College of Engineering

School of Aerospace Engineering

ME 6444: Nonlinear Systems

HW1

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Problem 1

Given the following vertical excitation system

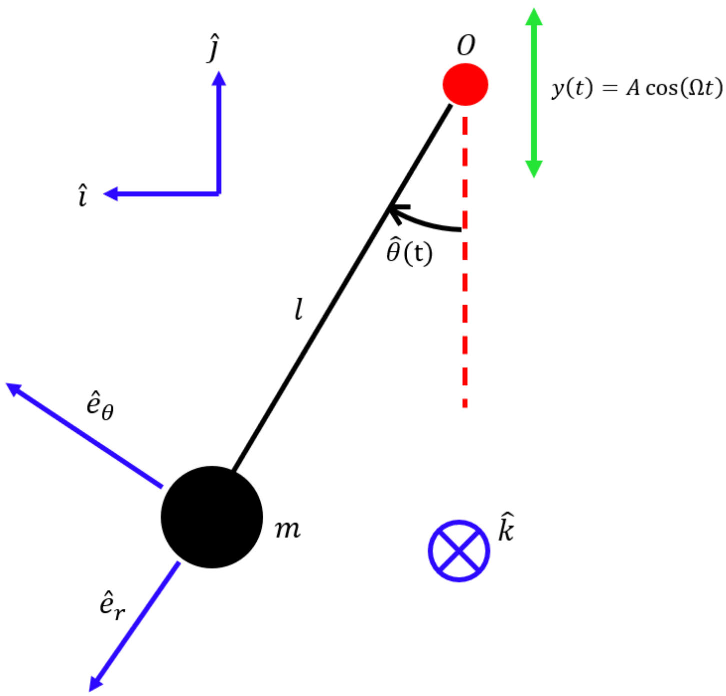

Figure 1: vertical excitation system

(a) Derive the EOM using Lagrange equations.

(b) Order EOM using $\theta \to \epsilon\theta$, $A \to \epsilon A$. Keep to and including $O(\epsilon)$.

Show that

$$\ddot{\theta} + \frac{1}{l}\left(g - \epsilon A\Omega^2 \cos(\Omega t)\right)\omega = 0.$$

Solution:
The Lagrange equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i.$$

The order n, variable q, and force Q will be expressed as

$$n = 1, \quad q_1 = \theta(t), \quad Q_1 = 0.$$

The kinematic energy (K.E.), T can be expressed as

$$T = \frac{1}{2} m\bar{v} \cdot \bar{v}.$$

Here the velocity of the mass will be

$$\begin{aligned}
\bar{v} &= \dot{y}\hat{j} + l\dot{\theta}\hat{e}_\theta \\
&= -A\Omega \sin(\Omega t)\hat{j} + l\dot{\theta}\left(\cos\theta\hat{i} + \sin\theta\hat{j}\right) \\
&= l\dot{\theta}\cos\theta\hat{i} + \left[l\dot{\theta}\sin\theta - A\Omega\sin(\Omega t)\right]\hat{j}
\end{aligned}$$

Given that

$$\begin{pmatrix}\hat{i} \\ \hat{j}\end{pmatrix} = \begin{pmatrix}\cos(\frac{\pi}{2}-\theta) & \sin(\frac{\pi}{2}-\theta) \\ -\sin(\frac{\pi}{2}-\theta) & \cos(\frac{\pi}{2}-\theta)\end{pmatrix}\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix}$$

$$\begin{pmatrix}\hat{i} \\ \hat{j}\end{pmatrix} = \begin{pmatrix}\sin\theta & \cos\theta \\ -\cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix}$$

$$\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix} = \begin{pmatrix}\sin\theta & -\cos\theta \\ \cos\theta & \sin\theta\end{pmatrix}\begin{pmatrix}\hat{i} \\ \hat{j}\end{pmatrix}$$

$$\begin{pmatrix}\hat{e}_r \\ \hat{e}_\theta\end{pmatrix} = \begin{pmatrix}\sin\theta\hat{i} - \cos\theta\hat{j} \\ \cos\theta\hat{i} + \sin\theta\hat{j}\end{pmatrix}.$$

Then

$$\begin{aligned}
T &= \frac{1}{2}m\left(l\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}m\left\{\left[l\dot{\theta}\sin\theta - A\Omega\sin(\Omega t)\right]\right\}^2 \\
&= \frac{1}{2}ml^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}ml^2\dot{\theta}^2\sin^2\theta - ml\dot{\theta}A\Omega\sin\theta\sin(\Omega t) + \frac{1}{2}mA^2\Omega^2\sin^2(\Omega t) \\
&= \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}A\Omega\sin\theta\sin(\Omega t) + \frac{1}{2}mA^2\Omega^2\sin^2(\Omega t)
\end{aligned}$$

The potential energy (P.E.), V becomes

$$V = mg\left(l - l\cos\theta + y\right)$$

Now we compute, $L = T - V$.

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - ml\dot{\theta}A\Omega\sin\theta\sin(\Omega t) + \frac{1}{2}mA^2\Omega^2\sin^2(\Omega t) - mg\left(l - l\cos\theta + y\right)$$

Then from Lagrange equation

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - mlA\Omega \sin\theta \sin(\Omega t)$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - mlA\Omega^2 \sin\theta \cos(\Omega t) - mlA\dot{\theta}\cos\theta \sin(\Omega t)$$

$$\frac{\partial L}{\partial \theta} = -ml\dot{\theta}A\Omega \cos\theta \sin(\Omega t) - mgl\sin\theta$$

Thus, the EOM becomes

$$\boxed{ml^2\ddot{\theta} - mlA\Omega^2 \sin\theta \cos(\Omega t) + mgl\sin\theta = 0.}$$

Now, if we assume the order to be

$$\theta \to \epsilon\theta, \quad A \to \epsilon A$$

and approximate the trigonometry as

$$\cos\theta \approx 1, \quad \sin\theta \approx \theta$$

we can modify the EOM in the following way

$$\epsilon ml^2\ddot{\theta} - ml(\epsilon A)\Omega^2(\epsilon\theta)\cos(\Omega t) + mgl\epsilon\theta = 0$$

$$\epsilon ml^2\ddot{theta} - ml\Omega^2 A\epsilon^2\theta\cos(\Omega t) + mgl\epsilon\theta = 0$$

$$\ddot{\theta} - \frac{1}{l}\Omega^2\epsilon A\theta\cos(\Omega t) + \frac{g}{l}\theta = 0.$$

Hence,

$$\boxed{\ddot{\theta} + \frac{1}{l}\left(g - \epsilon A\Omega^2\cos(\Omega t)\right)\theta = 0.}$$