7750: Mathematical Foundations of machine learning

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See end of last note for what we cover in the first half of today's lecture.

Today's focus: Vector spaces and subspaces

Vector space S over field $\mathbb{F}$ contains vectors addition rule '+' and scalar multiplication rule '.' such that:

'+' obeys commutativity and associativity

$x+y = y+x$ $\qquad$ $x+(y+z) = (x+y)+z.$

for all $x, y \in S$.

There is unique zero vector 0 s.t.

$$x + 0 = x \quad \forall x \in S.$$

For each $x \in S$, there is unique inverse element $-x$ s.t

$$x + (-x) = 0.$$

'$\cdot$' obeys (for all $a, b \in \mathbb{F}$ and $x, y \in S$),

distributivity over addition : $a(x+y) = ax + ay$.

associativity : $a \cdot (b \cdot x) = (ab) \cdot x$.

There is multiplicative identity of $\mathbb{F}$ (called 1)

s.t $1 \cdot x = x \quad \forall x \in S$.

and additive identity of $\mathbb{F}$ (called 0) s.t.

$$\underbrace{0}_{\in \mathbb{F}} \cdot x = \underbrace{0}_{\in S}.$$

Defining property : Closure under scalar multiplication and vector addition.

$$x, y \in S \Rightarrow ax + by \in S \quad \forall a, b \in \mathbb{F}$$

Examples

- $\mathbb{R}^N$: What is the field $\mathbb{F}$, and '+', '$\cdot$'?
- Bounded, continous functions on interval $[a, b]$ that are real valued, with

'+' = pointwise addition '$\cdot$' = pointwise mult by scalar

- Non-standard addition, standard multiplication

  $\mathbb{F} = \{0, 1\}$, addition modulo 2.

__Exercises:__ (Standard addition / mult. rules apply).

① Is $\mathcal{F} := \{f : [0,1] \to [0,1],\ f \text{ continuous}\}$

a vector space?

② Is $\mathcal{F} := \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is polynomial of degree at most } l\}$

a vector space?

③ Is the space of L-Lipschitz functions

on $\mathbb{R}$ a vector space?

Subspaces

These are just vector spaces T when viewed as members of a larger collection. S.

① Can a subspace be empty?

Must contain at least zero vector $\mathbb{0}$.

In other words, $\{0\}$ is subspace of any vector space.

(2) How to make $\{v_1, v_2, \dots, v_n\} \subseteq \mathbb{R}^d$ a subspace? I.e. design T s.t $v_i \in T$ and T is vector space.

Ex: Are the following subspaces?

- ✗ $S \in \mathbb{R}^N$, $T = \{v : v \text{ has at most 5 non-zeroes}\}$
- ✓ $S \in C([0,1])$, $T = \{\text{polynomials of degree} \leq l\}$

Linear combinations, spans, bases

Span of vectors $\{v_1, \dots, v_n\}$ is the collection of all possible linear combinations

$$\text{Span}(\{v_1, \dots, v_n\}) = \{a_1 v_1 + \dots + a_n v_n : a_1, \dots a_n \in \mathbb{F}\}.$$

Ex:
- If $S = \mathbb{R}^3$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, what is the $\text{Span}(\{v_1, v_2\})$? $\left[\left\{ \begin{pmatrix} \alpha \\ \beta \\ 0 \end{pmatrix}, \alpha, \beta \in \mathbb{R} \right\} \right]$
- What is span of $\left[M = \{b_0(x-k), k \in \mathbb{Z}\} \right]$ $\quad b_0(x) = \begin{cases} 1, & -\frac{1}{2} \leq x < 1/2 \\ 0, & \text{otherwise} \end{cases}$

Span($\mathcal{M}$) = functions that are piecewise const. between half-integers.

— A set of vectors $\{v_i\}_{i=1}^{n}$ is <u>linearly</u> <u>dependent</u> if $\exists\, a_1, \dots a_n$ s.t at least one non-zero s.t.

$$\sum_{i=1}^{n} a_i v_i = 0.$$

On the other hand, if

$$\sum_{i=1}^{n} a_i v_i \iff a_i = 0 \ \forall i,$$ then

$\{v_i\}_{i=1}^{n}$ are <u>linearly</u> <u>independent</u>.

<u>Ex</u>: Are $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $v_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

linearly dependent? Y

What if we add $v_4 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$? N (why?)

— A <u>basis</u> for a subspace T of S is a (countable) set of linearly indpt. vectors $\mathcal{B}$

s.t $\text{Span}(\mathcal{B}) = T$.

Ex. Is $\mathcal{B}$ unique?

- Fact (prove this for yourself): Every subspace has basis, all bases contain the same number of elements. The # of such elements is dimension.

Related ex. Every $x \in T$ has unique representation $x = \sum_i \alpha_i v_i$ if $\{v_i\}$ form a basis. Proved in class.

Ex: Is following a basis for $\mathbb{R}^3$?

- $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ✓

- $v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ✗

- What is a basis for all functions that are non-zero on $[0, 2]$ and piecewise const. between integers? [Construct 2 zeroth-order splines]