Chapter 3

The Greedy Method (貪進法)

Outline

- **3-1 Kruskal's algorithm Minimum spanning trees (MST)**
- **3.2 Prime's MST algorithm**
- **3-3 The single-source shortest path problem**
- **3-4 Linear Merge Algorithm**
- **3.5 The minimal cycle basis problem**
- **3.6 The 2-terminal one to any special channel routing problem**
- **3.7 The Minimum Cooperative Guards Problem for 1-Spiral Polygons Solved by the Greedy Method**

The greedy method

- Suppose that a problem can be solved by a sequence of decisions.
- The greedy method has that each decision is locally optimal. These locally optimal solutions will finally add up to a globally optimal solution.
- Only a few optimization problems can be solved by the greedy method.
- Greedy method at least produces a solution which is usually acceptable.

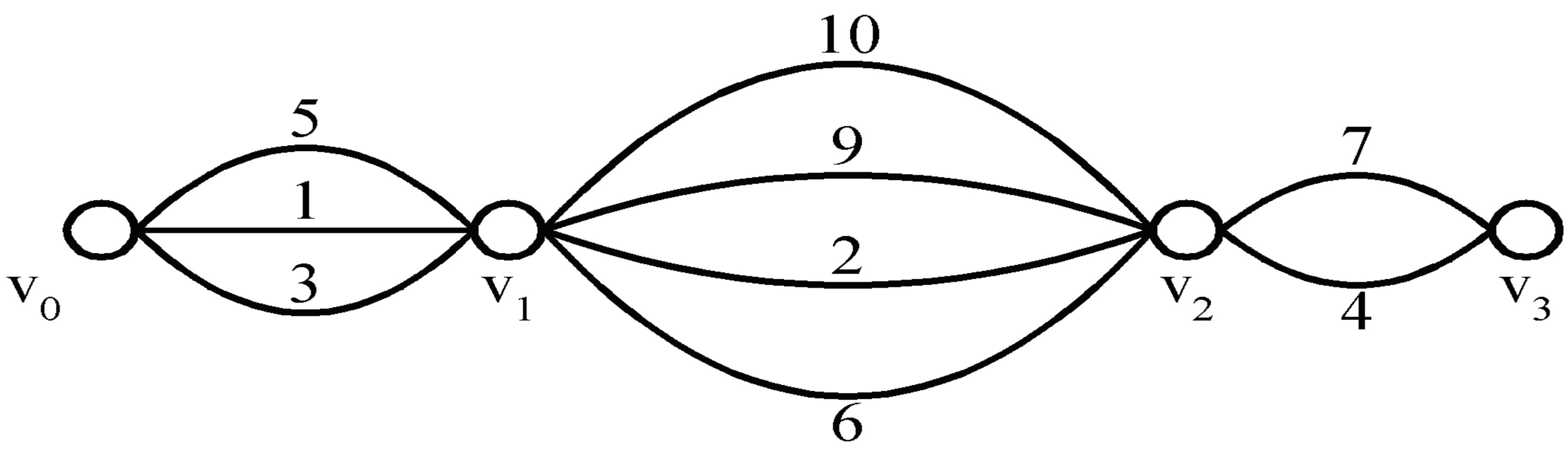
An simple example

- Problem: Pick ***k*** numbers out of n numbers such that the **sum of these k numbers is the largest**.
- Brute-and-force method: $\binom{N}{k}$
- Greedy method:
- Algorithm**:**

```
FOR i = 1 to k
    pick out the largest number and
    delete this number from the input.
ENDFOR
```

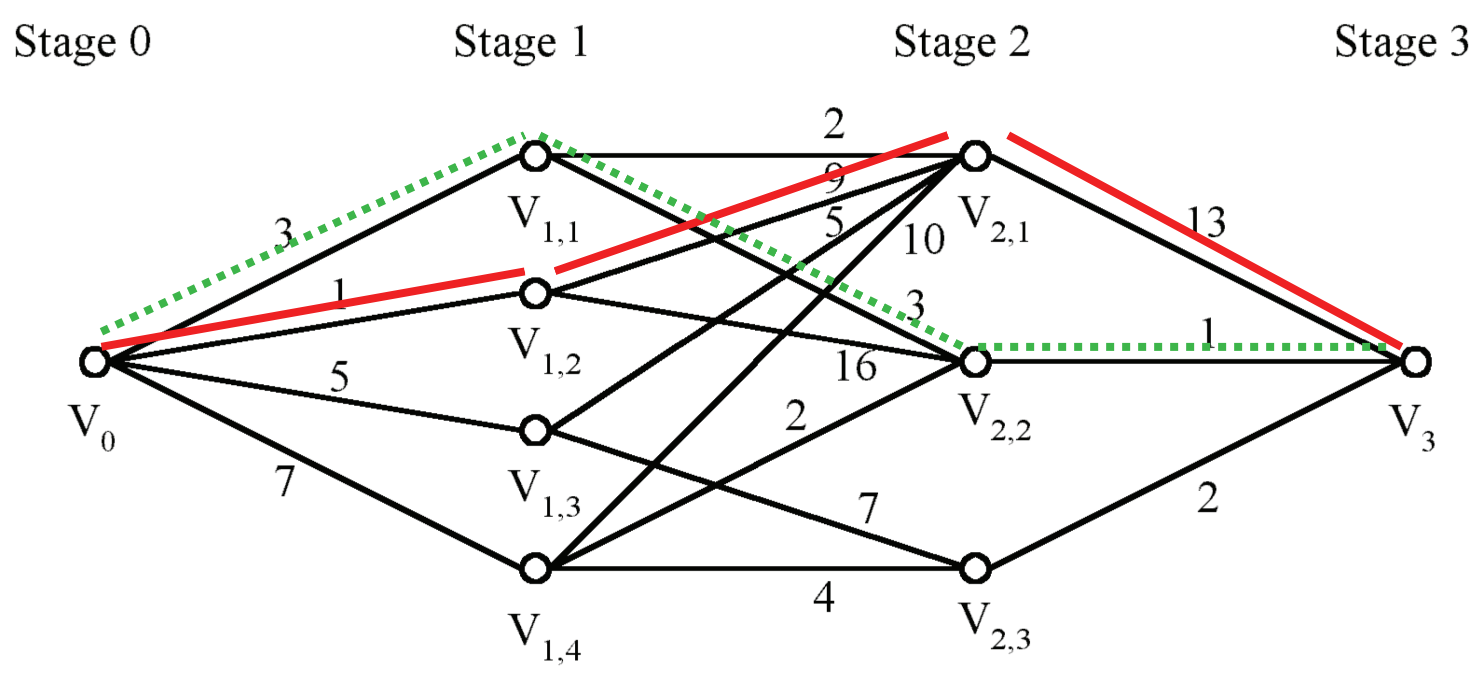
Shortest paths on a special graph

- Problem: Find a shortest path from v_0 to v_3.
- The greedy method can solve this problem.
- The shortest path: $1 + 2 + 4 = 7$.



Shortest paths on a multi-stage graph

- Problem: Find a shortest path from v_0 to v_3 in the multi-stage graph.

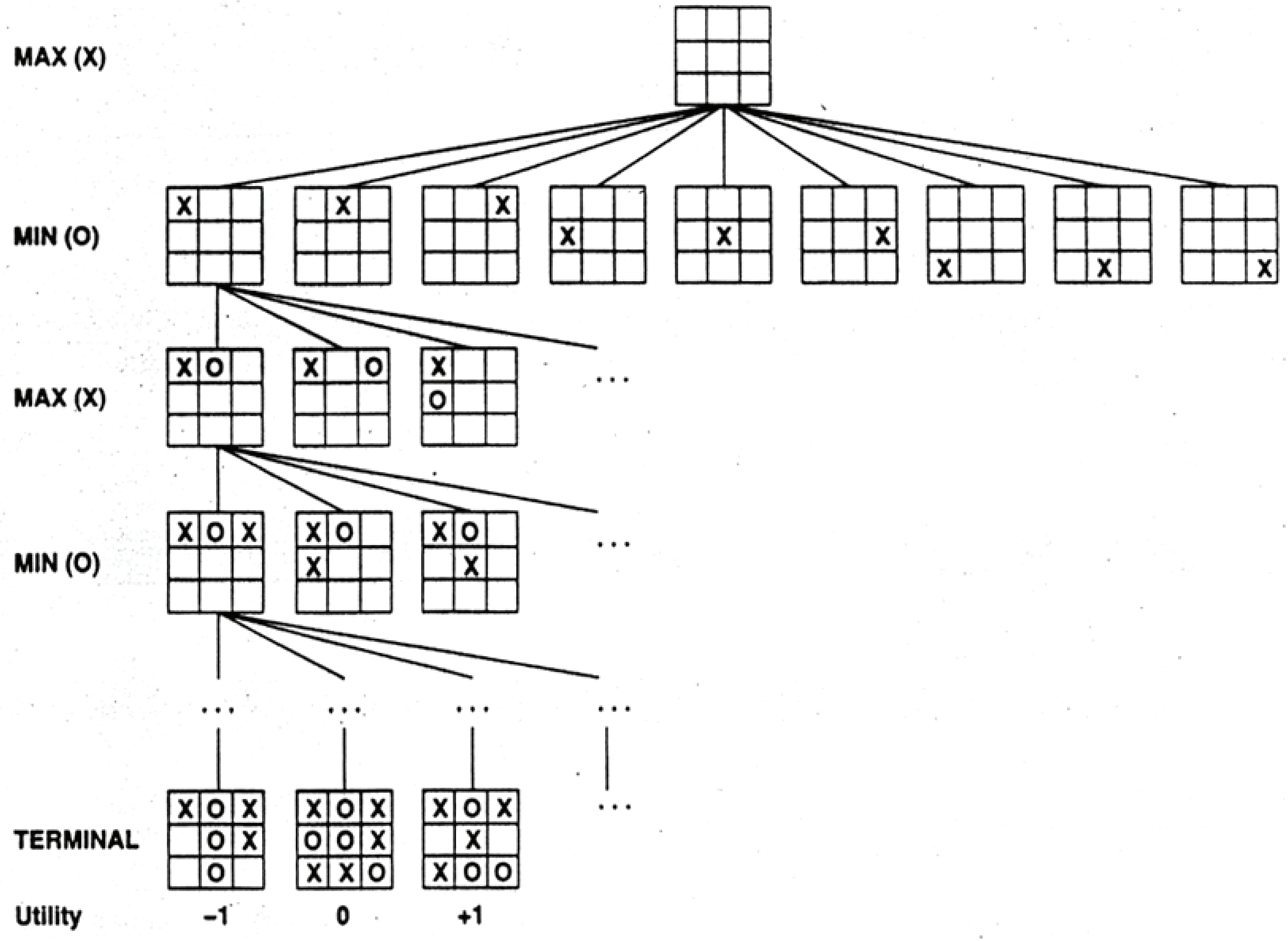


- Greedy method: $v_0 v_{1,2} v_{2,1} v_3 = 23$
- Optimal: $v_0 v_{1,1} v_{2,2} v_3 = 7$
- **The greedy method does not work.**

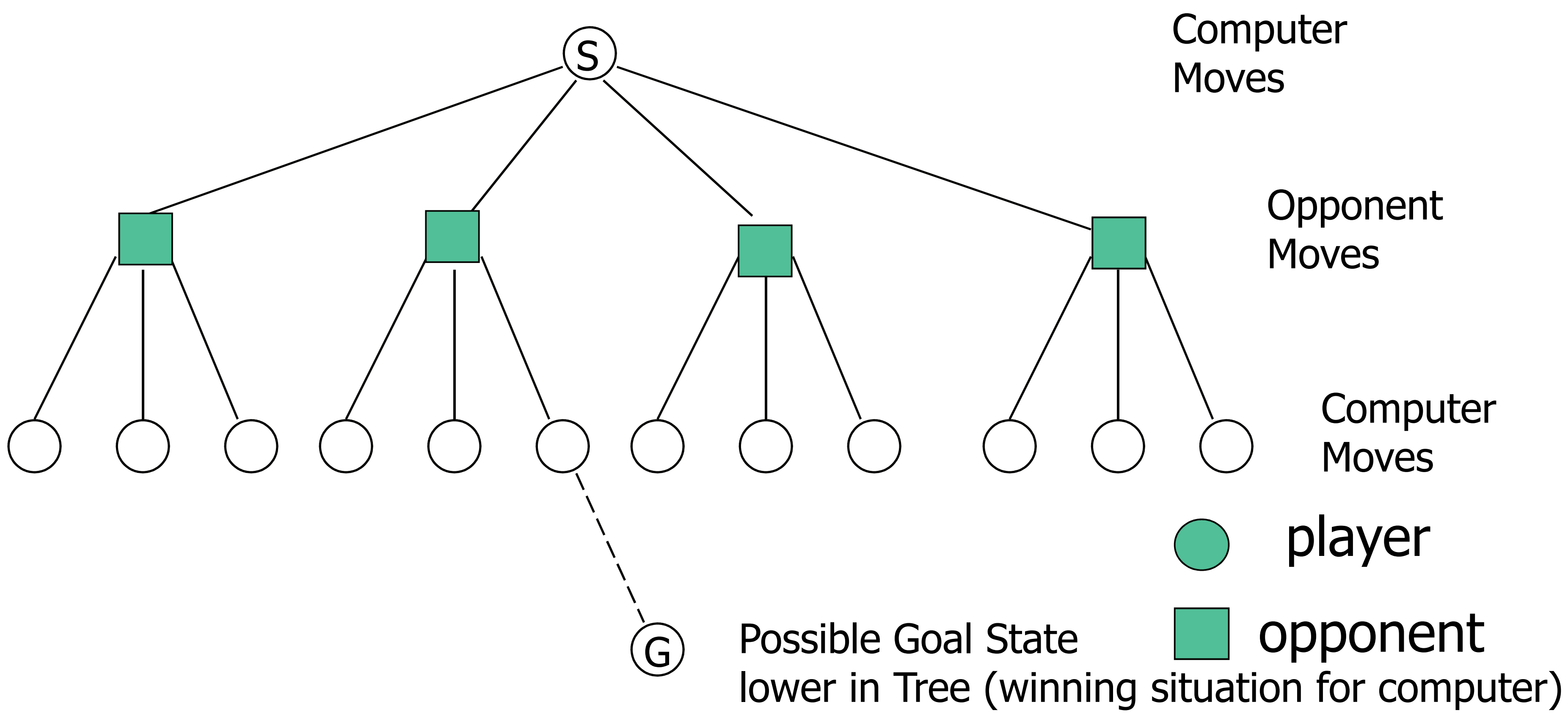
Game Playing & problem solving

- **Look ahead**.
  - Try to figure out most of the possible moves which he can make and then imagine how his opponent may react.
  - Must understand that his opponent will also look ahead.
  - When we make decisions, we often have to look ahead.
- **The greedy method**
  - **however, never does any work of looking ahead**.
  - **Greedy may fail to produce an optimal solution**.

Game Trees

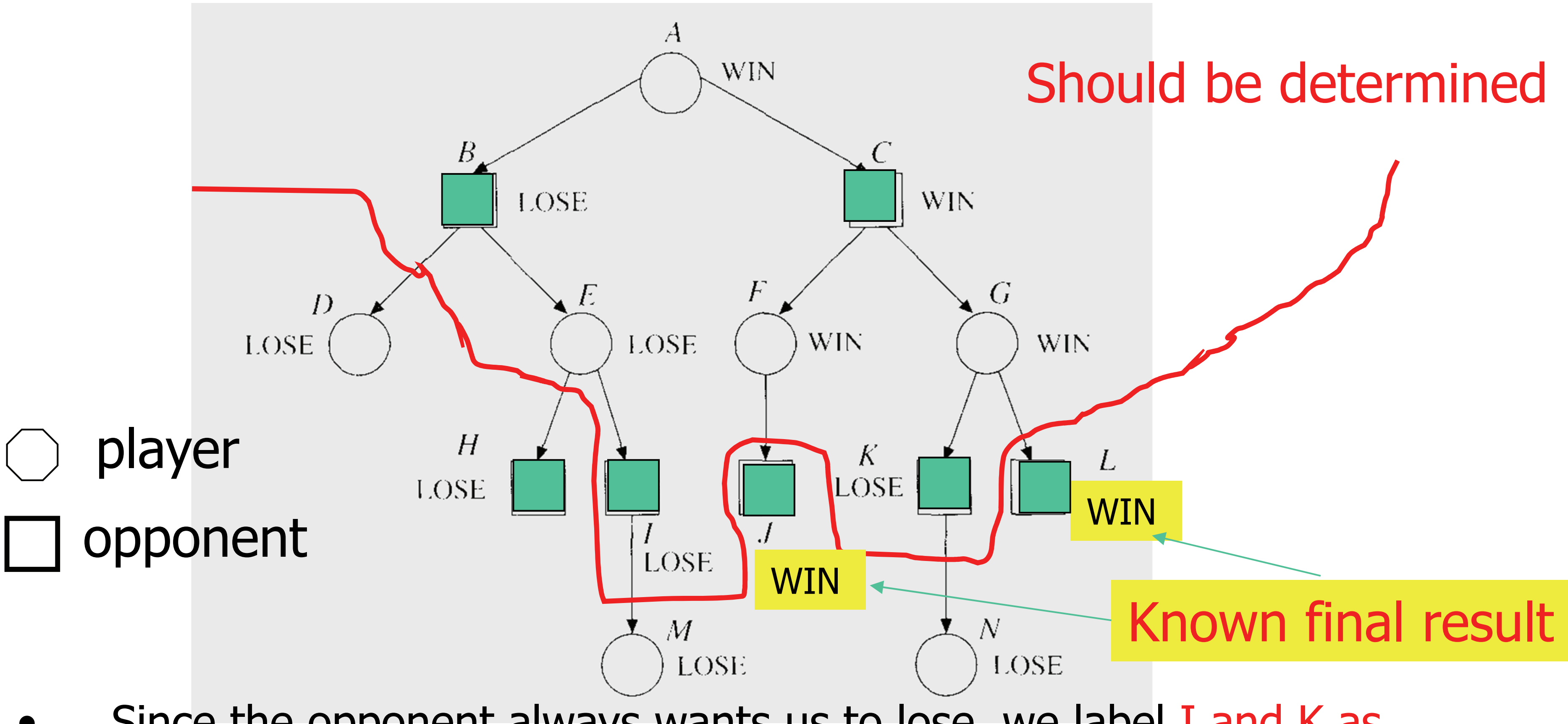


Game Tree Representation



- New aspect to search problem
  - there's an opponent we cannot control
  - how can we handle this?

End-Game Tree

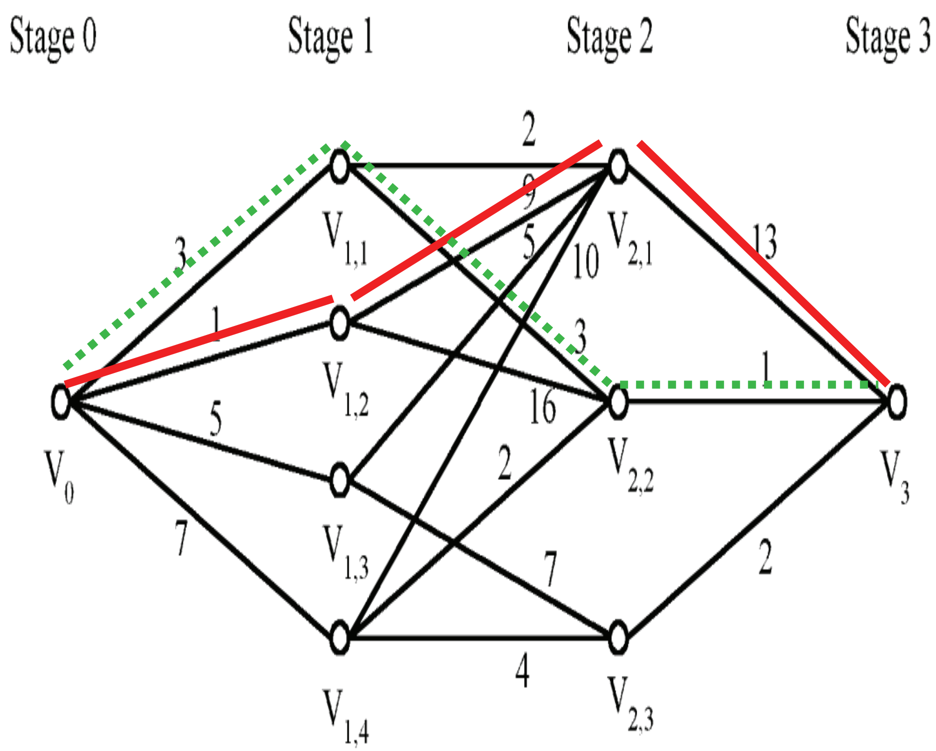


- Since the opponent always wants us to lose. we label I and K as LOSE: if any of these states is reached, we definitely will lose.
- Since we always want to win, we label F and G as WIN. Besides, we must label E as LOSE.
- Again, since the opponent always wants us to lose. we label B and C as LOSE and WIN respectively.
- Since we want to win. we label A as WIN.

Solution of the above problem

- $d_{min}(i,j)$: minimum distance between i and j.

$$d_{min}(v_0,v_3)=\min\begin{cases} 3+d_{min}(v_{1,1},v_3) \\ 1+d_{min}(v_{1,2},v_3) \\ 5+d_{min}(v_{1,3},v_3) \\ 7+d_{min}(v_{1,4},v_3) \end{cases}$$



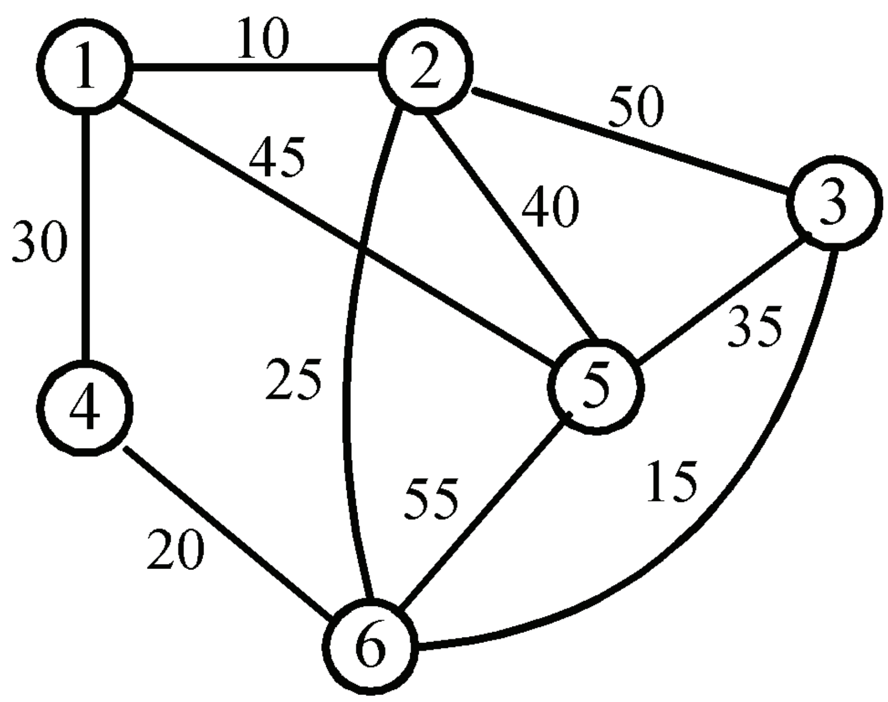
- This problem can be solved by the *dynamic programming method*.

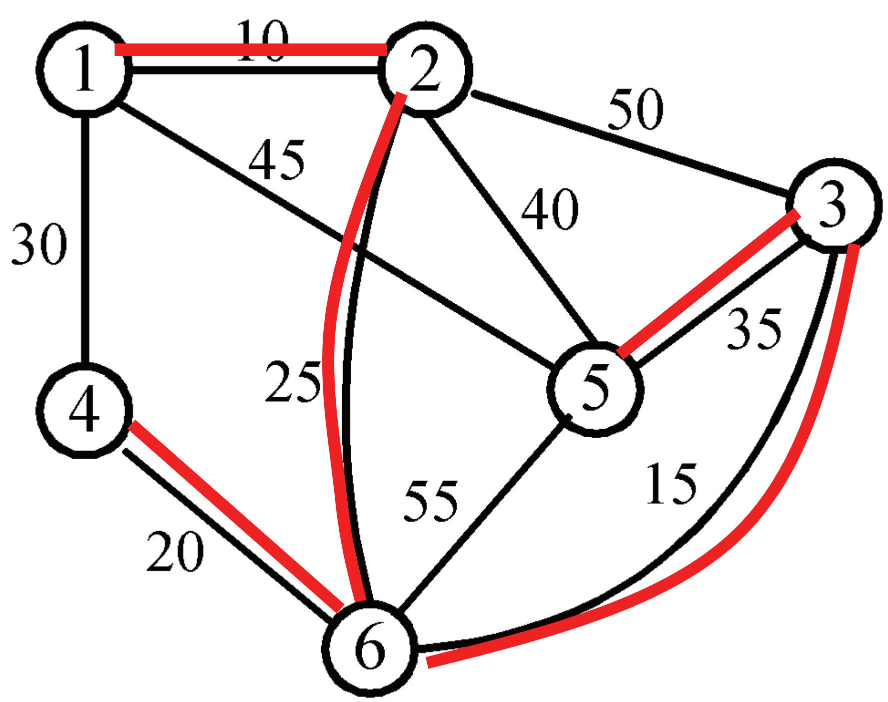
3-1 Minimum spanning trees (MST)

- It may be defined on Euclidean space points or on a graph.
- $G = (V, E)$: weighted connected undirected graph
- Spanning tree : $S = (V, T)$, $T \subseteq E$, undirected tree
- Minimum spanning tree (MST) : a spanning tree with the smallest total weight.

An example of MST

- A graph and one of its minimum costs spanning tree





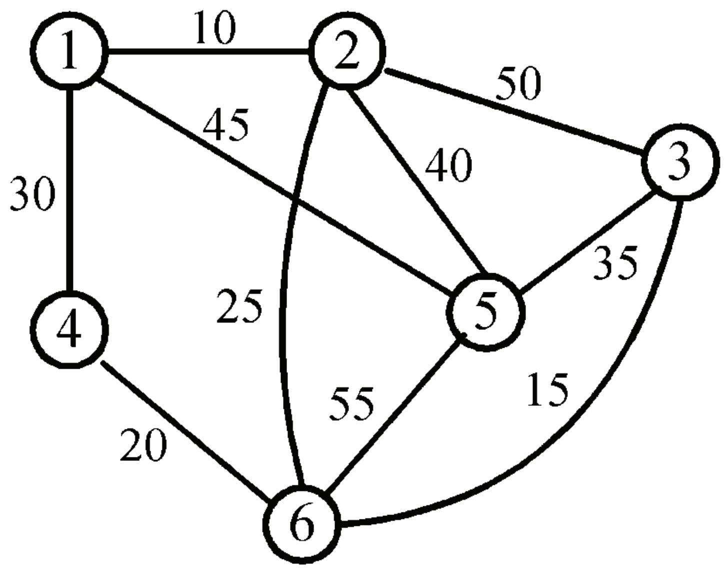
Kruskal's algorithm for finding MST

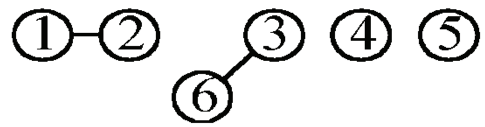
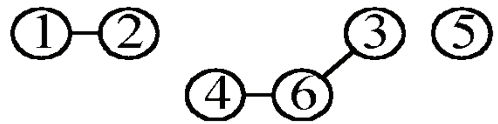
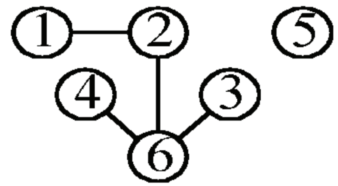
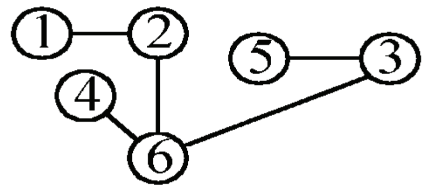
Step 1: Select the edge with the smallest weight edge from the set od edges. (may or may not sort all edges into non-decreasing order.)

Step 2: Add the next smallest weight edge to the forest if it will not cause a cycle. Discard the selected edge if otherwise.

Step 3: Stop if n-1 edges. Otherwise, go to Step 2.

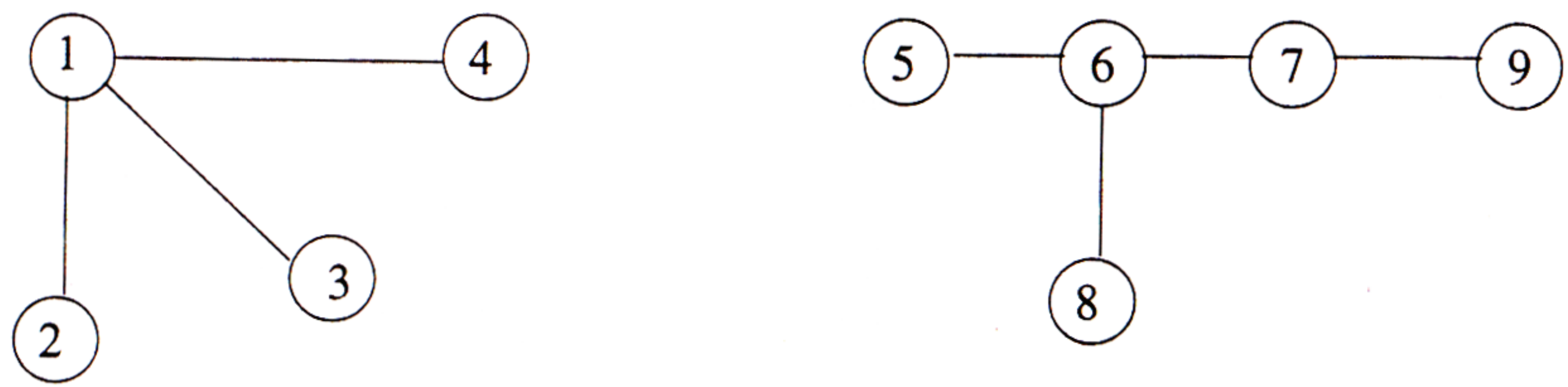
An example of Kruskal's algorithm



Edge	Cost	Spanning Forest
		
(1,2)	10	
(3,6)	15	
(4,6)	20	
(2,6)	25	
(1,4)	30	(reject)
(3,5)	35	

The details for constructing MST

- Use heap to maintain the smallest weight edge. (no need sorting)
- How do we check if a cycle is formed when a new edge is added?
  - By the SET and UNION method.
- A tree in the forest is used to represent a SET.
- If $(u, v) \in E$ and u, v are in the same set, then the addition of (u, v) will form a cycle.
- If $(u, v) \in E$ and $u \in S_1$, $v \in S_2$, then perform **UNION** of S_1 and S_2.
- To ***find*** an element in a set, find operation.
- Chapter 10 (amortized analysis) will show that both **union and find** operations will take **O(m)** steps.



A spanning forest

- If (3,4) is added, since 3 and 4 belong to the same set, a cycle will be formed.
- If (4,5) is added, since 4 and 5 belong to different sets, a new forest (1,2,3,4,5,6,7,8,9) is created.
- Thus we are always performing the union of two sets.

Time complexity

- Time complexity: $O(|E| \log|E|)$
  - Step 1: $O(|E| \log|E|)$
  - Step 2 & Step 3: $O(|E|\alpha(|E|,|V|))$

    Where α is the inverse of Ackermann's function.

Disjoint set 資料結構：

1. 一個維護所有 **disjoint dynamic sets** 組成的大集合 $S=\{S_1, S_2, \ldots, S_k\}$ 的資料結構。
2. 每個集合都被一個 **representative** 所代表，而 **representative** 是該集合中的某一個元素。
3. 此資料結構支援以下的指令：
   - **Make-Set(*x*)**: 創造一個新的集合 $\{x\}$
   - **Union(*x*, *y*):** 把兩個分別包含 x, y 的集合聯集起來
   - **Find-Set(*x*):** 傳回一個指標指向包含 x 的集合的 representative。

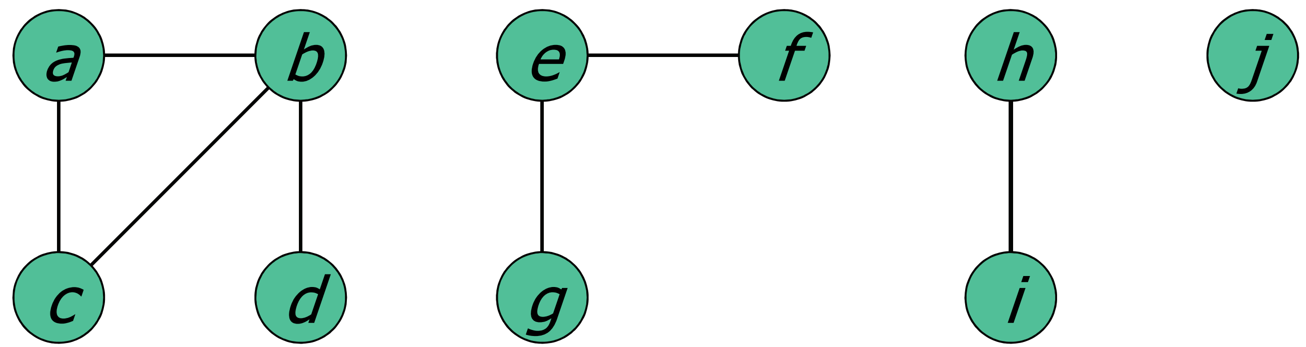
符號：

n: 所有 Make-Set 指令的總數。

m: 所有 Make-Set、Union 及 Find-Set 指令的總數

註： $m \geq n$ 且所有 Union 指令的總數最多不會超過 n-1。

disjoint-set 資料結構例子



(a)

Edge processed	Collection of disjoint sets									
initial sets	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$	$\{i\}$	$\{j\}$
(b,d)	$\{a\}$	$\{b,d\}$	$\{c\}$		$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$	$\{i\}$	$\{j\}$
(e,g)	$\{a\}$	$\{b,d\}$	$\{c\}$		$\{e,g\}$	$\{f\}$		$\{h\}$	$\{i\}$	$\{j\}$
(a,c)	$\{a,c\}$	$\{b,d\}$			$\{e,g\}$	$\{f\}$		$\{h\}$	$\{i\}$	$\{j\}$
(h,i)	$\{a,c\}$	$\{b,d\}$			$\{e,g\}$	$\{f\}$		$\{h,i\}$		$\{j\}$
(a,b)	$\{a,b,c,d\}$				$\{e,g\}$	$\{f\}$		$\{h,i\}$		$\{j\}$
(e,f)	$\{a,b,c,d\}$				$\{e,f,g\}$			$\{h,i\}$		$\{j\}$
(b,c)	$\{a,b,c,d\}$				$\{e,f,g\}$			$\{h,i\}$		$\{j\}$

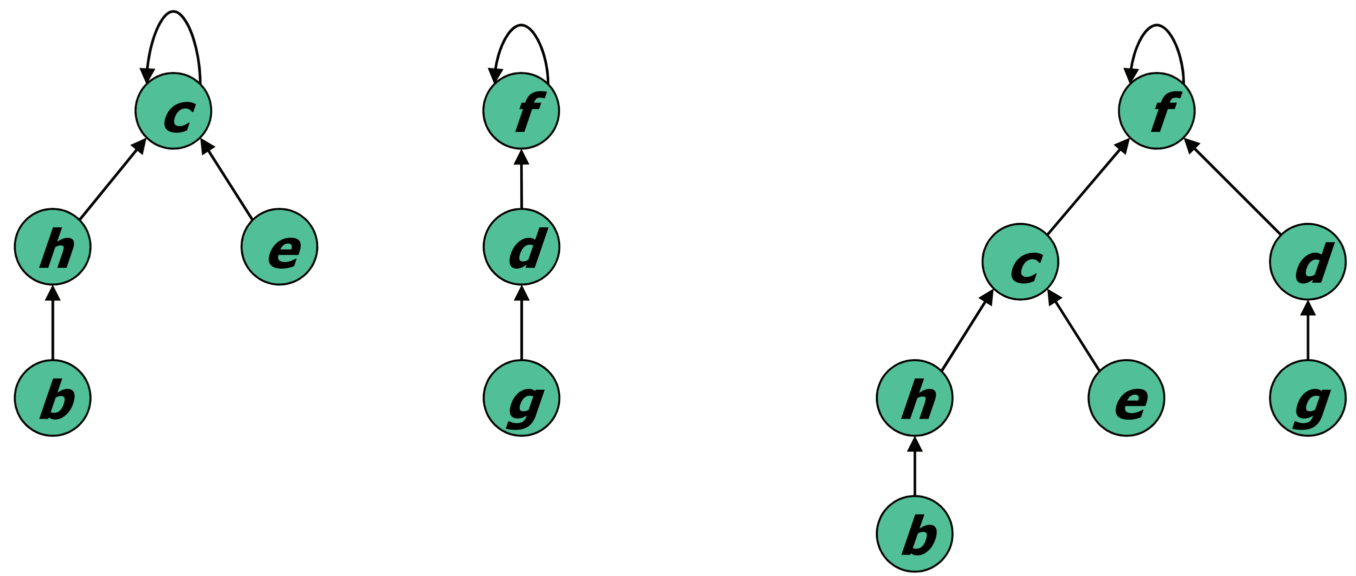
(b)

- $m = 2n$-1 個指令耗時 $O(n^2)$

Operation	Number of objects updated	
MAKE-SET(x_1)	**1**	
MAKE-SET(x_2)	**1**	
⋮	⋮	$O(n)$
MAKE-SET(x_n)	**1**	
UNION(x_1, x_2)	**1**	
UNION(x_2, x_3)	**2**	
UNION(x_3, x_4)	**3**	$O(1+2+...+n)$ $=O(n^2)$
⋮	⋮	
UNION(x_{n-1}, x_n)	***n*-1**	

- 每個指令的 amortized time 為 $O(n)$。

Disjoint-set forests



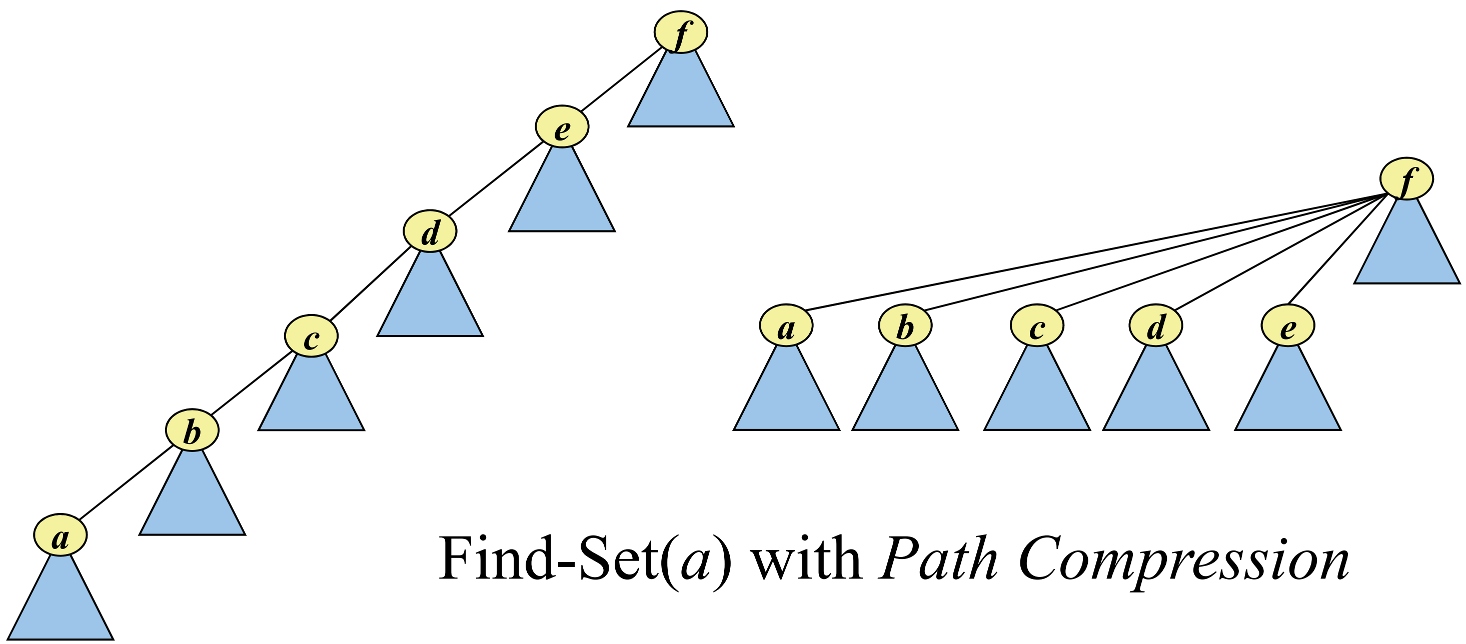
- Tree 的 root 為 representative。
- Make-Set(x): 耗時 $O(1)$
- Find-Set(x): 耗時 $O(h)$，h 為包含 x 的 tree 的高度。
  (找 $x \rightarrow$ root 的路徑)
- Union(x, y): x 的 root 指向 y 的 root。
  $\rightarrow$ 耗時 $O(h)$。

一些增進執行速度的技巧

1. ***Union by rank***: 由較小的 tree 的 root 指向較大的 tree 的 root (依照 tree 的高度區分)。

   ***rank*[*x*]**: x 的高度(由 x 到一 descendant leaf 的最長路徑的邊的數量)

2. ***Path compression***: 在執行 Find-Set(x) 對過程中，把所有find path 上的點指向 root。(不改變任何 rank 的值)



Find-Set(a) with *Path Compression*

Effect of the heuristics on the running time

1. 若只使用了 Union-by-rank， 簡單便可證明總共耗時 $O(m\lg n)$。
2. 若只使用了 Path-compression 可以證明 (在此不證) 總耗時為

$$\Theta(n + f \cdot (1+\log_{2+f/n} n)),$$

其中 n 是 Make-Set 指令的數量，且 f 是 Find-Set 指令的數量。

3. 當兩者皆用上時，在最壞的情況下共需耗時 $O(m\alpha(m,n))$。

註： $\alpha(m,n)$ 是 Ackermann's function 的反函數，其成長速率非常慢。在所有可能的應用中，$\alpha(m,n) \le 4$，因此在所有實際的應用上，我們可以把執行時間視為和 m 成正比。

Ackermann's function and its inverse

- 令 $g(i) = \left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}i$ 為 repeated exponentiation。(即 $g(4) = 2^{2^{2^{2}}}$)
- 函數 $\lg^* n = \min\{i \geq 0: \lg^{(i)} n \leq 1\}$ 本質上為 $g(i)$ 的反函數 (即 $\lg^* 2^{2^{2^{2}}} = 5$)

註： $\lg^* g(i) = i+1$。

- The Ackermann's function: 對整數 $i, j \geq 1$，

$$
\begin{array}{ll}
A(1, j) = 2^{j} & \text{for } j \geq 1 \\
A(i, 1) = A(i\text{-}1, 2) & \text{for } i \geq 2 \\
A(i, j) = A(i\text{-}1, A(i, j\text{-}1)) & \text{for } i, j \geq 2
\end{array}
$$

	$j = 1$	$j = 2$	$j = 3$	$j = 4$
$i = 1$	2^{1}	2^{2}	2^{3}	2^{4}
$i = 2$	2^{2}	$2^{2^{2}}$	$2^{2^{2^{2}}}$	$2^{2^{2^{2^{2}}}}$
$i = 3$	$2^{2^{2}}$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}16$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}16$	$\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}\left.2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\right\}16$

- 因對所有的 $j \ge 1$，$A(2, j) = 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}} \Big\} j = g(j)$ 所以
  當 $i \ge 2$，$A(i, j) \ge g(j)$。

- Ackermann's function 的反函數:

  $\alpha(m, n) = \min\{i \ge 1 : A(i, \lfloor m/n \rfloor) > \lg n\}$。

- $A(4, 1) = A(3, 2) = g(16) \approx 10^{80}$

- 由於 $\lg n < 10^{80}$，且 $A(4, \lfloor m/n \rfloor) \geq A(4, 1) \approx 10^{80}$，我們得知在實際應用上，$\alpha(m, n) \leq 4$。

- 在實際應用上，$\lg^* n \leq 5$（$n \leq 2^{65536}$）。

- 由於當 $i \geq 2$，$A(i, j) \geq g(j)$，因此 $\alpha(m, n) = O(\lg^* n)$。

註：

$$\xrightarrow{\text{increasing}}$$

$1, \quad \alpha, \quad \lg^*, \quad \lg\lg, \quad \lg, \quad n^{1/2}, \quad n, \quad n^k, \quad 2^n, \quad g, \quad A$

Ackermann's function

$$A(p, q) = \begin{cases} 2q, & p = 0 \\ 0,\ q = 0, & p \geq 1 \\ 2,\ p \geq 1, & q = 1 \\ A(p-1, A(p, q-1)), & p \geq 1,\ q \geq 2 \end{cases}$$

$$\Rightarrow A(p, q+1) > A(p, q),\quad A(p+1, q) > A(p, q)$$

$$A(3,4) = 2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}} \left.\vphantom{2^{2^{2^{2}}}}\right\} \text{65536 two's}$$

Expansion

$$
\begin{aligned}
A(1,2) &= A(0, A(1,1)) \\
&= A(0, A(0, A(1,0))) \\
&= A(0, A(0, A(0,1))) \\
&= A(0, A(0,2)) \\
&= A(0,3) \\
&= 4
\end{aligned}
$$

To demonstrate how $A(4,3)$'s computation results many steps and in a large number:

$$
\begin{aligned}
A(4,3) &= A(3, A(4,2)) \\
&= A(3, A(3, A(4,1))) \\
&= A(3, A(3, A(3, A(4,0)))) \\
&= A(3, A(3, A(3, A(3,1)))) \\
&= A(3, A(3, A(3, A(2, A(3,0))))) \\
&= A(3, A(3, A(3, A(2, A(2,1))))) \\
&= A(3, A(3, A(3, A(2, A(1, A(2,0)))))) \\
&= A(3, A(3, A(3, A(2, A(1, A(1,1)))))) \\
&= A(3, A(3, A(3, A(2, A(1, A(0, A(1,0))))))) \\
&= A(3, A(3, A(3, A(2, A(1, A(0, A(0,1)))))))
\end{aligned}
$$

Ackermann function

Values of $A(m, n)$

$m \backslash n$	0	1	2	3	4	n
0	1	2	3	4	5	$n+1$
1	2	3	4	5	6	$n+2$
2	3	5	7	9	11	$2n+3$
3	5	13	29	61	125	$2^{(n+3)}-3$
4	13	65533	$2^{65536}-3$	$2^{2^{65536}}-3$	$A(3, A(4, 3))$	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n+3 \text{ twos}}-3$
5	65533	$A(4, 65533)$	$A(4, A(5, 1))$	$A(4, A(5, 2))$	$A(4, A(5, 3))$	$A(4, A(5, n-1))$
6	$A(5, 1)$	$A(5, A(5, 1))$	$A(5, A(6, 1))$	$A(5, A(6, 2))$	$A(5, A(6, 3))$	$A(5, A(6, n-1))$

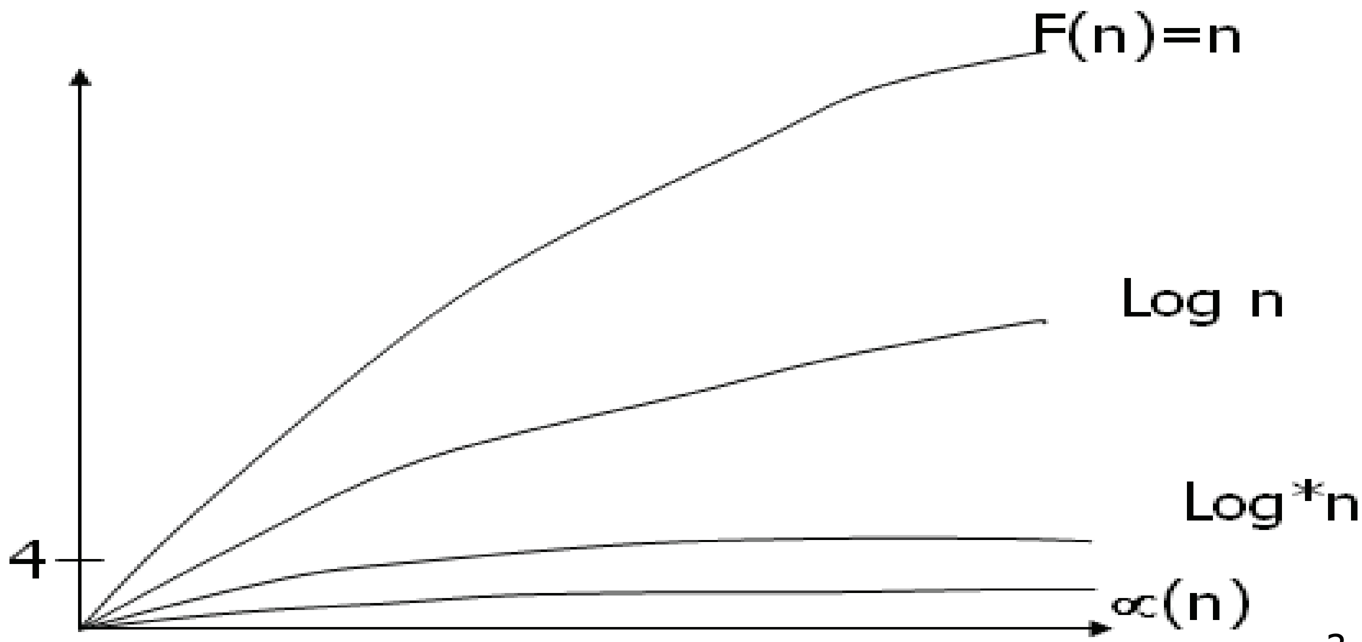
Inverse of Ackermann's function

- $\alpha(m, n) = \min\{Z \geq 1 \mid A(Z, 4\lceil m/n \rceil) > \log_2 n\}$

Practically, $A(3,4) > \log_2 n$

$\Rightarrow \alpha(m, n) \leq 3$

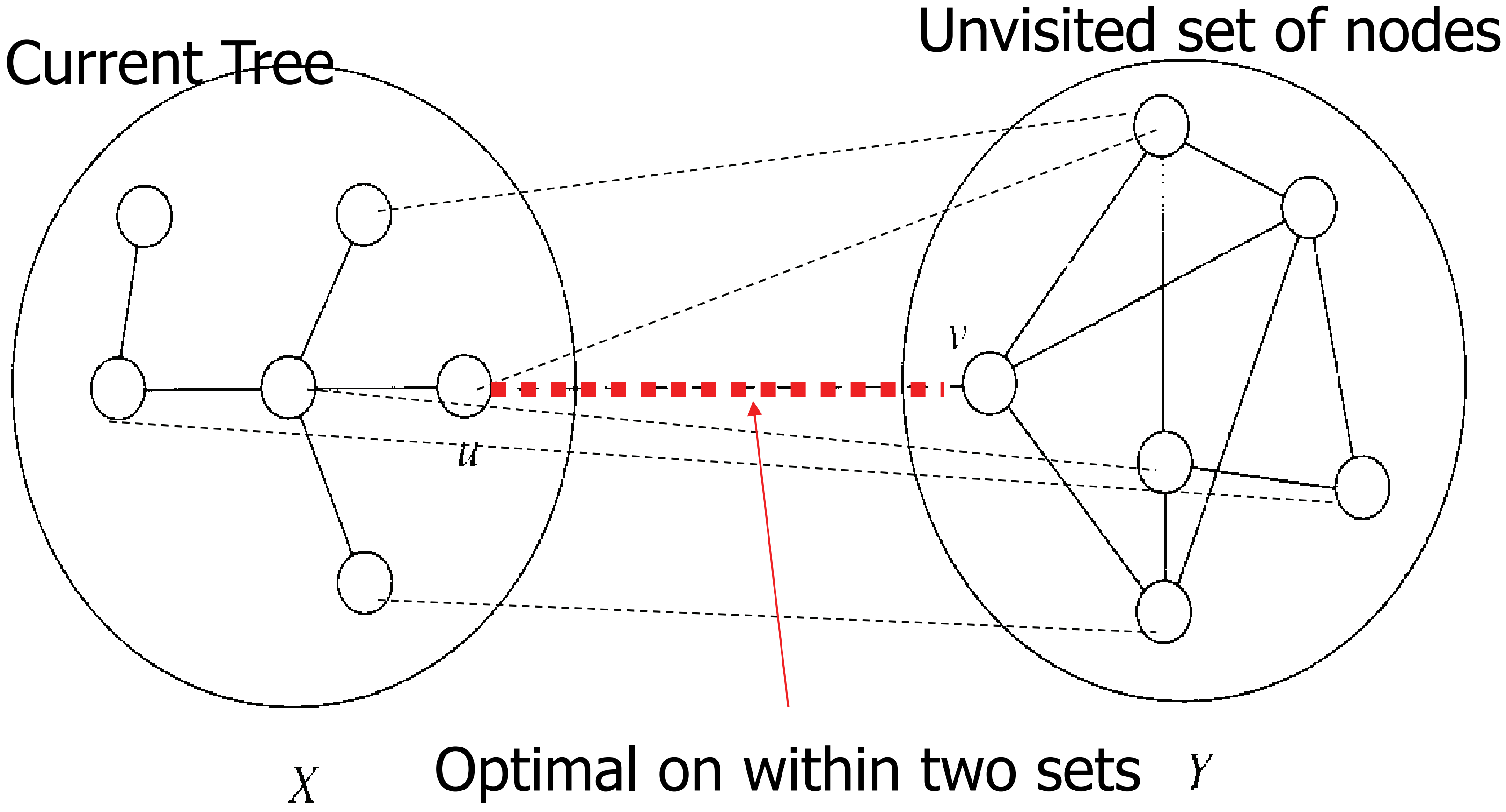
$\Rightarrow \alpha(m, n)$ is almost a constant.



3.2 Prime's MST algorithm

- Let **X** denote the set of vertices contained in the partially constructed minimal spanning tree.
- Let **Y=V-X**.
- The next edge (u, v) to be added is an edge between X and Y ($u \in X$ and $v \in Y$) with the **smallest weight**.
- V is added to X.
- **Can start with any vertex**.

FIGURE 3–10 An illustration of Prim's method.



Prim's algorithm for finding MST

Step 1: $x \in V$, Let $A = \{x\}$, $B = V - \{x\}$.

Step 2: Select $(u, v) \in E$, $u \in A$, $v \in B$ such that (u, v) has the smallest weight between A and B.

Step 3: Put (u, v) in the tree. $A = A \cup \{v\}$, $B = B - \{v\}$

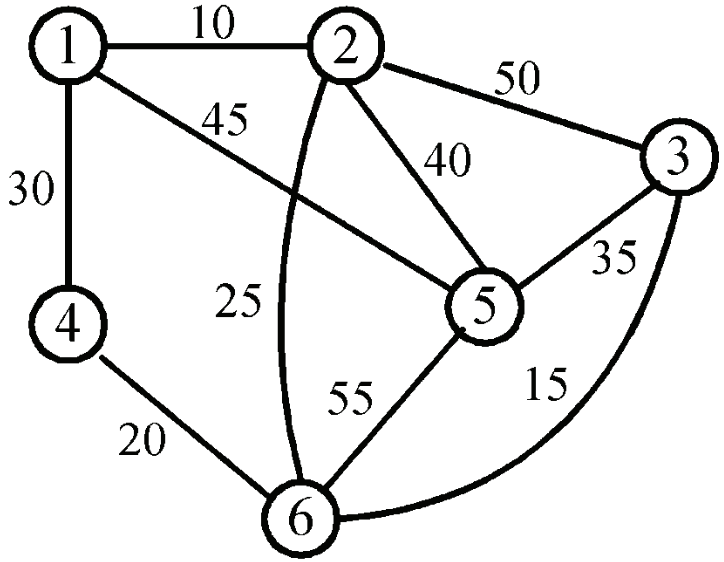
Step 4: If $B = \varnothing$, stop; otherwise, go to Step 2.

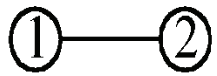
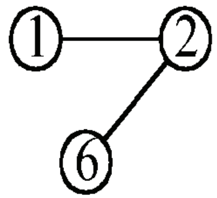
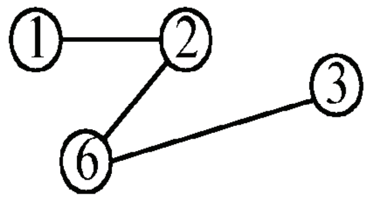
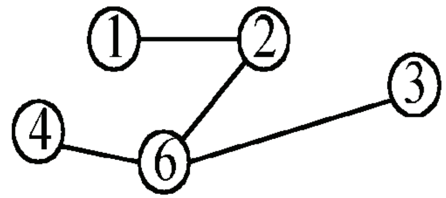
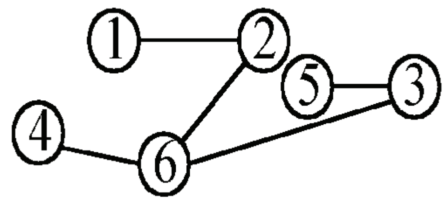
- Time complexity : $O(n^2)$, $n = |V|$.

  (see the example on the next page)

No test the cycle

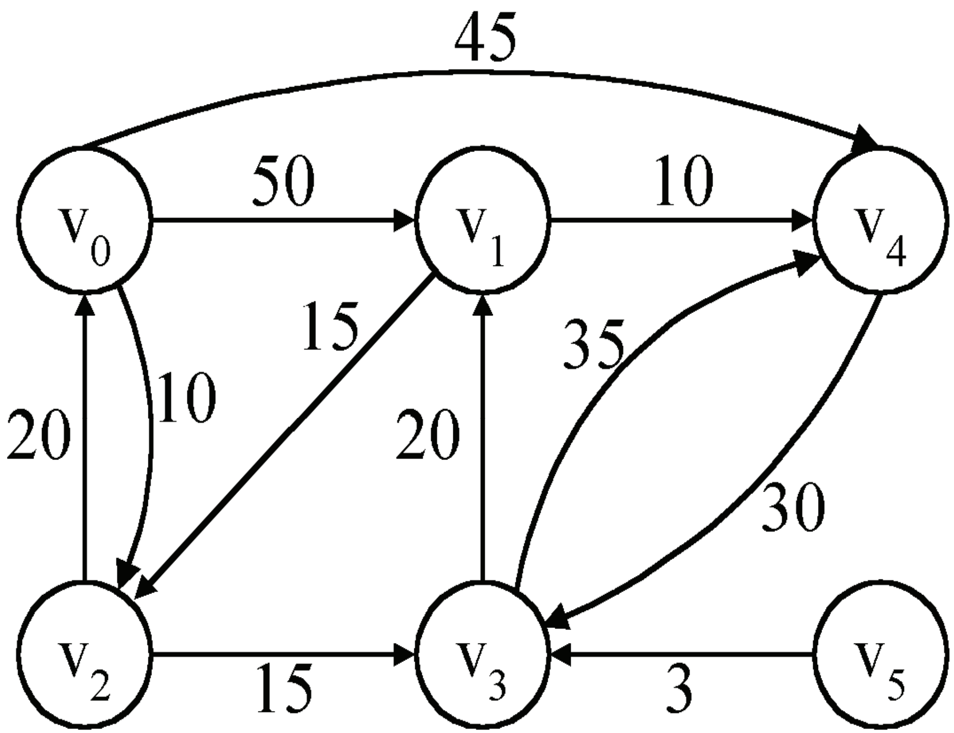
An example for Prim's algorithm



Edge	Cost	Spanning tree
(1,2)	10	
(2,6)	25	
(3,6)	15	
(6,4)	20	
(3,5)	35	

3-3 The single-source shortest path problem

- shortest paths from v_0 to all destinations



(a)

	Path	Length
1)	v_0v_2	10
2)	$v_0v_2v_3$	25
3)	$v_0v_2v_3v_1$	45
4)	v_0v_4	45

(b)

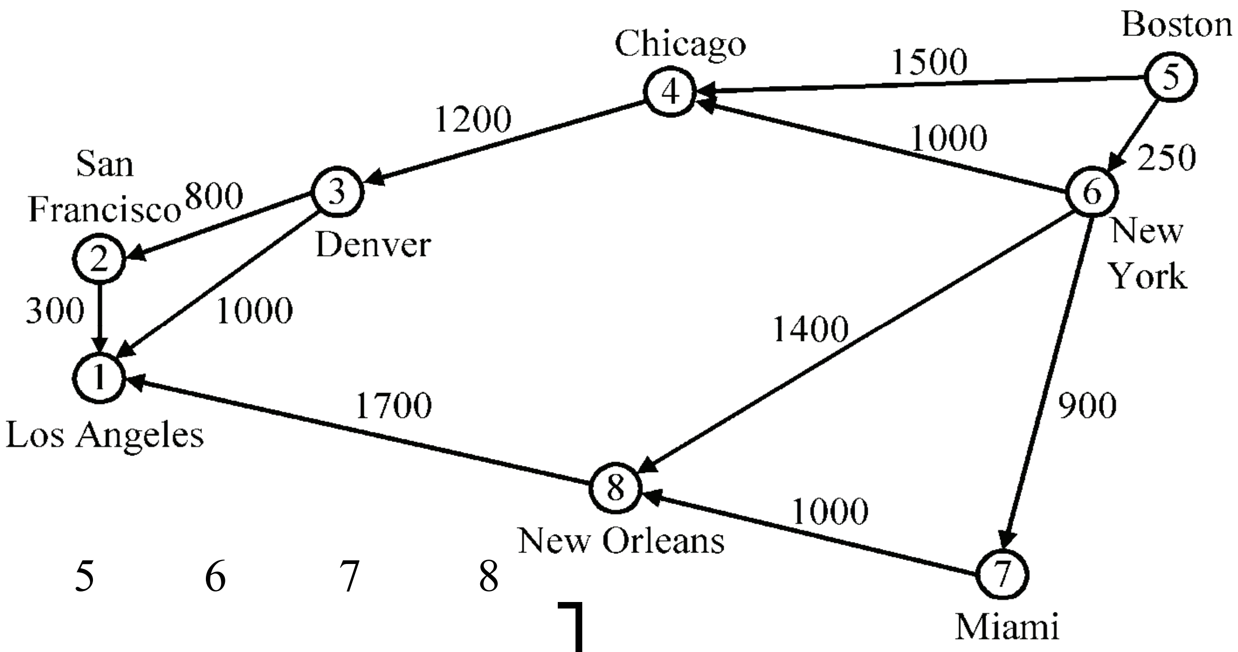
Algorithm 3–4 □ Dijkstra's algorithm to generate single-source shortest paths

Input: A directed graph $G = (V, E)$ and a source vertex v_0. For each edge $(u, v) \in E$, there is a non-negative number $c(u, v)$ associated with it. $|V| = n + 1$.

Output: For each $v \in V$, the length of a shortest path from v_0 to v.

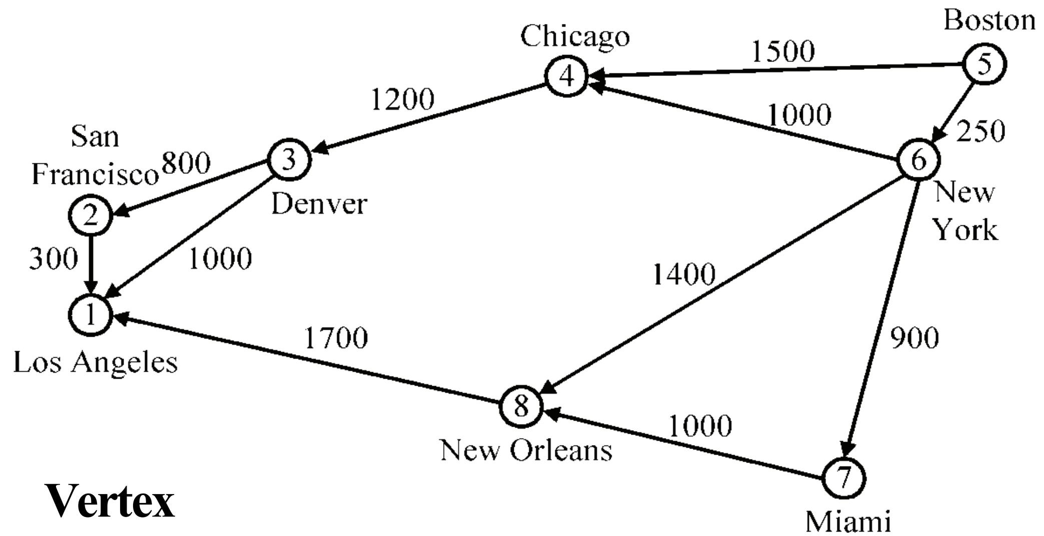
```
S := {v0}
For i := 1 to n do
Begin
    If (v0, vi) ∈ E then
        L(vi) := c(v0, vi)
    else
        L(vi) := ∞
End
For i := 1 to n do
Begin
    Choose u from V − S such that L(u) is the smallest
    S := S ∪ {u} (* Put u into S*)
    For all w in V − S do
        L(w) := min(L(w), L(u) + c(u, w))
End
```

Dijkstra's algorithm



Cost adjacency matrix. All entries not shown are $+\infty$.

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\left[\begin{array}{cccccccc}
0 & & & & & & & \\
300 & 0 & & & & & & \\
1000 & 800 & 0 & & & & & \\
 & & 1200 & 0 & & & & \\
 & & & 1500 & 0 & 250 & & \\
 & & & 1000 & & 0 & 900 & 1400 \\
 & & & & & & 0 & 1000 \\
1700 & & & & & & & 0
\end{array}\right]
\end{array}
$$



Iteration	S	Vertex Selected	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
Initial		----								
1	5	6	$+\infty$	$+\infty$	$+\infty$	1500	0	250	$+\infty$	$+\infty$
2	5,6	7	$+\infty$	$+\infty$	$+\infty$	1250	0	250	1150	1650
3	5,6,7	4	$+\infty$	$+\infty$	$+\infty$	1250	0	250	1150	1650
4	5,6,7,4	8	$+\infty$	$+\infty$	2450	1250	0	250	1150	1650
5	5,6,7,4,8	3	3350	$+\infty$	2450	1250	0	250	1150	1650
6	5,6,7,4,8,3	2	3350	3250	2450	1250	0	250	1150	1650
	5,6,7,4,8,3,2		3350	3250	2450	1250	0	250	1150	1650

- Time complexity : $O(n^2)$

The longest path problem*

- Can we use Dijkstra's algorithm to find the longest path from a starting vertex to an ending vertex in an acyclic directed graph?
- There are 3 possible ways to apply Dijkstra's algorithm:
  - Directly use "max" operations instead of "min" operations.
  - Convert all positive weights to be negative. Then find the shortest path.
  - Give a very large positive number M. If the weight of an edge is w, now M-w is used to replace w. Then find the shortest path.
- All these 3 possible ways would not work!
- **NP-complete problem**

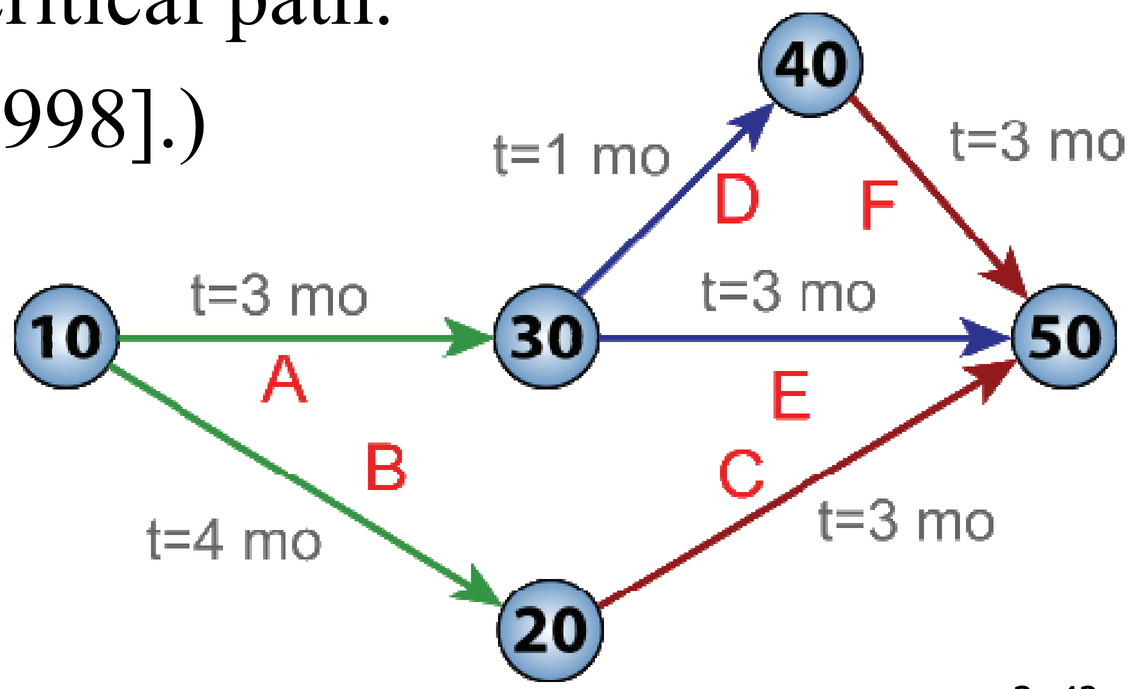
CPM for the longest path problem

- The longest path(critical path) problem can be solved by the critical path method (CPM) :

Step 1:Find a topological ordering.

Step 2: Find the critical path.

(see [Horiwitz 1998].)



Linear Merge Algorithm

- We are given two sorted lists L_1 and L_2, $L_1 = (a_1, a_2, \ldots, a_{n1})$ and $L_2 = (b_1, b_2, \ldots b_{,n2})$. L_1 and L_2 can be merged into one sorted list by applying the linear merge

Input: Two sorted lists, $L_1 = (a_1, a_2, \ldots, a_{n1})$ and $L_2 = (b_1, b_2, \ldots, b_{n2})$.

Output: A sorted list consisting of elements in L_1 and L_2.

```
Begin
  i : = 1          j : = 1
 do
  compare a_i and b_j
  if a_i > b_j then output b_j and j : = j + 1
              else  output a_i and i : = i + 1
 while (i ≤ n_1 and j ≤ n_2)
 if i > n_1 then output b_j , b_{j+1} , … , b_{n2} ,
            else  output a_i , a_{i+1} , … , a_{n1} .
End.
```

The worst case of merge sort

L_1: 1 3 5

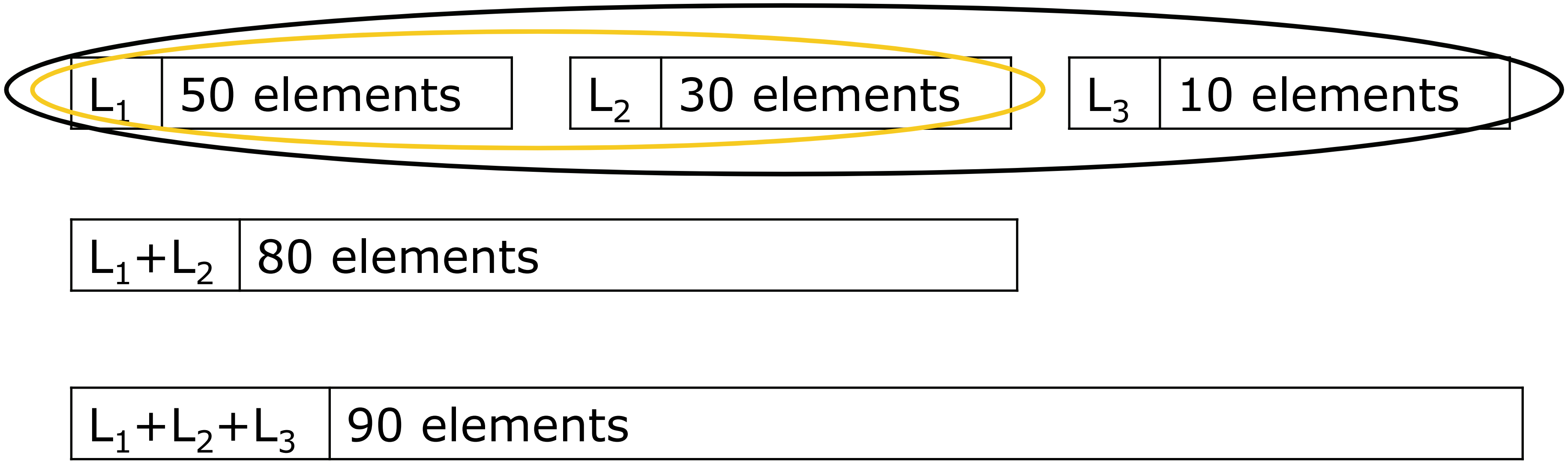
L_2: 2 4 6

The number of comparison required is m+n-1

3.4 The 2-way merging problem

- # of comparisons required for the linear 2-way merge algorithm is $m_1 + m_2 - 1$ where m_1 and m_2 are the lengths of the two sorted lists respectively.
- The problem: There are **n sorted lists**, each of length m_i. What is the optimal sequence of merging process to merge these n lists into one sorted list ?
- 50, 30, 10
- (1) 50+30-1=79, 80+10-1=89, total=79+89=168
- (2) 30+10-1=39, 40+50-1=89, total=39+89=128

2-WAY MERGE



$$\left.\begin{array}{l} 50+30-1 = 79 \\ 80+10-1 = 89 \end{array}\right\}$$ the number of comparisons required in this merging sequence is 168

2-WAY MERGE

L_1	50 elements	L_2	30 elements	L_3	10 elements

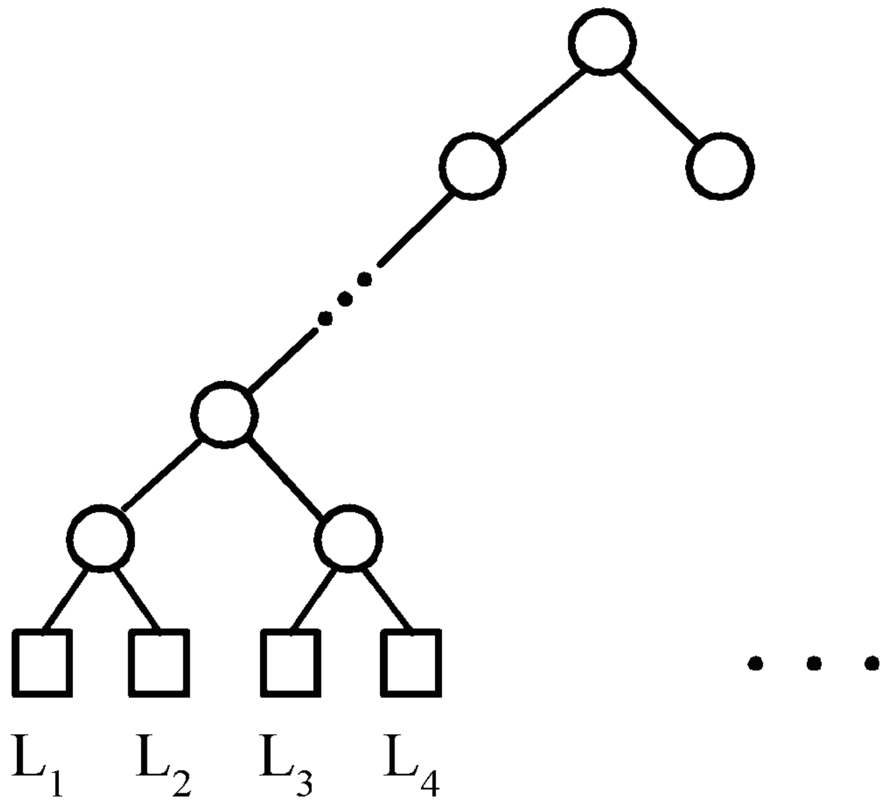
L_2+L_3	40 elements

$L_2+L_3+L_1$	90 elements

$$\left.\begin{array}{l} 30+10-1 = 39 \\ 40+50-1 = 89 \end{array}\right\}$$ the number of comparisons required is only 128

Extended binary trees

- Extended Binary Tree Representing a 2-way Merge



An Example Of Optimal 2-way Merge Problem

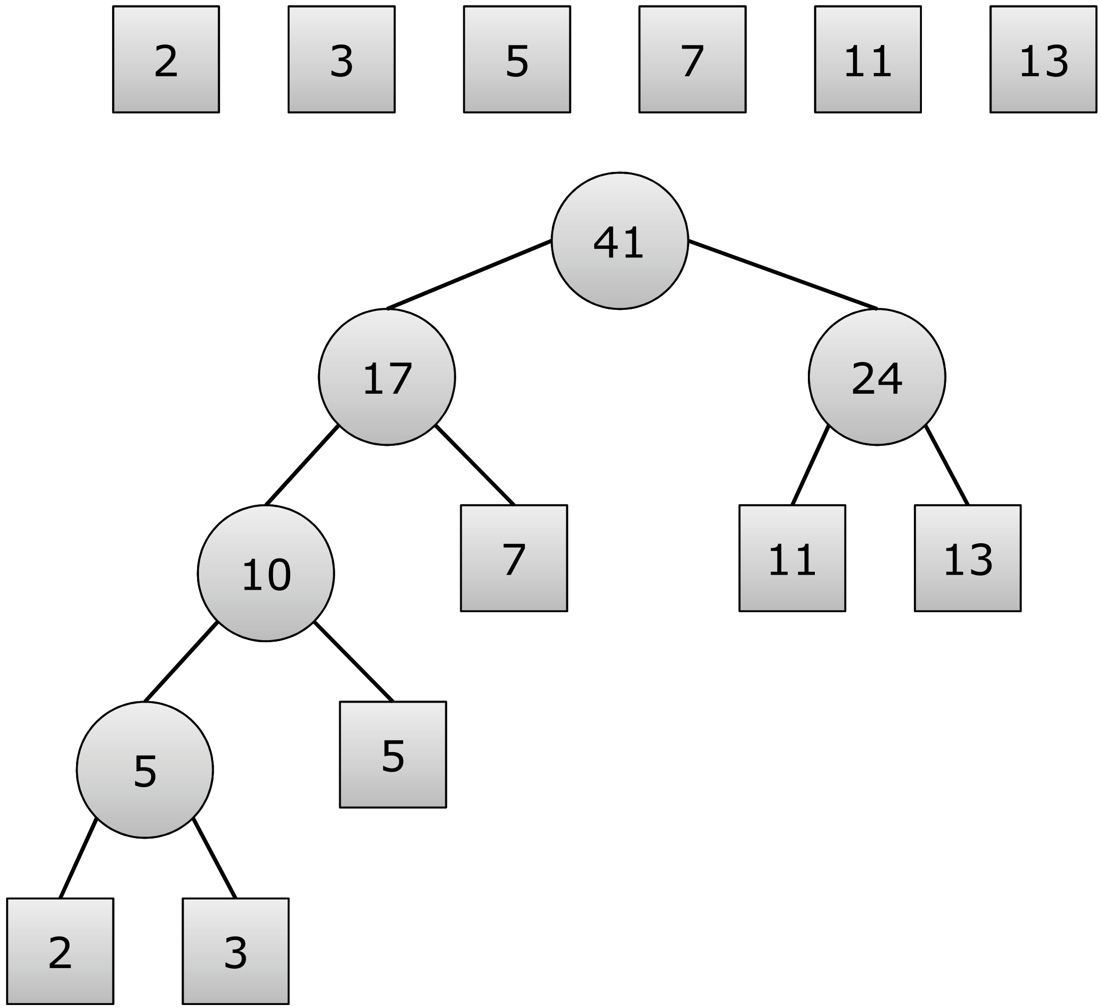
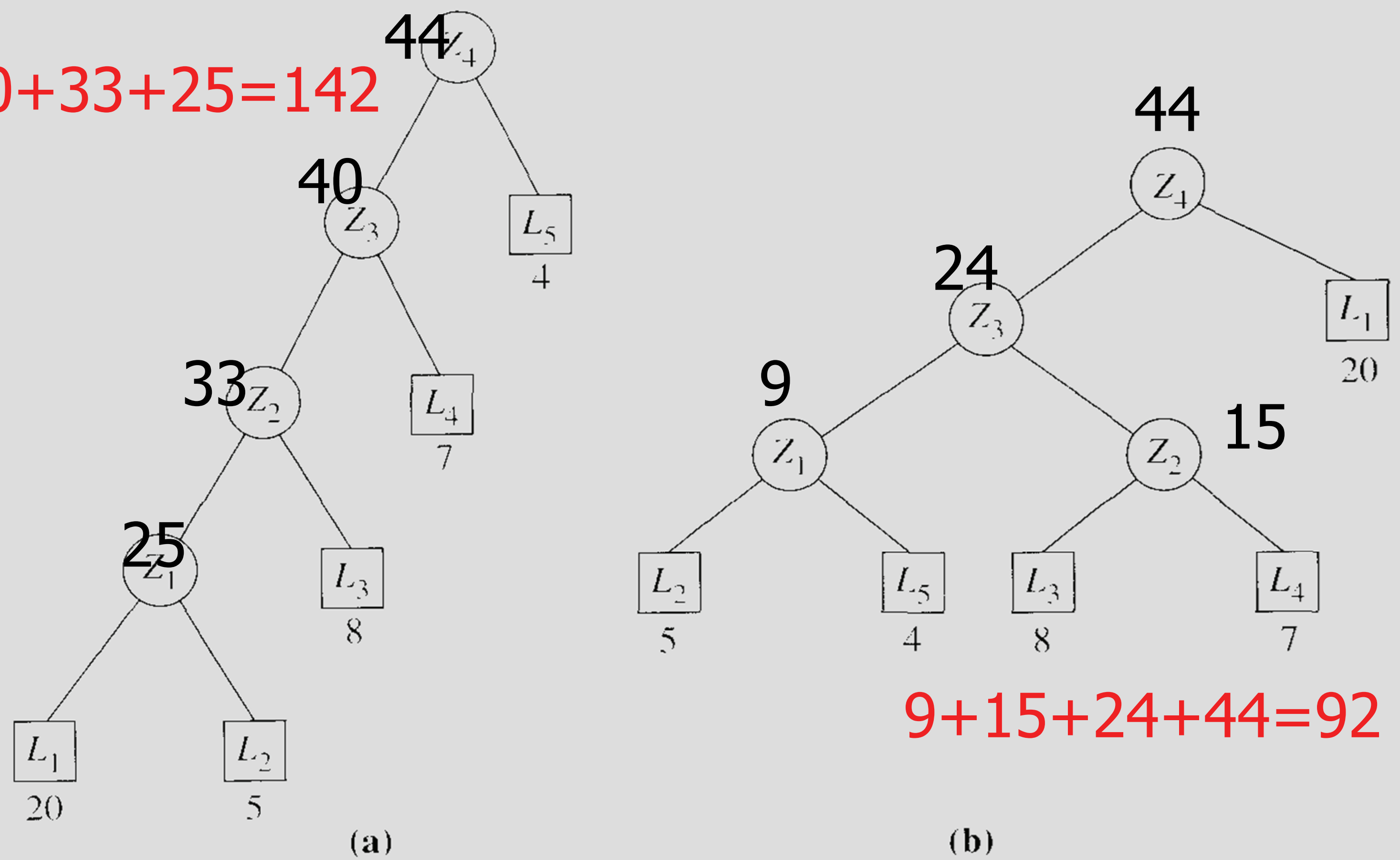


FIGURE 3–19 Different merging sequences.



44+40+33+25=142

9+15+24+44=92

Use n+m instead of n+m-1.

Optimal 2-way merge tree

Input: m sorted lists, L_i, $i = 1, 2, \ldots, m$, each L_i consisting of n_i elements.

Output: An optimal 2-way merge tree.

Step 1: Generate m trees, where each tree has exactly one node (external node) with weight n_i.

Step 2: Choose two trees T_1 and T_2 with minimal weights.

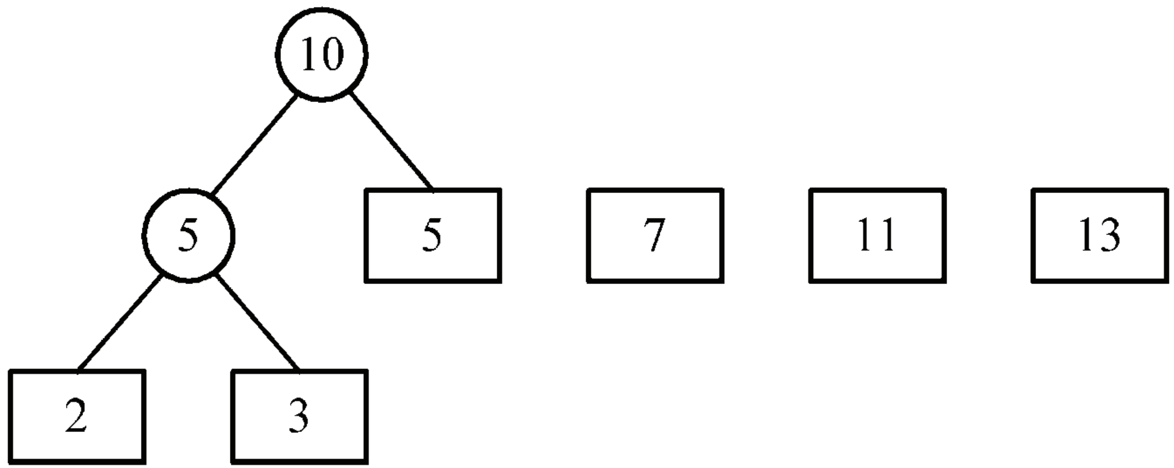
Step 3: Create a new tree T whose root has T_1 and T_2 as its subtrees and weight is equal to the sum of weights T_1 and T_2.

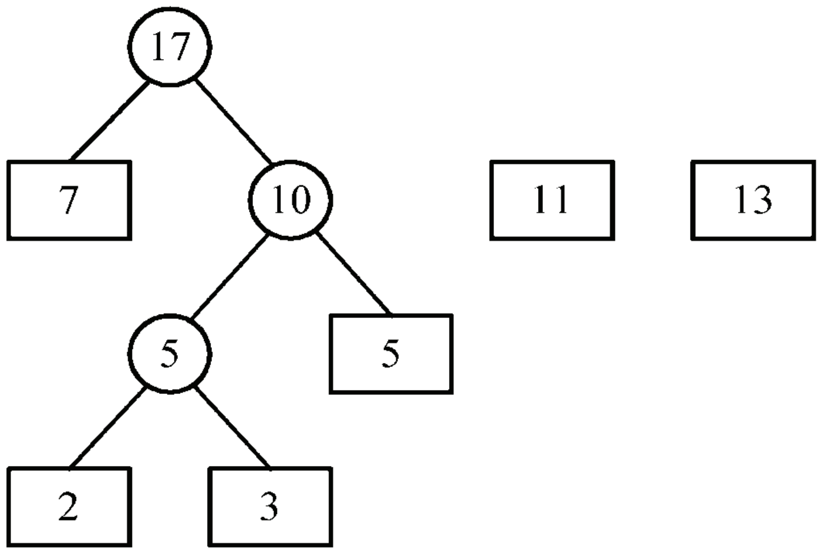
Step 4: Replace T_1 and T_2 by T.

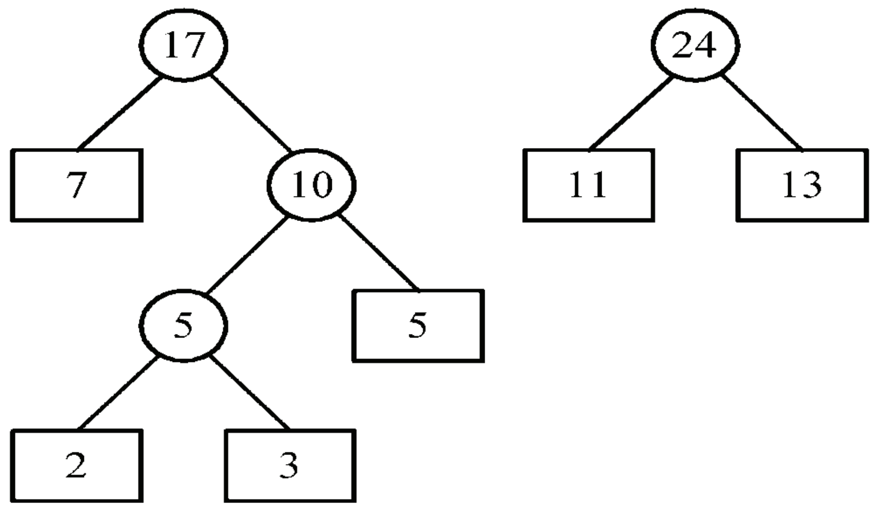
Step 5: If there is only one tree left, stop and return; otherwise, go to Step 2.

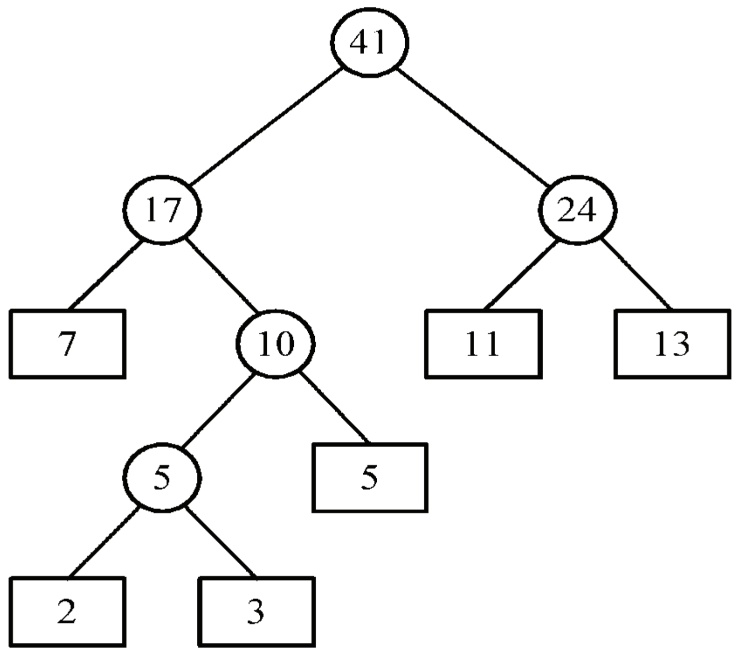
An example of 2-way merging

- Example: 6 sorted lists with lengths 2, 3, 5, 7, 11 and 13.







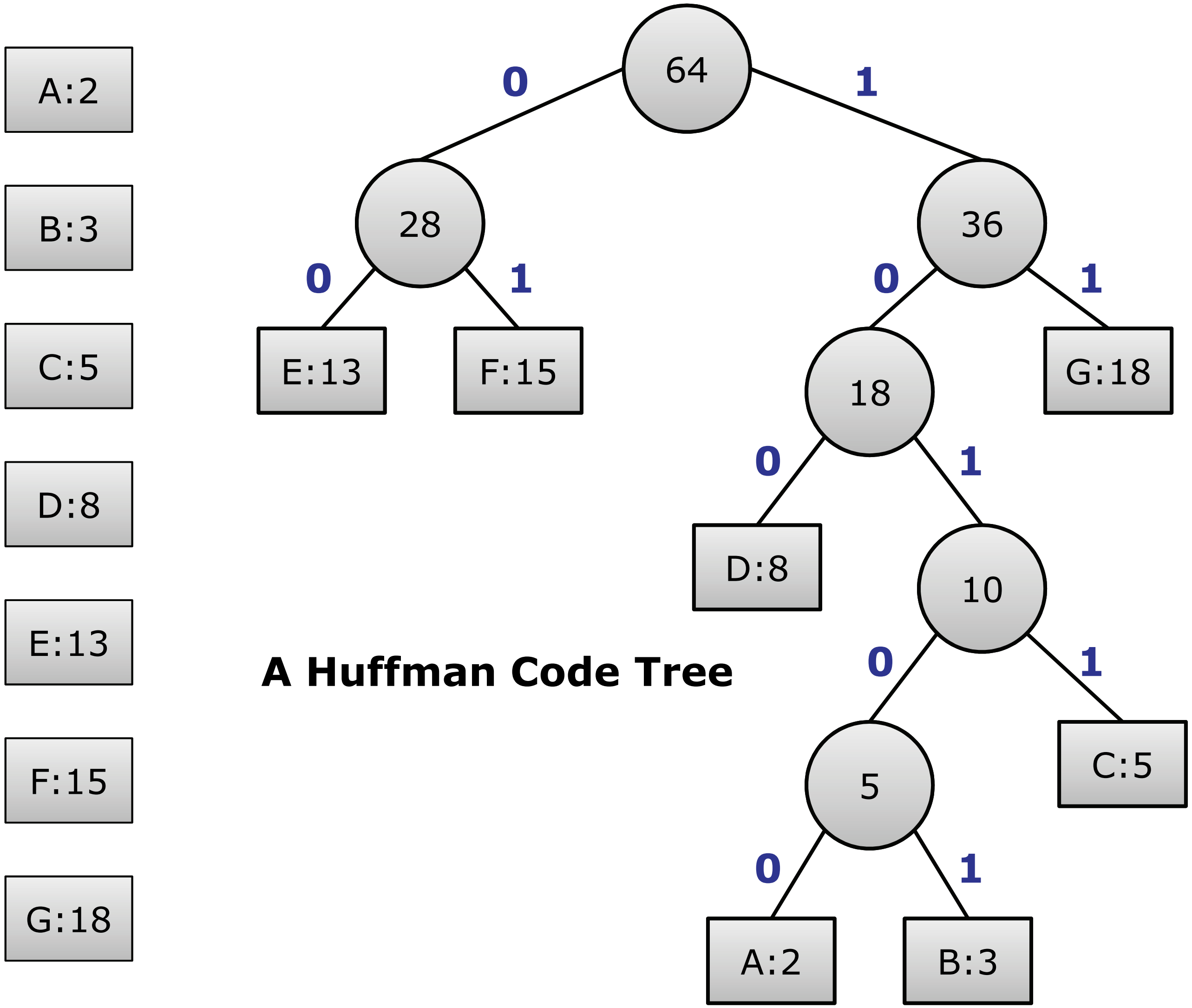


- Corrected proof by induction.
- Time complexity for generating an optimal extended binary tree:O(n log n) using min-heap,

Example Huffman codes

- In telecommunication, how do we represent a set of messages, each with an access frequency, by a sequence of 0's and 1's?
- To minimize the transmission and decoding costs, we may use short strings to represent more frequently used messages.
- This problem can by solved by using an extended binary tree which is used in the 2-way merging problem.

AN EXAMPLE OF HUFFMAN ALGORITHM



A Huffman Code Tree

Huffman Codes	
A	10100
B	10101
C	1011
D	100
E	00
F	01
G	11

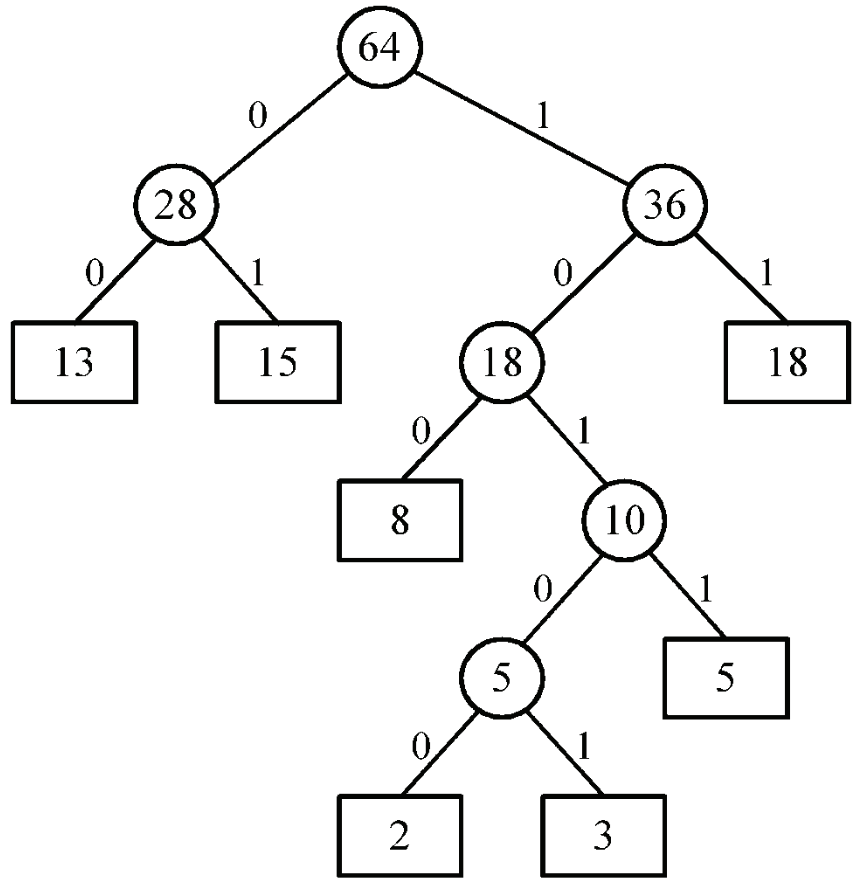
Symbols	frequencies	Huffman	ASCII	Reduced Bits
A	2	10100	1000001	4
B	3	10101	1000010	6
C	5	1011	1000011	15
D	8	100	1000100	32
E	13	00	1000101	65
F	15	01	1000110	75
G	18	11	1000111	90

An example of Huffman algorithm

- Symbols: A, B, C, D, E, F, G
  freq. : 2, 3, 5, 8, 13, 15, 18

- Huffman codes:
  A: 10100 B: 10101 C: 1011
  D: 100 E: 00 F: 01
  G: 11

- Decode:
  Given sequence
  110010010110010 1…



A Huffman code Tree

3.5 The minimal cycle basis problem

- 3 cycles:

  $A_1 = \{ab, bc, ca\}$

  $A_2 = \{ac, cd, da\}$

  $A_3 = \{ab, bc, cd, da\}$

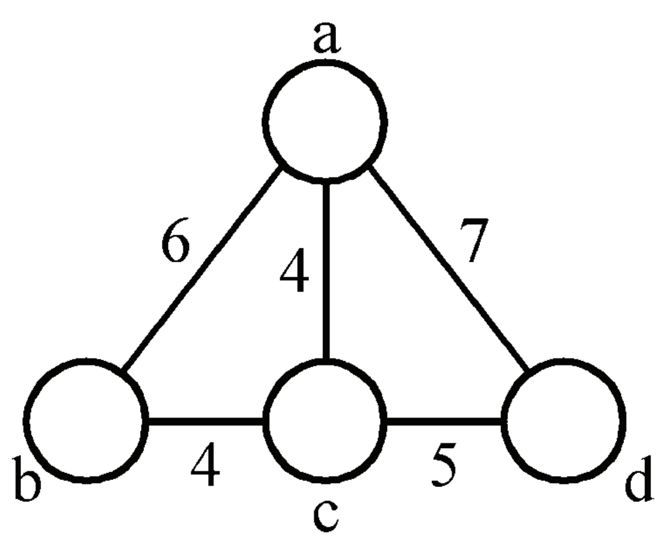
  where $A_3 = A_1 \oplus A_2$

  $(A \oplus B = (A \cup B)-(A \cap B))$

  $A_2 = A_1 \oplus A_3$

  $A_1 = A_2 \oplus A_3$

  Cycle basis : $\{A_1, A_2\}$ or $\{A_1, A_3\}$ or $\{A_2, A_3\}$



The minimal cycle basis problem

- **Def** : A cycle basis of a graph is a set of cycles such that every cycle in the graph can be generated by applying ring sum operator $\oplus$ on some cycles of this basis.
- Assume weighted graph.
- Weight of cycle is the total weight of all edges in this cycle.
- The weight of a cycle basis is the total weight of all cycle in the cycle basis.
- Minimal cycle basis : smallest total weight of all edges in this cycle.
- e.g. $\{A_1, A_2\}$

minimal cycle basis algorithm

- Algorithm for finding a minimal cycle basis:

  Step 1: Determine the size of the minimal cycle basis, demoted as k.

  Step 2: Find all of the cycles. Sort all cycles(by weight).

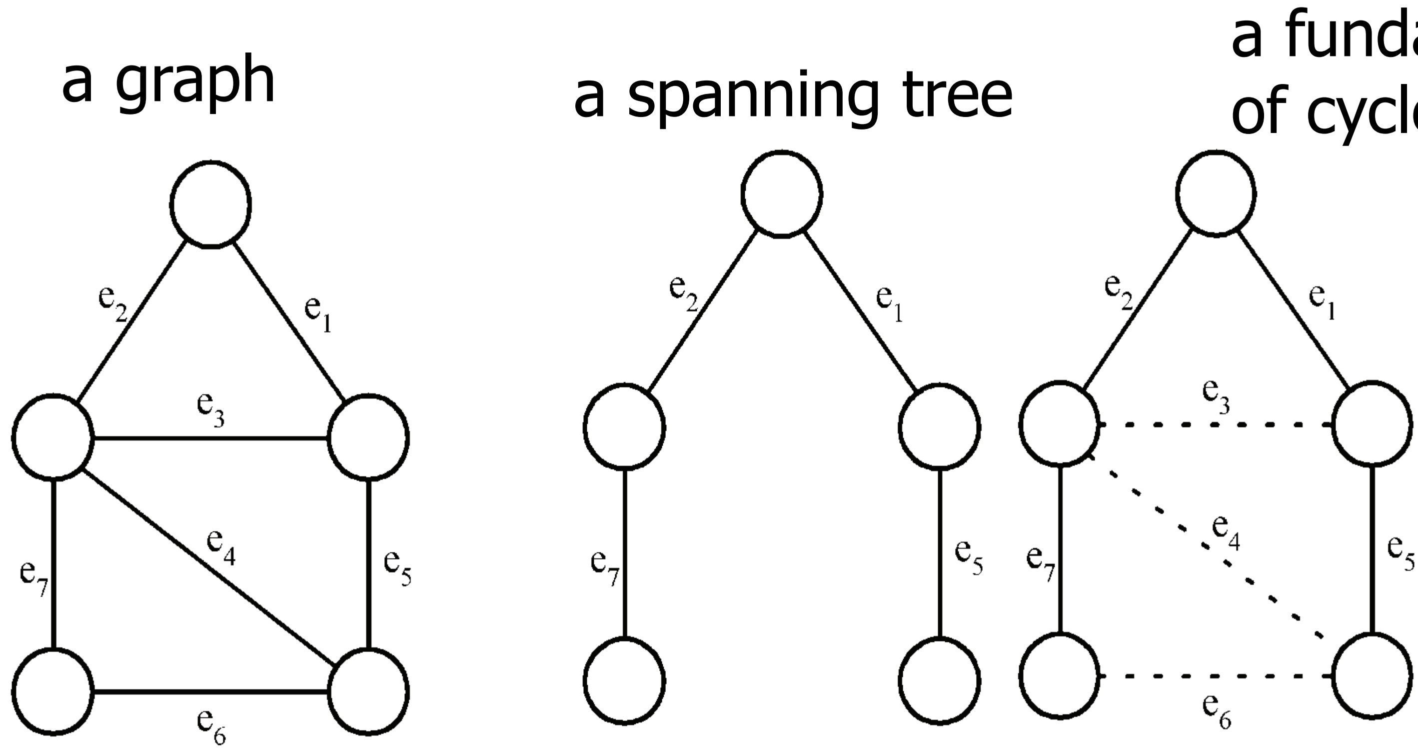
  Step 3: Add cycles to the cycle basis one by one. Check if the added cycle is already combination of some cycles already existing in the basis. If it is, delete this cycle.

  Step 4: Stop if the cycle basis has k cycles.

The minimal cycle basis problem – detail description

- Step 1 : (Determine the size of the minimal cycle basis, demoted as k)

A cycle basis corresponds to the fundamental set of cycles with respect to a spanning tree.



# of cycles in a cycle basis :

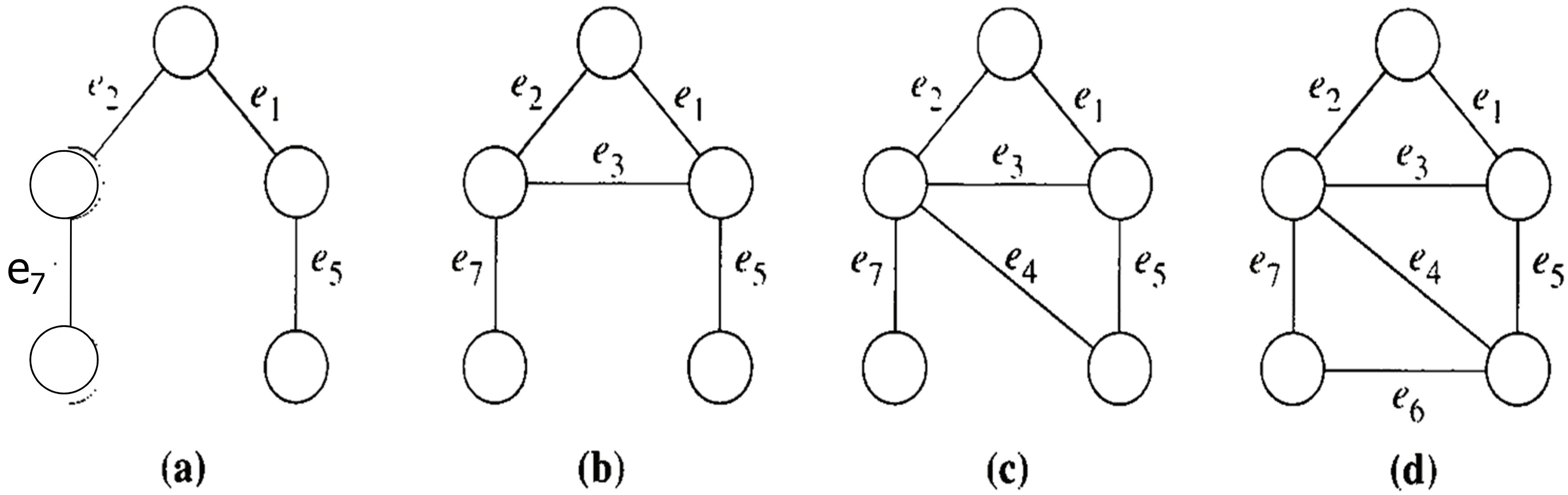
$= k$

$= |E| - (|V| - 1)$

$= |E| - |V| + 1$

Relationship of SPT and cycles

FIGURE 3–25 The relationship between a spanning tree and the cycles.



The number of independent cycles is equal to the number of edges which can be added to the spanning tree.
Which is the size of the cycle basis.
|E|-|V|+1

- Step 2:

  How to find all cycles in a graph?

  [Reingold, Nievergelt and Deo 1977]

  How many cycles in a graph in the worst case?

  In a complete digraph of n vertices and n(n-1) edges:

  $$\sum_{i=2}^{n} C_i^n (i-1)! > (n-1)!$$

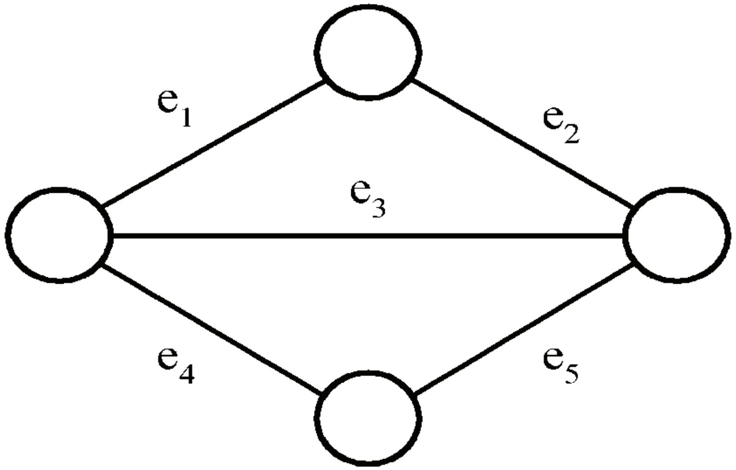
  i: cycle length

- Step 3:

  How to check if a cycle is a linear combination of some cycles?

  Using Gaussian elimination.

Gaussian elimination



- 2 cycles C_1 and C_2 are represented by a 0/1 matrix

$$\begin{array}{c} \\ C_1 \\ C_2 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & & \\ & & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- Add C_3

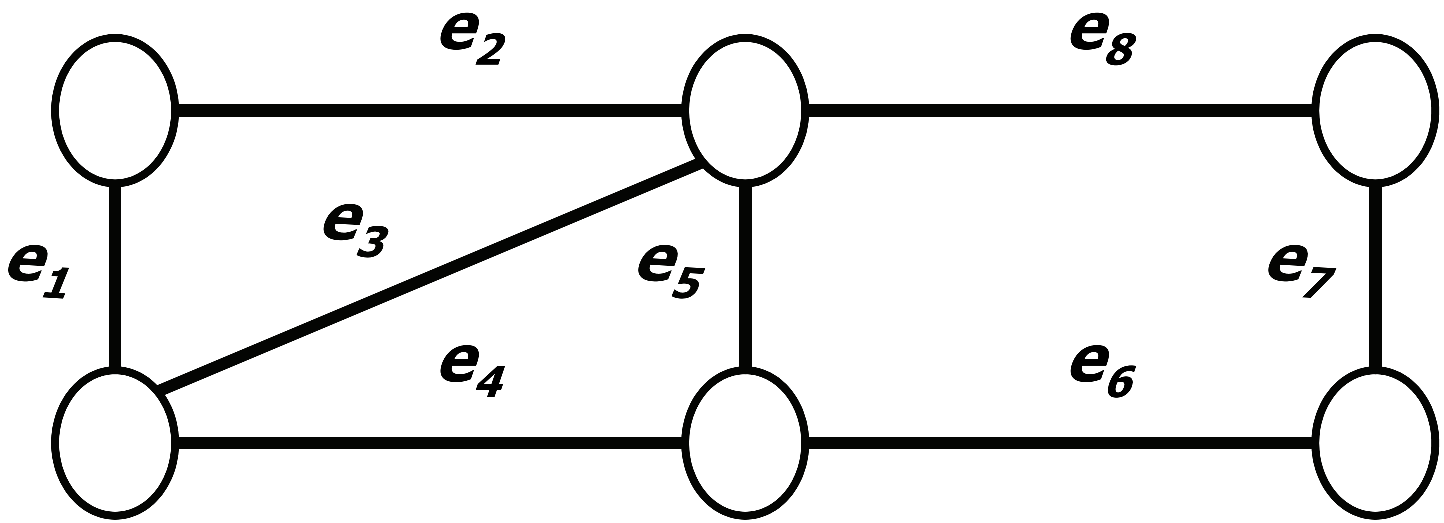
$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & & \\ & & 1 & 1 & 1 \\ 1 & 1 & & 1 & 1 \end{bmatrix} \end{array}$$

- $\oplus$ on rows 1 and 3

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & & \\ & & 1 & 1 & 1 \\ & & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$\oplus$ on rows 2 and 3 : empty

$\because C_3 = C_1 \oplus C_2$



$$C_1 = \{e_1, e_2, e_3\}$$

$$C_2 = \{e_3, e_5, e_4\}$$

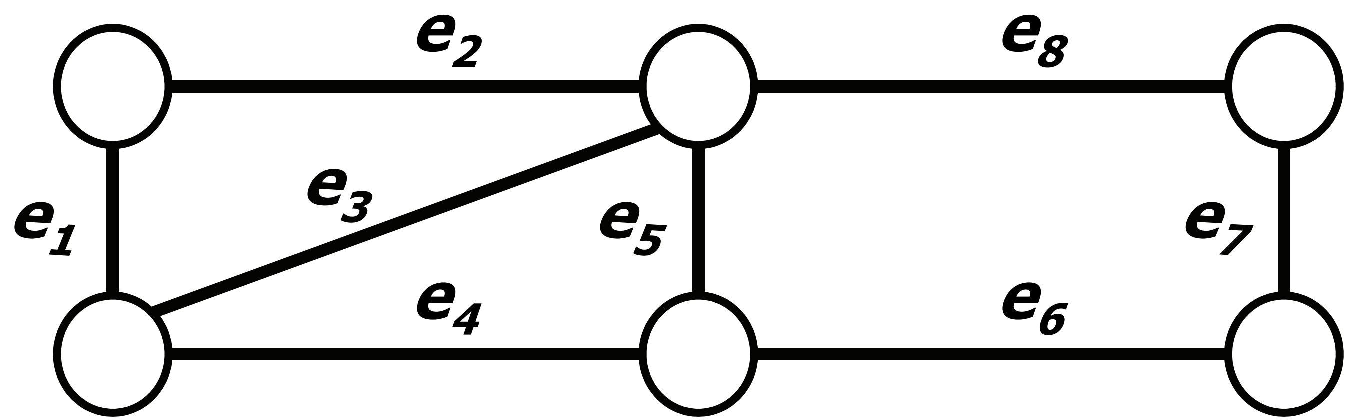
$$C_3 = \{e_2, e_5, e_4, e_1\}$$

$$C_4 = \{e_8, e_7, e_6, e_5\}$$

$$C_5 = \{e_3, e_8, e_7, e_6, e_4\}$$

$$C_6 = \{e_2, e_8, e_7, e_6, e_4, e_1\}.$$

Example

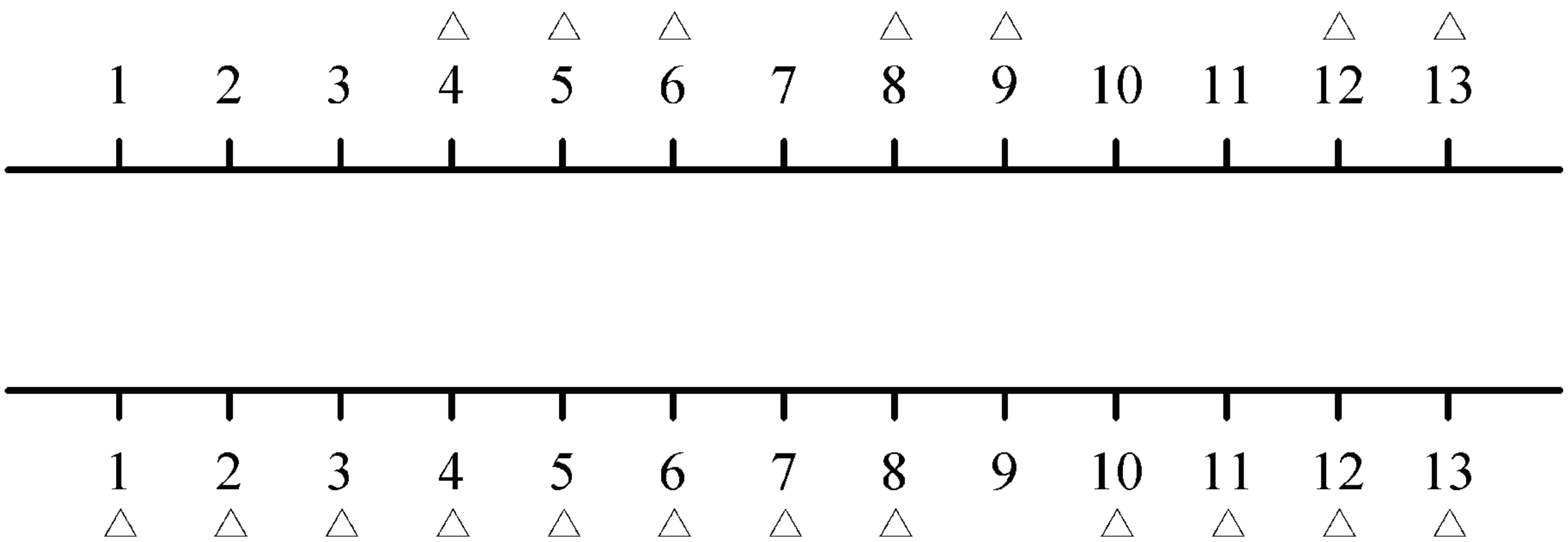


For this case, $|E| = 8$ and $|V| = 6$. Thus, $K = 8 - 6 + 1 = 3$. This greedy method will work as follows:

(1) C_1 is added.
(2) C_2 is added.
(3) C_3 is added and discarded because it is found that C_3 is a linear combination of C_1 and C_2.
(4) C_4 is added. Since now the cycle basis already has three cycles, we stop as $K = 3$. A minimum cycle basis is found to be $\{C_1, C_2, C_4\}$.

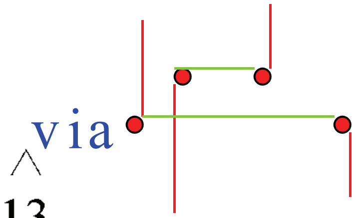
3.6 The 2-terminal one to any special channel routing problem

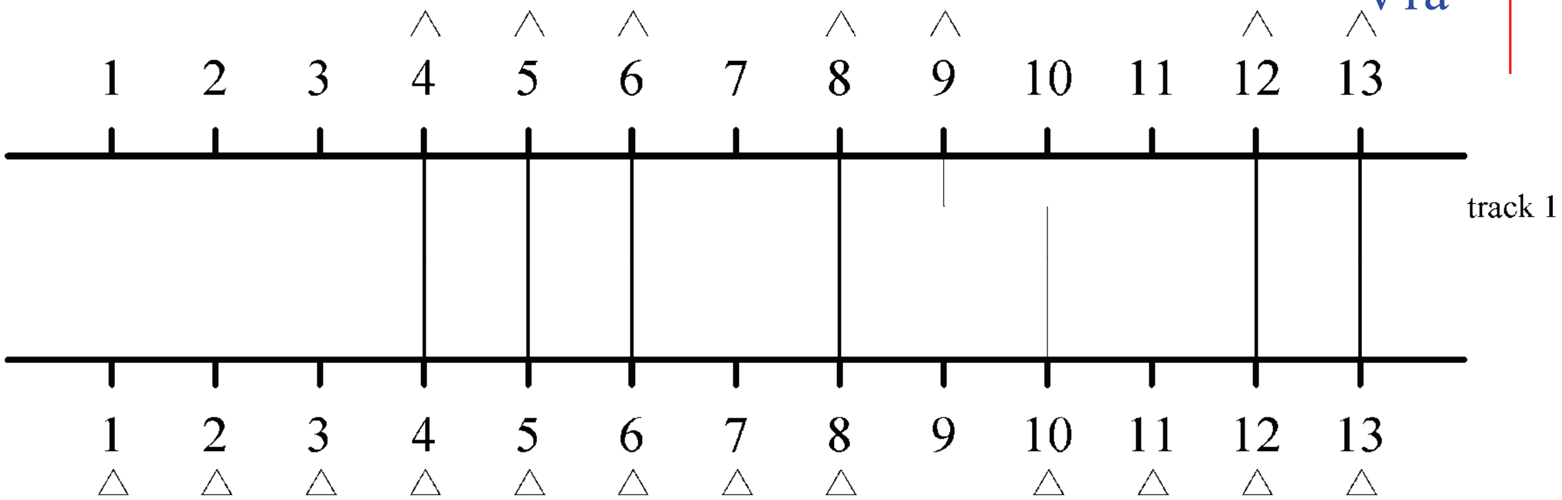
- Def: Given a set of terminals on the upper row and another set of terminals on the lower row, we have to connect each upper terminal to the lower row in a one to one fashion. This connection requires that # of tracks used is minimized.

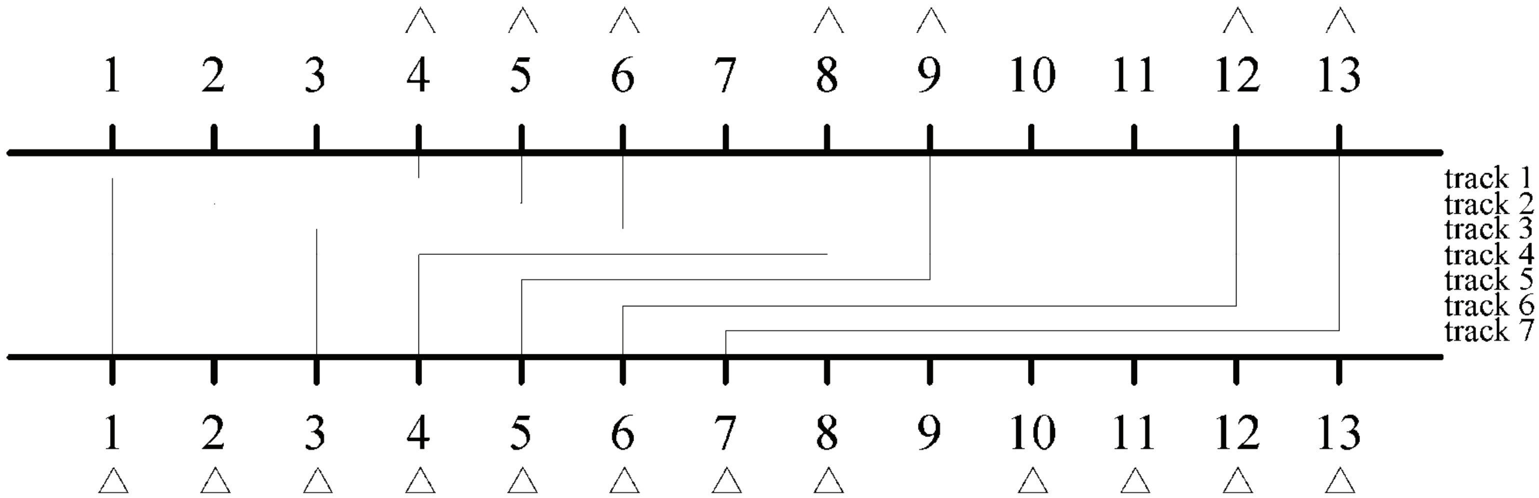


terminal

2 feasible solutions

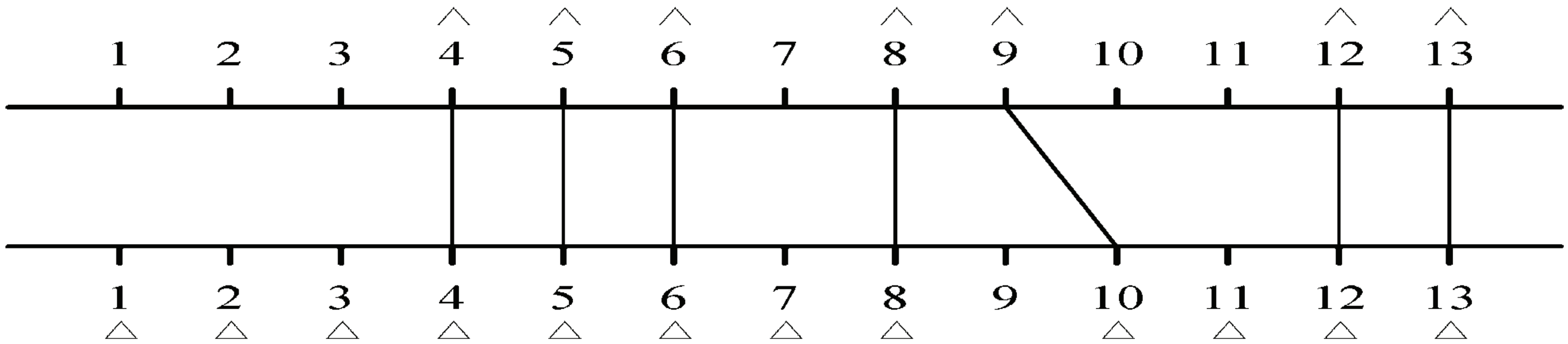




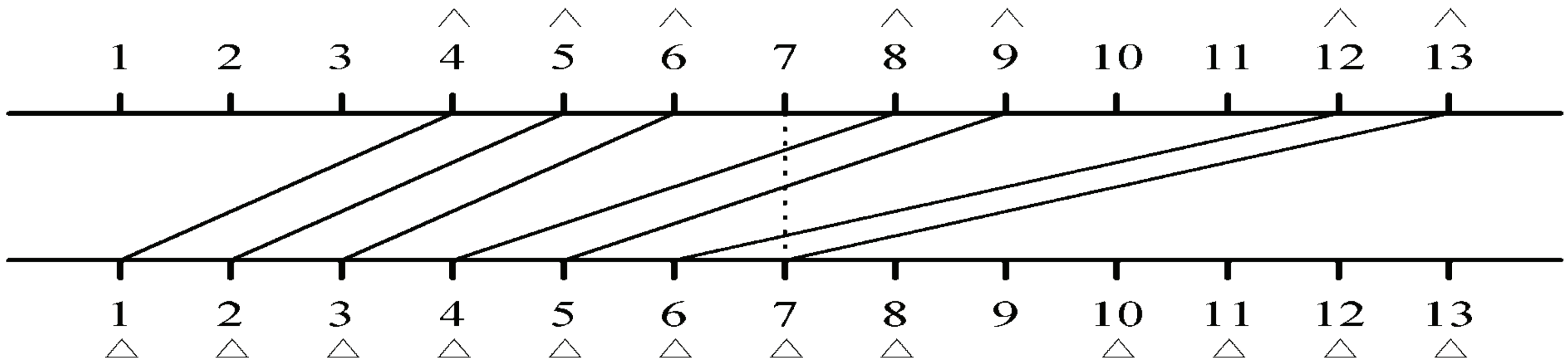


Redrawing solutions (local density)

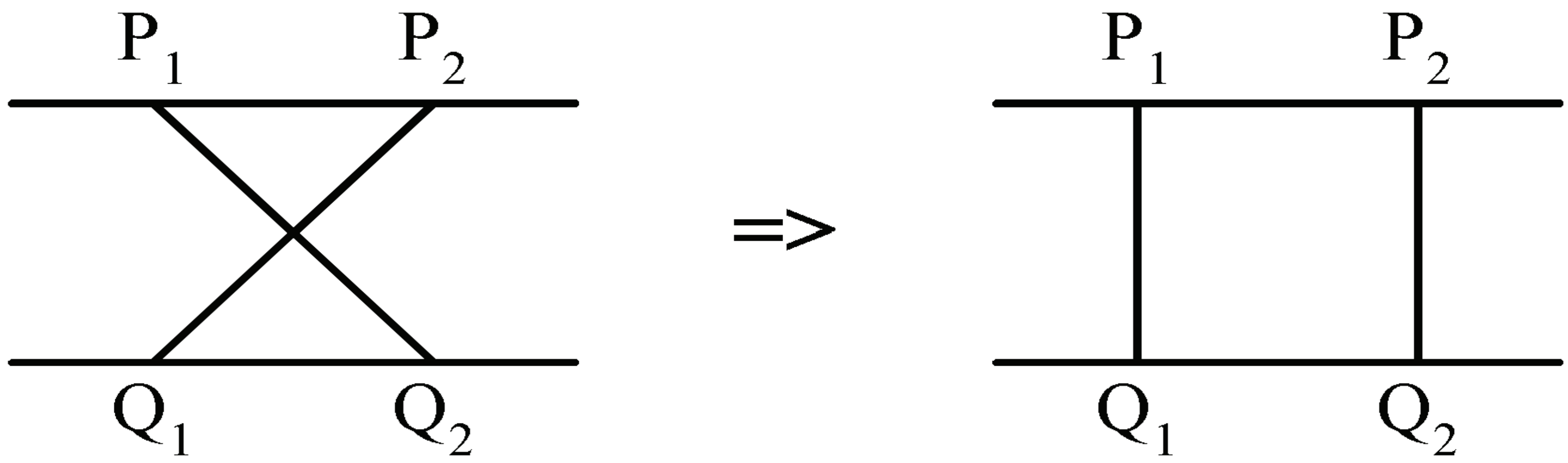
(a) Optimal solution



(b) Another solution

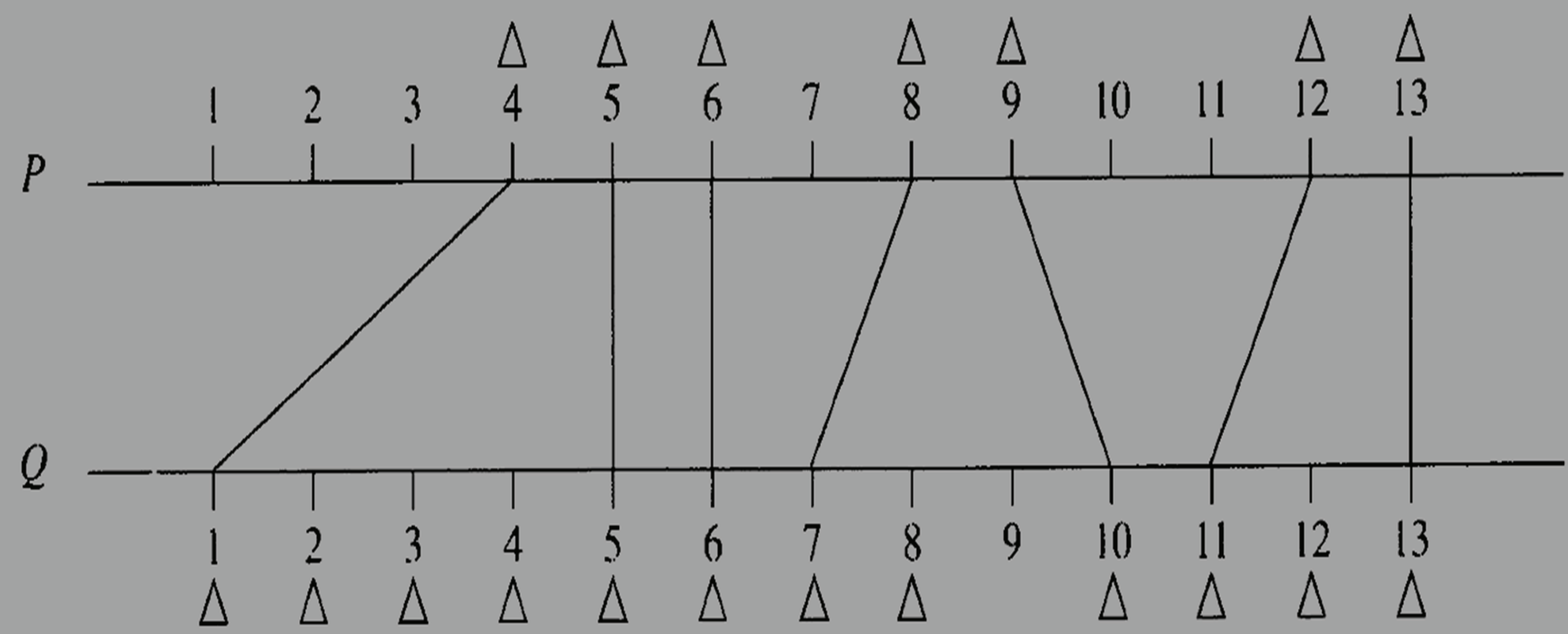


- At each point, the local density of the solution is # of lines the vertical line intersects.
- The density of a solution is the maximum local density.
- The problem: to minimize the density.
- **The density is a lower bound of # of tracks.**
- **Assume the minimum density is known.**
- Upper row terminals: $P_1, P_2, \ldots, P_n$ from left to right
- Lower row terminals: $Q_1, Q_2, \ldots, Q_m$ from left to right $m > n$.
- It would never have a **crossing connection**:



Assume minimum density is d=1

FIGURE 3–31 The problem instance in Figure 3–28 solved by the greedy algorithm.



Greedy Algorithm

- Suppose that we have a method to determine the minimum density, d, of a problem instance.
- The **greedy algorithm**:

Step 1 : P_1 is connected Q_1.

Step 2 : After P_i is connected to Q_j, we check whether P_{i+1} can be connected to Q_{j+1}. If the density is increased to d+1, try to connect P_{i+1} to Q_{j+2}.

Step 3 : Repeat Step2 until all P_i's are connected.

3.7 The Minimum Cooperative Guards Problem for 1-Spiral Polygons Solved by the Greedy Method

- The **minimum cooperative guards problem** is a variation of the **art gallery problem**,
- We are given a polygon, which represents an art gallery, and we are required to place a **minimum number of guards** in the polygon such that every point of the polygon is visible to at least one guard.
- The a n.

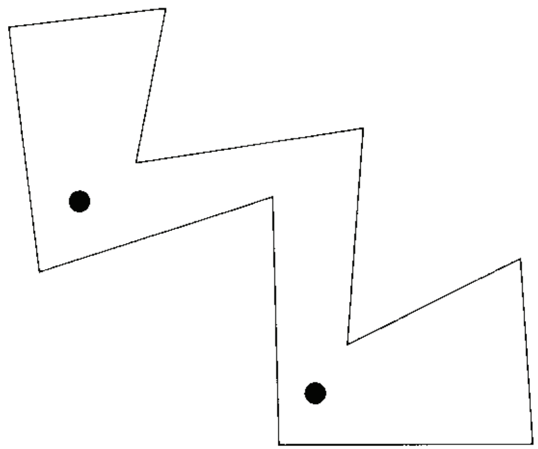
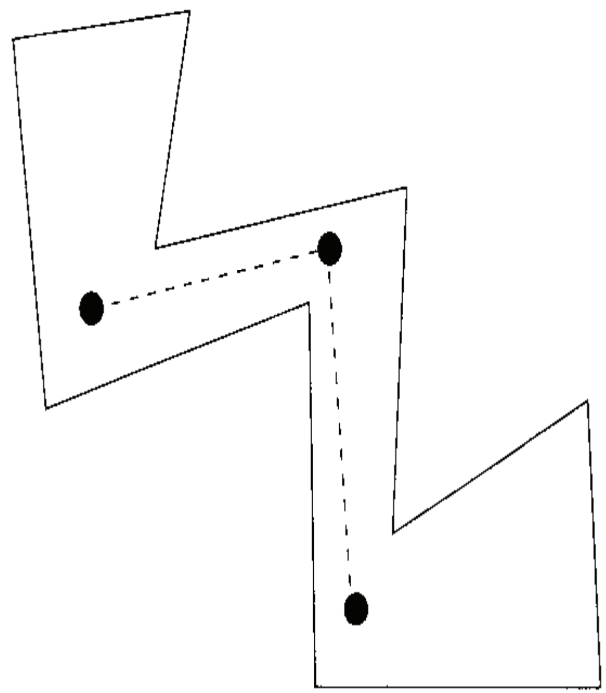


FIGURE 3–34 A solution of the art gallery problem.

Minimum cooperative guards problem

FIGURE 3–35 A solution of the minimum cooperative guards problem for the polygon in Figure 3–34.



NP-hard problem

The minimum cooperative guards problem

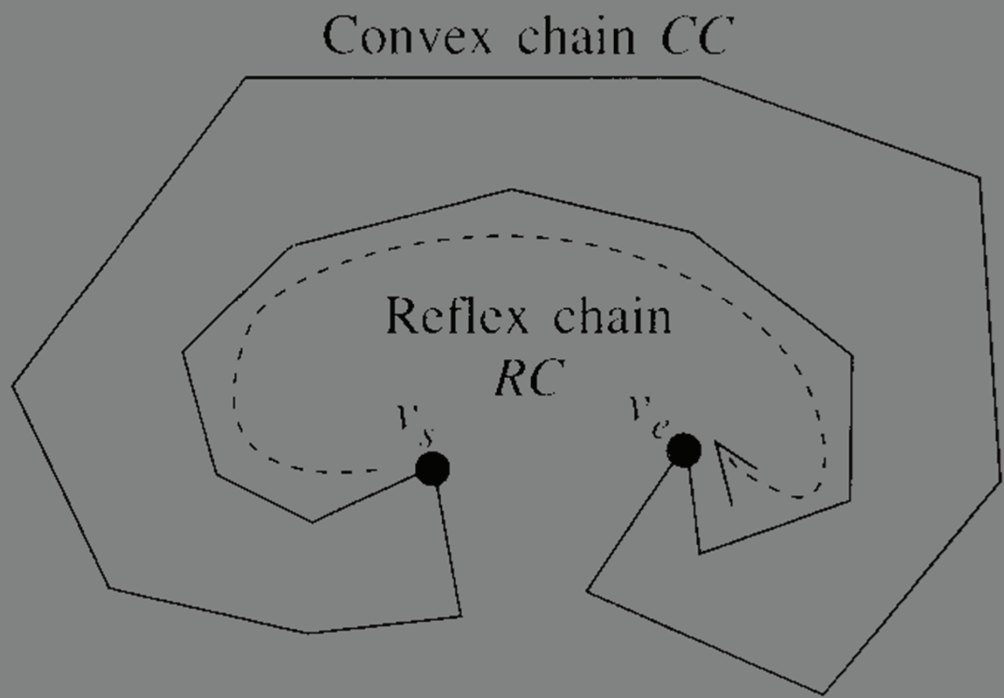
- The minimum cooperative guards problem puts more constraint on the art gallery problem.
- Note that it may be quite dangerous for a guard to be stationed in an art gallery if he cannot be seen by any other guards.
- We represent the relationship among guards by a visibility graph G of the guards.
- In G, every guard is represented by a vertex and there is an edge between two vertices if and only if the corresponding guards can see each other.
- In addition to finding a minimum set of guards who can see the given polygon, we further require that the visibility graph of these guards is connected.
- In other words, we require that **no guard is isolated and there is a path between every pair of guards.** We call such a problem the ***mininnun cooperative guards problem.***

1-spiral polygons

- It can again be shown that the minimum cooperative guards problem is NP-complete
- Thus, it is quite unlikely that this minimum cooperative guards problem can be solved by any polynomial algorithm on general polygons. But we shall show there is a greedy algorithm for this problem for 1-spiral (螺旋形的) polygons.
- Before defining 1-spiral polygons, let us first define reflex (convex) chains. ex (convex) chain, denoted as RC (CC), of a simple polygon is a chain of of this polygon if all of the vertices on this chain are reflex (convex) with respective to the interior of the polygon. We also stipulate that a reflex chain refers maximal reflex chain: that is, it is not contained in any other reflex chain.
- A 1-spiral polygon P is a simple polygon whose boundary can be partitioned into a reflex chain and a convex chain.

1-spiral polygons

FIGURE 3–36 A typical 1-spiral polygon.



Staring & ending regions

FIGURE 3–37 The starting and ending regions in 1-spiral polygons.

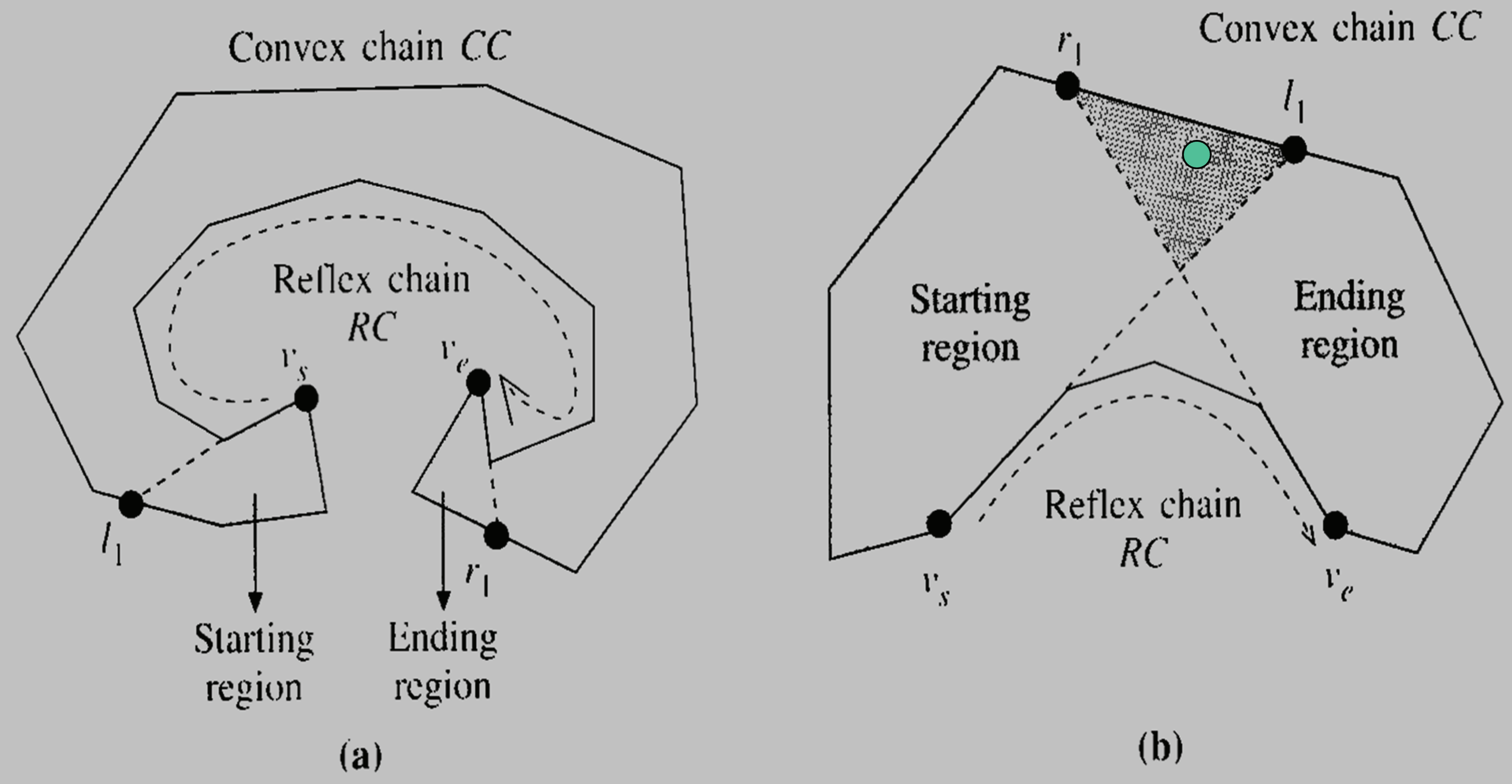
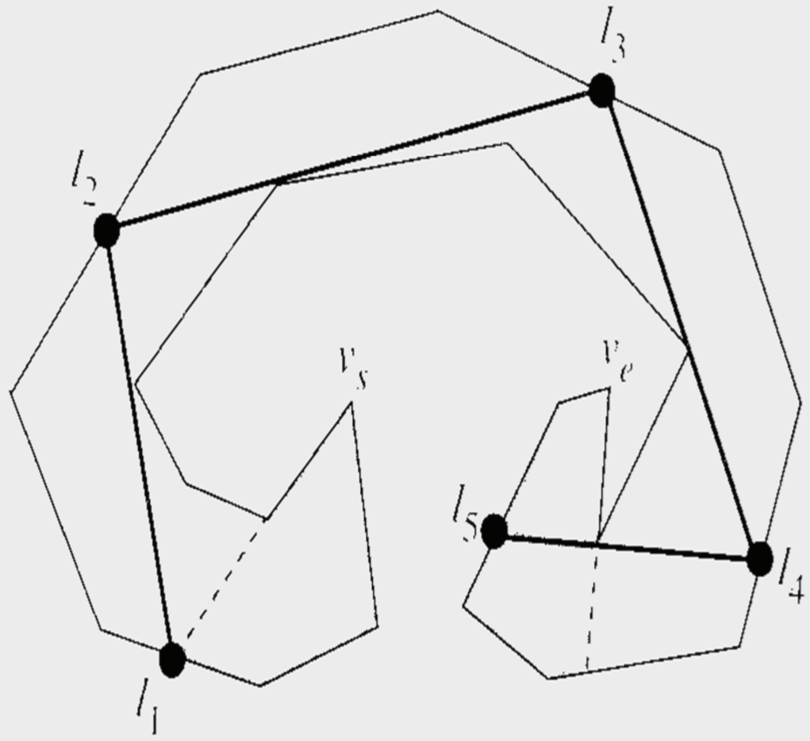
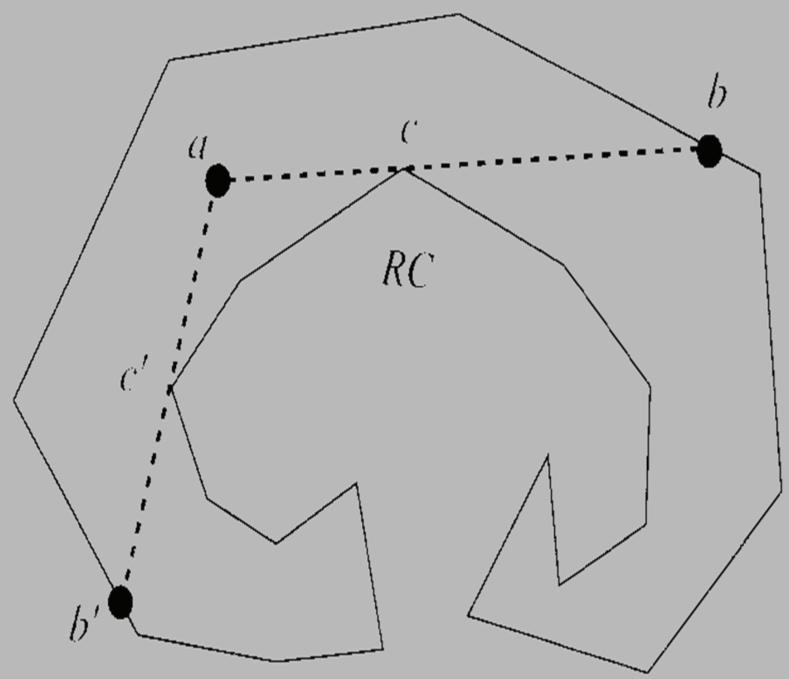


FIGURE 3–38 A set of guards $\{l_1, l_2, l_3, l_4, l_5\}$ in a 1-spiral polygon.



IGURE 3–39 The left and right supporting line segments with respect to a.



Supporting line segment

Algorithm 3–7 □ An algorithm to solve the minimum cooperative guards problem for 1-spiral polygons

Input: A 1-spiral polygon P.

Output: A set of points which is the solution of the minimum cooperative guards problem.

Step 1. Find the reflex chain RC and convex chain CC of P.

Step 2. Find the intersection points l_1 and r_1 of CC with the directed lines starting from v_s and v_e along the first and last edges of RC, respectively.

Step 3. Let $k = 1$.

While l_k is not on the ending region do

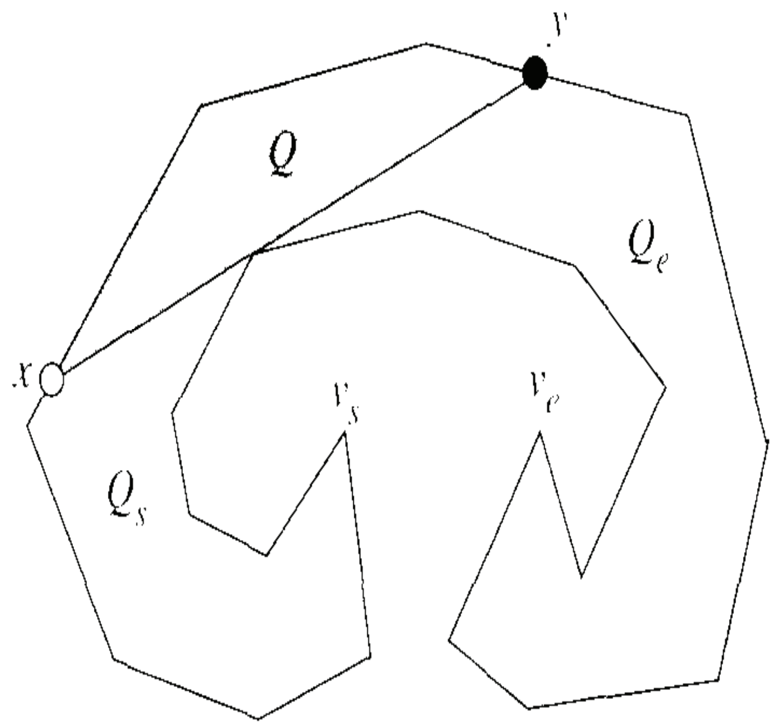
Draw the left tangent of l_k with respect to RC until it hits CC at l_{k+1}. ($\overline{l_k l_{k+1}}$ is a left supporting line segment with respect to l_k.)

Let $k = k + 1$.

End While.

Step 4. Report $\{l_1, l_2, \ldots, l_k\}$.

FIGURE 3–40 A supporting line segment $\overline{xy}$ and the regions Q, Q_s and Q_e.



Appendix: The knapsack problem

- n objects, each with a weight $w_i > 0$
  a profit $p_i > 0$
  capacity of knapsack: M

Maximize $\sum_{1 \le i \le n} p_i x_i$

Subject to $\sum_{1 \le i \le n} w_i x_i \le M$

$0 \le x_i \le 1, \ 1 \le i \le n$

The knapsack algorithm

- The greedy algorithm:

  Step 1: Sort p_i/w_i into nonincreasing order.

  Step 2: Put the objects into the knapsack according to the sorted sequence as possible as we can.

- e. g.

  $n = 3$, $M = 20$, $(p_1, p_2, p_3) = (25, 24, 15)$

  $(w_1, w_2, w_3) = (18, 15, 10)$

  Sol: $p_1/w_1 = 25/18 = 1.32$

  $p_2/w_2 = 24/15 = 1.6$

  $p_3/w_3 = 15/10 = 1.5$

  Optimal solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 1/2$