AE 6230 – HW1: Free Vibrations and Harmonic Excitation of SDOF Systems

Out: September 6, 2022; **Due:** September 13, 2022 by 11:59 PM ET in Canvas

Guidelines

- Read each question carefully before doing any work;
- If you find yourself doing pages of math, pause and consider if there is an easier approach;
- You can consult any relevant materials;
- You can discuss solution approaches with others, but your submission must be your own work;
- If you have doubts, please ask questions in class, during office hours, and/or Piazza (no questions via email);
- The solution to each question should concisely and clearly show the steps;
- Simplify your results as much as possible;
- Box the final answer for each question;
- If you use any code, please submit it with the solution.

Problem 1 – 40 points

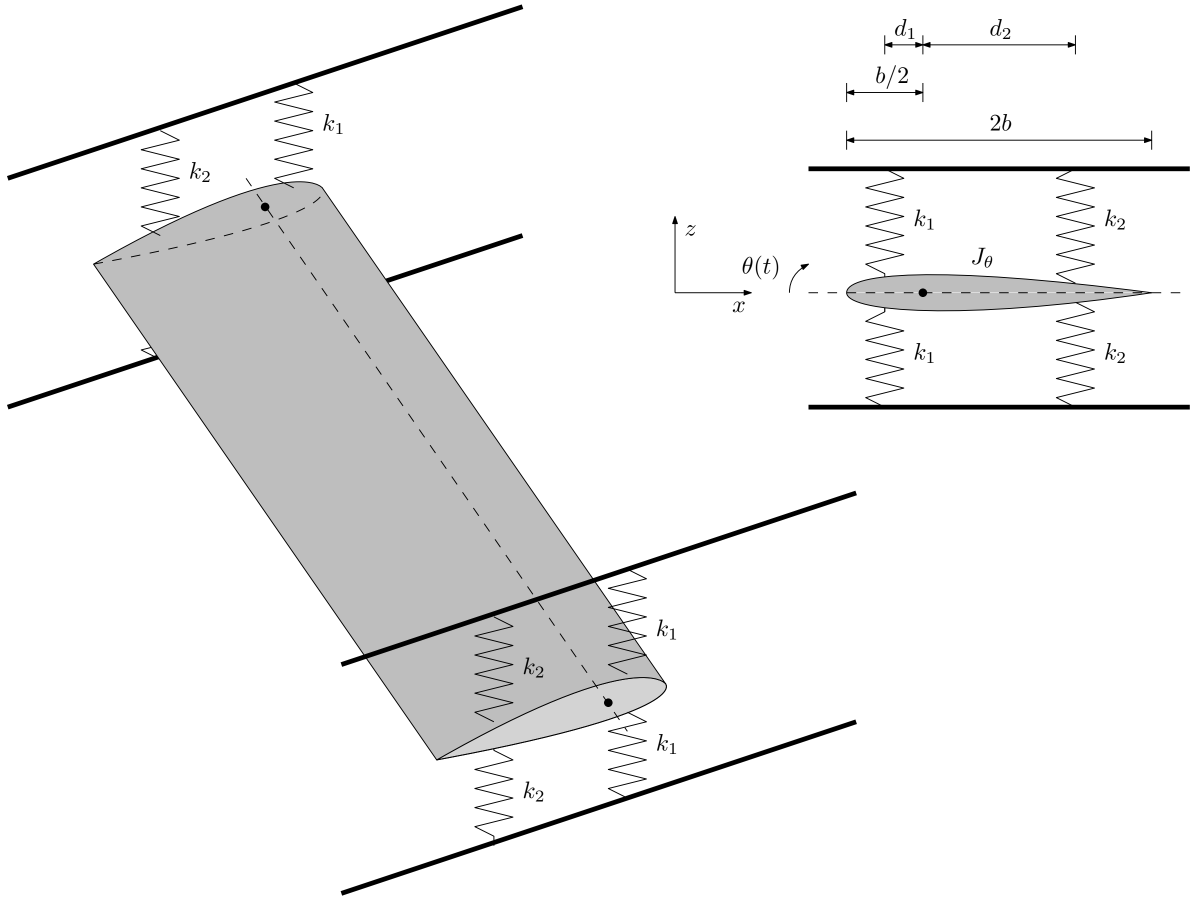


Figure 1: Schematic of wind-tunnel wing model.

Consider a uniform rigid wing mounted in a wind-tunnel test section (Fig. 1). The wing can pitch about the quarter-chord axis and the pitch motion is restrained by four springs on each end (near the wind-tunnel walls). The front springs have spring constant k_1 and are attached to the wing upper and lower surfaces at a distance d_1 ahead of the quarter chord (toward the leading edge); the rear springs have spring constant k_2 and connect to the wing at a distance d_2 downstream of the quarter chord (toward the trailing edge). The wing moment of inertia about the pitch axis is denoted by J_θ. Assuming the pitch angle θ as the degree of freedom (see the convention in Fig. 1) and neglecting the wing self-weight, answer the following questions:

1. After drawing the free-body diagram for the system
   (a) Derive the equation of motion for studying its free vibrations;
   (b) Determine the natural frequency ω_n and evaluate it for the parameters in Table 1;
2. Modify the attachment point of either the front or the rear springs to increase ω_n by 15%;
3. Assuming that four linear viscous dampers c_1 are added to the initial system (one for each front spring)
   (a) Find the minimum value of c_1 such that any free response satisfies

$$\delta = \ln \frac{x(t_1)}{x(t_2)} \geq 0.2 \tag{1}$$

   where δ is the logarithmic decrement and t_1 and $t_2 = t_1 + T$ are two consecutive oscillation peaks;
   (b) Evaluate the frequency of the damped motion ω_d and compare it with ω_n;
4. Considering the system with the viscous dampers
   (a) Determine the free response for the initial conditions $\theta_0 = 5$ deg and $\dot{\theta}_0 = 0$;
   (b) Plot the free response for $t \in [0, 5]$ s.

Table 1: Parameter values for Problem 1.

Parameter	Symbol	Value
Spring constant of the front springs	k_1	25 N/m
Spring constant of the rear springs	k_2	$0.75k_1$
Half chord	b	10 cm
Distance of the front springs from the pitch axis	d_1	$b/4$
Distance of the rear springs from the pitch axis	d_2	b
Moment of inertia about the pitch axis	J_θ	0.0004 kg·m^2

Recommendations and clarifications:

- **Question 1b:** only substitute the values in Table 1 at the end of the question to evaluate ω_n;
- **Question 2:** assume that the attachment points for all the front or all the rear springs change by the same amount and that you can only move one set of springs (either all the front or all the rear springs). The front springs must remain ahead of the pitch axis and the rear springs must remain downstream of the pitch axis;
- **Question 3:** answer this question for the initial system, that is, before the changes in Question 2. The viscous dampers connect to the upper and lower surfaces of the wing at the same distance d_1 from the pitch axis as the front springs;
- **Question 4a:** simplify your solution as appropriate;
- **Question 4b:** use a reasonable time step when plotting the free response and do not forget the units. Note that you can use the results from this question to check whether the logarithmic decrement is correct.

Problem 2 – 40 points

Consider the system in Problem 1 but now with the parameters in Table 2. The system is excited by a moment

$$M(t) = M_0 \sin \omega t \tag{2}$$

about the pitch axis, with zero initial conditions. Answer the following questions:

1. Plot the magnitude $|H(i\omega)|$ and phase lag $\phi(\omega)$ of the frequency response for $\omega/\omega_n \in [0, 4]$;
2. Using the complex response method
   (a) Determine the steady-state forced response;
   (b) Plot the steady-state forced response for $t \in [10T, 10T + 0.5]$ s where T is the period of the excitation;
3. Using the time-domain method
   (a) Determine the steady-state forced response;
   (b) Plot the steady-state forced response for $t \in [10T, 10T + 0.5]$ s;
   (c) Determine the complete forced response including the transient phase;
   (d) Plot the complete forced response and the transient terms for $t \in [0, 4]$ s.

Table 2: Parameters for Problem 2.

Parameter	Symbol	Value
Moment of inertia about pitch axis	J_θ	0.0004 kg·m^2
Natural frequency	ω_n	50 rad/s
Viscous damping factor	ζ	0.04
Excitation amplitude	M_0	0.1 Nm
Excitation frequency	ω	$0.5\omega_n$

Recommendations and clarifications:

- **Question 1:** use a reasonable frequency step when plotting the results and do not forget the units. Note that the phase lag is denoted by $\phi(\omega)$ instead of $\theta(\omega)$ as done in class to avoid confusion with $\theta(t)$;
- **Question 2 and 3:** use a reasonable time step when plotting the results and do not forget the units.

Problem 3 – 20 points

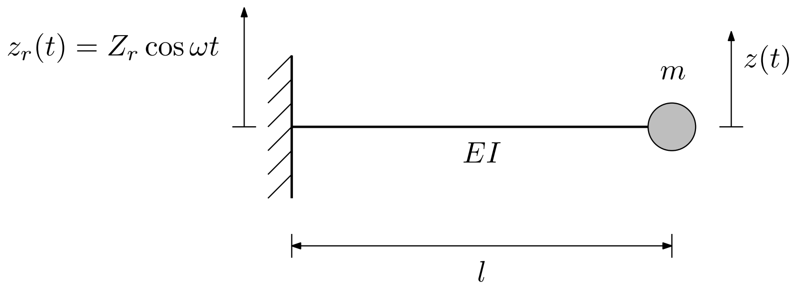


Figure 2: Schematic of a cantilevered beam in bending with a tip mass subject to harmonic motion of its root.

Consider the massless uniform isotropic cantilevered beam in bending in Fig. 2 with a tip mass. The beam root undergoes harmonic motion

$$z_r(t) = Z_r \cos \omega t \tag{3}$$

Very slight damping is present in the system such that, after a transient phase, the tip mass motion $z(t)$ contains only the excitation frequency ω. The impact of such slight damping on the amplitude and phase of $z(t)$ is assumed to be negligible. Answer the following questions:

1. After showing the free-body diagram, derive the equation of motion for the tip mass;
2. Considering the parameters in Table 3, determine
   (a) The natural frequency ω_n;
   (b) The maximum excitation frequency $\omega < \omega_n$ such that $|z(t) - z_e| \leq 1.1|Z_r|$ where z_e is the equilibrium position of the tip mass.

Table 3: Parameter values for Problem 3.

Parameter	Symbol	Value
Beam length	l	0.5 m
Beam bending stiffness	EI	5 N·m^2
Tip mass	m	0.5 kg

Recommendations and clarifications:

- **Question 2a:** only substitute the values in Table 3 at the end of the question to evaluate ω_n.

Problem 1 Solution – 40 points

Question 1a – 5 points

The four front (rear) springs in Fig. 1 operate in parallel because they experience the same elongation (in magnitude) when the wing pitches about its quarter-chord axis. To compute the restoring elastic force applied by each spring to the wing, we first determine the displacements of the front and rear attachment points, denoted by $h_1 = h_1(t)$ and $h_2 = h_2(t)$, respectively, in terms of the chosen degree of freedom $\theta = \theta(t)$. Using the convention for positive pitch angles in Fig. 1 and the small-angle assumption, we obtain

$$h_1(t) = d_1 \sin\theta(t) \approx d_1\theta(t) \qquad h_2(t) = d_2 \sin\theta(t) \approx d_2\theta(t) \tag{4}$$

where the directions of $h_1(t)$ and $h_2(t)$ are shown in Fig. 3(b) (the displacements are amplified in the figure for clarity). The small-angle assumption is required to solve this problem using the techniques seen in class that apply to linear systems. This assumption implies that forces and moments can be computed directly in the reference equilibrium configuration (zero pitch angle).

The restoring forces applied to the wing by the four front and rear springs, denoted by $F_1 = F_1(t)$ and $F_2 = F_2(t)$, respectively, are given by

$$F_1(t) = 4k_1h_1(t) = 4k_1d_1\theta(t) \qquad F_2(t) = 4k_2h_2(t) = 4k_2d_2\theta(t) \tag{5}$$

The directions of these forces are given in the free-body diagram in Figure 3(b). Note that $F_1(t)$ and $F_2(t)$ have *opposite directions* because they result from springs that are on different sides along the chord with respect to the pitch axis (pivot point). This results in restoring moments that are in the same direction, both opposing the pitch rotation, so that all springs contribute to the overall torsional stiffness of the system.

The equation of motion for studying free vibrations is derived by taking moments about the pitch axis, which is fixed. Specifically, we set the moments due to the forces (5) (with the appropriate signs) to equal the moment of inertia J_θ times the pitch angular acceleration $\ddot{\theta}(t)$. This gives

$$\begin{aligned} J_\theta\ddot{\theta}(t) &= -F_1(t)d_1 - F_2(t)d_2 \\ &= -4\big(k_1d_1^2 + k_2d_2^2\big)\,\theta(t) \end{aligned} \tag{6}$$

or

$$\ddot{\theta}(t) + \frac{4\big(k_1d_1^2 + k_2d_2^2\big)}{J_\theta}\theta(t) = 0 \tag{7}$$

This equation must be accompanied by the initial conditions

$$\theta(0) = \theta_0 \qquad \dot{\theta}(0) = \dot{\theta}_0 \tag{8}$$

Question 1b – 5 points

Equation (7) represents an harmonic oscillator and can be rewritten in the general form

$$\ddot{\theta}(t) + \omega_n^2\theta(t) = 0 \tag{9}$$

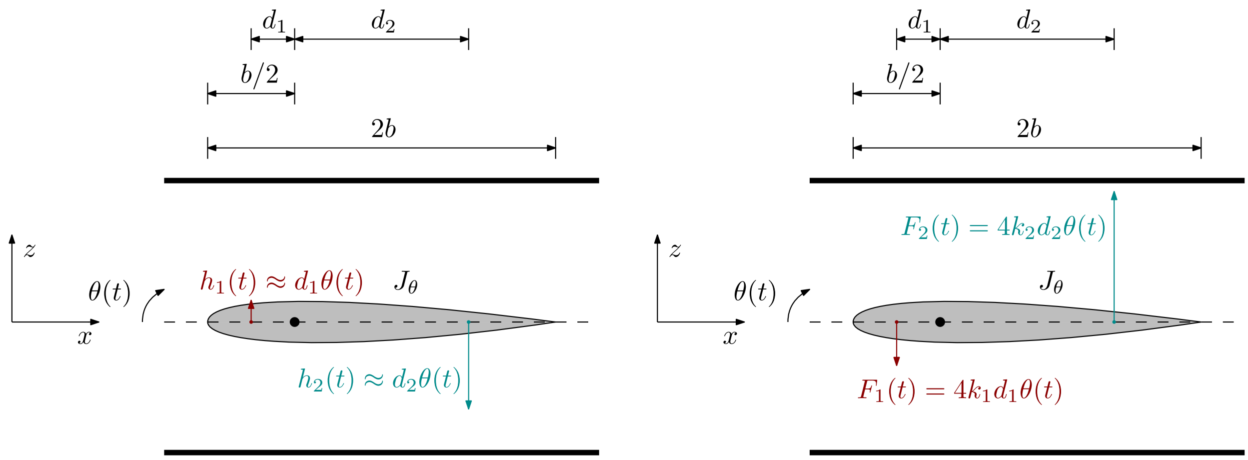
Comparing (7) and (9) shows that the natural frequency of the system is

$$\omega_n = \sqrt{\frac{k_\theta}{J_\theta}} = 2\sqrt{\frac{k_1d_1^2 + k_2d_2^2}{J_\theta}} \tag{10}$$

where $k_\theta = 4\left(k_1d_1^2 + k_2d_2^2\right)$ is the torsional stiffness constant of the spring arrangement in Fig. 1. Substituting the parameter values in Table 1, we obtain

$$\omega_n = 2\sqrt{\frac{25\,\mathrm{N/m}\cdot 0.025^2\,\mathrm{m}^2 + 0.75\cdot 25\,\mathrm{N/m}\cdot 0.1^2\,\mathrm{m}^2}{0.0004\,\mathrm{kg}\cdot\mathrm{m}^2}} = 45.0694\,\mathrm{rad/s} \tag{11}$$

The natural frequency has the expected units of rad/s like for the mass-spring system seen in class. However, the units of the inertia and stiffness properties of the system are different because in this case the degree of freedom of the system is a rotation rather than a position. An equation such as Eq. (7) can be used to study any harmonic-oscillator-like system that can be described by a single degree of freedom. Different problems will only differ in the meaning of the degree of freedom and in the expression that gives the natural frequency.



(a) Kinematics

(b) Free-body diagram

Figure 3: Kinematics and free-body diagram for the system in Fig. 1.

Question 2 – 5 points (which springs to move) + 5 points (new position)

To increase the natural frequency in Eq. (11) by 15% by moving either the front or the rear springs, we first determine the maximum frequency increase for each of these design modifications within the geometric constraints $d_1 \in [0, b/2]$ and $d_2 \in [0, 3b/2]$. When the front springs are at the leading edge ($d_1 = b/2$), the natural frequency becomes

$$\omega_n' = 2\sqrt{\frac{k_1(b/2)^2 + k_2 d_2^2}{J_\theta}} = 50\,\mathrm{rad/s} \tag{12}$$

Because ω_n' is about 10.94% larger than ω_n, we cannot meet the design requirement by moving the front springs. When the rear springs are at the trailing edge ($d_2 = 3b/2$), the natural frequency becomes

$$\omega_n'' = 2\sqrt{\frac{k_1 d_1^2 + k_2(3b/2)^2}{J_\theta}} = 66.1438\,\mathrm{rad/s} \tag{13}$$

Because ω_n'' is about 46.76% larger than ω_n, we can meet the requirement by moving the rear springs. Specifically, we can find the new value of d_2 by enforcing

$$1.15\omega_n = 2\sqrt{\frac{k_1 d_1^2 + k_2 d_2^2}{J_\theta}} \tag{14}$$

which gives

$$d_2 = \sqrt{\frac{1}{k_2}\left[\frac{J_\theta(1.15\omega_n)^2}{4} - k_1 d_1^2\right]} = 0.1162\,\mathrm{m} \tag{15}$$

This corresponds to shifting the rear springs toward the trailing edge by 16.16% compared with the initial position. It is also possible to meet the frequency requirement by moving the front springs behind the pitch axis, or by moving the front and rear springs both. However, the guidelines for Problem 1 specified to only move one set of springs and to keep each set of springs on the same side with respect to the pitch axis as they were initially.

Question 3a – 5 points

The logarithmic decrement associated with two consecutive peaks t_1 and $t_2 = t_1 + T$ is defined as

$$\delta = \ln\frac{\theta(t_1)}{\theta(t_1 + T)} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{16}$$

which gives

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \tag{17}$$

With the viscous dampers, Eq. (7) becomes

$$\ddot{\theta}(t) + \frac{4c_1 d_1^2}{J_\theta}\dot{\theta}(t) + \frac{4\left(k_1 d_1^2 + k_2 d_2^2\right)}{J_\theta}\theta(t) = 0 \tag{18}$$

or

$$\ddot{\theta}(t) + 2\zeta\omega_n\dot{\theta}(t) + \omega_n^2\theta(t) = 0 \tag{19}$$

The damping term in Eq. (18) is derived following the same rationale as for deriving the restoring elastic moment. Comparing Eqs. (18) and (19) shows that the viscous damping factor is

$$\zeta = \frac{c_\theta}{2J_\theta\omega_n} = \frac{2c_1 d_1^2}{J_\theta\omega_n} \tag{20}$$

where $c_\theta = 4c_1 d_1^2$ is the torsional viscous damping coefficient of the system with the four viscous dampers (which are in parallel, like the springs). Equations (17) and (20) allow us to solve for the viscous damping coefficient that each of the four dampers must have to achieve the desired logarithmic decrement:

$$c_1 = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}\frac{J_\theta\omega_n}{2d_1^2} \tag{21}$$

The minimum value of c_1 such that $\delta \geq 0.20$ is obtained by substituting $\delta = 0.20$ into Eq. (21), which gives

$$c_1 = 0.4588\,\mathrm{Ns/m} \tag{22}$$

This corresponds to a viscous damping factor $\zeta = 0.0318$. Note that c_1 has units of a translational viscous damping coefficient. The torsional viscous damping coefficient of the system (which was not asked) is

$$c_\theta = 4c_1 d_1^2 = 0.0011\,\mathrm{Nms/rad} \tag{23}$$

Question 3b – 5 points

The frequency of the damped motion is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 45.0466\,\mathrm{rad/s} \tag{24}$$

and its variation with respect to the undamped case is

$$\frac{\omega_d - \omega_n}{\omega_n} \times 100 = \left(\sqrt{1-\zeta^2} - 1\right) \times 100 = -0.05\% \tag{25}$$

The percentage decrease in the frequency is very slight, as expected for low damping levels ($\zeta \ll 1$).

Question 4a – 5 points

The general solution for the damped free vibrations of the system in Fig. 1 is

$$\theta(t) = e^{-\zeta\omega_n t}\left(\theta_0\cos\omega_d t + \frac{\dot{\theta}_0 + \zeta\omega_n\theta_0}{\omega_d}\sin\omega_d t\right) \tag{26}$$

For the case of zero initial pitch rate, Eq. (26) reduces to

$$\theta(t) = \theta_0 e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right) \tag{27}$$

or

$$\theta(t) = Ae^{-\zeta\omega_n t}\left(\cos\omega_d t - \phi\right) \tag{28}$$

with

$$A = \sqrt{\theta_0^2 + \frac{(\zeta\omega_n\theta_0 + \dot{\theta}_0)^2}{\omega_d^2}} = \theta_0\sqrt{1 + \frac{\zeta^2}{1-\zeta^2}} = \frac{\theta_0}{\sqrt{1-\zeta^2}} = 0.0873\,\mathrm{rad} = 5.0025\,\mathrm{deg} \tag{29}$$

and

$$\phi = \tan^{-1}\left(\frac{\zeta\omega_n\theta_0 + \dot{\theta}_0}{\theta_0\omega_d}\right) = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = 0.0318\,\mathrm{rad} = 1.8232\,\mathrm{deg} \tag{30}$$

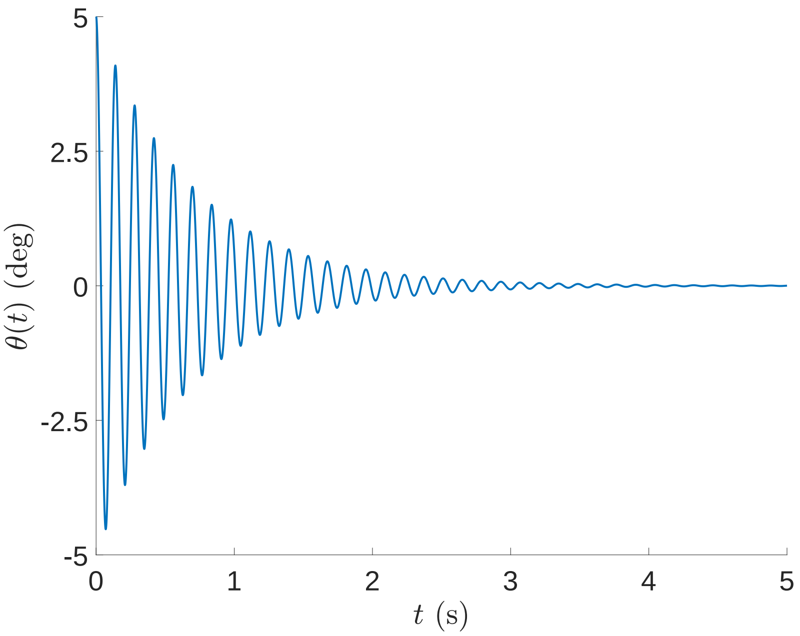


Figure 4: Free response of the system in Fig. 1 for Problem 1 Question 4b.

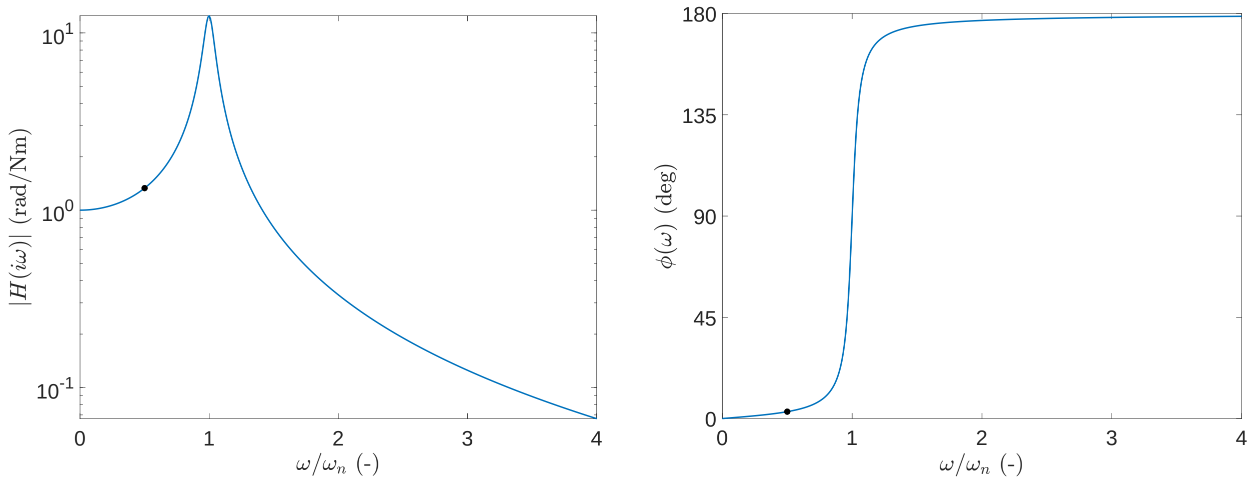
Question 4b – 5 points

To plot the free response, we must evaluate Eq. (27) by choosing an appropriate time step. This can be done considering that the period of the damped motion is

$$T = \frac{2\pi}{\omega_d} = 0.1394\,\mathrm{s} \tag{31}$$

Choosing $\Delta t = 0.001$ s ensures that the free oscillations are resolved very accurately. The free response is shown in Fig. 4. The logarithmic decrement approximated from the numerical results with the chosen time step confirms the desired damping level:

$$\delta = \ln\frac{\theta(0)}{\theta(T)} \approx 0.1998 \tag{32}$$



(a) Magnitude $|H(i\omega)|$ (b) Phase lag $\phi(\omega)$

Figure 5: Frequency response of the system in Fig. 1 for Problem 2 Question 1 (markers at $\omega = 0.5\omega_n$).

Problem 2 Solution – 40 points

Question 1 – 5 points (magnitude plot) + 5 points (phase lag plot)

The magnitude and phase lag of the frequency response function of the system in Fig. 1 are given by

$$|H(i\omega)| = \frac{1}{k_\theta} \frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2}} \qquad \phi(\omega) = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \tag{33}$$

where the torsional stiffness constant of the system is

$$k_\theta = J_\theta\, \omega_n^2 = 1\,\text{Nm/rad} \tag{34}$$

The torsion stiffness constant must be recomputed because the problem specifies the pitch moment of inertia and the natural frequency, which is different from the natural frequency obtained in Problem 1.

Figure 5 shows the frequency response plots. The black markers show the magnitude and phase lag at the excitation frequency $\omega = 0.5\omega_n$. The frequency response function has units of rad/Nm because the degree of freedom of the system is an angle, not a position like for the mass-damper-spring system seen in class. Because the torsional stiffness constant of the system equals one, the magnitude of $|H(i\omega)|$ is the same as the dynamic amplification factor. Note that the phase lag must be computed using the four-quadrant inverse tangent function (`atan2` function in MATLAB) that can return an angle between $-\pi$ and $+\pi$. That is because the frequency response function is a complex number, and the phase of a complex number may not necessarily lie between $-\pi/2$ and $+\pi/2$. Using the regular inverse tangent function would result in a discontinuity in the phase plot when the frequency ratio ω/ω_n reaches a unit value.

Question 2 – 5 points (steady-state response using complex response method) + 5 points (plot)

Using the complex response method, the steady-state response can be written as

$$\theta(t) = M_0 H(i\omega) e^{i\omega t} = M_0 |H(i\omega)| e^{i[\omega t - \phi(\omega)]} \tag{35}$$

To determine the steady-state response, we must evaluate the amplitude $|H(i\omega)|$ and the phase lag $\phi(\omega)$ at $\omega = 0.5\omega_n$. Because the excitation is a sine function, the steady-state response is the imaginary part of Eq. (35):

$$\theta(t) = M_0 |H(i\omega)| \sin\left[\omega t - \phi(\omega)\right] \tag{36}$$

From Fig. 5, we obtain

$$|H(i\omega)|\Big|_{\omega=0.5\omega_n} = 1.3314\,\text{rad/Nm} \qquad \phi(\omega)\Big|_{\omega=0.5\omega_n} = 0.0533\,\text{rad} = 3.0529\,\text{deg} \tag{37}$$

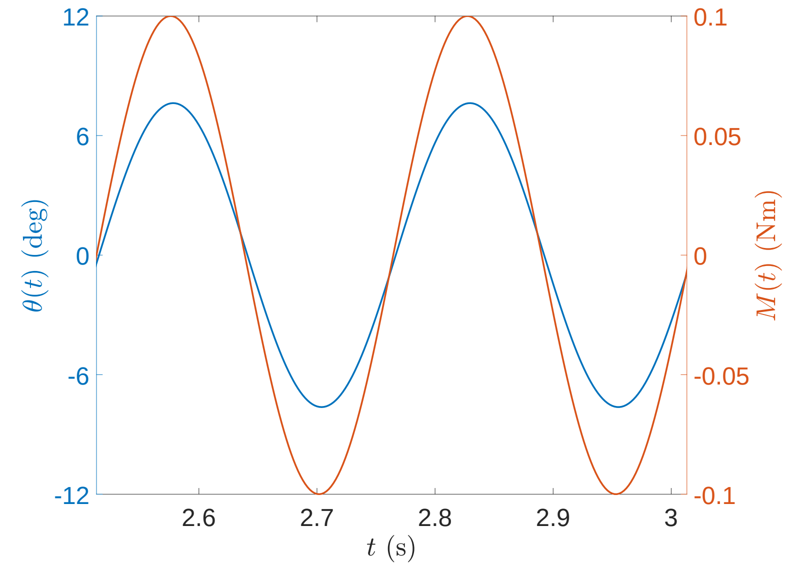


Figure 6: Harmonic excitation and steady-state response of the system in Fig. 1 for Problem 2 Question 2b.

The steady-state response (36), plotted in Fig. 6, has an amplitude $|\theta(t)| = M_0|H(i\omega)| = 0.1331$ rad $= 7.63$ deg and a phase lag of about 3.05 deg with respect to the excitation. Note that while it is possible to plot the steady-state response starting from $t = 0$ s, we must keep in mind that the steady-state response of a damped system subject to an harmonic excitation is the response after a sufficiently long time such that the transient terms have decayed.

Question 3a – 5 points

The steady-state response of the system obtained using the time-domain method must match the one computed in Question 2. The general solution for the forced response to an harmonic is assumed of the form

$$\theta(t) = e^{-\zeta\omega_n t}\left(A_1 \cos \omega_d t + A_2 \sin \omega_d t\right) + B_1 \cos \omega t + B_2 \sin \omega t \tag{38}$$

The last two terms, which compose the steady-state response, are the particular solution of

$$\ddot{\theta}(t) + 2\zeta\omega_n\dot{\theta}(t) + \omega_n^2\theta(t) = \frac{M_0}{J_\theta}\sin \omega t \tag{39}$$

To find the unknown coefficients B_1 and B_2, we enforce that the steady-state response (particular solution) satisfy Eq. (39). Using the harmonic balance method, this gives a 2×2 system of algebraic equations:

$$\begin{bmatrix} \omega_n^2 - \omega^2 & 2\zeta\omega_n\omega \\ -2\zeta\omega_n\omega & \omega_n^2 - \omega^2 \end{bmatrix} \begin{Bmatrix} B_1 \\ B_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_0/J_\theta \end{Bmatrix} \tag{40}$$

This system is not the same as obtained in class, which was for a cosine excitation. The solution of (40) is

$$\begin{aligned} B_1 &= -\frac{M_0}{k_\theta}\frac{2\zeta\omega/\omega_n}{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2} \\ B_2 &= \frac{M_0}{k_\theta}\frac{1-\omega^2/\omega_n^2}{\left(1-\omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2} \end{aligned} \tag{41}$$

Substituting $\omega/\omega_n = 0.5$, we obtain

$$B_1 = -0.0071\,\text{rad} \qquad B_2 = 0.1330\,\text{rad} \tag{42}$$

Question 3b – 5 points

Figure 7 shows the steady-state response obtained the time-domain approach on top of the solution from the complex response method. The results match, as they should. To demonstrate this analytically (which was not required), we note that the steady-state response from Question 2 can be rewritten as

$$\theta(t) = M_0|H(i\omega)| \sin\left[\omega t - \phi(\omega)\right] = M_0|H(i\omega)| \left[\sin \omega t \cos \phi(\omega) - \cos \omega t \sin \phi(\omega)\right] \tag{43}$$

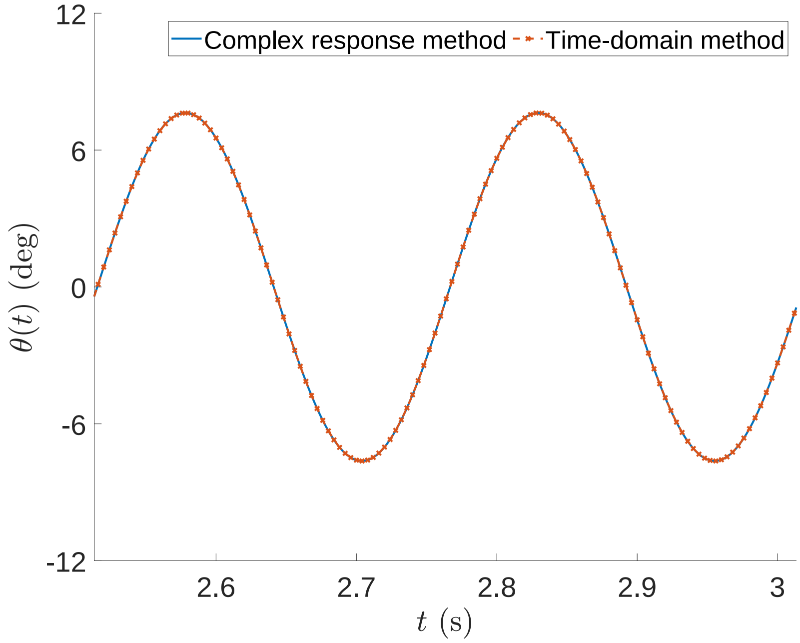


Figure 7: Steady-state response of the system in Fig. 1 for Problem 2 Question 3b.

Comparing Eq. (43) with the last two terms in Eq. (38) allows us to rewrite the coefficients B_1 and B_2 as

$$B_1 = -M_0|H(i\omega)|\sin\phi(\omega) \qquad B_2 = M_0|H(i\omega)|\sin\phi(\omega) \tag{44}$$

From these relations, we obtain

$$\begin{aligned} |H(i\omega)| &= \frac{1}{M_0}\sqrt{B_1^2 + B_2^2} = \frac{1}{k_\theta}\frac{1}{\sqrt{\left(1-\omega^2/\omega_n^2\right)^2 + (2\zeta\omega/\omega_n)^2}} \\ \phi(\omega) &= \tan^{-1}\left(-\frac{B_1}{B_2}\right) = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}\right) \end{aligned} \tag{45}$$

which are the known relations for the frequency response function used in Questions 1 and 2.

Question 3c – 5 points

To compute the complete forced response including the transient phase, we need to apply the zero initial conditions to the general solution in Eq. (38). This gives us two algebraic relations that we can solve for A_1 and A_2.

The condition of zero initial pitch angle gives

$$\theta(0) = A_1 + B_1 = 0 \qquad \longrightarrow \qquad A_1 = -B_1 = 0.0071\,\text{rad} \tag{46}$$

Using this result, Eq. (38) becomes

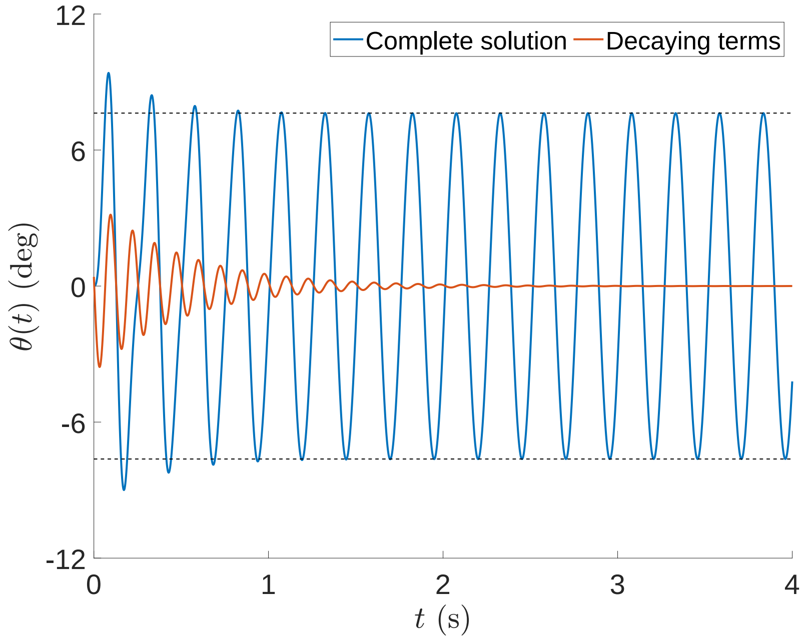
$$\theta(t) = e^{-\zeta\omega_n t}\left(A_2 \sin\omega_d t - B_1\cos\omega_d t\right) + B_1\cos\omega + B_2\sin\omega t \tag{47}$$

Taking the time derivative of Eq. (47) and applying the condition of zero initial pitch rate gives

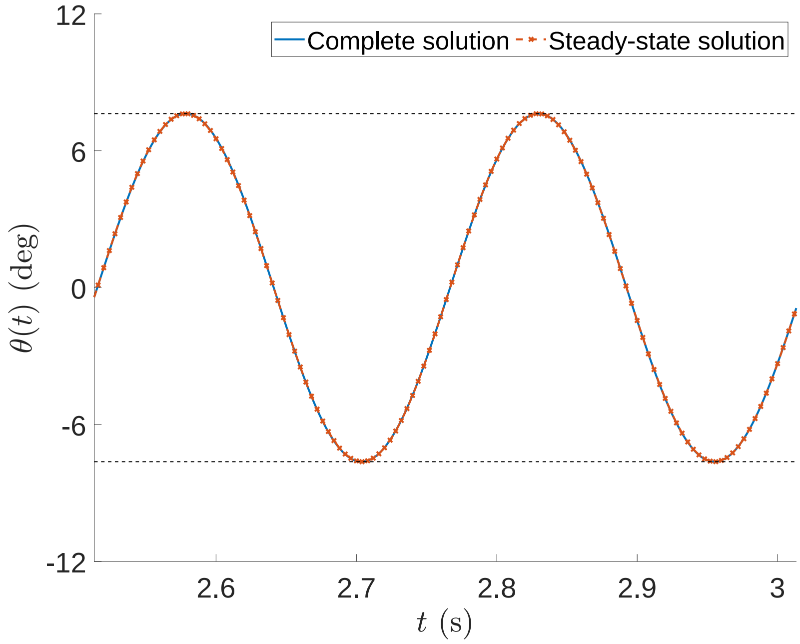
$$\dot{\theta}(0) = \zeta\omega_n B_1 + \omega B_2 + \omega_d A_2 = 0 \qquad \longrightarrow \qquad A_2 = -\frac{1}{\sqrt{1-\zeta^2}}\left(\zeta B_1 + \frac{\omega}{\omega_n}B_2\right) = -0.0662\,\text{rad} \tag{48}$$

Finally, the frequency of the damped motion is $\omega_d = \omega_n\sqrt{1-\zeta^2} = 49.96$ rad/s.

Substituting the problem parameters along with A_2, B_1, and B_2 into Eq. (47) gives the complete response of the system including the transient phase.



(a) Complete response and transient terms



(b) Zoom of complete response and steady-state response

Figure 8: Complete response of the system in Fig. 1 for the case in Problem 2 Question 3d.

Question 3d – 5 points

Figure 8(a) shows the complete response and Fig. 8(b) shows a zoom where the complete response including the transient terms is compared with the steady-state response. The decaying contribution is plotted separately in Fig. 8(a) to appreciate the duration of the transient phase. The horizontal dashed lines denote the steady-state response amplitude. We can observe that the transient phase has practically ended after ten oscillation periods, such that the complete response and the steady-state response overlap.

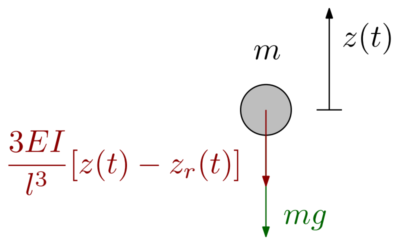


Figure 9: Free-body diagram for the mass in Fig. 3.

Problem 3 Solution – 20 points

Question 1 – 5 points (correct elastic force) + 5 points (equation of motion)

The massless beam acts as a linear spring with equivalent spring constant

$$k_{\text{eq}} = \frac{3EI}{l^3} \tag{49}$$

The equivalent spring constant (49) can be derived by computing the ratio between a static vertical force F applied at the beam tip and the resulting elastic displacement relative to the beam root (obtained from the beam in pure bending theory for a uniform beam). The elastic reaction applied to the tip mass due to the beam bending deflection is only sensitive to $z(t) - z_r(t)$. This is because the beam does not apply any restoring elastic force to the tip mass if it not deflected, that is, when $z(t) = z_r(t)$.

The equation of motion is derived by measuring $z(t)$ with respect to a reference frame that is fixed with the undeformed beam with no moving wall. In this case, the gravity load appears in the equation as a static load that is balanced by the elastic reaction due to the static beam deflection. Another possibility is to choose the reference frame so that its origin is offset by the static deflection due to the gravity load. In this case, the gravity load does not appear in the equation because it is already accounted for in the definition of the reference frame and corresponding meaning of $z(t)$. Both choices are given full points because the problem did not clarify what reference frame to choose.[1]

Considering the free-body diagram in Fig. 9, the equation of motion for the tip mass reads

$$m\ddot{z}(t) = -\frac{3EI}{l^3}\left[z(t) - z_r(t)\right] - mg \tag{50}$$

or

$$m\ddot{z}(t) + \frac{3EI}{l^3} z(t) = \frac{3EI}{l^3} z_r(t) - mg \tag{51}$$

where the moving wall introduces an harmonic excitation of the form

$$F(t) = F_0 \cos \omega t = \frac{3EI}{l^3} Z_r \cos \omega t = k_{\text{eq}} Z_r \cos \omega t \tag{52}$$

Note that the force applied to the mass due to the beam flexibility points downward in Fig. 9 because it provides a restoring effect.

Question 2a – 5 points

Equation (51) can be recast as

$$\ddot{z}(t) + \omega_n^2 z(t) = \omega_n^2 Z_r \cos \omega t - g \tag{53}$$

where, using the parameters in Table 3, the natural frequency is given by

$$\omega_n = \sqrt{\frac{3EI}{ml^3}} = 15.4919\,\text{rad/s} \tag{54}$$

The natural frequency is independent of the excitation. This is always the case if the system remains linear.

[1]Solutions where gravity has been neglected have also been given full points because the problem focused on the dynamics of the system and we only discussed how to deal with gravity briefly in class.

Question 2b – 5 points

In this question, we focus the attention on the dynamic contribution of the motion

$$\overline{z}(t) = z(t) - z_e \tag{55}$$

where

$$z_e = -g/\omega_n^2 \tag{56}$$

is the static equilibrium due to gravity with no moving wall. Using $z(t) = \overline{z} + z_e$, Eq. (51) becomes

$$\ddot{\overline{z}}(t) + \omega_n^2 \overline{z}(t) = \omega_n^2 Z_r \cos \omega t \tag{57}$$

This is the same equation that is obtained by neglecting gravity from the outset or by choosing the origin of the reference frame by the beam static deflection due to gravity.

Without damping, the solution of Eq. (57) contains both the natural frequency ω_n and the excitation frequency ω (see the derivation done in class). Assuming that slight damping is present in the system, we can expect the term associated with ω_n to vanish after a transient phase. At steady state, we assume the forced response as

$$\overline{z}(t) = A \cos \omega t \tag{58}$$

The impact of damping on amplitude and phase the response is neglected as specified in the Problem description. Substituting Eq. (58) into Eq. (53) gives

$$\overline{z}(t) = \frac{Z_r}{1 - \omega^2/\omega_n^2} \cos \omega t \tag{59}$$

To meet the requirement $|z(t) - z_e| = |\overline{z}(t)| \leq 1.1|Z_r|$, the excitation frequency $\omega < \omega_n$ must satisfy

$$\frac{1}{1 - \omega^2/\omega_n^2} \leq 1.1 \tag{60}$$

The maximum value of ω that meets this requirement is

$$\omega_{\max} = 0.3015\omega_n = 4.6710\,\mathrm{rad/s} \tag{61}$$