COLLEGE OF ENGINEERING
SCHOOL OF AEROSPACE ENGINEERING

AE 6511: OPTIMAL GUIDANCE AND CONTROLS

HW1

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Problem 1

For each of the following functions $f : \mathscr{D} \mapsto \mathbb{R}$ determine whether a minimum and/or an infimum of $f(\mathscr{D})$ exists and explain why or why not Weierstrass's theorem applies:

i) $\mathscr{D} = (-1, 1), f(x) = x^2$.

ii) $\mathscr{D} = (1.2], f(x) = \dfrac{1}{1-x}$.

iii) $\mathscr{D} = [0, 1], f(0) = 0, f(x) = 1, x \in (0, 1]$.

Solution:
i) For the function $f(x) = x^2$ there exists a minimum and infimum

$$\boxed{\min f = \inf f = f(0) = 0}$$

However, because $\mathscr{D}$ is not closed Weierstrass's theorem does not apply.

ii) For the function $f(x) = \dfrac{1}{1-x}$, $f(-1) = -\infty$ and this is not an element of $f(\mathscr{D})$. Thus, this f does not have a minimum or infimum. Moreover, because $\mathscr{D}$ is not closed Weierstrass's theorem does not apply.

iii) For this discontinuous function $f(x)$, there exists a minimum of

$$\boxed{\min f = f(0) = 0}.$$

However, because f is not continuous, the Weierstrass's theorem does not apply.

Problem 2

Determine $\text{fcone}(\mathscr{D}, (x_0, y_0))$ for the following sets $\mathscr{D} \subset \mathbb{R}^2$ and $(x_0, y_0) \in \mathscr{D}$.

i) $\mathscr{D} = \{(x, y) : y \geq 0\}$ and $(x_0, y_0) = (4, 0)$.

ii) $\mathscr{D} = \{(x, y) : x^2 + y^2 \leq 1\}$ and $(x_0, y_0) = (1, 0)$.

Solution:
i)

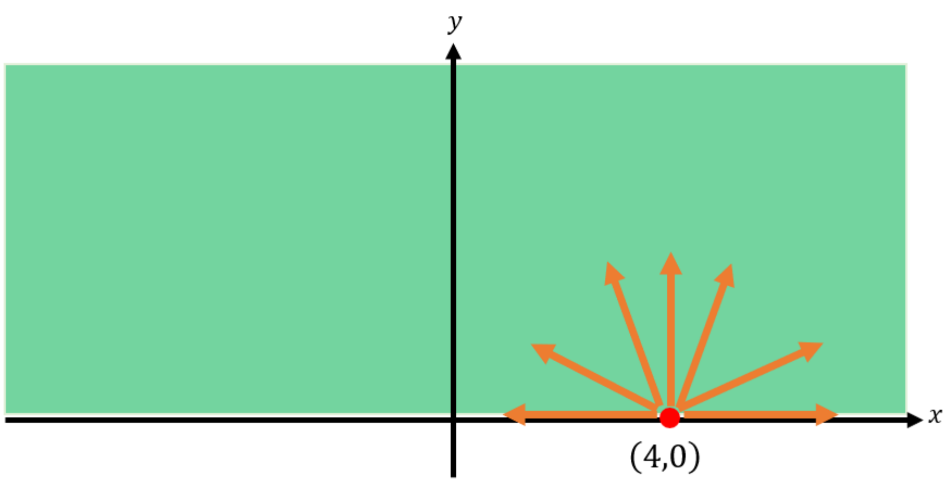

Figure 1: $\mathscr{D} = \{(x, y) : y \geq 0\}$

From Figure 1 we can see that the direction of the orange arrows are the directions the $fcone$ is allowed to move in. Hence,

$$\boxed{fcone(\mathscr{D}, (x_0, y_0)) = \{\xi_1 \in \mathbb{R}, \xi_2 \geq 0\}}$$

ii)

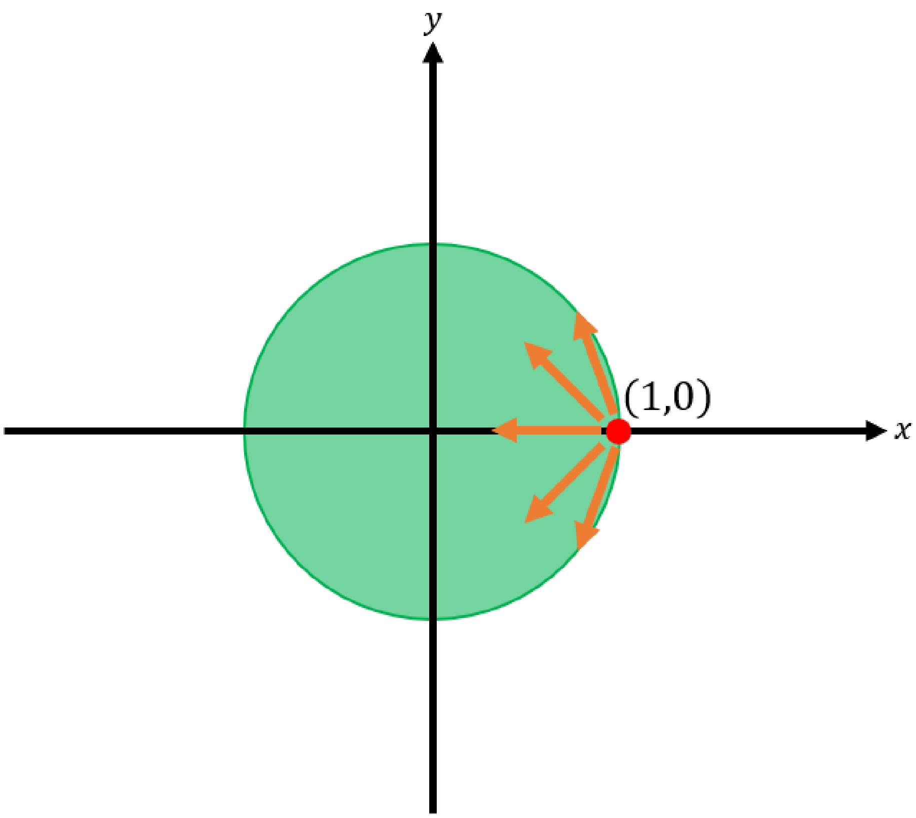

Figure 2: $\mathscr{D} = \{(x, y) : x^2 + y^2 \leq 1\}$

From Figure 2, we can see that the direction of the orange arrows are the directions the $fcone$ is allowed to move in. Hence,

$$\boxed{fcone(\mathscr{D}, (x_0, y_0)) = \{\xi_1 < 0, \xi_2 \in \mathbb{R}\}}$$

Problem 3

Let $f : \mathbb{R}^2 \mapsto \mathbb{R}$ be given by $f(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$. Evaluate $D_+ f((0,0); (\xi_1, \xi_2))$.

Solution:

$$\begin{aligned}
D_+ f((0,0); (\xi_1, \xi_2)) &= \lim_{\alpha \to 0^+} \frac{1}{\alpha} \left[f(0 + \alpha\xi_1, 0 + \alpha\xi_2) - f(0,0) \right] \\
&= \lim_{\alpha \to 0^+} \frac{1}{\alpha} \left[\sqrt{\alpha^2 \xi_1^2 + \alpha^2 \xi_2^2} \right] \\
&= \lim_{\alpha \to 0^+} \frac{|\alpha|}{\alpha} \sqrt{\xi_1^2 + \xi_2^2} \\
&= \sqrt{\xi_1^2 + \xi_2^2} > 0
\end{aligned}$$

Hence,

$$\boxed{\sqrt{\xi_1^2 + \xi_2^2}.}$$

Problem 4

Minimize the function $f : \mathscr{D} \mapsto \mathbb{R}$

$$f(x_1, x_2) = x_1^3 + x_2^3$$

where $\mathscr{D} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\}$.

Solution:
Take the first derivative and we get

$$f'(x_0) = \begin{bmatrix} 3x_1^2 \\ 3x_2^2 \end{bmatrix} \to (0,0) \text{ is on } bd(\mathscr{D})$$

The second derivative is

$$f''(x_0) = \begin{bmatrix} 6x_1 & 0 \\ 0 & 6x_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \to \text{ this is not a positive definite matrix}$$

Now we know that

$$f'(x_0)\xi = 0 \text{ where } \forall \xi \in fcone(\mathscr{D}, x_0) = \{\xi_1 \geq 0, \xi_2 \geq 0\}$$

and

$$\xi^T f''(x_0)\xi = \xi^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \xi = 0 \text{ where } \forall \xi \in fcone(\mathscr{D}, x_0)$$

This implies that (0,0) may be a local minimizer. Next, we check if there are any possible minimizers on the boundary. Let points on the boundary be $(\alpha, 0)$ and $(0, \beta)$ where $\alpha, \beta > 0$.

$$fcone(\mathscr{D}, (\alpha, 0)) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \in \mathbb{R}, \xi_2 \geq 0\}$$

$$\begin{aligned} f'(\alpha, 0)\xi &= 3x_1^2\xi - 3x_2^2\xi_2 \\ &= 3\alpha^2\xi_1 \end{aligned}$$

and $3\alpha^2\xi_1 \geq 0$ is not possible for $\forall \xi_1 \in \mathbb{R}$. Thus, not a candidate local minimizer. Similarly,

$$fcone(\mathscr{D}, (0, \beta)) = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1 \geq 0, \xi_2 \in \mathbb{R}\}$$

$$\begin{aligned} f'(0, \beta)\xi &= 3x_1^2\xi - 3x_2^2\xi_2 \\ &= -3\beta^2\xi_2 \end{aligned}$$

and $-3\beta^2\xi_2 \geq 0$ is not possible for $\forall \xi_1 \in \mathbb{R}$. Thus, not a candidate local minimizer. Hence, to minimize the given problem, we use the single candidate local minimizer of (0, 0). Hence, the minimization problem is solved as

$$\boxed{f(0,0) = 0}$$

From the plot below we can tell that $(0,0)$ is indeed the global minimizer.

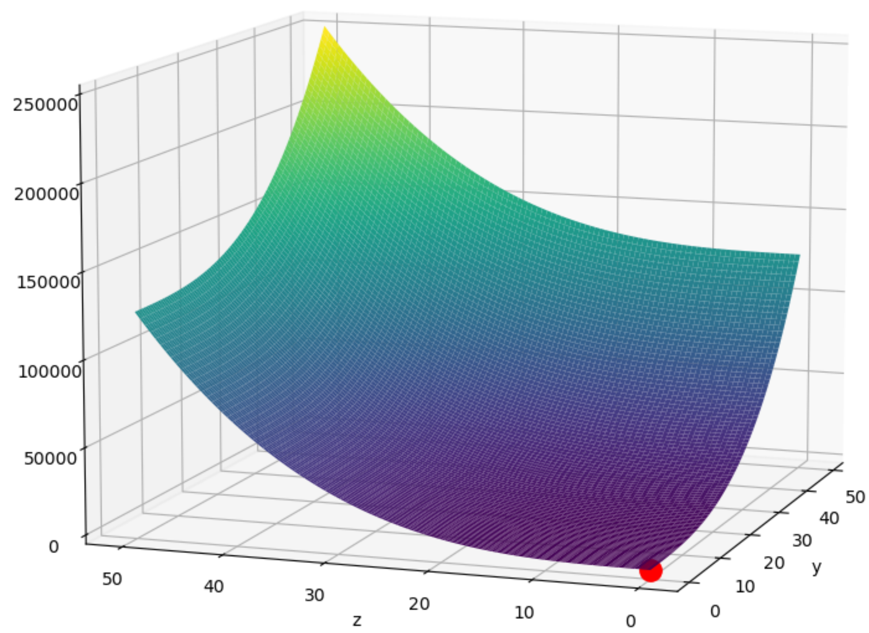

Figure 3: Plot of $f(x_1, x_2) = x_1^3 + x_2^3$ within the domain

The code to plot this in the subsection of Appendix: Problem 4: Python Code.

Problem 5

Assume a steady and level flight of an airplane and consider the propulsive thrust given by

$$T = \frac{1}{2}\rho V^2 S C_{D_{par}} + \frac{KW^2}{\frac{1}{2}\rho V^2 S},$$

where ρ is air density, V is aircraft velocity, $C_{D_{par}}$ is the zero-lift (parasitic) drag coefficient, K is the drag polar constant, and S is wing surface area. The drag coefficient C_D is given by the drag polar

$$C_D = C_{D_{par}} + KC_L^2,$$

and the lift coefficient is

$$C_L = \frac{W}{\frac{1}{2}\rho V^2 S}.$$

Consider the problem of finding the aircraft velocity V that minimizes the thrust T. Determine whether this problem is convex, and find all local and global minimizers and the corresponding values of T, C_L, C_D, and C_L/C_D.

Solution:
Keep in mind that all the constant parameters are positive. Since, the condition is a steady level flight we can assume that $V > 0$ due to the discontinuity at $V = 0$. Let $T : \mathscr{D} \mapsto \mathbb{R}$, $\mathscr{D} = \{V \in \mathbb{R} : 0 < V < \infty\}$, and let $T = T(V)$. If we take the second derivative of this function respect to the velocity we obtain the following.

$$T'(V) = \rho V S C_{D_{par}} - \frac{4KW^2}{\rho V^3 S}$$

$$T''(V) = \rho S C_{D_{par}} + \frac{12kW^2}{\rho V^4 S}$$

We can see that

$$\boxed{T''(V) > 0 \qquad \forall V \in \mathscr{D}.}$$

Hence, the function is strictly convex. If, the function $T(V)$ is strictly convex, we know that some V_0, which are critical points of the solution $T'(V) = 0$, will include a global minimizer. And for a strictly convex function the global minimizer also implies the solution to be a local minimizer. Thus,

$$\rho V S C_{D_{par}} - \frac{4KW^2}{\rho V^3 S} = 0$$

$$V^4 = \frac{4KW^2}{\rho^2 S^2 C_{D_{par}}}$$

$$V = \pm\left(\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D_{par}}}}\right)^{1/2}$$

$$\boxed{\therefore V_0 = \left(\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D_{par}}}}\right)^{1/2}.}$$

Plugging this into, $T(V)$ we obtain the global minimum.

$$T(V_0) = \frac{1}{2}\rho \pm \left(\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D_{par}}}}\right) SC_{D_{par}} + \frac{2KW^2}{\rho S\left(\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D_{par}}}}\right)}$$

$$= 2W\sqrt{KC_{D_{par}}}$$

$$\boxed{\therefore T_{min} = 2W\sqrt{KC_{D_{par}}}.}$$

Next, we compute the corresponding coefficients.

The lift coefficient:

$$\boxed{C_L(V_0) = \frac{W}{\frac{1}{2}\rho S\left(\frac{2W}{\rho S}\sqrt{\frac{K}{C_{D_{par}}}}\right)} = \sqrt{\frac{C_{D_{par}}}{K}}.}$$

The drag coefficient:

$$\boxed{C_D(V_0) = C_{D_{par}} + K\left(\frac{C_{D_{par}}}{K}\right) = 2C_{D_{par}}.}$$

The maximum lift to drag ratio:

$$\boxed{\frac{C_L}{C_D}(V_0) = \sqrt{\frac{C_{D_{par}}}{K}}/2C_{D_{par}} = \frac{1}{\sqrt{4C_{D_{par}}K}}.}$$

At steady level flight, we have $T = D$ and the graph is as follows for $V > 0$.

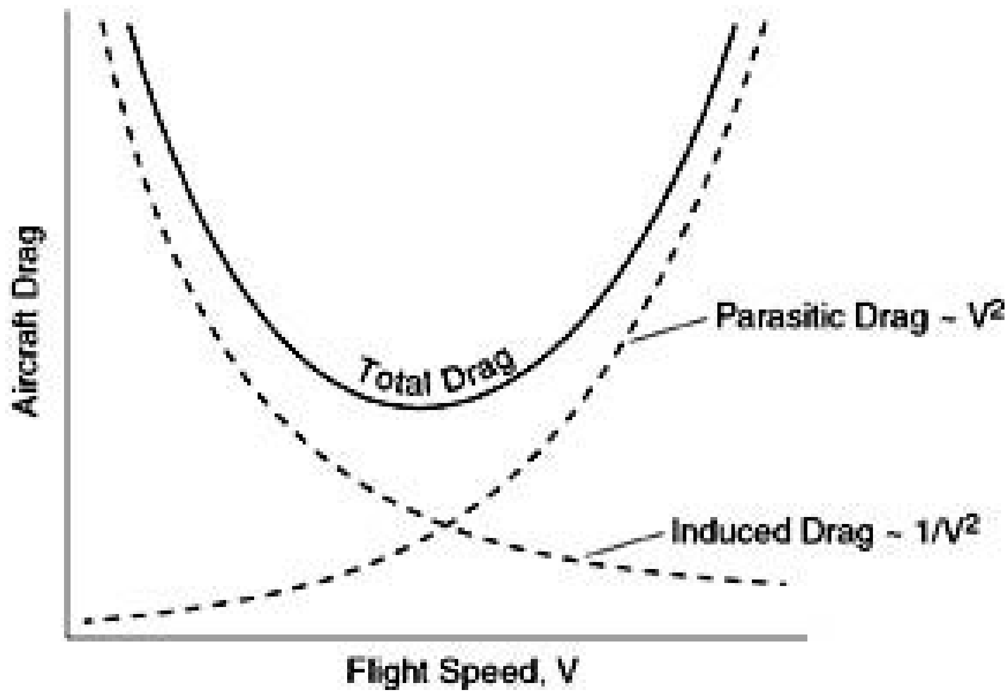

Figure 4: Drag curve at steady level flight. credit: MIT open courseware

This figure is from MIT courseware (https://ocw.mit.edu/ans7870/16/16.unified/propulsionS04/UnifiedPropulsion4/UnifiedPropulsion4.htm).

Problem 6

Consider the function $f : \mathbb{R}^2 \mapsto \mathbb{R}$ given by

$$f(y,z) = (z - py^2)(z - qy^2)$$

where $0 < p < q$.

(a) Show that $x_0 = (0,0)$ is a local minimizer of f along every line that passes through $(0,0)$, that is, for all $h \in \mathbb{R}^2$, the function $g(a) = f(x_0 + ah)$ is locally minimized by $a = 0$.

(b) Show that $f'(x_0) = 0$.

(c) Show that x_0 is <u>not</u> a local minimizer of f. (Hint: If $p < m < q$, then $f(y, my^2) < 0$ for $y \neq 0$ while $f(0,0) = 0$.).

(d) Plot the function f using $p = 1$, $q = 2$ to illustrate the fact that for this function x_0 is not a local minimizer even though x_0 is a local minimizer along every line through the origin.

This example demonstrates why working with the Gâteaux differential (which looks at the derivative of a function along one direction at a time) may lead to erroneous conclusions when solving optimization problems. This is more that a theoretical curiosity. A numerical algorithm based on screening potential minimizers by searching points where the Gâteaux differential zero will yield erroneous results. In this case, such an algoritm will return the origin as a strict local minimizer for this function whereas, as seen above, the origin is not a local minimizer.

Solution:
(a) Assuming the local minimizer of $x_0 = (0,0)$,

$$f(0,0) = 0.$$

Taking the first derivative of $f(y,z)$ we get

$$f'(y,z) = \begin{bmatrix} -2auz - 2pyz + 4pqy^3 & 2z - qy^2 - py^2 \end{bmatrix}.$$

If we take the second derivative we get

$$f''(y,z) = \begin{bmatrix} -2z - 2pz + 12pqy^2 & -2qy - 2py \\ -2qy - 2py & 2 \end{bmatrix}$$

and at (0, 0), we have $f''(0,0)$

$$f''(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

and this matrix is positive semi-definite. Hence, (0, 0) could be a local minimizer. Next, let $h \in \mathbb{R}^2$ and $g(a) = f(x_0 + ah)$.

$$\begin{aligned} g(a) &= f(ah) \\ &= (ah_2 - pa^2h_1^2)(ah_2 - qa^2h_1^2) \\ &= a^2(h_2 - pqh_1^2)(h_2 - qah_1^2). \end{aligned}$$

Also,

$$g'(a) = 2a(h_2 - pah_1^2)(h_2 - qah_q^2) + a^2(-ph_1^2)(h_2 - qah_1^2) + a^2(h_2 - pqh_1^2)(-qh_1^2).$$

Thus, $g'(0) = 0$. Furthermore,

$$\begin{aligned} g''(a) = {}& 2(h_2 - pah_1^2)(h_2 - qah_1^2) + 2a(-ph_1^2)(h_2 - qah_1^2) + 2a(h_2 - pah_1^2)(-qh_1^2) \\ &+ 2a(-ph_1^2)(h_2 - qah_1^2) + a^2(-ph_1^2)(-qh_1^2) + 2a(h_2 - pah_1^2)(-qh_1^2) + a^2(-ph_1^2)(-qh_1^2). \end{aligned}$$

Thus, $g''(0) = 2h_2^2 > 0$ if $h_2 \neq 0$. Then, if $h_2 = 0$, $g(a) = pqa^4h_1^4$, which is minimized at $a = 0$. Hence, $x_0 = (0,0)$ is a local minimizer of f along every line that passes through $(0,0)$.

(b) Taking the first derivative of $f(y,z)$ we get

$$f'(y,z) = \begin{bmatrix} -2auz - 2pyz + 4pqy^3 & 2z - qy^2 - py^2 \end{bmatrix}.$$

Then,

$$\begin{aligned} 2z - qy^2 - py^2 &= 0 \\ z &= \frac{p+q}{2}y^2 \end{aligned}$$

then substitute the z into the other component of $f'(y,z)$, and we obtain

$$\begin{aligned} -2y\frac{p+q}{2}y^2(p+q) + 4pqy^3 &= 0 \\ y^3(4pq - (p+q)^2) &= 0 \end{aligned}$$

Hence, $f'(y,z) = 0$ when $y = z = 0$. Thus, $f'(x_0) = 0$.

(c) Let $p < m < q$ then when $y \neq 0$ we compute

$$\begin{aligned} f(y, my^2) &= (my^2 - py^2)(my^2 - qy^2) \\ &= y^2(m-p)(m-q). \end{aligned}$$

Since, $m - p > 0$ and $m - q < 0$

$$y^2(m-p)(m-q) < 0.$$

Thus,

$$f(y, my^2) < f(x_0).$$

(d) The plot of f when $p = 1andq = 1$ is the following

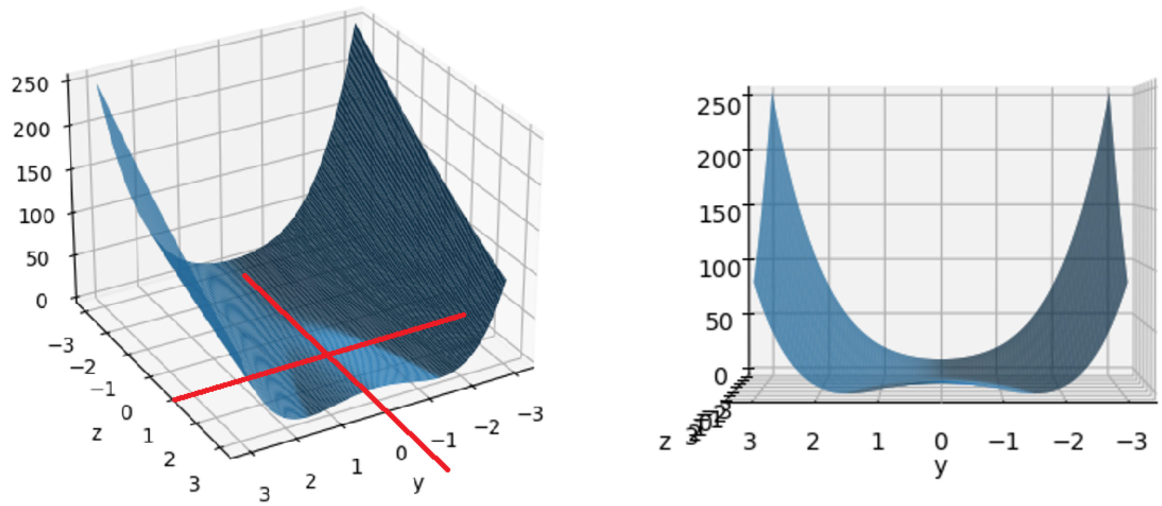

Figure 5: Surface plot of $f(y, z)$

The code to plot this is in the subsection of Appendix: Problem 6: Python Code.

Appendix

7.1 Problem 4: Python Code

```
import matplotlib.pyplot as plt
import numpy as np


y = np.linspace(0, 50, 100)
z = np.linspace(0, 50, 100)

Y, Z = np.meshgrid(y, z)

def f(a, b):
    return a**3 + b**3

fig = plt.figure()
ax = plt.axes(projection='3d')
ax.plot_surface(Y, Z, f(Y, Z), rstride=1, cstride=1,
                edgecolor='none', cmap='viridis')
ax.plot(0, 0, '.r', markersize=25)
ax.set_xlabel('y')
ax.set_ylabel('z')
plt.show()
```

7.2 Problem 6: Python Code

```
import matplotlib.pyplot as plt
import numpy as np


y = np.linspace(-3, 3, 100)
z = np.linspace(-3, 3, 100)

Y, Z = np.meshgrid(y, z)

def f(a, b):
```

```
    return (b - a**2) * (b - 2 * a**2)

fig = plt.figure()
ax = plt.axes(projection='3d')
ax.plot_surface(Y, Z, f(Y, Z), rstride=1, cstride=1,
                edgecolor='none')
ax.set_xlabel('y')
ax.set_ylabel('z')
plt.show()
```