) \delta u_{,x} + v_{,x} \delta v_{,x} + w_{,x} \delta w_{,x} \right) A \cdot \right] dx \, dt = 0$$

(1)

Integrating by parts, (K.E. in time; P.E. in space)

$$\boxed{\begin{aligned} -\rho A \ddot{u} + \left(\left(E_1\epsilon + E_2\epsilon^2 + \frac{T_0}{A}\right) A (1+u_{,x}) \right)_{,x} &= 0 \\ -\rho A \ddot{v} + \left(\left(E_1\epsilon + E_2\epsilon^2 + \frac{T_0}{A}\right) A v_{,x} \right)_{,x} &= 0 \\ -\rho A \ddot{w} + \left(\left(E_1\epsilon + E_2\epsilon^2 + \frac{T_0}{A}\right) A w_{,x} \right)_{,x} &= 0 \quad \boxed{0<x<\ell} \end{aligned}}$$

$$\boxed{\begin{aligned} A\cdot\left(E_1\epsilon + E_2\epsilon^2 + \frac{T_0}{A}\right)(1+u_{,x})\,\delta u \Big|_0^\ell &= 0 \\ A\cdot\left(E_1\epsilon + E_2\epsilon^2 + \frac{T_0}{A}\right) v_{,x}\,\delta v \Big|_0^\ell &= 0 \\ A\cdot\left(E_1\epsilon + E_2\epsilon^2 + \frac{T_0}{A}\right) w_{,x}\,\delta w \Big|_0^\ell &= 0 \end{aligned}}$$ BOUNDARY CONDITIONS

$$\epsilon = u_{,x} + \tfrac{1}{2}\left(u_{,x}^2 + v_{,x}^2 + w_{,x}^2\right)$$

b. Non-conservative → must use δW^{INT}

$\delta V = 0$ since even conservative portion treated as virtual work

$$\delta W^{INT} = -\int_0^\ell \sigma_{11} A\, \delta\epsilon_{11}\, dx$$

$$= -\int_0^\ell \left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \frac{T_0}{A}\right)\cdot\left((1+u_{,x})\delta u_{,x} + v_{,x}\delta v_{,x} + w_{,x}\delta w_{,x}\right) dx$$

H.P.

$$\int_{t_1}^{t_2}\int_0^\ell \left[\rho A\left(\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}\right) - \left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \frac{T_0}{A}\right)\cdot\left((1+u_{,x})\delta u_{,x} + v_{,x}\delta v_{,x} + w_{,x}\delta w_{,x}\right)\right] dx\,d$$

2

Integrate by parts ...

$$\boxed{\begin{aligned}
&-\rho A\ddot{u} + \left(\left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \tfrac{T_0}{A}\right)A(1+u_{,x})\right)_{,x} = 0 \\
&-\rho A\ddot{v} + \left(\left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \tfrac{T_0}{A}\right)A v_{,x}\right)_{,x} = 0 \\
&-\rho A\ddot{w} + \left(\quad \cdots \quad w_{,x}\right)_{,x} = 0 \\
&\hspace{8cm} \boxed{0 < x < \ell}
\end{aligned}}$$

$$\boxed{\begin{aligned}
&A\left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \tfrac{T_0}{A}\right)(1+u_{,x})\,\delta u\Big|_0^\ell = 0 \\
&A\left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \tfrac{T_0}{A}\right)v_{,x}\,\delta v\Big|_0^\ell = 0 \\
&A\left(E_1\epsilon + E_2\epsilon^2 + \alpha\dot{\epsilon} + \tfrac{T_0}{A}\right)w_{,x}\,\delta w\Big|_0^\ell = 0 \\
&\hspace{8cm} \boxed{\text{B.C.'s}}
\end{aligned}}$$

$$\epsilon = u_{,x} + \tfrac{1}{2}\left(u_{,x}^2 + v_{,x}^2 + w_{,x}^2\right)$$

2. Since $u(x,t)=c \ \forall x,t$ and $w(x,t)=0 \ \forall x,t$,

- we need to consider only the middle equations (i.e., $v(x,t)$ eqs.).

- Keeping only N.L. terms involving α simplifies equation to:

$$-\rho A \ddot{v} + T_0 v_{,xx} + \left((\alpha A v_{,x} \dot{v}_{,x}) v_{,x} \right)_{,x} = 0$$

or

$$-\rho A \ddot{v} + T_0 v_{,xx} + \underbrace{\left(\alpha A v_{,x}^2 \dot{v}_{,x} \right)_{,x}}_{} = 0$$

Purely NL damping

Solved numerically using Maple with a Galerkin procedure. Chose modes such that

$$v(x,t) = \phi(t) \sin\left(\frac{\pi x}{\ell}\right) + \beta(t) \sin\left(\frac{2\pi x}{\ell}\right)$$

Then,

$$\left.\begin{aligned} &\langle \epsilon_R, \sin\tfrac{\pi x}{\ell} \rangle = 0 \\ &\langle \epsilon_R, \sin\tfrac{2\pi x}{\ell} \rangle = 0 \end{aligned}\right\} \text{ yields ODEs}$$

Kelvin-Voigt Damping

```
> restart;
```

Here is the string with Kelvin Voigt damping --- **keeping NL terms associated with damping only -- terms that are (alpha*A*epsilon_dot*v,x),x**

```
> eq := -rho*A*diff(v(x,t),t,t) + T0*diff(v(x,t),x,x) + alpha*
  A*diff( diff(v(x,t),x)^2*diff(v(x,t),x,t)  , x);
```

$$eq := -\rho A \left(\frac{\partial^2}{\partial t^2} v(x,t)\right) + T0 \left(\frac{\partial^2}{\partial x^2} v(x,t)\right) + \alpha A \left(2 \left(\frac{\partial}{\partial x} v(x,t)\right) \left(\frac{\partial^2}{\partial x\, \partial t} v(x,t)\right) \left(\frac{\partial^2}{\partial x^2} v(x,t)\right) + \left(\frac{\partial}{\partial x} v(x,t)\right)^2 \left(\frac{\partial^3}{\partial x^2\, \partial t} v(x,t)\right)\right) \tag{2.1}$$

Applying Galerkin pocedure using a two-mode expansion using the pinned-pinned modes

```
> mode1_Eq:= int(subs(v(x,t)=phi(t)*sin(Pi*x/l)+beta(t)*sin(2*
  Pi*x/l),eq)*sin(Pi*x/l),x=0..l) = 0;
  mode2_Eq:= int(subs(v(x,t)=phi(t)*sin(Pi*x/l)+beta(t)*sin(2*
  Pi*x/l),eq)*sin(2*Pi*x/l),x=0..l) = 0;
```

$$mode1\_Eq := -\frac{1}{8}\,\frac{1}{l^3}\left(3\,\alpha A \pi^4 \phi(t)^2 \left(\frac{d}{dt}\phi(t)\right) + 4\,\rho A \left(\frac{d^2}{dt^2}\phi(t)\right) l^4 + 16\,\alpha A \pi^4 \phi(t) \left(\frac{d}{dt}\beta(t)\right)\beta(t) + 8\,\alpha A \pi^4 \beta(t)^2 \left(\frac{d}{dt}\phi(t)\right) + 4\, T0\, \pi^2 \phi(t)\, l^2\right) = 0$$

$$mode2\_Eq := -\frac{1}{2}\,\frac{1}{l^3}\left(4\,\alpha A \pi^4 \phi(t) \left(\frac{d}{dt}\phi(t)\right)\beta(t) + 2\,\alpha A \pi^4 \phi(t)^2 \left(\frac{d}{dt}\beta(t)\right) + \rho A \left(\frac{d^2}{dt^2}\beta(t)\right) l^4 + 12\,\alpha A \pi^4 \beta(t)^2 \left(\frac{d}{dt}\beta(t)\right) + 4\, T0\, \pi^2 \beta(t)\, l^2\right) = 0 \tag{2.2}$$

Convert to first-order form where x1=phi, y1=phi_dot and x2=beta, y2 = beta_dot

```
> eq1:= subs({diff(phi(t),t,t) = diff(y1(t),t),diff(beta(t),t,
  t) = diff(y2(t),t),phi(t)=x1(t),beta(t)=x2(t),diff(phi(t),t)=
  y1(t),diff(beta(t),t)=y2(t)},mode1_Eq);
  eq2:= subs({diff(phi(t),t,t) = diff(y1(t),t),diff(beta(t),t,
  t) = diff(y2(t),t),phi(t)=x1(t),beta(t)=x2(t),diff(phi(t),t)=
  y1(t),diff(beta(t),t)=y2(t)},mode2_Eq);
```

$$eq1 := -\frac{1}{8}\,\frac{1}{l^3}\left(3\,\alpha A \pi^4 x1(t)^2 y1(t) + 4\,\rho A \left(\frac{d}{dt} y1(t)\right) l^4 + 16\,\alpha A \pi^4 x1(t)\, y2(t)\, x2(t) + 8\,\alpha A \pi^4 x2(t)^2 y1(t) + 4\, T0\, \pi^2 x1(t)\, l^2\right) = 0$$

$$eq2 := -\frac{1}{2}\,\frac{1}{l^3}\left(4\,\alpha A \pi^4 x1(t)\, y1(t)\, x2(t) + 2\,\alpha A \pi^4 x1(t)^2 y2(t) + \rho A \left(\frac{d}{dt} y2(t)\right) l^4 \right. \tag{2.3}$$

$$+ 12\,\alpha A \pi^4 x2(t)^2 y2(t) + 4\, T0\, \pi^2 x2(t)\, l^2\Big) = 0$$

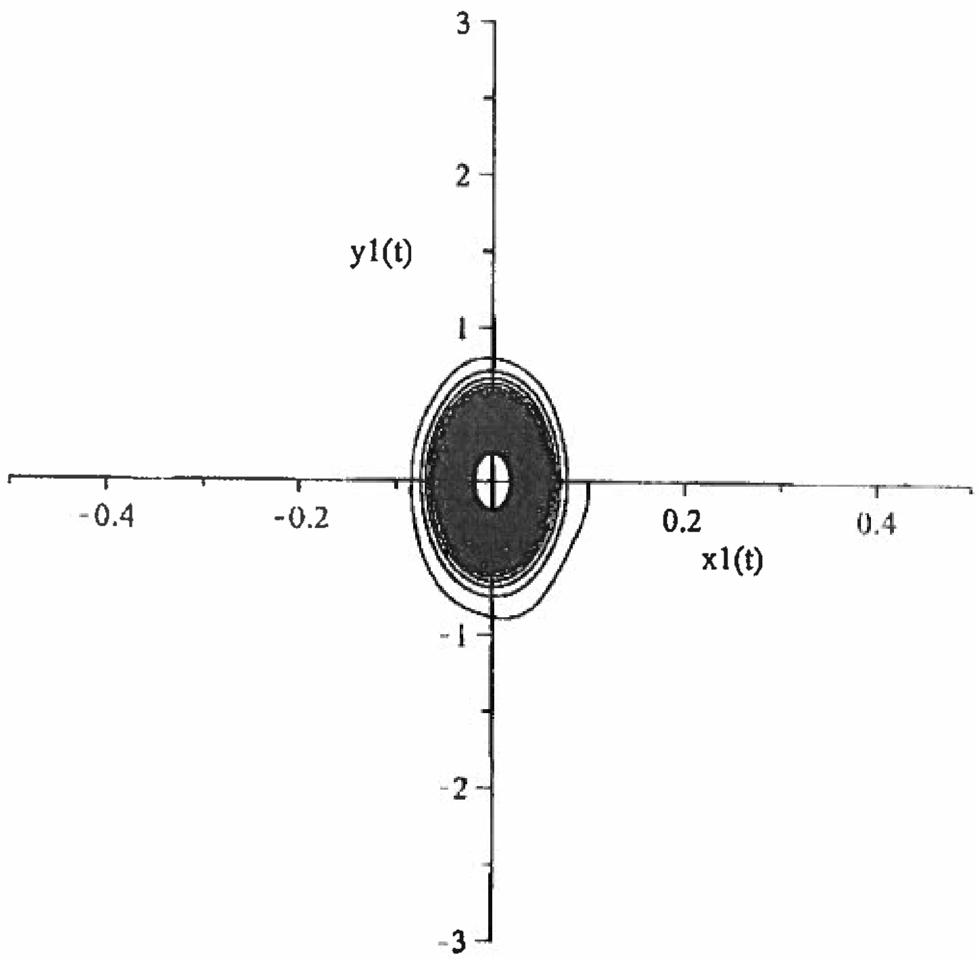
```
> l:=1; A:=1; rho:=1; T0:=10; alpha:=1.5;
```

$$l := 1$$
$$A := 1$$
$$\rho := 1$$
$$T0 := 10$$
$$\alpha := 1.5 \tag{2.4}$$

```
> with(DEtools):
```

This first scene is the phase plane for the first mode

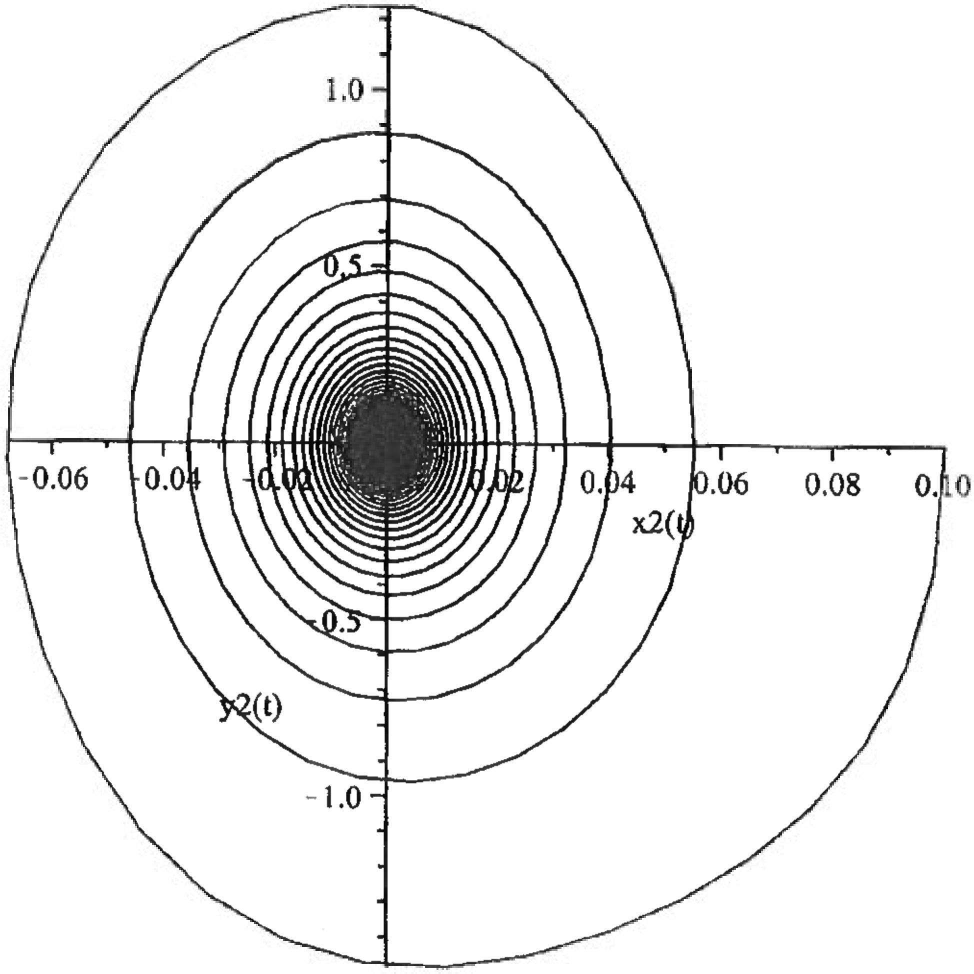
```
> DEplot([eq1,eq2,diff(x1(t),t)=y1(t),diff(x2(t),t)=y2(t)],[x1
  (t),y1(t),x2(t),y2(t)],t=0..100.0,[[x1(0)=0.1,y1(0)=0,x2(0)=
  0.1,y2(0)=0]],x1=-0.5..0.5,y1=-3..3,scene=[x1(t),y1(t)],
  stepsize=0.01,linecolor=black,thickness=1);
```



This second scene is the phase plane for the second mode

```
> DEplot([eq1,eq2,diff(x1(t),t)=y1(t),diff(x2(t),t)=y2(t)],[x1
  (t),y1(t),x2(t),y2(t)],t=0..100.0,[[x1(0)=0.1,y1(0)=0,x2(0)=
```

```
0.1,y2(0)=0]],x1=-0.5..0.5,y1=-3..3,scene=[x2(t),y2(t)],
stepsize=0.01,linecolor=black,thickness=1);
```



>

>