PURDUE UNIVERSITY

College of Engineering
School of Aeronautical and Astronautical Engineering

AAE 666: Nonlinear Dynamics, Systems, and Control

HW8

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Exercise 1

Obtain the describing function of the nonlinear function

$$\phi(y) = y^5$$

Solution:
Since $\phi(y)$ is an odd function, and if we subject this to a sinusoidal input we have

$$\phi(a\sin(\theta)) = a^5 \sin^5(\theta).$$

To find $\sin^5\theta$ we use the following procedure

$$\begin{gathered}
e^{(i\theta)n} = e^{ni\theta} \\
(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta) \\
(\cos(\theta) + i\sin(\theta))^5 = \cos(5\theta) + i\sin(5\theta) \\
\cos^5(\theta) + 5i\cos^4(\theta)\sin(\theta) - 10\cos^3(\theta)\sin^2(\theta) - 10i\cos^2(\theta)\sin^3(\theta) + \cdots \\
5\cos(\theta)\sin^4(\theta) + i\sin^5(\theta) = \cos(5\theta) + i\sin(5\theta)
\end{gathered}$$

By equating the imaginary terms we have

$$5\cos^4(\theta)\sin(\theta) - 10\cos^2(\theta)\sin^3(\theta) + \sin^5(\theta) = \sin(5\theta)$$

Solve this for $\sin^5(\theta)$

$$\begin{aligned}
\sin^5(\theta) &= \sin(5\theta) - 5\cos^4(\theta)\sin(\theta) + 10\cos^2(\theta)\sin^3(\theta) \\
\sin^5(\theta) &= \sin(5\theta) + 5\cos^2(\theta)\sin(\theta)\Big(2\sin^2(\theta) - \cos^2(\theta)\Big) \\
\sin^5(\theta) &= \sin(5\theta) + \frac{5}{2}\sin(2\theta)\cos(\theta)\Big(1 - \cos(2\theta) - \frac{1}{2} - \frac{1}{2}\cos(2\theta)\Big) \\
\sin^5(\theta) &= \sin(5\theta) + \frac{5}{2}\sin(2\theta)\cos(\theta)\Big(\frac{1}{2} - \frac{3}{2}\cos(2\theta)\Big) \\
\sin^5(\theta) &= \sin(5\theta) + \frac{5}{4}\sin(2\theta)\cos(\theta) - \frac{15}{4}\sin(2\theta)\cos(2\theta)\cos(\theta) \\
\sin^5(\theta) &= \sin(5\theta) + \frac{5}{4}\sin(2\theta)\cos(\theta) - \frac{15}{8}\sin(4\theta)\cos(\theta) \\
\sin^5(\theta) &= \sin(5\theta) + \frac{5}{8}\Big(\sin(3\theta) + \sin(\theta)\Big) - \frac{15}{16}\Big(\sin(5\theta) + \sin(3\theta)\Big) \\
\sin^5(\theta) &= \frac{5}{8}\sin(\theta) - \frac{5}{16}\sin(3\theta) + \frac{1}{16}\sin(5\theta).
\end{aligned}$$

From this we can deduce,

$$b_1(a) = \frac{5a^5}{8}.$$

Hence,

$$N(a) = \frac{b_1(a)}{a} = \frac{5a^4}{8}.$$

Exercise 2

Determine whether or not the following Duffing system has a periodic solution. Determine the approximate amplitude and period of all periodic solutions.

$$\ddot{y} - \dot{y} + y^3 = 0$$

Solution:
For this system we have

$$\begin{aligned} \ddot{y} - y &= u \\ u &= -\phi(y). \end{aligned}$$

Taking the Laplace transformation of the first equation we have

$$\begin{aligned} s^2\hat{y} - \hat{y} &= \hat{u} \\ \hat{G}(s) &= \frac{\hat{y}}{\hat{u}} = \frac{1}{s^2 - 1}. \end{aligned}$$

we know for $\phi(y) = y^3$

$$N(a) = \frac{3a^2}{4}$$

from **Example 152** on p. 188 of the Notes. Therefore, for some pair $a, \omega > 0$ using condition (15.7) we have

$$\begin{aligned} 1 + \frac{1}{(j\omega)^2 - 1}\left(\frac{3a^2}{4}\right) &= 0 \\ 1 - \frac{3a^2}{4\omega^2 + 4} &= 0 \\ \omega &= \frac{\sqrt{3a^2 - 4}}{2}. \end{aligned}$$

Hence, we predict that this system has periodic solutions of all amplitudes a and with approximate periods

$$T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{3a^2 - 4}}.$$

Exercise 3

Determine whether or not the following Duffing system has a periodic solution. Determine the approximate amplitude and period of all periodic solutions.

$$\ddot{y} + \mu(\frac{\dot{y}^3}{3} - \dot{y}) + y = 0$$

Solution:
For this system we have

$$\begin{aligned} \ddot{y} - \mu\dot{y} + y &= u \\ u &= -\phi(y) \\ \phi(y) &= -\frac{\mu}{3}\dot{y}^3 \end{aligned}$$

Taking the Laplace transformation of the first equation we have

$$\begin{aligned} s^2\hat{y} - \mu s\hat{y} + \hat{y} &= \hat{u} \\ \hat{G}(s) = \frac{\hat{y}}{\hat{u}} &= \frac{1}{s^2 - \mu s + 1}. \end{aligned}$$

for $\phi(y) = -\frac{\mu}{3}\dot{y}^3$ if $y(t) = a\sin(\omega t)$

$$\begin{aligned} \dot{y}^3 &= \Big(a\omega\cos(\omega t)\Big)^3 = a^3\omega^3\cos^3(\omega t) \\ &= a^3\omega^3\Big(\frac{3}{4}\cos(\omega t) + \frac{1}{4}\cos(3\omega t)\Big) \\ &\approx \frac{3a^3\omega^3}{4}\cos(\omega t) \\ &\approx \frac{3a^3\omega^3}{4}\sin(\omega t + \frac{\pi}{2}) \end{aligned}$$

Then,

$$\begin{aligned} b_1(a,\omega) &= \frac{3a^3\omega^3}{4}e^{j\frac{\pi}{2}} \\ &= \frac{3a^3\omega^3}{4}\Big(\cos(\frac{\pi}{2}) + j\sin(\frac{\pi}{2})\Big) \\ &= \frac{3ja^3\omega^3}{4} \end{aligned}$$

Thus, the describing function becomes

$$N(a,\omega) = \frac{3ja^2\omega^3}{4}$$

from **Example 152** on p. 188 of the Notes. Therefore, for some pair $a, \omega > 0$ using condition (15.7) we have

$$\begin{aligned} 1 + \frac{1}{(j\omega)^2 - \mu j\omega + 1}\Big(\frac{3ja^2\omega^3}{4}\Big) &= 0 \\ 4\omega^2 + 4j\mu\omega - 4 &= 3ja^2\omega^3 \\ (4\omega^2 - 4) + j(4\mu\omega - 3a^2\omega^3) &= 0. \end{aligned}$$

From this we get the following values

$$\omega = 1 \qquad a = \sqrt{\frac{4\mu}{3}}$$

Hence, we predict that this system has periodic solutions of all amplitudes $a = \sqrt{\frac{4\mu}{3}}$ and with approximate periods $T = 2\pi$.

Exercise 4

Use the describing function method to predict period solutions to

$$\dot{x}(t) = -x(t) - 2sgm(x(t-h))$$

Illustrate your results with numerical simulations.

Solution:
The system can be defined as

$$\begin{aligned}\dot{x}(t) &= -x(t) + u\\ u &= -\phi(y) = -2sgm(y)\\ y &= x(t-h).\end{aligned}$$

From the Notes we know that $N(a) = \dfrac{4}{\pi a}$ for $sgm(x)$, and therefore,

$$N(a) = \frac{8}{\pi a}.$$

Now we do the Laplace transform of the system

$$\begin{aligned}s\hat{x} &= -\hat{x} + \hat{u}\\ \hat{y} &= e^{-sh}\hat{x}\\ \therefore \hat{G}(s) &= \frac{\hat{y}}{\hat{u}} = \frac{e^{-sh}}{s+1}.\end{aligned}$$

Then

$$\begin{aligned}\hat{G}(j\omega) &= \frac{e^{-j\omega h}}{j\omega + 1} = \frac{e^{-j\omega h}(1-j\omega)}{1+\omega^2}\\ &= \frac{(1-j\omega)\Big(\cos(\omega h) - j\sin(\omega h)\Big)}{1+\omega^2}\\ &= \frac{\Big(\cos(\omega h) - \omega\sin(\omega h)\Big) - j\Big(\omega\cos(\omega h) + \sin(\omega h)\Big)}{1+\omega^2}.\end{aligned}$$

From

$$1 + \hat{G}(j\omega)N(a) = 0$$

$$1 + \frac{\Big(\cos(\omega h) - \omega\sin(\omega h)\Big) - j\Big(\omega\cos(\omega h) + \sin(\omega h)\Big)}{1+\omega^2}\left(\frac{8}{\pi a}\right) = 0$$
$$\Big(8\cos(\omega h) - 8\omega\sin(\omega h) + \pi a\omega^2 + \pi a\Big) - 8j\Big(\omega\cos(\omega h) + \sin(\omega h)\Big) = 0.$$

Thus, we can numerically compute an ω and a value that satisfies the equation above. That will give us a period solution for the system. Let $h = 1$, then we can numerically compute the ω value from the imaginary part

$$\begin{aligned} \omega\cos(\omega h) + \sin(\omega h) &= 0 \\ \tan(\omega h) &= -\omega \\ \tan(\omega) &= -\omega. \end{aligned}$$

We use MATLAB to numerically solve the smallest $\omega > 0$ value to be,

$$\omega = 2.0287577.$$

Next we plug the h and $omega$ value into the real part

$$8\cos(\omega h) - 8\omega\sin(\omega h) + \pi a\omega^2 + \pi a = 0$$

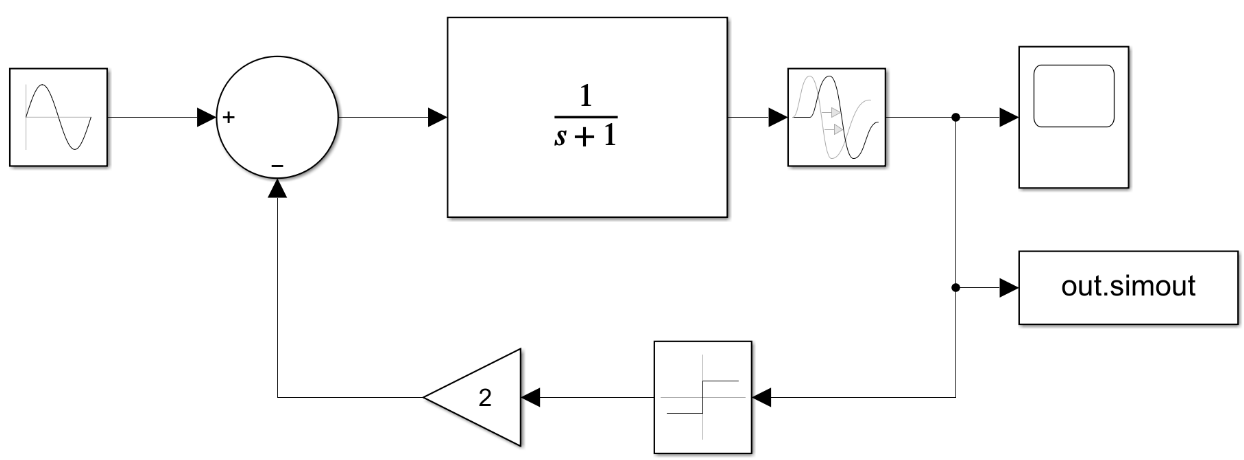
which gives us

$$a = 1.1259$$

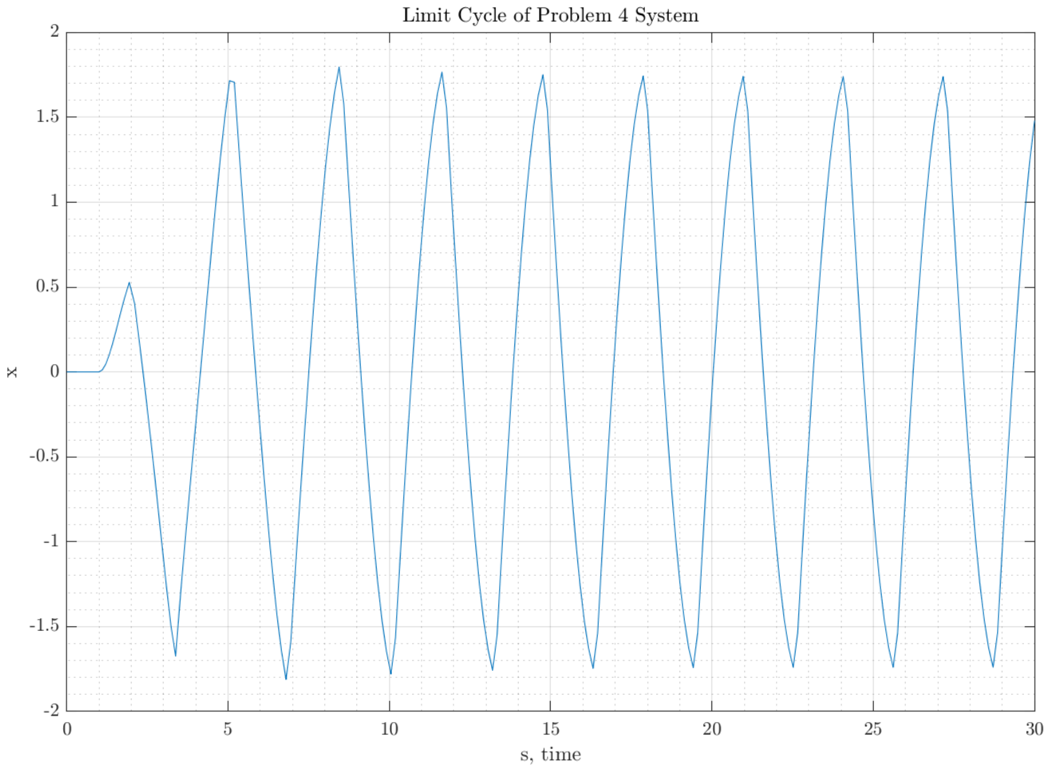
This means that we have a sinusoidal input with amplitude of $a = 0.5629$ and period of

$$T = \frac{2\pi}{\omega} = 3.0971$$

that gives a period solution to the system. Thus, we feed an input of $a\sin(\omega t)$ to the following system



and we obtain the following simulation.



We can see that the results give us a limit cycle for the system.

MATLAB CODE:

```
% AAE 666 HW9 PROBLEM 4 MATLAB CODE
% TOMOKI KOIKE

% Houskeeping commands
clear all; close all; clc;
%%
% Compute omega and a values for the describing function numerically
h = 1  % time delay constant
tol = 1e-6;  % tolerance
for omega = 0.1:0.0000001:10
    delta = abs(tan(omega) + omega);
    if delta < tol
        break
    end
end
format long;
```

```
disp(omega);
format;

% Compute a
syms a
assume(a, {'positive', 'real'});
eqn = 8*cos(omega*h) - 8*omega*sin(omega*h) + pi*a*omega^2 + pi*a == 0;
a = solve(eqn, a);
format long ;
disp(a);
format
a = double(a);
%%
% Period
T = 2*pi / omega
%%
% Simulate
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
out = sim('p4_signum.slx');

t = out.tout;
x = out.simout.signals.values;

fig = figure("Renderer","painters", "Position",[60 60 900 600]);
    plot(t, x);
    grid on; grid minor; box on;
    xlabel('s, time')
    ylabel('x')
    title('Limit Cycle of Problem 4 System')
saveas(fig, 'p4_result.png');
```

Exercise 5

Consider the double integrator

$$\ddot{q} = u$$

subject to a saturating PID controller

$$u = -k_P q - k_D \dot{q} - sat(\tilde{u}) \qquad \text{where} \qquad \tilde{u} = k_I \int q dq$$

(a) For $k_P = 1$ and $k_D = 2$ determine the largest of $k_I \leq 0$ for which the closed loop system is asymptotically stable about $q(t) \equiv 0$.

(b) For $k_P = 1$ and $k_D = 2$, use the describing function method to determine the smallest value $k_I \geq 0$ for which the closed loop system has a periodic solution.

Solution:
(a) For the system we define $x_1 = q$, $x_2 = \dot{q}$, and $x_3 = \int q$. Then the system can be represented by the following state space

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 2x_2 - sat(k_I x_3) \\
\dot{x}_3 &= x_1
\end{aligned}$$

Now if we assume that $-1 \leq \tilde{u} \leq 1$, this can be rewritten to be

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 - 2x_2 - k_I x_3 \\
\dot{x}_3 &= x_1.
\end{aligned}$$

The state matrix A for this linear system becomes

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & -2 & -k_I \\ 1 & 0 & 0 \end{bmatrix}.$$

The eigenvalues for this is

$$eig(A) = \begin{bmatrix} \frac{0.1111}{\sigma_1} + \sigma_1 - 0.6667 \\ -0.6667 + \frac{-0.0556-0.0962\,\mathrm{i}}{\sigma_1} + \sigma_1\,(-0.5000 + 0.8660\,\mathrm{i}) \\ -0.6667 + \frac{-0.0556+0.0962\,\mathrm{i}}{\sigma_1} + \sigma_1\,(-0.5000 - 0.8660\,\mathrm{i}) \end{bmatrix}$$

where

$$\sigma_1 = \left(\sqrt{(0.5000\,k_I - 0.0370)^2 - 0.0014} - 0.5000\,k_I + 0.0370\right)^{0.3333}$$

The real part for this is

$$\left[\begin{array}{c} Re(eig(A)) = \frac{0.1111\,\sigma_4}{|\sigma_5|^{0.3333}} + |\sigma_5|^{0.3333}\,\sigma_4 - 0.6667 \\ \mathrm{real}\,(-\sigma_3) + \mathrm{real}\,(0.8660\,\sigma_5{}^{0.3333}\,\mathrm{i}) - \sigma_1 - \sigma_2 - 0.6667 \\ \mathrm{real}\,(\sigma_3) + \mathrm{real}\,(-0.8660\,\sigma_5{}^{0.3333}\,\mathrm{i}) - \sigma_1 - \sigma_2 - 0.6667 \end{array}\right]$$

where

$$\sigma_1 = \frac{0.0556\,\sigma_4}{|\sigma_5|^{0.3333}}$$
$$\sigma_2 = 0.5000\,|\sigma_5|^{0.3333}\,\sigma_4$$
$$\sigma_3 = \frac{0.0962\,\mathrm{i}}{\sigma_5{}^{0.3333}}$$
$$\sigma_4 = \cos\left(0.3333\,\mathrm{angle}\,(\sigma_5)\right)$$
$$\sigma_5 = \sqrt{(0.5000\,k_I - 0.0370)^2 - 0.0014} - 0.5000\,k_I + 0.0370$$

We numerically solve the largest value using MATLAB (code provided) by plugging in a value and check if all the eigenvalues have negative real parts, and we figure out that the bounds for k_I is

$$0 \le k_I < 2.$$

MATLAB CODE:

```
% Houskeeping commands
clear all; close all; clc;
%%
% The system matrix A
syms k_I
assume(k_I, {'real', 'positive'});
A = [0, 1, 0; -1, -2, -k_I; 1, 0, 0];
%%
ev = eig(A)
ev_real = real(ev)
%%
inc = 0.0001;
for ki = 1.999:inc:2.1
    ev_real_vals = double(subs(ev_real, k_I, ki));
    if any(ev_real_vals > 0)
```

```
        break
    end
end
format long;
disp(ki-inc);
format;
```

(b) For this part we use the fact that we set $U = -\phi(y)$ and $\phi(y) = -sat(k_I x_3)$ which means that $y = k_I x_3$. Then we can the Laplace transform and get

$$s^2\hat{q} = -\hat{q} - 2s\hat{q} + \hat{U}$$
$$\hat{y} = \frac{k_I}{s}\hat{q}$$

which gives us

$$\hat{G}(s) = \frac{\hat{y}}{\hat{u}} = \frac{k_I}{s(s^2+2s+1)}.$$

Now,

$$\begin{aligned}\hat{G}(j\omega) &= \frac{k_I}{j\omega(-\omega^2+2j\omega+1)} \\ &= \frac{k_I\Big(-2\omega^2 - j\omega(1-\omega^2)\Big)}{4\omega^4+(\omega-\omega^3)^2}.\end{aligned}$$

From the **Notes** we know that the describing function for a saturation function becomes

$$N(a) = \begin{cases} 1 & if \quad 0 \le a \le 1 \\ \frac{2}{\pi}\Big[arcsin(\frac{1}{a}) + \frac{\sqrt{a^2-1}}{a^2}\Big] & if \quad 1 < a \end{cases}$$

If $0 \le a \le 1$ then $\omega = 0$ and is not adequate for out purpose, and therefore

$$1 + \hat{G}(j\omega)N(a) = 0$$
$$1 + \frac{k_I\Big(-2\omega^2 - j\omega(1-\omega^2)\Big)}{4\omega^4+(\omega-\omega^3)^2}N(a) = 0.$$

We separate this into the real and imaginary parts.

$$Real := 4\omega^4 + (\omega-\omega^3)^2 - 2k_I\omega^2 N(a)$$
$$Imag := -jk_I\omega(1-\omega^2)N(a)$$

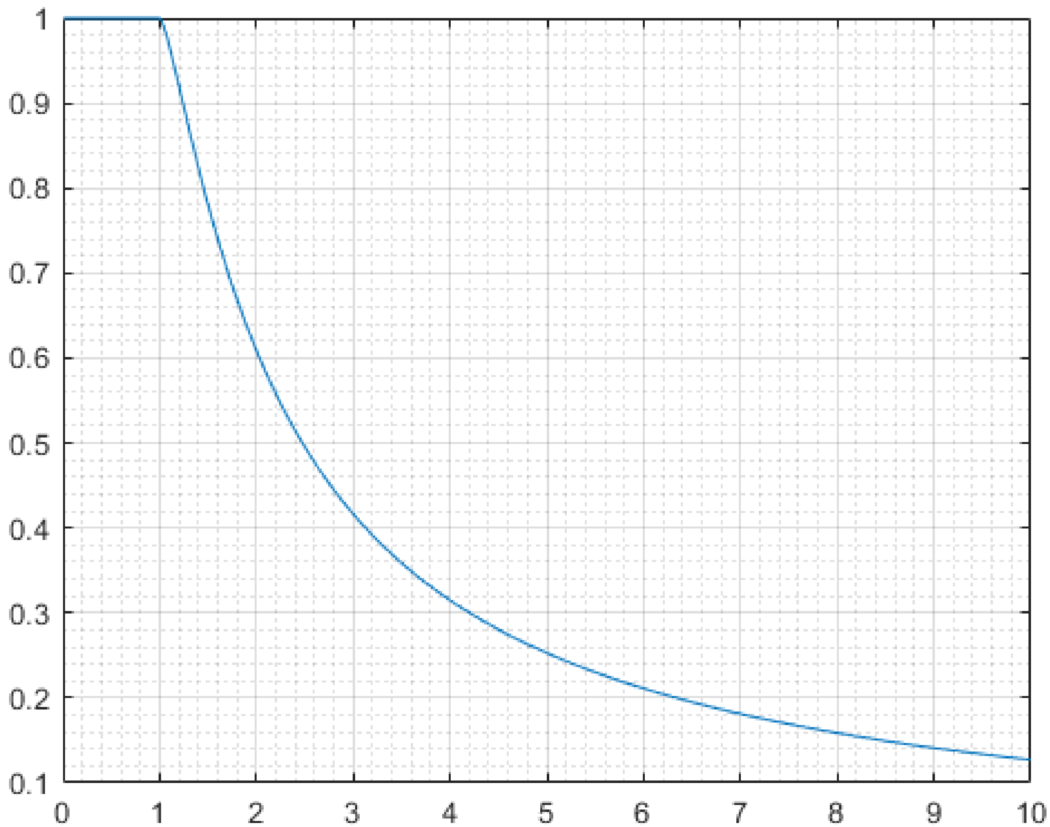
We solve this so that the imaginary and real parts are equal to 0. From the imaginary part we have

$$\omega = 1$$

since $N(a)$ at $0 \leq a \leq 1$ is a positive function. Then we solve the real part as

$$4 - 2k_I N(a) = 0.$$

Since, N(a) behaves as the following



the smallest $k_I \geq 0$ value will be when $N(a) = 1$ where $0 \leq a \leq 1$, and therefore,

$$k_I = 2.$$