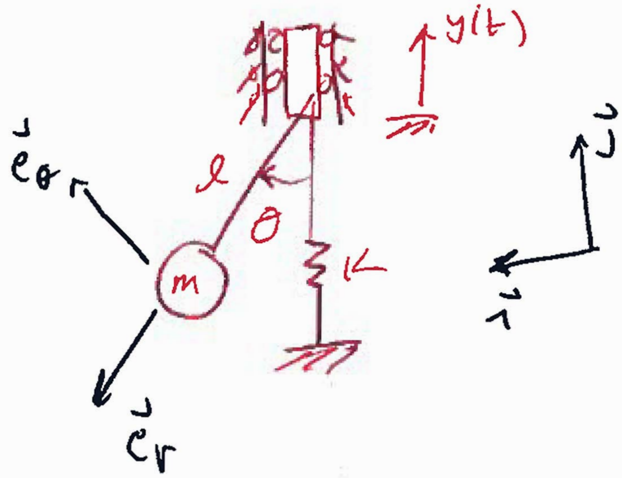
Problem 1

- Find EOM's using Lagrange's Eqs.



$q_1 = \theta, \ \dot{q}_1 = \dot{\theta}$

$q_2 = y, \ \dot{q}_2 = \dot{y}$

$\vec{i} = \cos\theta \vec{e}_\theta + \sin\theta \vec{e}_r \quad ; \quad \vec{j} = -\cos\theta \vec{e}_r + \sin\theta \vec{e}_\theta$

* velocity of mass (additive)

$$\vec{V} = l\dot{\theta}\vec{e}_\theta + \dot{y}\vec{j} = (l\dot{\theta} + \dot{y}\sin\theta)\vec{e}_\theta - \dot{y}\cos\theta \vec{e}_r$$

$$\vec{V}\cdot\vec{V} = (l\dot{\theta} + \dot{y}\sin\theta)^2 + (\dot{y}\cos\theta)^2$$

$$= \dot{y}^2 + (l\dot{\theta})^2 + 2l\dot{y}\dot{\theta}\sin\theta$$

KE

$$T = \frac{1}{2}m\left[\dot{y}^2 + (l\dot{\theta})^2 + 2l\dot{y}\dot{\theta}\sin\theta\right]$$

PE

$$V = mgl(1-\cos\theta) + mgy + \frac{1}{2}Ky^2$$

1^{st} EQN.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) - \frac{\partial L}{\partial \theta} = \cancel{Q_\theta} \rightarrow 0, \text{ NO NONCONSERVATIVE WORK}$$

$$L = \frac{1}{2}m\left[\dot y^2 + (l\dot\theta)^2 + 2l\dot y\dot\theta\sin\theta\right] - mgl(1-\cos\theta) - mgy - \frac{1}{2}ky^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta}\right) = \frac{d}{dt}\left(ml^2\dot\theta + ml\dot y\sin\theta\right)$$

$$= ml^2\ddot\theta + ml\ddot y\sin\theta + ml\dot y\dot\theta\cos\theta$$

$$\frac{\partial L}{\partial \theta} = ml\dot y\dot\theta\cos\theta - mgl\sin\theta$$

Thus,

$$ml^2\ddot\theta + ml\ddot y\sin\theta + mgl\sin\theta = 0$$

or

$$\boxed{\ddot\theta + \underbrace{\frac{\ddot y\sin\theta}{l}}_{\text{"extra" term not seen in a fixed pendulum}} + \frac{g}{l}\sin\theta = 0}$$

2^{nd} EQN.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = \cancelto{0}{Q_y} \text{, NO NONCONSERVATIVE WORK}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}\left(m\dot{y} + m\ell\dot{\theta}\sin\theta\right)$$

$$= m\ddot{y} + m\ell\ddot{\theta}\sin\theta + m\ell\dot{\theta}^2\cos\theta$$

$$\frac{\partial L}{\partial y} = -mg - ky$$

Thus

$$m\ddot{y} + m\ell\ddot{\theta}\sin\theta + m\ell\dot{\theta}^2\cos\theta + ky + mg = 0$$

or

$$\boxed{\ddot{y} + \underbrace{\ell\ddot{\theta}\sin\theta + \ell\dot{\theta}^2\cos\theta}_{\text{extra terms due to pendulum's swing motion}} + \frac{k}{m}y + g = 0}$$

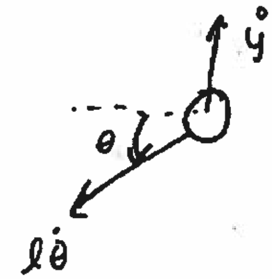
Note: I have measured y from zero spring stretch. If y is measured from static equilibrium, the term "+g" would be absent

First term is due to tangential acceleration; Second term due to normal acceleration

Problem 2

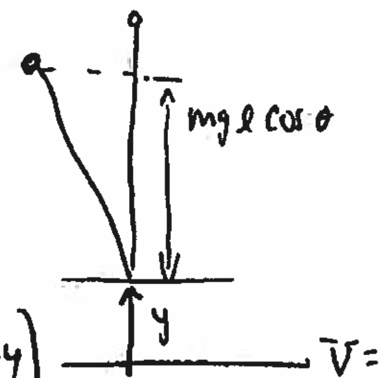
a.

$$v^2 = (-l\dot{\theta}\cos\theta)^2 + (-l\dot{\theta}\sin\theta + \dot{y})^2$$



$$T = \tfrac{1}{2} m v^2$$

$$V = mgl\cos\theta + mgy$$



$$L = T - V$$

$$= \tfrac{1}{2} m\left(l^2\dot{\theta}^2 - 2\dot{y}l\dot{\theta}\sin\theta + \dot{y}^2\right) - mg(l\cos\theta + y)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - m\ddot{y}l\sin\theta - m\dot{y}l\dot{\theta}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = -m\dot{y}l\dot{\theta}\cos\theta + mgl\sin\theta$$

$$\Downarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$ml^2\ddot{\theta} - m\ddot{y}l\sin\theta - mgl\sin\theta = 0$$

$$\ddot{\theta} + \left(\frac{-g}{l} - \frac{\ddot{y}}{l}\right)\sin\theta = 0$$

$$\boxed{\ddot{\theta} + \left(\frac{-g}{l} + \frac{A\Omega^2}{l}\cos\Omega t\right)\sin\theta = 0}$$

Stable when

$$\frac{A^2\Omega^2}{2gl} > 1$$

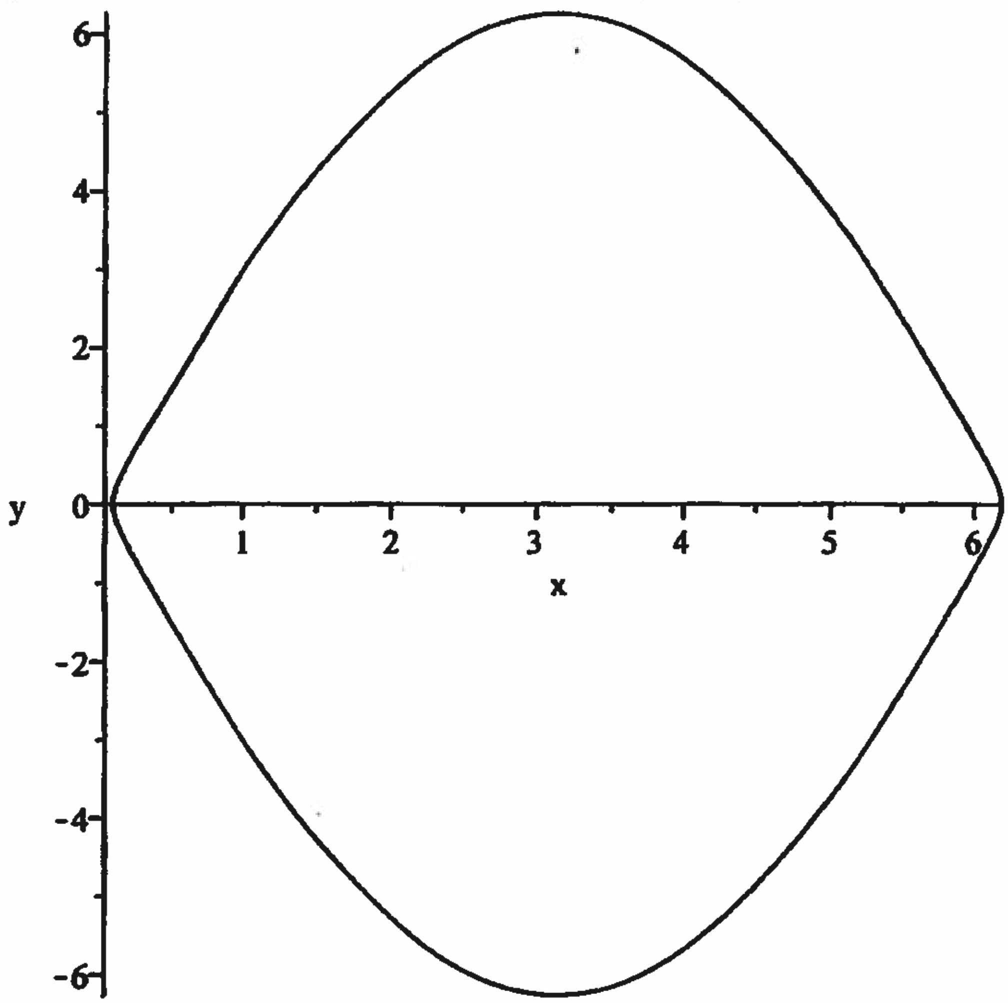
(Known Result)

```
> restart; with(DEtools):
> g:=9.81;l:=1;Omega:=0.01;A:=0.1;
```

$$g := 9.81$$
$$l := 1$$
$$\Omega := 0.01$$
$$A := 0.1 \tag{1}$$

Low frequency forcing - inverted pendulum does loop-over-loop

```
> DEplot([diff(x(t),t)=y(t),diff(y(t),t)=(g/l-A*Omega^2*sin(Omega*
  t))*sin(x(t))],[x(t),y(t)],t=0..30.0,[[x(0)=0.1,y(0)=0]],
  stepsize=0.001,linecolor=black,thickness=1);
```

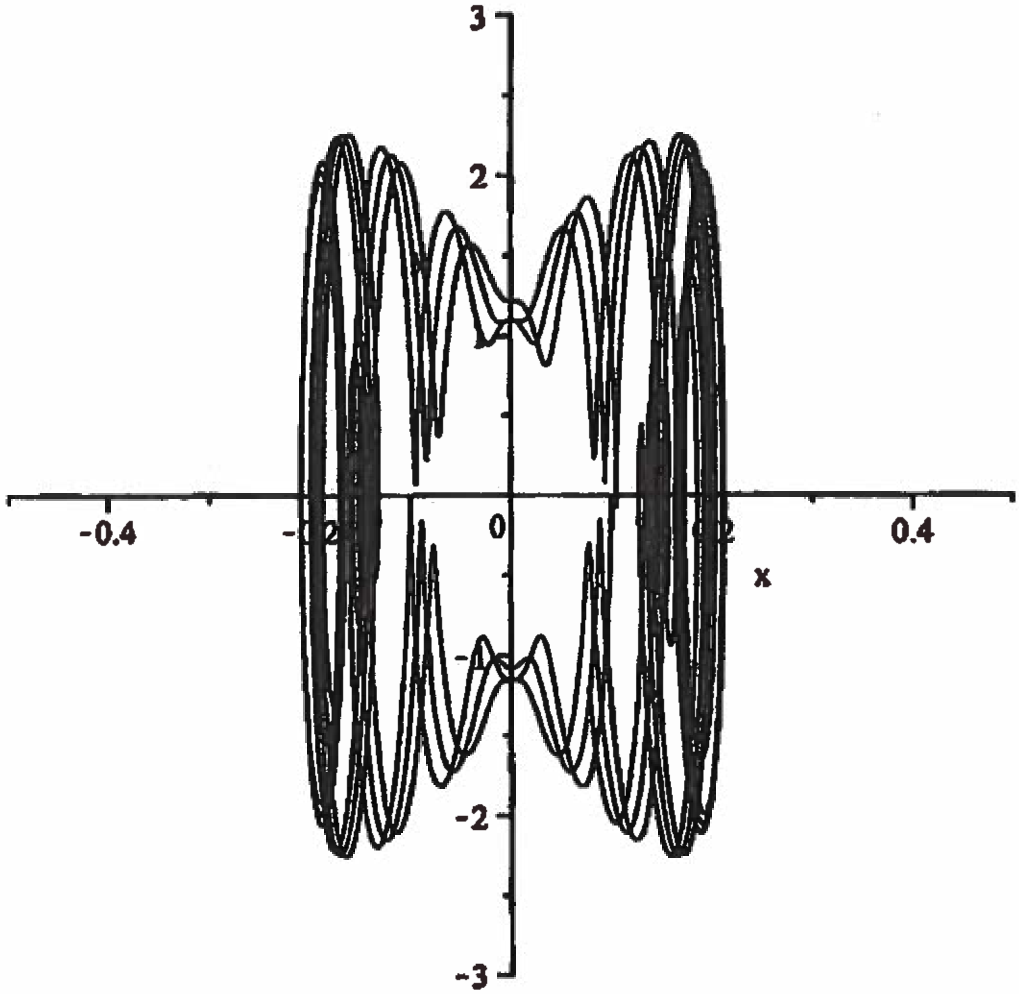



```
> g:=9.81;l:=1;Omega:=100;A:=0.1;
```

$$g := 9.81$$
$$l := 1$$
$$\Omega := 100$$
$$A := 0.1 \tag{2}$$

High frequency forcing - inverted pendulum stabilized with maximum angular displacement just over +- 0.2 radians

```
> DEplot([diff(x(t),t)=y(t),diff(y(t),t)=(g/l-A*Omega^2*sin(Omega*
t))*sin(x(t))],[x(t),y(t)],t=0..3.0,[[x(0)=0.1,y(0)=0]],x=-0.5.
.0.5,y=-3..3,stepsize=0.001,linecolor=black,thickness=1);
```



Problem 3

$$\delta L = \lim_{\epsilon \to 0} \Big[L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u})$$

$$- L(u, u', u'', \dot{u}) \Big]$$

where $\delta u = \epsilon\,\phi(x,t)$

By a Taylor Expansion about $\epsilon = 0$,

$$L(u+\delta u, u'+\delta u', u''+\delta u'', \dot{u}+\delta\dot{u}) =$$

$$\underbrace{L(\epsilon=0)}_{L(u,u',u'',\dot{u})} + \frac{\partial L}{\partial \epsilon}\bigg|_{\epsilon=0} \epsilon + \frac{1}{2}\frac{\partial^2 L}{\partial \epsilon^2}\bigg|_{\epsilon=0} \epsilon^2 + \cdots$$

Note: $$\frac{\partial L}{\partial \epsilon} = \frac{\partial L}{\partial(u+\epsilon\phi)}\frac{\partial(u+\epsilon\phi)}{\partial \epsilon} + \frac{\partial L}{\partial(u'+\epsilon\phi')}\frac{\partial(u'+\epsilon}{\partial \epsilon}$$

$$+ \frac{\partial L}{\partial(u''+\epsilon\phi'')}\frac{\partial(u''+\epsilon\phi'')}{\partial \epsilon}$$

$$+ \frac{\partial L}{\partial(\dot{u}+\epsilon\dot{\phi})}\frac{\partial(\dot{u}+\epsilon\dot{\phi})}{\partial \epsilon}$$

$$\left.\frac{\partial L}{\partial \epsilon}\right|_{\epsilon=0} = \frac{\partial L}{\partial u}\phi + \frac{\partial L}{\partial u'}\phi' + \frac{\partial L}{\partial u''}\phi'' + \frac{\partial L}{\partial \dot{u}}\dot{\phi}$$

$\frac{\partial^2 L}{\partial \epsilon^2}$ gets very messy:

$$\frac{\partial^2 L}{\partial \epsilon^2} = \frac{\partial}{\partial(u+\epsilon\phi)}\left[\frac{\partial L}{\partial(u+\epsilon\phi)}\frac{\partial(u+\epsilon\phi)}{\partial \epsilon} + \frac{\partial L}{\partial(u'+\epsilon\phi')}\frac{\partial(u'+\epsilon'\phi')}{\partial \epsilon}\right.$$

$$\left. + \frac{\partial L}{\partial(u''+\epsilon\phi'')}\frac{\partial(u''+\epsilon\phi'')}{\partial \epsilon} + \frac{\partial L}{\partial(\dot{u}+\epsilon\dot{\phi})}\frac{\partial(\dot{u}+\epsilon\dot{\phi})}{\partial \epsilon}\right]\frac{\partial(u+\epsilon}{\partial \epsilon}$$

$$+ \frac{\partial}{\partial(u'+\epsilon\phi')}\left[\text{ same as above }\right]\frac{\partial(u'+\epsilon\phi')}{\partial \epsilon}$$

$$+ \frac{\partial}{\partial(u''+\epsilon\phi'')}\left[\text{ same as above }\right]\frac{\partial(u''+\epsilon\phi'')}{\partial \epsilon}$$

$$+ \frac{\partial}{\partial(\dot{u}+\epsilon\dot{\phi})}\left[\text{ same as above }\right]\frac{\partial(\dot{u}+\epsilon\dot{\phi})}{\partial \epsilon}$$

$$\left.\frac{\partial^2 L}{\partial \epsilon^2}\right|_{\epsilon=0} = \left[\frac{\partial^2 L}{\partial u^2}\phi + \frac{\partial^2 L}{\partial u \partial u'}\phi' + \frac{\partial^2 L}{\partial u \partial u''}\phi'' + \frac{\partial^2 L}{\partial u \partial \dot{u}}\dot{\phi}\right]\phi$$

$$+ \left[\frac{\partial^2 L}{\partial u' \partial u}\phi + \frac{\partial^2 L}{\partial u'^2}\phi' + \frac{\partial^2 L}{\partial u' \partial u''}\phi'' + \frac{\partial^2 L}{\partial u' \partial \dot{u}}\dot{\phi}\right]\phi'$$

$$+ \left[\frac{\partial^2 L}{\partial u'' \partial u}\phi + \frac{\partial^2 L}{\partial u'' \partial u'}\phi' + \frac{\partial^2 L}{\partial u''^2}\phi'' + \frac{\partial^2 L}{\partial u'' \partial \dot{u}}\dot{\phi}\right]\phi''$$

$$+ \left[\frac{\partial^2 L}{\partial \dot{u} \partial u}\phi + \frac{\partial^2 L}{\partial \dot{u} \partial u'}\phi' + \frac{\partial^2 L}{\partial \dot{u} \partial u''}\phi'' + \frac{\partial^2 L}{\partial \dot{u}^2}\dot{\phi}\right]\dot{\phi}$$

Note: used $\frac{\partial}{\partial(u+\epsilon\phi)}\left(\frac{\partial(u+\epsilon\phi)}{\partial \epsilon}\right) = \frac{\partial}{\partial \epsilon}\left(\overbrace{\frac{\partial(u+\epsilon\phi)}{\partial(u+\epsilon\phi)}}^{1}\right)$

$$= 0$$

similarly

$$\frac{\partial}{\partial(u+\epsilon\phi)}\left(\frac{\partial(u'+\epsilon\phi')}{\partial \epsilon}\right) = \frac{\partial}{\partial \epsilon}\left(\frac{\partial(u'+\epsilon\phi')}{\partial(u+\epsilon\phi)}\right)$$

$$= \frac{\partial}{\partial \epsilon}\left(\cancel{\frac{\partial \phi'}{\partial a}}\right) = 0$$

$\ldots\ 0$

Back to δL

$$\delta L = \lim_{\epsilon \to 0} \Bigg[\frac{\partial L}{\partial u} \overbrace{\epsilon \phi}^{\delta u} + \frac{\partial L}{\partial u'} \overbrace{\epsilon \phi'}^{\delta u'} + \frac{\partial L}{\partial u''} \overbrace{\epsilon \phi''}^{\delta u''} + \frac{\partial L}{\partial \dot{u}} \overbrace{\epsilon \dot{\phi}}^{\delta \dot{u}}$$

$$+ O(\epsilon^2) \Bigg]$$

so, to first order

a. $$\boxed{\delta L = \frac{\partial L}{\partial u} \delta u + \frac{\partial L}{\partial u'} \delta u' + \frac{\partial L}{\partial u''} \delta u'' + \frac{\partial L}{\partial \dot{u}} \delta \dot{u}}$$

b. At second order

used $\epsilon^2 \phi^2 = \delta u^2$

$$\frac{1}{2}\delta^2 L = \frac{1}{2}\frac{\partial^2 L}{\partial \epsilon^2}\Big|_{\epsilon=0} \epsilon^2$$

$$\frac{1}{2}\delta^2 L = \frac{1}{2}\Big\{\Big[\frac{\partial^2 L}{\partial u^2}\delta u^2 + \frac{\partial^2 L}{\partial u \partial u'}\delta u \delta u' + \frac{\partial^2 L}{\partial u \partial u''}\delta u \delta u'' + \frac{\partial^2 L}{\partial u \partial \dot{u}}\delta u \delta \dot{u} + \dots \frac{\partial^2 L}{\partial \dot{u}^2}\delta \dot{\phi}^2\Big]\Big\}$$

above is clearly $\frac{1}{2}\delta(\delta L)$