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Variance Parameter Estimation for the Quantile Indicator Process

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We develop and evaluate estimators for asymptotic variance parameters related to the quantile indicator function process $\left\{1_{\{Y_k<y\}} : k \ge 1\right\}$ arising from a steady-state stochastic process $\{Y_k : k \ge 1\}$ The estimators under study are based on the methods of nonoverlapping batch quantiles (NBQ) and standardized time series (STS). We show that the bias terms of these estimators are of the order $O\left(m^{-1/4}\right)$, where m is the batch size the methods use. We show that a combined estimator incorporating the sum of NBQ and STS area estimators outperforms the individual component estimators in terms of asymptotic mean squared error. We illustrate our work via a Monte Carlo study involving several stochastic processes.

Key words: stationary process; quantile estimation; Geometric-Moment Contraction condition; variance parameter estimation; nonoverlapping batch quantiles; standardized time series.

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1. INTRODUCTION

Simulation is often used to analyze the characteristics of complicated stochastic processes. In the case of stationary systems, e.g., "warmed-up" simulations, one might be interested in estimating performance characteristics such as the mean or various quantiles. This article studies an emerging area of simulation output analysis—quantile estimation in steady-state simulations. For instance, what is the 95th percentile of a complicated queueing system's cycle-time distribution? Or what is the 0.90-quantile of the supply chain delivery delay of a much-needed part?

The general game plan in steady-state estimation can be described holistically as follows:

(a) (Basic point estimation.) Calculate the (obvious) point estimator for the unknown parameter of interest.

(b) (Variance estimation.) As a measure of risk, estimate the variance of the basic estimator from (a); one might then use the variance estimate to produce a confidence interval (CI) for the parameter.

For example, we would use the sample mean [resp., the appropriate sample quantile] to estimate the true mean [a particular quantile], and then provide an estimator for the variance of the sample mean [the sample quantile].

Of course, owing to the higher-order moments involved, the variance estimation task (b) is more complicated (and the subject of more research) than the basic point estimation task (a). In addition, with task (b) in mind, we note that the work on variance estimation for the sample mean is mature (for example, see Law 2015, Chapter 9, for a high-level survey on this area), while that for the quantile problem is not quite so developed. Only recently has quantile variance estimation come into vogue in the simulation literature, as researchers seek to study this important measure of risk. A reason for this contemporary interest is that sample quantile variance estimation is regarded as “harder to do” than sample mean variance estimation. In particular, sample quantile variance estimation typically requires more “work” (observations) to do an acceptable job than the analogous sample mean variance estimation problem (see, e.g., Dingeç et al. 2023b).

In order to review relavant archival material on quantile variance estimation, suppose that $\{Y_k : k \geq 1\}$ is a stationary sequence of observations, where stationary could mean are anything from independent and identically distributed (i.i.d.) observations (e.g., independent replications of Asian option pricing simulations) to the more-challenging case of autocorrelated random variables (r.v.’s) (e.g., consecutive cycle times in a queueing system). A number of papers in the simulation literature (for instance, Chu and Nakayama 2012, Calvin and Nakayama 2013, Dong and Nakayama 2017) consider the very special case in which the Y_k’s are i.i.d. r.v.’s and employ various tricks of the trade such as certain variance reduction techniques and standardized time series (STS) variance estimators (Schruben 1983) for better estimator and CI performance. **[More references? More details?]** There is also a substantial statistics literature involving empirical and quantile processes, even for the dependent case (see, for instance, the various classic references such as Csörgö and Révész 1981, Csörgö 1983, Shorack and Wellner 1986, Csörgö and Horváth 1993, Dehling et al. 2002, Wu 2005, del Barrio et al. 2007), though most of this work deals with estimator convergence properties and is not directly relevant to simulation applications such as those described in the current paper. **[VERY important to make sure that I haven’t made any false applicability claims and that I haven’t missed any obvious refs, especially w/rt the stats literature.]**

With real-world implementation in mind, a number of heuristic sequential procedures for quantile estimation have been developed over the years in the simulation literature, pertinent in the case where the Y_i’s are stationary: Alexopoulos et al. (2019) review several of these, including procedures given by Raatikainen (1990) and Chen and Kelton (2006, 2008). Notably, the Sequest method of Alexopoulos et al. (2019) and the Sequem method of Alexopoulos et al. (2015) have emerged as comparatively efficient in terms of quantile point estimation and quantile variance estimation. Lolos et al. (2022a,b) propose an improved sequential procedure—based on the work herein and to be described in the sequel.

The current paper builds on results from the following predecessors. Dingeç et al. (2023a) establish theory that provides sufficient conditions for quantile estimators from a stationary stochastic process to have finite variance parameters (defined below), while Dingeç et al. (2023b) go through a number of analytical and numerical examples validating the theory presented in the first paper and illustrating certain robustness results. Alexopoulos et al. (2023) propose new quantile variance estimators for sample quantiles based on the methods of nonoverlapping batch quantiles (NBQ) and STS—both of which incorporate batching—along with accompanying CIs for the quantiles; and, for all variance estimators under consideration, that paper establishes theoretically and empirically that the CIs are asymptotically valid as the batch size increases with the number of batches held fixed. Lolos et al. (2022a,b) use the former paper's results to propose a sequential quantile estimation procedure that is tested and validated on a large selection of interesting stochastic processes; and Lolos et al. (2022a,b) find that a version of the procedure incorporating a combined NBQ+STS variance estimator often outperforms its competitors in that it obtains valid CIs having comparatively modest widths.

We now outline what is coming up in the current paper in terms of both its organization and its contributions. §2 covers the notation and preliminaries that will be necessary to obtain our subsequent results. §3 [§4] presents our main results on the expected values of NBQ [STS] estimators of the asymptotic variance parameters related to (i) the steady-state quantile indicator function process, $\{1_{\{Y_k<y\}} : k \ge 1\}$, and (ii) the sample quantile estimator, denoted by $\widetilde{y}_p(n)$. We show that the bias terms of these variance parameter estimators are all at most of the order $O(m^{-1/4})$, where m is the batch size the methods use. §**??** gives a series of Monte Carlo examples involving stationary processes to compare the different estmators, in particular, in terms of the speed at which estimator bias disappears. To the best of our knowledge, this is the first paper in which such results appear. [It may turn out to be the case that the bias goes away faster than $O(m^{-1/4})$, so that the order we provide is conservative.] §6 concludes the article.

2. NOTATION AND PRELIMINARIES

Suppose that $\{Y_k : k \ge 1\}$ is a stationary stochastic process. For now, we assume that the marginal Y is a continuous r.v., so the cumulative distribution function (c.d.f.) $F(y) \equiv \Pr(Y \le y)$ is continuous, and there is a well-defined probability density function (p.d.f.) $f(y)$. For any $p \in (0,1)$, we define the p-quantile of Y by $y_p \equiv F^{-1}(p) \equiv \inf\{y : F(y) \ge p\}$.

The usual point estimator for y_p is the full-sample p-quantile $\widetilde{y}_p(n) \equiv Y_{(\lceil np \rceil)}$, where $Y_{(1)} \le \cdots \le Y_{(n)}$ are the order statistics from the sample $Y_1, Y_2, \ldots, Y_n$ and $\lceil \cdot \rceil$ denotes the ceiling (round up) function. We define the indicator r.v.'s $I_k(y) \equiv 1_{\{Y_k \le y\}}$ for $k \ge 1$ and $y \in \mathbb{R}$, as well as the corresponding sample mean $\overline{I}_n(y) \equiv n^{-1}\sum_{k=1}^{n} I_k(y)$, the later of which is an unbiased point estimator $F(y)$ for $y \in \mathbb{R}$.

We will invoke the following **Standing Assumptions** on $\{Y_k : k \geq 1\}$ and $\{I_k(y) : k \geq 1\}$ as we proceed.

Density-Regularity and Moment (DRM) Conditions The p.d.f. $f(y)$ is bounded and continuous for $y \in \mathbb{R}$; and at the quantile y_p to be estimated, we have $f(y_p) > 0$ and $f'(y_p)$ exists. Moreover, the marginal absolute moment $\mathrm{E}\big[|Y_0|^u\big] < \infty$ for some $u > 2$.

Geometric-Moment Contraction (GMC) Condition The process $\{Y_k : k \geq 1\}$ is defined by a function $\xi(\cdot)$ of a sequence of i.i.d. r.v.'s $\{\varepsilon_j : j \in \mathbb{Z}\}$ such that $Y_k = \xi(\ldots, \varepsilon_{k-1}, \varepsilon_k)$ for $k \geq 1$. Moreover, there exist constants $\psi > 0$, $C_\psi > 0$, and $r \in (0,1)$ such that for two independent sequences $\{\varepsilon_j : j \in \mathbb{Z}\}$ and $\{\varepsilon'_j : j \leq 0\}$ each consisting of i.i.d. r.v.'s distributed like ε_0, we have

$$\mathrm{E}\Big[\big|\xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) - \xi(\ldots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k)\big|^\psi\Big] \leq C_\psi r^k \text{ for } k \geq 0. \tag{1}$$

The DRM conditions ensure that the p.d.f. is well-behaved, especially in the neighborhood of a quantile of interest. The GMC condition is related to the temporal dependence structure of $\{Y_k : k \geq 1\}$ and is informally regarded as a proxy for the exponential decay of serial correlation. The GMC condition is satisfied by many interesting processes that come up in simulation applications, and in any case, the GMC condition can be used as an easy-to-check substitute for ϕ-mixing conditions that are typically required in simulation output analysis; see, for instance, Wu and Shao (2004), Shao and Wu (2007), Alexopoulos et al. (2023), and Dingeç et al. (2022, 2023a,c).

Here are some useful results consolidated from Dingeç et al. (2023a, §4) that we will use in the sequel (cf. related work in Doukhan 2018 and Xu 2021). [For now, we'll cite specific results from Dingeç et al. (2023a) but may have to change once we stabilize all of the papers. Also, the moment condition is trivially satisfied for the indicator process, so might not always need that.]

Theorem 1. *Suppose that $\{Y_k : k \geq 1\}$ satisfies the GMC and DRM conditions. Then we have the following.*

1. *[Dingeç et al. (2023a, Theorem 4)] For any $y \in \mathbb{R}$, $\{I_k(y) : k \geq 1\}$ satisfies the GMC condition.*
2. *Denote the autocovariance function of the indicator process by $R_{I(y)}(\ell) \equiv \mathrm{Cov}[I_k(y), I_{k+\ell}(y)]$ for lag $\ell \in \mathbb{Z}$ and $y \in \mathbb{R}$. Then*
   - *(i) [Dingeç et al. (2023a, Corollary 2(a,b))] $R_{I(y)}(\ell) = O(s^\ell)$ for some $s \in (0,1)$ and all $y \in \mathbb{R}$. Moreover, $\sum_{\ell \in \mathbb{Z}} |R_{I(y)}(\ell)| < \infty$, i.e., $\{I_k(y) : k \geq 1\}$ is short-range dependent (SRD).*
   - *(ii) [Dingeç et al. (2023a, Corollary 2(c))] For each real y, the indicator process $\big\{I_k(y) : k \geq 1\big\}$ has a finite variance parameter $\sigma^2_{I(y)} \equiv \lim_{n\to\infty} n\mathrm{Var}\big[\bar{I}_n(y)\big] = \sum_{\ell \in \mathbb{Z}} R_{I(y)}(\ell) \in [0, \infty)$.*
   - *(iii) [Dingeç et al. (2023a, Corollary 2(d))] The quantities $\gamma_{I(y),a} \equiv 2\sum_{\ell=1}^{\infty} \ell^a R_{I(y)}(\ell)$, for $a = 1, 2, \ldots$, are all finite.*
   - *(iv) [Dingeç et al. (2023a, Corollary 2(e) and Lemma 1)] Define the variance parameter corresponding to the p-quantile point estimator $\widetilde{y}_p(n)$ by $\sigma^2_{\widetilde{y}_p} \equiv \lim_{n\to\infty} n\mathrm{Var}\big[\widetilde{y}_p(n)\big]$. Then we have*

$$\sigma^2_{\widetilde{y}_p} = \frac{\sigma^2_{I(y_p)}}{f^2(y_p)} \in [0, \infty). \tag{2}$$

.

As we proceed, it will prove efficacious to incorporate the use of batching and what is known as the Bahadur representation of quantile estimators.

Batching and Bahadur Suppose we divide the n observations $Y_1, Y_2, \ldots, Y_n$ into $b \geq 1$ contiguous, nonoverlapping batches of size $m = n/b$ (assume m is an integer), so that the jth batch is $\{Y_{(j-1)m+1}, Y_{(j-1)m+2}, \ldots, Y_{jm}\}$, for $j = 1, 2, \ldots, b$. For any $y \in \mathbb{R}$ and $j = 1, 2, \ldots, b$, we compute the jth batch mean of the corresponding indicator r.v.'s, $\bar{I}_{j,m}(y) \equiv m^{-1} \sum_{\ell=1}^{m} I_{(j-1)m+\ell}(y)$; the order statistics $Y_{j,(1)} \leq Y_{j,(2)} \leq \cdots \leq Y_{j,(m)}$; and $\widehat{y}_p(j,m) \equiv Y_{j,(\lceil mp \rceil)}$, the jth batch quantile estimator (BQE) of y_p. The Bahadur representation (3) (Serfling 1980) is often used to bypass working with the BQEs directly, [Can someone remind me of the precise consitions that Bahadur requires?]

$$\widehat{y}_p(j,m) = y_p - \frac{\bar{I}_{j,m}(y_p) - p}{f(y_p)} + O_{\text{a.s.}}\left(\frac{\log^{3/2} m}{m^{3/4}}\right) \text{ as } m \to \infty \text{ for } j = 1, \ldots, b, \tag{3}$$

where if $\{\mathscr{U}_n : n \geq 1\}$ is a sequence of r.v.'s and $\{a_n : n \geq 1\}$ is a sequence of nonnegative constants, then $\mathscr{U}_n = O_{\text{a.s.}}(a_n)$ means there is an r.v. $\mathfrak{U}$ that is bounded a.s. (almost surely) such that $|\mathscr{U}_n| \leq \mathfrak{U} a_n$ for $n \geq 1$ a.s. (Wu 2005). Alexopoulos et al. (2019) prove the following BQE limit result.

Theorem 2. *If $\{Y_k : k \geq 1\}$ satisfies the GMC and DRM conditions, then [note that we no longer have to assume FCLT, since the FCLT follow from the assumptions by Theorem 1.3!]*

$$m^{1/2}\big[\widehat{y}_p(1,m) - y_p, \ldots, \widehat{y}_p(b,m) - y_p\big]^{\mathsf{T}} \underset{m\to\infty}{\Longrightarrow} \big[\sigma_{I(y_p)}/f(y_p)\big]\mathbf{Z}_b, \tag{4}$$

where $\mathbf{Z}_b \equiv [Z_1, Z_2, \ldots, Z_b]^{\mathsf{T}}$ for which $\{Z_i : i = 1, 2, \ldots, b\}$ are i.i.d. Nor$(0,1)$.

3. Expected Value of the Nonoverlapping Batch Quantile Estimators for $\sigma^2_{\widetilde{y}_p}$

We are interested in finding the bias of two classes of estimators for the variance parameter $\sigma^2_{\widetilde{y}_p}$ of the quantile process—NBQ and STS. In this section, we will give bias results for two different NBQ estimators; and in §4, we will derive the bias of an STS area estimator that also incorporates batching. We will give away the store right now by stating all of these estimators have bias terms that disappear approximately at rate $O(m^{-1/4})$ as the batch size m becomes large.

On the way to finding the bias of the two NBQ estimators for $\sigma^2_{\widetilde{y}_p}$, we will go through a recipe of intermediate calculations that will prove to be useful when we put everything together in §§3.7–3.8. In §3.1, we define the two NBQ estimators and present the statement (but not yet the proof) of the main Theorem 3. §3.2 uses the Bahadur representation (3) to neatly write the BQEs and their cousins in terms of the quantile-indicator process. §3.3 takes a side trip where we find simple order expressions for the expected value and variance of the Bahadur representation remainder terms, and §3.4 calculates several bounds on covariances involving those remainder terms. §3.5 finds expressions for the variances of three quantile point estimators of interest—$\widetilde{y}_p(n)$, $\widehat{y}_p(j,m)$, and $\overline{y}_p(b;m) \equiv b^{-1}\sum_{j=1}^{b}\widehat{y}_p(j,m)$. As a final intermediate step in our toils, §3.6 is devoted to the calculation of the expected value and variance of the difference $\widehat{y}_p(j,m) - \widetilde{y}_p(n)$. We will then be in position to prove Theorem 3 when we finally arrive at our §§3.7–3.8 destination.

3.1. Definitions of NBQ Estimators and Statement of Theorem 3

As alluded to above, there are actually *two* NBQ estimators for $\sigma^2_{\widetilde{y}_p}$ both of which resemble the sample variance of the BQEs for the p-quantile y_p. The first is centered at the sample mean of the BQEs, $\overline{y}_p(b,m) = b^{-1}\sum_{j=1}^{b}\widehat{y}_p(j,m)$, while the second is centered at the full-sample point estimator $\widetilde{y}_p(n)$ for y_p. To be explicit, define the (first) NBQ estimator for $\sigma^2_{\widetilde{y}_p}$ by

$$\mathcal{N}_{\widetilde{y}_p}(b,m) \equiv \frac{m}{b-1}\sum_{j=1}^{b}\big(\widehat{y}_p(j,m) - \overline{y}_p(b,m)\big)^2 = \frac{m}{b-1}\left[\sum_{j=1}^{b}\widehat{y}_p^{\,2}(j,m) - b\overline{y}_p^{\,2}(b,m)\right];$$

and define the (second) NBQ estimator for $\sigma^2_{\widetilde{y}_p}$ by

$$\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m) \equiv \frac{m}{b-1}\sum_{j=1}^{b}\big(\widehat{y}_p(j,m) - \widetilde{y}_p(n)\big)^2.$$

Here is the main theorem that we will prove on the NBQ estimators. Note that in what follows, the number of batches b is always fixed, so that for any reasonable function $\zeta(\cdot)$, we have $O(\zeta(m)) = O(\zeta(n))$, and we will often use these terms interchangeably. Moreover, we do not necessarily strive for the tightest inequalities in the subsections below, especially if those improved inequalities would anyhow be overwhelmed by others as we proceed.

Theorem 3. *Suppose that* $\{Y_k : k \geq 1\}$ *satisfies the DRM and GMC Standing Assumptions [and maybe whatever is needed for Bahadur, but I think that's about all we need]. Further assume that the number of nonoverlapping batches b is fixed. Then the expected values of the two NBQ estimators are of the form*

$$\mathrm{E}\big[\mathcal{N}_{\widetilde{y}_p}(b,m)\big] = \sigma^2_{\widetilde{y}_p} + O\Big(m^{-1/4}\log^3 m\Big) \quad \textit{and} \quad \mathrm{E}\big[\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)\big] = \sigma^2_{\widetilde{y}_p} + O\Big(m^{-1/4}\log^3 m\Big) \quad \textit{as } m \to \infty. \tag{5}$$

Thus, both $\mathrm{E}\big[\mathcal{N}_{\widetilde{y}_p}(b,m)\big]$ and $\mathrm{E}\big[\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)\big]$ are asymptotically unbiased estimators for $\sigma^2_{\widetilde{y}_p}$ as the batch size $m \to \infty$. We establish background results in §§3.2–3.6, before completing Theorem 3's proof in §§3.7–3.8.

3.2. Bahadur Representation of Quantile Estimators

By the Bahadur representation (3) (with the batch size m replaced by the full sample size n), we have

$$\widetilde{y}_p(n) = y_p - \frac{\overline{I}_n(y_p) - p}{f(y_p)} + Q_n, \tag{6}$$

where $Q_n = O_{\text{a.s.}}\big(n^{-3/4}\log^{3/2} n\big)$ and we recall that $\overline{I}_n(y) = n^{-1}\sum_{k=1}^n I_k(y) = n^{-1}\sum_{i=1}^n 1_{\{Y_i \le y\}}$ for all $y \in \mathbb{R}$. Similarly, for batches $j = 1, 2, \ldots, b$, we have

$$\widehat{y}_p(j,m) \;=\; y_p - \frac{\overline{I}_{j,m}(y_p) - p}{f(y_p)} + Q_{j,m}, \tag{7}$$

where $Q_{j,m} = O_{\text{a.s.}}\big(m^{-3/4}\log^{3/2} m\big)$ and $\overline{I}_{j,m}(y) \equiv m^{-1}\sum_{k=1}^m I_{(j-1)m+k}(y)$ for all $y \in \mathbb{R}$. Then we can write the sample mean of the BQEs as

$$\overline{y}_p(b;m) \;=\; \frac{1}{b}\sum_{j=1}^b \widehat{y}_p(j,m) \;=\; y_p - \frac{\frac{1}{b}\sum_{j=1}^b \overline{I}_{j,m}(y_p) - p}{f(y_p)} + \frac{1}{b}\sum_{j=1}^b Q_{j,m} \;=\; y_p - \frac{\overline{I}_n(y_p) - p}{f(y_p)} + \overline{Q}_m, \tag{8}$$

where $\overline{Q}_m \equiv b^{-1}\sum_{j=1}^b Q_{j,m}$ and $\overline{I}_n(y_p) = b^{-1}\sum_{j=1}^b \overline{I}_{j,m}(y_p)$. We will also eventually need the fact that

$$\widehat{y}_p(j,m) - \widetilde{y}_p(n) \;=\; \frac{\overline{I}_n(y_p)}{f(y_p)} - \frac{\overline{I}_{j,m}(y_p)}{f(y_p)} + Q_{j,m} - Q_n \quad \text{for } j = 1, 2 \ldots, b. \tag{9}$$

3.3. Expected Value and Variance of the Bahadur Remainder Term

The following results will be useful when we finally get around to simplifying our final expressions for $\mathrm{E}\big[\mathcal{N}_{\widetilde{y}_p}(b,m)\big]$ and $\mathrm{E}\big[\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)\big]$ in §§3.7–3.8.

Lemma 1. *Let $\{X_n : n \ge 1\}$ be a sequence of r.v.'s and $\{a_n : n \ge 1\}$ be a sequence of nonnegative constants.*

$$\textit{If } X_n \;=\; O_{\text{a.s.}}(a_n), \;\; \textit{then } \mathrm{E}[X_n] \;=\; O(a_n) \;\; \textit{and } \mathrm{Var}(X_n) \;=\; O(a_n^2). \tag{10}$$

Proof: If $X_n = O_{\text{a.s.}}(a_n)$, then there exist r.v.'s $\mathscr{U} \in \mathbb{R}^+$ and $\mathscr{V} \in \mathbb{Z}^+$ and real constants $M_{\mathscr{U}}$ and $M_{\mathscr{V}}$ such that

$$0 < \mathscr{U} \le M_{\mathscr{U}} \quad \text{and} \quad 0 < \mathscr{V} \le M_{\mathscr{V}} \text{ a.s.} \tag{11}$$

and

$$|X_n| \le \mathscr{U} a_n \quad \text{for} \quad n \ge \mathscr{V} \quad \text{a.s.} \tag{12}$$

Let $X_n^+ \equiv \max\{X_n, 0\}$ and $X_n^- \equiv -\min\{X_n, 0\}$ respectively denote the positive and negative parts of X_n for $n \ge 1$. Equations (11) and (12) imply that

$$X_n \in \big[-M_{\mathscr{U}} a_n,\, M_{\mathscr{U}} a_n\big] \;\text{ and }\; X_n^+, X_n^- \in \big[0,\, M_{\mathscr{U}} a_n\big] \;\text{ for }\; n \ge M_{\mathscr{V}} \text{ a.s.} \tag{13}$$

So by the definition of the expectations $\mathrm{E}[X_n^+]$, $\mathrm{E}[X_n^-]$, and $\mathrm{E}[X_n]$, we have

$$\mathrm{E}[X_n^+],\, \mathrm{E}[X_n^-] \in \big[0,\, M_{\mathscr{U}} a_n\big] \;\text{ and }\; \mathrm{E}[X_n] \equiv \mathrm{E}[X_n^+] - \mathrm{E}[X_n^-] \in \big[-M_{\mathscr{U}} a_n,\, M_{\mathscr{U}} a_n\big] \;\text{ for }\; n \ge M_{\mathscr{V}} \tag{14}$$

(Shiryaev 2016, pp. 218–219). Thus, Equation (14) implies that

$$|\mathrm{E}[X_n]| \le M_{\mathscr{U}} a_n \;\text{ for }\; n \ge M_{\mathscr{V}}. \tag{15}$$

In addition, Popoviciu's inequality (Bhatia and Davis 2000, p. 353) implies that

$$\mathrm{Var}[X_n] \;\le\; \frac{(2M_{\mathscr{U}} a_n)^2}{4} \;=\; M_{\mathscr{U}}^2 a_n^2 \;\text{ for }\; n \ge M_{\mathscr{V}}. \tag{16}$$

Expression (10) follows immediately from Equations (15) and (16). ∎

Example 1. For the Bahadur remainder term Q_n, we have

$$\mathrm{E}[Q_n] = O\big(n^{-3/4}\log^{3/2} n\big) \quad \text{and} \quad \mathrm{Var}[Q_n] = O\big(n^{-3/2}\log^{3} n\big). \quad \square \tag{17}$$

3.4. Bounds on Several Covariances Involving Remainder Terms

It is a straightforward algebraic fact that

$$\sigma^2_{I(y),n} \equiv n\mathrm{Var}\big[\bar{I}_n(y)\big] = R_{I(y)}(0) + 2\sum_{\ell=1}^{n-1}\left(1-\frac{\ell}{n}\right)R_{I(y)}(\ell).$$

Theorem 1.2(i)'s finding that $R_{I(y)}(\ell) = O(s^\ell)$ is sufficient to invoke Aktaran-Kalaycı et al. (2007, Corollary 2), which in turn yields

$$\sigma^2_{I(y),n} = \sigma^2_{I(y)} - \frac{\gamma_{I(y),1}}{n} + O(s^n) = \sigma^2_{I(y)} + O(1/n), \tag{18}$$

where Theorem 1.2(ii) and (iii) respectively yield $\sigma^2_{I(y)} \in [0,\infty)$ and $|\gamma_{I(y),1}| < \infty$ (so all of the terms above are well-defined).

This easy little result immediately allows us to establish bounds for several covariances involving the Bahadur error terms, which we will now catalog for future use. To begin, we have

$$\begin{aligned}
\big|\mathrm{Cov}\big[\bar{I}_n(y), Q_n\big]\big| &\le \sqrt{\mathrm{Var}\big[\bar{I}_n(y)\big]\mathrm{Var}\big[Q_n\big]} \quad \text{(Cauchy–Schwarz)} \\
&= \sqrt{\left[\frac{\sigma^2_{I(y)}}{n} + O(n^{-2})\right] O\big(n^{-3/2}\log^3 n\big)} \quad \text{(by Equations (18) and (17) in that order)} \\
&= O\big(n^{-5/4}\log^{3/2} n\big).
\end{aligned} \tag{19}$$

Similarly, and taking into account that b is fixed so that $O(m) = O(n)$,

$$\big|\mathrm{Cov}\big[\bar{I}_{j,m}(y), Q_n\big]\big| \le \sqrt{\left[\frac{\sigma^2_{I(y)}}{m} + O(m^{-2})\right] O\big(n^{-3/2}\log^3 n\big)} = O\big(m^{-5/4}\log^{3/2} m\big). \tag{20}$$

And again similarly,

$$\big|\mathrm{Cov}\big[\bar{I}_n(y), Q_{j,m}\big]\big| = O\big(m^{-5/4}\log^{3/2} m\big). \tag{21}$$

Thus, by the triangle inequality and Equation (21), we have

$$\big|\mathrm{Cov}\big[\bar{I}_n(y), \bar{Q}_m\big]\big| \le \frac{1}{b}\sum_{j=1}^{b}\big|\mathrm{Cov}\big[\bar{I}_n(y), Q_{j,m}\big]\big| = O\big(m^{-5/4}\log^{3/2} m\big). \tag{22}$$

Finally, we have

$$\begin{aligned}
\big|\mathrm{Cov}\big[Q_{j,m}, Q_n\big]\big| &\le \sqrt{\mathrm{Var}\big[Q_{j,m}\big]\,\mathrm{Var}\big[Q_n\big]} \quad \text{(Cauchy–Schwarz)} \\
&= O\big(m^{-3/4}\log^{3/2} m\, n^{-3/4}\log^{3/2} n\big) \quad \text{(by Equation (17))} \\
&= O\big(m^{-3/2}\log^{3} m\big).
\end{aligned} \tag{23}$$

3.5. Some Variance Calculations on Quantile Point Estimators

We now derive simple expressions for the variances of the three quantile point estimators $\widetilde{y}_p(n)$, $\widehat{y}_p(j,m)$, and $\overline{y}_p(b;m)$. First of all, by Equation (6), we have

$$\begin{aligned}
\mathrm{Var}\big[\widetilde{y}_p(n)\big] &= \mathrm{Var}\left[y_p - \frac{\overline{I}_n(y_p)-p}{f(y_p)} + Q_n\right] \\
&= \frac{\mathrm{Var}\big[\overline{I}_n(y_p)\big]}{f^2(y_p)} + \mathrm{Var}\big[Q_n\big] - \frac{2\,\mathrm{Cov}\big[\overline{I}_n(y_p), Q_n\big]}{f(y_p)} \\
&= \left[\frac{\sigma^2_{I(y_p)}}{nf^2(y_p)} + O(1/n^2)\right] + O\big(n^{-3/2}\log^3 n\big) + O\big(n^{-5/4}\log^{3/2} n\big) \\
&\quad \text{(by Equations (18), (17), and (19) in that order)} \\
&= \frac{\sigma^2_{\widetilde{y}_p}}{n} + O\big(n^{-5/4}\log^3 n\big). \qquad (24)
\end{aligned}$$

By Equation (7) and calculations analogous to those above, we have for $j = 1, 2, \ldots, b$,

$$\mathrm{Var}\big[\widehat{y}_p(j,m)\big] = \mathrm{Var}\left[y_p - \frac{\overline{I}_{j,m}(y_p)-p}{f(y_p)} + Q_{j,m}\right] = \frac{\sigma^2_{\widetilde{y}_p}}{m} + O\big(m^{-5/4}\log^3 m\big). \qquad (25)$$

Similarly, by Equation (8),

$$\begin{aligned}
\mathrm{Var}\big[\overline{y}_p(b;m)\big] &= \mathrm{Var}\left[y_p - \frac{\overline{I}_n(y_p)-p}{f(y_p)} + \overline{Q}_m\right] \\
&= \frac{\mathrm{Var}\big[\overline{I}_n(y_p)\big]}{f^2(y_p)} + \mathrm{Var}\big[\overline{Q}_m\big] - \frac{2\,\mathrm{Cov}\big[\overline{I}_n(y_p), \overline{Q}_m\big]}{f(y_p)} \\
&= \left[\frac{\sigma^2_{I(y_p)}}{nf^2(y_p)} + O(1/n^2)\right] + O\big(n^{-3/2}\log^3 n\big) + O\big(m^{-5/4}\log^{3/2} m\big) \\
&\quad \text{(by Equations (18), (17), and (22) in that order)} \\
&= \frac{\sigma^2_{\widetilde{y}_p}}{n} + O\big(m^{-5/4}\log^3 m\big). \qquad (26)
\end{aligned}$$

3.6. Some Final Intermediate Expected Value and Variance Calculations

We go through one last set of steps before undertaking the proof of Theorem 3 in §§3.7–3.8. In particular, in this subsection, we will derive expressions for the expected value and variance of $\widehat{y}_p(j,m) - \widetilde{y}_p(n)$.

First, by Equation (9), we have

$$\begin{aligned}
\mathrm{E}\Big[\widehat{y}_p(j,m) - \widetilde{y}_p(n)\Big] &= \mathrm{E}\left[\frac{\overline{I}_n(y_p)}{f(y_p)} - \frac{\overline{I}_{j,m}(y_p)}{f(y_p)} + Q_{j,m} - Q_n\right] \\
&= \frac{p}{f(y_p)} - \frac{p}{f(y_p)} + \mathrm{E}\big[Q_{j,m}\big] - \mathrm{E}\big[Q_n\big] \\
&= O\Big(m^{-3/4}\log^{3/2} m\Big) \quad \text{(by Equation (17) applied twice).} \qquad (27)
\end{aligned}$$

We now work on several intermediate results. First,

$$
\begin{aligned}
\left|\sum_{i=1,i\neq j}^{b} \mathrm{Cov}\big[\bar{I}_{j,m}(y),\,\bar{I}_{i,m}(y)\big]\right| &\le 2\sum_{i=2}^{b}\Big|\mathrm{Cov}\big[\bar{I}_{1,m}(y),\,\bar{I}_{i,m}(y)\big]\Big| \\
&\le \frac{2}{m^2}\sum_{i=2}^{b}\sum_{k=1}^{m}\sum_{\ell=1}^{m}\big|\mathrm{Cov}\big[I_k(y),\,I_{(i-1)m+\ell}(y)\big]\big| \\
&\le \frac{2}{m^2}\sum_{i=2}^{b}\sum_{k=1}^{m}\sum_{\ell=1}^{m}\big|R_{I(y)}\big((i-1)m+\ell-k\big)\big| \\
&\le \frac{C}{m^2}\sum_{i=2}^{b}\sum_{k=1}^{m}\sum_{\ell=1}^{m} s^{(i-1)m+\ell-k} \quad \text{(by the GMC, for large-enough } C\text{)} \\
&= \frac{C}{m^2}\sum_{i=1}^{b-1} s^{im}\sum_{k=1}^{m} s^{-k}\sum_{\ell=1}^{m} s^{\ell} \\
&= \frac{C}{m^2}\,\frac{s^m(1-s^{m(b-1)})}{1-s^m}\,\frac{s^{-1}(1-s^{-m})}{1-s^{-1}}\,\frac{s(1-s^m)}{1-s} \\
&= O(1/m^2). \qquad (28)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathrm{Cov}\big[\bar{I}_{j,m}(y),\,\bar{I}_{n}(y)\big] &= \frac{1}{b}\sum_{i=1}^{b}\mathrm{Cov}\big[\bar{I}_{j,m}(y),\,\bar{I}_{i,m}(y)\big] \\
&= \frac{1}{b}\left[\mathrm{Var}\big[\bar{I}_{1,m}(y)\big] + \sum_{i=1,i\neq j}^{b}\mathrm{Cov}\big[\bar{I}_{j,m}(y),\,\bar{I}_{i,m}(y)\big]\right] \\
&= \frac{1}{b}\left[\left(\frac{\sigma^2_{I(y)}}{m} + O(1/m^2)\right) + O(1/m^2)\right] \quad \text{(by (18) with } m \text{ instead of } n\text{, and (28))} \\
&= \frac{\sigma^2_{I(y)}}{n} + O(1/m^2). \qquad (29)
\end{aligned}
$$

By Equations (6) and (7), we have

$$
\begin{aligned}
\mathrm{Cov}\big[\widehat{y}_p(j,m),\,\widetilde{y}_p(n)\big] &= \mathrm{Cov}\left[y_p - \frac{\bar{I}_{j,m}(y_p)-p}{f(y_p)} + Q_{j,m},\; y_p - \frac{\bar{I}_n(y_p)-p}{f(y_p)} + Q_n\right] \\
&= \mathrm{Cov}\left[\frac{\bar{I}_{j,m}(y_p)}{f(y_p)} - Q_{j,m},\; \frac{\bar{I}_n(y_p)}{f(y_p)} - Q_n\right] \\
&= \frac{\mathrm{Cov}\big[\bar{I}_{j,m}(y_p),\,\bar{I}_n(y_p)\big]}{f^2(y_p)} - \frac{\mathrm{Cov}\big[\bar{I}_{j,m}(y_p),\,Q_n\big]}{f(y_p)} \\
&\quad - \frac{\mathrm{Cov}\big[Q_{j,m},\,\bar{I}_n(y_p)\big]}{f(y_p)} + \mathrm{Cov}(Q_{j,m},\,Q_n) \\
&= \left[\frac{\sigma^2_{I(y_p)}}{nf^2(y_p)} + O(1/m^2)\right] + O\big(m^{-5/4}\log^{3/2} m\big) + O\big(m^{-5/4}\log^{3/2} m\big) + O\big(m^{-3/2}\log^{3} m\big)
\end{aligned}
$$

$$\text{(by Equations (29), (20), (21), and (23) in that order)}$$
$$= \frac{\sigma^2_{\widetilde{y}_p}}{n} + O\big(m^{-5/4}\log^3 m\big). \tag{30}$$

Finally, we calculate

$$\begin{aligned}
\mathrm{Var}\big[\widehat{y}_p(j,m) - \widetilde{y}_p(n)\big] &= \mathrm{Var}\big[\widehat{y}_p(j,m)\big] + \mathrm{Var}\big[\widetilde{y}_p(n)\big] - 2\,\mathrm{Cov}\big[\widehat{y}_p(j,m),\, \widetilde{y}_p(n)\big] \\
&= \left[\frac{\sigma^2_{\widetilde{y}_p}}{m} + O\big(m^{-5/4}\log^3 m\big)\right] + \left[\frac{\sigma^2_{\widetilde{y}_p}}{n} + O\big(n^{-5/4}\log^3 n\big)\right] - 2\left[\frac{\sigma^2_{\widetilde{y}_p}}{n} + O\big(m^{-5/4}\log^3 m\big)\right] \\
&\qquad \text{(by Equations (25), (24), and (30) in that order)} \\
&= \frac{(b-1)\sigma^2_{\widetilde{y}_p}}{n} + O\big(m^{-5/4}\log^3 m\big). \qquad (31)
\end{aligned}$$

3.7. Calculation of $\mathrm{E}\big[\mathcal{N}_{\widetilde{y}_p}(b,m)\big]$

Here we calculate the expected value of $\mathcal{N}_{\widetilde{y}_p}(b,m)$—the “first” NBQ estimator of the variance parameter $\sigma^2_{\widetilde{y}_p}$. Noting that

$$\mathrm{E}\big[\overline{y}_p(b;m)\big] = \frac{1}{b}\sum_{j=1}^{b}\mathrm{E}\big[\widehat{y}_p(j,m)\big] = \mathrm{E}\big[\widehat{y}_p(j,m)\big], \tag{32}$$

we have

$$\begin{aligned}
\mathrm{E}\big[\mathcal{N}_{\widetilde{y}_p}(b,m)\big] &= \frac{mb}{b-1}\left[\mathrm{E}\big[\widehat{y}^2_p(j,m)\big] - \mathrm{E}\big[\overline{y}^2_p(b;m)\big]\right] \\
&= \frac{mb}{b-1}\left[\mathrm{Var}\big[\widehat{y}_p(j,m)\big] + \big(\mathrm{E}\big[\widehat{y}_p(j,m)\big]\big)^2 - \mathrm{Var}\big[\overline{y}_p(b;m)\big] - \Big(\mathrm{E}\big[\overline{y}_p(b;m)\big]\Big)^2\right] \\
&= \frac{mb}{b-1}\Big[\mathrm{Var}\big[\widehat{y}_p(j,m)\big] - \mathrm{Var}\big[\overline{y}_p(b;m)\big]\Big] \quad \text{(by Equation (32))} \qquad (33) \\
&= \frac{mb}{b-1}\left\{\left[\frac{\sigma^2_{\widetilde{y}_p}}{m} + O\big(m^{-5/4}\log^3 m\big)\right] - \left[\frac{\sigma^2_{\widetilde{y}_p}}{n} + O\big(m^{-5/4}\log^3 m\big)\right]\right\} \quad \text{(by (25) and (26))} \\
&= \sigma^2_{\widetilde{y}_p} + O\big(m^{-1/4}\log^3 m\big). \qquad (34)
\end{aligned}$$

3.8. Calculation of $\mathrm{E}\big[\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)\big]$

We complete the proof by calculating the expected value of the “second” NBQ estimator, $\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)$.

$$\begin{aligned}
\mathrm{E}\Big[\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)\Big] &= \frac{m}{b-1}\sum_{j=1}^{b}\mathrm{E}\Big[\big(\widehat{y}_p(j,m) - \widetilde{y}_p(n)\big)^2\Big] \\
&= \frac{m}{b-1}\sum_{j=1}^{b}\Big\{\mathrm{Var}\big[\widehat{y}_p(j,m) - \widetilde{y}_p(n)\big] + \big(\mathrm{E}\big[\widehat{y}_p(j,m) - \widetilde{y}_p(n)\big]\big)^2\Big\} \\
&= \frac{m}{b-1}\sum_{j=1}^{b}\left\{\left[\frac{(b-1)\sigma^2_{\widetilde{y}_p}}{n} + O\big(m^{-5/4}\log^3 m\big)\right] + \Big(O\big(m^{-3/4}\log^{3/2} m\big)\Big)^2\right\} \quad \text{(by (31) and (27))} \\
&= \sigma^2_{\widetilde{y}_p} + O\big(m^{-1/4}\log^3 m\big). \quad \blacksquare \qquad (35)
\end{aligned}$$

REMARK 1. Equations (34) and (35) imply that both batch quantile estimators $\mathcal{N}_{\widetilde{y}_p}(b,m)$ and $\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m)$ have slightly worse than $O(m^{-1/4})$ bias, which decreases quite slowly—at least compared to variance estimators in the context of mean estimation where the corresponding batch means variance estimator bias terms are at worst $O(1/m)$ (Aktaran-Kalaycı et al. 2007).

4. Expected Value of the Standardized Time Series Area Estimator for $\sigma^2_{\widetilde{y}_p}$

The area estimator is discussed and motivated extensively in Alexopoulos et al. (2023), where we present the theory necessary to construct asymptotically valid CIs for the quantile y_p. In that former paper, we merely need the expected value of the area estimator to converge to $\sigma^2_{\widetilde{y}_p}$ as the sample size grows, but we do not study the rate. In this section of the current paper, we derive more-precise results on the expected value of the area estimator.

§4.1 presents notation that will be used in the sequel. Based on the entire sample $\{Y_1, Y_2, \ldots, Y_n\}$ of size n, we define two STS processes—one associated with the indicator process and one associated with the quantile-estimator process. We also formally define the associated STS area estimator for $\sigma^2_{I(y)}$ as well as the STS area estimator for $\sigma^2_{\widetilde{y}_p}$. §4.2 derives our main results on the expected value of the area estimators, where we show that those for $\sigma^2_{I(y)}$ and $\sigma^2_{\widetilde{y}_p}$ have bias of at most $O(n^{-1/4})$. In §4.3, we study versions of the area estimator that incorporate batching. In particular, the (batched) STS area estimator for $\sigma^2_{\widetilde{y}_p}$ is based on the average of individual area estimators for $\sigma^2_{\widetilde{y}_p}$ calculated from each of b contiguous batches, each of which is itself comprised of m observations. Batching typically reduces estimator variance at the potential expense of extra bias, a phenomenon borne out by the Monte Carlo analysis upcoming in §5. We also show how to combine the STS estimator with the NBQ estimator from §3, which has the advantage of reducing estimator variance without incurring a further significant increase in bias.

4.1. Definitions and Notation

For now, we will regard $\{Y_1, Y_2, \ldots, Y_n\}$ as a single large batch of observations. The STS indicator and quantile-estimator processes are respectively defined as

$$T_{I(y),n}(t) \equiv \frac{\lfloor nt \rfloor}{\sigma_{I(y)}\sqrt{n}}\left[\bar{I}_{\lfloor nt \rfloor}(y) - \bar{I}_n(y)\right] \quad \text{for } n \geq 1 \text{ and } t \in [0,1] \tag{36}$$

and

$$T_{\widetilde{y}_p,n}(t) \equiv \frac{\lfloor nt \rfloor}{\sigma_{\widetilde{y}_p}\sqrt{n}}\left[\widetilde{y}_p(\lfloor nt \rfloor) - \widetilde{y}_p(n)\right] \quad \text{for } p \in (0,1),\ n \geq 1, \text{ and } t \in [0,1], \tag{37}$$

where the cumulative sample average $\bar{I}_{\lfloor nt \rfloor}(y) = \sum_{k=1}^{\lfloor nt \rfloor} I_k(y)/\lfloor nt \rfloor$, with $\bar{I}_0(y) \equiv 0$; and $\widetilde{y}_p(\lfloor nt \rfloor)$ is the point estimator of the p-quantile y_p based on the partial sample $\{Y_1, Y_2, \ldots, Y_{\lfloor nt \rfloor}\}$, i.e., if $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(\lfloor nt \rfloor)}$ denote the order statistics from the partial sample, then $\widetilde{y}_p(\lfloor nt \rfloor) \equiv Y_{(\lceil \lfloor nt \rfloor p \rceil)}$, where $Y_{(0)} \equiv 0$. This expression is less daunting than it looks since in practice, t is of the form $\frac{k}{n}$, in which case $\widetilde{y}_p(\lfloor nt \rfloor) = \widetilde{y}_p(k) = Y_{(\lceil kp \rceil)}$.

Informally speaking, the STS $\{T_{I(y),n}(t)\}$ is "traditional" in that it only involves sample averages, so that its analysis is greatly aided by results that are already in the literature. On the other hand, the STS $\{T_{\widetilde{y}_p,n}(t)\}$ is less straightforward since it involves a quantile process, forcing us to be a little more careful in the subsequent analysis.

Going forward, suppose that $w(t)$ is a continuous function on $t \in [0,1]$ satisfying $\int_0^1 \int_0^1 w(s)w(t)\big(\min(s,t) - st\big)\,\mathrm{d}s\,\mathrm{d}t = 1$; this is a necessary condition to ensure the asymptotic unbiasedness of the variance estimators to be studied below. We define $\alpha_k \equiv kw(\frac{k}{n})$ for $k = 1, 2, \ldots, n-1$ and $\alpha_n \equiv -\sum_{k=1}^{n-1} \alpha_k$. We note that $\sum_{k=1}^{n} \alpha_k = 0$; and moreover, since $w(t)$ is continuous on a closed interval, it is clear that

$$|\alpha_k| = O(n) \text{ for } k = 1, 2, \ldots, n-1, \text{ and } \alpha_n = O(n^2). \tag{38}$$

We define the following "area" functionals associated, respectively, with $\{T_{I(y),n}(t)\}$ and $\{T_{\widetilde{y}_p,n}(t)\}$,

$$A_{I(y)}(w;n) \;\equiv\; \frac{\sigma_{I(y)}}{n} \sum_{k=1}^{n} w(\tfrac{k}{n})\, T_{I(y),n}(\tfrac{k}{n}) \quad \text{for } y \in \mathbb{R} \text{ and } n \geq 1 \tag{39}$$

and

$$A_{\widetilde{y}_p}(w;n) \;\equiv\; \frac{\sigma_{\widetilde{y}_p}}{n} \sum_{k=1}^{n} w(\tfrac{k}{n})\, T_{\widetilde{y}_p,n}(\tfrac{k}{n}) \quad \text{for } p \in (0,1) \text{ and } n \geq 1. \tag{40}$$

We say that $A^2_{I(y)}(w;n)$ is the STS weighted area estimator for $\sigma^2_{I(y)}$, and $A^2_{\widetilde{y}_p}(w;n)$ is the STS weighted area estimator for $\sigma^2_{\widetilde{y}_p}$.

4.2. Expected Value of the STS Area Estimator

The analysis of these estimators begins with an old result applied to the indicator process to establish that the STS area estimator $\mathrm{E}\big[A^2_{I(y)}(w;n)\big]$ is asymptotically unbiased for $\sigma^2_{I(y)}$ as $n \to \infty$.

Lemma 2. *Under the Standing Assumptions, we have* $\mathrm{E}\big[A^2_{I(y)}(w;n)\big] = \sigma^2_{I(y)} + O(1/n)$.

Proof: Under the GMC condition, Theorem 1.2(iii) implies that $|\gamma_{I(y),1}| < \infty$; and this is sufficient to invoke Goldsman et al. (1990, Corollary 4.2), which immediately implies the result. ∎

The following finding is this section's main theorem establishing that $\mathrm{E}\big[A^2_{\widetilde{y}_p}(w;n)\big]$ is asymptotically unbiased for $\sigma^2_{\widetilde{y}_p}$, though the bias converges to zero at a dawdling $O(n^{-1/4})$ rate.

Theorem 4. *Under the Standing Assumptions [and maybe whatever else is needed for Bahadur], we have* $\mathrm{E}\big[A^2_{\widetilde{y}_p}(w;n)\big] = \sigma^2_{\widetilde{y}_p} + O(n^{-\frac{1}{4}+\varepsilon})$ *for* $\varepsilon > 0$.

Before proving the theorem, we prepare with several lemmas that we will need. The first relates the two area functionals.

Lemma 3. *Using the notation of the Bahadur representation (6), we have*

$$A_{\widetilde{y}_p}(w;n) \;=\; -\frac{A_{I(y_p)}(w;n)}{f(y_p)} + \frac{1}{n^{3/2}} \sum_{\ell=1}^{n} \alpha_\ell Q_\ell. \tag{41}$$

Proof: Starting with Equations (37) and (40), we have

$$\begin{aligned}
A_{\widetilde{y}_p}(w;n) &= \frac{1}{n^{3/2}}\sum_{k=1}^{n} k\,w(\tfrac{k}{n})\big[\,\widetilde{y}_p(k)-\widetilde{y}_p(n)\,\big] \\
&= \frac{1}{n^{3/2}}\left\{\sum_{k=1}^{n-1} k\,w(\tfrac{k}{n})\,\widetilde{y}_p(k)-\left[\sum_{k=1}^{n-1} k\,w(\tfrac{k}{n})\right]\widetilde{y}_p(n)\right\} \\
&= \frac{1}{n^{3/2}}\sum_{k=1}^{n}\alpha_k\widetilde{y}_p(k) \\
&= \frac{1}{n^{3/2}}\sum_{k=1}^{n}\alpha_k\left(y_p-\frac{\overline{I}_k(y_p)-p}{f(y_p)}+Q_k\right) \quad \text{(by Bahadur Equation (6))} \\
&= \frac{1}{n^{3/2}}\sum_{k=1}^{n}\alpha_k\left(\frac{\overline{I}_n(y_p)-\overline{I}_k(y_p)}{f(y_p)}+Q_k\right) \quad \left(\text{since } \textstyle\sum_{k=1}^{n}\alpha_k=0\right),
\end{aligned}$$

and the result follows from Equations (36) and (39). ∎

Lemma 4. *Under the Standing Assumptions [and maybe a Bahadur assumption?], we have*

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\mathrm{Cov}\big[\overline{I}_i(y),Q_j\big]\right| = O\big(n^{\frac{11}{4}+\varepsilon}\big). \tag{42}$$

Proof: The Cauchy–Schwarz inequality and then Equations (18) and (17) in that order imply that

$$\left|\mathrm{Cov}\big[\overline{I}_i(y),Q_j\big]\right| \le \sqrt{\mathrm{Var}\big[\overline{I}_i(y_p)\big]}\sqrt{\mathrm{Var}[Q_j]} = O\big(i^{-\frac{1}{2}}j^{-\frac{3}{4}}\log^{\frac{3}{2}}j\big) = O\big(i^{-\frac{1}{2}}j^{-\frac{3}{4}+\varepsilon}\big) \quad \text{for any } \varepsilon>0. \tag{43}$$

Equation (38) gives bounds on the α_k's and guarantees a large-enough generic constant C for which

$$\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\mathrm{Cov}\big[\overline{I}_i(y),Q_j\big]\right| \\
&= \left|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\alpha_i\alpha_j\mathrm{Cov}\big[\overline{I}_i(y),Q_j\big]+\sum_{i=1}^{n-1}\alpha_i\alpha_n\mathrm{Cov}\big[\overline{I}_i(y),Q_n\big]+\sum_{j=1}^{n-1}\alpha_n\alpha_j\mathrm{Cov}\big[\overline{I}_n(y),Q_j\big]+\alpha_n^2\,\mathrm{Cov}\big[\overline{I}_n(y),Q_n\big]\right| \\
&\le C\left[n^2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|\mathrm{Cov}\big[\overline{I}_i(y),Q_j\big]\right|+n^3\sum_{i=1}^{n-1}\left|\mathrm{Cov}\big[\overline{I}_i(y),Q_n\big]\right|+n^3\sum_{j=1}^{n-1}\left|\mathrm{Cov}\big[\overline{I}_n(y),Q_j\big]\right|+n^4\left|\mathrm{Cov}\big[\overline{I}_n(y),Q_n\big]\right|\right] \\
&\le C\left[n^2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}i^{-\frac{1}{2}}j^{-\frac{3}{4}+\varepsilon}+n^3\sum_{i=1}^{n-1}i^{-\frac{1}{2}}n^{-\frac{3}{4}+\varepsilon}+n^3\sum_{j=1}^{n-1}n^{-\frac{1}{2}}j^{-\frac{3}{4}+\varepsilon}+n^4n^{-\frac{1}{2}}n^{-\frac{3}{4}+\varepsilon}\right] \quad \text{(by Equation (43))} \\
&\le C\left[n^2n^{\frac{1}{2}}n^{\frac{1}{4}+\varepsilon}+n^3n^{\frac{1}{2}}n^{-\frac{3}{4}+\varepsilon}+n^3n^{-\frac{1}{2}}n^{\frac{1}{4}+\varepsilon}+n^4n^{-\frac{1}{2}}n^{-\frac{3}{4}+\varepsilon}\right] \quad \text{(integral approximations of sums)} \\
&= O\big(n^{\frac{11}{4}+\varepsilon}\big).\ ∎
\end{aligned}$$

Lemma 5. *Under the Standing Assumptions [and maybe a Bahadur assumption?], we have*

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\mathrm{E}\big[Q_iQ_j\big]\right| = O\big(n^{\frac{5}{2}+\varepsilon}\big). \tag{44}$$

Proof: First of all, Cauchy–Schwarz and Equation (17) imply that, for any $\epsilon > 0$, we have

$$\left|\mathrm{E}[Q_iQ_j]\right| \le \sqrt{\mathrm{E}[Q_i^2]\mathrm{E}[Q_j^2]} = \sqrt{O\big(i^{-\frac{3}{2}}\log^3 i\big)O\big(j^{-\frac{3}{2}}\log^3 j\big)} = O\big(i^{-\frac{3}{4}+\frac{\varepsilon}{2}}j^{-\frac{3}{4}+\frac{\varepsilon}{2}}\big). \tag{45}$$

By Equation (45) and symmetry,

$$\begin{aligned}
\left|\sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j\mathrm{E}[Q_iQ_j]\right| &= \left|\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\alpha_i\alpha_j\mathrm{E}[Q_iQ_j] + 2\sum_{i=1}^{n-1}\alpha_i\alpha_n\mathrm{E}[Q_iQ_n] + \alpha_n^2\,\mathrm{E}[Q_n^2]\right| \quad \text{(by Equation (38))}\\
&\le C\left[n^2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\left|\mathrm{E}[Q_iQ_j]\right| + 2n^3\sum_{i=1}^{n-1}\left|\mathrm{E}[Q_iQ_n]\right| + n^4\,\mathrm{E}[Q_n^2]\right]\\
&\le C\left[n^2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1}i^{-\frac{3}{4}+\frac{\varepsilon}{2}}j^{-\frac{3}{4}+\frac{\varepsilon}{2}} + 2n^3\sum_{i=1}^{n-1}i^{-\frac{3}{4}+\frac{\varepsilon}{2}}n^{-\frac{3}{4}+\frac{\varepsilon}{2}} + n^4n^{-\frac{3}{2}+\varepsilon}\right] \quad \text{(by Equation (45))}\\
&\le C\left[n^2n^{\frac{1}{4}+\frac{\varepsilon}{2}}n^{\frac{1}{4}+\frac{\varepsilon}{2}} + 2n^3n^{\frac{1}{4}+\frac{\varepsilon}{2}}n^{-\frac{3}{4}+\frac{\varepsilon}{2}} + n^4n^{-\frac{3}{2}+\varepsilon}\right] \quad \text{(integral approximations of sums)}\\
&= O\big(n^{\frac{5}{2}+\varepsilon}\big). \quad \blacksquare
\end{aligned}$$

At this point, we are finally ready to prove the main theorem.

Proof (of Theorem 4): Starting from Equation (41), we obtain

$$\begin{aligned}
\mathrm{E}\big[A_{\tilde{y}_p}^2(w;n)\big] &= \frac{\mathrm{E}\big[A_{I(y_p)}^2(w;n)\big]}{f^2(y_p)} - \frac{2}{n^{3/2}f(y_p)}\mathrm{E}\left[A_{I(y_p)}(w;n)\sum_{\ell=1}^{n}\alpha_\ell Q_\ell\right] + \frac{1}{n^3}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\alpha_k\alpha_\ell\mathrm{E}[Q_kQ_\ell]\\
&= \frac{\sigma_{I(y_p)}^2}{f^2(y_p)} + O(1/n) + \frac{2}{n^{3/2}f(y_p)}\mathrm{E}\left[\frac{\sigma_{I(y_p)}}{n}\sum_{k=1}^{n}w\big(\tfrac{k}{n}\big)T_{I(y_p),n}\big(\tfrac{k}{n}\big)\sum_{\ell=1}^{n}\alpha_\ell Q_\ell\right] + O\big(n^{-\frac{1}{2}+\varepsilon}\big)\\
&\qquad \text{(by Lemma 2, Equation (39), and Equation (44))}\\
&= \sigma_{\tilde{y}_p}^2 + \frac{2}{n^3f(y_p)}\mathrm{E}\left[\sum_{k=1}^{n}\alpha_k\big(\bar{I}_n(y_p) - \bar{I}_k(y_p)\big)\sum_{\ell=1}^{n}\alpha_\ell Q_\ell\right] + O\big(n^{-\frac{1}{2}+\varepsilon}\big) \quad \text{(by Equation (36))}\\
&= \sigma_{\tilde{y}_p}^2 - \frac{2}{n^3f(y_p)}\mathrm{E}\left[\sum_{k=1}^{n}\alpha_k\big(\bar{I}_k(y_p) - p\big)\sum_{\ell=1}^{n}\alpha_\ell Q_\ell\right] + O\big(n^{-\frac{1}{2}+\varepsilon}\big) \quad \Big(\text{since } \textstyle\sum_{k=1}^{n}\alpha_k = 0\Big)\\
&= \sigma_{\tilde{y}_p}^2 - \frac{2}{n^3f(y_p)}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\alpha_k\alpha_\ell\mathrm{E}\big[\big(\bar{I}_k(y_p) - p\big)Q_\ell\big] + O\big(n^{-\frac{1}{2}+\varepsilon}\big)\\
&= \sigma_{\tilde{y}_p}^2 - \frac{2}{n^3f(y_p)}\sum_{k=1}^{n}\sum_{\ell=1}^{n}\alpha_k\alpha_\ell\mathrm{Cov}\big[\bar{I}_k(y_p), Q_\ell\big] + O\big(n^{-\frac{1}{2}+\varepsilon}\big)\\
&= \sigma_{\tilde{y}_p}^2 + O\big(n^{-\frac{1}{4}+\varepsilon}\big) + O\big(n^{-\frac{1}{2}+\varepsilon}\big) \quad \text{(by Equation (42)).} \quad \blacksquare
\end{aligned}$$

4.3. Batched and Combined Estimators

Without going into too much of a notational deep dive, we can easily adopt the area estimators for use in a batching environment—a estimator for each batch. To this end, recall that the we divide our n observations $\{Y_1, Y_2, \ldots, Y_n\}$ into b contiguous batches, each of size m (assuming $n = bm$). Using the notation developed in

§§2 and 4.1, we now define the various batched versions of the standardized time series and their associated estimators.

For batches $j = 1, 2, \dots, b$, the STS indicator and quantile-estimator processes are given by

$$T_{I(y),j,m}(t) \equiv \frac{\lfloor mt \rfloor}{\sigma_{I(y)}\sqrt{m}}\left[\bar{I}_{j,\lfloor mt \rfloor}(y) - \bar{I}_{j,m}(y)\right] \quad \text{for } m \geq 1 \text{ and } t \in [0,1] \tag{46}$$

and

$$T_{\widetilde{y}_p,j,m}(t) \equiv \frac{\lfloor mt \rfloor}{\sigma_{\widetilde{y}_p}\sqrt{m}}\left[\widetilde{y}_p(j,\lfloor mt \rfloor) - \widetilde{y}_p(j,m)\right] \quad \text{for } p \in (0,1),\ m \geq 1, \text{ and } t \in [0,1]. \tag{47}$$

We define the following batched area functionals, analogous to their full-sample counterparts in §4.1,

$$A_{I(y)}(w; j, m) \;\equiv\; \frac{\sigma_{I(y)}}{m}\sum_{k=1}^{m} w(\tfrac{k}{m})\, T_{I(y),j,m}(\tfrac{k}{m}) \quad \text{for } j = 1, 2, \dots, b,\ y \in \mathbb{R}, \text{ and } m \geq 1 \tag{48}$$

and

$$A_{\widetilde{y}_p}(w; j, m) \;\equiv\; \frac{\sigma_{\widetilde{y}_p}}{m}\sum_{k=1}^{m} w(\tfrac{k}{m})\, T_{\widetilde{y}_p,i,m}(\tfrac{k}{m}) \quad \text{for } j = 1, 2, \dots, b,\ p \in (0,1), \text{ and } m \geq 1. \tag{49}$$

The batched STS weighted area estimators for $\sigma^2_{I(y)}$ and $\sigma^2_{\widetilde{y}_p}$ are respectively given by

$$\mathcal{A}_{I(y)}(w; b, m) \;\equiv\; \frac{1}{b}\sum_{j=1}^{b} A^2_{I(y)}(w; j, m) \quad \text{and} \quad \mathcal{A}_{\widetilde{y}_p}(w; b, m) \;\equiv\; \frac{1}{b}\sum_{j=1}^{b} A^2_{\widetilde{y}_p}(w; j, m).$$

The next results on the expected values of the batched area estimators are immediate from Lemma 2 and Theorem 4.

Corollary 1. *Under the Standing Assumptions [maybe add assumptions for Bahadur?], we have*

$$\mathrm{E}\big[\mathcal{A}_{I(y)}(w; b, m)\big] \;=\; \sigma^2_{I(y)} + O(1/m) \quad \textit{and} \quad \mathrm{E}\big[\mathcal{A}_{\widetilde{y}_p}(w; b, m)\big] \;=\; \sigma^2_{\widetilde{y}_p} + O(m^{-\frac{1}{4}+\varepsilon}) \ \textit{ for } \varepsilon > 0.$$

Based on Theorem 3 and Corollary 1, we can combine the NBQ and STS estimators. Define the following estimators for $\sigma^2_{\widetilde{y}_p}$. [I suppose that we could do the same for $\sigma^2_{I(y)}$, though in this case the NBQ component has $O(m^{-1/4})$ bias while the area estmator only has $O(1/m)$ bias — likely due to the way in which the NBM's bias was proven (we can probably do better).]

$$C_{\widetilde{y}_p}(w; b, m) \;\equiv\; \frac{(b-1)\mathcal{N}_{\widetilde{y}_p}(b, m) + b\mathcal{A}_{\widetilde{y}_p}(w; b, m)}{2b-1}$$

and

$$\widetilde{C}_{\widetilde{y}_p}(w; b, m) \;\equiv\; \frac{(b-1)\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b, m) + b\mathcal{A}_{\widetilde{y}_p}(w; b, m)}{2b-1}.$$

Then we immediately have

Corollary 2. *Under the Standing Assumptions [maybe add assumptions for Bahadur?], we have*

$$\mathrm{E}\big[C_{\widetilde{y}_p}(w; b, m)\big] \;=\; \sigma^2_{\widetilde{y}_p} + O(m^{-\frac{1}{4}+\varepsilon}) \ \textit{ for } \varepsilon > 0 \quad \textit{and} \quad \mathrm{E}\big[\widetilde{C}_{\widetilde{y}_p}(w; b, m)\big] \;=\; \sigma^2_{\widetilde{y}_p} + O(m^{-\frac{1}{4}+\varepsilon}) \ \textit{ for } \varepsilon > 0.$$

Why are we so interested in the batched and combined estimator results of Corollaries 1 and 2? The answer stems from Dingeç et al. (2023a, Theorems 2 and 5) who show that the GMC condition is sufficient to apply certain functional central limit theorems that Alexopoulos et al. (2023) subsequently use to obtain

$$\begin{aligned}
\mathcal{N}_{\widetilde{y}_p}(b,m) &\underset{m\to\infty}{\Longrightarrow} \frac{\sigma^2_{\widetilde{y}_p}\,\chi^2(b-1)}{b-1}\\
\widetilde{\mathcal{N}}_{\widetilde{y}_p}(b,m) &\underset{m\to\infty}{\Longrightarrow} \frac{\sigma^2_{\widetilde{y}_p}\,\chi^2(b-1)}{b-1}\\
\mathcal{A}_{\widetilde{y}_p}(w;b,m) &\underset{m\to\infty}{\Longrightarrow} \frac{\sigma^2_{\widetilde{y}_p}\,\chi^2(b)}{b}\\
\mathcal{C}_{\widetilde{y}_p}(w;b,m) &\underset{m\to\infty}{\Longrightarrow} \frac{\sigma^2_{\widetilde{y}_p}\,\chi^2(2b-1)}{2b-1}\\
\widetilde{\mathcal{C}}_{\widetilde{y}_p}(w;b,m) &\underset{m\to\infty}{\Longrightarrow} \frac{\sigma^2_{\widetilde{y}_p}\,\chi^2(2b-1)}{2b-1},
\end{aligned}$$

where $\chi^2(\nu)$ denotes a χ^2 r.v. having ν degrees of freedom (d.f.). Alexopoulos et al. (2023) then derive confidence intervals for y_p of the familiar generic form

$$y_p \in \widehat{y}_p \pm t_{\alpha/2,\nu}\,\widehat{\sigma}_{\widetilde{y}_p}/\sqrt{n},$$

where $t_{\alpha/2,\nu}$ is the $(1-\frac{\alpha}{2})$ quantile of the Student t-distribution with ν d.f.; the point estimator $\widehat{y}_p$ for y_p is either $\widetilde{y}_p(n)$ or $\overline{y}_p(b,m)$; and $\widehat{\sigma}^2_{\widetilde{y}_p}$ is one of the various estimators for $\sigma^2_{\widetilde{y}_p}$ studied in Alexopoulos et al. (2023) and the current paper.

Of course, it is well known that the more d.f. the better—resulting in lower variance of the estimators for $\sigma^2_{\widetilde{y}_p}$, tighter and less-variable CIs, etc. However, the familiar bias/variance trade-off is also in play: For a fixed sample size n, an increase in b (which results in lower variance) forces a smaller batch size m (typically resulting in higher bias). These issues will be addressed by a Monte Carlo study coming up in §5.

5. Experimental Results

In §5.1 we conduct Monte Carlo (MC) experiments to illustrate the performance of our various estimators for $\widehat{\sigma}^2_{\widetilde{y}_p}$. We carry out this work on various stationary processes of interest: (i) a first-order autoregressive (AR(1)) process; (ii) an autorgressive-to-Pareto (ARTOP) process; and (iii) the waiting-time process from an M/M/1 queueing system. §5.2 concerns the selection of weights for the area estimator. [Note that *we have absolutely no theoretical support or even conclusive empirical evidence for weight selection.* Also, note that AR(1) should go before ARTOP (obviously). I also put M/M/1 third, because that's what we've been doing in our other work.].

5.1. Monte Carlo Evaluation for M/M/1, AR(1), and ARTOP Processes

[Add more information for the examples.]

Example 2. Consider the Gaussian AR(1) time-series defined by the linear regression model $Y_k = \mu + \phi(Y_{k-1} - \mu) + \epsilon_k$ for $k \geq 1$, where $\phi \in (-1, 1)$ is the autoregressive parameter, the residuals $\{\epsilon_k : k \geq 1\}$ are i.i.d. $\text{Nor}(0, \sigma_\epsilon^2)$, and Y_0 is the initial state of the process. The steady-state marginal distribution of this process is $\text{Nor}\big(\mu, \sigma_\epsilon^2/(1-\phi^2)\big)$. In this experiment we take $Y_0 \sim \text{Nor}(0, 1)$, $\phi = 0.9$, and $\sigma_\epsilon^2 = 1 - \phi^2 = 0.19$; hence the process is stationary with a standard normal marginal distribution. The computation of the variance parameter $\sigma^2_{\widetilde{y}_p}$ in Equation (**??**) is detailed in Dingeç et al. (2023b). [This discussion can be written more succinctly. if we're only gonna use N(0,1) observations, no need to define a more-general AR(1)!]

Example 3. ARTOP process with $\gamma = 1, \theta = 2.1,$ and $\beta = 0.995$

Example 4. Consider an M/M/1 queueing system with arrival rate $\lambda = 0.8$, service rate $\omega = 1$ (traffic intensity $\rho = 0.9$) and first-in, first-out (FIFO) service discipline. Let Y_k be the time spent by the kth entity in queue (prior to service). The steady-state c.d.f. of Y_k is

$$F(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 - \rho & \text{if } y = 0, \\ 1 - \rho e^{-\omega(1-\rho)y} & \text{if } y > 0; \end{cases} \tag{50}$$

hence the steady-state distribution of Y_k has mean $\mu = \rho/(\omega - \lambda) = 4$, and the quantiles of this distribution are readily computed by inverting Equation (51). The steady-state distribution of Y_k is distinctly nonnormal, having an atom at zero, an exponential tail, and a skewness of $2(3 - 3\rho + \rho^2)/[\rho^{1/2}(2-\rho)^{3/2}] \approx 2.109$. The variance parameter $\sigma^2_{I(y_p)}$ of the indicator process was computed from Blomqvist (1967, Equation (22)). After some algebra, we [REF?] obtained the following analytical expression for the asymptotic variance parameter:

$$\sigma^2_{\widetilde{y}_p} = \frac{1}{\omega^2(1-\rho)^4}\left\{\frac{[-2 + p(3-\rho) + 2\rho](1+\rho)}{1-p} - 4\rho\,\ell\text{n}\Big(\frac{\rho}{1-p}\Big)\right\}.$$

[The following content is the discussion of the tables, which needs quite a bit of work.]

In Tables 1–6 we present experimental results comparing the following estimators: (i) the STS-Area $\mathscr{A}^2_{b,m}$; (ii) the "first" BM $\mathcal{N}_1(b,m)$; (iii) the "second" BM $\mathcal{N}_2(b,m)$; (iv) the "Combo 1" based on $\mathcal{N}_1(b,m)$; and (v) the "Combo 2" based on $\mathcal{N}_2(b,m)$. For this experimental analysis we used 2,500 independent replications and we considered three examples:

In each table, column 1 contains the values of p, x_p, and σ^2 (the latter quantity is set in bold red type); column 2 contains the value of $\mathscr{L} = \log_2(m)$; columns 3, 7, 11, 15, and 19 respectively contain the average values of the selected variance-parameter estimators; columns 4, 8, 12, 16, and 20 respectively contain the average bias of the selected variance-parameter estimators; while columns 5, 9, 13, 17, and 21 respectively contain the standard deviations of the selected variance-parameter estimators. Finally, columns 6, 10, 14, 18, and 22 contain respectively the root-mean-square error (RMSE).

The first goal of this analysis was to verify the convergence of the respective variance estimators to the true value in column 1. It is clear that this was the case for every variance estimator in Tables 1–6 as the average values of the selected variance-parameter estimators approached the true value and the respective biases went to zero. However, the columns that contain the average bias reveal that the batched STS Area Estimator had larger bias compared to the "first" BM $\mathcal{N}_1(b,m)$ and the "second" BM $\mathcal{N}_2(b,m)$, especially for small batch sizes.

The comparison between the "first" BM $\mathcal{N}_1(b,m)$ and the "second" BM $\mathcal{N}_2(b,m)$ variance estimators did not yield a clear winner as there were situations where $\mathcal{N}_1(b,m)$ performed better than $\mathcal{N}_1(b,m)$ and others where it performed worse. Overall, for the examples that we considered there was a tendency for the "first" BM $\mathcal{N}_1(b,m)$ to perform slightly better. It is important to note that any potential advantages of either of these estimators significantly diminished as the batch size increased to values greater than 2^{17}.

The "Combo 1" and the "Combo 2" variance estimators outperformed all the others with respect to the standard deviation of the selected variance parameters and the RMSE. This was expected due to the larger number of degrees of freedom associated with the combined variance estimators. Again, for similar reasons as the ones explained in the paragraph above there was not a clear winner between the two.

Table 1 Experimental results for a stationary waiting-time process in an M/M/1 queueing system with traffic intensity $\rho = 0.8$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} = 7, 8, \ldots, 20$.

p (x_p) Var. Par.	$\mathcal{L}$	Batched STS Area Estimator				NBQ V2 Estimator				NBQ V1 Estimator				Combined V2 Estimator				Combined V1 Estimator			
		Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE
0.5	7	812.2	177.2	529.5	558.4	1566.0	931.0	1395.1	1677.2	1451.3	816.3	1266.3	1506.6	1183.1	548.1	848.0	1009.8	1126.7	491.7	786.7	927.7
(2.3500)	8	1409.7	774.7	1128.4	1368.7	1708.9	1073.9	1576.5	1907.6	1608.1	973.1	1470.1	1763.0	1556.9	921.9	1143.3	1468.7	1507.3	872.3	1095.1	1400.1
635.0	9	1697.4	1062.4	1537.4	1868.8	1366.9	731.9	1405.9	1585.0	1308.9	673.9	1340.9	1500.7	1534.8	899.8	1210.1	1508.0	1506.2	871.2	1183.6	1469.6
	10	1489.4	854.4	1440.2	1674.6	928.3	293.3	626.1	691.4	903.9	268.9	602.7	660.0	1213.3	578.3	869.1	1043.9	1201.3	566.3	862.0	1031.4
	11	1110.6	475.6	808.9	938.4	769.7	134.7	343.3	368.8	758.6	123.6	334.8	356.9	942.9	307.9	484.1	573.7	937.4	302.4	481.6	568.7
	12	836.0	201.0	352.9	406.1	698.9	63.9	221.6	230.7	693.7	58.7	218.4	226.2	768.5	133.5	227.5	263.8	766.0	131.0	226.4	261.6
	13	729.9	94.9	236.3	254.6	664.2	29.2	187.6	189.9	661.6	26.6	186.3	188.2	697.5	62.5	157.2	169.2	696.3	61.3	156.8	168.3
	14	682.8	47.8	192.7	198.6	643.7	8.7	170.3	170.5	642.4	7.4	169.7	169.9	663.5	28.5	132.1	135.2	662.9	27.9	131.9	134.8
	15	654.3	19.3	176.3	177.3	640.1	5.1	167.0	167.1	639.4	4.4	166.7	166.7	647.3	12.3	122.6	123.2	647.0	12.0	122.5	123.1
	16	646.4	11.4	166.6	167.0	639.1	4.1	163.1	163.2	638.8	3.8	163.0	163.0	642.8	7.8	116.9	117.2	642.7	7.7	116.9	117.1
	17	639.1	4.1	161.9	162.0	640.0	5.0	163.5	163.6	639.8	4.8	163.5	163.5	639.5	4.5	114.4	114.5	639.5	4.5	114.4	114.4
	18	638.9	3.9	159.6	159.7	634.7	-0.3	159.3	159.3	634.6	-0.4	159.2	159.2	636.8	1.8	112.4	112.5	636.8	1.8	112.4	112.5
	19	639.4	4.4	163.0	163.1	632.8	-2.2	164.1	164.1	632.8	-2.2	164.1	164.1	636.2	1.2	114.5	114.5	636.2	1.2	114.5	114.5
	20	632.5	-2.5	157.6	157.6	637.6	2.6	163.6	163.6	637.6	2.6	163.6	163.6	635.0	0.0	112.2	112.2	635.0	0.0	112.2	112.2
0.75	7	1125.4	-2173.3	682.2	2277.9	2527.5	-771.2	1727.4	1891.8	2497.1	-801.6	1692.4	1872.7	1815.3	-1483.4	1060.7	1823.6	1800.3	-1498.4	1043.6	1826.0
(5.8158)	8	2351.4	-947.3	1727.0	1969.7	4027.8	729.1	2721.3	2817.2	3940.8	642.1	2624.7	2702.1	3176.3	-122.4	1960.0	1963.8	3133.5	-165.2	1911.9	1919.1
3298.7	9	3785.6	486.9	2732.4	2775.4	5064.1	1765.4	3548.2	3963.1	4940.3	1641.6	3416.8	3790.7	4414.7	1116.0	2682.6	2905.5	4353.8	1055.1	2620.3	2824.7
	10	4853.0	1554.3	3419.9	3756.6	4798.4	1499.7	3211.7	3544.6	4707.5	1408.8	3111.5	3415.6	4826.1	1527.4	2831.1	3216.9	4781.4	1482.7	2787.7	3157.5
	11	4992.9	1694.2	3657.9	4031.2	4113.1	814.4	2162.2	2310.5	4066.2	767.5	2113.7	2248.8	4560.0	1261.3	2449.3	2755.0	4536.9	1238.2	2431.2	2728.4
	12	4242.5	943.8	2046.1	2253.3	3703.1	404.4	1329.6	1389.7	3681.0	382.3	1312.2	1366.7	3977.1	678.4	1361.7	1521.3	3966.2	667.5	1355.5	1511.0
	13	3819.2	520.5	1402.5	1495.9	3466.6	167.9	1036.7	1050.2	3456.0	157.3	1029.9	1041.8	3645.7	347.0	944.1	1005.9	3640.5	341.8	941.7	1001.8
	14	3547.5	248.8	1045.6	1074.8	3366.1	67.4	905.0	907.5	3360.9	62.2	902.2	904.3	3458.3	159.6	726.8	744.1	3455.7	157.0	725.8	742.6
	15	3412.5	113.8	936.5	943.4	3345.8	47.1	878.2	879.5	3343.1	44.4	876.9	878.0	3379.7	81.0	652.7	657.8	3378.4	79.7	652.3	657.1
	16	3356.4	57.7	873.3	875.2	3337.2	38.5	861.3	862.2	3335.9	37.2	860.7	861.5	3347.0	48.3	617.7	619.6	3346.3	47.6	617.5	619.3
	17	3332.1	33.4	859.7	860.4	3327.3	28.6	839.2	839.7	3326.6	27.9	838.9	839.3	3329.7	31.0	605.0	605.8	3329.4	30.7	604.9	605.7
	18	3316.1	17.4	814.8	815.0	3312.8	14.1	829.5	829.6	3312.5	13.8	829.3	829.4	3314.5	15.8	578.2	578.4	3314.3	15.6	578.1	578.3
	19	3310.2	11.5	838.5	838.6	3306.4	7.7	856.2	856.3	3306.2	7.5	856.1	856.2	3308.3	9.6	593.9	594.0	3308.2	9.5	593.9	594.0
	20	3292.4	-6.3	813.3	813.4	3316.4	17.7	853.0	853.2	3316.3	17.6	853.0	853.2	3304.2	5.5	580.6	580.6	3304.1	5.4	580.6	580.6

Table 2 Experimental results for a stationary waiting-time process in an M/M/1 queueing system with traffic intensity $\rho = 0.8$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} = 7, 8, \ldots, 20$.

		Batched STS Area Estimator				NBQ V2 Estimator				NBQ V1 Estimator				Combined V2 Estimator				Combined V1 Estimator			
p (x_p) **Var. Par.**	$\mathcal{L}$	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE
0.95	7	1884	-30596	1038	30614	5839	-26641	5100	27125	3606	-28874	2032	28945	3830	-28650	2749	28781	2731	-29749	1347	29779
(13.8629)	8	4611	-27869	2770	28007	9082	-23398	5698	24081	7382	-25098	3787	25382	6811	-25669	3651	25927	5974	-26506	2867	26660
32480	9	9503	-22977	6386	23848	14877	-17603	7458	19117	13965	-18515	6732	19701	12147	-20332	6009	21202	11699	-20781	5751	21562
	10	16816	-15664	12658	20139	24106	-8374	11553	14268	23806	-8673	11450	14364	20403	-12077	10560	16042	20256	-12224	10535	16138
	11	26142	-6338	19292	20306	34970	2491	18686	18851	34807	2327	18554	18700	30486	-1994	16199	16322	30406	-2074	16133	16266
	12	33519	1039	22209	22233	39307	6828	22823	23822	39103	6623	22503	23457	36367	3887	19361	19747	36267	3787	19213	19583
	13	37166	4686	18578	19160	37129	4649	18450	19027	37006	4526	18232	18785	37148	4668	15334	16029	37087	4607	15238	15919
	14	36801	4321	17075	17613	34516	2036	11262	11445	34464	1984	11218	11393	35676	3196	11496	11932	35651	3171	11478	11908
	15	35003	2524	12155	12414	33469	990	9686	9736	33444	964	9669	9717	34249	1769	8505	8687	34236	1756	8498	8677
	16	33714	1235	10240	10314	33148	668	8827	8852	33135	655	8821	8845	33436	956	7193	7257	33429	949	7191	7253
	17	33065	585	8831	8851	32693	213	8363	8365	32686	206	8360	8363	32882	402	6177	6190	32878	399	6177	6189
	18	32996	516	8343	8359	32556	76	8460	8461	32552	72	8459	8459	32779	300	5911	5918	32778	298	5910	5918
	19	32564	84	8239	8239	32570	90	8315	8315	32568	89	8314	8315	32567	87	5859	5859	32566	86	5858	5859
	20	32462	-18	7978	7978	32593	113	8318	8318	32592	112	8317	8318	32526	47	5705	5705	32526	46	5705	5705
0.99	7	2255	-189006	1170	189010	16809	-174452	17586	175336	3846	-187415	2095	187426	9416	-181845	8861	182060	3038	-188223	1424	188229
(21.9101)	8	6001	-185260	3245	185288	26575	-164686	26786	166850	8256	-183005	4024	183049	16125	-175136	13839	175682	7111	-184150	3177	184178
191261	9	13214	-178047	7919	178223	38974	-152287	32980	155817	17133	-174128	7526	174290	25890	-165371	18023	166351	15143	-176118	6765	176248
	10	27618	-163643	17701	164597	53585	-137676	30487	141011	34675	-156586	14418	157248	40395	-150866	20481	152249	31091	-160170	14065	160787
	11	54707	-136554	37687	141659	80128	-111133	37863	117406	67239	-124021	28074	127159	67216	-124045	31916	128086	60874	-130387	28645	133497
	12	92769	-98492	66687	118945	123088	-68173	56787	88726	117742	-73519	54211	91345	107688	-83573	52972	98947	105057	-86204	52186	100769
	13	135781	-55480	93623	108826	179440	-11821	87475	88270	177898	-13363	87261	88278	157264	-33997	76579	83786	156506	-34755	76546	84067
	14	179612	-11649	128352	128879	218074	26814	117142	120172	217213	25952	116265	119126	198538	7277	106052	106302	198114	6853	105703	105925
	15	204722	13461	110567	111384	213377	22116	109486	111697	212818	21557	108149	110276	208981	17720	91854	93547	208706	17445	91319	92971
	16	209709	18448	106715	108298	202611	11350	77118	77949	202367	11106	76595	77396	206216	14955	75154	76628	206096	14835	74931	76385
	17	203576	12315	70787	71850	195545	4284	55570	55735	195433	4172	55513	55670	199624	8363	48901	49611	199569	8308	48876	49577
	18	199607	8346	57126	57732	193633	2372	53765	53818	193573	2312	53742	53791	196667	5406	41550	41900	196638	5377	41538	41885
	19	196113	4852	52085	52310	192648	1387	51992	52010	192617	1356	51983	52001	194408	3147	38005	38135	194392	3131	38001	38130
	20	193492	2231	49780	49830	193054	1794	48724	48757	193036	1775	48720	48752	193277	2016	35609	35666	193268	2007	35607	35664

Table 3 Experimental results of an ARTOP process with $\gamma = 1, \theta = 2.1$, and $\beta = 0.995$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} = 7, 8, \ldots, 20$.

p (x_p) Var. Par.	$\mathcal{L}$	Batched STS Area Estimator Avg.	Bias	Std. Dev.	RMSE	NBQ V2 Estimator Avg.	Bias	Std. Dev.	RMSE	NBQ V1 Estimator Avg.	Bias	Std. Dev.	RMSE	Combined V2 Estimator Avg.	Bias	Std. Dev.	RMSE	Combined V1 Estimator Avg.	Bias	Std. Dev.	RMSE
0.5	7	101.6	-19.8	447.0	447.4	299.3	178.0	983.7	999.7	269.4	148.0	906.7	918.7	198.9	77.5	623.2	628.0	184.2	62.8	586.9	590.3
(1.3911)	8	198.5	77.2	651.3	655.9	327.8	206.5	714.2	743.4	296.8	175.4	662.3	685.1	262.2	140.8	603.7	619.9	246.9	125.5	581.0	594.4
121.4	9	322.5	201.1	1078.2	1096.8	325.6	204.3	850.9	875.1	298.0	176.6	809.7	828.7	324.0	202.7	743.2	770.3	310.4	189.1	728.3	752.5
	10	370.4	249.0	2242.1	2255.9	225.3	103.9	198.7	224.3	208.4	87.1	184.7	204.1	299.0	177.6	1148.9	1162.5	290.7	169.3	1147.7	1160.1
	11	250.3	129.0	243.5	275.6	171.9	50.6	92.8	105.7	162.9	41.6	86.7	96.1	211.7	90.4	140.8	167.3	207.3	86.0	139.2	163.6
	12	192.2	70.8	116.1	136.0	144.9	23.5	56.4	61.1	140.2	18.9	53.7	56.9	168.9	47.5	70.0	84.6	166.6	45.2	69.2	82.7
	13	156.2	34.9	68.1	76.5	133.5	12.1	44.1	45.7	131.1	9.8	42.7	43.8	145.0	23.7	43.8	49.8	143.9	22.5	43.4	48.9
	14	138.3	17.0	46.4	49.4	128.1	6.7	37.8	38.4	126.9	5.5	37.1	37.6	133.3	11.9	31.9	34.0	132.7	11.3	31.6	33.6
	15	129.6	8.2	38.0	38.9	124.9	3.6	34.5	34.7	124.3	3.0	34.2	34.3	127.3	5.9	26.3	26.9	127.0	5.6	26.2	26.8
	16	125.7	4.3	34.4	34.7	124.1	2.7	32.4	32.5	123.8	2.4	32.2	32.3	124.9	3.5	24.2	24.4	124.7	3.4	24.1	24.3
	17	122.6	1.3	31.7	31.7	122.5	1.2	31.9	32.0	122.4	1.0	31.9	31.9	122.6	1.2	22.3	22.4	122.5	1.2	22.3	22.3
	18	122.6	1.3	31.6	31.6	122.5	1.2	31.1	31.1	122.4	1.1	31.1	31.1	122.6	1.2	22.1	22.1	122.5	1.2	22.1	22.1
	19	121.3	0.0	30.3	30.3	122.2	0.8	31.1	31.1	122.2	0.8	31.1	31.1	121.7	0.4	21.8	21.8	121.7	0.4	21.8	21.8
	20	122.1	0.7	31.0	31.0	121.4	0.0	30.5	30.5	121.4	0.0	30.5	30.5	121.7	0.4	21.9	21.9	121.7	0.4	21.9	21.9
0.75	7	223.7	-428.6	2080.3	2124.0	799.7	147.4	3461.1	3464.2	769.9	117.6	3305.7	3307.7	507.1	-145.2	2479.0	2483.2	492.5	-159.8	2404.2	2409.5
(1.9351)	8	519.1	-133.2	2332.2	2336.0	1155.8	503.5	3689.9	3724.1	1112.7	460.3	3533.4	3563.3	832.4	180.1	2707.8	2713.8	811.2	158.9	2633.6	2638.4
652.3	9	1044.8	392.5	4834.3	4850.2	1477.7	825.4	5288.2	5352.3	1420.5	768.2	5101.8	5159.3	1257.8	605.5	4113.0	4157.3	1229.7	577.4	4033.9	4075.0
	10	1988.2	1335.9	33099.7	33126.6	1318.0	665.7	2936.4	3010.9	1268.6	616.3	2830.3	2896.6	1658.4	1006.1	16985.5	17015.3	1634.1	981.8	16975.2	17003.5
	11	1315.0	662.7	3118.3	3188.0	935.6	283.3	632.1	692.7	909.3	256.9	606.0	658.2	1128.3	476.0	1684.1	1750.1	1115.3	463.0	1678.6	1741.3
	12	1029.1	376.8	913.9	988.6	779.8	127.5	341.5	364.5	766.3	114.0	330.9	350.0	906.4	254.1	531.7	589.3	899.8	247.5	528.6	583.7
	13	845.0	192.6	435.7	476.4	719.4	67.0	255.3	264.0	712.4	60.1	250.7	257.8	783.2	130.8	278.9	308.0	779.7	127.4	277.2	305.1
	14	744.2	91.8	256.0	272.0	688.1	35.7	212.5	215.5	684.5	32.2	210.5	212.9	716.6	64.2	178.7	189.9	714.8	62.5	178.0	188.6
	15	693.6	41.3	205.0	209.1	671.9	19.6	190.6	191.6	670.2	17.8	189.6	190.4	683.0	30.7	145.9	149.1	682.1	29.8	145.6	148.6
	16	673.8	21.5	181.9	183.2	668.8	16.5	179.3	180.1	667.9	15.6	178.8	179.5	671.4	19.1	130.6	132.0	670.9	18.6	130.4	131.8
	17	659.9	7.6	171.2	171.4	659.7	7.4	173.3	173.4	659.2	6.9	173.1	173.2	659.8	7.5	123.3	123.5	659.6	7.2	123.2	123.5
	18	658.6	6.3	169.0	169.2	659.5	7.2	174.7	174.8	659.3	7.0	174.5	174.7	659.0	6.7	123.0	123.2	658.9	6.6	122.9	123.1
	19	653.2	0.8	165.0	165.0	658.7	6.4	170.1	170.3	658.6	6.3	170.1	170.2	655.9	3.6	119.0	119.1	655.8	3.5	119.0	119.1
	20	656.1	3.8	169.6	169.7	657.4	5.1	165.1	165.2	657.4	5.1	165.1	165.2	656.8	4.4	119.3	119.4	656.7	4.4	119.3	119.4

Table 4 Experimental results of an ARTOP process with $\gamma = 1, \theta = 2.1$, and $\beta = 0.995$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} = 7, 8, \ldots, 20$.

		Batched STS Area Estimator				NBQ V2 Estimator				NBQ V1 Estimator				Combined V2 Estimator				Combined V1 Estimator			
p (x_p) **Var. Par.**	$\mathcal{L}$	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE
0.95	7	668	-12533	5867	13839	3157	-10044	14408	17563	2597	-10604	13634	17272	1893	-11308	9516	14780	1617	-11584	9185	14783
(4.1643)	8	2370	-10831	17984	20993	6986	-6215	32553	33141	6571	-6630	32197	32872	4641	-8560	21543	23181	4437	-8764	21376	23102
13201	9	5132	-8069	16862	18693	15200	1999	72014	72042	14861	1660	70781	70801	10086	-3115	39342	39465	9920	-3281	38758	38897
	10	46958	33757	1749162	1749488	31114	17913	372607	373037	30534	17333	360870	361286	39162	25961	1065953	1066269	38876	25675	1060252	1060563
	11	23951	10750	77224	77969	27023	13822	66947	68359	26578	13377	64979	66342	25463	12262	67137	68248	25244	12043	66246	67332
	12	27807	14606	77432	78798	23532	10331	34623	36131	23153	9952	33674	35114	25704	12503	48888	50461	25517	12316	48570	50107
	13	26155	12954	69351	70550	17562	4361	10935	11772	17382	4181	10672	11462	21927	8726	37065	38078	21838	8637	36988	37983
	14	19033	5832	14175	15327	15090	1889	6331	6607	15012	1811	6250	6507	17093	3892	8591	9431	17054	3853	8563	9390
	15	15890	2689	7031	7528	14168	967	4620	4720	14131	930	4594	4687	15043	1842	4664	5014	15024	1823	4653	4997
	16	14544	1343	4646	4837	13787	586	3987	4030	13767	566	3976	4017	14172	971	3277	3417	14162	961	3272	3410
	17	13785	584	3807	3851	13505	304	3655	3668	13496	295	3651	3663	13647	446	2730	2766	13643	442	2728	2764
	18	13560	359	3617	3635	13453	252	3597	3606	13448	247	3594	3603	13507	306	2615	2632	13505	304	2614	2631
	19	13403	202	3452	3458	13301	100	3427	3428	13299	98	3426	3427	13353	152	2489	2493	13352	151	2488	2493
	20	13297	96	3447	3448	13364	163	3414	3418	13362	161	3413	3417	13330	129	2437	2440	13329	128	2436	2440
0.99	7	1061	-213216	8256	213375	15922	-198355	79647	213748	4032	-210245	21183	211309	8374	-205903	42129	210169	2523	-211754	13995	212216
(8.9615)	8	5149	-209128	36627	212311	28450	-185827	119126	220732	13379	-200898	71217	213147	16615	-197662	69166	209414	9199	-205078	47770	210568
214277	9	13456	-200821	63685	210677	55124	-159153	238687	286882	38664	-175613	197717	264446	33960	-180317	130912	222828	25860	-188417	112048	219216
	10	63961	-150316	774835	789280	341692	127415	10855637	10856384	325339	111062	10632413	10632993	200622	-13655	5721547	5721564	192576	-21701	5611750	5611792
	11	149196	-65081	1008149	1010248	359839	145562	5863407	5865214	350163	135886	5717035	5718649	252846	38569	3121658	3121896	248085	33808	3050694	3050881
	12	389911	175634	2852610	2858012	542259	327982	4877664	4888678	532850	318573	4740608	4751300	464876	250599	3644191	3652798	460246	245969	3579002	3587444
	13	524018	309741	4731970	4742097	521692	307415	1533489	1563998	512116	297839	1480442	1510104	522874	308597	2861910	2878500	518162	303885	2845187	2861370
	14	540122	325845	1636833	1668951	389527	175250	563378	590007	383525	169248	545855	571492	466020	251743	973150	1005185	463066	248789	967598	999070
	15	433001	218724	829837	858178	282290	68013	167006	180324	279890	65613	163576	176245	358842	144565	451512	474091	357661	143384	450683	472942
	16	323008	108731	323937	341698	249888	35611	105017	110891	248792	34515	103842	109428	287028	72751	182098	196092	286489	72212	181769	195588
	17	266306	52029	146212	155193	230223	15946	73093	74812	229724	15447	72753	74375	248551	34274	88058	94493	248305	34028	87933	94288
	18	236186	21909	77113	80165	224068	9791	64519	65258	223826	9549	64376	65081	230223	15946	53780	56095	230104	15827	53724	56006
	19	225507	11230	63720	64702	217850	3573	60228	60334	217728	3451	60162	60261	221739	7462	45877	46480	221679	7402	45851	46445
	20	217120	2843	57690	57760	218651	4374	56898	57065	218588	4311	56869	57032	217873	3596	41702	41857	217842	3565	41691	41843

Table 5 Experimental results for the AR(1) process with $\phi = 0.9$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{7, 8, \ldots, 20\}$.

		Batched STS Area Estimator				NBQ V2 Estimator				NBQ V1 Estimator				Combined V2 Estimator				Combined V1 Estimator			
p (x_p) **Var. Par.**	$\mathcal{L}$	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE
0.5	7	16.961	-3.898	5.004	6.343	19.183	-1.676	4.870	5.150	19.125	-1.733	4.861	5.160	18.054	-2.804	3.544	4.520	18.026	-2.833	3.541	4.534
(0.0000)	8	19.025	-1.833	5.116	5.435	19.831	-1.027	5.041	5.144	19.794	-1.065	5.033	5.144	19.422	-1.437	3.616	3.891	19.403	-1.455	3.613	3.895
20.858	9	19.939	-0.920	5.389	5.467	20.304	-0.554	5.250	5.279	20.281	-0.577	5.246	5.277	20.119	-0.740	3.826	3.897	20.107	-0.751	3.825	3.898
	10	20.370	-0.489	5.340	5.362	20.671	-0.187	5.239	5.242	20.657	-0.201	5.237	5.241	20.518	-0.340	3.759	3.775	20.511	-0.347	3.758	3.774
	11	20.638	-0.221	5.141	5.145	20.697	-0.161	5.183	5.185	20.688	-0.170	5.181	5.184	20.667	-0.191	3.614	3.619	20.662	-0.196	3.614	3.619
	12	20.751	-0.107	5.309	5.310	20.705	-0.154	5.197	5.200	20.699	-0.160	5.196	5.199	20.728	-0.130	3.667	3.669	20.725	-0.133	3.667	3.669
	13	20.525	-0.334	5.292	5.303	20.809	-0.050	5.327	5.327	20.805	-0.054	5.326	5.326	20.664	-0.194	3.738	3.743	20.662	-0.196	3.738	3.743
	14	20.813	-0.045	5.165	5.165	20.893	0.034	5.374	5.374	20.890	0.032	5.374	5.374	20.852	-0.006	3.751	3.751	20.851	-0.007	3.750	3.750
	15	20.660	-0.199	5.137	5.141	20.936	0.078	5.356	5.356	20.934	0.076	5.355	5.356	20.796	-0.063	3.716	3.717	20.795	-0.064	3.716	3.717
	16	20.797	-0.061	5.233	5.233	21.028	0.170	5.197	5.200	21.027	0.168	5.197	5.200	20.911	0.052	3.698	3.698	20.910	0.052	3.698	3.698
	17	20.682	-0.176	5.228	5.231	20.907	0.049	5.286	5.286	20.907	0.048	5.285	5.286	20.793	-0.065	3.680	3.681	20.793	-0.066	3.680	3.681
	18	20.918	0.060	5.254	5.254	20.997	0.138	5.430	5.432	20.996	0.138	5.430	5.432	20.957	0.098	3.763	3.765	20.956	0.098	3.763	3.764
	19	20.815	-0.044	5.171	5.171	20.909	0.050	5.314	5.314	20.908	0.050	5.314	5.314	20.861	0.003	3.734	3.734	20.861	0.002	3.734	3.734
	20	20.930	0.072	5.387	5.388	20.908	0.049	5.226	5.226	20.907	0.049	5.226	5.226	20.919	0.061	3.730	3.730	20.919	0.061	3.730	3.730
0.75	7	18.803	-4.055	6.246	7.447	21.135	-1.723	5.431	5.698	20.804	-2.054	5.321	5.704	19.950	-2.908	4.263	5.160	19.787	-3.071	4.236	5.232
(0.6745)	8	21.005	-1.853	6.095	6.371	21.999	-0.859	5.629	5.694	21.825	-1.033	5.567	5.662	21.494	-1.364	4.292	4.503	21.409	-1.449	4.275	4.514
22.858	9	22.127	-0.731	6.260	6.303	22.393	-0.465	5.829	5.847	22.295	-0.563	5.797	5.824	22.258	-0.600	4.310	4.352	22.209	-0.648	4.299	4.348
	10	22.317	-0.541	5.974	5.998	22.771	-0.087	5.776	5.777	22.718	-0.140	5.760	5.761	22.541	-0.317	4.152	4.164	22.515	-0.343	4.146	4.160
	11	22.733	-0.125	5.810	5.811	22.798	-0.060	5.779	5.780	22.768	-0.089	5.771	5.771	22.765	-0.093	4.042	4.043	22.750	-0.108	4.039	4.041
	12	22.912	0.054	5.749	5.749	22.719	-0.139	5.761	5.763	22.703	-0.155	5.757	5.759	22.817	-0.041	4.071	4.071	22.809	-0.049	4.070	4.070
	13	22.654	-0.204	5.884	5.888	22.808	-0.050	5.863	5.863	22.799	-0.059	5.860	5.861	22.730	-0.128	4.150	4.152	22.725	-0.133	4.149	4.151
	14	22.887	0.029	5.779	5.779	22.844	-0.014	5.832	5.832	22.838	-0.019	5.831	5.831	22.866	0.008	4.099	4.099	22.863	0.005	4.098	4.098
	15	22.771	-0.087	5.801	5.802	22.775	-0.083	5.868	5.868	22.771	-0.087	5.867	5.867	22.773	-0.085	4.140	4.141	22.771	-0.087	4.140	4.140
	16	22.787	-0.071	5.718	5.718	22.938	0.080	5.810	5.810	22.936	0.078	5.809	5.810	22.862	0.004	4.107	4.107	22.860	0.003	4.107	4.107
	17	22.682	-0.176	5.707	5.710	22.881	0.023	5.845	5.845	22.880	0.022	5.845	5.845	22.780	-0.078	4.024	4.025	22.779	-0.079	4.024	4.025
	18	22.875	0.018	5.654	5.654	23.007	0.149	5.876	5.877	23.006	0.148	5.875	5.877	22.940	0.082	4.106	4.107	22.940	0.082	4.106	4.107
	19	22.844	-0.014	5.593	5.593	23.025	0.168	5.801	5.803	23.025	0.167	5.801	5.803	22.933	0.075	4.044	4.045	22.933	0.075	4.044	4.045
	20	22.972	0.115	5.779	5.780	22.810	-0.048	5.725	5.726	22.810	-0.048	5.725	5.725	22.893	0.035	4.030	4.030	22.892	0.034	4.030	4.030

Table 6 Experimental results for the AR(1) process with $\phi = 0.9$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{7, 8, \ldots, 20\}$.

p (x_p) Var. Par.	$\mathcal{L}$	Batched STS Area Estimator Avg.	Bias	Std. Dev.	RMSE	NBQ V2 Estimator Avg.	Bias	Std. Dev.	RMSE	NBQ V1 Estimator Avg.	Bias	Std. Dev.	RMSE	Combined V2 Estimator Avg.	Bias	Std. Dev.	RMSE	Combined V1 Estimator Avg.	Bias	Std. Dev.	RMSE
0.95	7	30.912	-7.353	14.120	15.920	33.950	-4.315	9.300	10.252	31.275	-6.990	8.311	10.860	32.407	-5.858	9.081	10.806	31.091	-7.174	8.907	11.437
(1.6449)	8	36.479	-1.786	15.062	15.167	36.504	-1.761	9.666	9.825	35.610	-2.655	9.306	9.677	36.491	-1.774	9.600	9.762	36.051	-2.213	9.545	9.798
38.265	9	37.594	-0.671	13.516	13.533	37.580	-0.685	10.070	10.093	37.028	-1.237	9.842	9.920	37.587	-0.678	8.791	8.818	37.315	-0.950	8.754	8.805
	10	37.812	-0.453	11.414	11.423	38.109	-0.156	10.105	10.106	37.698	-0.567	9.961	9.977	37.958	-0.307	7.855	7.861	37.756	-0.509	7.820	7.836
	11	38.386	0.121	10.799	10.799	37.984	-0.281	9.849	9.853	37.804	-0.461	9.783	9.793	38.188	-0.077	7.425	7.425	38.099	-0.166	7.405	7.407
	12	38.662	0.397	10.384	10.392	38.054	-0.211	9.797	9.799	37.986	-0.279	9.769	9.773	38.363	0.098	7.214	7.215	38.330	0.065	7.206	7.206
	13	38.104	-0.161	10.008	10.009	38.085	-0.180	9.928	9.930	38.040	-0.225	9.911	9.913	38.094	-0.171	7.141	7.143	38.072	-0.193	7.136	7.138
	14	38.306	0.041	9.899	9.899	38.326	0.061	9.737	9.738	38.291	0.026	9.726	9.726	38.316	0.051	7.016	7.017	38.299	0.034	7.013	7.013
	15	38.422	0.157	9.894	9.895	38.115	-0.150	9.956	9.957	38.098	-0.167	9.950	9.951	38.271	0.006	7.035	7.035	38.262	-0.003	7.034	7.034
	16	38.226	-0.038	9.943	9.943	38.255	-0.010	9.836	9.837	38.246	-0.019	9.834	9.834	38.240	-0.025	7.091	7.091	38.236	-0.029	7.090	7.090
	17	38.153	-0.112	9.532	9.532	38.418	0.153	9.878	9.879	38.412	0.147	9.877	9.878	38.283	0.019	6.806	6.806	38.280	0.016	6.805	6.805
	18	38.451	0.186	9.582	9.584	38.743	0.478	9.878	9.890	38.738	0.473	9.877	9.888	38.595	0.330	6.887	6.895	38.592	0.328	6.886	6.894
	19	38.399	0.134	9.496	9.497	38.682	0.417	9.760	9.769	38.679	0.414	9.759	9.768	38.538	0.273	6.748	6.754	38.537	0.272	6.748	6.754
	20	38.819	0.554	9.716	9.731	38.306	0.041	9.747	9.748	38.304	0.039	9.747	9.747	38.566	0.301	6.788	6.795	38.566	0.301	6.788	6.794
0.99	7	43.023	-38.589	19.888	43.412	59.836	-21.776	19.167	29.010	40.193	-41.419	10.956	42.844	51.296	-30.316	14.885	33.773	41.630	-39.981	12.478	41.883
(2.3263)	8	63.544	-18.067	29.612	34.688	64.789	-16.823	18.939	25.332	56.424	-25.188	15.541	29.597	64.157	-17.455	19.384	26.085	60.040	-21.571	18.620	28.496
81.612	9	69.221	-12.391	33.086	35.330	74.505	-7.107	21.540	22.682	68.185	-13.426	19.802	23.925	71.821	-9.791	21.641	23.753	68.712	-12.900	21.333	24.930
	10	76.350	-5.261	31.978	32.408	79.563	-2.048	22.842	22.933	76.832	-4.780	22.154	22.664	77.931	-3.680	21.385	21.699	76.587	-5.025	21.288	21.873
	11	81.773	0.161	29.693	29.694	81.200	-0.411	21.763	21.767	80.209	-1.403	21.445	21.491	81.491	-0.121	19.890	19.891	81.003	-0.609	19.845	19.855
	12	83.965	2.353	26.111	26.217	81.832	0.220	21.628	21.629	81.558	-0.054	21.525	21.525	82.915	1.304	17.379	17.428	82.780	1.169	17.357	17.396
	13	82.641	1.029	23.842	23.864	81.407	-0.205	21.678	21.679	81.241	-0.370	21.616	21.619	82.034	0.422	16.566	16.571	81.952	0.341	16.549	16.553
	14	82.426	0.815	22.724	22.739	81.423	-0.188	20.838	20.839	81.307	-0.305	20.796	20.798	81.933	0.321	15.450	15.453	81.876	0.264	15.436	15.439
	15	81.767	0.155	21.163	21.163	81.482	-0.129	20.288	20.288	81.401	-0.211	20.263	20.264	81.627	0.015	14.903	14.903	81.587	-0.025	14.895	14.895
	16	82.122	0.511	21.256	21.262	81.094	-0.518	20.445	20.452	81.030	-0.582	20.427	20.436	81.616	0.005	14.667	14.668	81.585	-0.027	14.662	14.662
	17	81.788	0.177	20.670	20.671	81.524	-0.087	20.606	20.606	81.494	-0.117	20.599	20.599	81.658	0.047	14.482	14.482	81.644	0.032	14.480	14.480
	18	81.523	-0.088	21.343	21.343	82.320	0.709	20.759	20.771	82.299	0.687	20.754	20.766	81.916	0.304	14.887	14.890	81.905	0.293	14.886	14.888
	19	82.083	0.471	20.924	20.929	82.607	0.995	20.864	20.888	82.594	0.982	20.862	20.885	82.341	0.729	14.732	14.750	82.334	0.723	14.731	14.749
	20	82.971	1.360	20.830	20.875	82.194	0.582	20.854	20.862	82.185	0.573	20.852	20.860	82.589	0.977	14.409	14.442	82.584	0.973	14.409	14.442

Table 7 Experimental results for the AR(1) process with $\phi = 0.9$. All estimates are based on 1,000,000 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{4, 5, \ldots, 14\}$.

		Batched STS Area Estimator				NBQ V2 Estimator				NBQ V1 Estimator				Combined V2 Estimator				Combined V1 Estimator			
p (x_p) **Var. Par.**	$\mathcal{L}$	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE
0.5	4	3.733	-17.125	1.403	17.183	10.347	-10.511	2.952	10.918	10.239	-10.619	2.923	11.014	6.987	-13.871	1.711	13.976	6.934	-13.924	1.696	14.027
(0.0000)	5	8.274	-12.584	2.994	12.935	14.648	-6.210	3.866	7.315	14.548	-6.310	3.842	7.388	11.411	-9.447	2.557	9.787	11.361	-9.497	2.545	9.832
20.858	6	13.501	-7.357	4.459	8.603	17.604	-3.254	4.531	5.578	17.523	-3.335	4.513	5.612	15.520	-5.338	3.275	6.263	15.480	-5.378	3.266	6.292
	7	17.032	-3.826	5.055	6.340	19.174	-1.684	4.891	5.173	19.117	-1.741	4.880	5.181	18.086	-2.772	3.577	4.525	18.058	-2.800	3.571	4.538
	8	18.908	-1.950	5.232	5.583	19.998	-0.860	5.089	5.161	19.962	-0.896	5.082	5.161	19.445	-1.413	3.678	3.940	19.427	-1.431	3.675	3.944
	9	19.891	-0.967	5.276	5.363	20.430	-0.428	5.192	5.210	20.407	-0.451	5.188	5.208	20.156	-0.702	3.717	3.783	20.145	-0.713	3.716	3.783
	10	20.382	-0.476	5.284	5.306	20.646	-0.212	5.240	5.245	20.632	-0.226	5.238	5.243	20.512	-0.346	3.730	3.746	20.505	-0.353	3.729	3.746
	11	20.644	-0.214	5.276	5.280	20.749	-0.109	5.272	5.274	20.740	-0.118	5.271	5.272	20.696	-0.162	3.733	3.737	20.691	-0.167	3.733	3.737
	12	20.770	-0.088	5.261	5.262	20.803	-0.055	5.283	5.283	20.797	-0.061	5.282	5.283	20.786	-0.072	3.729	3.729	20.783	-0.075	3.728	3.729
	13	20.819	-0.039	5.251	5.251	20.831	-0.027	5.289	5.289	20.827	-0.031	5.289	5.289	20.825	-0.033	3.728	3.728	20.823	-0.035	3.728	3.728
	14	20.849	-0.009	5.243	5.244	20.838	-0.020	5.290	5.290	20.836	-0.022	5.289	5.289	20.844	-0.014	3.723	3.723	20.843	-0.015	3.723	3.723
0.75	4	3.914	-18.944	1.493	19.003	11.775	-11.083	3.543	11.635	10.391	-12.467	2.984	12.819	7.782	-15.076	1.990	15.206	7.101	-15.757	1.754	15.854
(0.6745)	5	8.883	-13.975	3.432	14.390	16.161	-6.697	4.402	8.015	15.144	-7.714	4.030	8.703	12.464	-10.394	2.918	10.796	11.964	-10.894	2.798	11.248
22.858	6	14.747	-8.111	5.388	9.738	19.386	-3.472	5.077	6.150	18.786	-4.072	4.872	6.349	17.030	-5.828	3.850	6.985	16.735	-6.123	3.800	7.207
	7	18.755	-4.103	6.149	7.393	21.105	-1.753	5.441	5.716	20.774	-2.084	5.330	5.723	19.911	-2.947	4.219	5.146	19.748	-3.110	4.192	5.220
	8	20.825	-2.033	6.175	6.501	21.974	-0.884	5.627	5.696	21.793	-1.065	5.567	5.668	21.390	-1.468	4.244	4.491	21.301	-1.557	4.228	4.505
	9	21.869	-0.989	6.057	6.137	22.422	-0.436	5.719	5.736	22.325	-0.533	5.688	5.713	22.141	-0.717	4.204	4.264	22.093	-0.765	4.194	4.263
	10	22.382	-0.476	5.939	5.958	22.645	-0.213	5.762	5.766	22.592	-0.266	5.745	5.752	22.512	-0.346	4.156	4.170	22.486	-0.372	4.151	4.167
	11	22.649	-0.209	5.863	5.867	22.750	-0.108	5.788	5.789	22.721	-0.137	5.779	5.781	22.699	-0.159	4.128	4.131	22.684	-0.174	4.125	4.129
	12	22.776	-0.082	5.813	5.814	22.796	-0.062	5.791	5.792	22.780	-0.078	5.787	5.787	22.786	-0.072	4.107	4.107	22.778	-0.080	4.105	4.106
	13	22.823	-0.035	5.778	5.778	22.827	-0.031	5.797	5.797	22.817	-0.041	5.794	5.794	22.825	-0.033	4.100	4.100	22.820	-0.038	4.099	4.099
	14	22.850	-0.008	5.757	5.757	22.837	-0.021	5.798	5.798	22.831	-0.027	5.796	5.796	22.844	-0.014	4.086	4.086	22.841	-0.017	4.086	4.086

Table 8 Experimental results for the AR(1) process with $\phi = 0.9$. All estimates are based on 1,000,000 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{4, 5, \ldots, 14\}$.

		Batched STS Area Estimator				NBQ V2 Estimator				NBQ V1 Estimator				Combined V2 Estimator				Combined V1 Estimator			
p (x_p) **Var. Par.**	$\mathcal{L}$	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE	Avg.	Bias	Std. Dev.	RMSE
0.95	4	7.038	-31.227	2.615	31.336	17.631	-20.634	6.718	21.700	11.105	-27.160	3.262	27.355	12.250	-26.015	3.678	26.273	9.039	-29.226	2.262	29.313
(1.6449)	5	13.907	-24.358	5.761	25.030	24.086	-14.179	7.833	16.199	17.431	-20.834	4.758	21.370	18.916	-19.349	5.066	20.001	15.641	-22.624	4.067	22.986
38.265	6	21.593	-16.672	9.515	19.196	30.209	-8.056	8.812	11.939	24.425	-13.840	6.529	15.303	25.833	-12.432	6.875	14.207	22.987	-15.278	6.279	16.518
	7	30.909	-7.356	13.764	15.606	33.901	-4.364	9.393	10.358	31.227	-7.038	8.391	10.951	32.381	-5.884	8.940	10.702	31.066	-7.199	8.758	11.337
	8	36.676	-1.589	15.281	15.364	36.458	-1.807	9.895	10.058	35.583	-2.682	9.541	9.911	36.569	-1.696	9.751	9.897	36.138	-2.127	9.701	9.932
	9	37.470	-0.795	13.427	13.451	37.515	-0.750	9.898	9.927	36.978	-1.287	9.686	9.771	37.492	-0.773	8.761	8.795	37.228	-1.037	8.721	8.782
	10	37.740	-0.525	11.728	11.739	37.949	-0.316	9.830	9.835	37.543	-0.722	9.685	9.712	37.842	-0.423	7.883	7.894	37.643	-0.622	7.847	7.871
	11	38.220	-0.045	10.903	10.903	38.110	-0.155	9.784	9.785	37.928	-0.337	9.714	9.720	38.166	-0.099	7.451	7.451	38.076	-0.189	7.431	7.433
	12	38.444	0.179	10.415	10.417	38.186	-0.079	9.756	9.756	38.116	-0.149	9.727	9.728	38.317	0.052	7.206	7.206	38.283	0.018	7.197	7.197
	13	38.372	0.107	10.039	10.040	38.234	-0.031	9.736	9.736	38.188	-0.077	9.718	9.719	38.304	0.039	7.029	7.029	38.281	0.016	7.023	7.023
	14	38.350	0.085	9.833	9.833	38.250	-0.015	9.731	9.731	38.216	-0.049	9.720	9.720	38.301	0.036	6.933	6.933	38.284	0.019	6.929	6.929
0.99	4	7.038	-74.574	2.615	74.620	36.661	-44.951	15.764	47.635	11.105	-70.507	3.262	70.582	21.615	-59.997	8.002	60.529	9.039	-72.573	2.262	72.608
(2.3263)	5	17.427	-64.185	6.634	64.527	45.853	-35.759	18.161	40.107	18.135	-63.477	4.976	63.672	31.414	-50.198	9.869	51.159	17.775	-63.837	4.530	63.997
81.612	6	35.243	-46.369	13.766	48.370	48.614	-32.998	17.788	37.487	28.187	-53.425	7.580	53.960	41.822	-39.790	11.864	41.521	31.771	-49.841	8.585	50.575
	7	43.220	-38.392	19.835	43.213	59.552	-22.060	19.132	29.200	40.076	-41.536	10.893	42.940	51.257	-30.355	14.874	33.803	41.673	-39.939	12.465	41.839
	8	63.915	-17.697	30.127	34.940	64.789	-16.823	19.307	25.608	56.400	-25.212	15.826	29.768	64.345	-17.267	19.637	26.149	60.217	-21.395	18.885	28.538
	9	68.306	-13.306	32.349	34.979	74.779	-6.833	21.552	22.609	68.435	-13.177	19.759	23.750	71.491	-10.121	21.495	23.758	68.370	-13.242	21.167	24.968
	10	76.838	-4.774	32.670	33.017	79.376	-2.236	22.527	22.638	76.711	-4.901	21.846	22.389	78.087	-3.525	21.755	22.039	76.776	-4.836	21.665	22.198
	11	81.323	-0.289	29.337	29.338	80.940	-0.672	22.000	22.010	79.971	-1.641	21.697	21.759	81.135	-0.477	19.626	19.632	80.658	-0.954	19.576	19.600
	12	83.407	1.795	26.351	26.412	81.610	-0.002	21.443	21.443	81.340	-0.272	21.337	21.339	82.523	0.911	17.746	17.769	82.390	0.778	17.721	17.739
	13	82.902	1.290	23.820	23.854	81.656	0.044	21.099	21.099	81.489	-0.123	21.038	21.038	82.289	0.677	16.305	16.319	82.207	0.595	16.288	16.299
	14	82.521	0.909	22.366	22.384	81.644	0.032	20.931	20.931	81.532	-0.080	20.894	20.894	82.090	0.478	15.512	15.519	82.035	0.423	15.501	15.506

5.2. Selection of Area Estimator Weights

The limited experimentation in Alexopoulos et al. (2023) with the constant weight $w_0(\cdot)$ did not consider the alternatives $w_2(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ (Goldsman et al. 1990) and $\{w_{\cos,\ell}(t) = \sqrt{8}\pi\ell\cos(2\pi\ell t)\colon \ell = 1, 2, \ldots\}$ (Foley and Goldsman 1999), which are tailored to the estimation of the steady-state mean of the base process $\{Y_k : k \geq 1\}$ and yield first-order unbiased estimators for the respective variance parameter $\varphi^2 = \lim_{n\to\infty} n\mathrm{Var}(\overline{Y}_n)$. In particular, the STS area estimators for φ^2 obtained from the orthonormal sequence $\{w_{\cos,\ell}(\cdot) : \ell = 1, 2, \ldots\}$ are asymptotically unbiased as $m \to \infty$ for fixed b; hence they can be averaged to yield an estimator with smaller variance.

At this junction we wish to recall a few findings regarding the bias of the estimators of φ^2 in the last paragraph. The main competitor of the STS area estimators for φ^2 is the nonoverlapping-batch-means (NBQ) estimator $\mathscr{N}_{b,m} \equiv \frac{m}{b-1}\sum_{j=1}^{b}(\overline{Y}_{j,m} - \overline{Y}_n)^2$, where $\overline{Y}_{j,m}$ the sample average from batch j. (Notice that the NBQ estimator $m\widetilde{S}^2_{b,m}$ is an analogue of $\mathscr{N}_{b,m}$.) Aktaran-Kalaycı et al. (2007) obtained detailed expressions for the expected value of various estimators of φ^2, including the ones mentioned in this section. Specifically, the NBQ estimator has first-order bias equal to $-\gamma_1(b+1)/n$, where $\gamma_1 \equiv 2\sum_{k=1}^{\infty} k\mathrm{Cov}(Y_1, Y_{1+k})$ (assuming that the infinite series is summable). Analytical results for the two stochastic processes in Examples 5 and 6 below reveal that, for fixed b, the STS area estimator of φ^2 based on the quadratic weight $w_2(\cdot)$ has more prominent bias than the NBQ estimator $\mathscr{N}_{b,m}$ for very small batch sizes m until it "catches up" as m increases, and eventually outperforms the NBQ estimator with regard to the rate of convergence to φ^2. Further, Example 1 in Alexopoulos et al. (2016) (corresponding to the Example 6 below) illustrates that for processes with positive autocorrelation and fixed (b, m), the bias of the estimator for φ^2 can become more pronounced as j increases. (Of course, this effect diminishes as m increases.)

Unfortunately, the derivation of detailed fine-tuned expressions for the expectation of the estimators of the variance parameter $\sigma^2 = \lim_{n\to\infty} n\mathrm{Var}\big[\widetilde{y}_p(n)\big]$ is very challenging. First, all estimators of σ^2 are inherently biased, partly due to the Bahadur representation. Second, such calculations involve joint moments of order statistics, which are often hard to obtain even for i.i.d. processes; and this task is compounded in the presence of autocorrelation. Based on our ongoing research we conjecture that the bias of all estimators for σ^2 under study is approximately $O(m^{-1/4})$.

A key question is: do the properties of the STS area estimators based on the weights $w_2(t) = \sqrt{840}(3t^2 - 3t + 1/2)$ (Goldsman et al. 1990) and $\{w_{\cos,\ell}(t) = \sqrt{8}\pi\ell\cos(2\pi\ell t)\colon \ell = 1, 2, \ldots\}$ carry over to the quantile-estimation setting? The following two examples attempt to provide a preliminary answer with regard to the small-sample bias of the variance estimators $m\widetilde{S}^2_{b,m}$ and $\mathscr{A}^2_{b,m}(w)$ corresponding to the weight functions $w_0(\cdot)$, $w_2(\cdot)$, and $\{w_{\cos,\ell}(\cdot) : \ell = 1, 2\}$.

[Note that we've al;readfy described the AR(1), ARTOP, and M/M/1 processes, so make sure that the material below isn't repetitive.]

EXAMPLE 5. Consider the Gaussian first-order autoregressive [AR(1)] time-series defined by the linear regression model $X_i = \mu_X + \phi(X_{i-1} - \mu_X) + \epsilon_i$ for $i \geq 1$, where $\phi \in (-1, 1)$ is the autoregressive parameter, the residuals $\{\epsilon_i : i \geq 1\}$ are i.i.d. $N(0, \sigma_\epsilon^2)$, and X_0 is the initial state of the process. The steady-state marginal distribution of this process is $N[\,\mu_X, \sigma_\epsilon^2/(1-\phi^2)\,]$. In this experiment we take $X_0 \sim N(0,1)$, $\phi = 0.9$, and $\sigma_\epsilon^2 = 1 - \phi^2 = 0.19$; hence the process is stationary with a standard normal marginal distribution. The computation of the variance parameter σ^2 in Equation (**??**) is detailed in Dingeç et al. (2023b).

Figure 1 displays plots of the five estimators ($m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\cos,1}$ ("STS Cos,1"), and $w_{\cos,2}$ ("STS Cos,2")) and of σ^2 computed from 2,500 independent replications for a fixed batch count $b = 32$, values $p \in \{0.75, 0.9, 0.95, 0.99, 0.995\}$, and increasing batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$. Figure 2 displays plots of the respective estimated relative bias (as a percentage) of the variance estimators.

An examination of Figures 1–3 reveals the following findings: (i) All variance estimators converge to the value σ^2, as anticipated by theory. Indeed, for $m = 2^{20}$ all averages are within 2% of σ^2. (ii) The NBQ variance estimator $m\widetilde{S}^2_{b,m}$ typically has the lowest small-sample estimated bias; this is illustrated best for $p = 0.99$ or 0.995. (iii) There is no evidence in this experiment that any of the alternative weights $w_2(\cdot)$ and $\{w_{\cos,\ell} : \ell = 1, 2\}$ induces a variance estimator with lower small-sample bias than $w_0(\cdot)$. Although for $p = 0.995$ the estimator $\mathscr{A}_{b,m}(w_0)$ has the most-pronounced estimated bias at $m = 2^{10}$, it catches up to the NBQ estimator near $m = 2^{12}$, while the STS area estimators corresponding to $w_2(\cdot)$ and $\{w_{\cos,\ell} : \ell = 1, 2\}$ bounce from negative to excessive positive estimated bias before settling near σ^2 for $m \approx 2^{17}$. (iv) Among the five competing estimators of σ^2, the NBQ estimator appears to exhibit the quickest convergence to a small neighborhood of σ^2 (within 2%) followed by $\mathscr{A}_{b,m}(w_0)$.

EXAMPLE 6. Consider an M/M/1 queueing system with arrival rate $\lambda = 0.8$, service rate $\omega = 1$ (traffic intensity $\rho = 0.9$) and first-in, first-out (FIFO) service discipline. Let X_i be the time spent by the ith entity in queue (prior to service). The steady-state c.d.f. of X_i is

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 - \rho & \text{if } x = 0, \\ 1 - \rho e^{-\omega(1-\rho)x} & \text{if } x > 0; \end{cases} \tag{51}$$

hence the steady-state distribution of X_i has mean $\mu_X = \rho/(\omega - \lambda) = 4$, and the quantiles of this distribution are readily computed by inverting Equation (51). The steady-state distribution of X_i is distinctly nonnormal, having an atom at zero, an exponential tail, and a skewness of $2(3 - 3\rho + \rho^2)/[\rho^{1/2}(2-\rho)^{3/2}] \approx 2.109$. The variance parameter $\sigma^2_{I(x_p)}$ of the indicator process was computed from Blomqvist (1967, Equation (22)). After some algebra, we obtained the following analytical expression for the asymptotic variance parameter:

$$\sigma^2 = \frac{1}{\omega^2(1-\rho)^4}\left\{\frac{[-2 + p(3-\rho) + 2\rho](1+\rho)}{1-p} - 4\rho\ln\left(\frac{\rho}{1-p}\right)\right\}.$$

Figure 1 Estimated expected values of the variance estimators $m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\cos,1}$ ("STS Cos,1"), and $w_{\cos,2}$ ("STS Cos,2") for selected marginal quantiles of the AR(1) process in Example 5 with correlation coefficient $\phi = 0.9$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

In this example we generated stationary sample paths $\{X_k : k \geq 1\}$ by sampling X_1 from the c.d.f. in Equation (51). Figures 4–6 depict the experimental results based on 2,500 independent replications for a fixed batch count $b = 32$, values $p \in \{0.5, 0.75, 0.9, 0.95, 0.99, 0.995\}$, and increasing batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

For this test process, the dominance of the NBQ estimator (primarily) and the STS area estimator $\mathscr{A}^2_{b,m}(w_0)$ (secondarily) over their competitors with regard to the rate of convergence to a narrow neighborhood of σ^2 (say, within 2%) is more evident than in Example 5.

Figure 2 Estimated percent relative bias of the variance estimators $m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\cos,1}$ ("STS Cos,1"), and $w_{\cos,2}$ ("STS Cos,2") for selected marginal quantiles of the stationary AR(1) process in Example 5 with correlation coefficient $\phi = 0.9$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

Based on the limited experimentation in Examples 5 and 6 and the early stage of our theoretical study of the bias of the aforementioned variance estimators, which may eventually lead to better weight functions adapted to quantile estimation, we will use the constant weight $w_0(\cdot)$ in our sequential procedure.

6. Conclusions

APPENDIX

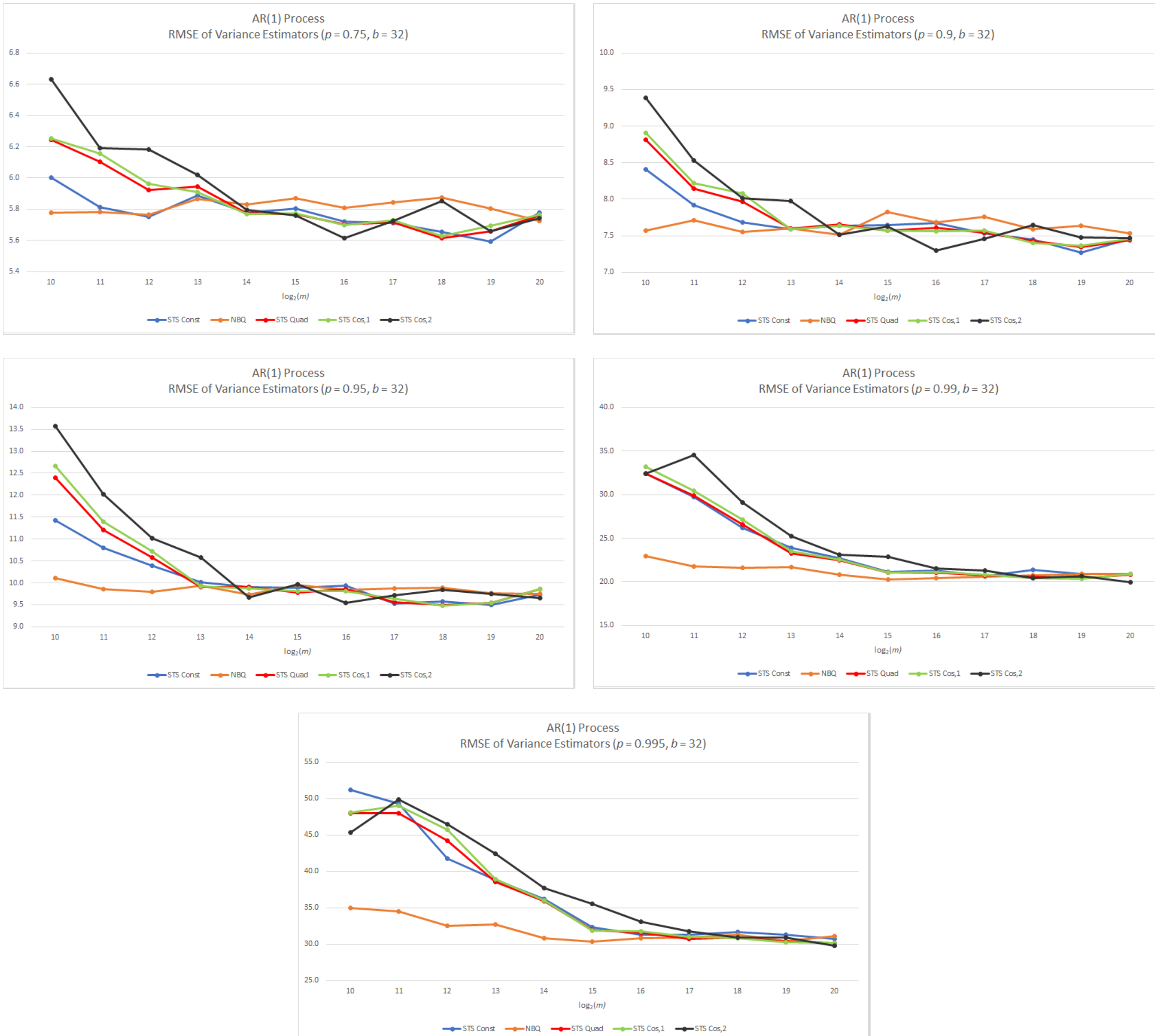

Figure 3 Estimated Root Mean Square Error (RMSE) of the variance estimators $m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\cos,1}$ ("STS Cos,1"), and $w_{\cos,2}$ ("STS Cos,2") for selected marginal quantiles of the stationary AR(1) process in Example 5 with correlation coefficient $\phi = 0.9$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

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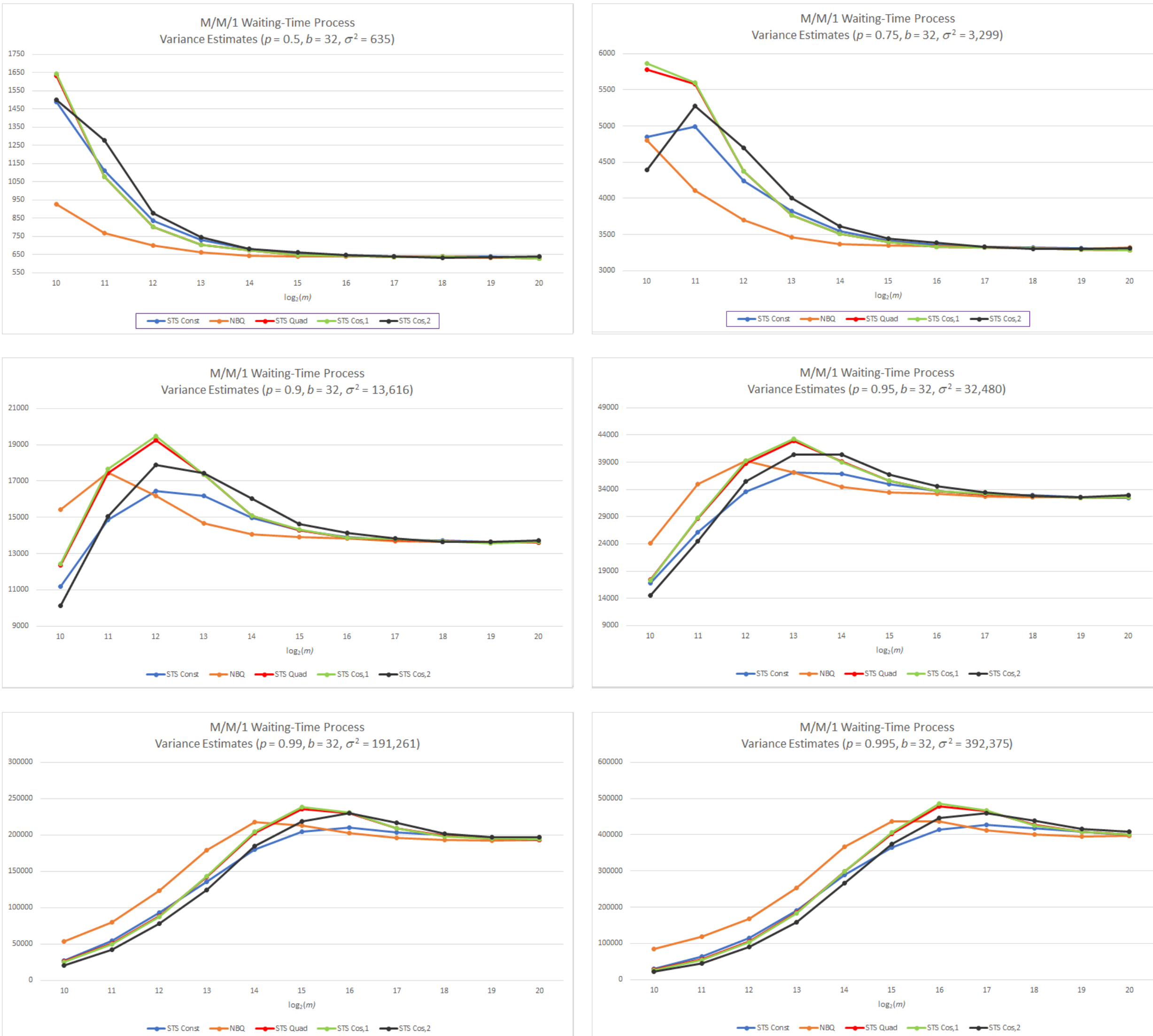

Figure 4 Estimated expected values of the variance estimators $m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\text{cos},1}$ ("STS Cos,1"), and $w_{\text{cos},2}$ ("STS Cos,2") for selected marginal quantiles of the stationary waiting-time process in the M/M/1 queueing system in Example 6 with traffic intensity $\rho = 0.8$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

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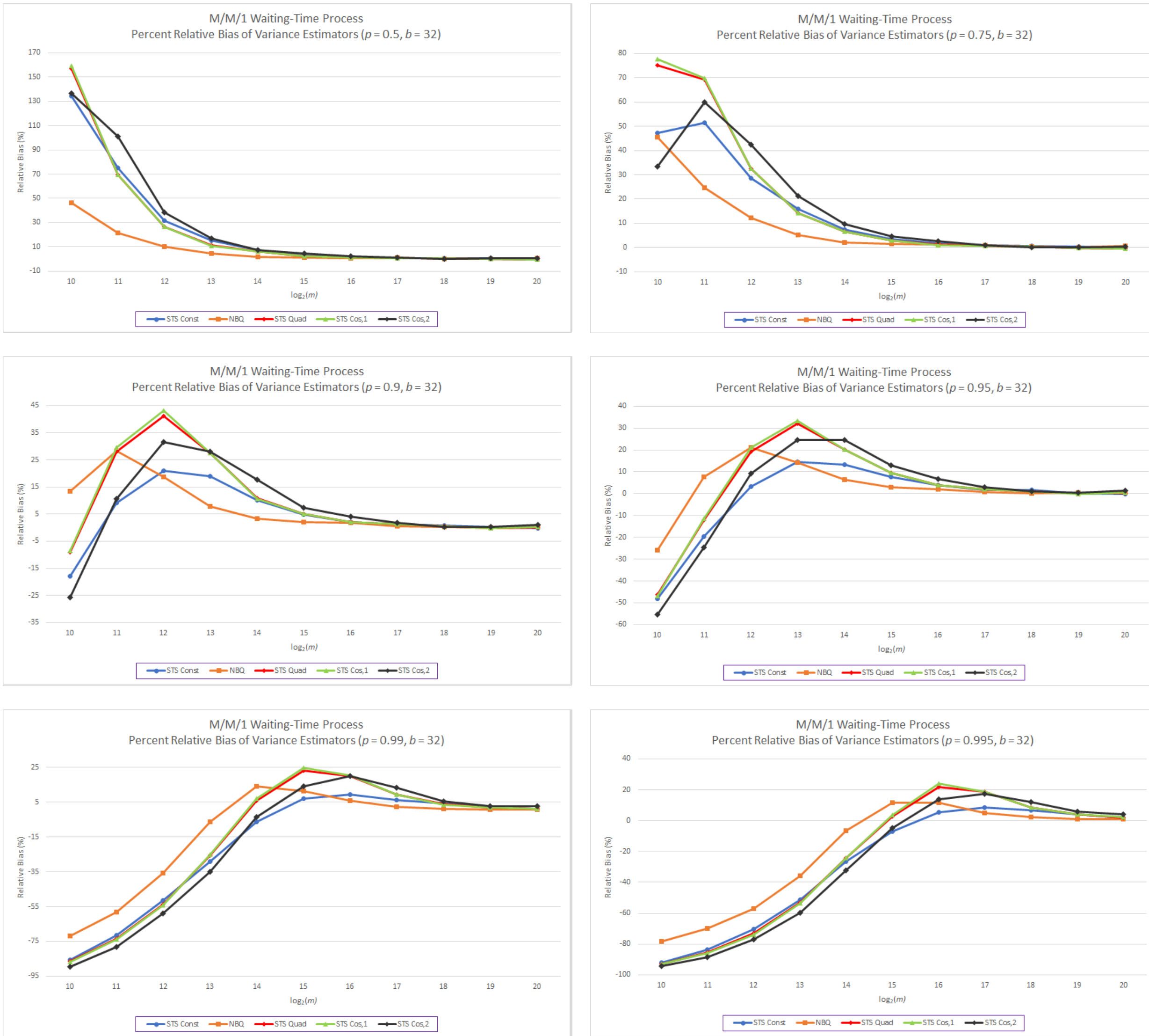

Figure 5 Estimated percent relative bias of the variance estimators $m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\cos,1}$ ("STS Cos,1"), and $w_{\cos,2}$ ("STS Cos,2") for selected marginal quantiles of the stationary waiting-time process in the M/M/1 queueing system in Example 6 with traffic intensity $\rho = 0.8$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

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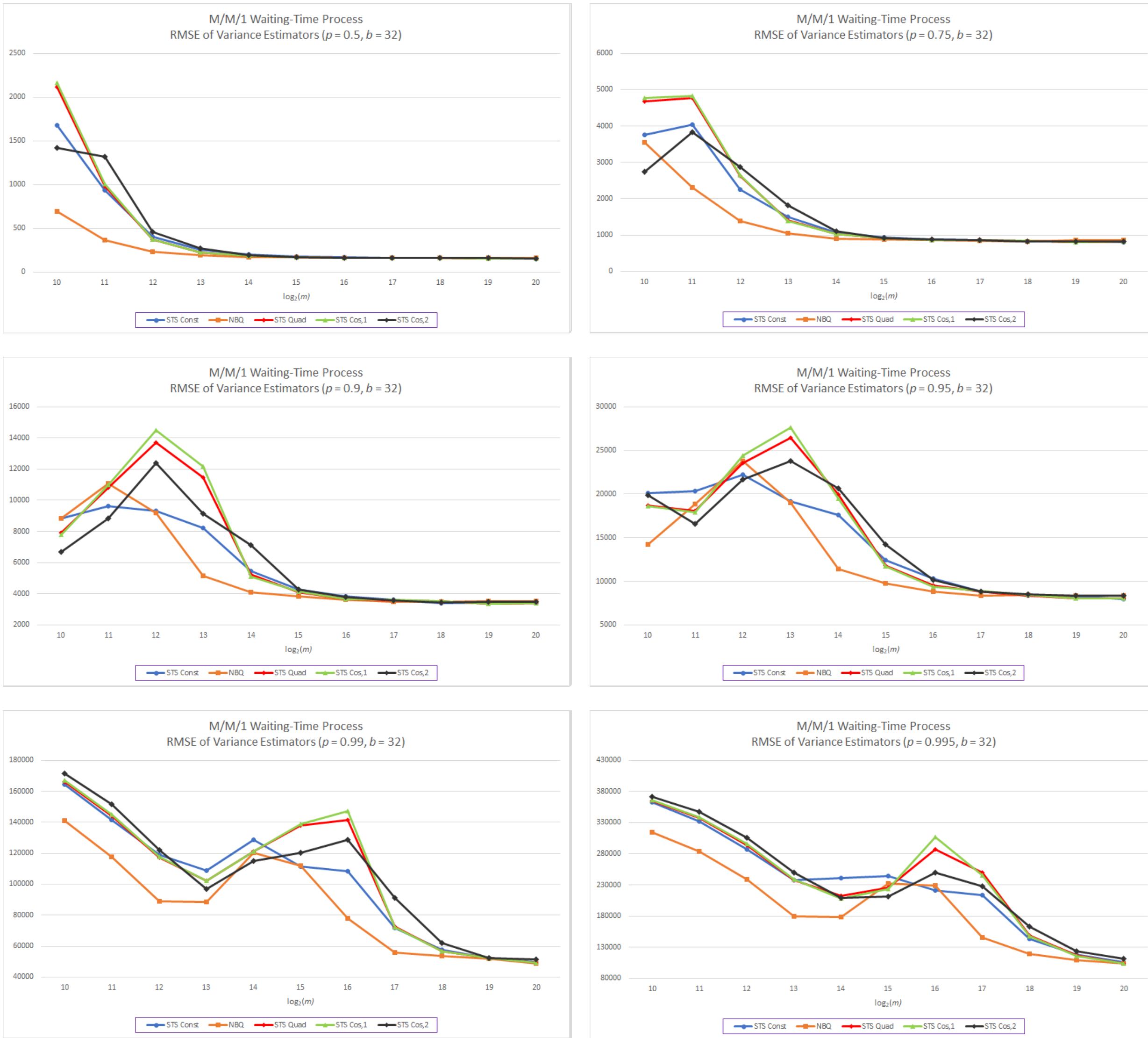

Figure 6 Estimated RMSE of the variance estimators $m\widetilde{S}^2_{b,m}$ ("NBQ") and $\mathscr{A}^2_{b,m}(w)$ for the weight functions w_0 ("STS Const"), w_2 ("STS Quad"), $w_{\cos,1}$ ("STS Cos,1"), and $w_{\cos,2}$ ("STS Cos,2") for selected marginal quantiles of the stationary waiting-time process in the M/M/1 queueing system in Example 6 with traffic intensity $\rho = 0.8$. All estimates are based on 2,500 independent replications with $b = 32$ batches and batch sizes $m = 2^{\mathcal{L}}$, $\mathcal{L} \in \{10, 11, \ldots, 20\}$.

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