Lecture: Adjacency Matrix

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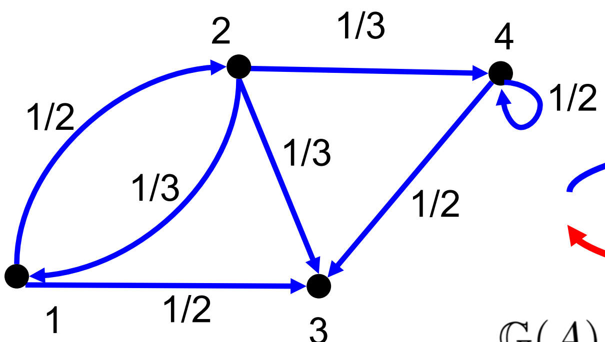
Review

Graph and Matrix



an *n*-node **graph** $\mathbb{G}$

Adjacency Matrix $A_{\mathbb{G}}$

$\mathbb{G}(A)$ Graph of an non-negative matrix.

$$\begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

an n×n **matrix A**

There exists a directed edge $i \to j$ $\Longleftrightarrow$ $A_{ij} > 0$

- **Adjacency Matrix** of an n-node graph is an n×n **matrix A**

$$A_{ij} = \begin{cases} w_{ij}, & i \to j; \\ 0, & \text{otherwise} \end{cases} \qquad\qquad j \to i$$

- Given an non-negative matrix $A \in \mathbb{R}^{n \times n}$,

the ***graph of a matrix*** A *is* a directed graph of ***n*** nodes such that there exists a directed **edge** $i \to j$ with the weight A_{ij} if and only if $A_{ij} > 0.$

Compact Form for Consensus Algorithms

- **Consensus:**

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t) = \sum_{j=1}^{m} w_{ij} x_j(t) \qquad w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \qquad \sum_{j=1}^{m} w_{ij} = 1$$

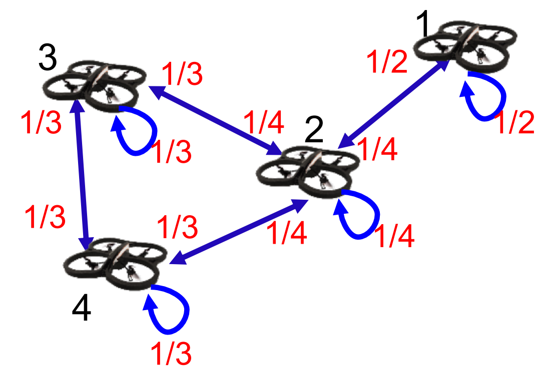
$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

$$x_1(t+1) = \frac{1}{2}x_1(t) + \frac{1}{2}x_2(t)$$

$$x_2(t+1) = \frac{1}{4}x_1(t) + \frac{1}{4}x_2(t) + \frac{1}{4}x_3(t) + \frac{1}{4}x_4(t)$$

$$x_3(t) = \frac{1}{3}x_2(t) + \frac{1}{3}x_3(t) + \frac{1}{3}x_4(t)$$

$$x_4(t) = \frac{1}{3}x_2(t) + \frac{1}{3}x_3(t) + \frac{1}{3}x_4(t)$$



- **Compact Form:** $x(t+1) = Ax(t)$

$$A_{ij} = \begin{cases} w_{ij}, & i \to j; \\ 0, & \text{otherwise} \end{cases}$$

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

- *A* is the **adjacency Matrix** of the underlying graph (with different choices of weights for different applications)

$$x(t) = A^t x(0)$$

Powers of Adjacency Matrices

Non-negative matrices

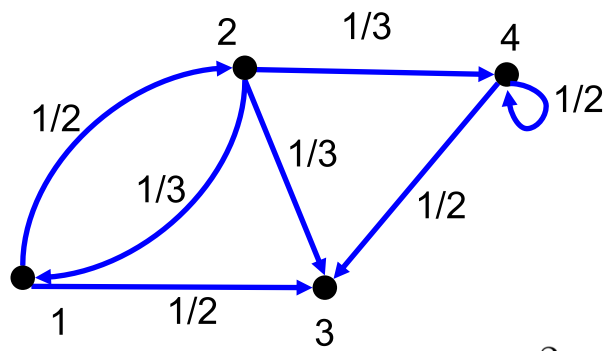
a path of length 2 from i to j
(i,k),(k,j)

$(A^2)_{ij}$ =(ith row of A)*(jth column of A) ≥ 0

> 0 If there exists a k such that $A_{ik}A_{kj} > 0$

=0 otherwise

$$= \sum_{k=1}^{m} A_{ik}A_{kj}$$



$$A = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

$(A^2)_{13} = ?$

$\frac{1}{2}$ $\frac{1}{3}$

Directed calculation: $\frac{1}{6}$

Paths of length 2 from 1 to 3: (1,2), (2,3)

$(A^k)_{ij} =$ Sum of weights of all paths of length k from i to j.

=number of paths for unweighted graphs.

$(A^k)_{ij} > 0$ if and only if there exists a path of length k from i to j in its **unweighted** graph $\mathbb{G}(A)$.

$(A^4)_{14} > 0??$ $(A^{1000})_{14} > 0??$

$(A^k)_{ij} > 0$ $\Longleftrightarrow$ there exists a path of length k from i to j in its **unweighted** graph $\mathbb{G}(A)$.

Given a matrix $A \in \mathbb{R}^{m \times m}$. Suppose the graph of A is **strongly connected** with **self-arcs**.

Prove: A is **primitive**. $A^k > 0$, where *k* is larger than the graph diameter.

There exists a path between any node *i* and any node *j* of length unknown.

Self-Arcs:

$$\underbrace{(i, i_1), (i_1, i_2), ..., (i_q, j) \quad (j, j), (j, j), ..., (j, j)}$$

A path from any node i to any node j of length m

$$(A^m)_{ij} > 0$$

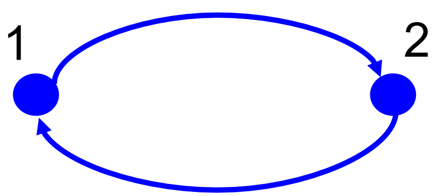
$$A^m > 0$$

A is **primitive**.

Connections between Graphs and its Adjacency Matrices

$\mathbb{G}$ is **strongly connected** ⟺ *A* is **irreducible.** $\quad \sum_{i=0}^{n-1} A^i > 0$

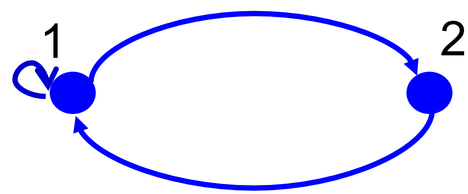
$\mathbb{G}$ is **strongly connected** and aperiodic ⟺ *A* is **primitive** $\quad A^k > 0$



Strongly Connected

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

irreducible



Strongly Connected and Aperiodic

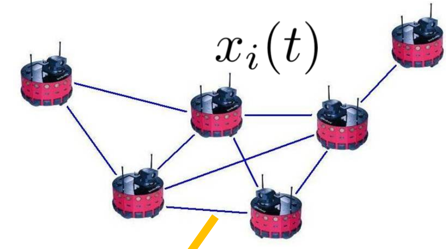
$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

primitive

Distributed Algorithm for Consensus

$$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$$

$$w_{ij} = \begin{cases} > 0, & j \in \mathcal{N}_i \\ 0, & \text{otherwise.} \end{cases} \qquad \sum_{j=1}^{m} w_{ij} = 1$$



$$\boxed{x(t+1) = Ax(t)}$$

Graph of A is strongly connected and aperiodic

If A is also Primitive

A is row stochastic

Gershgorin Circle Theorem

Perron - Frobenius Theorem

$A\mathbf{1} = \mathbf{1}$

1 is the largest eigenvalue in magnitude.

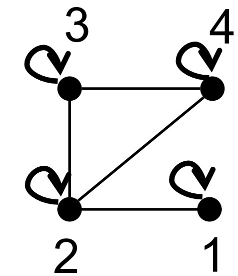
1 is a simple eigenvalue

all the other eigenvalues are with magnitude strictly less than 1

$$\lim_{t \to \infty} A^t = \mathbf{1} w'$$

$$\lim_{t \to \infty} x(t) = \mathbf{1} w' x(0)$$

Checking whether a Matrix is **primitive by Graph Theory**



- **Consensus:** $x_i(t+1) = \frac{1}{|\mathcal{N}_i|} \sum_{j \in \mathcal{N}_i} x_j(t)$ $\quad\mathbb{G}^*$ Strongly connected with self-arcs.

$x(t+1) = Ax(t)$ A is primitive?

$\mathbb{G}(A)$

$$A = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$i \to j \text{ in } \mathbb{G}^* \Longleftrightarrow j \in \mathcal{N}_i \text{ in } \mathbb{G}^* \Longleftrightarrow A_{ij} > 0 \Longleftrightarrow i \to j \text{ in } \mathbb{G}(A)$$

$\mathbb{G}(A) = \mathbb{G}^*$ is strongly connected with self-arcs. Thus A is primitive.

- **Metropolis:** $x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$ $\quad w_{ij} > 0 \quad \sum_{j \in \mathcal{N}_i} w_{ij} = 1$

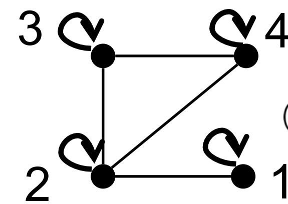
$x(t+1) = Nx(t)$ $\quad\mathbb{G}(N) = \mathbb{G}^*$

$$N = \begin{bmatrix} 3/4 & 1/4 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/4 & 5/12 & 1/3 \\ 0 & 1/4 & 1/3 & 5/12 \end{bmatrix}$$

- **Periodic Gossiping.** One neighbor pair of agents *i* and *j* ***gossip*** at time *t*:

$$x_i(t+1) = x_j(t+1) = \frac{1}{2}(x_i(t) + x_j(t)) \qquad x_k(t+1) = x_k(t)$$

Periodic Gossip Sequence: (1,2), (2,3), (2,4), (3,4),…..



$\mathbb{G}^*$ Strongly connected with self-arcs.

$$x(4(k+1)) = Mx(4k)$$

$$M = M_4 M_3 M_2 M_1$$

Gossip Pairs:

(3,4)

$$M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

(2,4)

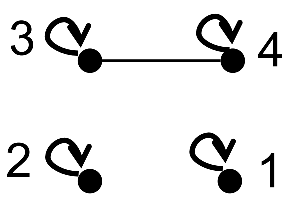
$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

(2,3)

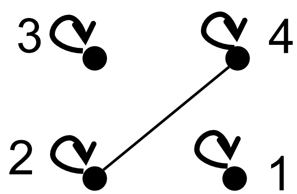
$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1,2)

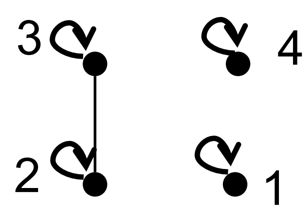
$$M_1 = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



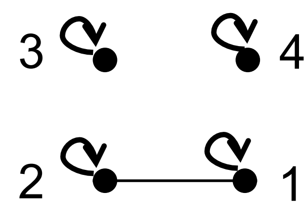
$\mathbb{G}(M_4)$



$\mathbb{G}(M_3)$



$\mathbb{G}(M_2)$



$\mathbb{G}(M_1)$

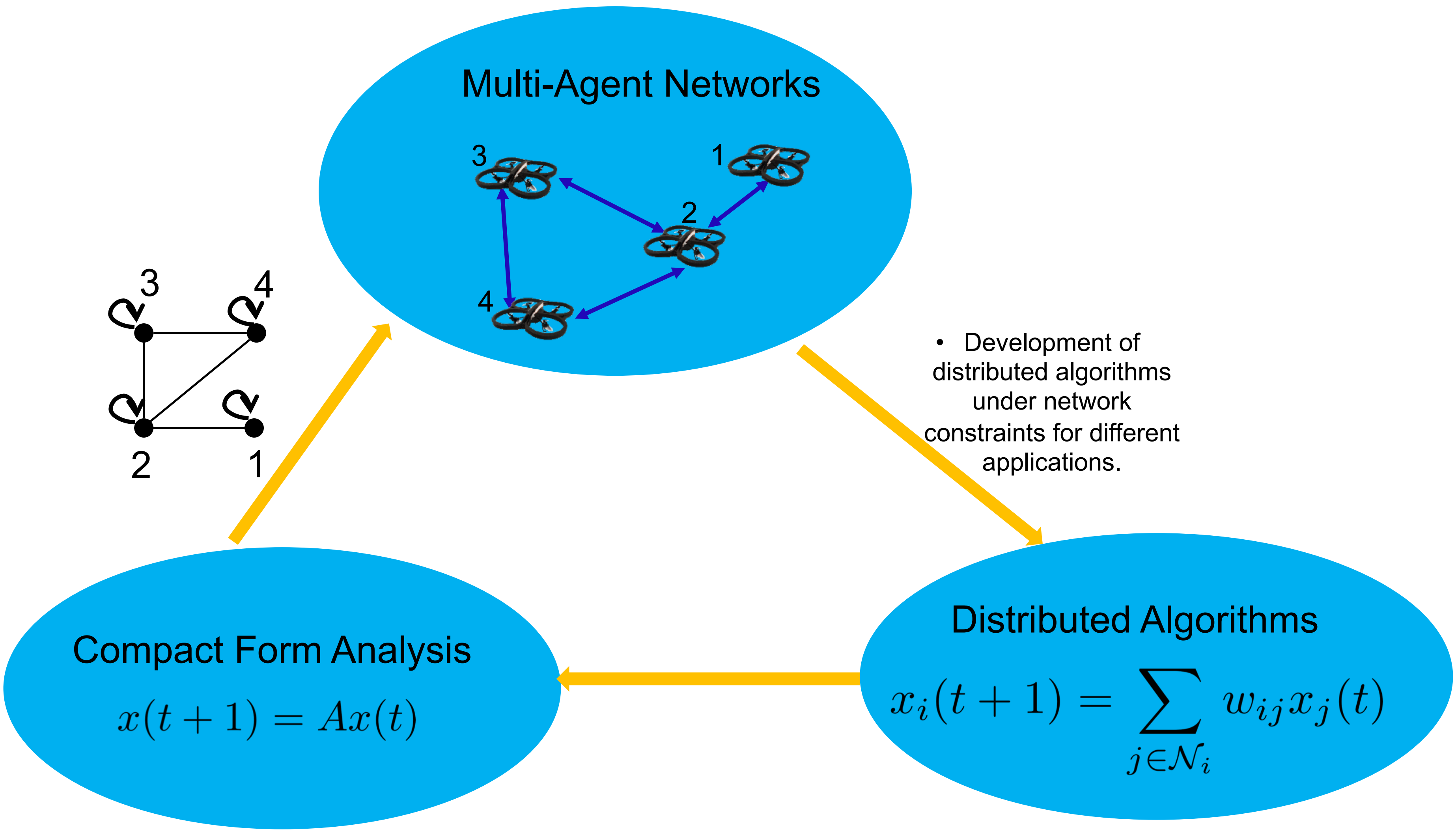
The union of these graphs is $\mathbb{G}^*$

$\mathbb{G}(M_2) \circ \mathbb{G}(M_1) = \mathbb{G}(M_2 M_1)$

M is primitive? $\mathbb{G}(M) = \mathbb{G}(M_4 M_3 M_2 M_1)$

$$= \mathbb{G}(M_4) \circ \mathbb{G}(M_3) \circ \mathbb{G}(M_2) \circ \mathbb{G}(M_1)$$

Is strongly connected with self-arcs.


Multi-Agent Networks
3
1
2
4
3
4
2
1
• Development of distributed algorithms under network constraints for different applications.
Distributed Algorithms
$x_i(t+1) = \sum_{j \in \mathcal{N}_i} w_{ij} x_j(t)$
Compact Form Analysis
$x(t+1) = Ax(t)$