

# The Lazy Engineer and Recommendation Subgraphs\*

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## Abstract

Recommendations are central to the utility of many of the popular websites such as YouTube, Facebook and Quora, as well as many e-commerce store websites. Such sites usually have a large list of candidate pages that could serve as recommendations from the current page based on relevance. To inter-connect the website for efficient traversal it is critical to subselect a few recommendations on each page from the list of all candidates.

In our formulation of the problem, the set of pages on the site is divided into a few popular pages where surfers arrive and the remaining large number of pages that should be recommended from the popular pages. The natural bipartite graph of potential recommendations is a candidate supergraph. Given the limited space to display recommendations a subgraph that has bounded outdegree  $c$  on the popular pages must be chosen. Our goal is to maximize the number of undiscovered pages that have indegree at least  $a$ . We introduce this as the  $(c, a)$ -recommendation subgraph problem.

To solve such problems optimally at web-scale graph sizes, implementations would involve writing distributed matching algorithms. Instead, in this work, we study the effectiveness of a lazy engineer solving the recommendation subgraph problem. We investigate the cases when the candidate supergraph is a random graph under two models: the fixed-degree model where every node on the left has exactly  $d$  random neighbors on the right, as well as the standard Erdős-Renyi model with expected degree  $d$  on the left side. We show that for most reasonable parameters of the models, the lazy engineer would have found solutions very close to optimal. We further show the conditions under which a perfect recommendation subgraph, a generalization of perfect matchings, exists. Lastly, to more realistically model web graphs we propose generalizations of the random graph models using topic taxonomies, varying subgraph edge densities and edge weights that simulate traffic patterns. Surprisingly, the lazy engineer would still have found solutions near optimal under different parameters.

## 1 Introduction

One of the great benefits of the web as a useful source of hyperlinked information comes from the careful choices made in crafting the recommendations that link a given page to closely related pages. Even though this advantage was identified well before the www was in place by Bush [?], it continues to persist even today. Smooth site navigation is enabled by carefully chosen recommendations to closely related pages. Thus, the presence of recommendations is an integral feature of several popular websites that are known to have high user engagement as reflected in long sessions. Examples range from entertainment sites like YouTube that recommend a set of videos on the right for every video watched by a user, information sites like Quora that recommend a set of related

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questions on every question page, to retail sites like Amazon that recommend similar products on every product page.

In this work, we analyze a recommendation system that can be built offline in two stages: the first decides on candidate recommendations for each item based on relevance and retrieves a large candidate set of  $d$  items [1, 15, 16]. The second stages prunes it to  $c < d$  such that globally we ensure that the resulting recommendation subgraph is ‘efficient’ for traversal.

We can represent this notion of recommendations by using a directed graph. A vertex is simply an item and a directed edge  $(u, v)$  is a recommendation from  $u$  to  $v$ . Under this graph model for the above problem, if  $c = 1$  and we require the chosen recommendation subgraph be strongly connected then our problem reduces to Hamilton cycle, an NP-complete problem [3].

## 1.1 Bipartite Recommendation Subgraphs

We can generalize the above graph recommendation problem by using a directed bipartite graphs. We can use one partition  $L$  to represent the set of items for which we are required to suggest recommendations. We can use the other partition  $R$  to represent the set of items that are potentially recommended. Note that a single item can be represented in both  $L$  and  $R$  if needed. Now the input to the problem is this directed graph  $G$  where each vertex has  $d$  recommendations. The output required is a subgraph  $G'$  where each vertex in  $L$  has  $c < d$  recommendations. The simplest goal is to minimize the number of vertices that have in-degree less than an integer  $a$ . We call this the  $(c, a)$ -recommendation subgraph problem. Note that if  $a = c = 1$  this is simply the problem of maximum bipartite matching [14]. If  $a = 1$  and  $c > 1$ , we obtain a  $b$ -matching problem, that can be converted to the usual bipartite matching problem. [8]

**Implementation Issues** Over the past few years, the second and third authors have implemented complex graph algorithms in cutting-edge web-technology companies including Google, Facebook and BloomReach. There are two key hurdles in making such graph algorithms practical. The first is that the method used must be very simple to implement, debug and deploy. The second is that the method must scale gracefully with larger sizes.

Matching algorithms require linear memory and super-linear run-time neither of which scale well. Note that the Facebook graph has over a billion vertices [] and hundreds of billions of edges [], YouTube has XXX videos and YYY recommendations [] in total. Even an e-commerce website with 100M product pages and 100 recommendation candidates per product would require easily over 160GB in main memory to run these algorithms without using graph compression techniques that add additional technical difficulties in development and maintenance. In practice offline problem instances are solved by using distributed computing such as map-reduce [4]. However, efficient map-reduce algorithms for graph problems are notoriously difficult and we are back to building a complicated system.

## 1.2 Algorithms and Analyses

**The Lazy Engineer** These reasons motivated the authors to investigate the “lazy” approach of choosing a very simple (almost any) set of recommendation to see if they would produce near optimal solutions at least under realistic scenarios in practice. In practice, this would immediately imply that a very simple back-end system can be built in many cases without the need for a complicated algorithm. If the thresholds provided by our analysis are insufficient in quality, then the designer can consider implementing the classical algorithms.

To be more specific about practical scenarios we briefly describe input web graph characteristics to give an idea for the sizes of  $L$ ,  $R$ ,  $c$  and  $a$  to expect. It is well known, and can be verified by prior experiences that a small number of ‘head’ pages contribute to a significant amount of the traffic while a long tail of the remaining pages contribute to the rest [5, 10]. For example demonstrated by one of the prior measurements [13] it is not unreasonable to expect that 50% of the traffic is contributed by less than 1% (a few thousands) of the web pages and a large number of tail pages (a few hundreds of thousands) contributing the other half of the traffic. This implies that in practice  $L$  can be up to two orders of magnitude smaller than  $R$ . By observing recommendations of Quora, Amazon, YouTube and our work in building these technologies in practice, typical values for  $c$  range from 5 to 20 recommendations per page. Values of  $a$  are the hardest to nail down but one can think of it as something that can range from 1 to 10 roughly.

**Recommendation Graph Models** The setting for investigating the effectiveness of the lazy engineer is provided by using a random graph model for the recommendation supergraph. Since the lazy algorithm involves choosing *any* set of  $c$  recommendations, the natural random graph model that permits analysis of this method is the *fixed-degree* model where every node on the left has recommendations to a random subset of size exactly  $d$  nodes in the right. This model is similar to the another model called  $d$ -out, where vertices from both sides of the bipartition send  $d$  edges to the other side of the bipartition [7].

We study this model in Section 2.1. Our main result identifies the range of parameters involving  $c, a, l = |L|$  and  $r = |R|$  where the lazy algorithm is very effective. In addition to showing that it is a  $(1 - \frac{1}{e})$ -approximation algorithm in expectation, we also get much better bounds for the expected performance for a wide range of realistic parameters and also prove high probability bounds on its performance. We extend the models later to more realistic graphs including a hierarchical, cartesian and weighted graph models.

**The Greedy Algorithm** While the lazy algorithm chooses any set of  $c$  recommendations, the natural greedy algorithm will need some work: scanning the nodes on the right that must be discovered, we look to see if there are  $a$  neighbors from the left that have not exhausted their bound of  $c$  edges in the subgraph, and if so, use them to add this node to the discovered set. We study this algorithm in Section 2.2. We show the easy result that Greedy gives a  $\frac{1}{a+1}$ -approximation, and also do an average case analysis. However, we use the usual Erdős-Renyi model [6] rather than the fixed-degree model for this analysis.

In the subsequent Section 2.3, we compare and plot the worst-case performance of Greedy against the average case expected performance of the random or lazy algorithm, as a basis for our later computational comparisons in a similar vein.

**Optimal Recommendation Subgraphs** Finally we turn to the question of whether the maximum number of nodes allowed by the degree constraints can be covered (at least  $a$  times) by a recommendation subgraph: we say that there is an perfect  $(c, a)$ -recommendation subgraph in this case. Under the usual Erdős-Renyi model, we use existing results on the existence of a perfect matching to characterize the edge probability over which there exist such optimal recommendation subgraphs, by using a subset partitioning method for the analysis. By relaxing the condition of optimality of matching (i.e. perfect matchings) to disallowing short augmenting paths in these methods, we also turn the above results into  $(1 - \epsilon)$ -approximation algorithms trading off the running time, for appropriately dense random graphs as before.

## 2 Algorithms for Recommendation Subgraphs

In this section, we analyze the lazy algorithm of choosing any set of  $c$  recommendations, and the slightly more interesting greedy algorithm for finding a  $(c, a)$ -recommendation subgraph. To do this, we first introduce the fixed-degree random graph model for the candidate supergraph of recommendations.

### 2.1 Fixed Degree Model

In this model, we assume that a bipartite graph  $G = (L, R, E)$  is generated probabilistically by the following procedure. Each vertex  $v \in L$ , uniformly samples a set of  $d$  neighbors from  $R$ . For convenience let  $|L| = l$ ,  $|R| = r$  and  $k = l/r$ . From  $G$ , we sample a subgraph  $H$  where for each vertex  $u \in L$  a set of  $c$  neighbors are sampled uniformly from set of incident vertices. The following theorem derives a lower bound on the expected solution.

**Theorem 1.** *Let  $S$  be the random variable denoting the number of vertices  $v \in R$  such that  $\deg_H(v) \geq a$ . Then*

$$\mathbb{E}[S] \geq r \left( 1 - e^{-ck + \frac{a-1}{r}} \frac{(ck)^a - 1}{ck - 1} \right)$$

where the expectation is over the randomness of  $G$  and  $H$ .

*Proof.* Let  $X_{uv}$  be the indicator variable of the event that the edge  $uv$  ( $u \in L$ ,  $v \in R$ ) is in the subgraph that we picked and set  $X_v = \sum_{u \in U} X_{uv}$  so that  $X_v$  represents the degree of the vertex  $v$  in our subgraph. Because our algorithm uniformly subsamples a uniformly random selection of edges, we can assume that  $H$  was generated the same way as  $G$  but sampled  $c$  instead of  $d$  edges for each vertex  $u \in L$ . So  $X_{uv}$  is a Bernoulli random variable. Using the trivial bound  $\binom{n}{i} \leq n^i$  on binomial coefficients we get:

$$\begin{aligned} \Pr[X_v < a] &= \sum_{i=0}^{a-1} \binom{cl}{i} \left(1 - \frac{1}{m}\right)^{cl-i} \left(\frac{1}{r}\right)^i \\ &\leq \sum_{i=0}^{a-1} \left(\frac{cl}{r}\right)^i \left(1 - \frac{1}{r}\right)^{cl-i} \\ &\leq \left(1 - \frac{1}{r}\right)^{cl-(a-1)} \sum_{i=0}^{a-1} (ck)^i \\ &\leq \left(1 - \frac{1}{r}\right)^{cl-(a-1)} \frac{(ck)^a - 1}{ck - 1} \\ &\leq e^{-ck + \frac{a-1}{r}} \frac{(ck)^a - 1}{ck - 1} \end{aligned}$$

Letting  $Y_v = [X_v \geq a]$ , we now see that

$$\mathbb{E}[S] = \mathbb{E} \left[ \sum_{v \in R} Y_v \right] \geq r \left( 1 - e^{-ck + \frac{a-1}{r}} \frac{(ck)^a - 1}{ck - 1} \right)$$

□

**Theorem 2.** *The above sampling algorithm gives a  $(1 - \frac{1}{e})$ -factor approximation to the  $(c, 1)$ -graph recommendation problem in expectation.*

*Proof.* The size of the optimal solution is bounded above by both the number of edges in the graph and the number of vertices in  $R$ . The former of these is  $cl = ckr$  and the latter is  $r$ , which shows that  $OPT \leq r \max(ck, 1)$ . Therefore, by simple case analysis the approximation ratio in expectation is at least

$$\frac{1 - \exp(-ck)}{\min(ck, 1)} \geq 1 - \frac{1}{e}$$

□

For the  $(c, 1)$ -recommendation subgraph problem the approximation obtained by this sampling approach can be much better for certain values of  $ck$ . In particular, if  $ck > 1$ , then the approximation ratio is  $1 - \exp(-ck)$ , which approaches 1 as  $ck \rightarrow \infty$ . In particular, if  $ck = 3$ , then the solution will be at least 95% as good as the optimal solution even with our trivial bounds. Similarly, when  $ck < 1$ , the approximation ratio is  $(1 - \exp(-ck))/ck$  which also approaches 1 as  $ck \rightarrow 0$ . In particular, if  $ck = 0.1$  then the solution will be at 95% as good as the optimal solution. The case when  $ck = 1$  therefore represents the worst case outcome for this model where we only guarantee 63% optimality. The graph below shows the approximation ratio as a function of  $ck$ .

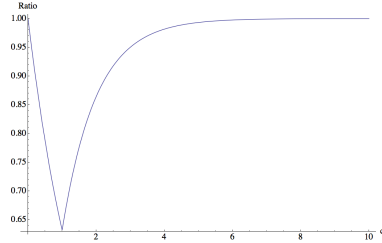


Figure 1: Approximation Ratio as a function of  $ck$

For the general  $(c, a)$ -recommendation subgraph problem, if  $ck > a$ , then the problem is easy on average. This is in comparison to the trivial estimate of  $cl$ . For a fixed  $a$ , a random solution gets better as  $ck$  increases because the decrease in  $e^{-ck}$  more than compensates for the polynomial in  $ck$  next to it. However, in the more realistic case  $ck < a$ , we need to use the trivial estimate of  $ckr/a$ , and the analysis for  $a = 1$  does not extend here. The following table shows how large  $ck$  needs to be for the solution to be 95% optimal for different values of  $a$ .

$a$	1	2	3	4	5
$ck$	3.00	4.74	7.05	10.01	13.48

Figure 2: The required  $ck$  to obtain 95% optimality

We close out this section by showing that the main result that holds in expectation also holds with high probability. While Chernoff bounds are usually stated for independent variables, the variant below holds for any number of pairwise non-positively correlated variables.

**Theorem 3.** [2] Let  $X_1, \dots, X_n$  be non-positively correlated variables. If  $X = \sum_{i=1}^n X_i$ , then for any  $\delta \geq 0$

$$\Pr[X \geq (1 + \delta)E[X]] \leq \left( \frac{e^\delta}{(1 + \delta)^{1+\delta}} \right)^{E[X]}$$

Using this we can convert our expectation result to one that holds with high probability.

**Theorem 4.** *Let  $S$  be the random variable denoting the number of vertices  $v \in R$  such that  $\deg_H(v) \geq a$ . Then  $S \leq r(1 - 2\exp(-ck))$  with probability at most  $(e/4)^{r(1-\exp(-ck))}$ .*

*Proof.* We can write  $S$  as  $\sum_{v \in R} 1 - X_v$  where  $X_v$  is the indicator variable that denotes that  $X_v$  is matched. Note that the variables  $1 - X_v$  for each  $v \in R$  are non-positively correlated. In particular, if  $N(v)$  and  $N(v')$  are disjoint, then  $1 - X_v$  and  $1 - X_{v'}$  are independent. Otherwise,  $v$  not claiming any edges can only increase the probability that  $v'$  gets an edge from any vertex  $u \in N(v) \cap N(v')$ , so the variables would be negatively correlated. Furthermore, note that the expected size of  $S$  is  $r(1 - \exp(-ck))$  by Theorem 1. Therefore, we can apply Theorem 3 with  $\delta = 1$  and obtain the result.  $\square$

## 2.2 The Greedy Algorithm

The results in the previous section concentrated on producing nearly optimal solutions in expectation. In this section, we will show that it is possible to obtain good solutions regardless of the model that generated the recommendation subgraph.

We analyze the following natural greedy algorithm. Consider each vertex in  $R$  in some arbitrary order and if there is some  $v \in R$  that has  $a$  neighbors in  $L$  all of which have degree  $< c$ , add the edges to these neighbors to  $H$ . If there are any ties about the nodes to be picked either in the selection of  $v$  or its neighbors, we can break ties arbitrarily.

**Theorem 5.** *The greedy algorithm achieves a  $1/(a + 1)$ -approximation ratio for the  $(c, a)$ -graph recommendation problem.*

*Proof.* Let  $R_{\text{GREEDY}}, R_{\text{OPT}} \subseteq R$  be the set of vertices that have degree  $\geq a$  in the greedy and optimal solutions respectively. Note that any  $v \in R_{\text{OPT}}$  along with neighbors  $\{u_1, \dots, u_a\}$  forms a set of candidate edges that can be used by the greedy algorithm. So we can consider  $R_{\text{OPT}}$  as a candidate pool for  $R_{\text{GREEDY}}$ . Each move that the greedy algorithm makes might make some of the candidates infeasible, but as long as the candidate pool is not depleted, the greedy algorithm can continue adding vertices to its solution. Each time the greedy algorithm claims some vertex  $v \in R$  with edges to  $\{u_1, \dots, u_a\}$ , we obviously have to remove  $v$  from the candidate pool. If any  $u_i$  was saturated (i.e. had degree  $c$ ) in the optimal solution, we would also need to remove an arbitrary vertex  $v_i \in R$  adjacent to  $u_i$  in the optimal solution. In other words, by using an edge of  $u_i$ , we force it to not use an edge it used to some other  $v_i$ , which might cause the degree of  $v_i$  to go below  $a$ . Therefore, at each step of the greedy algorithm, we have to remove at most  $a + 1$  vertices from the candidate pool. Since our candidate pool has size  $\text{OPT}$ , the greedy algorithm can not stop before it has added  $\text{OPT}/(a + 1)$  vertices to the solution.  $\square$

Note that this approximation guarantee is as good as we can expect. In particular, if we set  $a = 1$  then we obtain the familiar  $1/2$ -approximation of the greedy algorithm for matchings. As in matchings, randomizing the order in which the vertices are processed still leaves a constant factor gap in the size of the solution as in the matchings case [12].

Despite this result, the greedy algorithm fares much better when we give it the same expectation treatment we have given the sampling algorithm. The following theorem shows that just as maximal matchings are likely to be close to being maximum in sufficiently dense random graphs proving the near optimality of the greedy algorithm for the  $(c, a)$ -recommendation subgraph problem.

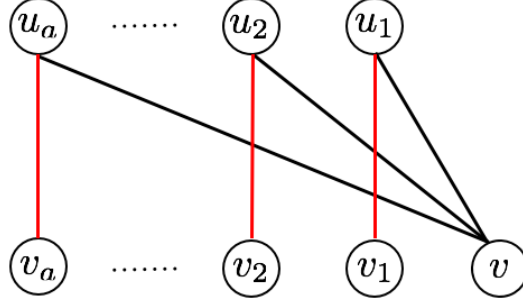


Figure 3: This diagram shows one step of the greedy algorithm. When  $v$  claims edges to  $u_1, \dots, u_a$ , it potentially removes  $v_1, \dots, v_a$  from the pool of candidates that are available. The potentially invalidated edges are shown in red.

**Theorem 6.** Let  $G = (L, R, E)$  be a graph drawn from the  $G_{l,r,p}$ . If  $S$  is the size of the  $(c, a)$ -recommendation subgraph produced by the greedy algorithm, then:

$$E[S] \geq r - \frac{a(lp)^{a-1}}{p}$$

*Proof.* Note that if edges are generated uniformly, we can consider the graph as being revealed to us one vertex at a time as the greedy algorithm runs. In particular, consider the event  $X_i$  that the greedy algorithm matches the  $(i+1)^{th}$  vertex it inspects. While,  $X_{i+1}$  is dependent on  $X_1, \dots, X_i$ , the worst condition for  $X_{i+1}$  is when all the previous  $i$  vertices were from the same vertices in  $L$ , which are now not available for matching the  $(i+1)^{th}$  vertex. The maximum number of such invalidated vertices is at most  $\lceil i/c \rceil$ . Therefore, the probability that fewer than  $a$  of the at least  $l - \lceil i/c \rceil$  available vertices have an edge to this vertex is at most  $\Pr[Y \sim \text{Bin}(l - \frac{i}{c}, p) : Y < a]$ . Using the approximation that

$$\Pr[Y \sim \text{Bin}(l - \frac{ia}{c}, p) : Y < a] \leq a(1-p)^{l - \frac{ia}{c} - a + 1} (lp)^{a-1}$$

and summing over all the  $X_i$  using the linearity of expectation, we obtain the following inequality:

$$\begin{aligned} E[S] &\geq r - \sum_{i=0}^{r-1} E[\neg X_i] \\ &\geq r - \sum_{i=0}^{r-1} \Pr[Y \sim \text{Bin}(l - \frac{ia}{c}, p) : Y < a] \\ &\geq r - \sum_{i=0}^{r-1} a(1-p)^{l - \frac{ia}{c} - a + 1} [lp - \frac{iap}{c}]^{r-1} \\ &\geq r - a(lp)^{r-1} \sum_{i=0}^{r-1} (1-p)^{l - \frac{ia}{c} - a + 1} \geq r - \frac{a(lp)^{a-1}}{p} \end{aligned}$$

□

### 2.3 Comparing the Worst-Case and Average-Case Approximations

It is instructive to compare the performance of the random sampling algorithm with that of greedy. With high probability, the approximation ratio of the randomized algorithm is  $(1 - \exp(-ck))/(1 - \exp(-dk))$  while the approximation ratio of the greedy algorithm is always at least  $1/2$ . In the following graphs  $f \geq 1$  represents a value such that  $d = fc$  and we use the values  $k = 0.1$  and  $k = 0.2$  respectively since the case that is practical for our purposes is when  $l \ll r$ . In the graphs below, the red curve shows the approximation ratio for the model used in Section 2.1 and the blue curve shows the worst case approximation ratio described in section 5. As we would expect, the sampling algorithm performs outperforms the greedy algorithm significantly when  $k$  and more importantly  $c$  gets larger.

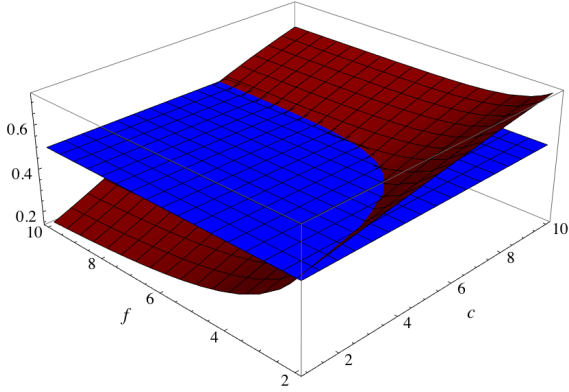


Figure 4: Approximation ratios when  $k=0.1$

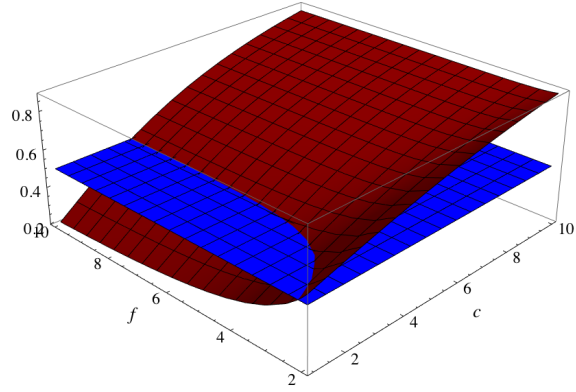


Figure 5: Approximation ratios when  $k=0.2$

### 2.4 Existence of Optimal Recommendation Subgraphs

Let  $G = (L, R, E)$  be a bipartite graph. We define a *perfect*  $(c, a)$ -recommendation subgraph on  $G$  to be a subgraph  $H$  such that  $\deg_H(u) \leq c$  for all  $u \in L$  and  $\deg_H(v) = a$  for  $\min(|R|, |L|c/a)$  of the vertices in  $R$ . In this section we will prove a sufficient condition for perfect  $(c, a)$ -recommendation subgraphs to exist in a bipartite graph where edges are sampled uniformly and independently with probability  $p$ . Our result relies on a characterization of perfect matchings [11].

**Theorem 7.** *Let  $G = (L, R, E)$  be a bipartite graph with  $|L| = |R| = n$  and  $G$  is drawn from  $G_{n,n,p}$ . If  $p \geq \frac{\log n - \log \log n}{n}$ , then as  $\lim_{n \rightarrow \infty}$  probability that  $G$  has a perfect matching approaches 1.*

We will prove that a perfect  $(c, a)$ -recommendation subgraph exists in random graphs with high probability by building it up from  $a$  matchings which must exist with high probability if  $p$  is sufficiently high. In the next theorem, we show that  $p$  only needs to be  $\Theta(\frac{\log n}{n})$  for this to succeed. Furthermore, this decomposition into matchings motivates an algorithm that can return a solution that comes within  $(1 - \epsilon)$  of the optimal solution and can be run in almost linear time. While the theorem is stated for the case when  $a \leq c$ , it applies equally well to the  $a > c$  case by partitioning  $L$  instead of  $R$  in the following proof.

**Theorem 8.** *Let  $G = (L, R, E)$  be a random graph and let  $|L| = l$ ,  $|R| = r$  and  $k = l/r$ . If  $G$  is drawn from  $G_{l,r,p}$  with  $p \geq a \frac{\log l - \log \log l}{l}$  and  $kc \geq a$ , then the probability that  $G$  has a perfect  $(c, a)$ -recommendation subgraph tends to 1 as  $l, r \rightarrow \infty$ .*



*Proof.* Given the size and the degree constraints of  $L$ , at most  $lc/a$  vertices in  $R$  can obtain degree  $a$  in a  $(c, a)$ -recommendation subgraph. We can therefore restrict  $R$  to an arbitrary subset  $R'$  of itself of size  $lc/a$ . Next, we pick an enumeration of the vertices in  $R' = \{v_0, \dots, v_{lc/a-1}\}$  and add these vertices into  $a$  subsets by defining  $R_i = \{v_{(i-1)l/a}, \dots, v_{(i-1)l/a+l}\}$  for each  $1 \leq i \leq c$  where the arithmetic in the indices is done modulo  $lc/a$ . Note both  $R_i$  and all of the  $L_i$  have size  $l$ .

Using these new sets we define the graphs  $G_i$  on the bipartitions  $(L, R_i)$ . Since the sets  $R_i$  are intersecting, we cannot define the graphs  $G_i$  to be induced subgraphs. However, note that each vertex  $v \in R$  falls into exactly  $a$  of these subsets. Therefore, we can uniformly assign each edge in  $G$  to one of  $a$  graphs among  $\{G_1, \dots, G_c\}$  it can fall into, and make each of those graphs a random graph. In fact, while the different  $G_i$ s are coupled, taken in isolation we can consider any single  $G_i$  to be drawn from the distribution  $G_{l,l,p/a}$  since  $G$  was drawn from  $G_{l,r,p}$ . Since  $p/a \geq (\log l - \log \log l)/l$  by assumption, we can now say that by Theorem 7, the probability that  $G_i$  has no perfect matching is  $o(1)$ .

Considering  $c$  to be fixed, by a union bound we can now conclude that except for a  $o(1)$  probability, each one of the  $G_i$  has a perfect matching. By superimposing all of these perfect matchings, we can see that every vertex in  $R'$  has degree  $a$ . Since each vertex in  $L$  is in exactly  $c$  matchings, each vertex in  $L$  has degree  $c$ . It follows that except for a  $o(1)$  probability there exists a  $(c, a)$ -recommendation subgraph in  $G$ .  $\square$

## 2.5 Not-So-Lazy Algorithm Using Perfect Matchings

The above result now enables us to design a near linear time algorithm with a  $(1 - \epsilon)$  approximation guarantee to the  $(c, a)$ -recommendation subgraph problem by leveraging combinatorial properties of matchings.

**Theorem 9.** *Let  $G = (L, R, E)$ ,  $|L| = l$ ,  $|R| = r$ , and  $G$  is drawn from  $G_{l,r,p}$  where  $p \geq a \frac{\log l - \log \log l}{l}$ . Then there exists an algorithm that can find a  $(1 - \epsilon)$ -approximation in time  $O(\frac{|E|c^2}{\epsilon})$  with probability  $1 - o(1)$ .*

*Proof.* As with the proof of the previous theorem, we can arbitrarily restrict  $R$  to a subset of size  $lc/a$  and divide  $R$  into sets  $R_1, \dots, R_c$  such that each vertex in  $R$  is contained in exactly  $a$  of the sets. Using the previous theorem, we know that each of the graphs  $G_i$  has a perfect matching with high probability. These perfect matchings can be approximated to a  $1 - \epsilon/c$  factor by finding matchings that do not have augmenting paths of length  $\geq 2c/\epsilon$ . This can be done for each  $G_i$  in  $O(|E|c/\epsilon)$  time. Furthermore, the union of unmatched vertices makes up an at most  $c(\epsilon/c)$  fraction of  $R'$ .  $\square$

## 3 More Expressive Models of Recommendation Graphs

Having described the fixed-degree model in detail above, we can now use it as a template for the analysis of three more richer, more expressive and realistic models. In practice, websites can not pick edges uniformly at random from all the other pages on the site. There are natural clusters of pages so that pages within the clusters are more closely related than those outside.

This is the motivation to extend our previous analysis to the cases when the underlying graph has substructure. In Section 3.1, we analyze the *hierarchical tree model* which models graphs that contain natural tree like clusters. In Section 3.3, we generalize to the *cartesian product model*

where edge density may vary across different pairs of categories. Finally, in Section 3.3 we study the *weighted model* which assigns weights to the graph so that we can incorporate traffic patterns across a website.

### 3.1 Hierarchical Tree Model

We will assume that we are given a bipartite graph  $G = (L, R, E)$ . The vertex sets  $L$  and  $R$  are the leaf sets of two full binary trees  $T_L$  and  $T_R$  of depth  $D$  where there is a one-to-one correspondence between the subtrees of these two trees. We also assume that each branching in both  $T_L$  and  $T_R$  splits the nodes evenly into the two subtrees. As in the previous sections, we set  $|L|/|R| = k$ , and assume that this ratio still holds if we take any subtree on the left and its corresponding subtree on the right. For simplicity of notations, we will use a subtree and its leaf set interchangeably. We assume that the trees are fixed in advance but  $G$  is generated probabilistically according to the following procedure. Let  $u \in L$  and  $T_L^0, \dots, T_L^{D-1}$  be the subtrees it belongs at depths  $0, \dots, D-1$ . Also, let  $T_R^0, \dots, T_R^{D-1}$  be the subtrees on the right that correspond to these trees on the left. We let  $u$  make an edge to  $d_{D-1}$  of the vertices in  $T_R^{D-1}$ ,  $d_{D-2}$  edges to the vertices in  $T_R^{D-1} \setminus T_R^{D-2}$  and so on. Each vertex is chosen with uniform probability and we let  $d = d_0 + \dots + d_{D-1}$ .

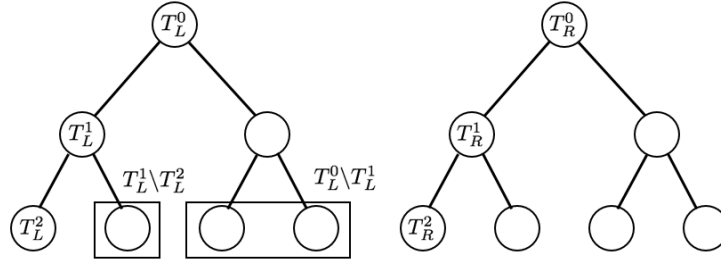


Figure 6: This diagram shows the notation we use for this model and the 1-to-1 correspondence of subtrees

Our goal now is to find a  $b$ -matching [8] in this graph that is close to optimal in expectation. That is, our degree upper and bounds on vertices in  $L$  is  $c$  and 1 respectively. Let  $c = c_0 + \dots + c_{D-1}$  similar to how we defined  $d$ . To combine the analysis of the randomness of the algorithm and the randomness of the graph, the algorithm will pick  $c_i$  edges uniformly from among the  $d_i$  edges going to each level of the subtree. This enables us to think of the subgraph our algorithm finds as being generated by the graph generation process, but with fewer neighbors selected. With this model and parameters in place, we can prove the following theorem.

**Theorem 10.** *Let  $S$  be the subset of edges  $v \in R$  such that  $\deg_H(v) \geq 1$ . Then*

$$E[S] \geq r(1 - \exp(-ck))$$

where the expectation is over  $G$  and  $H$ .

*Proof.* Let  $v \in R$  and let  $T_L^{D-1}, T_L^{D-2} \setminus T_L^{D-1}, \dots, T_L^0 \setminus T_L^1$  be the sets it can take edges from. Since  $T_L$  and  $T_R$  split perfectly evenly at each node the vertices in these sets will be chosen from  $r_{D-1}, r_{D-1}, r_{D-2}, \dots, r_1$  vertices in  $R$  as neighbors, where  $r_i$  is the size of subtree of the right tree rooted at depth  $i$ . Furthermore, each of these sets described above have size  $l_{D-1}, l_{D-1}, l_{D-2}, \dots, l_1$  respectively, where  $l_i$  is the size of a subtree of  $T_L$  rooted at depth  $i$ . It follows that the probability that  $v$  does not receive any edges at all is at most

$$\begin{aligned}
\Pr[\neg X_v] &= \left(1 - \frac{1}{r_{D-1}}\right)^{c_0 l_{D-1}} \prod_{i=1}^{D-1} \left(1 - \frac{1}{r_i}\right)^{c_{D-i} l_i} \\
&\leq \exp\left(-\frac{l_{D-1}}{r_{D-1}} c_0\right) \prod_{i=1}^{D-1} \exp\left(-\frac{l_i}{r_i} c_{D-i}\right) \\
&= \exp(-(c_0 + \dots + c_{D-1})k) \\
&= \exp(-ck)
\end{aligned}$$

Since this is an indicator variable, it follows that

$$\mathbb{E}[S] = \mathbb{E}\left[\sum_{v \in R} X_v\right] \geq r(1 - \exp(-ck))$$

□

Note that this is the same result as we obtained for the fixed degree model in Section 2.1. In fact, the approximation guarantees when  $ck \ll 1$  or  $ck \gg 1$  hold exactly as before.

The sampling of  $H$  can be done algorithmically because we separated out the edge generation process at a given depth from the edge generation process at deeper subtrees. There is no ambiguity as to why an edge is in the underlying graph. That is, if we superimpose  $T_L$  and  $T_R$ , then an edge between  $u_l \in L$  and  $v_r \in R$  must have come from an edge generated at the lowest common ancestor of  $u_l$  and  $v_r$ . So the algorithm can actually sample intelligently and in the same way that the graph was generated in the first place. Also note that we do not have to assume that the trees  $T_L$  and  $T_R$  are binary. We only need the trees to be regular and evenly divided at each vertex since the proof only relies on the proportions of the sizes of the subtrees in  $T_L$  and  $T_R$ .

### 3.2 Cartesian Product Model

We can extend the analysis in Section 2.1 in a way orthogonal to the hierarchical tree model. We assume that  $L$  has been partitioned into  $t$  subsets  $L_1, \dots, L_t$  and that  $R$  has been partitioned into  $t'$  subsets  $R_1, \dots, R_{t'}$ . For convenience, we let  $|L_i| = l_i$  and  $|R_i| = r_i$ . Given this suppose that for each  $1 \leq i \leq t$  and each  $1 \leq j \leq t'$ ,  $G[L_i, R_j]$  is an instance of the fixed degree model with  $d = d_{ij}$ . We assume that for all  $i$ , we have  $\sum_{j=1}^{t'} d_{ij} = d$  for some fixed  $d$ . Also assume that we have fixed in advance  $c_{ij}$  for each  $1 \leq i \leq t$  and  $1 \leq j \leq t'$  that satisfy  $\sum_{j=1}^{t'} c_{ij} = c$  for all  $i$  for some fixed  $c$ . To sample  $H$  from  $G$ , we sample  $c_{ij}$  neighbors for each  $u_i \in L_i$  from  $R_i$ . Letting  $S$  be the set of vertices in  $v \in R$  that satisfy  $\deg_H(v) \geq 1$ , we can show the following:

**Theorem 11.** *With  $S$ ,  $G$  and  $H$  defined as above, we have*

$$\mathbb{E}[S] \geq r - \sum_{j=1}^{t'} r_j \exp\left(-\sum_{i=1}^t c_{ij} \frac{l_i}{r_j}\right)$$

where the expectation is over  $G$  and  $H$ .

*Proof.* Let  $v_j \in R_j$  be an arbitrary vertex and let  $X_{v_j}$  be the indicator variable for the event that  $\deg_H(v_i) \geq 1$ . The probability that none of the neighbors of some  $u_i \in R_i$  is  $v_j$  is exactly  $(1 - \frac{1}{r_j})^{c_{ij}}$ . It follows that the probability that the degree of  $v_j$  in the subgraph  $H[L_i, R_j]$  is 0 is at most  $(1 - \frac{1}{r_j})^{c_{ij}l_i}$ . Considering this probability over all  $R_j$  gives us:

$$\Pr[X_{v_i} = 0] = \prod_{i=1}^t \left(1 - \frac{1}{r_j}\right)^{c_{ij}l_i} \leq \exp\left(-\sum_{i=1}^t c_{ij} \frac{l_i}{r_j}\right)$$

By linearity of expectation  $E[S] = \sum_{i=1}^{t'} r_i E[X_{v_i}]$ , so it follows that

$$E[S] \geq \sum_{j=1}^{t'} r_j \left(1 - \exp\left(-\sum_{i=1}^t c_{ij} \frac{l_i}{r_j}\right)\right) = r - \sum_{j=1}^{t'} r_j \exp\left(-\sum_{i=1}^t c_{ij} \frac{l_i}{r_j}\right)$$

□

This model is interesting because it can capture a broader set of recommendation subgraphs than the fixed degree model. However, it is difficult to estimate how good a solution will be without knowing the sizes of the sets in the partitions. We note that we obtain the approximation guarantee of  $(1 - \exp(-ck))$  provided that  $l_i/r_j = k$  for all  $i$  and  $j$  where  $k$  is some fixed constant. Another interesting point about this model and the algorithm we described for sampling  $H$  is that we are free to set the  $c_{ij}$  as we see fit. In particular,  $c_{ij}$  can be chosen to maximize the approximation guarantee in expectation we obtained above using gradient descent or other first order methods prior to running the recommendation algorithm to increase the quality of the solution.

### 3.3 Weighted Model

The fixed degree model of Section 2.1 is a simple and convenient model, but the assumption that all recommendations hold the same weight is unrealistic. This motivates fixing the graph to be the complete bipartite graph  $K_{l,r}$ , and giving the edges i.i.d weights with mean  $\mu$ . We modify the objective function accordingly, so that we count only the vertices in  $R$  which have weight  $\geq 1$ . If we assume that  $ck\mu \geq 1 + \epsilon$  for some  $\epsilon > 0$ , then the naive sampling solution we outlined in Section 2.1 still performs exceptionally well. If we let  $S$  be the size of the solution produced by this algorithm we have:

**Theorem 12.** *Let  $G = K_{l,r}$  be a bipartite graph where the edges have i.i.d. weights and come from a distribution with mean  $\mu$  that is supported on  $[0, b]$ . If the algorithm from Section 2.1 is used to sample a subgraph  $H$  from  $G$ , then*

$$E[S] = \sum_{v \in R} E[X_v] = r \left(1 - \exp\left(-\frac{2l\epsilon^2}{b^2}\right)\right)$$

*Proof.* For each edge  $uv \in G$ , let  $W_{uv}$  be its random weight,  $Y_{uv}$  be the indicator for the event  $uv \in H$  and define  $X_{uv} = Y_{uv}W_{uv}$ . Since weights and edges are sampled by independent processes, we have  $E[X_{uv}] = E[W_{uv}]E[Y_{uv}]$  for all edges. Since  $c$  edges out of  $r$  are picked for each vertex,  $E[Y_{uv}] = \frac{c}{r}$ , so  $E[X_{uv}] = \frac{c}{r}\mu$ . Therefore, the expected weight coming into a vertex  $v \in R$  would be

$$E[X_v] = \sum_{u \in L} E[X_{uv}] = \frac{cl\mu}{r} = ck\mu$$

However,  $X_{uv}$  for each  $u$  are i.i.d random variables. Since by assumption  $ck\mu = 1 + \epsilon$ , by Hoeffding bounds [9] we can obtain

$$\Pr[X_v \leq 1] = \Pr[X_v - \mathbb{E}[X_v] \geq \epsilon] \leq \exp\left(-\frac{2l\epsilon^2}{b^2}\right)$$

By linearity of expectation we can now get the result in the theorem

$$\mathbb{E}[S] = \sum_{v \in R} \mathbb{E}[X_v] = r \left(1 - \exp\left(-\frac{2l\epsilon^2}{b^2}\right)\right)$$

□

There are two things to note about this variant. The first is that since the variables  $X_v$  are negatively correlated, our results in 2.3 can be readily extended to the results of this section. The second is that the condition that  $W_{uv}$  are i.i.d is not necessary to obtain the full effect of the analysis. Indeed, the only place in the proof where the fact that  $W_{uv}$  are i.i.d is when we argued that  $X_{uv}$  is large with high probability by a Hoeffding bound. For the bound to apply, it is sufficient to assume that  $W_{uv}$  for all  $v$  are independent. In particular, it is possible that  $W_{uv}$  for all  $u$  are inter-dependent. This allows us to assume an weight distribution that depends on the strength of the recommender and the relevance of the recommendation separately.

## 4 Conclusions

In this paper we proposed several different models for explaining how recommendation subgraphs arise probabilistically and how optimization problems on these graphs can be solved, but there is much more that can be done both on the theoretical and the empirical fronts. On the theory side the biggest open problem is the hardness of the general  $(c, a)$ -recommendation subgraph problem. For specific values of  $c$  and  $a$  we know polynomial time algorithms. For e.g.,  $a = 1$  leads to either bipartite matching or fractional matching which can be solved in polynomial time but the hardness of the general problem remains open. On the empirical side, the graph models that we introduced can be made richer. For example, the hierarchical graphs require the trees to be balanced and of the same size which is constraining. The edge weights on the weighted graph is sampled from a distribution as opposed to being arbitrary. We expect the research community and our future work to follow-up along both directions in the future.

## References

- [1] Gediminas Adomavicius and Alexander Tuzhilin. Toward the next generation of recommender systems: A survey of the state-of-the-art and possible extensions. *IEEE Trans. on Knowl. and Data Eng.*, 17(6):734–749, June 2005.
- [2] Anne Auger and Benjamin Doerr. *Theory of Randomized Search Heuristics: Foundations and Recent Developments*. Series on Theoretical Computer Science. World Scientific Publishing Company, 2011.
- [3] Thomas H. Cormen, Charles E. Leiserson, Ronald L Rivest, and Clifford Stein. MIT press, 2001.

- [4] Jeffrey Dean and Sanjay Ghemawat. Mapreduce: simplified data processing on large clusters. In *OSDI04: Proceedings of the sixth conference on symposium on operating systems design and implementation*. USENIX Association, 2004.
- [5] Bowei Du, Michael Demmer, and Eric Brewer. Analysis of www traffic in cambodia and ghana. In *Proceedings of the 15th international conference on World Wide Web*, WWW '06, pages 771–780. ACM, 2006.
- [6] P. Erdős and A. Rényi. On random graphs, I. *Publicationes Mathematicae (Debrecen)*, 6:290–297, 1959.
- [7] Alan Frieze and Boris Pittel. Perfect matchings in random graphs with prescribed minimal degree. In *Mathematics and Computer Science III*, pages 95–132. Springer, 2004.
- [8] Harold Gabow. An efficient reduction technique for degree-constrained subgraph and bidirected network flow problems. In *Proceedings of the fifteenth annual ACM symposium on Theory of computing*, STOC '83, pages 448–456. ACM, 1983.
- [9] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American statistical association*, 58(301):13–30, 1963.
- [10] Bernardo A Huberman and Lada A Adamic. Internet: growth dynamics of the world-wide web. *Nature*, 401(6749):131–131, 1999.
- [11] Svante Janson, Tomasz Luczak, and Andrzej Rucinski. *Random Graphs*. Wiley Series in Discrete Mathematics and Optimization. Wiley, 2011.
- [12] Richard Karp, Umesh Vazirani, and Vijay Vazirani. An optimal algorithm for on-line bipartite matching. In *Proceedings of the twenty-second annual ACM symposium on Theory of computing*, pages 352–358. ACM, 1990.
- [13] Chetan Kumar, John B Norris, and Yi Sun. Location and time do matter: A long tail study of website requests. *Decision Support Systems*, 47(4):500–507, 2009.
- [14] L. Lovász and M.D. Plummer. *Matching theory*. North-Holland mathematics studies. Akadémiai Kiadó, 1986.
- [15] Paul Resnick and Hal R Varian. Recommender systems. *Communications of the ACM*, 40(3):56–58, 1997.
- [16] J. Ben Schafer, Joseph Konstan, and John Riedi. Recommender systems in e-commerce. In *Proceedings of the 1st ACM conference on Electronic commerce*, EC '99, pages 158–166. ACM, 1999.