Homework #6 Due: November 19, 2021

1. Study the material of Chapter 7 ("Linear-Quadratic Optimal Control") posted on Canvas.

2. Write a MATLAB code based on the document uploaded to Canvas to solve the following problem. A ship is located at the point $(x_0, y_0) = (-20, 0)$ ml at time $t = 0$ when it encounters a medical emergency and it has to reach the shore as soon as possible. It is known that there is a small city at the location $(x_1, y_1) = (-15, 35.5)$ ml with a medical center. As the captain of the ship, you are to determine the fastest possible route to the city. It is assumed that the speed of the ship with respect to the water is constant, $v = 15\,\mathrm{ml/hr}$. You also know the speed and direction of the sea currents in the area, which are given to you from a meteorological satellite as $\vec{v}_c = u(x,y)\,\hat{\mathbf{i}} + v(x,y)\,\hat{\mathbf{j}}$.

   (a) Derive the necessary conditions for the optimal control strategy, and calculate the optimal path and the time to reach the city, assuming that the currents are constant, given by

   $$\vec{v}_c = 2\,\hat{\mathbf{i}} - 6\,\hat{\mathbf{j}}$$

   Plot the optimal path in the $x - y$ plane along with the vectors showing the direction of the currents.

   (b) When you are about to start your dash to the shore, you learn that the doctor in the medical center will be able to fly by helicopter to any point at the shore to pick up the patient. Find the new optimal path and the time to reach the shore, assuming that the contour of the shoreline is known to be

   $$\psi(x,y) = 25 - 0.25x - 0.002x^3 - y = 0$$

   Plot the optimal path in the $x - y$ plane along with the vectors showing the direction of the currents.

   (c) An update of the meteorological data from the satellite shows that strong winds have developed in the area and that the currents have changed significantly. The new currents are

   $$\vec{v}_c = -(y - 50)\,\hat{\mathbf{i}} + 2(x - 15)\,\hat{\mathbf{j}}$$

   Recalculate the optimal control and plot the optimal path in the $x - y$ plane along with the vectors showing the direction of the currents. Plot the optimal steering angle history $\theta^*(t)$.

   In all cases, plot the Hamiltonian and verify that remains zero for all time.

3. Investigate the existence of conjugate points in the closed interval $[0, t_\mathrm{f}]$ for all values of $t_\mathrm{f} > 0$, for the problem

   $$\min\ J(u) = \int_0^{t_\mathrm{f}} [t^2 + x^2(t) + u^2(t)]\,\mathrm{d}t$$

   where $\dot{x}(t) = u(t)$ and the boundary conditions are $x(0) = 0$ and $x(t_\mathrm{f}) = 0$.

4. Show that the matrix Riccati differential equation

   $$-\dot{P}(t) = A^\mathsf{T} P(t) + P(t)A - P(t)BR_2^{-1}B^\mathsf{T}P(t)$$

   with initial condition

   $$P(0) = P_0$$

where P_0 is nonsingular, has the solution

$$P(t) = e^{-A^{\mathsf{T}}t}\left(P_0^{-1} - \int_0^t e^{-As} B R_2^{-1} B^{\mathsf{T}} e^{-A^{\mathsf{T}}s}\,\mathrm{d}s\right)^{-1} e^{-At}.$$

5. Consider the harmonic oscillator

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_1(t) + u(t)\end{aligned}$$

with the energy performance measure

$$J(u) = \int_0^{t_\mathrm{f}} u^2(t)\,\mathrm{d}t$$

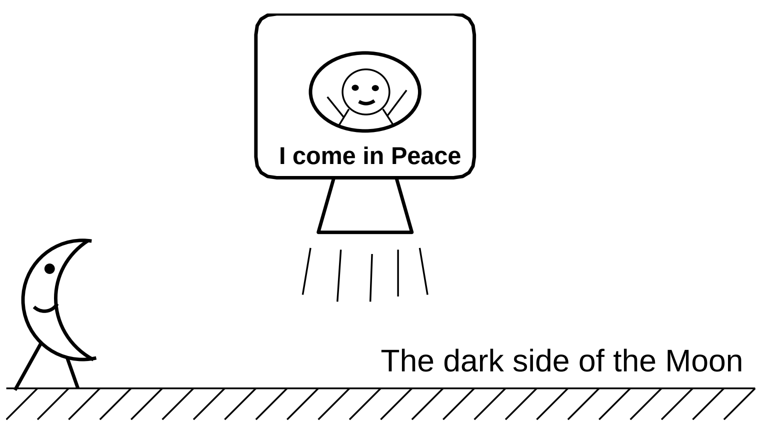
Let $x_1(0) = 1$ and $x_2(0) = 2$, and require that $x_1(t_\mathrm{f}) = 0$ and $x_2(t_\mathrm{f}) = 0$. Consider three cases $t_\mathrm{f} = 10, 5, 1$. For each case, simulate the controlled system with the minimum energy controller and compute the control effort as measured by

$$\left(\int_0^{t_\mathrm{f}} u^2(t)\mathrm{d}t\right)^{\frac{1}{2}}, \quad \int_0^{t_\mathrm{f}} |u(t)|\mathrm{d}t, \quad \max_{t\in[0,t_\mathrm{f}]} |u(t)|$$

6. Consider the problem of a spacecraft attempting to make a soft landing on the moon using the minimum amount of fuel. The equations of motion are

$$\begin{aligned}\dot{h} &= v\\ \dot{v} &= -g + \frac{u}{m}\\ \dot{m} &= -cu\end{aligned}$$

where m denotes the mass, h the altitude, v the vertical velocity, and u is the thrust of the spacecraft's engine, c a constant, and g the gravity acceleration of the moon (considered to be constant). The control u is restricted so that $0 \le u(t) \le u_{\max} = 1$.



The boundary conditions are

$$h(0) = h_0, \quad v(0) = v_0, \quad m(0) - m_{\mathrm{net}} - m_0 = 0$$
$$h(T) = v(T) = 0$$

where m_{net} denotes the mass of the spacecraft without fuel, h_0 the initial height, v_0 the initial velocity, m_0 the initial amount of fuel, and T is the (given) time taken for touchdown. Solve this optimal control problem, that is, minimize

$$J(u) = \int_0^T u(t)\,\mathrm{d}t$$

subject to the previous dynamic and boundary constraints.

7. Consider the following problem

$$\min\ J(u) = \frac{1}{2}\int_0^T (u^2 - \alpha x^2)\,\mathrm{d}t$$

subject to

$$\dot{x} = x + u, \qquad x(0) = x(T) = 0$$

   (a) Find the extremal control and the extremal trajectory for this problem for $\alpha > 1$.

   (b) Investigate the existence (and location) of conjugate points for $\alpha = 2$ and $T = \pi$.
   Hint: Investigate the accessory minimization problem.

8. Use the Maximum Principle to solve the following optimal control problem

$$\min \int_0^{\frac{\pi}{6}} \left(\frac{u^2(t)}{\cos(t)} - u(t) \right) \mathrm{d}t$$

subject to

$$\dot{x}(t) = -u(t)$$

with boundary conditions

$$x(0) = 0, \qquad x(\frac{\pi}{6}) = -\frac{1}{8}$$

and control constraint $u(t) \geq 0$.

You may use the fact that $\cos(t) > 0$ on $t \in [0, \frac{\pi}{6}]$.