College of Engineering
School of Aeronautics and Astronautics

AAE 421
Flight Dynamics and Controls

HW 5
Modern Control Theory

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Problem 1 (20pts)

For the differential equations that follow, rewrite the equations in the state-space formulation.

(1)

$$\ddot{c} + 2\zeta\omega_n\dot{c} + \omega_n^2 c = r$$

Let,

$$x_1 \coloneqq c, \qquad x_2 \coloneqq \dot{c}, \qquad u \coloneqq r$$

Then,

$$\begin{pmatrix} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -\omega_n^2 x_1 - 2\zeta\omega_n x_2 + u \end{pmatrix}$$

Thus, the state space representation becomes

$$\dot{x} \quad = \quad \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \ .$$

(2)

$$\ddot{\theta} + 3\dot{\theta} + 2\dot{\alpha} + 5\alpha = -6\delta_e$$

$$\dot{\alpha} + 4\alpha - 15\dot{\theta} = -3\delta_e$$

Reorganize the equation as

$$\ddot{\theta} = -3\dot{\theta} - 2\dot{\alpha} - 5\alpha - 6\delta_e$$

$$\dot{\alpha} = -4\alpha + 15\dot{\theta} - 3\delta_e$$

Plug $\dot{\alpha}$ of the second equation into the first equation and we obtain

$$\ddot{\theta} = -33\dot{\theta} + 3\alpha$$

$$\dot{\alpha} = -4\alpha + 15\dot{\theta} - 3\delta_e$$

Let,

$$x_1 \coloneqq \theta, \qquad x_2 \coloneqq \dot{\theta}, \qquad x_3 \coloneqq \alpha, \qquad \mathrm{u} \coloneqq \delta_\mathrm{e}$$

Then,

$$\begin{pmatrix} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -33x_2 + 3x_3 \\ \dot{x}_3 & = & 15x_2 - 4x_3 - 3u \end{pmatrix}.$$

Thus, the state space representation becomes

$$\dot{x} \quad = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -33 & 3 \\ 0 & 15 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} u\ .$$

Problem 2 (20 pts)

The transfer functions for a feedback control system follow. Determine the states space equations for the closed-loop system.

(1)

$$G(s) = \frac{k}{s(s+2)(s+3)}, \qquad H(s) = 1$$

The transfer function of the closed loop feedback system is

$$J(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{k}{s(s+2)(s+3) + k} = \frac{k}{s^3 + 5s^2 + 6s + k} .$$

Thus, the state space realization becomes

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -6 & -5 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = [k \quad 0 \quad 0], \qquad D = 0$$

Thus, the state space equations are

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -6 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [k \quad 0 \quad 0]x .$$

(2)

$$G(s) = \frac{k}{s(s^2 + 8s + 10)}, \qquad H(s) = 1$$

The transfer function of the closed loop feedback system is

$$J(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{k}{s(s^2 + 8s + 10) + k} = \frac{k}{s^3 + 8s^2 + 10s + k} .$$

Thus, the state space realization becomes

$$\mathrm{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -8 & -10 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad C = [k \quad 0 \quad 0], \qquad D = 0$$

Thus, the state space equations are

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -k & -8 & -10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [k \quad 0 \quad 0]x \ .$$

Problem 3 (20 pts)

The state space equations are given as follows:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\mathrm{u}$$

$$\mathrm{y} = \begin{bmatrix}1 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

(1) Determine if the system is controllable.

The A and B matrices of this system are

$$A = \begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 2\end{bmatrix}, \qquad B = \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$

Then the controllability matrix becomes

$$Q_c = \begin{bmatrix}B & AB & A^2B\end{bmatrix} \ .$$

Since,

$$AB = \begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

$$A^2B = \begin{bmatrix}1 & 2 & -1\\ 0 & 1 & 0\\ 1 & -4 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 3\end{bmatrix}$$

$$Q_c = \begin{bmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 0 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -3\\ 0 & 1 & 3\\ 0 & 0 & 0\end{bmatrix}$$

$$\therefore rank(Q_c) = 2 \neq 3 \ .$$

This system is uncontrollable.

(2) Determine if the system is observable.

The A and C matrices of this system are

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$$

Then the observability matrix becomes

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} .$$

Since

$$\mathrm{CA} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix}$$

$$\mathrm{C}A^2 = \begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 0 \end{bmatrix}$$

$$Q_o = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -3 & 1 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore rank(Q_o) = 3 \ .$$

Thus, the system is observable.

Problem 4 (20 pts)

The longitudinal equations for an airplane having poor handling qualities follow. Use state feedback to provide stability augmentation so that the augmented aircraft has the following short- and long-period (phugoid) characteristics:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}-0.01 & 0.1 & 0 & -32.2\\ -0.40 & -0.8 & 180 & 0\\ 0 & -0.003 & -0.5 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta \theta\end{bmatrix} + \begin{bmatrix}0\\ -10\\ -2.8\\ 0\end{bmatrix}\delta_e$$

$$\zeta_{sp} = 0.6, \qquad \omega_{n,sp} = 3.0\, rad/s$$

$$\zeta_{p} = 0.05, \qquad \omega_{n,p} = 0.1\, rad/s$$

Use the Ackermann's formula to design the feedback gain to locate the closed-loop eigenvalues.

We know that …

The longitudinal dynamics of a plane are described by

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{w}\\ \Delta\dot{q}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}X_u & X_w & 0 & -g\\ Z_u & Z_w & u_0 & 0\\ M_u & M_w & M_q & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta \theta\end{bmatrix} + \begin{bmatrix}X_{\delta_e}\\ Z_{\delta_e}\\ M_{\delta_e}\\ 0\end{bmatrix}\Delta\delta_e \ .$$

The short period mode has the approximate 2-D dynamics described by:

$$\begin{bmatrix}\Delta\dot{w}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}Z_w & u_0\\ M_w & M_q\end{bmatrix}\begin{bmatrix}\Delta w\\ \Delta q\end{bmatrix} + \begin{bmatrix}Z_{\delta_e}\\ M_{\delta_e}\end{bmatrix}\Delta\delta_e$$

$$\begin{bmatrix}\Delta\dot{w}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}-0.8 & 180\\ -0.003 & -0.5\end{bmatrix}\begin{bmatrix}\Delta w\\ \Delta q\end{bmatrix} + \begin{bmatrix}-10\\ -2.8\end{bmatrix}\Delta\delta_e \ .$$

and the phugoid mode has the approximate 2-D dynamics described by:

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}X_u & -g\\ -Z_u/u_0 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}X_{\delta_e}\\ Z_{\delta_e}/u_0\end{bmatrix}\Delta\delta_e$$

$$\begin{bmatrix}\Delta\dot{u}\\ \Delta\dot{\theta}\end{bmatrix} = \begin{bmatrix}-0.01 & -32.2\\ 0.0022 & 0\end{bmatrix}\begin{bmatrix}\Delta u\\ \Delta\theta\end{bmatrix} + \begin{bmatrix}0\\ -0.0556\end{bmatrix}\Delta\delta_e$$

Short Period Mode:

The desired poles are

$$s = -\zeta_{sp}\omega_{sp} \pm j\omega_{sp}\sqrt{1-\zeta_{sp}^2} = -1.8 \pm 2.4j \ .$$

Now we use a MATLAB function that conducts Ackermann's formula step-by-step, which is the one as follows.

```
function res = ackermannMethod(A, B, dp, tol)
    % Function that computes the eigVal placement with Ackermann's method
    % A: system A matrix
    % B: system B matrix
    % dp: array of desired poles
    % tol: tolerance

    %Checking for user inputed tolerance
    if nargin == 3
        %using default value
        tol = 2;
    elseif nargin > 4
        error('Too many inputs.')
    elseif nargin < 3
        error('Too few inputs.')
    end

    sz = size(A); n = sz(1);
    % Step-1
    ad_s = poly(dp);
    % Step-2
    ad_A = 0; idx = 1;
    for i = n:-1:0
        ad_A = ad_A + A^i * ad_s(idx);
        idx = idx + 1;
    end
    % Step-3
    Qc = ctrb(A,B);
    % Step-4
    e = zeros(1, n); e(end) = 1;
    K = e*inv(Qc)*ad_A;
    % Step-5
    Ad = A - B * K;
    for p = eig(Ad)'
       ct = 0;
       for pp = dp
           if round(p,tol) == round(pp,tol)
               ct = ct + 1;
           end
       end
       if ct == 0
           error('The gains do not produce the desired poles.');
       end
    end

    % Results
    res.check = 1;
    res.K = K;
    res.Ad = Ad;
    res.Qc = Qc;
    res.DA = ad_A;
```

```
    res.Ds = ad_s;
end
```

From the desired poles we know that

$$a_d(s) = s^2 + 3.6s + 9\,.$$

Then plug in the A matrix

$$a_d(A_{sp}) = A_{sp}^2 + 3.6A_{sp} + 9 = \begin{bmatrix} 6.22 & 414 \\ -0.0069 & 6.91 \end{bmatrix}.$$

The controllability matrix becomes

$$Q_c = [B_{sp} \quad A_{sp}B_{sp}] = \begin{bmatrix} -10 & -496 \\ -2.8 & 1.43 \end{bmatrix}$$

where

$$A_{sp} = \begin{bmatrix} -0.8 & 180 \\ -0.003 & -0.5 \end{bmatrix}, \qquad B_{sp} = \begin{bmatrix} -10 \\ -2.8 \end{bmatrix}$$

Then,

$$\mathrm{K_{sp}} = [0 \quad 1]Q_c^{-1}a_d(A_{sp}) = \; [-0.0125 \quad -0.7769].$$

REMINDER: I use the convention $u = -Kx$.

We verify the results by using

$$A_d = A_{sp} - B_{sp}\mathrm{K} = \begin{bmatrix} -0.9246 & 172.2308 \\ -0.0379 & -2.6754 \end{bmatrix}$$

and the eigenvalues of this new A_d matrix is the same as the desired poles.

Phugoid Mode:

The desired poles are

$$s = -\zeta_p\omega_p \pm j\omega_p\sqrt{1-\zeta_p^2} = -0.0050 \pm 0.0999j\ \ .$$

From the desired poles we know that

$$a_d(s) = s^2 + 0.01s + 0.01\,.$$

Then plug in the A matrix

$$a_d(A_p) = A_p^2 + 0.01A_p + 0.01 = \begin{bmatrix} 0.0816 & 5.5511e-17 \\ 3.3881e-21 & 0.0816 \end{bmatrix}.$$

The controllability matrix becomes

$$Q_c = [B_p \quad A_pB_p] = \begin{bmatrix} 0 & -1.7889 \\ -0.0556 & 0 \end{bmatrix}$$

where

$$A_p = \begin{bmatrix} -0.01 & -32.2 \\ 0.0022 & 0 \end{bmatrix}, \qquad B_p = \begin{bmatrix} 0 \\ -0.0556 \end{bmatrix}$$

Then,

$$\mathrm{K} = [0 \quad 1]Q_c^{-1}a_d(A_{sp}) = [-0.0456 \quad -3.1031e-17] \approx [-0.0456 \quad 0] \ .$$

REMINDER: I use the convention $u = -Kx$.

We verify the results by using

$$A_d = A_{sp} - B_{sp}\mathrm{K} = \begin{bmatrix} -0.01 & -32.2 \\ -3.1056e-4 & -1.7239e-18 \end{bmatrix}$$

and the eigenvalues of this new A_d matrix is the same as the desired poles.

Problem 5 (20 pts)

The rolling motion of an aerospace vehicle is given by these state equations:

$$\begin{bmatrix} \dot{\delta}_a \\ p \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -1/\tau & 0 & 0 \\ L_{\delta_a} & L_p & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_a \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 1/\tau \\ 0 \\ 0 \end{bmatrix} \delta_v$$

Where δ_a, p, ϕ, and δ_v are the aileron deflection angle, roll rate, and voltage input to the aileron actuator motor. Note that in this problem the aileron angle is considered a state and the control voltage, δ_v is the input. Determine the optimal control law that minimizes the performance index, J, as follows:

$$J = \int_0^\infty (x'Qx + u'Ru)dt$$

Where

$$Q = \begin{bmatrix} 1/\delta_{a_{max}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\phi_{max}^2 \end{bmatrix}, \qquad R = \left[1/\delta_{v_{max}}^2\right]$$

For the problem, assuming the following:

$$\tau = 0.1\ s, \qquad L_{\delta_a} = 30\ s^{-2}$$

$$L_p = -1.0\ rad/s, \qquad \delta_{a_{max}} = \pm 25° = 0.436\ \text{rad}$$

$$\phi_{max} = \pm 45° = 0.787\ rad, \qquad \delta_{v_{max}} = 10\ V$$

For this problem we will use MATLAB. First we setup the provided parameters and matrices.

```
% Given parameters
tau = 0.1;
L_da = 30;
L_p = -1.0;
d_amax = 0.436;
phi_max = 0.787;
d_vmax = 10;

% System A and B matrices
A = [-1/tau, 0, 0; L_da, L_p, 0; 0, 1, 0];
B = [1/tau; 0; 0];

% Weighting matrices Q and R
```

```
Q = diag([1/d_amax, 0, 1/phi_max^2]);
R = [1/d_vmax^2];
```

Then we run "lqr()" to get the LQR gains

```
% Obtain the LQR gains
K = lqr(A, B, Q, R);
```

Then the optimal control law of this system is to use a state variable feedback with the feedbacks generated with the LQR method

$$u = -Kx$$

$$\dot{x} = (A - BK)x$$

where

$$K = [14.7931 \quad 3.1775 \quad 12.7065] \; .$$