P1: A BP

Wednesday, January 8, 2020 10:57 PM

Mathematical Background

In the mathematical formulation and solution of problems in mechanics, we encounter numerous useful mathematical quantities. Some are already familiar concepts. For example, many of your courses have discussed physical phenomena through the use of ***vector methods***. This is a particularly useful approach because, in the application of physical laws, the results must be independent of coordinate system. Also, vector notation provides a compact method of expressing complicated results. You have previously developed proficiency with many definitions and vector operations. It will be useful in this course to expand your prior development of vector approaches and introduce new quantities and notation as well. We will consider topics of specific use in the analysis of the dynamics of spacecraft. Some review is initially included for notational clarity; then, alternative algebraic definitions of vectors and other associated quantities are discussed.

Gibbsian Vectors

The three-dimensional vector analysis we use today is essentially in the form developed around 1880-1882 by Josiah Willard Gibbs (1839-1903). Much of the current vector notation originated with an English electrical engineer, Oliver Heaviside (1850-1925), dating from about 1893. Gibbs and Heaviside actually utilized modified versions of an approach by William Rowan Hamilton (1805-1865) that utilized four-element entities (quaternions) for operations in three-dimensional space. Gibbs argued that three-element quantities (vectors) were more efficient and could be extended to higher dimensions. The Gibbsian vector approach was very controversial at the time, but was eventually accepted and put into general practice by physicists and engineers.

Vector Definitions

Gibbsian vectors are usually defined in terms of 3-D Euclidian geometry:

Vectors are directed line segments that add commutatively $(\bar{u}+\bar{v}=\bar{v}+\bar{u})$ and associatively $\left[\bar{u}+(\bar{v}+\bar{w})=(\bar{u}+\bar{v})+\bar{w}\right]$ according to the parallelogram rule

Operational Rules:

(i) dot product $\longrightarrow$ $\bar{u}\bullet\bar{v}=|\bar{u}||\bar{v}|\cos(\bar{u},\bar{v})$

commutative $u \cdot v = v \cdot u$

(ii) cross product $\longrightarrow$ $\bar{u}\times\bar{v}=|\bar{u}||\bar{v}|\sin(\bar{u},\bar{v})\hat{e}$ $\hat{e}_n \perp u, v$

commutative $u \cdot v = v \cdot u$

(ii) cross product ➡ $\bar{u} \times \bar{v} = |\bar{u}||\bar{v}|\sin(\bar{u},\bar{v})\,\hat{e}_n$ $\hat{e}_n \perp u, v$

anticommutative $u \times v = -u \times v$

Useful Identities

(i) $\bar{u} \times \bar{v} \bullet \bar{w} = \bar{u} \bullet \bar{v} \times \bar{w}$

(ii) $\bar{u} \times (\bar{v} \times \bar{w}) = (\bar{u} \bullet \bar{w})\bar{v} - (\bar{u} \bullet \bar{v})\bar{w}$

A3

convenient to use a dextral orthonormal triad

Notation and Operations

$\bar{v} = v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3$

component form

$\hat{e}$ vector basis

$v_1, v_2, v_3 \Longrightarrow$ scalar components measure numbers

subscript form

$\bar{v} = \sum_{j=1}^{3} v_j\hat{e}_j = v_j\hat{e}_j \Rightarrow$ summation convention

Kronecker delta (represents the result of a dot product)

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$ $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$

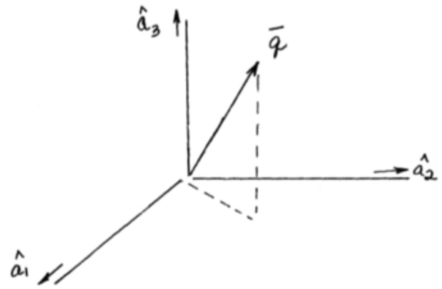
Permutation symbol (represents the result of a cross product)

$$\varepsilon_{\alpha\beta\gamma} = \begin{cases} 1 & \alpha\beta\gamma \text{ cyclic} \\ 0 & \text{repeated index} \\ -1 & \alpha\beta\gamma \text{ anticyclic} \end{cases}$$

$\boxed{\hat{e}_\alpha \times \hat{e}_\beta = \varepsilon_{\alpha\beta\gamma}\,\hat{e}_\gamma}$

Coordinate Transformations

relationships between sets of unit vectors



$\bar{q} = q_1\hat{a}_1 + q_2\hat{a}_2 + q_3\hat{a}_3$

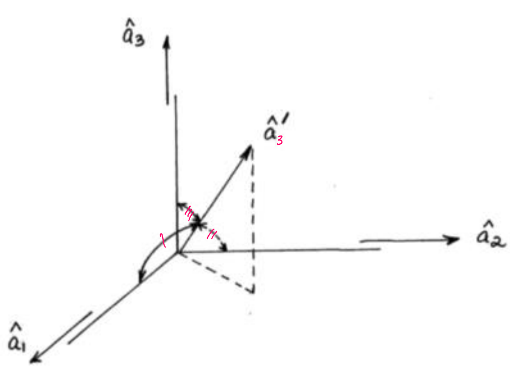
$$\left.\begin{aligned} \therefore\ q_1 &= \bar{q} \bullet \hat{a}_1 \\ q_2 &= \bar{q} \bullet \hat{a}_2 \\ q_3 &= \bar{q} \bullet \hat{a}_3 \end{aligned}\right\}$$

i: not repeated

$q_i = q \cdot \hat{a}_i$

represents 3 diff. eqn.

$\bar{q} = (\bar{q} \bullet \hat{a}_1)\hat{a}_1 + (\bar{q} \bullet \hat{a}_2)\hat{a}_2 + (\bar{q} \bullet \hat{a}_3)\hat{a}_3$



$\hat{a}_1' = (\hat{a}_1' \bullet \hat{a}_1)\hat{a}_1 + (\hat{a}_1' \bullet \hat{a}_2)\hat{a}_2 + (\hat{a}_1' \bullet \hat{a}_3)\hat{a}_3$

$\hat{a}_2' = (\hat{a}_2' \bullet \hat{a}_1)\hat{a}_1 + (\hat{a}_2' \bullet \hat{a}_2)\hat{a}_2 + (\hat{a}_2' \bullet \hat{a}_3)\hat{a}_3$

$\hat{a}_3' = \underbrace{(\hat{a}_3' \bullet \hat{a}_1)}_{|\hat{a}_3'||\hat{a}_1|\cos(\hat{a}_3',\hat{a}_1) = \cos(\hat{a}_3',\hat{a}_1)}\hat{a}_1 + (\hat{a}_3' \bullet \hat{a}_2)\hat{a}_2 + (\hat{a}_3' \bullet \hat{a}_3)\hat{a}_3$

$\hat{a}_1' = \cos(\hat{a}_1', \hat{a}_1)\hat{a}_1 + \cos(\hat{a}_1', \hat{a}_2)\hat{a}_2 + \cos(\hat{a}_1', \hat{a}_3)\hat{a}_3$

$\hat{a}_2' = \cos(\hat{a}_2', \hat{a}_1)\hat{a}_1 + \cos(\hat{a}_2', \hat{a}_2)\hat{a}_2 + \cos(\hat{a}_2', \hat{a}_3)\hat{a}_3$

$|\hat{a}_3||\hat{a}_1|\cos(\hat{a}_3,\hat{a}_1) = \cos(\hat{a}_3,\hat{a}_1)$

direction cosines

$$\hat{a}_1' = \cos(\hat{a}_1', \hat{a}_1)\hat{a}_1 + \cos(\hat{a}_1', \hat{a}_2)\hat{a}_2 + \cos(\hat{a}_1', \hat{a}_3)\hat{a}_3$$
$$\hat{a}_2' = \cos(\hat{a}_2', \hat{a}_1)\hat{a}_1 + \cos(\hat{a}_2', \hat{a}_2)\hat{a}_2 + \cos(\hat{a}_2', \hat{a}_3)\hat{a}_3$$
$$\hat{a}_3' = \cos(\hat{a}_3', \hat{a}_1)\hat{a}_1 + \cos(\hat{a}_3', \hat{a}_2)\hat{a}_2 + \cos(\hat{a}_3', \hat{a}_3)\hat{a}_3$$

$$\left.\begin{aligned} \hat{a}_1' &= \ell_{11}\,\hat{a}_1 + \ell_{12}\,\hat{a}_2 + \ell_{13}\,\hat{a}_3 \\ \hat{a}_2' &= \ell_{21}\,\hat{a}_1 + \ell_{22}\,\hat{a}_2 + \ell_{23}\,\hat{a}_3 \\ \hat{a}_3' &= \ell_{31}\,\hat{a}_1 + \ell_{32}\,\hat{a}_2 + \ell_{33}\,\hat{a}_3 \end{aligned}\right\}$$

component format

subscript format

$\hat{a}_i' = l_{ij} \cdot \hat{a}_j$

i.e. $l_{32} = \cos(\hat{a}_3', \hat{a}_2)$

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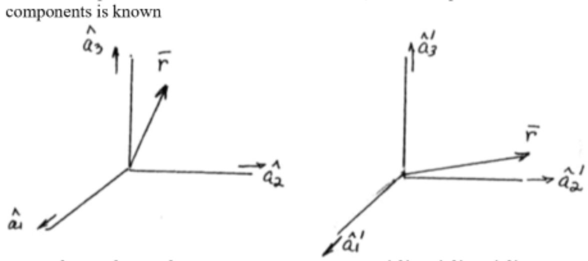
Extension: a given vector $\overline{q}$ or $\overline{r}$ can be expressed in terms of different vector bases; components differ depending on the set of unit vectors selected

If relationship between unit vector bases is known, relationship between components is known



$\overline{r} = r_1\,\hat{a}_1 + r_2\,\hat{a}_2 + r_3\,\hat{a}_3$ $\qquad$ $\overline{r} = r_1'\,\hat{a}_1' + r_2'\,\hat{a}_2' + r_3'\,\hat{a}_3'$

$$\overline{r} = r_1'\,\hat{a}_1' + r_2'\,\hat{a}_2' + r_3'\,\hat{a}_3'$$
$$\overline{r} = r_1'\left(\ell_{11}\,\hat{a}_1 + \ell_{12}\,\hat{a}_2 + \ell_{13}\,\hat{a}_3\right) + r_2'\left(\ell_{21}\,\hat{a}_1 + \ell_{22}\,\hat{a}_2 + \ell_{23}\,\hat{a}_3\right) + r_3'\left(\ell_{31}\,\hat{a}_1 + \ell_{32}\,\hat{a}_2 + \ell_{33}\,\hat{a}_3\right)$$

$$\overline{r} = \left(\ell_{11}\,r_1' + \ell_{21}\,r_2' + \ell_{31}\,r_3'\right)\hat{a}_1 + \left(\ell_{12}\,r_1' + \ell_{22}\,r_2' + \ell_{32}\,r_3'\right)\hat{a}_2 + \left(\ell_{13}\,r_1' + \ell_{23}\,r_2' + \ell_{33}\,r_3'\right)\hat{a}_3$$

$$\begin{aligned} r_1 &= \ell_{11}\,r_1' + \ell_{21}\,r_2' + \ell_{31}\,r_1' \\ r_2 &= \ell_{12}\,r_1' + \ell_{22}\,r_2' + \ell_{32}\,r_3' \\ r_3 &= \ell_{13}\,r_1' + \ell_{23}\,r_2' + \ell_{33}\,r_3' \end{aligned} \qquad \begin{aligned} r_1' &= \ell_{11}\,r_1 + \ell_{12}\,r_2 + \ell_{13}\,r_3 \\ r_2' &= \ell_{21}\,r_1 + \ell_{22}\,r_2 + \ell_{23}\,r_3 \\ r_3' &= \ell_{31}\,r_1 + \ell_{32}\,r_2 + \ell_{33}\,r_3 \end{aligned}$$

Different formats to represent information

$$\boxed{\begin{aligned} r_a &= l_{ba}\,r_b' \\ r_i' &= l_{ij}\,r_j \end{aligned}}$$

format

$C_{32} = \cos(\hat{a}_3', \hat{a}_2)$

transformation / rotation matrix.

matrix format

$$L = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

$\overline{r}$ defined as a column vector

$$\begin{bmatrix} r_1' \\ r_2' \\ r_3' \end{bmatrix} = \underbrace{\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}}_{\ell} \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \longrightarrow$$

column vector matrix format

$\ell \neq \ell^T$

$\overline{r}$ defined as a row vector

$$\begin{bmatrix} r_1' & r_2' & r_3' \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r \end{bmatrix} \overbrace{\begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}}^{\ell^T} \longrightarrow$$

row vector matrix format.

$$\begin{bmatrix} r_1' & r_2' & r_3' \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$C = \ell^T$

$$\left.\begin{array}{ll} \text{column format} & \overline{r}' = L\,\overline{r} \\ \text{row format} & \overline{r}' = \overline{r}\,C \end{array}\right\}$$

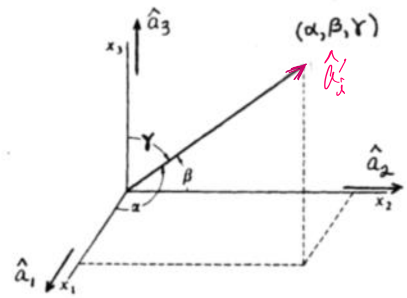
$$\begin{bmatrix} \hat{a}_1' \\ \hat{a}_2' \\ \hat{a}_3' \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \hat{a}_1 \\ \hat{a}_2 \\ \hat{a}_3 \end{bmatrix}$$

$$\begin{bmatrix} \hat{a}_1' & \hat{a}_2' & \hat{a}_3' \end{bmatrix} = \begin{bmatrix} \hat{a}_1 & \hat{a}_2 & \hat{a}_3 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Properties of Rotation Matrices

Properties of L (or C) —— ℓ_{ij} (or C_{ij}) —— are based on two-trigonometric results:

1



$$\hat{a}_1' = (\hat{a}_1' \cdot \hat{a}_1)\hat{a}_1 + (\hat{a}_1' \cdot \hat{a}_2)\hat{a}_2 + (\hat{a}_1' \cdot \hat{a}_3)\hat{a}_3$$

$$\hat{a}_1' = \cos(\hat{a}_1', \hat{a}_1)\hat{a}_1 + \cos(\hat{a}_1', \hat{a}_2)\hat{a}_2 + \cos(\hat{a}_1', \hat{a}_3)\hat{a}_3$$

$$\hat{a}_1' = \underbrace{\cos\alpha\, \hat{a}_1 + \cos\beta\, \hat{a}_2 + \cos\gamma\, \hat{a}_3}_{\text{magnitude of } 1}$$

unit magnitude

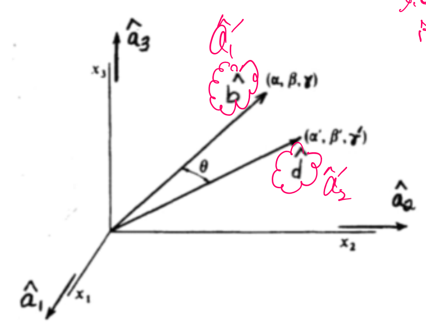
$$\boxed{\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1}$$

$\cos(\hat{a}_1', \hat{a}_1)$ $\cos(\hat{a}_1', \hat{a}_2)$ $\cos(\hat{a}_1', \hat{a}_3)$

$\cos\alpha = \ell_{11}$ $\cos\beta = \ell_{12}$ $\cos\gamma = \ell_{13}$

$$\therefore \left.\begin{aligned} \ell_{11}^2 + \ell_{12}^2 + \ell_{13}^2 = 1 \\ \ell_{21}^2 + \ell_{22}^2 + \ell_{23}^2 = 1 \\ \ell_{31}^2 + \ell_{32}^2 + \ell_{33}^2 = 1 \end{aligned}\right\} \quad \boxed{\ell_{ij}\,\ell_{kj} = 1,\ i = k}$$

2



i.e.
if $\hat{a}_1' = \hat{b}$
$\hat{a}_2' = \hat{d}$
$\theta \to 90°$
$\cos\theta = 0$

$$\hat{b} = \cos\alpha\,\hat{a}_1 + \cos\beta\,\hat{a}_2 + \cos\gamma\,\hat{a}_3$$
$$\hat{d} = \cos\alpha'\,\hat{a}_1 + \cos\beta'\,\hat{a}_2 + \cos\gamma'\,\hat{a}_3$$
$$\hat{b} \bullet \hat{d} = |\hat{b}||\hat{d}|\cos\theta = \cos\theta$$

$$\boxed{\hat{b} \bullet \hat{d} = \cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'}$$

assume $\perp$ unit vectors

Let $\hat{b} = \hat{a}_1'$ $\quad \hat{d} = \hat{a}_2'$ $\quad \Rightarrow \quad \theta = 90°$

Also then $\cos\alpha = \ell_{11}$ $\cos\alpha' = \ell_{21}$
$\cos\beta = \ell_{12}$ $\cos\beta' = \ell_{22}$
$\cos\gamma = \ell_{13}$ $\cos\gamma' = \ell_{23}$

$$\begin{aligned} \cos 90° = \quad 0 &= \ell_{11}\ell_{21} + \ell_{12}\ell_{22} + \ell_{13}\ell_{23} \\ 0 &= \ell_{21}\ell_{31} + \ell_{22}\ell_{32} + \ell_{23}\ell_{33} \\ 0 &= \ell_{11}\ell_{31} + \ell_{12}\ell_{32} + \ell_{13}\ell_{33} \end{aligned}$$

$\Rightarrow \boxed{\ell_{ij}\,\ell_{kj} = 0 \quad i \neq k}$

Note: relationships based on the assumption that unit vectors are mutually perpendicular

1 + 2

Summarize $\boxed{\ell_{ij}\,\ell_{kj} = \delta_{ik}}$ $\begin{cases} = 1 \\ = 0 \end{cases}$ check if it violates

(1) + (2)

Summarize $\boxed{\ell_{ij}\,\ell_{kj} = \delta_{ik}}$

$\begin{cases} = 1 \\ = 0 \end{cases}$ check if it violates

9 direction cosines

6 relationships between them

such systems are orthogonal

this eqn. reflects the orthogonality condition

Orthogonality condition used to develop other characteristics of transformation matrices. Orthogonality condition written in terms of matrices:

$$\begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{identity matrix or Unity Matrix} \Rightarrow U}$$

$$L\,L^T = U$$

Note: Inverse of a matrix (L^{-1}) defined as the matrix which, when multiplied by original matrix produces U

↓

$$L\,L^{-1} = U$$

↓

$$\begin{cases} L^T = L^{-1} \\ C^T = C^{-1} \end{cases}$$

Dyadics

Once concepts of vectors and transformation matrices are clear, other useful quantities can be exploited

Recall → many quantities in mechanics are well represented by 3-element vectors

Examples: position, velocity, acceleration, ang. velocity ang. momentum.

Note that each quantity in the list can be expressed in components; values for the components depend on the vector basis

Each quantity is associated with a particular vector basis; can rewrite quantity in terms of a different vector basis

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \Rightarrow \bar{\bar{I}} = I_{11}\hat{a}_1\hat{a}_1 + I_{12}\hat{a}_1\hat{a}_2 + I_{13}\hat{a}_1\hat{a}_3 + I_{21}\hat{a}_2\hat{a}_1 + I_{22}\hat{a}_2\hat{a}_2 + I_{23}\hat{a}_2\hat{a}_3 + \ldots + I_{33}\hat{a}_3\hat{a}_3$$

define component "directions"

$$\bar{\bar{H}} = H_{123}\hat{a}_1\hat{a}_2\hat{a}_3$$

transformation from $\hat{a}$ to $\hat{b}$ as a col vector

$$\underset{\hat{b}}{[r]} = \underset{\hat{b}\cdot\hat{a}}{[\ell]}\,\underset{\hat{a}}{[r]}$$ **a key relationship**

transformations from one vector basis to another

col vector format

dyads $\hat{a}_1\hat{a}_1$ $\hat{a}_1\hat{a}_2$ $\hat{a}_2\hat{a}_1$

triads $\hat{a}_1\hat{a}_2\hat{a}_3$

⋮

polyads $\hat{a}_1\hat{a}_2\hat{a}_3\hat{a}_4\cdots\hat{a}_n$

$$\bar{r}' = L\,\bar{r} \qquad \text{OR} \qquad \bar{r}' = \bar{r}\,C$$
$$\bar{v}' = L\,\bar{v} \qquad\qquad \bar{v}' = \bar{v}\,C$$
$$\bar{H}' = L\,\bar{H} \qquad\qquad \bar{H}' = \bar{H}\,C$$
$$\vdots \qquad\qquad\qquad \vdots$$

Some quantities that are useful in mechanics cannot be represented by a 3-element vector; yet, they are still associated with a particular vector basis

Example: Set of moments and products of inertia

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

We already know that any inertia matrix is associated with

(i) particular point

(ii) particular vector basis **

We already know that any inertia matrix can be transformed to another vector basis through the use of the similarity transformation

Define: matrix I associated with vector basis $\hat{u}$
matrix I' associated with vector basis $\hat{u}'$

$$\begin{bmatrix} \hat{u}_1' \\ \hat{u}_2' \\ \hat{u}_3' \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix}$$

$$\begin{bmatrix} \hat{u}_1' \\ \hat{u}_2' \\ \hat{u}_3' \end{bmatrix} = L \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \hat{u}_3 \end{bmatrix}$$

In 340 we used the notation: $L \Leftrightarrow \underset{\hat{u}' \bullet \hat{u}}{\left[\ell \right]}$

* identity

$$\varepsilon_{ijk}\,\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$[\hat{u}'] = L[\hat{u}]$

Transform a vector

$$\underset{u'}{\{v\}} = \underset{\hat{u}' \cdot \hat{u}}{[\ell]} \underset{\hat{u}}{\{v\}}$$

Transform a matrix

$$\underset{u'}{[I]} = \underset{\hat{u}' \cdot \hat{u}}{[\ell]} \underset{\hat{u}}{[I]} [\ell]^T$$

similarity transformation

$I' = LTL^T$ or $I' = C^T I C$

$$\begin{bmatrix} I'_{11} & I'_{12} & I'_{13} \\ I'_{21} & I'_{22} & I'_{23} \\ I'_{31} & I'_{32} & I'_{33} \end{bmatrix} = \begin{bmatrix} \ell_{11} & \ell_{12} & \ell_{13} \\ \ell_{21} & \ell_{22} & \ell_{23} \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix} \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \ell_{11} & \ell_{21} & \ell_{31} \\ \ell_{12} & \ell_{22} & \ell_{32} \\ \ell_{13} & \ell_{23} & \ell_{33} \end{bmatrix}$$

Represents 9 equations (one for each I'_{ij})

We would like to have more flexibility

scalar e | element

Vectors: matrix format $\underset{\hat{e}}{[t_1 \ t_2 \ t_3]}$ $\quad \underset{\hat{e}}{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}}$

component format $\bar{v} = v_1\,\hat{e}_1 + v_2\,\hat{e}_2 + v_3\,\hat{e}_3$

Inertia matrix: matrix format $\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$

component format ???

component format – corresponding to vector component format; use dyadic representation to carry vector basis explicitly

$\bar{\bar{I}} = I_{11}\hat{e}_1\hat{e}_1 + I_{12}\hat{e}_1\hat{e}_2 + I_{13}\hat{e}_1\hat{e}_3 + I_{21}\hat{e}_2\hat{e}_1$

component format – corresponding to vector component format; use dyadic representation to carry vector basis explicitly

$$\bar{\bar{I}} = I_{11}\hat{e}_1\hat{e}_1 + I_{12}\hat{e}_1\hat{e}_2 + I_{13}\hat{e}_1\hat{e}_3 + I_{21}\hat{e}_2\hat{e}_1 + I_{22}\underbrace{\hat{e}_2\hat{e}_2}_{\text{dyads}} + I_{23}\hat{e}_2\hat{e}_3 + I_{31}\hat{e}_3\hat{e}_1 + I_{32}\hat{e}_3\hat{e}_2 + I_{33}\hat{e}_3\hat{e}_3$$

9 elements ⟶ 9 terms in expressions (each term is scalar + dyad)

27 elements

$$\bar{\bar{\bar{K}}} = K_{abc}\,\hat{e}_a\hat{e}_b\hat{e}_c$$

dyad – two vectors sitting together; no intervening operation like dot/cross

NOT commutative $\bar{u}\,\bar{v} \neq \bar{v}\,\bar{u}$ $\quad \mathbb{V} = v_i\hat{n}_i$

Summation convention $\left\{ I_{ij}\hat{n}_i\hat{n}_j \right.$

$$\mathbb{U}\mathbb{V} = (u_1\hat{n}_1 + u_2\hat{n}_2 + u_3\hat{n}_3)(v_1\hat{n}_1 + v_2\hat{n}_2 + v_3\hat{n}_3)$$
$$= u_1v_1\hat{n}_1\hat{n}_1 + u_1v_2\hat{n}_1\hat{n}_2 + u_1v_3\hat{n}_1\hat{n}_3 + \ldots$$
$$= u_iv_j\hat{n}_i\hat{n}_j$$

Use of dyadics extends to any quantity that we have traditionally represented as a matrix. One advantage of dyadic format is that dyads and dyadics operate on vectors using the operational rules similar to those for vectors

Example: Angular Momentum

$\mathbb{W} = \omega_i\hat{e}_i$

$$\bar{H} = H_1\,\hat{e}_1 + H_2\,\hat{e}_2 + H_3\,\hat{e}_3$$

To evaluate $\bar{H}$ typically uses matrix format

$$\underset{\hat{u}}{\{H\}} = \underset{\hat{u}}{[I]}\underset{\hat{u}}{\{\omega\}} \qquad \textbf{OR} \qquad [H] = I\,[\omega]$$

$$\underbrace{\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}}_{\text{IH vector for basis } \hat{u}} = \underbrace{\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}}_{\text{associated w/ basis } \hat{u}} \underbrace{\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}}_{\text{ang. vel. of rigid body w.r.t inertial frame; basis } \hat{u}}$$

$$\mathbb{H} = \bar{\bar{I}}\cdot\mathbb{W} \qquad \bar{H} = \bar{\bar{I}}\cdot\bar{\omega}$$

Note: $\underset{\hat{u}'}{\{H\}} = \underset{\hat{u}' \bullet \hat{u}}{[\ell]} \underset{\hat{u}}{\{H\}}$ **OR** $[H'] = L[H]$

Utilizing inertia dyadic

$\bar{H} = \bar{\bar{I}} \bullet \bar{\omega}$ dot product between a dyadic and a vector is a vector
Note: in general

$$\begin{aligned}
\bar{u} \bullet \bar{v} &= (u_1\hat{e}_1 + u_2\hat{e}_2 + u_3\hat{e}_3) \bullet (v_1\hat{e}_1 + v_2\hat{e}_2 + v_3\hat{e}_3) \\
&= u_1v_1(\hat{e}_1 \bullet \hat{e}_1) + u_1v_2(\hat{e}_1 \bullet \hat{e}_2) + u_1v_3(\hat{e}_1 \bullet \hat{e}_3) \\
&\quad + u_2v_1(\hat{e}_2 \bullet \hat{e}_1) + u_2v_2(\hat{e}_2 \bullet \hat{e}_2) + u_2v_3(\hat{e}_2 \bullet \hat{e}_3) \\
&\quad + u_3v_1(\hat{e}_3 \bullet \hat{e}_1) + u_3v_2(\hat{e}_3 \bullet \hat{e}_2) + u_3v_3(\hat{e}_3 \bullet \hat{e}_3)
\end{aligned}$$

$\bar{u} \bullet \bar{v} =$

$$\begin{aligned}
\bar{\bar{I}} \bullet \bar{\omega} &= (I_{11}\hat{e}_1\hat{e}_1 + I_{12}\hat{e}_1\hat{e}_2 + I_{13}\hat{e}_1\hat{e}_3 + I_{21}\hat{e}_2\hat{e}_1 + I_{22}\hat{e}_2\hat{e}_2 + I_{23}\hat{e}_2\hat{e}_3 \\
&\quad + I_{31}\hat{e}_3\hat{e}_1 + I_{32}\hat{e}_3\hat{e}_2 + I_{33}\hat{e}_3\hat{e}_3) \bullet (\omega_1\hat{e}_1 + \omega_2\hat{e}_2 + \omega_3\hat{e}_3) \\
&= I_{11}\omega_1\hat{e}_1(\hat{e}_1 \bullet \hat{e}_1) + I_{11}\omega_2\hat{e}_1(\hat{e}_1 \bullet \hat{e}_2) + I_{11}\omega_3\hat{e}_1(\hat{e}_1 \bullet \hat{e}_3) + \\
&\quad I_{12}\omega_1\hat{e}_1(\hat{e}_2 \bullet \hat{e}_1) + I_{12}\omega_2\hat{e}_1(\hat{e}_2 \bullet \hat{e}_2) + I_{12}\omega_3\hat{e}_1(\hat{e}_2 \bullet \hat{e}_3) + \\
&\quad I_{13}\omega_1\hat{e}_1(\hat{e}_3 \bullet \hat{e}_1) + I_{13}\omega_2\hat{e}_1(\hat{e}_3 \bullet \hat{e}_2) + I_{13}\omega_3\hat{e}_1(\hat{e}_3 \bullet \hat{e}_3) + \ldots
\end{aligned}$$

$\bar{\bar{I}} \bullet \bar{\omega} =$

$$\boxed{\begin{aligned}\overline{\overline{I}} \bullet \overline{\omega} = \; & \left(I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3\right)\hat{e}_1 \\ & + \left(I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3\right)\hat{e}_2 \\ & + \left(I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3\right)\hat{e}_3\end{aligned}}$$

Let $\underset{\hat{a}}{I} = \begin{bmatrix} 200 & 150 & 0 \\ 150 & 300 & 0 \\ 0 & 0 & 500 \end{bmatrix}$ kg-m^2

$$\overline{\omega} = 3\hat{a}_2 + 2\hat{a}_3 \text{ rad/s}$$

$$\begin{aligned}\overline{\overline{I}} \bullet \overline{\omega} = & \left(200\hat{a}_1\hat{a}_1 + 150\hat{a}_1\hat{a}_2 + 150\hat{a}_2\hat{a}_1 + 300\hat{a}_2\hat{a}_2 + \right. \\ & \left. + 500\hat{a}_3\hat{a}_3\right) \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right) \quad \text{kg-m}^2/\text{s} \\ = & \; 200\hat{a}_1\hat{a}_1 \bullet \left(3\hat{a}_2 + 2\hat{a}_3\right) \\ & 150\hat{a}_1\hat{a}_2 \bullet \left(3\hat{a}_2 + 2\hat{a}_3\right) \\ & 150\hat{a}_2\hat{a}_1 \bullet \left(3\hat{a}_2 + 2\hat{a}_3\right) \\ & 300\hat{a}_2\hat{a}_2 \bullet \left(3\hat{a}_2 + 2\hat{a}_3\right) \\ & 500\hat{a}_3\hat{a}_3 \bullet \left(3\hat{a}_2 + 2\hat{a}_3\right)\end{aligned}$$

$\overline{H} =$ kg-m^2/s

Example: Rotational Kinetic Energy

scalar $T = \frac{1}{2}\left\{\underset{\hat{a}}{\omega}\right\}^T \left[\underset{\hat{a}}{I}\right]\left\{\underset{\hat{a}}{\omega}\right\} = \frac{1}{2}\{\omega\}^T\{H\}$

$$= \frac{1}{2}\{\omega_1 \quad \omega_2 \quad \omega_3\}\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}\begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix}$$

$$T = \frac{1}{2}\bar{\omega} \bullet \bar{H}$$

$$\boxed{T = \frac{1}{2}\omega \cdot \bar{\bar{I}} \cdot \omega}$$

Kinetic Energy

Previous example

$$T = \frac{1}{2}\left(3\hat{e}_2 + 2\hat{e}_3\right) \bullet \left(200\hat{a}_1\hat{a}_1 + 150\hat{a}_1\hat{a}_2 + 150\hat{a}_2\hat{a}_1 + 300\hat{a}_2\hat{a}_2 + \right.$$
$$\left. + 500\hat{a}_3\hat{a}_3\right) \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right)$$

$$\begin{aligned} 2T = \; & \left(3\hat{e}_2 + 2\hat{e}_3\right) \bullet 200\hat{a}_1\hat{a}_1 \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right) \\ & + \left(3\hat{e}_2 + 2\hat{e}_3\right) \bullet 150\hat{a}_1\hat{a}_2 \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right) \\ & + \left(3\hat{e}_2 + 2\hat{e}_3\right) \bullet 150\hat{a}_2\hat{a}_1 \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right) \\ & + \left(3\hat{e}_2 + 2\hat{e}_3\right) \bullet 300\hat{a}_2\hat{a}_2 \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right) \\ & + \left(3\hat{e}_2 + 2\hat{e}_3\right) \bullet 500\hat{a}_3\hat{a}_3 \bullet \left(3\hat{e}_2 + 2\hat{e}_3\right) \quad \text{kg-m}^2/\text{s}^2 \end{aligned}$$

$$T = 2350 \;\; \text{kg-m}^2/\text{s}^2$$