Analysis Fundamentals: A Primer

As we have discussed, a norm imbues a vector space with a sense of distance. Distances, in turn, allow us to define basic topological concepts, including convergence of sequences and series, continuous functions, etc. In this short set of notes, we review some of the main concepts of analysis, in the context of normed linear spaces. We will do this with an end-goal in mind, though: we will establish the following fundamental result from functional analysis.

Proposition 1 *The space of continuous functions on the interval $[a, b]$ with the norm*

$$\|\boldsymbol{f}\|_\infty = \sup_{t\in[a,b]} |f(t)|, \tag{1}$$

is a Banach space.

In our review of analysis below, we will introduce key concepts that make sense of all the words above, then we will prove some fundamental results, then we will use these results to prove the proposition.

A *Banach space* is a linear vector space with a norm that is "complete". It should be clear that the space of continuous functions on an interval is a linear vector space, and you should be able to quickly see that the norm defined above is valid (see Section 2 below if you are unsure of what is meant by sup above).

So all the work is showing that this space of functions is "complete" when distances are measured using $\|\cdot\|_\infty$. This word "complete" means that every sequence of elements in this space whose terms get closer to one another as the sequence progresses actually converges to something in the space. This is formalized as "every Cauchy sequence is a convergent sequence". To understand what this means, we need to say carefully what it means for a sequence of vectors to converge, and then define carefully what we mean by "Cauchy sequence".

In what follows below, $\mathcal{S}$ is a linear vector space equipped with the generic norm $\|\cdot\|$.

1 Convergent sequences and Cauchy sequences

Definition 1 *A sequence of vectors $\{\boldsymbol{x}_n\}_{n=1}^\infty \subset \mathcal{S}$ is said to* **converge** *if there exists a $\boldsymbol{x}^\star \in \mathcal{S}$ such that for every $\epsilon > 0$, there exists an N_ϵ such that*

$$\|\boldsymbol{x}_n - \boldsymbol{x}^\star\| < \epsilon \quad \textit{for all} \quad n \geq N_\epsilon.$$

We write

$$\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^\star.$$

This is a very straightforward extension of the definition you learned in Calculus, only with $\mathcal{S}$ as the real numbers $\mathbb{R}$ (one dimensional) and the absolute value $|\cdot|$ as the norm.

Some quick examples (you can prove these statements at home):

1. $\mathcal{S} = \mathbb{R}^2$, and $\|\cdot\|$ be the standard Euclieuan norm (ℓ_2). Set

$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{for some } 0 < \theta < 2\pi.$$

   Then for any $\boldsymbol{x}_0$, the sequence $\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1}$ does not converge.

2. Fix $0 < \alpha < 1$. Then the sequence $\boldsymbol{x}_n = \alpha\boldsymbol{A}\boldsymbol{x}_{n-1}$, for $\boldsymbol{A}$ as above, converges to $\mathbf{0}$ for any $\boldsymbol{x}_0$.

3. Fix $\alpha > 1$. Then the sequence $\boldsymbol{x}_n = \alpha\boldsymbol{A}\boldsymbol{x}_{n-1}$, for $\boldsymbol{A}$ as above, does not converge for any $\boldsymbol{x}_0$.

4. Let $\boldsymbol{A}$ be a square matrix that has eigenvalues whose magnitudes are less than 1. Then $\boldsymbol{x}_n = \boldsymbol{A}\boldsymbol{x}_{n-1}$ converges (to $\mathbf{0}$) for every $\boldsymbol{x}_0$.

5. Consider the sequence of functions $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ on $[0,1]$ with

$$x_n(t) = \begin{cases} 0, & 0 \leq t \leq \frac{1}{2} - \frac{1}{2n}, \\ nt + (1-n)/2, & \frac{1}{2} - \frac{1}{2n} \leq t \leq \frac{1}{2} + \frac{1}{2n} \\ 1, & \frac{1}{2} + \frac{1}{2n} \leq t \leq 1. \end{cases} \tag{2}$$

   (If you sketch this function, you will see that it increases on a line from 0 to 1 on an interval of length $1/n$ centered at $t = 1/2$.) Then taking $\|\cdot\|$ as the L_2 norm,

$$\|\boldsymbol{x} - \boldsymbol{y}\| = \left(\int_0^1 |x(t) - y(t)|^2 \, dt\right)^{1/2}, \tag{3}$$

   $\{\boldsymbol{x}_n\}$ converges (to a step function $\boldsymbol{x}^\star$ which is 0 for $t \leq 1/2$ and 1 for $t > 1/2$.

6. Consider the same sequence of functions above, but now let $\|\cdot\|$ be the sup-norm defined in (1). Then $\{\boldsymbol{x}_n\}$ does not converge. (Note that for the $\boldsymbol{x}^\star$ in the last part, $\|\boldsymbol{x}_n - \boldsymbol{x}^*\|_\infty = 1$ for all n.)

The last two examples above show us something critical: that the norm that comes with the space can play a critical role in determining which sequences converge and which ones do not. This issue is one of the differences between analysis in $\mathbb{R}^N$ and analysis in spaces of functions. In $\mathbb{R}^N$, it turns out that all valid norm are "topologically equivalent"; if a sequence converges in one norm, it will converge (to the same point) in all other norms. So in the first four examples above, it did not really matter that we were using the Euclidean norm. For spaces of functions, different norms can certainly make a difference.

Our next definition gives us another notion of convergence.

Definition 2 *A sequence $\{\boldsymbol{x}_n\}$ is called a* **Cauchy sequence** *if for every $\epsilon > 0$ there exists an N_ϵ such that*

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \leq \epsilon \quad \textit{for all} \quad n, m \geq N_\epsilon.$$

It is tempting to think that Cauchy sequences and convergent sequences are the same thing. But the difference between them becomes apparent when there are things "missing" from the space the sequence lies in.

For example, consider the set $\mathcal{X} = (0, 1]$ with the standard distance (absolute value) and the sequence $x_n = 1/n,\ n = 1, 2, \ldots$; each of these x_n are in $\mathcal{X}$. It should also be clear that $\{x_n\}$ is a Cauchy sequence (take $N_\epsilon = \lfloor 1/\epsilon \rfloor$ ... e.g. for $n, m \geq 10$, we know that $|x_n - x_m| \leq 1/10$). It does not, however, converge in $\mathcal{X}$; there is no $x^\star \in \mathcal{X}$ that meets the criteria of Definition 1. It is because 0 is missing from $\mathcal{X}$; $\{x_n\}$ converges just fine in $[0, 1]$.

There are very rich spaces of functions that are missing points in this very same manner. Let $\mathcal{S}$ be the space of continuous functions with the L_2 norm in (3), and consider the sequence of functions $\{x_n(t)\}$ defined in (2). As

$$\int_0^1 |x_n(t) - x_m(t)|^2 \, dt \ \leq \ \frac{1}{n}, \quad \text{for all } m \geq n,$$

it is clear that this sequence is Cauchy. But this sequence does not converge in $\mathcal{S}$ with the L_2 norm; the limit point is a discontinuous function and hence not in $\mathcal{S}$.

One of the key ways we think about approximating arbitrary functions is by constructing a sequence of other functions with prescribed structure that get closer and closer to our target. For this to all make sense, we want to make sure that everything we "converge to" (in the Cauchy sense) is actually there in the space. That motivates the following definitions.

Definition 3 *Let $\mathcal{S}$ be a vector space equipped with the norm $\|\cdot\|$. We call $\mathcal{S}$* **complete** *if every Cauchy sequence in $\mathcal{S}$ converges in to a point in $\mathcal{S}$. A complete normed linear space is also called a* **Banach space**[1].

Note that the examples above show that the space of continuous functions on an interval $[a, b]$ is not complete under the L_2 norm, and hence not a Banach space. We will show below that it is complete under the $\|\cdot\|_\infty$ norm.

There is also a related definition for inner product space.

Definition 4 *Let $\mathcal{S}$ be a vector space equipped with an inner product $\langle \cdot, \cdot \rangle$. We call $\mathcal{S}$ a* **Hilbert space**[2] *if it is complete under the norm induced by this inner product.*

We close this section with two more statements about how Cauchy sequences behave that we will use later on, and then finally one of the central results in the analysis of the reals: that *every* sequence inside an interval has to have a convergent subsequence.

Proposition 2 *Let $\{\boldsymbol{x}_n\}_{n=1}^\infty$ be a Cauchy sequence in $(\mathcal{S}, \|\cdot\|)$. If $\{\boldsymbol{x}_n\}$ has a convergent subsequence in $\mathcal{S}$, then $\{\boldsymbol{x}_n\}$ converges in $\mathcal{S}$.*

[1] Stefan Banach was a Polish mathematician who was integral (pun!) in developing the theory of linear operators in early 20th century.

[2] David Hilbert was probably the highest-profile mathematician of the early 20th century. In 1900, he gave an address where he outlined the 23 most important problems open in mathematics; it set a large part of the research agenda for math for 50 years.

Proof Let $\boldsymbol{x}^*$ be the limit point of the convergent subsequence. Fix $\epsilon > 0$. We know that there exists an N such that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \leq \epsilon/2, \quad \text{for all} \ \ n, m \geq N.$$

Since $\{\boldsymbol{x}_n\}$ has a convergent subsequence, we also know that there is a $p \geq N$ such that $\|\boldsymbol{x}_p - \boldsymbol{x}^*\| \leq \epsilon/2$, and so

$$\|\boldsymbol{x}_n - \boldsymbol{x}_p\| \leq \epsilon/2, \quad \text{for all } n \geq N.$$

Thus for all $n \geq N$,

$$\|\boldsymbol{x}_n - \boldsymbol{x}^*\| \leq \|\boldsymbol{x}_n - \boldsymbol{x}_p\| + \|\boldsymbol{x}_p - \boldsymbol{x}^*\| \leq \epsilon,$$

and so $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}^*$. ∎

Proposition 3 *Let $\{\boldsymbol{x}_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(\mathcal{S}, \|\cdot\|)$. Then $\{\boldsymbol{x}_n\}$ is bounded: there exists an M such that*

$$\|\boldsymbol{x}_n\| \leq M \quad \textit{for all } n \geq 1.$$

Proof This is straightforward. Since $\{\boldsymbol{x}_n\}$ is Cauchy, there exsists and N such that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\| \leq 1, \quad \text{for all} \ \ m, n \geq N,$$

so by the reverse triagnle equality, $\|\boldsymbol{x}_n\| \leq \|\boldsymbol{x}_N\| + 1$ for all $n \geq N$. Thus we can take

$$M = \max\left\{\|\boldsymbol{x}_1\|, .., \|\boldsymbol{x}_{N-1}\|, \|\boldsymbol{x}_N\| + 1\right\}.$$

∎

The next Proposition, which again is one of the central results in analysis, is stated in the specialized scenario of real numbers using (the norm) $|x - y|$ to measure distance. It can, however, be generalized to far broader settings. This requires, though, building up a lot more mathematical machinery (in particular, the abstract notion of a compact set). These notions are useful, but we will not need them to achieve the goal stated at the beginning of these notes.

Proposition 4 *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of real numbers in the interval $[a, b]$, that is $a \leq x_n \leq b$ for all n. Then $\{x_n\}$ has a convergent subsequence.*

Proof For convenience, and without loss of generality, we consider $\{x_n\} \subset [0, 1]$ — any sequence in $[a, b]$ can be transformed into a sequence in $[0, 1]$ by subtracting a from each element then dividing by $b - a$, the convergent subsequence found as below and then transformed back. Define the dyadic intervals as

$$D_0 = [0, 1], \quad D_{j,k} = [2^{-j}k, 2^{-j}(k + 1)], \quad j \geq 1, k = 0, 1, \ldots, 2^j - 1.$$

Note that $D_{j+1,2k} \subset D_{j,k}$, $D_{j+1,2k+1} \subset D_{j,k}$ and $D_{j+1,2k} \cup D_{j+1,2k+1} = D_{j,k}$.

We construct a subsequence $\{x'_j\}_{j=1}^{\infty}$ as follows.

1. Since $\{x_n\}$ has an infinite number of terms, at least one of[3] $\{x_n\} \cap D_{1,0}$ or $\{x_n\} \cap D_{1,1}$ must also be infinite. If $\{x_n\} \cap D_{1,0}$, take $\kappa_1 = 0$ and $b_j = 0$; otherwise take $\kappa_1 = 1$ and $b_j = 1$. Take
$$n_1 = \text{smallest } n \geq 1 \text{ such that } x_n \in D_{1,\kappa_1},$$
and set $x'_1 = x_{n_1}$.

2. For $j = 2, 3, \ldots$, we know that since $\{x_n\} \cap D_{j-1,\kappa_{j-1}}$ is infinite, at least one of $\{x_n\} \cap D_{j,\kappa_{j-1}}$ or $\{x_n\} \cap D_{j,\kappa_{j-1}+1}$ is also infinite. If $\{x_n\} \cap D_{j,\kappa_{j-1}}$ is infinite, take $\kappa_j = 2\kappa_{j-1}$ and $b_j = 0$; otherwise take $\kappa_j = 2\kappa_{-1} + 1$ and $b_j = 1$. Take
$$n_j = \text{smallest } n \geq n_{j-1} + 1 \text{ such that } x_n \in D_{j,\kappa_j},$$
and set $x'_j = x_{n_j}$.

Now let x^* be the number with binary expansion specified by the $\{b_j\}$ above; that is, take
$$x^* = b_1/2 + b_2/4 + b_3/8 + \cdots = \sum_{j \geq 1} 2^{-j} b_j.$$
Since the b_j are binary, it should be clear that the sum above converges, and, crucially, x^* is in the interval of interest $[0, 1]$. By construction, $x^* \in D_{j,\kappa_j}$ for all $j \geq 0$ and so since x'_j is also in D_{j,κ_j}, we have
$$|x'_j - x^*| \leq 2^{-j}.$$
Thus $\{x'_j\}$ converges with $\lim_{j\to\infty} x'_j = x^*$. ■

2 Continuous and bounded functions

Before proceeding further, we should recall the exact definition of a continuous function.

Definition 5 *A function $f : \mathcal{R} \to \mathbb{R}$ is called* **continuous** *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that*
$$|x_1 - x_2| < \delta \quad \Rightarrow \quad |f(x_1) - f(x_2)| \leq \epsilon.$$

The following proposition codifies the main feature of continuous functions: we can pull limit operations from outside the function argument to inside.

Proposition 5 *Let f be a continuous function on $\mathbb{R}$, and let $\{x_n\}$ be a sequence with $\lim_{n\to\infty} x_n = x^\star$. Then $\lim_{n\to\infty} f(x_n) = f(x^\star)$.*

Proof Fix $\epsilon > 0$. Since f is continuous, we that there exists a $\delta > 0$ such that $|f(x_n) - f(x^\star)| < \epsilon$ whenever $|x_n - x^\star| < \delta$. We also know that there exists N_δ such that $|x_n - x^\star\| < \delta$ for $n \geq N_\delta$. Thus
$$n \geq N_\delta \quad \Rightarrow \quad |f(x_n) - f(x^\star)|,$$

[3] Here we are using the notation $\{x_n\} \cap S$ to mean the subsequence of $\{x_n\}$ with all the terms outside of S removed.

and so

$$\lim_{n\to\infty} f(x_n) = f(x^\star).$$

■

We will sometimes use the following notation:

Definition 6 *We write the space of continuous functions on the interval* $[a, b]$ *as* $\mathcal{C}[a, b]$.

It should be clear that $\mathcal{C}[a, b]$ is a linear vector space.

We should also make perfectly clear what we mean by 'sup' (suprememum) and its counter part 'inf' (infimum). These are more refined versions the maximum and minimum of a set. To see why this is necessary, consider the following. When a set $\mathcal{X}$ is finite, e.g.

$$\mathcal{X} = \{3, -1, -1/2, 14, 8\},$$

talking about the minimum and maximum of $\mathcal{X}$ is not at all controversial; above $\min \mathcal{X} = -1$ and $\max \mathcal{X} = 14$. But when $\mathcal{X}$ has an infinite number of elements, things are trickier. For example, the set

$$\mathcal{X} = \left\{\frac{1}{n},\ n = 1, 2, 3, \ldots\right\}, \tag{4}$$

has no minimum. You are tempted to say that 0 is the minimum, but 0 is not a member of that set. Similarly, the set

$$\mathcal{X} = \left\{1 - e^{-n},\ n = 1, 2, 3, \ldots\right\}, \tag{5}$$

has no maximum.

Rather than just giving up In the situations above, we can gain a little clarity by adding a little nuance. The **infimum** of a set $\mathcal{X}$ is the greatest lower bound. In the example (4) above, it is clear that 0 is smaller than every member of $\mathcal{X}$, and 0 is the largest number for which this is true. Thus, while the minimum does not exist, we can safely write

$$\inf\left\{\frac{1}{n},\ n = 1, 2, 3, \ldots\right\} = 0.$$

The **supremum** of a set $\mathcal{X}$ is the smallest upper bound. So for the example in (5), the maximum does not exist, but

$$\sup\left\{1 - e^{-n},\ n = 1, 2, 3, \ldots\right\} = 1.$$

The definition of the norm we are interested in in (1) above refers to the supremum of a function, while above we defined it in the context of a set. In these settings, we understand that the supremum of a function over an interval is simply the sup of its image. That is, when we write

$$\sup_{t\in[a,b]} |f(t)|,$$

what we mean is

$$\sup \mathcal{Y}, \quad \text{where} \;\; \mathcal{Y} = \{y \in \mathbb{R} \;:\; y = |f(x)| \text{ for some } x \in [a, b]\}.$$

With this usage, we can say things like

$$\inf_{x\in\mathbb{R}}\ e^{-x} = 0,$$

and

$$\sup_{t\in\mathbb{R}}\ \tan^{-1}(t) = \frac{\pi}{2}.$$

There is also no inconsistencies if we use sup (or inf) when the maximum (or minimum) is well-defined, e.g.

$$\sup_{t\in[0,1]}\ t - t^2 = \max_{t\in[0,1]}\ t - t^2 = \frac{1}{4}.$$

Exercise. Now that we are completely comfortable with the definition

$$\|\boldsymbol{f}\|_\infty = \sup_{t\in[a,b]}\ |f(t)|,$$

you should convince yourself that $\|\cdot\|_\infty$ is a valid norm for on $\mathcal{C}[a,b]$ by writing out a formal proof.

It is also an interesting fact, and one that we will use below, that functions that are continuous on an interval also have to be bounded.

Proposition 6 *Let $f(t)$ be a continuous function on an interval $[a,b]$. Then f is bounded; there exists a finite M such that*

$$\sup_{t\in[a,b]} |f(t)| \leq M.$$

Proof We prove this by contradiction. Suppose that f is not bounded; then for every $k = 1, 2, \ldots$, we know the set $\{t \in [a,b] \ :\ |f(t)| > k\}$ is not empty. Create the sequence $\{\tau_k\}_{k\geq 1}$ as

$$\tau_k = \inf\left\{t \in [a,b] \ :\ |f(t)| > k\right\}.$$

We know from Propostion 4 that $\{\tau_k\}$ has a convergent subsequence; call this convergent subsequence $\{\tau'_k\}$. But while $\{\tau'_k\}$ converges, $\{f(\tau'_k\}$ does not, as it is unbounded in k. Thus $\boldsymbol{f}$ cannot be continuous.

■

3 The Banach space of continuous functions

With all the pieces we need in place, we turn to our main result. Recall that $\mathcal{C}[a,b]$ is the set of continuous (and so by Proposition 6 bounded) functions $f(t)$ on the interval $[a,b]$. Also recall that

$$\|\boldsymbol{f}\|_\infty = \sup_{t\in[a,b]} |f(t)|.$$

Proposition 7 *The set of continuous functions equipped with the* sup*-norm, $(\mathcal{C}[a,b], \|\cdot\|_\infty)$, is a Banach space.*

Proof As we have mentioned before, it should be clear that $\mathcal{C}[a,b]$ is a linear space, and that $\|\cdot\|_\infty$ is a valid norm on this space. So all we need to prove is that every Cauchy sequence in $\mathcal{C}[a,b]$ converges to an element of $\mathcal{C}[a,b]$. Let $\{\boldsymbol{x}_n\}_{n=1}^\infty$ be a Cauchy sequence in $\mathcal{C}[a,b]$; for every $\epsilon > 0$, there exists an N such that

$$\|\boldsymbol{x}_n - \boldsymbol{x}_m\|_\infty \leq \epsilon \quad \text{for all } n, m \geq N.$$

This is a pointwise bound on the functions $\boldsymbol{x}_n$ and $\boldsymbol{x}_m$; that is, when $n, m \geq N$

$$|x_n(t) - x_m(t)| \leq \epsilon \quad \text{for all } t \in [a,b]. \tag{6}$$

Moreover, we know that since $\{\boldsymbol{x}_n\}$ is Cauchy, it is also bounded, meaning that there is an M such that

$$|x_n(t)| \leq M \quad \text{for all } n \text{ and } t \in [a,b].$$

Fix t. The uniform bounds above tell us that $\{x_n(t)\}$ is a Cauchy sequence in the interval $[-M, M]$. By Proposition 4, $\{x_n(t)\}$ has a convergent subsequence, and by Proposition 2 we thus know that $\{x_n(t)\}$ converges; i.e. there exists an $x^*(t) \in [-M, M]$ such that

$$x^*(t) = \lim_{n\to\infty} x_n(t).$$

As M is independent of t, we know that $|x^*(t)|$ is also uniformly bounded by M.

Now that we know that $x_n(t) \to x^*(t)$ pointwise, we can show that this convergence is uniform. Again, we can fix $\epsilon > 0$ and choose N so that (6) holds for all $n, m \geq N$. Taking $m \to \infty$ (and so $x_m(t) \to x^*(r)$) we have

$$|x_n(t) - x^*(t)| \leq \epsilon \quad \text{for all } t \in [a,b], \tag{7}$$

for $n \geq N$.

It remains to show that the function $x^*(t)$ is continuous. Fix $\epsilon > 0$, and choose n large enough so that (7) holds with $\epsilon/3$. Because $x_n(t)$ is continuous, there is a $\delta > 0$ such that

$$|t_1 - t_2| \leq \delta \Rightarrow |x_n(t_1) - x_n(t_2)| \leq \epsilon/3,$$

and also

$$|x^*(t_1) - x^*(t_2)| \leq |x^*(t_1) - x_n(t_1)| + |x_n(t_1) - x_n(t_2)| + |x_n(t_2) - x^*(t_2)| \leq \epsilon.$$

$\blacksquare$