Solution: Relative Motion of Two Bodies

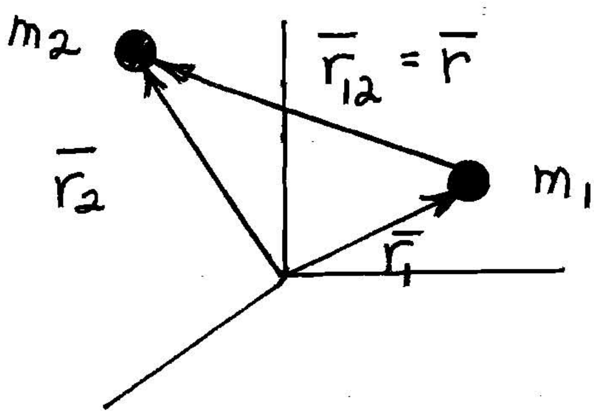
Solve $\ddot{\bar{r}} + \dfrac{\mu}{r^3}\bar{r} = \bar{0}$

Method 1 : **Classical Derivation**

I. Observations from angular momentum

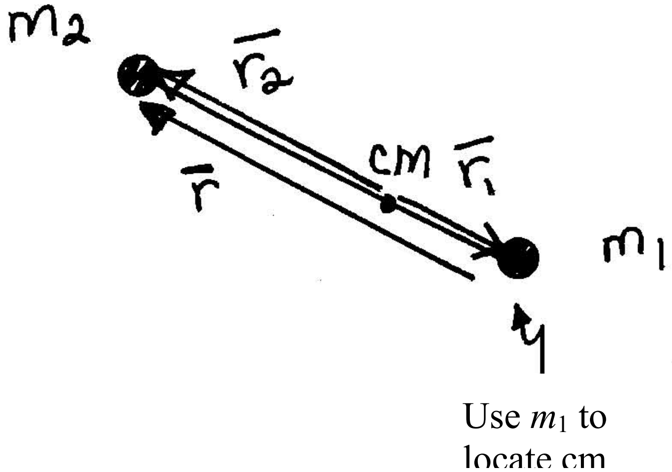
1. n-body problem – angular momentum of system

$$\sum_{i=1}^{n} m_i \left(\bar{r}_i \times \dot{\bar{r}}_i \right) = \bar{C}_3 \quad \text{constant vector}$$



Let $n = 2$

System linear momentum conserved



$$(m_1+m_2)\bar{r}_{cm} = m_2\bar{r}$$

$$\bar{r}_{cm} = \frac{m_2}{m_1+m_2}\bar{r}$$

$\downarrow$

Sub back into equation for $\bar{C}_3$

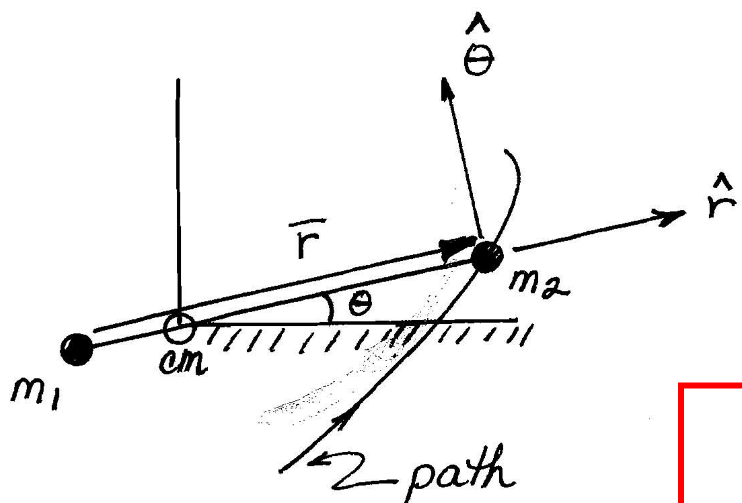
$$\bar{C}_3 = m_1\left(\frac{-m_2}{m_1+m_2}\bar{r} \times \frac{-m_2}{m_1+m_2}\dot{\bar{r}}\right) + m_2\left(\frac{m_1}{m_1+m_2}\bar{r} \times \frac{m_1}{m_1+m_2}\dot{\bar{r}}\right)$$

Note: $\dot{\bar{r}} = \frac{{}^i d\,\bar{r}}{dt}$ **relative velocity**

2. $\overline{h} = \overline{r} \times \dot{\overline{r}} = \text{constant}$ $\Bigg\{$

Invariable plane –

3. Represent $\overline{\boldsymbol{h}}$ in scalar component / magnitude form



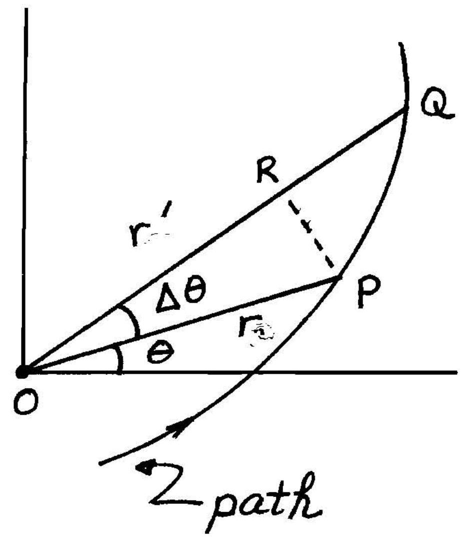
$\overline{r} = r\,\hat{r}$

$\dot{\overline{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$

$\left|\overline{h}\right| = \left|\overline{r} \times \dot{\overline{r}}\right| = r^2\,\dot{\theta}$

4. ***h*** related to areal velocity

[Kepler III. Line joining planet to Sun sweeps out equal areas in equal times.]



Actually already known from ***h***

(Assume motion in a plane)

ΔA represents area of triangle ***OPQ*** swept over by radius vector in interval Δt

Area triangle = $\frac{1}{2}$ (base) (height)

$$\Delta A = \frac{1}{2}(r')(r\sin\Delta\theta) = \frac{r' r \sin\Delta\theta}{2}$$

$$\frac{\Delta A}{\Delta t} = \frac{r' r}{2}\frac{\sin\Delta\theta}{\Delta\theta}\frac{\Delta\theta}{\Delta t}$$

As $\Delta\theta$ diminishes, ratio of area of triangle to that of sector approaches unity as a limit

$\Longrightarrow$ limit of $\dfrac{\sin\Delta\theta}{\Delta\theta}$ is unity

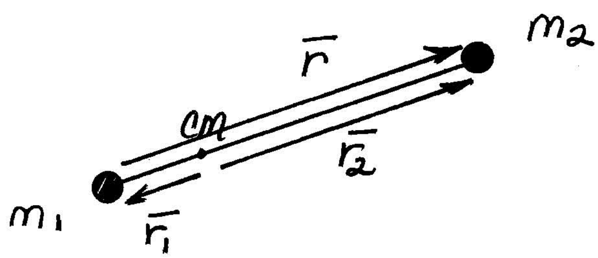
Passing to the limit $\Delta t \to 0$

II. Observations from energy

1. gravity field is <u>conservative</u>

$$T - U = C_4$$

2. write equation $T - U = C_4$ in a more convenient form



$$\bar{r}_1 = \frac{-m_2}{m_1 + m_2}\bar{r} \qquad \dot{\bar{r}}_1 = \frac{-m_2}{m_1 + m_2}\dot{\bar{r}}$$

$$\bar{r}_2 = \frac{m_1}{m_1 + m_2}\bar{r} \qquad \dot{\bar{r}}_2 = \frac{m_1}{m_1 + m_2}\dot{\bar{r}}$$

$$T = \frac{1}{2}\sum_{i=1}^{n} m_i\, \bar{v}_i^P \bullet \bar{v}_i^P$$

$$T = \frac{1}{2}m_1\left(\dot{\bar{r}}_1 \bullet \dot{\bar{r}}_1\right) + \frac{1}{2}m_2\left(\dot{\bar{r}}_2 \bullet \dot{\bar{r}}_2\right)$$

$$T = \frac{1}{2}m_1\left(\frac{-m_2}{m_1 + m_2}\dot{\bar{r}} \bullet \frac{-m_2}{m_1 + m_2}\dot{\bar{r}}\right) + \frac{1}{2}m_2\left(\frac{m_1}{m_1 + m_2}\dot{\bar{r}} \bullet \frac{m_1}{m_1 + m_2}\dot{\bar{r}}\right)$$

$$U = \frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}}$$

$$U = \frac{1}{2} G \left(\frac{m_1 m_2}{r} + \frac{m_2 m_1}{r} \right) = \frac{G m_1 m_2}{r}$$

$$T - U = \frac{1}{2} \left(\dot{\bar{r}} \cdot \dot{\bar{r}} \right) \frac{m_1 m_2}{m_1 + m_2} - \frac{G m_1 m_2}{r} = C_4$$

$\Downarrow$

Multiply by $\dfrac{m_1 + m_2}{m_1 m_2}$

$$\frac{1}{2} \left(\dot{\bar{r}} \bullet \dot{\bar{r}} \right) - \frac{G(m_1 + m_2)}{r} = C_4 \frac{(m_1 + m_2)}{m_1 m_2}$$

Define $\bar{v} = \dot{\bar{r}} = \dfrac{{}^I d\bar{r}}{dt}$ $\longleftarrow$ base point moves!!!

$\bar{v} = \dot{\bar{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$v^2 = \left|\dot{\bar{r}}\right|^2 = \dot{r}^2 + r^2\dot{\theta}^2$

$\Downarrow$

$$\frac{v^2}{2} - \frac{G(m_1 + m_2)}{r} = C_4 \frac{(m_1 + m_2)}{m_1 m_2} = \mathcal{E}$$ "energy"

Let $\mu = G(m_1 + m_2)$

$\Downarrow$

$\boxed{}$

General U: $\mathcal{E} = \dfrac{v^2}{2} - U'$

III. Using known constants ($h; \mathcal{E}$), vector 2nd-order DE

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \bar{0}$$

has been replaced by two 1st-order scalar differential equations in the dependent variables r, θ

Note: only 2 dependent variables because motion takes place in a plane (polar components simplifies problem)

$$h = r^2\dot{\theta}$$

$$\mathcal{E} = \frac{1}{2}v^2 - U' \quad = \quad \frac{1}{2}\underbrace{\left(\dot{r}^2 + r^2\dot{\theta}^2\right)} - U'$$

Solution ?

Expression for ***r*** ?

$$h = r^2 \frac{d\theta}{dt} \quad \Longrightarrow \quad dt = \frac{r^2}{h} d\theta$$ can remove time

$$\mathcal{E} = \frac{1}{2}\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\} - U'$$

$$2\mathcal{E} = \left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + r^2\left(\frac{d\theta}{d\theta}\frac{h}{r^2}\right)^2 - 2U'$$

$$= \left(\frac{h}{r^2}\frac{dr}{d\theta}\right)^2 + \frac{h^2}{r^2} - 2U'$$

$$\frac{2}{h^2}[\mathcal{E} + U'] = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \longleftarrow$$ differential equation

integrate?

Introduce new variable

$$\varsigma = \frac{1}{r} = r^{-1} \qquad \frac{d\varsigma}{d\theta} = -r^{-2}\frac{dr}{d\theta}$$

$$\Downarrow$$

$$\frac{2}{h^2}[\mathcal{E} + U'] = \left(\frac{d\varsigma}{d\theta}\right)^2 + \varsigma^2$$

$$\Downarrow$$

$$\frac{d\varsigma}{d\theta} = \pm\sqrt{\frac{2}{h^2}(\mathcal{E} + U') - \varsigma^2}$$

$$d\theta = \frac{d\varsigma}{\pm\sqrt{\frac{2}{h^2}\left(\mathcal{E}+U'\right)-\varsigma^2}}$$

3 parameter family of orbits

fcn of $\varsigma \rightarrow U' = \frac{\mu}{r} = \mu\,\varsigma$

Integrate

$$\theta = \cos^{-1}\frac{\varsigma - \frac{\mu}{h^2}}{\sqrt{\frac{\mu^2}{h^4}+\frac{2\mathcal{E}}{h^2}}} \quad + \quad \omega$$

$$\frac{1}{r} = \frac{\mu}{h^2} + \sqrt{\frac{\mu^2}{h^4}+\frac{2\mathcal{E}}{h^2}}\ \cos\left(\theta-\omega\right)$$

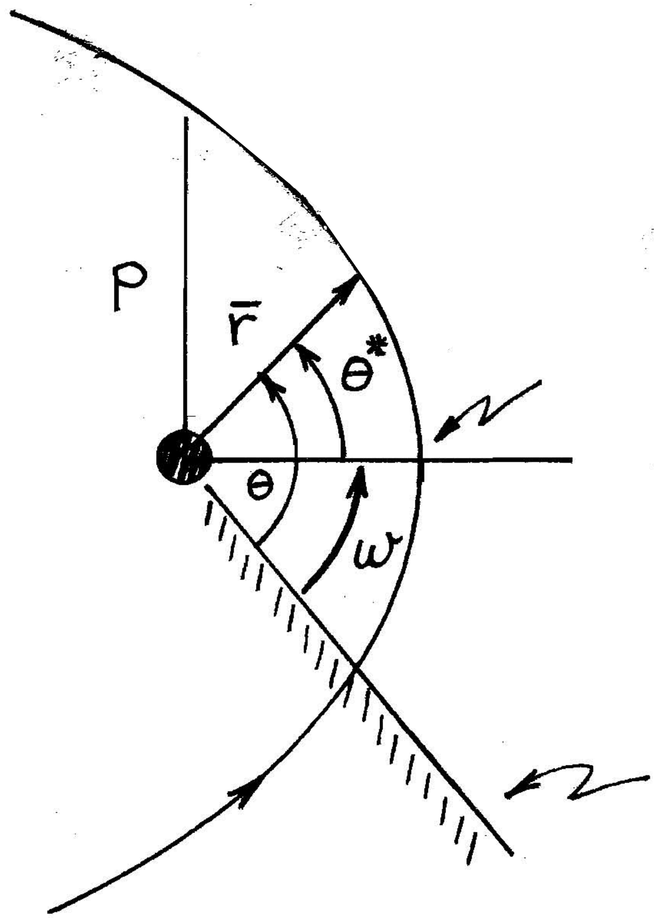
Standard polar equation of a conic section referred to the focus as the origin

where $\quad p = \frac{h^2}{\mu}$

$$e = \sqrt{1+\frac{2\mathcal{E}h^2}{\mu^2}}$$

ellipse, parabola, hyperbola

Conic Section



$h = \sqrt{\mu p}$ $\qquad \mathcal{E} = -\dfrac{\mu^2}{2h^2}\left(1-e^2\right)$

define $\quad a = \dfrac{p}{\left(1-e^2\right)}$ $\qquad$ semimajor axis

$$\Rightarrow \quad \mathcal{E} = -\frac{\mu}{2a} \quad \Rightarrow \quad -\frac{\mu}{2a} = \frac{v^2}{2} - \frac{\mu}{r}$$

<u>Method 2</u> : **Direct Vector Derivation**

$$\ddot{\bar{r}} + \frac{\mu}{r^3}\bar{r} = \bar{0} \qquad \text{equation of relative motion for two-body system}$$

$$\bar{h} = \bar{r} \times \dot{\bar{r}} \qquad \text{constant}$$

Start with some observations concerning vectors

$$\frac{{}^I d}{dt}\left(\frac{\bar{r}}{r}\right) = \frac{\dot{\bar{r}}}{r} - \frac{\bar{r}}{r^2}\dot{r} = \frac{r^2\dot{\bar{r}} - r\dot{r}\,\bar{r}}{r^3}$$

$$\left[\text{Note:} \quad \dot{\bar{r}} = \frac{{}^I d\,\bar{r}}{dt} \qquad \dot{r} = \frac{d\,r}{dt}\right]$$

Rewrite

$$\frac{{}^I d\hat{r}}{dt} = \frac{(\bar{r} \bullet \bar{r})\dot{\bar{r}} - (\bar{r} \bullet \dot{\bar{r}})\,\bar{r}}{r^3}$$

Identity:

$$\begin{array}{ccc} \bar{A} \times (\bar{B} \times \bar{C}) & = & (\bar{A} \bullet \bar{C})\,\bar{B} - (\bar{A} \bullet \bar{B})\,\bar{C} \\ \bar{r} \quad\;\; \dot{\bar{r}} \quad \bar{r} & & (\bar{r} \bullet \bar{r})\,\dot{\bar{r}} - (\bar{r} \bullet \dot{\bar{r}})\,\bar{r} \end{array}$$

$$\frac{{}^I d\,\hat{r}}{dt} = -\frac{\bar{r} \times (\bar{r} \times \dot{\bar{r}})}{r^3} = \frac{(\bar{r} \times \dot{\bar{r}}) \times \bar{r}}{r^3}$$

$$= \qquad\qquad = \bar{h} \times \underbrace{\frac{\bar{r}}{r^3}}_{\text{sub}}$$

$$\dot{\hat{r}} = \bar{h} \times \left(-\frac{\ddot{\bar{r}}}{\mu} \right) \quad \text{OR} \quad \dot{\hat{r}} = \frac{\ddot{\bar{r}} \times \bar{h}}{\mu}$$

constant

constant

Integrate once

$$\hat{r} = \frac{\dot{\bar{r}} \times \bar{h}}{\mu} + \text{integration constant} \quad = \frac{\dot{\bar{r}} \times \bar{h}}{\mu} - \bar{e}$$

$\bar{e}$ in $\Big\{$

Dot product with $\bar{r}$

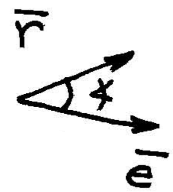
$$\bar{r} \bullet \hat{r} = \bar{r} \bullet \frac{\dot{\bar{r}} \times \bar{h}}{\mu} - \bar{r} \bullet \bar{e}$$

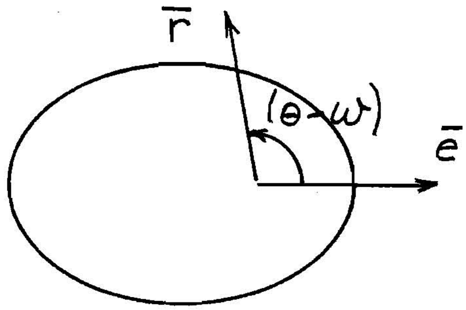
$$\left[\quad \text{Identity} \quad \bar{A} \bullet \bar{B} \times \bar{C} = \bar{C} \bullet \bar{A} \times \bar{B} \quad \right]$$

$$r = \frac{\bar{h}}{\mu} \bullet \underbrace{\left(\bar{r} \times \dot{\bar{r}} \right)}_{\bar{h}} - \bar{r} \bullet \bar{e}$$

$$r = \frac{h^2}{\mu} - \bar{r} \bullet \bar{e}$$

$$r + \bar{r} \bullet \bar{e} = h^2 / \mu$$





$$\boxed{r + re\cos(\theta - \omega) = h^2 / \mu}$$

Note: $\bar{h}$ → 3 constants

$\bar{h}$ normal to plane of motion

$\bar{e}$ always IN plane of motion

$\bar{h}$ and $\bar{e}$ are NOT independent so only represents 5 arbitrary constants

But together $\bar{h}$, $\bar{e}$ determine size, shape, and orientation of conic with respect to focus

6th constant related to <u>time</u>, i.e., orbital position relative to periapsis