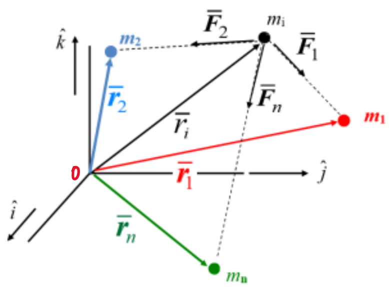
N – Body Problem

Write an expression for force acting on one body due to existence of multiple other bodies

Assume: Gravity is the only force acting
"System" of n – bodies (masses m_1, m_2, …, m_n)
All masses spherically symmetric

$$|\bar{f}_2| = \frac{Gm_1m_2}{r^2}$$



force on m_i due to m_n:

$$\bar{F}_n = -\frac{Gm_im_n}{r_{ni}^3}\bar{r}_{ni} \quad \text{vector } n \to i$$

mag

$$\bar{r}_{ni} = \bar{r}_i - \bar{r}_n$$

Sum all forces

$$\bar{F}_T = -\frac{Gm_im_1}{r_{1i}^3}\bar{r}_{1i} - \frac{Gm_im_2}{r_{2i}^3}\bar{r}_{2i} + \ldots - \frac{Gm_im_n}{r_{ni}^3}\bar{r}_{ni}$$

$$\left(\text{does NOT include} \quad -\frac{Gm_im_i}{r_{ii}^3}\bar{r}_{ii}\right)$$

Force Model $\boxed{\bar{F}_T = -Gm_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ji}^3}\bar{r}_{ji}}$

Using this force model, write EOM from Newton II

$${}^{I}\frac{d}{dt}(m_i\bar{v}_i) = \bar{F}_{Total}$$ Newton

Note: only true if derivative wrt an inertial frame

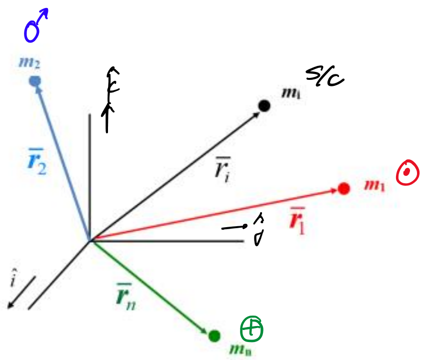
$$m_i\left({}^{I}\frac{d\bar{v}_i}{dt}\right) + \bar{v}_i\cancel{\frac{dm_i}{dt}} = \bar{F}_T$$

Acceleration as seen in the inertial frame

Assume m_i constant

$$\boxed{m_i\frac{d^2\bar{r}_i}{dt^2} = -G\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{m_im_j}{r_{ji}^3}\bar{r}_{ji}}$$

Vector of Equation of Motion for how particle i moves over time



$$m_i \ddot{\bar r}_i = -G \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_i m_j}{r_{ji}^3}\,\bar r_{ji}$$ 2nd order vector DE

Let $m_i \to$ s/c, $m_1 \to$ Sun, $m_2 \to$ Mars, $m_n \to$ Earth, Jupiter, Mercury, Uranus, ….

$$\underbrace{m_i \frac{d^2\bar r_i}{dt}}_{} \quad \underbrace{m_i \ddot{\bar r}_i}_{\text{s/c}} = -\underbrace{\frac{G m_i m_1}{r_{1i}^3}\bar r_{1i}}_{\text{sun}} - \underbrace{\frac{G m_i m_2}{r_{2i}^3}\bar r_{2i}}_{\text{Mars}} - \underbrace{\sum_{\substack{j=3\\ j\neq i}}^{n} \frac{G m_i m_i}{r_{ji}^3}\bar r_{ji}}_{\text{Earth, Venus, Jupiter}}$$

. . .

Alternative formulation using potential function, U:

$$\boxed{m_i \frac{d^2 r_i}{dt^2} = \nabla_i U}$$

$\bar r_i = x_i \hat i + y_i \hat j + z_i \hat k$

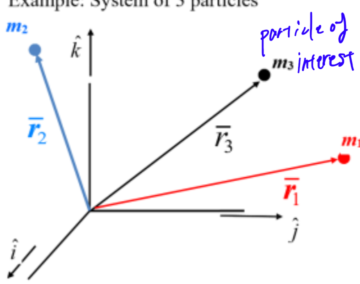
where $\bar\nabla_i \to$ vector gradient operator

$$\nabla_i(\,) = \hat i \frac{\partial}{\partial x_i}(\,) + \hat j \frac{\partial}{\partial y_i}(\,) + \hat k \frac{\partial}{\partial z_i}(\,)$$

$U \to$ gravitational potential (scalar)

$$\boxed{U = \frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_i m_j}{r_{ij}}}$$

Example: System of 3 particles



relative vector

$$\boxed{\begin{aligned}\bar r_{12} &= \bar r_2 - \bar r_1\\ \bar r_{13} &= \bar r_3 - \bar r_1\end{aligned}}$$

$$\boxed{\begin{aligned}\bar r_1 &= x_1\hat i + y_1\hat j + z_1\hat k\\ \bar r_2 &= x_2\hat i + y_2\hat j + z_2\hat k\\ \bar r_3 &= x_3\hat i + y_3\hat j + z_3\hat k\end{aligned}}$$

base point

Force (total) on m_1 ➡ $\bar F_T = -G \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_i m_j}{r_{ji}^3}\bar r_{ji}$ general expression

$$\overline{F}_1 = G\left(\frac{m_1 m_2}{r_{12}^3}\overline{r}_{12} + \frac{m_1 m_3}{r_{13}^3}\overline{r}_{13}\right) = m_1 \frac{d^2\overline{r}_1}{dt^2}$$

Alternate expression

$$\overline{F}_1 = \overline{\nabla}_1 U \qquad \text{where} \qquad U = \frac{1}{2} G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_i m_j}{r_{ij}}$$

magnitude

$$U = \frac{1}{2} G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_1}{r_{21}} + \frac{m_2 m_3}{r_{23}} + \frac{m_3 m_1}{r_{31}} + \frac{m_3 m_2}{r_{32}}\right)$$

3 particle system

$$U = G\left(\frac{m_1 m_2}{r_{12}} + \frac{m_1 m_3}{r_{13}} + \frac{m_2 m_3}{r_{23}}\right)$$

relative distances

DOF? Coordinates used to describe configuration?

$\overline{r}_1, \ \overline{r}_2, \ \overline{r}_3$

→ All quantities in U must be written in terms of the independent variables

$$\overline{r}_{12} = \overline{r}_2 - \overline{r}_1 = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$
$$\overline{r}_{13} = \overline{r}_3 - \overline{r}_1 = (x_3 - x_1)\hat{i} + (y_3 - y_1)\hat{j} + (z_3 - z_1)\hat{k}$$
$$\overline{r}_{23} = \overline{r}_3 - \overline{r}_2 = (x_3 - x_2)\hat{i} + (y_3 - y_2)\hat{j} + (z_3 - z_2)\hat{k}$$

relative vectors in terms of independent

where

$$r_{12} = |\overline{r}_{12}| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$
$$r_{13} = |\overline{r}_{13}| = \left[(x_3 - x_1)^2 + (y_3 - y_1)^2 + (z_3 - z_1)^2\right]^{1/2}$$

$$\frac{\partial U}{\partial x_1} = Gm_1 m_2 \frac{\partial (r_{12}^{-1})}{\partial x_1} + Gm_1 m_3 \frac{\partial (r_{13}^{-1})}{\partial x_1} + Gm_2 m_3 \frac{\partial (r_{23}^{-1})}{\partial x_1}$$
$$= \frac{Gm_1 m_2}{r_{12}^3}(x_2 - x_1) + \frac{Gm_1 m_3}{r_{13}^3}(x_3 - x_1)$$

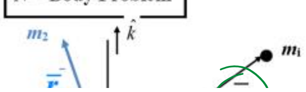
$$\frac{\partial U}{\partial y_1} = \ldots$$
$$\frac{\partial U}{\partial z_1} = \ldots$$

$$\overline{\nabla}_1 U = \left\{\frac{Gm_1 m_2}{r_{12}^3}(x_2 - x_1) + \frac{Gm_1 m_3}{r_{13}^3}(x_3 - x_1)\right\}\hat{i}$$
$$\left\{\frac{Gm_1 m_2}{r_{12}^3}(y_2 - y_1) + \frac{Gm_1 m_3}{r_{13}^3}(y_3 - y_1)\right\}\hat{j}$$
$$+\left\{\frac{Gm_1 m_2}{r_{12}^3}(z_2 - z_1) + \frac{Gm_1 m_3}{r_{13}^3}(z_3 - z_1)\right\}\hat{k}$$

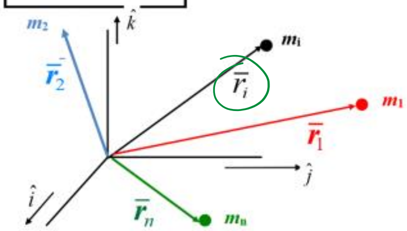
$$\overline{\nabla} U = \frac{Gm_1 m_2}{r_{12}^3}\overline{r}_{12} + \frac{Gm_1 m_3}{r_{13}^3}\overline{r}_{13}$$

$$\overline{F}_1 = -\frac{Gm_1 m_2}{r_{12}^3}\overline{r}_{21} - \frac{Gm_1 m_3}{r_{13}^3}\overline{r}_{31}$$

N – Body Problem



N – Body Problem



Vector Equation of Motion for m_i

$$\boxed{m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \bar{r}_{ji}} \leftarrow \bar{r}_i - \bar{r}_j$$

→ 6 scalar first-order differential equations for m_i } solvable? — math model?

Observations concerning solution $\bar{r}_i(t)$:

1. Independent variable –

   Dependent vars – scalar components pos + vel for each $m_i (\bar{r}_i, \dot{\bar{r}}_i)$

   note: $\bar{r}_i$ have inertially fixed base pts.

2. Time history $\bar{r}_j(t)$ NOT KNOWN

   m_j affected by m_i ; motion of m_i changes force on m_j → changes acceleration on m_j → changes position of m_j

   → scalar components of $\bar{r}_j$, $\dot{\bar{r}}_j$ are also unknown dependent variables

   # of dep var > # of eqns

   ⇓

   unsolvable

3. Add additional equations so no. of equations = no. of unknowns

   ⇓

   Need 6 scalar, first-order differential equations for each particle in the system

   ⇓

   $6n$ first-order (scalar) differential equations are necessary equations nonlinear and coupled } complete set solve?

4. For every first-order differential equation that appears, a complete analytical solution requires the ability to analytically integrate the DE

   If you can integrate a differential equation, you have an integral of the motion (note that a constant appears) $6n$ scalar 1st order ⇒ $6n$ constants

   Given a coupled set of differential equations, increasingly difficult to integrate (may try lots of approaches …)
   But must be accomplished for a complete solution.

   We have $6n$ equations in $6n$ dependent variables
   We need $6n$ integrals of the motion or $6n$ constants to solve our system of differential equations $n=3 \Rightarrow 18$ consts

5. To date, we only know how to obtain 10 integrals
   ⇒ The n-body problem is NOT completely solvable

note: assume only 2 bodies (S/C ⊕)

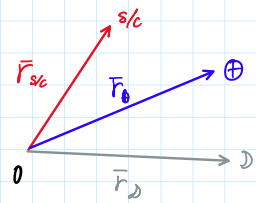
$n=2 \rightarrow 12$ integrals to solve

~~two-body~~ problem not solvable

Known since Euler's time (1707-1783)
Nothing new since

Ten Known Integrals

Can't integrate individual equations directly but collect equations into certain combinations -- allows integration of some of the new equations



$$m_{s/c} \ddot{\bar{r}}_{s/c} = -\frac{G m_{s/c} m_\oplus}{r_{\oplus s/c}^3} \bar{r}_{\oplus s/c} - \frac{G m_{s/c} m_☽}{r_{☽ s/c}^3} \bar{r}_{☽ s/c}$$

$$= -\frac{G m_{s/c} m_\oplus}{r_{\oplus s/c}^3} (\bar{r}_{s/c} - \bar{r}_\oplus) - \frac{G m_{s/c} m_☽}{r_{☽ s/c}^3} (\bar{r}_{s/c} - \bar{r}_☽)$$

1 2nd order vector DE = 6 1st order scalar DE

6 scalar DE → independent variable : t
dependent variables?

$$\underbrace{\bar{r}_{s/c}, \dot{\bar{r}}_{s/c}}_{6}, \underbrace{\bar{r}_\oplus}_{3}, \underbrace{\bar{r}_☽}_{3}$$

⇓

12 dep var

6 eqns = 12 vars

X cannot solve !

⇓

$m_\oplus \ddot{\bar{r}}_\oplus = \ldots$

$m_☽ \ddot{\bar{r}}_☽ = \ldots$

⇓

18 scalar DE
and 18 dep vars

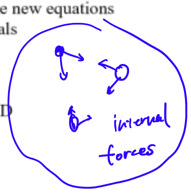
⇓

match !

Ten Known Integrals

Can't integrate individual equations directly but collect equations into certain combinations -- allows integration of some of the new equations
Aided immensely by physical significance of the integrals

1. $\boxed{\text{Linear Momentum}}$



Conserved for system ← no external forces in FBD

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \underbrace{(\bar{r}_i - \bar{r}_j)}_{\bar{r}_{ji}}$$

To get total $\bar{p}$, add up all equations

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i = \underbrace{-G \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} (\bar{r}_i - \bar{r}_j)}$$

zero because terms appear
in form $(\bar{r}_1 - \bar{r}_2) + (\bar{r}_2 - \bar{r}_1)$

$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i = \bar{0}$ now we can integrate twice

Integrate twice

$$\underbrace{\sum_{i=1}^{n} m_i \bar{r}_i}_{m\bar{r}_{cm}} = \bar{C}_1 t + \bar{C}_2 \rightarrow 2 \text{ vector consts}$$

↓

6 scalar consts

↑

$m = \sum_{i=1}^{n} m_i$

Note: $\bar{p} = \left(\sum_{i=1}^{n} m_i \dot{\bar{r}}_i \right) =$ constant $\bar{C}_1$

$\bar{p} = m \bar{V}_{cm} = \bar{C}_1$ → mag + dir

center of mass has const vel

2. $\boxed{\text{Angular Momentum}}$

Conserved for system ← no external forces (or moments) in FBD

$$m_i \ddot{\bar{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ji}^3} \underbrace{(\bar{r}_i - \bar{r}_j)}_{\bar{r}_{ji}}$$

Vector cross with $\bar{r}_i$; add up all equations

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \times \bar{r}_i = \sum_{i=1}^{n} G \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_i m_j}{r_{ij}^3} (\bar{r}_j - \bar{r}_i) \times \bar{r}_i$$

$$(\bar{r}_j \times \bar{r}_i) - \underbrace{(\bar{r}_i \times \bar{r}_i)}_{\text{zero}}$$

$$(\bar{r}_1 \times \bar{r}_2) + (\bar{r}_2 \times \bar{r}_1)$$

$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \times \bar{r}_i = \bar{0}$ ⟸ Equation we can integrate

Integrate once

$\sum_{i=1}^{n} m_i (\bar{r}_i \times \dot{\bar{r}}_i) = \bar{C}_3$ < const in mag, dir

Total angular momentum of a system of n particles → constant in magnitude AND direction

Can define significant surface: **invariable plane**

plane contains cm whose normal coincides with ang. momentum vector $\bar{C}_3$

10 consts
$\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$, energy

3. $\boxed{\text{Total Energy}}$

Conserved for system ← internal forces derivable from potential (conservative)
so system conservative

DE → $m_i \ddot{\bar{r}}_i = \bar{\nabla}_i U$

$\Sigma_1, \Sigma_2, \Sigma_3$, energy

3. Total Energy

Conserved for system $\leftarrow$ internal forces derivable from potential — conservative

so system conservative

DE $\rightarrow$ $m_i \ddot{\bar{r}}_i = \bar{\nabla}_i U$

Scalar dot product with $\dot{\bar{r}}_i$; add up all equations

$$\sum_{i=1}^{n} m_i \ddot{\bar{r}}_i \cdot \dot{\bar{r}}_i = \underbrace{\sum_{i=1}^{n} \nabla_i U \cdot \dot{\bar{r}}_i}$$ $\leftarrow$ associated

$\bar{r}_i = x\hat{i} + y\hat{j} + z\hat{k}$

$\dot{\bar{r}}_i = \frac{{}^{I}d\bar{r}_i}{dt} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}$

$$\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right)$$

$$\sum_{i=1}^{n} m_i \frac{d}{dt}\left(\frac{1}{2}\dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right) = \boxed{\frac{dU}{dt}}$$ $\leftarrow$ definition total derivative

$$\frac{d}{dt}\left[\sum_{i=1}^{n} m_i \left(\frac{1}{2} m_i \dot{\bar{r}}_i \cdot \dot{\bar{r}}_i\right)\right] = \frac{dU}{dt}$$ $\leftarrow$ can integrate!

Total kinetic Energy = Σ KE for each particle

$$\frac{d}{dt}T = \frac{d}{dt}U \quad \text{OR} \quad \frac{d}{dt}T - \frac{d}{dt}U = 0 \Leftarrow \text{Equation we can integrate}$$

$$\frac{dT}{dt} = \frac{dU}{dt}$$

Integrate once

$$T - U = C_4 \quad \text{(scalar)}$$

Kinetic Energy + potential Energy = const.

potential Energy = − grav. Energy