AAE 567 Spring 2018 Homework 2 Solutions

February 22, 2018

2.2.2 Problem 1

We are given that $\mathbf{x}$ is a uniform random variable over $[0, 10]$, $\mathbf{v}$ is a uniform random variable over $[0, 4]$, $\mathbf{x}$ and $\mathbf{v}$ are independent random variables, $\mathbf{y} = \mathbf{x} + \mathbf{v}$, and $\mathcal{H}_3$ is the subspace spanned by $\{1, \mathbf{y}, \mathbf{y}^2\}$. We want to compute the optimal estimate $\hat{x} = P_{\mathcal{H}_3} x$ and the error $E(x - \hat{x})^2$. We will then compare this result to the conditional expectation $E(x|y)$.

Because x is uniform over $[0, 10]$, we have $f_{\mathbf{x}}(x) = \frac{1}{10}$ on the interval $[0, 10]$ and 0 elsewhere. Similarly, $f_{\mathbf{v}}(v) = \frac{1}{4}$ on the interval $[0, 4]$ and 0 elsewhere. By consulting Lemma 12.3.1, we obtain

$$
\begin{aligned}
f_{\mathbf{x},\mathbf{y}}(x, y) = f_{\mathbf{x}}(x) f_{\mathbf{v}}(y - x) &= \frac{1}{40} \qquad \text{if } 0 \le x \le 10 \text{ and } 0 \le y - x \le 4 \\
&= 0 \qquad \text{otherwise.}
\end{aligned}
$$

Also from the lemma we have $f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(x, y) dx$. For this we have 3 regions for $\mathbf{y}$: $[0, 4]$, $[4, 10]$, and $[10, 14]$. We obtain

$$
\begin{aligned}
f_{\mathbf{y}}(y) &= \int_0^y \frac{1}{40} dx = \frac{y}{40} \text{ if } 0 \le y \le 4 \\
&= \int_{y-4}^{y} \frac{1}{40} dx = \frac{1}{10} \text{ if } 4 \le y \le 10 \\
&= \int_{y-4}^{10} \frac{1}{40} dx = \frac{14 - y}{40} \text{ if } 10 \le y \le 14 \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

Finally, it is noted that $f_{\mathbf{y}}(y)$ is positive and the area under $f_{\mathbf{y}}(y)$ equals one. So $f_{\mathbf{y}}(y)$ is indeed a density function.

We can now use the expression for $f_{\mathbf{y}}(y)$ to obtain the conditional density function as defined in Lemma 12.3.1:

$$f_{\mathbf{x}|\mathbf{y}}(x|y) = \frac{f_{\mathbf{x}}(x) f_{\mathbf{v}}(y-x)}{f_{\mathbf{y}}(y)} = \frac{1}{40 f_{\mathbf{y}}(y)}$$

where the last equality holds for all values of x and y for which the probability density functions are non-zero. We obtain:

$$\begin{aligned}
f_{\mathbf{x}|\mathbf{y}}(x|y) &= \frac{1}{40}\frac{40}{y} = \frac{1}{y} && \text{if } 0 \leq x \leq y \text{ and } 0 \leq y \leq 4 \\
&= \frac{1}{40} 10 = \frac{1}{4} && \text{if } y-4 \leq x \leq y \text{ and } 4 \leq y \leq 10 \\
&= \frac{1}{40}\frac{40}{14-y} = \frac{1}{14-y} && \text{if } y-4 \leq x \leq 10 \text{ and } 10 \leq y \leq 14
\end{aligned}$$

From section 12.2 equation (2.2) we have the conditional expectation given by $E(\mathbf{x}|\mathbf{y} = y) = \int_{-\infty}^{\infty} x f_{\mathbf{x}|\mathbf{y}}(x|y) dx$. Substituting in the known conditional probability density function, we obtain

$$\begin{aligned}
E(\mathbf{x}|\mathbf{y} = y) &= \int_0^y \frac{x}{y} dx = \frac{y}{2} && \text{if } 0 \leq y \leq 4 \\
&= \int_{y-4}^{y} \frac{x}{4} dx = y - 2 && \text{if } 4 \leq y \leq 10 \\
&= \int_{y-4}^{10} \frac{x}{14-y} dx = \frac{y+6}{2} && \text{if } 10 \leq y \leq 14.
\end{aligned}$$

Let $g = \begin{bmatrix} 1 & \mathbf{y} & \mathbf{y}^2 \end{bmatrix}^*$. Recall that the orthogonal projection $P_{\mathcal{H}_3} x = R_{xg} R_g^{-1} g$ and $E(x - \hat{x})^2 = R_x - R_{xg} R_g^{-1} R_{gx}$. For arbitrary vectors a and b, we have $R_a = E(aa^*)$ and $R_{ab} = E(ab^*)$. Then R_g is the 3x3 matrix given by

$$R_g = E \begin{bmatrix} 1 \\ \mathbf{y} \\ \mathbf{y}^2 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{y} & \mathbf{y}^2 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{y} & \mathbf{y}^2 \\ \mathbf{y} & \mathbf{y}^2 & \mathbf{y}^3 \\ \mathbf{y}^2 & \mathbf{y}^3 & \mathbf{y}^4 \end{bmatrix}.$$

Using the fact that x and v are independent random variables we can compute $E\mathbf{y}^k = E(\mathbf{x} + \mathbf{v})^k$ for $k = 1, 2, 3, 4$. Notice that R_x is a scalar given by

$$R_x = Ex^2 = \int_0^{10} \frac{x^2}{10} dx = \frac{100}{3}.$$

Moreover, R_{xg} is a row vector of length three, that is,

$$R_{xg} = E\begin{bmatrix}\mathbf{x} & \mathbf{xy} & E\mathbf{xy}^2\end{bmatrix} = \begin{bmatrix}E\mathbf{x} & E\mathbf{x}(\mathbf{x}+\mathbf{v}) & \mathbf{x}(\mathbf{x}+\mathbf{v})^2\end{bmatrix}.$$

Evaluating the previous matrices yields

$$\begin{aligned} R_g &= \begin{bmatrix} 1 & 7 & \frac{176}{3} \\ 7 & \frac{176}{3} & 546 \\ \frac{176}{3} & 546 & \frac{81568}{15} \end{bmatrix} \\ R_{xg} &= \begin{bmatrix} 5 & \frac{130}{3} & 410 \end{bmatrix} \\ R_x &= \frac{100}{3} \end{aligned}$$

Using $\widehat{\mathbf{x}} = R_{xg}R_g^{-1}g$, we now find the estimate and the error covariance:

$$\begin{aligned} \hat{x} &= \frac{25}{29}\mathbf{y} - \frac{30}{29} \\ E(x-\hat{x})^2 &= R_x - R_{xg}R_g^{-1}R_{gx} = \frac{100}{87} \end{aligned}$$

Note that including the $\mathbf{y}^2$ term does not add any accuracy to the estimate, as the coefficient for this term is 0.

Note also that both the conditional expectation and the projection onto $\mathcal{H}_3$ are linear, but while the projection is linear over the entire domain [0,10], the conditional expectation is defined piecewise over three separate intervals. Plotting both the conditional expectation and $P_{\mathcal{H}_3}\mathbf{x}$ together we obtain Figure 1

The following Matlab code solves this problem:

```
%2.2.2 #1
syms x v y
R_g = zeros(3,3);
for i=1:3
    for j = 1:3
        R_g(i,j) = int(y^(i+j-1)/40,y,0,4) + int(y^(i+j-2)/10,y,4,10) ...
            + int(y^(i+j-2)*(14-y)/40,y,10,14);
    end
end
Ex = int(x/10,x,0,10);
Ex2 = int(x^2/10,x,0,10);
Ex3 = int(x^3/10,x,0,10);
```

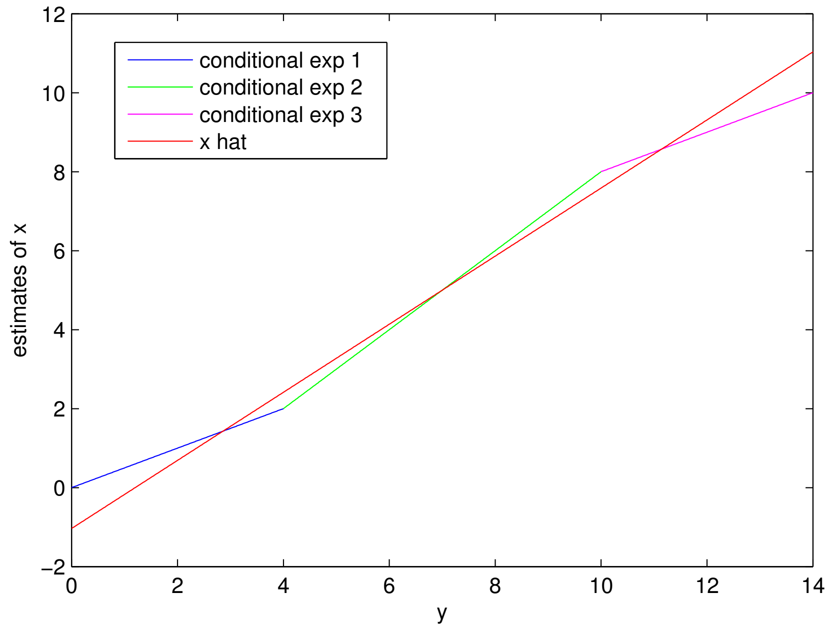

Figure 1: Plot for y on the interval $[0, 14]$ of the conditional expectation $E(x|y)$ and the optimal estimate $P_{\mathcal{H}_3}x$ for 2.2.2 Exercise 1

```
Ev = int(v/4,v,0,4);
Ev2 = int(v^2/4,v,0,4);
R_fg = [Ex, Ex2+Ex*Ev, Ex3+2*Ex2*Ev+Ex*Ev2];

alphas1 = R_fg/R_g;
est_err1 = Ex2 - R_fg/R_g*R_fg';

figure(1)
y1 = 0:.01:4;
y2 = 4:.01:10;
y3 = 10:.01:14;
y_full = 0:.01:14;
plot(y1,y1/2,'b')
hold on
plot(y2,y2-2,'g')
plot(y3,(y3+6)/2,'m')
plot(y_full,alphas1(1)*y_full/y_full+alphas1(2)*y_full+alphas1(3)*y_full.^2,'r')
legend('conditional exp 1', 'conditional exp 2', 'conditional exp 3', 'x hat')
xlabel('y')
ylabel('estimates of x')
hold off
```

2.2.2 Problem 2

We are given that $\mathbf{y}$ is a uniform random variable on $[0,1]$ and $\mathbf{x} = e^{\mathbf{y}}$. We also have that $\mathcal{H}$ is the subspace spanned by $\{1,\ \mathbf{y},\ \mathbf{y}^2\}$. We want to compute the optimal estimate $\hat{x} = P_{\mathcal{H}}x$ and the error $E(x - \hat{x})$. We will also show that $E(x|\mathbf{y} = y) = e^y$ and plot both $\hat{x}$ and e^y to compare them.

We have $P_{\mathcal{H}}x = R_{xg}R_g^{-1}g$ with $g = [1\ \mathbf{y}\ \mathbf{y}^2]^{tr}$, $R_g = E(gg^*)$, and $R_{xg} = [E(x1)\ E(x\mathbf{y})\ E(x\mathbf{y}^2)]$. For this particular g we have $R_{i,j} = E(\mathbf{y}^{i+j-2})$. We first compute the components of R_g:

$$
\begin{aligned}
E1 &= 1 \\
E\mathbf{y} &= \int_0^1 y dy = \frac{1}{2} \\
E\mathbf{y}^2 &= \int_0^1 y^2 dy = \frac{1}{3} \\
E\mathbf{y}^3 &= \int_0^1 y^3 dy = \frac{1}{4} \\
E\mathbf{y}^4 &= \int_0^1 y^4 dy = \frac{1}{5}
\end{aligned}
$$

We also have the components of R_{xg} given by:

$$
\begin{aligned}
E\mathbf{x}1 &= \int_0^1 e^y dy = e - 1 \\
E\mathbf{x}\mathbf{y} &= \int_0^1 e^y y dy = 1 \\
E\mathbf{x}\mathbf{y}^2 &= \int_0^1 e^y y^2 dy = e - 2
\end{aligned}
$$

Putting this all together, we obtain

$$
\hat{x} = \begin{bmatrix} e-1 & 1 & e-2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ \mathbf{y} \\ \mathbf{y}^2 \end{bmatrix}
$$

which results in

$$
\hat{x} \approx 1.0130 + 0.8511\mathbf{y} + 0.8392\mathbf{y}^2
$$

The error is given by $E(x-\hat{x})^2 = R_x - R_{xg}R_g^{-1}R_{gx} = Ex^2 - R_{xg}R_g^{-1}R_{xg}^{tr}$. We obtain

$$
\begin{aligned}
E(x-\hat{x})^2 &= \int_0^1 e^{2y} dy - \begin{bmatrix} e-1 & 1 & e-2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}^{-1} \begin{bmatrix} e-1 \\ 1 \\ e-2 \end{bmatrix} \\
&\approx 2.7835 \times 10^{-5}
\end{aligned}
$$

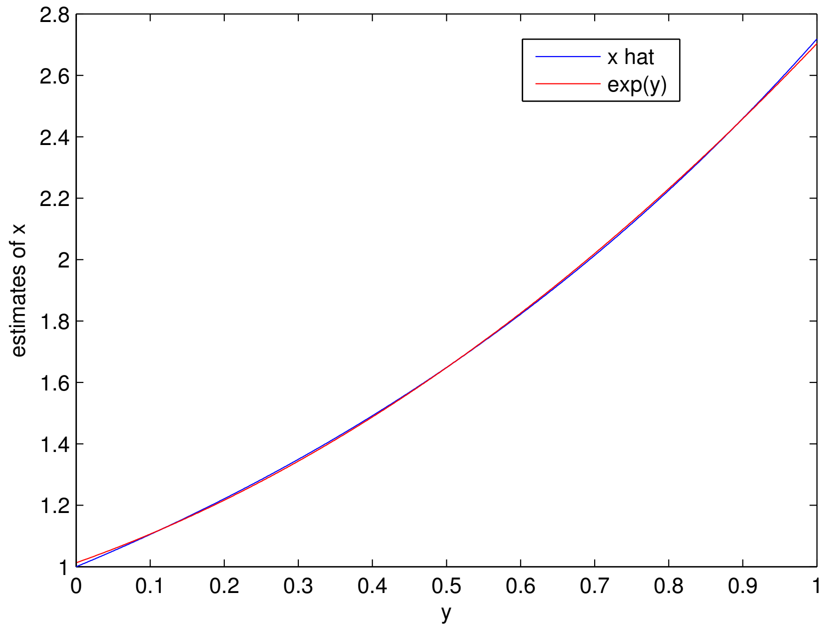

Figure 2: Plot for y on the interval $[0, 14]$ of the conditional expectation $E(x|y)$ and the optimal estimate $P_{mathcal H_3} x$ for 2.2.2 Exercise 2

From theorem 12.2.1, the conditional expectation of x given y, $\hat{g}(\mathbf{y}) = E(\mathbf{x}|\mathbf{y})$ is the function of y which gives the infimum (and in this case minimum) of $E|\mathbf{x} - g(\mathbf{y})|^2$. Since $\mathbf{x} = e^{\mathbf{y}}$, clearly we have zero error when $g(\mathbf{y}) = e^{\mathbf{y}}$, so we must have $E(\mathbf{x}|\mathbf{y}) = e^{\mathbf{y}}$

The conditional expectation has zero error since it is the exact expression that it attempts to estimate, but the error for the projection is also very low with magnitude 10^{-5}. As can be seen in Figure 2, the two curves are closely matched.

The following Matlab code solves this problem:

```
%2.2.2 #2
R_g = zeros(3,3);
R_fg = zeros(1,3);
for i = 1:3
    R_fg(i) = int(exp(y)*y^(i-1),y,0,1);
    for j = 1:3
        R_g(i,j) = int(y^(i+j-2),y,0,1);
```

```
    end
end

alphas2 = R_fg/R_g;
est_err = eval(int(exp(2*y),y,0,1) - R_fg/R_g*R_fg');

z = 0:.001:1;
x_hat = alphas(1)*z./z + alphas(2)*z + alphas(3) * z.^2;
ey = exp(z);
figure(2)
plot(z,ey,'b')
hold on
plot(z,x_hat,'r')
ylabel('estimates of x')
xlabel('y')
legend('x hat','exp(y)')
```

1 2.2.2 Problem 3

We are given a random variable $\mathbf{x}$ with finite $E|\mathbf{x}|^2$ and $\mathcal{H} = \text{span}\,\{2\}$, and we wish to find $\hat{\mathbf{x}} = P_{\mathcal{H}}\mathbf{x}$ and $E|\mathbf{x} - \hat{\mathbf{x}}|^2$.

We begin by denoting $\mu_x = E\mathbf{x}$, the mean of $\mathbf{x}$. We note that $R_f = E|\mathbf{x}|^2$, $R_g = 4$, and $R_{fg} = R_{gf} = 2\mu_x$.

We then have

$$\begin{aligned} \hat{\mathbf{x}} &= P_{\mathcal{H}}\mathbf{x} && (1)\\ &= R_{fg}R_g^{-1}g && (2)\\ &= 2\mu_x\frac{1}{4}2 && (3)\\ &= \mu_x && (4) \end{aligned}$$

and

$$\begin{aligned} E|\mathbf{x} - \hat{\mathbf{x}}|^2 &= R_f - R_{fg}R_{gf}R_g^{-1} && (5)\\ &= E|\mathbf{x}|^2 - (\mu_x 2)(2\mu_x)/4 && (6)\\ &= E|\mathbf{x}|^2 - \mu_x^2 && (7) \end{aligned}$$