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Consensus Based Distributed Motion Planning on a Sphere*

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***Abstract*— In this paper, consensus theory and semidefinite programming techniques are applied for planning of multiple collision free trajectories, for a team of communicating vehicles whose motions are constrained to evolve on the surface of a sphere. Such algorithms have applications in planetary-scale motion control for mobile sensing networks in air and space. Based on the communication graph for each vehicle, each vehicle synthesizes a time-varying Laplacian-like matrix $\mathfrak{L}^i$. The set of Laplacian-like matrices are used individually in a distributed manner to drive given initial positions of the vehicles to consensus positions on the sphere. Collision avoidance and formation configurations are realized via the concept of semidefinite programming. For each vehicle, the problem is coded as a set of linear matrix inequalities (LMI), augmented with a number of constraints, and solved by semidefinite programming (SDP). We also provide Lyapunov-based stability analysis, together with simulation results to demonstrate the effectiveness of the approach.**

I. INTRODUCTION

This paper presents an approach to constrained control of multiple vehicles navigating on the surface of a sphere, based on consensus theory and constrained attitude control (CAC) via semidefinite programming. Such algorithms have applications in planetary-scale mobile sensing networks in: air [1]; sea e.g. in remote-sensing and persistent sensing at ocean-basin scales [2], [3]; space navigation and satellite cluster positioning [4], [5].

Most of the previous work on multi-vehicle motion planning has focused on two-dimensional, e.g. [6], [7], and three-dimensional motion planning, e.g. [8], [9], [10]. Two-dimensional path planning is limited to the plane, while three-dimensional models are useful for planning motion control in volumetric 3D space. Both path planning models are limited when the motion is constrained to evolve on a sphere. The works [11] and [12] are beginning the research into the important area of distributed control on a sphere, and therefore there is the need to explore the topic further.

The work in [11] is based on the works of Justh and Krishnaprasad [6], [13], where a geometric approach to the gyroscopic control of vehicle motion in planar and three-dimensional particle models was developed for formation acquisition and control with collision avoidance in free space. They found that for their unconstrained gyroscopic control system on $SE(3)$, there are three possible types of relative equilibria: (i) parallel motion with arbitrary spacing; (ii) circular motion with a common radius, axis of rotation, direction of rotation and arbitrary along-axis spacing; (iii) helical motion with a common radius, axis of rotation, direction of rotation, along-axis speed (pitch) and arbitrary along-axis spacing.

The control system developed in [11] conforms to the number (ii) type of relative equilibrium described above. That is, the control system is capable of circular motion of particles, with a common radius, axis of rotation and direction of rotation. This means that, at steady state, all of the particles converge to a circular pattern on the sphere while moving in the same direction.

The geometric approach to the gyroscopic control of vehicle motion developed in [6] and [13] is effective in formation control of multiple systems in unconstrained spaces, and for formations that conform only to the relative equilibria described above. However, the approach cannot be applied to the more general motion control problem involving: (i) constrained spaces which contain static obstacles such as clutter; (ii) speed constrained vehicles; and (iii) other formations which are different from the three relative equilibria described above. This motivates the development of an alternative approach presented in this paper.

In this work, a new approach is presented to the general problem of constrained path planning on the sphere with avoidance of collisions, by using consensus and the concept of constrained attitude control (CAC) [14], implemented by semidefinite programming (SDP). Each individual vehicle communicates with its close neighbors within its sensor view and uses the Laplacian matrix of the communication graph $\mathbf{L}$ in a semidefinite program to plan consensus trajectories on the sphere. Then the concept of CAC is used to incorporate collision avoidance, by maintaining specified minimum distances between vehicles. By contrast with the approach of [11], [12], this algorithm can be used for motion control in both constrained and unconstrained spaces on the sphere, e.g. for planning consensus trajectories around static obstacles or noncooperative moving obstacles on the sphere. In addition, the approach can be applied to constrained vehicle motion, where the velocities cannot be constant. Moreover, formations that are different from circular motion are possible on the sphere. The approach may also yield faster trajectories than the gyroscopic approach.

The rest of the paper is organized as follows. The problem statement is presented in Section II, and the solution and convergence analysis are in Section III. Simulation results are in Section IV, and conclusion follows in Section V. Frequently used notation in this paper are listed in Table 1.

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TABLE I
FREQUENTLY USED NOTATION

α^i	Minimum angular separation from obstacle number i
β^{ij}	Minimum angular separation between vehicles i and j
C	The consensus space for $\mathbf{x}$, $C = \{\mathbf{x} \mid x^1 = x^2 =, \cdots, = x^n\}$
i	Vehicle number i
$\mathbf{I}_n$	The $n \times n$ identity matrix
$\mathbf{L}$	Laplacian matrix
$\mathfrak{L}, \mathfrak{L}^i$	Laplacian-like stochastic matrix
n	Number of vehicles
$\mathbf{0}$	A vector consisting of all zeros
$\otimes$	Kronecker multiplication operator
θ^{ij}	Angle between vehicles i and j
q	Quaternion vector, $q = [q_1\ q_2\ q_3\ \vert q_4]^T$
$\mathbf{R}$	Rotation matrix
$SE(3)$	Special Euclidean group
S^m	The set of $m \times m$ positive definite matrices
$\mathbf{u}, \dot{\mathbf{x}}$	Stacked vector of n control inputs
$\dot{x}^i$	Control input of vehicle i
x^i	Position vector of vehicle i
$\mathbf{x}$	Stacked vector of n position vectors
x^j_{obs}	Obstacle vector number j
φ^{ij}	Angle between vehicle i and obstacle j
Λ	A positive definite matrix variable, $\Lambda \in S^3$

II. Problem Statement

Given a set of communicating vehicles, randomly positioned on the surface of a sphere, with initial states $x^i(t_0) \in \mathbb{R}^3$, $i = 1, \cdots, n$, a set of obstacles $x^j_{obs} \in \mathbb{R}^3$, $j = 1, \cdots, m$, and the Laplacian matrix of their communication graph $\mathbf{L}$, our concern is to drive $x^i(t_0)$ to a consensus position $x^c = x(t_f)$, or to a consensus formation, while satisfying collision avoidance and unit norm constraints. Note that $x(t_f)$ needs not be known a priori to any of the vehicles.

To illustrate the problem, consider Figure 1. The positions of the vehicles on the unit sphere centered on $\mathbf{0}$, are indicated by the respective unit vectors x^i originating from $\mathbf{0}$. The obstacle positions are indicated by the unit vectors x^i_{obs}. The angle between vehicles i and j is θ^{ij}, and the angle between vehicle i and obstacle k is φ^{ik}. The motion control problem is to drive all x^i to a consensus position or to a formation on the sphere, while avoiding each other and also avoiding the x^i_{obs} along the way, and constantly staying on the sphere.

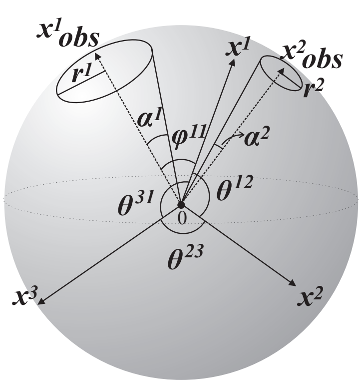

Fig. 1. Constrained position control on a sphere.

Observe that this path planning problem is akin to an attitude stabilization problem. Essentially, any kind of motion on a sphere is a rotation of some vector by some angle, about some axis, and the angle and axis defines a quaternion vector.

The problem stated above has two major parts: *consensus* and *collision avoidance*. The consensus part is that of driving the positions of the vehicles to a consensus position on the sphere. If communication is maintained between the vehicles for all time then the consensus position is usually the centroid of the initial positions, meaning that the vehicles should eventually rendezvous to a single point on the sphere.

The problem of collision avoidance is resolved by applying CAC, which will be described in Section III.B. Formations are realized by defining and maintaining minimum angular separations between the vehicles using the same concept of CAC.

III. Solution

In this section, a solution is developed to the problem stated in Section II. The solution involves four intermediate steps: (i) synthesis of position consensus on a sphere; (ii) formulation of CAC based collision avoidance; (iii) formulation of CAC based formation control; (iv) formulation of collision free arbitrary reconfigurations on a sphere.

A. Synthesis of Position Consensus on a Sphere

By semidefinite programming (SDP), the problem under consideration can be coded as a set of linear matrix inequalities (LMI), and solved for consensus position trajectories on the sphere, using available optimization software tools, such as Sedumi [15] and Yalmip [16]. Note that the main problem at this stage is to find a feasible sequence of consensus trajectories for each vehicle on the sphere, which satisfies norm and avoidance constraints. Therefore, rather than state the objective function as a minimization or maximization problem, we state the objective function as the discrete time version of a semidefinite consensus dynamics, which will be augmented with an arbitrary number of constraints which will be introduced as we proceed.

It is assumed that vehicle i can sense *at least one* vehicle within its communication range. Suppose w is the number of vehicles in the communication neighborhood of vehicle i, a Laplacian-like stochastic matrix $\mathfrak{L}^i$ is chosen for each i, so that x^i are driven to consensus, while maintaining unit norm constraint. To obtain $\mathfrak{L}^i$, we begin by defining a semidefinite matrix variable, Λ^i for each i

$$\begin{aligned}\mathfrak{L}^i(t) &= \left[w\Lambda^i_1(t)\ \ -\Lambda^i_2(t)\cdots-\Lambda^i_w(t)\right],\\ \dot{x}^i(t) &= -\,\mathfrak{L}^i(t)\left[x^T_1(t)\ x^T_2(t)\ \cdots x^T_w(t)\right]^T,\end{aligned} \tag{1}$$

where $x^T_1(t), \cdots, x^T_w(t)$ are the position vectors of vehicles that i is communicating with at time t.

For the purpose of analysis, we present the collective description as

$$\mathfrak{L}(t) = \underbrace{\begin{bmatrix}\Lambda^1(t) & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & \Lambda^n(t)\end{bmatrix}}_{\Lambda(t)}\underbrace{\begin{bmatrix}l_{11}\mathbf{I}_3 & \dots & l_{1n}\mathbf{I}_3\\ \vdots & \ddots & \vdots\\ l_{n1}\mathbf{I}_3 & \dots & l_{nn}\mathbf{I}_3\end{bmatrix}}_{\Gamma=\mathbf{L}\otimes\mathbf{I}_3},$$

where l are elements of $\mathbf{L}$, $\Lambda^i(t) \succ 0\, \forall i, \forall t$[1] and $\mathbf{L} = [l_{ij}]$ $(i,j = 1,\ldots,n)$ is the collective Laplacian matrix.

A collective semidefinite consensus protocol is therefore given as

$$\dot{\mathbf{x}}(t) = -\mathfrak{L}(t)\mathbf{x}(t), \tag{2}$$

The Euler's first order discrete time equivalents of (1) and (2) are

$$x^i_{k+1} = x^i_k - \Delta t \mathfrak{L}^i(t) x^i_k, \tag{3}$$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \dot{\mathbf{x}}_k = \mathbf{x}_k - \Delta t \mathfrak{L}(t)\mathbf{x}_k. \tag{4}$$

Each vehicle builds a SDP in which (3) is included as the dynamics constraint, which will be augmented with a number of convex constraints that we will define as we proceed.

For the vehicles to remain on the sphere, norm constraints will be defined for each i as

$$(x^i)^T_k (x^i_{k+1} - x^i_k) = 0. \tag{5}$$

Essentially, (5) is the discrete version of $x^i(t)^T \dot{x}^i(t) = 0$ or $\mathbf{x}(t)^T \dot{\mathbf{x}}(t) = 0$, which guarantees $x^i(t)^T x^i(t) = 1$ or $\mathbf{x}(t)^T \mathbf{x}(t) = n$ (for n vehicles) as long as $\left\|x^i(0)\right\| = 1\ \forall\ i$.

Equation (3) drives the positions $x^i(0)$ to consensus, and (5) forces them to remain on the sphere.

Theorem 1 *The consensus strategy* $\dot{\mathbf{x}}(t) = -\mathbf{L}\mathbf{x}(t)$, *achieves global consensus asymptotically for* $\mathbf{L}$ *if and only if the associated (static) communication graph of* $\mathbf{L}$ *has a spanning tree* [17].

Proof: See [17] page 10.

Based on the proof of Theorem 1, we develop the proof convergence of (2).

Theorem 2 *The time varying system* (2) *achieves consensus if* $\mathbf{L}$ *is connected.*

Proof: First note that if $\mathbf{x}$ belongs to the *consensus space* $\mathscr{C} = \{\mathbf{x}\,|\,x^1 = x^2 =, \cdots, = x^n\}$, then $\dot{\mathbf{x}} = \mathbf{0}$. $\mathscr{C}$ is the nullspace of $\mathfrak{L}(t)$, i.e. the set of all $\mathbf{x}$ such that $\mathfrak{L}(t)\mathbf{x} = 0$. Thus, once $\mathbf{x}$ enters $\mathscr{C}$ it stays there. Assume that $\mathbf{x} \notin \mathscr{C}$, and consider a Lyapunov candidate function $V = \mathbf{x}^T \Gamma \mathbf{x}$. Note that $V > 0$ unless $\mathbf{x} \in \mathscr{C}$. Then,

$$\begin{aligned}\dot{V} =& \mathbf{x}^T \Gamma \dot{\mathbf{x}} + \dot{\mathbf{x}}^T \Gamma \mathbf{x},\\ =& -\mathbf{x}^T \Gamma \mathfrak{L}(t)\mathbf{x} - \mathbf{x}^T \mathfrak{L}(t)^T \Gamma \mathbf{x},\\ =& -2\mathbf{x}^T \Gamma \Lambda(t) \Gamma \mathbf{x}\\ =& -2y^T \Lambda(t) y < 0,\end{aligned} \tag{6}$$

where $y = \Gamma\mathbf{x} \neq 0$ for $\mathbf{x} \notin \mathscr{C}$. Therefore, $\mathbf{x}$ approaches a point in $\mathscr{C}$ as t tends to ∞. This proves the claim. Equation 6 is true as long as $\mathbf{L}$ is nonempty.

B. Formulation of CAC based Collision Avoidance

To incorporate collision avoidance, we apply the concept of CAC. To illustrate, consider Figure 1. It is desired that the time evolution of the position vectors $x^1(t)$, $x^2(t)$ and $x^3(t)$ should avoid two constraint regions around x^1_{obs} and x^2_{obs}, defined by cones, whose base radius are r^1 and r^2, respectively.

Let the angle between vehicles i and j be θ^{ij}, and let the angle between vehicle i and obstacle k be φ^{ik}. Then the requirements for collision avoidance are: $\varphi^{11}(t) \geq \alpha^1$, $\varphi^{21}(t) \geq \alpha^1$, $\varphi^{31}(t) \geq \alpha^1$, and $\varphi^{12}(t) \geq \alpha^2$, $\varphi^{22}(t) \geq \alpha^2$, $\varphi^{32}(t) \geq \alpha^2$, $\forall\ t \in [t_0, t_f]$. The equivalent constraints can be written as:

$$x^1(t)^T x^1_{obs} \leq \cos\alpha^1, \tag{7}$$

$$x^2(t)^T x^1_{obs} \leq \cos\alpha^1, \tag{8}$$

$$x^3(t)^T x^1_{obs} \leq \cos\alpha^1, \tag{9}$$

$$x^1(t)^T x^2_{obs} \leq \cos\alpha^2, \tag{10}$$

$$x^2(t)^T x^2_{obs} \leq \cos\alpha^2, \tag{11}$$

$$x^3(t)^T x^2_{obs} \leq \cos\alpha^2. \tag{12}$$

In this decentralized approach, there is no central optimization. Thus, each vehicle solves only the constraint problems involving itself and the vehicles within its communication neighborhood. However, the constraints will be converted to LMI form in order to include them into the respective SDPs. This can be done using the Schur complement formula [18], which states that the inequality

$$\mathbf{S}\mathbf{R}^{-1}\mathbf{S}^T - \mathbf{Q} \leq 0;\ \mathbf{Q} = \mathbf{Q}^T,\ \mathbf{R} = \mathbf{R}^T,\ \mathbf{R} > 0,$$

is equivalent to, and can be represented by the LMI

$$\begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{R}\end{bmatrix} \geq 0.$$

For any vehicle/obstacle pair, let $\mathbf{x}^{ij}(t) = [x^i(t)^T \quad x^j_{obs}(t)^T]^T$, then

$$\mathbf{x}^{ij}(t)^T \underbrace{\begin{bmatrix}\mathbf{0}_3 & \mathbf{I}_3\\ \mathbf{I}_3 & \mathbf{0}_3\end{bmatrix}}_{\mathbf{G}} \mathbf{x}^{ij}(t) \leq 2\cos\alpha. \tag{13}$$

Since $\mathbf{G}$ is not positive definite, we add a positive number $\mu \geq 1$ to shift its eigenvalues and make it positive definite. Thus (13) may be written as

$$\mathbf{x}^{ij}(t)^T \left(\mu \mathbf{I}_6 + \begin{bmatrix}\mathbf{0}_3 & \mathbf{I}_3\\ \mathbf{I}_3 & \mathbf{0}_3\end{bmatrix}\right) \mathbf{x}^{ij}(t) \leq 2(\cos\alpha + \mu).$$

Let $\mathbf{M} = \left(\mu \mathbf{I}_6 + \begin{bmatrix}\mathbf{0}_3 & \mathbf{I}_3\\ \mathbf{I}_3 & \mathbf{0}_3\end{bmatrix}\right)^{-1}$, then by the Schur complement the LMI equivalent of (7) is

$$\begin{bmatrix}2(\mu + \cos\alpha^1) & \mathbf{x}^{11}(t)^T\\ \mathbf{x}^{11}(t) & \mathbf{M}\end{bmatrix} \geq 0, \tag{14}$$

where $\mathbf{x}^{11}(t) = [x^1(t)^T \quad x^1_{obs}(t)^T]^T$.

Thus, for the constraints (7)-(12), the equivalent LMI constraints are

$$\begin{bmatrix}2(\mu + \cos\alpha^j) & \begin{bmatrix}x^i(k+1)\\ x^j_{obs}\end{bmatrix}^T\\ \begin{bmatrix}x^i(k+1)\\ x^j_{obs}\end{bmatrix} & \mathbf{M}\end{bmatrix} \geq 0,$$

$i = 1,2,3,\ j = 1,2.$

[1]The symbol $\Lambda \succ 0$ means that Λ is a positive definite matrix.

C. Formulation of CAC based Formation Control

To obtain formation patterns, relative spacing is defined between individual vehicles using the method presented above, by specifying a minimum angular separation of β^{ij} between any two vehicles i and j. The set of avoidance constraints that will result in the desired formation pattern are then defined as $\theta^{ij} \geq \beta^{ij}\ \forall\ i,j$. To include intervehicle collision avoidance for n vehicles, the avoidance requirements result in extra $P(n-2) = \frac{n!}{(n-2)!}$ constraints, which are included along with the static obstacle avoidance constraints such as (7)-(12). Therefore, in addition to two static obstacle avoidance constraints for each vehicle (such as (7) and (10) for vehicle 1), each vehicle has two more intervehicle collision avoidance constraints such as

$$\begin{bmatrix} 2(\mu+\cos\beta^{ij}) & \begin{bmatrix} x^i(k+1) \\ x^j(k+1) \end{bmatrix}^T \\ \begin{bmatrix} x^i(k+1) \\ x^j(k+1) \end{bmatrix} & \mathbf{M} \end{bmatrix} \geq 0, \tag{15}$$

$\forall\ i,j\ (i \neq j)$.

Putting together Section III.A - Section III.C, the optimization problem of finding a feasible sequence of consensus trajectories may be posed as a SDP, as follows. Given the set of initial positions $x^i(t_0),\ (i=1,\cdots,n)$, and the plant equation (1) for each vehicle, find a feasible sequence of trajectories that converges to consensus at steady state, and satisfies the following constraints:

$$\begin{aligned} & x^i_{k+1} = x^i_k - \Delta t \mathfrak{L}^i(t) x^i_k, \\ & (x^i)^T_k (x^i_{k+1} - x^i_k) = 0, \\ & \begin{bmatrix} 2(\mu+\cos\alpha^j) & \begin{bmatrix} x^i_{k+1} \\ x^j_{obs} \end{bmatrix}^T \\ \begin{bmatrix} x^i_{k+1} \\ x^j_{obs} \end{bmatrix} & \mathbf{M} \end{bmatrix} \geq 0, \\ & \begin{bmatrix} 2(\mu+\cos\beta^{ij}) & \begin{bmatrix} x^i_{k+1} \\ x^j_{k+1} \end{bmatrix}^T \\ \begin{bmatrix} x^i_{k+1} \\ x^j_{k+1} \end{bmatrix} & \mathbf{M} \end{bmatrix} \geq 0, \end{aligned}$$

where x^i_{k+1} and $\Lambda^i_k \in \mathfrak{L}^i(t)$ are the optimization variables and are declared as SDP variables. The role of Λ^i_k is to shape the trajectories x^i_{k+1} in order to satisfy norm and avoidance constraints.

D. Formulation of Collision free Reconfigurations on a Sphere

Consider a more traditional reconfiguration problem in which several vehicles have to change their positions relative to that of a set of virtual leaders, whose positions may be static or time-varying. Each vehicle is connected to its corresponding virtual leader via a leader-follower digraph. An example topology for three vehicles is shown in Figure 2. In the figure, the vertices in broken circles are the virtual leaders' states, while the vertices with solid circles correspond to the real vehicles' states. There are three separate leader follower digraphs (with pointing arrows), which are not connected with each other, and there is an undirected graph which enables the vehicles to communicate. This graph provides intervehicle communication which is used to detect and avoid potential collisions.

Let the state of a virtual leader vehicle i be $x^i_v(t)$, then the corresponding leader-follower Laplacian matrix for each vehicle pair $\mathbf{x}^i_*(t) = [x^i_v(t)^T \quad x^i(t)^T]^T$ is

$$\mathbf{L}^i(t) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

and the collective dynamics of $\mathbf{x}^i_*(t)$ is

$$\dot{\mathbf{x}}^i_*(t) = -\begin{bmatrix} \Lambda^i(t) & \mathbf{0} \\ \mathbf{0} & \Lambda^i_v(t) \end{bmatrix} \left(\mathbf{L}^i(t) \otimes \mathbf{I}_3\right) \mathbf{x}^i_*(t).$$

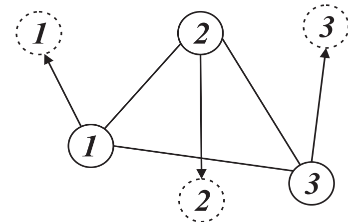

Fig. 2. Multiple virtual leaders graph topology with an undirected topology.

Simulation result for applying this strategy to the collision free reconfiguration of three vehicles will be shown in the next section.

IV. Simulation Results

In this section, three simulation results are presented for distributed control on a sphere. The first experiment is to test rendezvous on the sphere, i.e. convergence without avoidance constraints. The second experiment is for formation acquisition on the sphere with collision avoidance. The third experiment is to test arbitrary reconfigurations on the sphere with collision avoidance. Three different communication topologies, shown in Figure 3, are used to solve different problems in the experiments. In the figure, Topology 2 (left) is a fully connected communication graph with no leader, Topology 3 (center) is a line communication graph with one leader, node 1, and Topology 4 (right) is a cyclic communication graph with no leader.

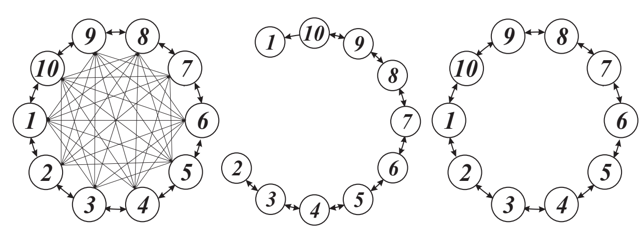

Fig. 3. Topology 2 (left), Topology 3 (center) and Topology 4 (right).

Sedumi [15], and Yalmip [16], were used for solving all the optimization problems in this section. The simulations were done with Matlab R2009a on an Intel® Core(TM)2

Duo P8600 @ 2.40GHz with 2 GB RAM, running Windows 7.

A. Rendezvous on a Sphere without Avoidance

In this simulation, ten vehicles will converge to a consensus position on the sphere, using Topology 2. The initial positions are:

$$
\begin{aligned}
x^1(0) =& [0.3417 \; 0.5555 \; 0.7581]^T \\
x^2(0) =& [0.4960 \; -0.1270 \; -0.8589]^T \\
x^3(0) =& [-0.3045 \; -0.9497 \; 0.0730]^T \\
x^4(0) =& [0.5735 \; 0.7952 \; 0.1967]^T \\
x^5(0) =& [-0.8005 \; -0.3867 \; -0.4580]^T \\
x^6(0) =& [-0.3727 \; -0.7372 \; 0.5637]^T \\
x^7(0) =& [0.0355 \; -0.5117 \; -0.8585]^T \\
x^8(0) =& [-0.6553 \; -0.7428 \; -0.1371]^T \\
x^9(0) =& [0.9188 \; -0.2446 \; -0.3094]^T \\
x^{10}(0) =& [-0.0261 \; -0.8773 \; -0.4792]^T
\end{aligned}
$$

The final consensus position obtained from this experiment is $x^c = [0.3818 \; -0.8794 \; -0.2845]^T$. The result is presented in Figure 4.

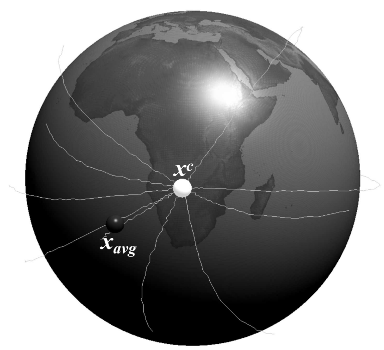

Fig. 4. Ten-vehicle rendezvous on a sphere without avoidance constraints.

B. Formation Acquisition on a Sphere with Avoidance

In this simulation, ten vehicles will converge to a formation on the sphere. To realize the formation, they should maintain a relative spacing with each other while also avoiding a static obstacle. For the static obstacle avoidance, $\alpha = 30^\circ$, and to maintain the relative spacing between the vehicles, $\beta^{ij} = 20^\circ \; \forall \; i,j = 1,\cdots,10, \; i \neq j$. The initial positions are the same as those used in the previous experiment. The result for Topology 2 is shown in Figure 5 (left), while Figure 5 (right) shows the result obtained using Topology 4.

C. Collision Free Reconfiguration with Avoidance of No-fly Zones

In this simulation, three flying vehicles (e.g. UAVs), are required to fly from their initial positions to given final positions. The initial positions are: $x_0^1 = [0.8659 \; 0 \; -$

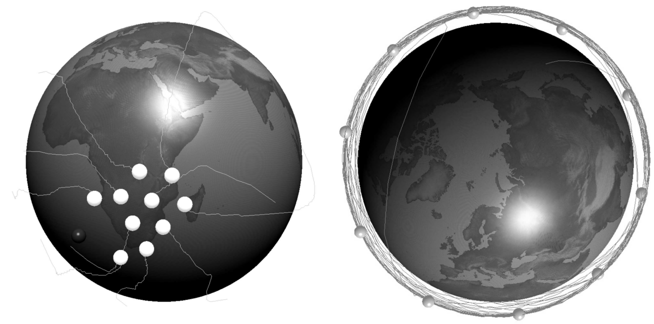

Fig. 5. Ten-vehicle formation acquisition using Topology 2 (left), and using Topology 4 (right).

$0.4999]^T$, $x_0^2 = [0.4165 \; -0.5721 \; 0.7071]^T$, $x_0^3 = [-0.5878 \; -0.809 \; 0]^T$. The desired final positions are: $x_f^1 = [-0.4330 \; -0.7499 \; 0.4999]^T$, $x_f^2 = [-0.2939 \; -0.9045 \; -0.309]^T$, $x_f^3 = [0.9393 \; -0.3052 \; 0.1564]^T$. For intervehicle collision avoidance, they are required to maintain a minimum safety distance of $r = \cos 10^\circ$ units. Five no-fly zones are imposed on the vehicles at the following positions: $x_{obs}^1 = [0.5237 \; -0.7208 \; 0.454]^T$, $x_{obs}^2 = [0.2939 \; -0.9045 \; -0.309]^T$, $x_{obs}^3 = [0 \; -0.9877 \; 0.1564]^T$, $x_{obs}^4 = [0.5878 \; -0.809 \; 0]^T$, $x_{obs}^5 = [0 \; -0.9511 \; 0.309]^T$. The radius of the no-fly zones are equal to r, therefore $\beta^{ij} = \alpha^j = 10^\circ \; \forall \; i,j \; (i \neq j)$ for this simulation. The result in Figure 6 shows obstacle avoidance while the result in Figure 7 shows intervehicle collision avoidance.

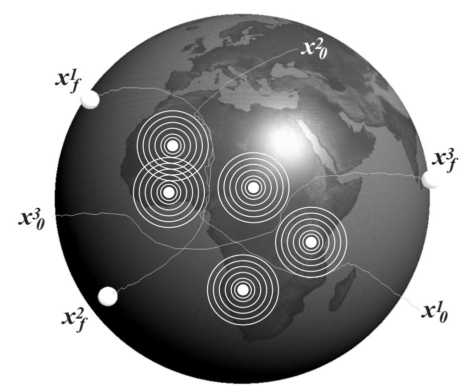

Fig. 6. Three-vehicle reconfiguration with collision avoidance and avoidance of no-fly zones.

V. Conclusions

In this paper, a solution is provided for constrained navigation of multiple vehicles on the sphere, by applying consensus and SDP optimization techniques. The algorithm implements distributed path planning on a sphere with collision avoidance, which is directly useful for motion control of omnidirectional flying vehicles, or vehicles with zero turn radius (e.g. rotorcraft and spacecraft). However, for a flying vehicle with nonzero turn radius, the algorithm can be used as a rough path planner, and a trajectory smoothing algorithm

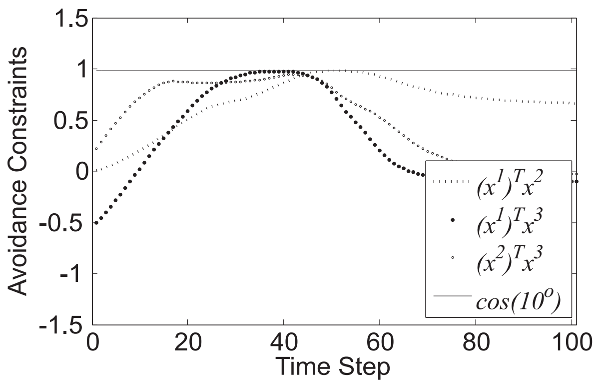

Fig. 7. Avoidance constraints graph for Figure 6.

can be employed to generate smooth trajectories from the rough path.

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