Due: November 7, 2021

Assignment #5

1. A rocket is launched from the origin (0,0) with velocity $u(0)$ parallel to the x-axis and $v(0)$ parallel to the y-axis. Assuming a constant thrust, we wish to find the thrust direction $\theta(t)$ for minimum time to the point $(x_\mathrm{f}, 0)$. The equations of motion can be written as

$$\dot{u}(t) = \cos\theta(t)$$
$$\dot{v}(t) = \sin\theta(t)$$
$$\dot{x}(t) = u(t)$$
$$\dot{y}(t) = v(t)$$

   (a) Show that the minimum final time is the smallest positive real root of the quartic equation

$$t_\mathrm{f}^4 - 4t_\mathrm{f}^2 + 8x_\mathrm{f}\cos\gamma_0 t_\mathrm{f} - 4x_\mathrm{f}^2 = 0$$

   where $x(t_\mathrm{f}) = x_\mathrm{f}, u(0) = \cos\gamma_0$ and $v(0) = \sin\gamma_0$.

   (b) What is the optimal thrust direction profile $\theta^\star(t)$ in terms of t_f, γ_0 and x_f?

2. Consider the following system of differential equations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u \end{aligned}$$

subject to initial conditions $x_1(0) = x_2(0) = 0$. We want to find an optimal control $u(t)$ to minimize the cost

$$\mathcal{J} = \frac{1}{2}\int_0^{t_f} u^2(t)\,\mathrm{d}t$$

for the following cases.

   (a) When $t_f = 3$ and the final state is given by $x(3) = [1\ 2]^T$.

   (b) When $t_f = 3$, $x_1(3) = 1$ and $x_2(3)$ is not specified.

   (c) When $t_f = 3$ and the final state is not specified explicitly, but we would like it to be as close as possible to $x(3) = [1\ 2]^T$. If the optimal trajectory you get is not "sufficiently" close to the required value, how would you modify the cost to achieve the desired accuracy? How does the overall cost change when the optimal trajectories satisfy the boundary condition at t_f with increasing accuracy? Compare these results with the case (a) above.

   (d) When $t_f = 3$ and the final state should be on the line $2x_1 + 5x_2 = 20$.

   (e) When t_f is not specified and $x(t_f)$ should be on the (moving) line

$$2x_1 + 5x_2 = 20 + \frac{t^2}{2}$$

For each of these cases, derive the necessary conditions for the optimal control.

Solve *analytically* the necessary conditions in order to calculate the optimal control and the corresponding optimal trajectories for each case. Plot the optimal trajectories vs. time and the optimal paths in the $x_1 - x_2$ plane. Verify that the boundary conditions are satisfied. Verify that the Hamiltonian is constant along the optimal trajectories. What is the optimal cost for each case?

3. Consider the system

$$\dot{x} = \cos\theta + u(y)$$
$$\dot{y} = \sin\theta$$

where

$$u(y) = -\alpha(3y - y^3)$$

(a) Compute the minimum-time paths from $x(0) = y(0) = 1$ to the origin. In particular, show that the optimal strategy for $\theta(t)$ satisfies the following differential equation

$$\dot{\theta} = 3\alpha(1-y^2)\cos^2\theta.$$

What role does α play in the solution?

(b) Find (numerically) the solution for $\alpha = 0.2$. Hint: First show that the initial and final values of θ are related by the expression $\sec\theta(0) - 2\alpha = \sec\theta(t_f)$.

4. Write a MATLAB code based on the document uploaded to Canvas to solve the following problem. A ship is located at the point $(x_0, y_0) = (-20, 0)$ ml at time $t = 0$ when it encounters a medical emergency and it has to reach the shore as soon as possible. It is known that there is a small city at the location $(x_1, y_1) = (-15, 35.5)$ ml with a medical center. As the captain of the ship, you are to determine the fastest possible route to the city. It is assumed that the speed of the ship with respect to the water is constant, $v = 15\,\text{ml/hr}$. You also know the speed and direction of the sea currents in the area, which are given to you from a meteorological satellite as $\vec{v}_c = u(x,y)\,\hat{\mathbf{i}} + v(x,y)\,\hat{\mathbf{j}}$.

(a) Derive the necessary conditions for the optimal control strategy, and calculate the optimal path and the time to reach the city, assuming that the currents are constant, given by

$$\vec{v}_c = 2\hat{\mathbf{i}} - 6\hat{\mathbf{j}}$$

Plot the optimal path in the $x - y$ plane along with the vectors showing the direction of the currents.

(b) When you are about to start your dash to the shore, you learn that the doctor in the medical center will be able to fly by helicopter to any point at the shore to pick up the patient. Find the new optimal path and the time to reach the shore, assuming that the contour of the shoreline is known to be

$$\psi(x,y) = 25 - 0.25x - 0.002x^3 - y = 0$$

Plot the optimal path in the $x - y$ plane along with the vectors showing the direction of the currents.

(c) An update of the meteorological data from the satellite shows that strong winds have developed in the area and that the currents have changed significantly. The new currents are

$$\vec{v}_c = -(y-50)\,\hat{\mathbf{i}} + 2(x-15)\,\hat{\mathbf{j}}$$

Recalculate the optimal control and plot the optimal path in the $x - y$ plane along with the vectors showing the direction of the currents. Plot the optimal steering angle history $\theta^*(t)$.

In all cases, plot the Hamiltonian and verify that remains zero for all time.

5. A man has a quantity of savings $S > 0$ at a bank. He has no other income and he is trying to find a way to spend all his money of the next time period $[0, t_\mathrm{f}]$ in order to maximize his enjoyment. Assume that its instantaneous rate of enjoyment is

$$E = 2\sqrt{r}$$

where r is the spending rate of his fortune. Future enjoyment is counted less today, so he will try to maximize

$$J(r) = \int_0^{t_\mathrm{f}} \exp(-\beta t) E(t) \mathrm{d}t = \int_0^{t_\mathrm{f}} 2 \exp(-\beta t) \sqrt{r(t)} \mathrm{d}t.$$

In the meantime, the bank gives him some interest proportional to its total capital $x(t)$. This gives

$$\dot{x}(t) = \alpha x(t) - r(t)$$

with boundary conditions $x(0) = S$ and $x(t_\mathrm{f}) = 0$. Assume that $\alpha > \beta > \alpha/2 > 0$.

Find the optimal spending policy $r(t)$ and the optimal capital history $x(t)$.