Properties of Quadratic-Form Estimators for Stationary Processes Satisfying the Geometric-Moment Contraction Condition

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Outline

Introduction — Analysis of Simulation Output is Interesting

Steps in a Simulation Study:

- Preliminary Analysis of the System
- Model Building
- Verification & Validation
- Experimental Design & Simulation Runs
- Statistical Analysis of Output Data
- Implementation

Input processes driving a simulation are random variables (e.g., interarrival times, service times, and breakdown times).

Must regard the output from the simulation as random.

Runs of the simulation only yield *estimates* of measures of system performance (e.g., the mean customer waiting time).

These estimators are themselves random variables, and are therefore subject to sampling error.

Sampling error must be taken into account to make valid inferences concerning system performance.

Lots of measures you could be interested in:

- Means — what is the mean customer waiting time?

  Means aren't enough. If I have one foot in boiling water and one foot in freezing water, on average, I'm fine. So. . . .

- Variances — how much is the waiting time liable to vary?
- Quantiles
  - What's the 90th percentile of the length of COVID infectiousness?
  - What's the 10% quantile of a retirement portfolio value?
  - What's the 99% quantile of the customer in a certain steady-state queue?
- Success Probabilities — will my job be completed on time?

Would like *point estimators* and *confidence intervals* for all of the above.

Problem: Simulations almost never produce raw output that is independent and identically distributed (i.i.d.) normal data.

Example: Customer waiting times from a queueing system…

(1) Are not independent — typically, they are serially correlated. If one customer at the post office waits in line a long time, then the next customer is also likely to wait a long time.

(2) Are not identically distributed. Customers showing up early in the morning might have a much shorter wait than those who show up just before closing time.

(3) Are not normally distributed — they are usually skewed to the right (and are certainly never less than zero).

Archetypal Example on Serial Correlation: Suppose that $Y_1, Y_2, \ldots, Y_n$ are identically distributed but *not independent*. Is the sample mean $\bar{Y}_n \equiv \frac{1}{n}\sum_{i=1}^n Y_i$ a good estimator for $\mu = \mathrm{E}[Y_i]$?

$\mathrm{E}[\bar{Y}_n] = \frac{1}{n}\sum_{i=1}^n \mathrm{E}[Y_i] = \mu$, so it's unbiased. ☺

On the other hand, define the *covariance function*, $R_k \equiv \mathrm{Cov}(Y_1, Y_{1+k})$, $k = 0, 1, 2, \ldots$. Can show (later) that

$$\mathrm{Var}(\bar{Y}_n) \quad = \quad \frac{1}{n}\left[R_0 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)R_k\right].$$

This is a problem, since the "classical" confidence interval (CI) for the mean μ requires i.i.d. observations and $\mathrm{Var}(\bar{Y}_n) = \mathrm{Var}(Y_i)/n = R_0/n$. Those extra covariances are Trouble! ☹

What do we mean by "Trouble"?

If you ignore all of those extra covariance terms $R_1, R_2, \ldots$ and they happen to be positive (typical for queueing simulations of customer waiting times), then you will think that $\text{Var}(\bar{Y}_n)$ is a lot smaller than it actually is.

In other words, you'll be a lot more confident about the estimator $\bar{Y}_n$ (and any conclusions you reach) than you ought to be. Specifically, you'll end up with:

- A highly biased estimator for $\text{Var}(\bar{Y}_n)$ (way too low)
- Confidence intervals having coverage probability $\ll 1 - \alpha$

The point is that *it's difficult to apply "classical" statistical tools to analyze simulation output* — in large part due to the presence of serial correlation.

What We've Been Doing for Years and Years: Give methods to perform valid statistical analysis of output from discrete-event computer simulations.

Why all the fuss?

- Beware — improper statistical analysis can invalidate all results
- Tremendous applications if you can get it right
- Lots of cool research problems out there

Types of Simulations

Generally speaking, there are two types of simulations with respect to output analysis: Finite-Horizon (Terminating) and Steady-State simulations.

Finite-Horizon Simulations: The termination of a finite-horizon simulation takes place at a specific time or is caused by the occurrence of a specific event. Examples:

- Mass transit system during rush hour.
- A disease makes its way through a population over a one-year period.
- Distribution system over one month.
- Production system until a set of machines breaks down.
- Start-up phase of any system — stationary or nonstationary.

Steady-State (Stationary) Simulations: Study the *long-run* behavior of a system. A performance measure is called a *steady-state parameter* if it is a characteristic of the equilibrium distribution of an output stochastic process.

Examples:

- Continuously operating communication system where the objective is the computation of the mean delay of a packet in the long run.
- Distribution system over a long period of time.
- Many Markov chains.

In this talk, we'll focus on steady-state simulations.

What's coming up?

In §2, we'll go over some math background that we'll refer to occasionally.

§3 deals with the batch means method of point and confidence interval estimation for the steady-state mean and the variance of the sample mean.

§4 concerns the assumptions underlying the method of batch means — mixing vs. the Geometric-Moment Contraction Condition (GMCC).

§5 presents analytical results for a first-order autoregressive [AR(1)] time-series process when the batch means variance estimator is used.

§6 discusses results analogous work for quadratic-form estimators.

§7 gives some conclusions and suggestions for future work.

Outline

Some Math Background

We'll look at a few examples illustrating the fact that things turn out a little differently when you don't have i.i.d. observations.

Working Assumptions: For the remainder of this talk, suppose that $Y_1, Y_2, \ldots, Y_n$ are stationary — *identically distributed* with mean μ, but *not necessarily independent*.

Such an assumption applies in the context of steady-state simulation.

Let's get properties of the sample mean and variance — these guys perform very well in the i.i.d. case — but you have to be careful when using them in simulation output analysis.

Properties of the Sample Mean: First, $\mathrm{E}[\bar{Y}_n] = \frac{1}{n}\sum_{i=1}^n \mathrm{E}[Y_i] = \mu$, so the sample mean is still unbiased for μ. ☺

Now recall the *covariance function*, $R_k \equiv \mathrm{Cov}(Y_1, Y_{1+k})$, $\forall k$. Then

$$\begin{aligned}
\mathrm{Var}(\bar{Y}_n) &= \mathrm{Cov}(\bar{Y}_n, \bar{Y}_n) \\
&= \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \mathrm{Cov}(Y_i, Y_j) \\
&= \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n R_{|i-j|} \qquad (1) \\
&= \frac{1}{n}\left[R_0 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)R_k\right]. \qquad (2)
\end{aligned}$$

How did we go from (1) to (2)?

Just add up the terms in the following matrix of covariances.

$$\begin{pmatrix} R_0 & R_1 & R_2 & \cdots & R_{n-3} & R_{n-2} & R_{n-1} \\ R_1 & R_0 & R_1 & \cdots & R_{n-4} & R_{n-3} & R_{n-2} \\ R_2 & R_1 & R_0 & \cdots & R_{n-5} & R_{n-4} & R_{n-3} \\ & & \vdots & \vdots & \vdots & & \\ R_{n-3} & R_{n-4} & R_{n-5} & \cdots & R_0 & R_1 & R_2 \\ R_{n-2} & R_{n-3} & R_{n-4} & \cdots & R_1 & R_0 & R_1 \\ R_{n-1} & R_{n-2} & R_{n-3} & \cdots & R_2 & R_1 & R_0 \end{pmatrix}$$

The result follows because there are n R_0 terms, $2(n-1)$ R_1 terms, $2(n-2)$ R_2 terms, …, and $2(n-(n-1))$ R_{n-1} terms.

Equation (2) is important — it relates the variance of the sample mean to the covariances of the process. With this in mind, define

$$\sigma_n^2 \equiv n\mathrm{Var}(\bar{Y}_n) = R_0 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)R_k.$$

We also define the related *variance parameter*,

$$\sigma^2 \equiv \lim_{n\to\infty}\sigma_n^2 \overset{\star}{=} R_0 + 2\sum_{k=1}^{\infty}R_k = \sum_{k=-\infty}^{\infty}R_k,$$

where $=^\star$ holds if the R_k's decrease to 0 quickly as $k \to \infty$.

The variance parameter σ^2 is so pretty and turns up all over the place.

Notice that if the Y_i's are i.i.d., then for all $k \neq 0$, we have $R_k = 0$, in which case we have the familiar result $\sigma^2 = \sigma_n^2 = R_0 = \mathrm{Var}(Y_1)$.

But in the dependent case, $\sigma^2 = \sum_{k=-\infty}^{\infty} R_k$ adds in the effects of the covariances. In queueing applications, the covariances are positive and $\sigma^2 \doteq \sigma_n^2 \gg \mathrm{Var}(Y_1)$, which may be bigger than you'd think.

The ratio $\sigma_n^2/\mathrm{Var}(Y_1)$ is sort of the number of Y_i's needed to obtain the information equivalent to one "independent" observation.

⚠ Warning: $\sigma_n^2 \gg \mathrm{Var}(Y_1)$ causes the classical CI for the mean μ to misbehave. Stay tuned.

Example: The first-order autoregressive process is defined by

$$Y_i = \phi Y_{i-1} + \varepsilon_i, \quad \text{for } i = 1, 2, \ldots,$$

where $-1 < \phi < 1$, $Y_0 \sim \text{Nor}(0, 1)$, and the ε_i's are i.i.d. $\text{Nor}(0, 1 - \phi^2)$ random variables (rv's) that are independent of Y_0.

Can easily show that the Y_i's are all Nor(0,1) and $R_k = \phi^{|k|}$, $\forall k$.

After a little algebra, one can also show that

$$\sigma^2 = \sum_{k=-\infty}^{\infty} \phi^{|k|} = \frac{1 + \phi}{1 - \phi}.$$

So, e.g., for $\phi = 0.9$, we have $\sigma^2 = 19$.

The ratio $\sigma^2 / \text{Var}(Y_1) = 19$, indicating a great loss of info. $\square$

Properties of the Sample Variance:

$$S_Y^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_n)^2.$$

If $Y_1, Y_2, \ldots, Y_n$ are i.i.d., then S_Y^2 is unbiased for $R_0 = \mathrm{Var}(Y_1)$. Moreover, S_Y^2 is also unbiased for $\sigma_n^2 = n\mathrm{Var}(\bar{Y}_n) = R_0$ and $\sigma^2 = \lim_{n\to\infty} \sigma_n^2 = R_0$. ☺

But if the Y_i's are dependent, then S_Y^2 may not be such a great estimator for $\mathrm{Var}(Y_1)$, σ_n^2, or σ^2. ☹

In fact, it can be shown (won't go through details here) that

$$\mathrm{E}[S_Y^2] < \mathrm{Var}(Y_1) \ll n\mathrm{Var}(\bar{Y}_n). \quad \square$$

Thus, one should *not* use S_Y^2/n to estimate $\mathrm{Var}(\bar{Y}_n)$.

So what happens if you dare to use it?

Here's the classical $100(1-\alpha)\%$ CI for the mean μ of i.i.d. normal observations with unknown variance:

$$\mu \in \bar{Y}_n \pm t_{\alpha/2,n-1}\sqrt{S_Y^2/n},$$

where $t_{\alpha/2,n-1}$ is a t-distribution quantile.

Since $\mathrm{E}[S_Y^2/n] \ll \mathrm{Var}(\bar{Y}_n)$, the CI will have true coverage $\ll 1-\alpha$! Oops! This is why you have to be really careful with correlated data!

Outline

The Batch Means Variance Estimator

Continue to assume that we have on hand one looonnng stationary (steady-state) simulation output, $Y_1, Y_2, \ldots, Y_n$ with mean μ.

Goal: We'll obtain point and CI's for μ.

The point estimator for μ is easy — it's our unbiased old friend, $\bar{Y}_n$.

A confidence interval for μ takes more work. This requires estimation of $\mathrm{Var}(\bar{Y}_n)$ since the CI will typically be of the form

$$\mu \in \bar{Y}_n \pm t_{\alpha/2,\nu}\sqrt{\widehat{\mathrm{Var}}(\bar{Y}_n)},$$

where $\widehat{\mathrm{Var}}(\bar{Y}_n)$ is an estimator of $\mathrm{Var}(\bar{Y}_n)$, and $t_{\alpha/2,\nu}$ is an appropriate t-distribution quantile with associated degrees of freedom ν.

Instead of $\text{Var}(\bar{Y}_n)$, we can just as well estimate the *variance parameter*,

$$\sigma^2 = \sum_{k=-\infty}^{\infty} R_k,$$

which pops up all over the place: simulation output analysis, Brownian motions, financial engineering applications, etc.

Several methods are available in the simulation literature for estimating σ^2...

- (Nonoverlapping) Batch Means
- Overlapping Batch Means
- Standardized Time Series
- Regenerative Estimation

Each method requires different assumptions to work properly. This is where the GMCC will finally come in to play (in a few minutes).

The method of *batch means* (BM) is often used to estimate σ^2 and to calculate confidence intervals for μ.

Idea: Divide one long simulation run into a number of contiguous *batches*, and then appeal to a central limit theorem (CLT) to assume that the resulting batch sample means are approximately i.i.d. normal.

In particular, suppose that we partition $Y_1, Y_2, \ldots, Y_n$ into b nonoverlapping, contiguous batches, each consisting of m observations (assume that $n = bm$).

$$\underbrace{Y_1, \ldots, Y_m}_{\text{batch } 1}, \underbrace{Y_{m+1}, \ldots, Y_{2m}}_{\text{batch } 2}, \ldots, \underbrace{Y_{(b-1)m+1}, \ldots, Y_{bm}}_{\text{batch } b}$$

The ith batch mean is the sample mean of the m observations from batch $i = 1, 2, \ldots, b$,

$$\bar{Y}_{i,m} \;\equiv\; \frac{1}{m}\sum_{j=1}^{m} Y_{(i-1)m+j}.$$

$$\underbrace{Y_1,\ldots,Y_m}_{\underbrace{\text{batch 1}}_{\bar{Y}_{1,m}}},\underbrace{Y_{m+1},\ldots,Y_{2m}}_{\underbrace{\text{batch 2}}_{\bar{Y}_{2,m}}},\ldots,\underbrace{Y_{(b-1)m+1},\ldots,Y_{bm}}_{\underbrace{\text{batch b}}_{\bar{Y}_{b,m}}}$$

The batch means are correlated for small m, but for large m, a CLT says that

$$\bar{Y}_{1,m},\ldots,\bar{Y}_{b,m} \;\approx\; \text{i.i.d. Nor}(\mu, \text{Var}(\bar{Y}_{i,m})) \;\approx\; \text{Nor}(\mu, \sigma^2/m).$$

We define the *batch means estimator* for

$$\sigma^2 = \lim_{n\to\infty} n\mathrm{Var}(\bar{Y}_n) = \lim_{m\to\infty} m\mathrm{Var}(\bar{Y}_{1,m})$$

as

$$\widehat{V}_{\mathrm{B}} \equiv \frac{m}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m}-\bar{Y}_n)^2 \Rightarrow \frac{\sigma^2\chi^2(b-1)}{b-1} \quad \text{as } m\to\infty,$$

where "$\Rightarrow$" denotes convergence in distribution.

Is $\widehat{V}_{\mathrm{B}}$ a good estimator of σ^2? Assuming uniform integrability[1] of $\widehat{V}_{\mathrm{B}}$,

$$\mathrm{E}[\widehat{V}_{\mathrm{B}}] \doteq \frac{\sigma^2}{b-1}\mathrm{E}[\chi^2(b-1)] = \sigma^2 \quad \text{and} \quad \mathrm{Var}[\widehat{V}_{\mathrm{B}}] \doteq 2\sigma^4/(b-1).$$

So $\widehat{V}_{\mathrm{B}}$ is asymptotically *consistent* for σ^2 as m and $b\to\infty$. ☺

[1] Don't ask

Now the *batch means confidence interval* for μ.

Since the batch means $\bar{Y}_{1,m}, \ldots, \bar{Y}_{b,m} \approx$ i.i.d. Nor$(\mu, \sigma^2/m)$ for large m, we get the following approximate $100(1-\alpha)\%$ CI for μ:

$$\mu \ \in \ \bar{Y}_n \pm t_{\alpha/2,b-1}\sqrt{\widehat{V}_{\mathrm{B}}/n}.$$

And it really delivers nice CI's for μ with coverage $\doteq 1-\alpha$ provided that...

- The batch size m is large enough. [Lots of literature on that.]
- Certain other conditions hold. [See next section.]

Example: Suppose we want to estimate the expected average waiting time for a set of $n = 1000$ customers at Space Mountain starting at noon one fine day. Assuming the system is in steady state, divide the data into $b = 5$ contiguous batches of $m = 200$ customers. The resulting batch means are:

i	1	2	3	4	5
$\bar{Y}_{i,m}$	3.2	4.3	5.1	4.2	4.6

Then (take my word for it) $\bar{Y}_n = 4.28$ and $\widehat{V}_\mathrm{B} = 97.4$. For level $\alpha = 0.05$, we have $t_{0.025,4} = 2.78$, and thus the following 95% CI for the expected average waiting time for the customers:

$$\mu \in 4.28 \pm (2.78)\sqrt{97.4/1000} = [3.41, 5.15]. \quad \Box$$

Outline

Those Nasty Assumptions — Mixing vs. GMCC

An Admission: We kind of glossed over the assumptions necessary to assure that $\widehat{V}_{\mathrm{B}} \Rightarrow \sigma^2\chi^2(b-1)/(b-1)$. This has to be handled a little carefully.

Old vs. New Assumptions

- Moment and Mixing (M&M) Assumptions (old, boring, hard-to-use)
- Geometric-Moment Contraction Condition (new, exciting, easier-to-use)

Goal: Replace M&M with GMCC!

What is Mixing?

For sigma-fields ("fancy sets of events") $\mathcal{A}$ and $\mathcal{B}$, define

$$\phi(\mathcal{A},\mathcal{B}) \;\equiv\; \sup \big|P(B|A) - P(B)\big|,$$

where the sup ("max") is over all events $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $P(A) > 0$.

Let $\mathcal{M}_i^j$ be the sigma-field generated by the observations $\{Y_k\}_{k=i}^j$ for $i \le j$.

We say that $\{Y_k\}$ is φ-*mixing* if $\varphi(\ell) \equiv \sup_{j\in\mathbb{Z}} \phi\big(\mathcal{M}_{-\infty}^j, \mathcal{M}_{j+\ell}^\infty\big) \to 0$ as $\ell \to \infty$.

In other words, for all $j \ge 1$ and $\ell \ge 1$, and any events $A \in \mathcal{M}_{-\infty}^j$ and $B \in \mathcal{M}_{j+\ell}^\infty$, we have $\big|P(B|A) - P(B)\big| \le \varphi(\ell) \to 0$ as $\ell \to \infty$.

What is the Problem with Mixing?

Informally, we can interpret φ-mixing to mean that events in the far future are approximately independent of those in the past.

But I have no idea how to check whether or not $Y_1, Y_2, \ldots$ is φ-mixing in practice!

In addition, when you use mixing to prove certain results, you may have to assume **KraZy** stuff. For instance, in an old paper establishing that $\widehat{V}_{\mathrm{B}}$ has certain properties, Chien et al. required the nutty, restrictive M&M assumptions $\mathrm{E}[Y_k^{28}] < \infty$ and $\varphi(\ell) = O(\ell^{-15})$!

How are you possibly supposed to verify those conditions for real data sets? No way today! ☹

The GMCC Comes to the Rescue

We FINALLY get around to stating the GMCC (first studied by Wu), which is used as a proxy for the rate at which serial dependence fades away.

Consider two independent sequences $\{\varepsilon_j : j \in \mathbb{Z}\}$ and $\{\varepsilon_j' : j \leq 0\}$, each consisting of i.i.d. rv's distributed like ε_0. Suppose the stationary process $\{Y_k\}$ is defined by a function $\xi(\cdot)$ such that $Y_k = \xi(\ldots, \varepsilon_{k-1}, \varepsilon_k)$ for $k \geq 0$; and define the "coupled" process $Y_k' \equiv \xi(\ldots, \varepsilon_{-1}', \varepsilon_0', \varepsilon_1, \ldots, \varepsilon_k)$ for $k \geq 0$.

We say that $\{Y_k\}$ satisfies the GMCC if there exist constants $\psi > 0$, $\mathcal{C}_\psi > 0$, and $r_\psi \in (0,1)$ such that

$$\mathrm{E}\big[\big|Y_k - Y_k'\big|^\psi\big] \leq \mathcal{C}_\psi r_\psi^k \quad \text{for } k \geq 0.$$

In this set-up, the rv's Y_k and Y_k' are identically distributed for all $k \geq 0$.

Moreover, although Y_0 and Y_0' are initialized independently, we see that for $k \geq 1$ the use of common random numbers $(\varepsilon_1, \ldots, \varepsilon_k)$ means that Y_k and Y_k' are dependent and in fact tend to geometrically converge toward each other as k increases.

At first glance, this looks a little nasty, but in the simulation environment, you can tell if the GMCC holds by the use of empirical testing procedures — because you can control the primitive random numbers (the ε_j's and ε_j''s) driving the simulation.

I ♥ the GMCC

Here are some things that the GMCC can do better than M&M conditions. (Won't bore you with the proofs.)

- The GMCC is easier to check in practice than nasty M&M conditions.
- Many processes satisfy the GMCC, but aren't φ-mixing, e.g., the AR(1)!
- A GMCC process has an exponentially decaying covariance function R_k.
- The GMCC can be used to establish cool stuff like $\mathrm{Cov}(Y_iY_j, Y_kY_\ell) = O(r^{\ell-i})$ for $i \le j \le k \le \ell$ and $r \in (0,1)$.
- It is *much, much easier*[2] to use the GMCC to establish that
  - $\mathrm{E}[\widehat{V}_\mathrm{B}] = \sigma^2 + O(m^{-1})$
  - $\mathrm{Var}(\widehat{V}_\mathrm{B}) = 2\sigma^4/(b-1) + O\big((bm)^{-1}\big)$
  - $\widehat{V}_\mathrm{B} \Rightarrow \sigma^2\chi^2(b-1)/(b-1)$ as $m \to \infty$ (and then CIs!)

[2] Don't need uniform integrability or a corvée of algebra

Outline

Example: AR(1) Results for Batch Means

A whirlwind tour showing that the GMCC results work for the AR(1).

First of all, take my word for it that the AR(1) is GMCC.

Recall that $\sigma^2 = (1+\phi)/(1-\phi)$.

In addition, after a LOT of algebra, it can be shown that

$$\begin{aligned}
\mathrm{Cov}(Y_iY_j, Y_kY_\ell) &= \varphi^{\ell+k-j-i} = O(r^{\ell-i}) \quad \text{for } i \le j \le k \le \ell \quad \checkmark \\
\mathrm{E}\big[\widehat{V}_{\mathrm{B}}\big] &= \sigma^2 - \frac{2\phi(b+1)}{n(1-\phi)^2} + O(\phi^m/m) \quad \checkmark \\
\mathrm{Var}\big(\widehat{V}_{\mathrm{B}}\big) &= \frac{2\sigma^4}{b-1} + O\big((bm)^{-1}\big) \quad \checkmark
\end{aligned}$$

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Quadratic-Form Variance Estimators

Again divide $\{Y_1, Y_2, \ldots, Y_n\}$ into b contiguous batches of size m.

Define the quadratic-form estimator $Q_d(m)$ for σ^2 from batch d by

$$Q_d(m) \equiv \sum_{i=1}^{m} \sum_{j=1}^{m} q_{ij}(m) Y_{(d-1)m+i} Y_{(d-1)m+j} \quad \text{for } 1 \le d \le b.$$

Then average these to provide an overall batched quadratic-form estimator,

$$\bar{Q}(b,m) \equiv \frac{1}{b} \sum_{d=1}^{b} Q_d(m).$$

The $q_{ij}(m)$'s are "magic" constants designed to ensure $\mathrm{E}\big[\bar{Q}(b,m)\big] \doteq \sigma^2$.

Example: The STS area estimator is a batched quadratic-form estimator with

$$q_{ij}(m) = \frac{12}{m^3 - m}\Big(\frac{m+1}{2} - i\Big)\Big(\frac{m+1}{2} - j\Big).$$

By design of the $q_{ij}(m)$'s, can easily show that

$$\begin{aligned} \mathrm{E}\big[\bar{Q}(b,m)\big] &= R_0 + 2\sum_{j=1}^{m-1} R_j + \frac{2}{m^3 - m}\sum_{j=1}^{m-1}\big(-3jm^2 + 2j^3 + j\big)R_j \\ &= \sigma^2 \quad \text{as } m \to \infty, \end{aligned}$$

since the exponential decay of the R_j's implies that $\sum_{j=1}^{\infty} j^3 |R_j| < \infty$.

Can also show (mercifully, not here) that

$$\mathrm{Var}\big(\bar{Q}(b,m)\big) = \frac{2\sigma^4}{b} + O(m^{-2}).$$

Outline

Conclusions and Future Work

Showed that

- The GMCC is easier to understand and use than nasty M&M conditions.
- The GMCC performs as advertised, yielding variance estimators with the proper asymptotic means and variances.

Lots of interesting stuff to think about…

- Theoretical wish list
  - Use our results for estimation of other performance measures such as quantiles
  - Continue to build a library of stochastic processes that can be proven to be GMCC
- Implementation items
  - Conduct a Monte Carlo study involving additional estimators and additional performance metrics
  - Provide easy-to-use software to conduct data analysis for GMCC processes

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