College of Engineering
School of Aeronautics and Astronautics

AAE 564
System Analysis and Synthesis

Homework 1
State Space Representation of Dynamic Systems

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Exercise 1 Consider a system described by a single n^{th}-order linear differential equation of the form

$$q^{(n)} + a_{n-1}q^{(n-1)} + \dots a_1\dot{q} + a_0 q = u$$

where $q(t) \in \mathbb{R}$ and $q^{(n)} := \frac{d^n q}{dt^n}$. By appropriate definition of state variables, obtain a first order state space description of this system.

Say, $x_0 = q$, $x_1 = \dot{q}$, $x_2 = \ddot{q}$, $\dots$, $x_{n-1} = q^{(n-1)}$, $x_n = q^{(n)}$

then

$$\begin{aligned} x_1 &= \dot{x}_0 \\ x_2 &= \dot{x}_1 \\ &\vdots \\ x_{n-1} &= \dot{x}_{n-2} \\ x_n &= \dot{x}_{n-1} \end{aligned}$$

and using the given equation

$$q^{n} + a_{n-1}q^{(n-1)} + \dots a_1\dot{q} + a_0 q = u$$

$$\Longrightarrow \dot{x}_{n-1} = -(a_{n-1}x_{n-1} + \dots a_1 x_1 + a_0 x_0) + u$$

Thus, the first order state space description of this system is

$$\boxed{\underbrace{\begin{bmatrix} \dot{x}_{n-1} \\ \dot{x}_{n-2} \\ \dot{x}_{n-3} \\ \vdots \\ \dot{x}_3 \\ \dot{x}_2 \\ \dot{x}_1 \\ \dot{x}_0 \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & 1 & & \vdots \\ 0 & 0 & & \ddots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}}_{n\times n} \underbrace{\begin{bmatrix} x_{n-1} \\ x_{n-2} \\ x_{n-3} \\ \vdots \\ x_3 \\ x_2 \\ x_1 \\ x_0 \end{bmatrix}}_{n\times 1} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{n\times 1} u}$$

Exercise 2 By appropriate definition of state variables, obtain a first order state space description of the following systems where q_1 and q_2 are real scalars.

(i)

$$\begin{aligned} 2\ddot{q}_1 + \ddot{q}_2 + \sin q_1 &= 0 \\ \ddot{q}_1 + 2\ddot{q}_2 + \sin q_2 &= 0 \end{aligned}$$

(ii)

$$\begin{aligned} \ddot{q}_1 + \dot{q}_2 + q_1^3 = 0 \\ \dot{q}_1 + \dot{q}_2 + q_2^3 = 0 \end{aligned}$$

(i) Say $q_1 = x_1$, $\dot{q}_1 = x_2$, $\ddot{q}_1 = x_3$

then $x_2 = \dot{x}_1$, $x_3 = \dot{x}_2$

Say $q_2 = y_1$, $\dot{q}_2 = y_2$, $\ddot{q}_2 = y_3$

then $y_2 = \dot{y}_1$, $y_3 = \dot{y}_2$

and

$$\begin{cases} 2x_3 + y_3 + \sin x_1 = 0 \\ x_3 + 2y_3 + \sin y_1 = 0 \end{cases}$$

$$\Downarrow$$

$$3x_3 + 2\sin x_1 - \sin y_1 = 0$$

$$\therefore \quad x_3 = -\frac{2}{3}\sin x_1 + \frac{1}{3}\sin y_1$$

also

$$3y_3 + 2\sin y_1 - \sin x_1 = 0$$

$$\therefore \quad y_3 = \frac{1}{3}\sin x_1 - \frac{2}{3}\sin y_1$$

thus,

$$\boxed{\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \\ \dot{y}_2 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2\sin x_1}{x_1} & 0 & \frac{\sin y_1}{y_1} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{\sin x_1}{x_1} & 0 & -\frac{2\sin y_1}{y_1} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ y_2 \\ y_1 \end{bmatrix}}$$

(ii) Say $q_1 = x_1$, $\dot{q}_1 = x_2$, $\ddot{q}_1 = x_3$

then $x_2 = \dot{x}_1$, $x_3 = \dot{x}_2$

Say $q_2 = y_1$, $\dot{q}_2 = y_2$, $\ddot{q}_2 = y_3$

then $y_2 = \dot{y}_1$, $y_3 = \dot{y}_2$

and from $\dot{q}_1 + \dot{q}_2 + q_2^3 = 0$

take the derivative of this

$$\ddot{q}_1 + \ddot{q}_2 + 3\dot{q}_2 q_2^2 = 0$$

then,

$$\begin{cases} \ddot{q}_1 + \dot{q}_2 + q_1^3 = 0 \\ \ddot{q}_1 + \ddot{q}_2 + 3\dot{q}_2 q_2^2 = 0 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} x_3 + y_2 + x_1^3 = 0 \\ x_3 + y_3 + 3y_2 y_1^2 = 0 \end{cases}$$

$$\Downarrow$$

$$x_3 = -x_1^3 - y_2$$

$$y_3 = x_1^3 + y_2(3y_1^2 + 1)$$

thus,

$$\boxed{\begin{bmatrix} \dot{x}_2 \\ \dot{x}_1 \\ \dot{y}_2 \\ \dot{y}_1 \end{bmatrix} = \begin{bmatrix} 0 & -x_1^2 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & x_1^2 & 3y_1^2 + 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ x_1 \\ y_2 \\ y_1 \end{bmatrix}}$$

Exercise 3 Obtain a state-space description of the following system.

$$\begin{aligned}\ddot{q}_1 + \dot{q}_2 + \sin q_1 &= u\\ \dot{q}_2 + q_1 + q_2 &= 0\\ y &= q_1 + q_2\end{aligned}$$

Say $q_1 = a_1,\ \dot{q}_1 = a_2,\ \ddot{q}_1 = a_3$

$q_2 = b_1,\ \dot{q}_2 = b_2,\ \ddot{q}_2 = b_3$

then $a_2 = \dot{a}_1,\ a_3 = \dot{a}_2,\ b_2 = \dot{b}_1,\ b_3 = \dot{b}_2$

from the given system equation

$$\begin{cases} a_3 + b_2 + \sin a_1 = u \\ b_2 + a_1 + b_1 = 0 \quad \leftarrow \text{take derivative} \\ y = a_1 + b_1 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{a}_2 = a_3 = -\sin a_1 - b_2 + u \\ \dot{b}_2 = -a_2 - b_2 \\ y = a_1 + b_1 \end{cases}$$

thus,

$$\boxed{\begin{gathered}\begin{bmatrix}\dot{a}_2\\ \dot{a}_1\\ \dot{b}_2\\ \dot{b}_1\end{bmatrix} = \begin{bmatrix}0 & -\frac{\sin a_1}{a_1} & -1 & 0\\ 1 & 0 & 0 & 0\\ -1 & 0 & -1 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}a_2\\ a_1\\ b_2\\ b_1\end{bmatrix} + \begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}u \\ y = \begin{bmatrix}0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}a_2\\ a_1\\ b_2\\ b_1\end{bmatrix}\end{gathered}}$$

Exercise 4 Consider the discrete-time system described by

$$q(k+3) + 7q(k+2) + q(k+1) + 6q(k) + 7u(k) = 0$$

Obtain a state space description of this system.

say $x_0(k) = q(k)$, $x_1(k) = q(k+1)$
$x_2(k) = q(k+2)$, $x_3(k) = q(k+3)$

then $x_0(k+1) = x_1(k)$
$x_1(k+1) = x_2(k)$
$x_2(k+1) = x_3(k) = -7q(k+2) - q(k+1) - 6q(k) - 7u(k)$
$= -7x_2(k) - x_1(k) - 6x_0(k) - 7u(k)$

thus,

$$\boxed{\begin{bmatrix} x_2(k+1) \\ x_1(k+1) \\ x_0(k+1) \end{bmatrix} = \begin{bmatrix} -7 & -1 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_2(k) \\ x_1(k) \\ x_0(k) \end{bmatrix} + \begin{bmatrix} -7 \\ 0 \\ 0 \end{bmatrix} u(k)}$$

Exercise 5 Obtain a state space representation of the following system:

$$q(k+n) + a_{n-1}q(k+n-1) + \ldots + a_1 q(k+1) + a_0 q(k) = 0$$

where $q(k) \in \mathbb{R}$.

Say, $q(k) = x_0(k)$, $q(k+1) = x_1(k)$, $\ldots$, $q(k+n-1) = x_{n-1}(k)$,

$$q(k+n) = x_n(k)$$

Then

$$\begin{aligned} x_0(k+1) &= x_1(k) \\ x_1(k+1) &= x_2(k) \\ &\vdots \\ x_{n-2}(k+1) &= x_{n-1}(k) \\ x_{n-1}(k+1) &= x_n(k) \end{aligned}$$

Using the given equation

$$q(k+n) = -a_{n-1}q(k+n-1) - \ldots - a_1 q(k+1) - a_0 q(k)$$

$$\therefore\ x_{n-1}(k+1) = x_n(k) = -a_{n-1}x_{n-1}(k) - \ldots - a_1 x_1(k) - a_0 x_0(k)$$

Thus,

$$\boxed{\underset{n\times 1}{\begin{bmatrix} x_{n-1}(k+1) \\ x_{n-2}(k+1) \\ x_{n-3}(k+1) \\ \vdots \\ x_3(k+1) \\ x_2(k+1) \\ x_1(k+1) \\ x_0(k+1) \end{bmatrix}} = \underset{n\times n}{\begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 \cdots & 0 & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & \vdots & 1 & & 0 \\ 0 & 0 & & \ddots & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & 1 & 0 \end{bmatrix}} \underset{n\times 1}{\begin{bmatrix} x_{n-1}(k) \\ x_{n-2}(k) \\ x_{n-3}(k) \\ \vdots \\ x_3(k) \\ x_2(k) \\ x_1(k) \\ x_0(k) \end{bmatrix}}}$$

Exercise 6 Obtain a state description of the following system:

$$q_1(k+2) + q_2(k+1) + q_1(k) = u(k)$$
$$q_1(k+2) - q_2(k+1) + q_2(k) = 0$$
$$y(k) = q_1(k+1) + q_2(k)$$

Say $a_0(k) = q_1(k)$, $a_1(k) = q_1(k+1)$, $a_2(k) = q_1(k+2)$
$b_0(k) = q_2(k)$, $b_1(k) = q_2(k+1)$, $b_2(k) = q_2(k+2)$

Then $a_0(k+1) = a_1(k)$, $a_1(k+1) = a_2(k)$
$b_0(k+1) = b_1(k)$, $b_1(k+1) = b_2(k)$

From the given equations

$$\begin{cases} q_1(k+2) + q_2(k+1) + q_1(k) = u(k) & \cdots ① \\ q_1(k+2) - q_2(k+1) + q_2(k) = 0 & \cdots ② \\ y(k) = q_1(k+1) + q_2(k) \end{cases}$$

$$① + ② \Rightarrow 2q_1(k+2) = -q_1(k) - q_2(k) + u(k)$$
$$① - ② \Rightarrow 2q_2(k+2) = -q_1(k) + q_2(k) + u(k)$$

$$\Rightarrow \begin{cases} q_1(k+2) = -\frac{1}{2}q_1(k) - \frac{1}{2}q_2(k) + \frac{1}{2}u(k) \\ q_2(k+2) = -\frac{1}{2}q_1(k) + \frac{1}{2}q_2(k) + \frac{1}{2}u(k) \\ y(k) = q_1(k+1) + q_2(k) \end{cases}$$

$$\Rightarrow \begin{cases} a_1(k+1) = -\frac{1}{2}a_0(k) - \frac{1}{2}b_0(k) + \frac{1}{2}u(k) \\ b_1(k+1) = -\frac{1}{2}a_0(k) + \frac{1}{2}b_0(k) + \frac{1}{2}u(k) \\ y(k) = a_1(k) + b_0(k) \end{cases}$$

Thus

$$\boxed{\begin{aligned}\begin{bmatrix} a_1(k+1) \\ a_0(k+1) \\ b_1(k+1) \\ b_0(k+1) \end{bmatrix} &= \begin{bmatrix} 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1(k) \\ a_0(k) \\ b_1(k) \\ b_0(k) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} u(k) \\ y(k) &= \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1(k) \\ a_0(k) \\ b_1(k) \\ b_0(k) \end{bmatrix}\end{aligned}}$$

Exercise 7 Recall the two pendulum cart example in the notes. Consider the following parameter sets

	m_0	m_1	m_2	l_1	l_2	g
P1	2	1	1	1	1	1
P2	2	1	1	1	0.99	1
P3	2	1	0.5	1	1	1
P4	2	1	1	1	0.5	1

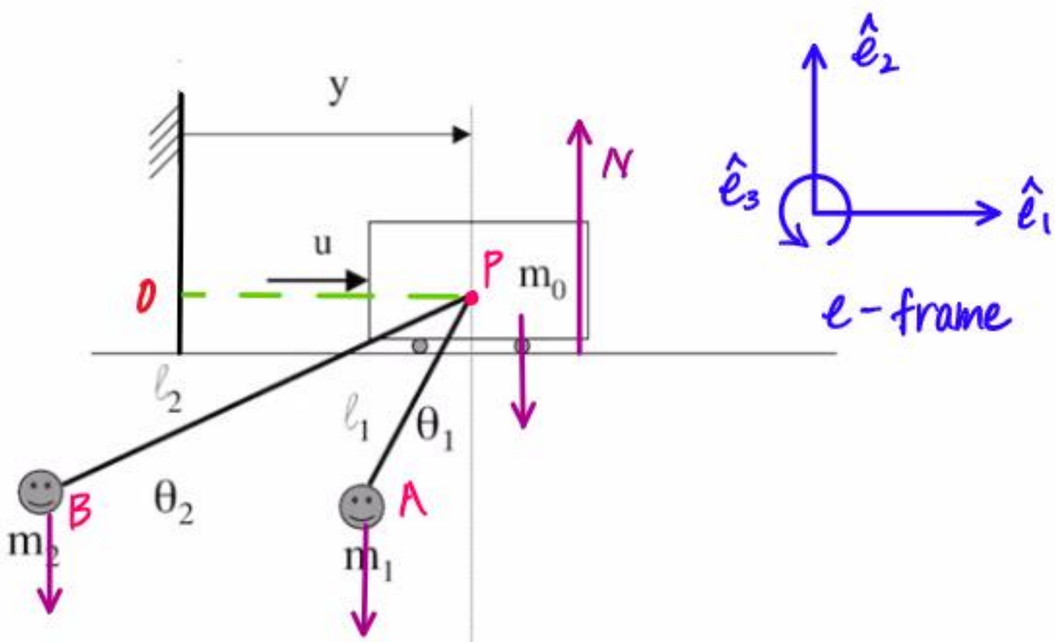
and initial conditions,

	y	θ_1	θ_2	$\dot{y}$	$\dot{\theta}_1$	$\dot{\theta}_2$
IC1	0	$-10°$	$10°$	0	0	0
IC2	0	$10°$	$10°$	0	0	0
IC3	0	$-90°$	$90°$	0	0	0
IC4	0	$-90.01°$	$90°$	0	0	0
IC5	0	$100°$	$100°$	0	0	0
IC6	0	$100.01°$	$100°$	0	0	0
IC7	0	$179.99°$	$0°$	0	0	0

Simulate the system with $u = 0$ using the following combinations:

$$P1: \quad IC1, IC2, IC3, IC7$$
$$P4: \quad IC1, IC2, IC3, IC4$$

Derivation:



cart

$$ {}^{e}\bar{r}^{OP} = y\,\hat{e}_1 \longrightarrow {}^{e}\bar{v}^{OP} = \dot{y}\,\hat{e}_1 \longrightarrow {}^{e}\bar{a}^{OP} = \ddot{y}\,\hat{e}_1 $$

pendulum 1 $\qquad$ θ_1 is rotating in $\hat{e}_3$ direction

$$ {}^{e}\bar{r}^{OA} = y\,\hat{e}_1 - l_1 \sin\theta_1\,\hat{e}_1 - l_1\cos\theta_1\,\hat{e}_2 $$

$$ \begin{aligned} {}^{e}\bar{v}^{OA} &= \frac{{}^{e}d}{dt}\left({}^{e}\bar{r}^{OA}\right) \\ &= \dot{y}\,\hat{e}_1 - l_1\dot{\theta}_1\cos\theta_1\,\hat{e}_1 + l_1\dot{\theta}_1\sin\theta_1\,\hat{e}_2 \end{aligned} $$

$$ \begin{aligned} {}^{e}\bar{a}^{OA} &= \frac{{}^{e}d}{dt}\left({}^{e}\bar{a}^{OA}\right) \\ &= \ddot{y}\,\hat{e}_1 - l_1\ddot{\theta}_1\cos\theta_1\,\hat{e}_1 + l_1\dot{\theta}_1^2\sin\theta_1\,\hat{e}_1 + l_1\ddot{\theta}_1\sin\theta_1\,\hat{e}_2 + l_1\dot{\theta}_1^2\cos\theta_1\,\hat{e}_2 \end{aligned} $$

pendulum 2 $\qquad$ same process as pendulum 1

$$ {}^{e}\bar{a}^{OB} = \ddot{y}\,\hat{e}_2 - l_2\ddot{\theta}_2\cos\theta_2\,\hat{e}_1 + l_2\dot{\theta}_2^2\sin\theta_2\,\hat{e}_1 + l_2\ddot{\theta}_2\sin\theta_2\,\hat{e}_2 + l_2\dot{\theta}_2^2\cos\theta_2\,\hat{e}_2 $$

$$ \Rightarrow \quad m_0\,{}^{e}\bar{a}^{OP} + m_1\,{}^{e}\bar{a}^{OA} + m_2\,{}^{e}\bar{a}^{OB} = \Sigma \bar{F} $$

RHS $\quad u\,\hat{e}_1 - \cancel{m_0 g}\,\hat{e}_2 - \cancel{m_1 g}\,\hat{e}_2 - \cancel{m_2 g}\,\hat{e}_2 + \cancel{N}\,\hat{e}_2$

LHS
$$ \begin{aligned} & m_0\ddot{y}\,\hat{e}_1 \\ & + m_1\left[\left(\ddot{y} - l_1\ddot{\theta}_1\cos\theta_1 + l_1\dot{\theta}_1^2\sin\theta_1\right)\hat{e}_1 + \left(l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1\right)\hat{e}_2\right] \\ & + m_2\left[\left(\ddot{y} - l_2\ddot{\theta}_2\cos\theta_2 + l_2\dot{\theta}_2^2\sin\theta_2\right)\hat{e}_1 + \left(l_2\ddot{\theta}_2\sin\theta_2 + l_2\dot{\theta}_2^2\cos\theta_2\right)\hat{e}_2\right] \end{aligned} $$

$$ \Rightarrow \hat{e}_1 \quad \boxed{\begin{aligned} (m_0+m_1+m_2)\ddot{y} - m_1 l_1\cos\theta_1\,\ddot{\theta}_1 - m_2 l_2\cos\theta_2\,\ddot{\theta}_2 & \\ + m_1 l_1\dot{\theta}_1^2\sin\theta_1 + m_2 l_2\dot{\theta}_2^2\sin\theta_2 &= u \end{aligned}} \quad \cdots (1) $$

$\hat{e}_2$ $m_1(l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1) + m_2(l_2\ddot{\theta}_2\sin\theta_2 + l_2\dot{\theta}_2^2\cos\theta_2) = 0$

$\because m_1 \neq 0 \quad m_2 \neq 0 \qquad \neq 0$

$$\boxed{l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1 = l_2\ddot{\theta}_2\sin\theta_2 + l_2\dot{\theta}_2^2\cos\theta_2 = 0} \cdots ②$$

$\therefore$

From conservation of angular momentum

pendulum 1

$$\Sigma\,{}^e\bar{M}^{PA} = \frac{d}{dt}{}^e\bar{H}^{PA} = \frac{d}{dt}\left({}^e\bar{r}^{PA} \times m_1 \frac{{}^ed}{dt}{}^e\bar{r}^{PA}\right)$$

$$= \frac{d}{dt}\Big[(y\hat{e}_1 - l_1\sin\theta_1\hat{e}_1 - l_1\cos\theta_1\hat{e}_2)$$

$$\times\, m_1(\dot{y}\hat{e}_1 - l_1\dot{\theta}_1\cos\theta_1\hat{e}_1 + l_1\dot{\theta}_1\sin\theta_1\hat{e}_2)\Big]$$

$$= m_1\frac{d}{dt}\Big(-l_1^2\dot{\theta}_1\sin^2\theta_1\hat{e}_3 - l_1^2\dot{\theta}_1\cos^2\theta_1\hat{e}_3$$

$$+ y l_1\dot{\theta}_1\sin\theta_1\hat{e}_3 + \dot{y}l_1\cos\theta_1\hat{e}_3\Big)$$

$$= m_1\frac{d}{dt}\Big[(-l_1^2\dot{\theta}_1)\underbrace{(\sin^2\theta_1 + \cos^2\theta_1)}_{1}$$

$$+ \cancel{\dot{y}l_1\dot{\theta}_1\sin\theta_1} + y l_1\ddot{\theta}_1\sin\theta_1 + y l_1\dot{\theta}_1^2\cos\theta_1$$

$$+ \ddot{y}l_1\cos\theta_1 - \cancel{\dot{y}l_1\dot{\theta}_1\sin\theta_1}\Big]\hat{e}_3$$

$$= m_1\frac{d}{dt}\Big[(-l_1^2\dot{\theta}_1) + \ddot{y}l_1\cos\theta_1 + y\underbrace{(l_1\ddot{\theta}_1\sin\theta_1 + l_1\dot{\theta}_1^2\cos\theta_1)}_{0\ \because ②}\Big]\hat{e}_3$$

$$= -m_1 l_1^2\ddot{\theta}_1 + m_1 l_1\cos\theta_1\ddot{y}$$

and $\Sigma\,{}^e\bar{M}^{PA} = m_1 l_1 g\sin\theta_1$ momentum by gravity

Thus,

$$\boxed{-m_1 l_1\cos\theta_1\ddot{y} + m_1 l_1^2\ddot{\theta}_1 + m_1 l_1 g\sin\theta_1 = 0} \cdots ③$$

Similarly, pendulum 2 becomes

$$\boxed{-m_2 l_2 \cos\theta_2 \ddot{y} + m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 g \sin\theta_2 = 0} \cdots (4)$$

Thus, from (1), (3), (4) the motion can be described as

$$\boxed{\begin{array}{l} (m_0+m_1+m_2)\ddot{y} - m_1 l_1 \cos\theta_1 \ddot{\theta}_1 - m_2 l_2 \cos\theta_2 \ddot{\theta}_2 + m_1 l_1 \sin\theta_1 \dot{\theta}_1^2 + m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 = u \\ \qquad -m_1 l_1 \cos\theta_1 \ddot{y} + m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g \sin\theta_1 = 0 \\ \qquad -m_2 l_2 \cos\theta_2 \ddot{y} + m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 g \sin\theta_2 = 0 \end{array}}$$

The system equation for the double pendulum cart system is

$$\begin{array}{rcl} (m_0 + m_1 + m_2)\ddot{y} - m_1 l_1 cos\theta_1 \ddot{\theta}_1 - m_2 l_2 cos\theta_2 \ddot{\theta}_2 + m_1 l_1 sin\theta_1 \dot{\theta}_1^2 + m_2 l_2 sin\theta_2 \dot{\theta}_2^2 & & = u \\ -m_1 l_1 cos\theta_1 \ddot{y} + m_1 l_1^2 \ddot{\theta}_1 \quad + \quad m_1 l_1 g sin\theta_1 & & = 0 \\ -m_2 l_2 cos\theta_2 \ddot{y} + m_2 l_2^2 \ddot{\theta}_2 \quad + \quad m_2 l_2 g sin\theta_2 & & = 0 \end{array}$$

We simulate this system of equations using MATLAB for given initial conditions and input parameters. The given initial conditions and input parameters are organized in the following table.

	m_0	m_1	m_2	l_1	l_2	g	u
$P1$	2	1	1	1	1	1	0
$P2$	2	1	1	1	0.99	1	0
$P3$	2	1	0.5	1	1	1	0
$P4$	2	1	1	1	0.5	1	0

	y	θ_1	θ_2	$\dot{y}$	$\dot{\theta}_1$	$\dot{\theta}_2$
$IC1$	0	-10°	10°	0	0	0
$IC2$	0	10°	10°	0	0	0
$IC3$	0	-90°	90°	0	0	0
$IC4$	0	-90.01°	90°	0	0	0
$IC5$	0	100°	100°	0	0	0
$IC6$	0	100.01°	100°	0	0	0
$IC7$	0	179.99°	0°	0	0	0

Simulations:

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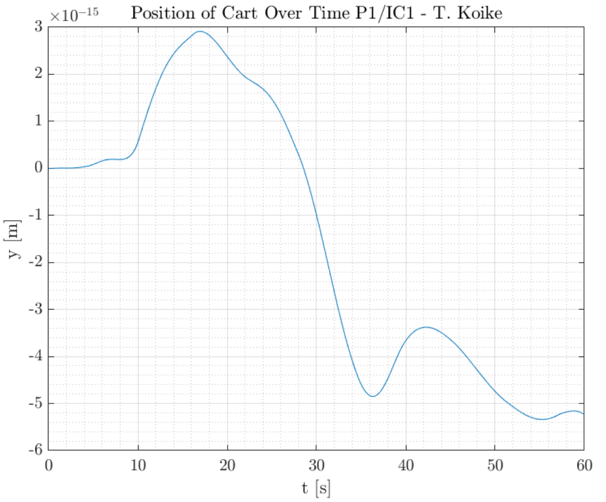
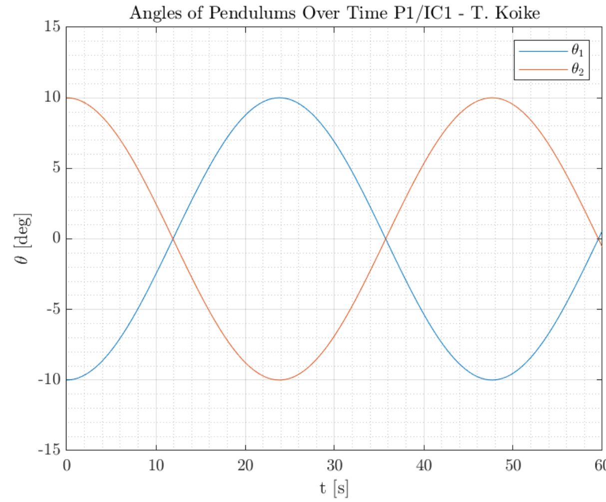
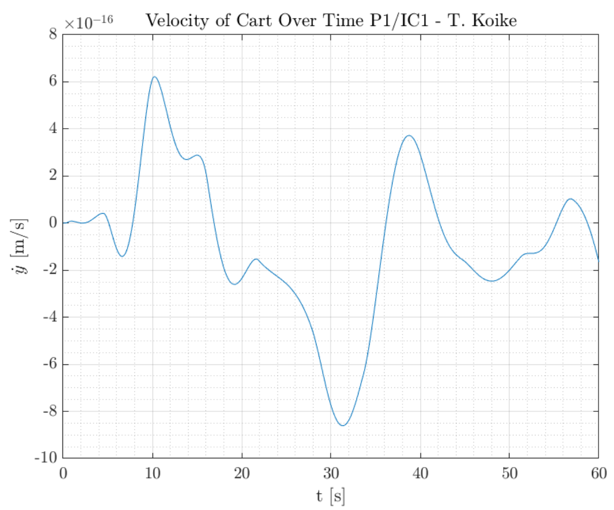
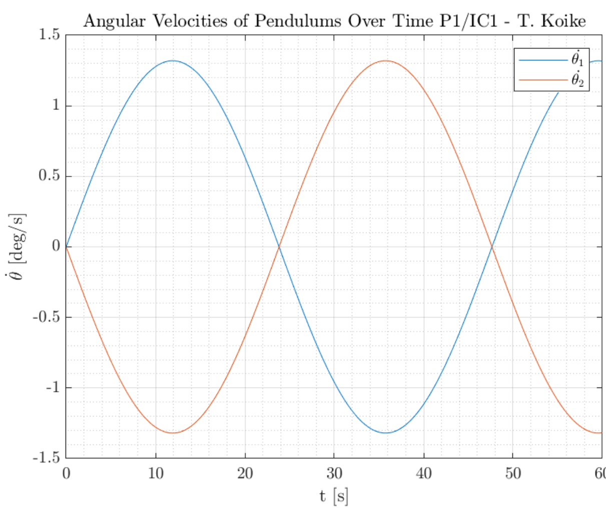
P1 and IC1

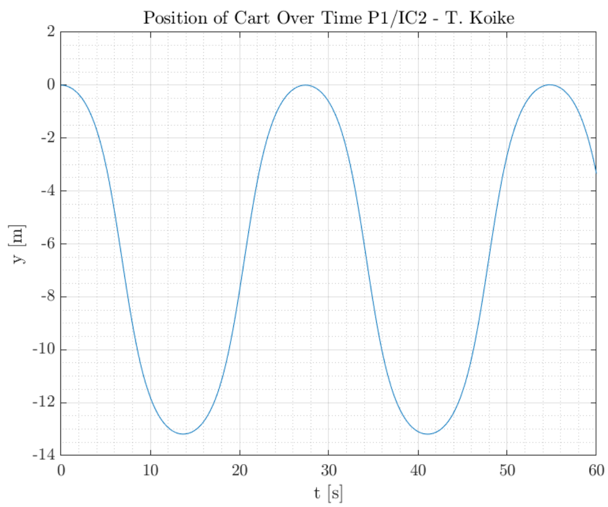


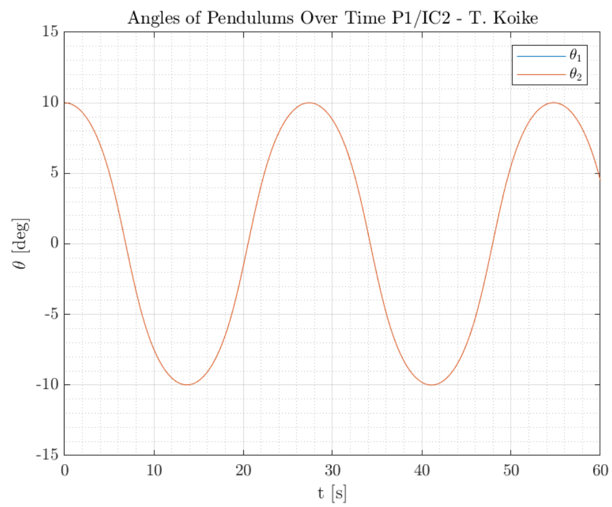


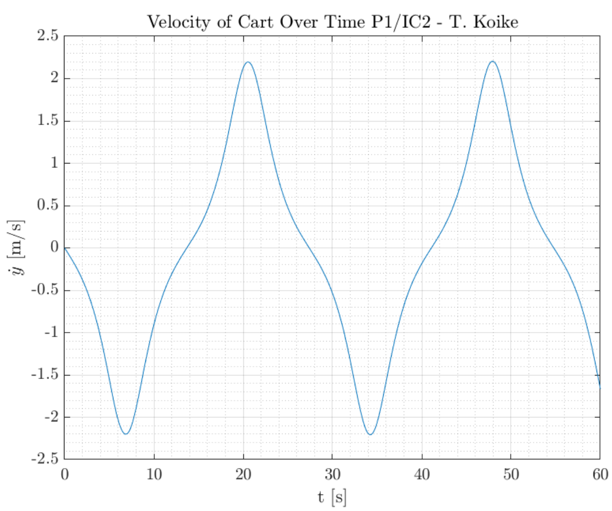


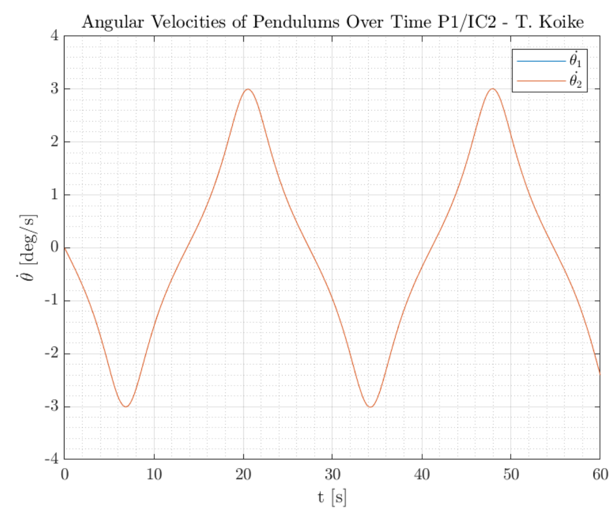


P1 and IC2

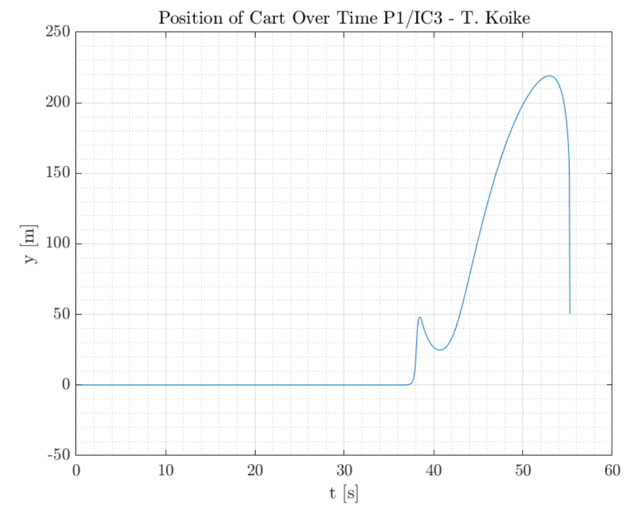

Position of Cart Over Time P1/IC2 - T. Koike
y [m]
t [s]

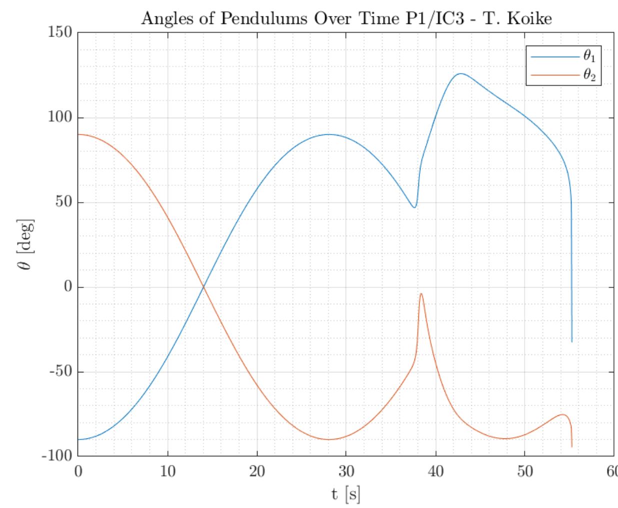

Angles of Pendulums Over Time P1/IC2 - T. Koike
θ_1
θ_2
θ [deg]
t [s]

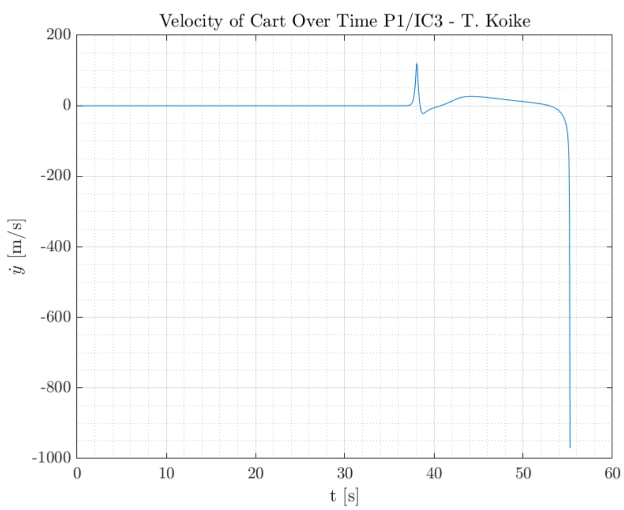

Velocity of Cart Over Time P1/IC2 - T. Koike
$\dot{y}$ [m/s]
t [s]

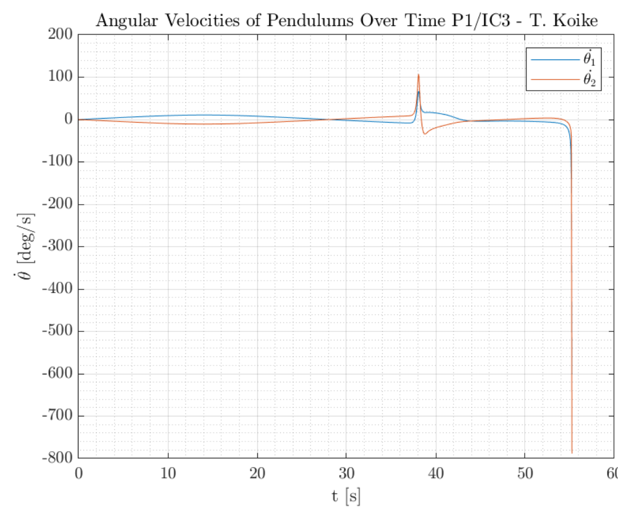

Angular Velocities of Pendulums Over Time P1/IC2 - T. Koike
$\dot{\theta}_1$
$\dot{\theta}_2$
$\dot{\theta}$ [deg/s]
t [s]

P1 and IC3

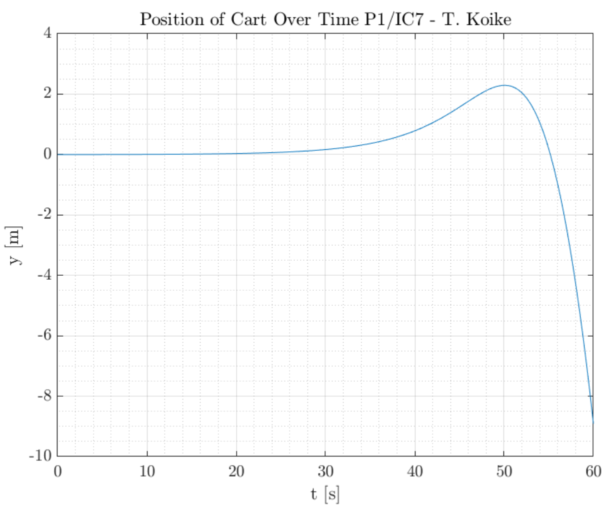

Position of Cart Over Time P1/IC3 - T. Koike
y [m]
t [s]

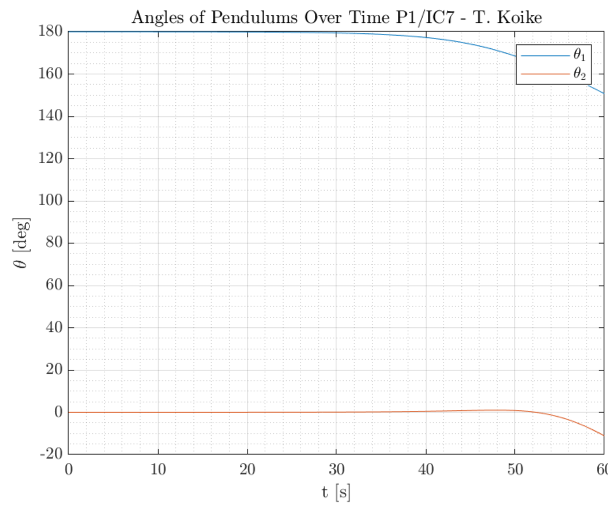

Angles of Pendulums Over Time P1/IC3 - T. Koike
θ [deg]
t [s]

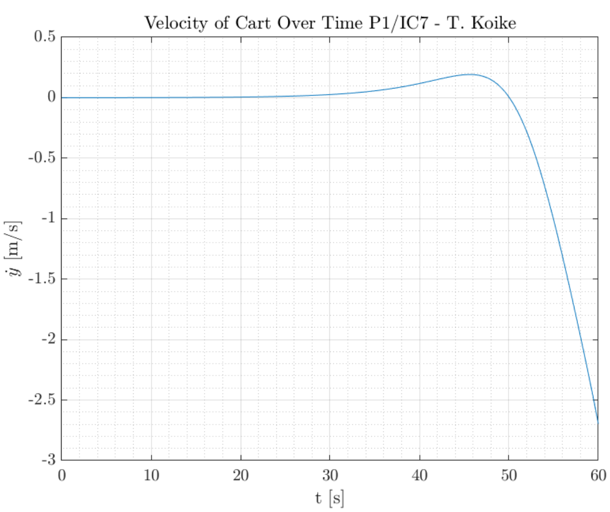

Velocity of Cart Over Time P1/IC3 - T. Koike
ẏ [m/s]
t [s]

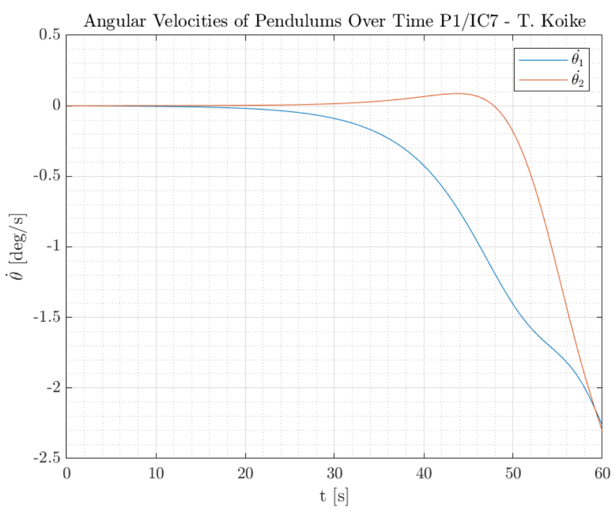

Angular Velocities of Pendulums Over Time P1/IC3 - T. Koike
θ̇ [deg/s]
t [s]

P1 and IC7

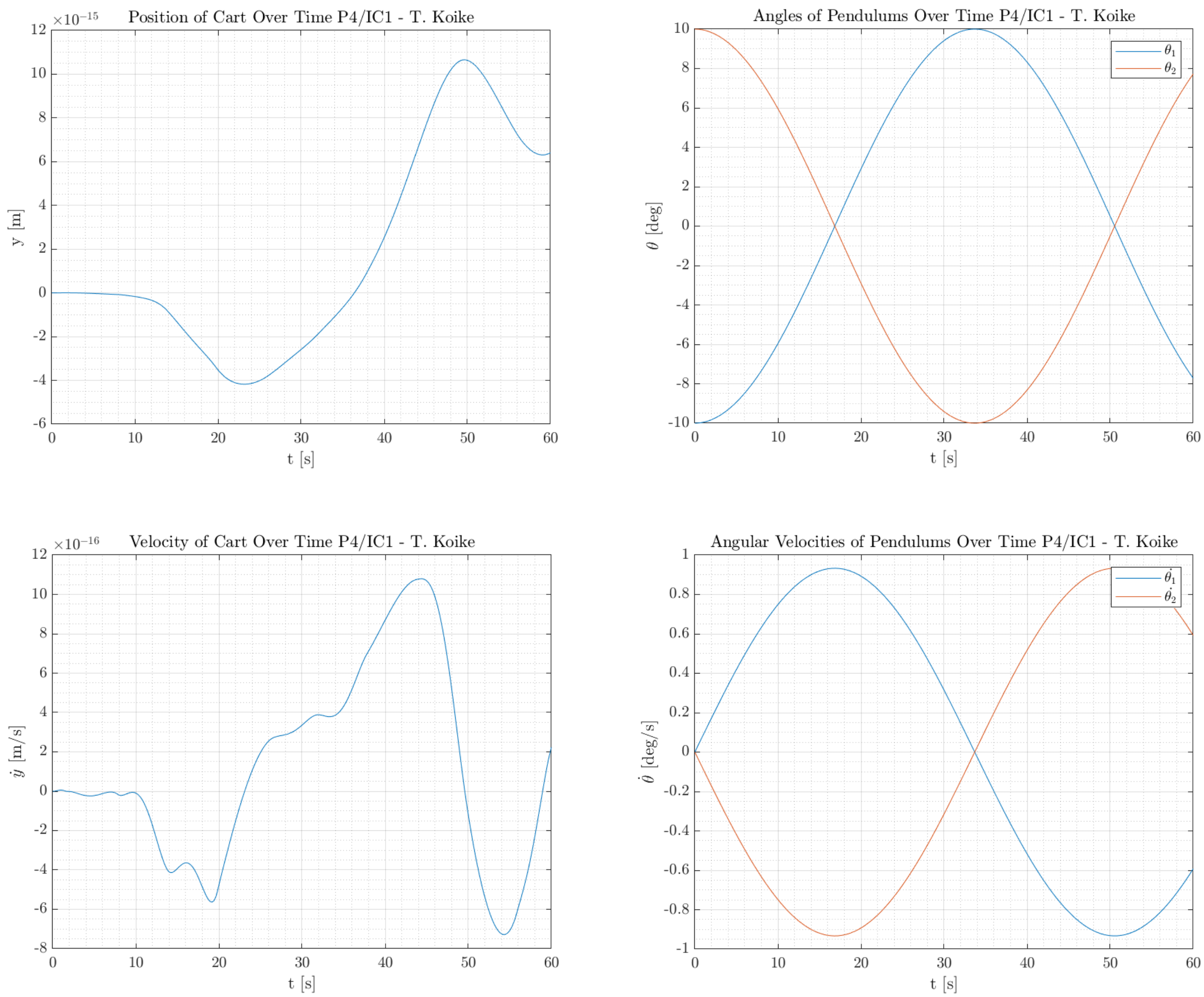

Position of Cart Over Time P1/IC7 - T. Koike
y [m]
t [s]


Angles of Pendulums Over Time P1/IC7 - T. Koike
θ [deg]
t [s]
θ1
θ2

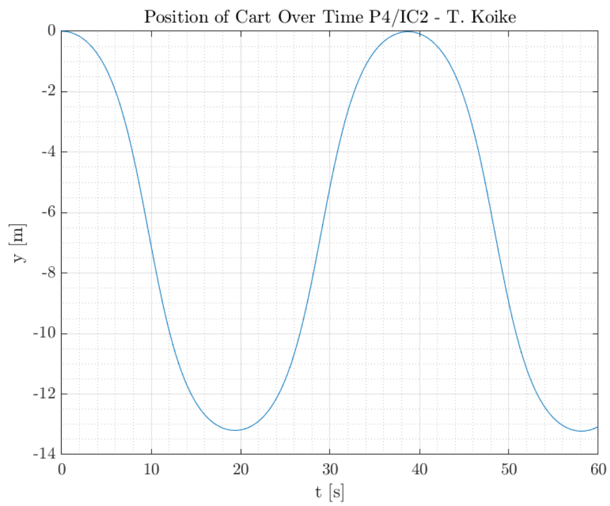

Velocity of Cart Over Time P1/IC7 - T. Koike
ẏ [m/s]
t [s]

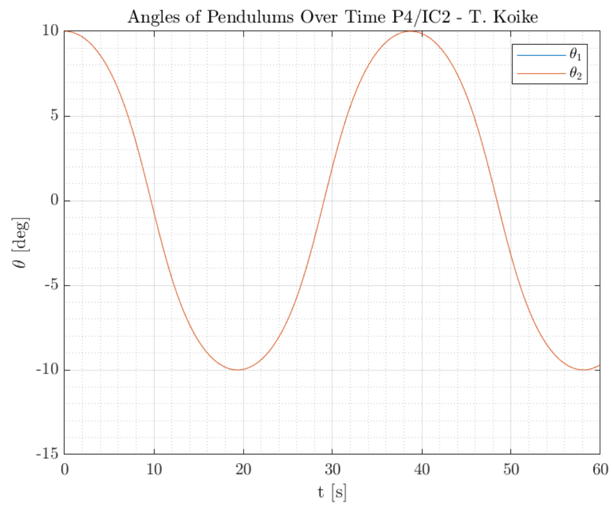

Angular Velocities of Pendulums Over Time P1/IC7 - T. Koike
θ̇ [deg/s]
t [s]
θ̇1
θ̇2

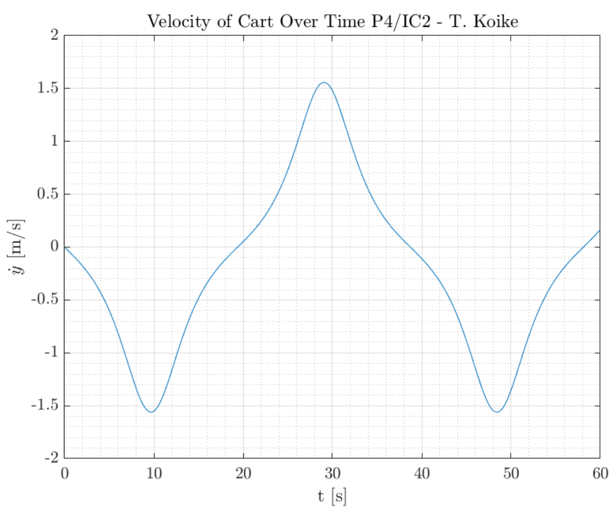
P4 and IC1

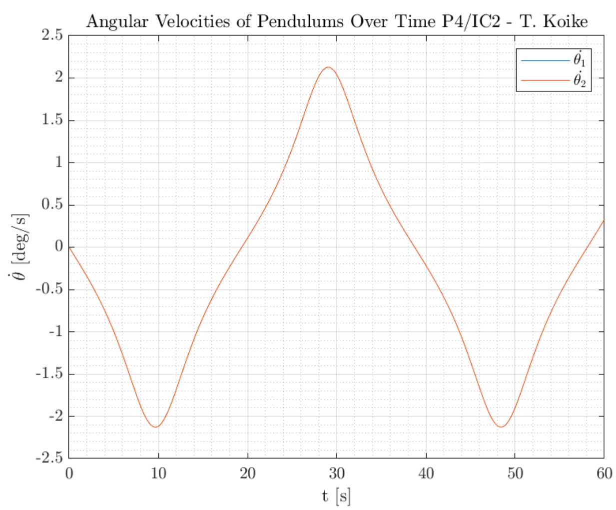

Position of Cart Over Time P4/IC1 - T. Koike
×10^-15
y [m]
t [s]
Angles of Pendulums Over Time P4/IC1 - T. Koike
θ [deg]
θ1
θ2
t [s]
Velocity of Cart Over Time P4/IC1 - T. Koike
×10^-16
ẏ [m/s]
t [s]
Angular Velocities of Pendulums Over Time P4/IC1 - T. Koike
θ̇ [deg/s]
θ̇1
θ̇2
t [s]

P4 and IC2

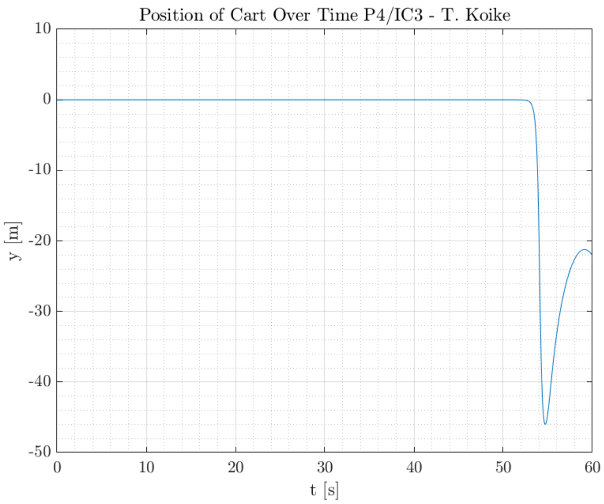

Position of Cart Over Time P4/IC2 - T. Koike
y [m]
t [s]

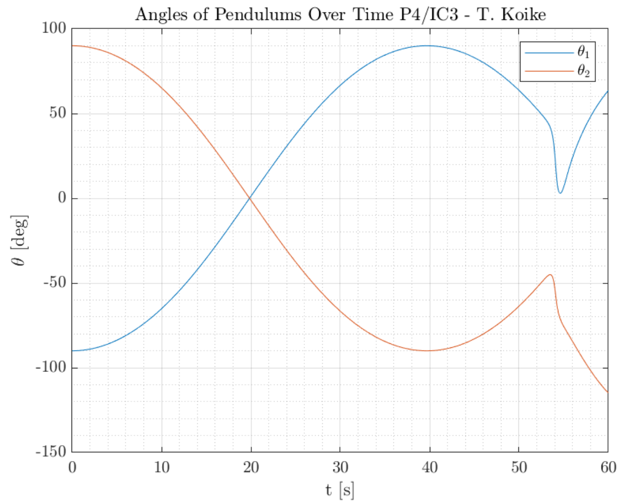

Angles of Pendulums Over Time P4/IC2 - T. Koike
θ_1
θ_2
θ [deg]
t [s]

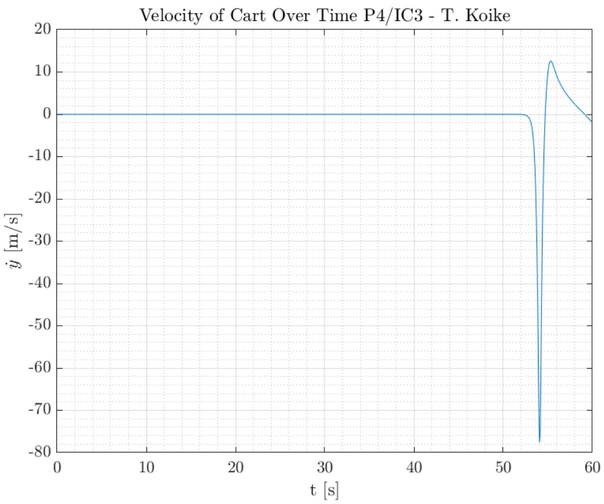

Velocity of Cart Over Time P4/IC2 - T. Koike
$\dot{y}$ [m/s]
t [s]

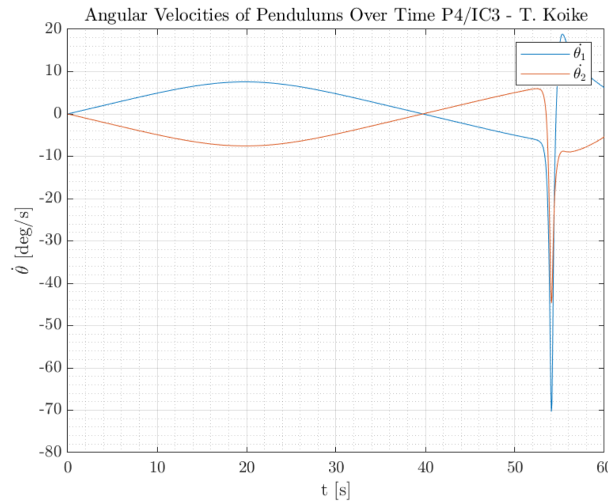

Angular Velocities of Pendulums Over Time P4/IC2 - T. Koike
$\dot{\theta}_1$
$\dot{\theta}_2$
$\dot{\theta}$ [deg/s]
t [s]

P4 and IC3

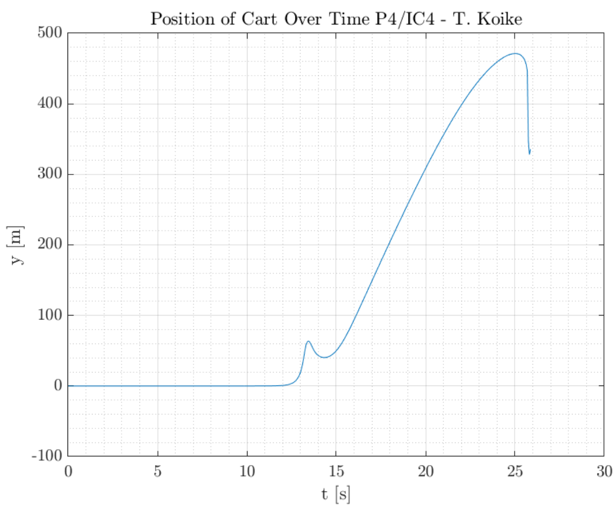

Position of Cart Over Time P4/IC3 - T. Koike
y [m]
t [s]

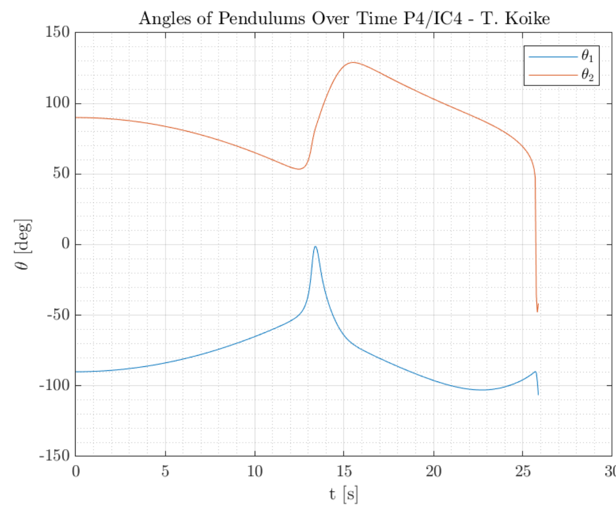

Angles of Pendulums Over Time P4/IC3 - T. Koike
θ_1
θ_2
θ [deg]
t [s]

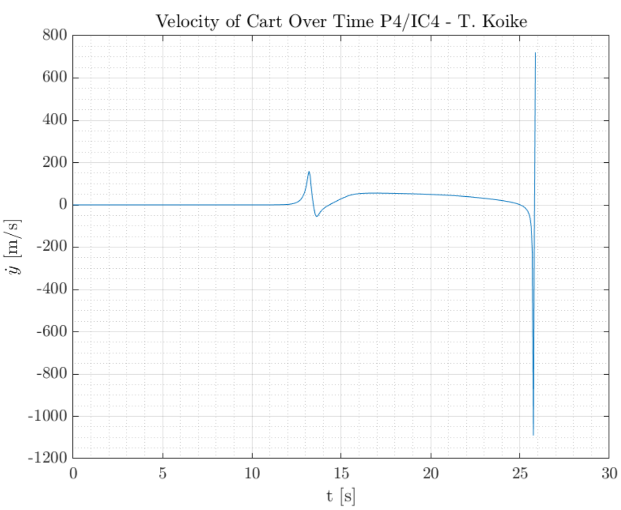

Velocity of Cart Over Time P4/IC3 - T. Koike
$\dot{y}$ [m/s]
t [s]

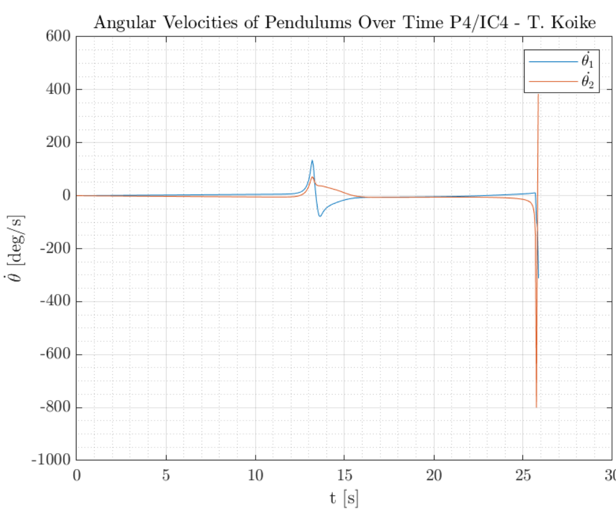

Angular Velocities of Pendulums Over Time P4/IC3 - T. Koike
$\dot{\theta}_1$
$\dot{\theta}_2$
$\dot{\theta}$ [deg/s]
t [s]

P4 and IC4


Position of Cart Over Time P4/IC4 - T. Koike
y [m]
t [s]


Angles of Pendulums Over Time P4/IC4 - T. Koike
θ_1
θ_2
θ [deg]
t [s]


Velocity of Cart Over Time P4/IC4 - T. Koike
$\dot{y}$ [m/s]
t [s]


Angular Velocities of Pendulums Over Time P4/IC4 - T. Koike
$\dot{\theta}_1$
$\dot{\theta}_2$
$\dot{\theta}$ [deg/s]
t [s]

Appendix

MATLAB Code

AAE 564 HW 1

Author: Tomoki Koike

```
clear all; close all; clc;
fdir = 'C:\Users\Tomo\Desktop\studies\2020-
Fall\AAE564\matlab_simulink\outputs\hw1';
set(groot, 'defaulttextinterpreter','latex');
set(groot, 'defaultAxesTickLabelInterpreter','latex');
set(groot, 'defaultLegendInterpreter','latex');
```

Simulation

```
% Simulate this system for given initial conditions

% Initial conditions
IC1 = [0, -10, 10, 0, 0, 0];
IC2 = [0, 10, 10, 0, 0, 0];
IC3 = [0, -90, 90, 0, 0, 0];
IC4 = [0, -90.01, 90, 0, 0, 0];
IC5 = [0, 100, 100, 0, 0, 0];
IC6 = [0, 100.01, 100, 0, 0, 0];
IC7 = [0, 179.99, 0, 0, 0, 0];

% P1
m0 = 2;
m1 = 1;
m2 = 1;
l1 = 1;
l2 = 1;
g = 1;
u = 0;

t_span = linspace(0, 60, 2^10);  % time span
% opts = odeset('RelTol',1e-6,'AbsTol',1e-7);  % option for ode
```

```
% P1 and IC1, IC2, IC3, IC7
IC = [IC1; IC2; IC3; IC7];
counter = 1;
for i = [1, 2, 3, 7]
    [t, q] = ode45(@(t, x) double_pendulum_system(t, x, m0, m1, m2, l1, l2, g,
u), t_span, IC(counter,:));
    plot_simulation(t, q, fdir, "P1&IC"+num2str(i));
    counter = counter + 1;
end
```

```
% P4
m0 = 2;
m1 = 1;
```

```
m2 = 1;
l1 = 1;
l2 = 1;
g = 0.5;
u = 0;
```

```
% P4 and IC1, IC2, IC3, IC4
IC = [IC1; IC2; IC3; IC4];
counter = 1;
for i = [1, 2, 3, 4]
    [t, q] = ode45(@(t, x) double_pendulum_system(t, x, m0, m1, m2, l1, l2, g,
u), t_span, IC(counter,:));
    plot_simulation(t, q, fdir, "P4&IC"+num2str(i));
    counter = counter + 1;
end
```

Functions

```
function dxdt = double_pendulum_system(t, x, m0, m1, m2, l1, l2, g, u)
    dxdt = zeros(6, 1);  % Preallocate the derivative vector

    % Set the variables
    y = x(1);
    theta1 = x(2);
    theta2 = x(3);
    y_dot = x(4);
    theta1_dot = x(5);
    theta2_dot = x(6);

    % Create matrices to simplify the calculations
    M = [      m0 + m1 + m2, -m1*l1.*cosd(theta1), -m2*l2.*cosd(theta2);
         -m1*l1.*cosd(theta1),            m1*l1.^2,                    0;
         -m2*l2.*cosd(theta2),                   0,            m2*l2.^2 ];

    G = [(m1*l1.*sind(theta1).*theta1_dot.^2 +
m2*l2.*sind(theta2).*theta2_dot.^2); m1*l1*g.*sind(theta1);
m2*l2*g.*sind(theta2)];

    W = [1; 0; 0];

    dxdt(1) = y_dot;
    dxdt(2) = theta1_dot;
    dxdt(3) = theta2_dot;
    dxdt(4:end) = M \ (W * u - G);
end

function plot_simulation(t, q, file_dir, file_name_str)
    P_IC = split(file_name_str, "&");
    % Plot position of cart vs time
    fig1 = figure(1);
    plot(t, q(:, 1))
    title("Position of Cart Over Time "+P_IC(1)+"/"+P_IC(2)+ " - T. Koike")
    xlabel('t [s]')
```

```
    ylabel('y [m]')
    grid on; grid minor; box on;
    saveas(fig1, fullfile(file_dir, "y_"+file_name_str+".png"));

    % Plot the angles of pendulums vs time
    fig2 = figure(2);
    plot(t, q(:, 2))
    hold on
    plot(t, q(:, 3))
    hold off
    title("Angles of Pendulums Over Time "+P_IC(1)+"/"+P_IC(2)+ " - T. Koike")
    xlabel('t [s]')
    ylabel('$\theta$ [deg]')
    legend('$\theta_1$', '$\theta_2$')
    grid on; grid minor; box on;
    saveas(fig2, fullfile(file_dir, "theta1&2_"+file_name_str+".png"));

    % Plot the velocity of cart vs time
    fig3 = figure(3);
    plot(t, q(:, 4))
    title("Velocity of Cart Over Time "+P_IC(1)+"/"+P_IC(2)+ " - T. Koike")
    xlabel('t [s]')
    ylabel('$\dot{y}$ [m/s]')
    grid on; grid minor; box on;
    saveas(fig3, fullfile(file_dir, "ydot_"+file_name_str+".png"));

    % Plot the angular velocities vs time
    fig4 = figure(4);
    plot(t, q(:, 5))
    hold on
    plot(t, q(:, 6))
    hold off
    title("Angular Velocities of Pendulums Over Time "+P_IC(1)+"/"+P_IC(2)+ " -
T. Koike")
    xlabel('t [s]')
    ylabel('$\dot{\theta}$ [deg/s]')
    legend('$\dot{\theta_1}$', '$\dot{\theta_2}$')
    grid on; grid minor; box on;
    saveas(fig4, fullfile(file_dir, "thetadot1&2_"+file_name_str+".png"));
end
```