Problem #1

$$\begin{cases} \dot{x}_1 = x_2 + x_3 \\ \dot{x}_2 = x_1 + u \\ \dot{x}_3 = x_3 \\ y_1 = x_1 + x_2 \\ y_2 = x_3 \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad D = 0$$

$$CA = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$CA^2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$Q_0 = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \text{rank}(Q_0) = 2 < 3$

Thus, this system is $\boxed{\text{unobservable}}$

Problem #2

$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + x_3 + u \\ \dot{x}_3 = x_1 + x_2 \\ y = 3x_1 + x_2 + x_3 \end{cases}$$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \quad D = 0$$

(a)

$$CA = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$$

$$CA^2 = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$$

$$Q_0 = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \operatorname{rank}(Q_0) = 2 < 3$

This system is $\boxed{\text{unobservable}}$

(b)

$$sI - A = \begin{pmatrix} s & 0 & 0 \\ -1 & s & -1 \\ -1 & -1 & s \end{pmatrix}$$

$$\det(sI - A) = s \begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} = s(s^2 - 1) = 0 \Rightarrow s = 0, \pm 1$$

$s = 0$ PBH

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \\ 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

eigenvalue of 0 is observable

$s = 1$

$$\text{PBH} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \\ 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

eigenvalue 1 is observable

$s = -1$

$$\text{PBH} \begin{pmatrix} -1 & 0 & 0 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \\ 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

eigenvalue $\boxed{\underline{-1} \text{ is unobservable}}$

CamScannerでスキャン

Tomoki Koike

Problem #3

$$\begin{cases} \dot{x}_1 = x_2 + x_3 + 3u \\ \dot{x}_2 = x_3 + u \\ \dot{x}_3 = x_2 + u \end{cases} \qquad A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

(a)

$$AB = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$A^2B = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$Q_c = \begin{pmatrix} B & AB & A^2B \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 3 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{rank}(Q_c) = 2 < 3$$

This system is $\boxed{\text{uncontrollable}}$

(b)

$$sI - A = \begin{pmatrix} s & -1 & -1 \\ 0 & s & -1 \\ 0 & -1 & s \end{pmatrix}$$

$$\det(sI-A) = s\begin{vmatrix} s & -1 \\ -1 & s \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ 0 & s \end{vmatrix} - \begin{vmatrix} 0 & s \\ 0 & -1 \end{vmatrix}$$

$$= s(s^2-1) = 0 \rightarrow s = 0, \pm 1$$

$s = 0$

$$\text{PBH} \quad \begin{pmatrix} 0 & -1 & -1 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

eigenvalue 0 is controllable

$s=1$

$$\text{PBH} \begin{pmatrix} 1 & -1 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

eigenvalue 1 is controllable

$s=-1$

$$\text{PBH} \begin{pmatrix} -1 & -1 & -1 & 3 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 1 & -3 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

eigenvalue $\boxed{-1 \text{ is uncontrollable}}$

Problem #4

$$\begin{cases} x_1(k+1) = x_2(k) + x_3(k) \\ x_2(k+1) = x_3(k) \\ x_3(k+1) = x_2(k) + u(k) \end{cases} \quad A_d = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \; B_d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

if $T = 1$ (sample time)

since from problem #3 we know that A_d, B_d gives a stabilizable system we can choose arbitrary poles.

We will choose 0 as our poles.

now let $K = (k_1 \; k_2 \; k_3)$

$A_d + B_d K$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (k_1 \; k_2 \; k_3)$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ k_1 & k_2+1 & k_3 \end{pmatrix}$$

$sI - A_d - B_d K$

$$= \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ k_1 & k_2+1 & k_3 \end{pmatrix} = \begin{pmatrix} s & -1 & -1 \\ 0 & s & -1 \\ -k_1 & -k_2-1 & s-k_3 \end{pmatrix}$$

$\det(sI - A_d - B_d K)$

$$= s \begin{vmatrix} s & -1 \\ -k_2-1 & s-k_3 \end{vmatrix} + \begin{vmatrix} 0 & -1 \\ -k_1 & s-k_3 \end{vmatrix} - \begin{vmatrix} 0 & s \\ -k_1 & -k_2-1 \end{vmatrix}$$

$$= s[s(s-k_3)-(k_2+1)] - k_1 - k_1 s$$
$$= s^2(s-k_3) - (k_1+k_2+1)s - k_1$$
$$= s^3 - k_3 s^2 - (k_1+k_2+1)s - k_1$$

if selected poles are all zeros

$$\begin{aligned} k_3 &= 0 \\ k_1+k_2+1 &= 0 \\ k_1 &= 0 \end{aligned} \Rightarrow \begin{pmatrix} k_1 = 0 \\ k_2 = -1 \\ k_3 = 0 \end{pmatrix}$$

Then state feedback controller becomes

$$u = Kx = (0 \quad -1 \quad 0)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\boxed{u = -x_2(k)}$$

Problem #5

$$\ddot{y} - 2y = \ddot{u} - u$$

$$\xrightarrow{\mathcal{L}} \quad s^2\hat{y} - 2\hat{y} = s^2\hat{u} - \hat{u}$$

$$\hat{G} = \frac{\hat{y}}{\hat{u}} = \frac{s^2-1}{s^2-2} = \frac{s^2-2+1}{s^2-2} = \frac{1}{s^2-2} + 1$$

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad D = 1$$

let $K = (k_1 \;\; k_2)$ & $L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$

$$A + BK = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}(k_1 \;\; k_2) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 2+k_1 & k_2 \end{pmatrix}$$

$$\det(sI - A - BK) = \begin{vmatrix} s & -1 \\ -2-k_1 & s-k_2 \end{vmatrix} = s(s-k_2) - 2 - k_1 = s^2 - k_2 s - k_1 - 2$$

for this to be asymptotically stable

$$k_2 < 0 \quad \text{and} \quad -k_1 - 2 > 0$$

$$\therefore \; k_2 < 0 \quad \text{and} \quad k_1 < -2$$

$$A + LC = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}(1 \;\; 0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} l_1 & 0 \\ l_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} l_1 & 1 \\ l_2+2 & 0 \end{pmatrix}$$

$$\det(sI - A - LC) = \begin{vmatrix} s-l_1 & -1 \\ -l_2-2 & s \end{vmatrix} = s(s-l_1) - l_2 - 2$$

$$= s^2 - l_1 s - l_2 - 2$$

for this to be asymptotically stable

$$l_1 < 0 \quad \text{and} \quad -l_2 - 2 > 0$$

$$\therefore l_1 < 0 \quad \text{and} \quad l_2 < -2$$

Thus,

$K = (k_1 \ k_2)$ where $k_2 < 0$ and $k_1 < -2$

$L = \begin{pmatrix} l_1 \\ l_2 \end{pmatrix}$ where $l_1 < 0$ and $l_2 < -2$

The controller is

$$\dot{\hat{x}} = [A + BK + L(C + DK)]\hat{x} - Ly$$

$$u = K\hat{x}$$

Problem#6

$$\begin{cases} \dot{x}_1 = -x_1 \\ \dot{x}_2 = -2x_2 \\ z = x_1 + x_2 \end{cases} \quad A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad C = (1 \ \ 1) \quad D = 0$$

$$Q = C'C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1 \ \ 1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

solve Lyapunov equation

$$PA + A'P + Q = 0 \quad \text{where} \quad P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}$$

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -P_{11} & -2P_{12} \\ -P_{12} & -2P_{22} \end{pmatrix} + \begin{pmatrix} -P_{11} & -P_{12} \\ -2P_{12} & -2P_{22} \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

$$\Rightarrow \quad \begin{aligned} -2P_{11} + 1 &= 0 \\ -3P_{12} + 1 &= 0 \\ -4P_{22} + 1 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} P_{11} &= \frac{1}{2} \\ P_{12} &= \frac{1}{3} \\ P_{22} &= \frac{1}{4} \end{aligned}$$

$$\therefore P = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}$$

now

$$\int_0^\infty z(t)^2 dt = x(0)' P x(0) = (1 \ \ 1)\begin{pmatrix} \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \left(\frac{5}{6} \ \ \frac{7}{12}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{10}{12} + \frac{7}{12} = \frac{17}{12} \qquad \boxed{\frac{17}{12}}$$