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INFORMS JOURNAL ON COMPUTING
Vol. 00, No. 0, Xxxxx 0000, pp. 000–000
ISSN 0899-1499, EISSN 1526-5528

Submitted to *INFORMS Journal on Computing*

Geometric-Moment Contraction of Stationary Processes and Their Indicator Processes: Simulation-Based Benchmark Examples

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Abstract. We present methodology and numerical results for enabling simulation-based performance evaluation of point and confidence-interval estimators of the mean and given quantiles of a stationary benchmark process $\{Y_k : k \geq 0\}$ that satisfies the Geometric-Moment Contraction condition and has a bounded marginal density with a bounded derivative. In two 2024 papers, Dingeç et al. derived basic properties of the autocovariance functions of such a benchmark process and its indicator process $\{I_k(y) : k \geq 0\}$ for given y, where $I_k(y)$ is the binary indicator of the event $Y_k \leq y$; and they verified the finiteness of the following variance parameters: (a) the limit of $n\mathrm{Var}\big[\overline{Y}_n\big]$ as n increases, where $\overline{Y}_n$ is the mean of the sequence $\{Y_1, \ldots, Y_n\}$; (b) the analogous limit of $n\mathrm{Var}\big[\overline{I}_n(y)\big]$; and (c) the analogous limit of $n\mathrm{Var}\big[\widetilde{y}_p(n)\big]$, where $\widetilde{y}_p(n)$ is the empirical p-quantile of the sequence in (a) for given p in $(0, 1)$. To identify and study benchmark processes for effectively stress-testing proposed mean and quantile estimators, we derive exact (or numerically tractable) expressions for the relevant autocovariance functions and variance parameters of the following processes: the standard first-order autoregressive (AR(1)) process; an AR(1) process with Cauchy innovations; an Autoregressive-to-Pareto process; and the queue-waiting-time process in an M/M/1 queueing system. For each process, we also calculate the autocorrelations between the nonoverlapping batch means at all lags and verify that those quantities tend to zero with increasing batch size.

Key words: Stationary stochastic process; Geometric-Moment Contraction condition; simulation analysis; batch-means method; first-order autoregressive process; autoregressive-to-Pareto process; M/M/1 queue-waiting-time process.

Ver. GMC1B-v122 || June 7, 2025 || 16:29.

1. Introduction

Given a stationary stochastic process $\{Y_k : k \geq 0\}$ representing the output of a steady-state discrete-time random process such as a simulation whose initialization involves the random variable (r.v.) Y_0, we study the estimation of the following characteristics of the process: the marginal mean $\mu \equiv \mathrm{E}[Y_0]$; the marginal variance $\mathrm{Var}[Y_0]$; the marginal cumulative distribution function (c.d.f.) $F(y) \equiv \Pr(Y_0 \leq y)$ for $y \in \mathbb{R}$; and a user-specified p-quantile $y_p \equiv F^{-1}(p) \equiv \inf\{y : F(y) \geq p\}$, where $p \in (0,1)$. Such estimation problems are central in the analysis and evaluation of production, service, and other real-world systems. Examples include the estimation of: the mean cycle time taken over all successive items in a production plant; the probability that an individual item's cycle time will exceed a contractual value; and the smallest upper bound on the wait time in a queue that a certain fraction of individual items will experience.

To estimate the mean, variance, c.d.f., and quantiles, we typically proceed by collecting the first $n \geq 1$ consecutive outputs of the simulation $\{Y_k : k = 1, \ldots, n\}$ that were generated after the start of the simulation, when Y_0 was initialized; and then we compute the following point estimators: (a) the sample mean $\overline{Y}_n \equiv n^{-1} \sum_{k=1}^{n} Y_k$ as an estimator for μ; (b) the empirical c.d.f. $F_n(y) \equiv \overline{I}_n(y) \equiv n^{-1} \sum_{k=1}^{n} I_k(y)$ as an estimator for $F(y)$, where $I_k(y) \equiv \mathbf{1}_{\{Y_k \leq y\}}$ and the indicator function $\mathbf{1}_{\mathscr{E}} \equiv 1$ if event $\mathscr{E}$ occurs, whereas $\mathbf{1}_{\mathscr{E}} \equiv 0$ otherwise; and (c) the empirical p-quantile $\widetilde{y}_p(n) \equiv Y_{(\lceil np \rceil)}$ as an estimator for y_p, where $Y_{(1)} \leq \cdots \leq Y_{(n)}$ are the sample's order statistics and $\lceil \cdot \rceil$ denotes the ceiling function.

Of course, the point estimators $\overline{Y}_n$, $\overline{I}_n(y)$, and $\widetilde{y}_p(n)$ are themselves r.v.'s and hence exhibit variability that needs to be taken into account as part of any analysis. In order to address this uncertainty, it is of interest to verify the existence of, and then to estimate, the following *variance parameters*: (a) $\sigma_Y^2 \equiv \lim_{n\to\infty} n\mathrm{Var}\big[\overline{Y}_n\big]$ associated with $\{Y_k : k \geq 0\}$, (b) $\sigma_{I(y)}^2 \equiv \lim_{n\to\infty} n\mathrm{Var}\big[\overline{I}_n(y)\big]$ associated with the indicator process $\{I_k(y) : k \geq 0\}$, and (c) $\sigma_{\widetilde{y}_p}^2 \equiv \lim_{n\to\infty} n\mathrm{Var}\big[\widetilde{y}_p(n)\big]$ corresponding to the empirical p-quantile estimator $\widetilde{y}_p(n)$.

The analysis of the underlying properties of σ_Y^2, $\sigma_{I(y)}^2$, and $\sigma_{\widetilde{y}_p}^2$ is often based on certain moment and mixing assumptions (Billingsley 1999) that are typically difficult to validate in practice (Wu 2005, Alexopoulos et al. 2019, 2024). With this issue in mind, Alexopoulos et al. (2019) and Dingeç et al. (2024a,b) invoke different assumptions that are arguably more intuitive and more-easily validated. Notably, the *Geometric-Moment Contraction* (GMC) condition of Wu (2005) is a serviceable alternative to the aforementioned mixing conditions (also see, e.g., Doukhan 2018).

Geometric-Moment Contraction Condition The stationary process $\{Y_k : k \geq 0\}$ is defined by a function $\xi(\cdot)$ of a sequence of independent and identically distributed (i.i.d.) r.v.'s $\{\varepsilon_j : j \in \mathbb{Z}\}$ for $\mathbb{Z} \equiv \{0, \pm1, \pm2, \ldots\}$, where $Y_k = \xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$ for $k \geq 0$. Moreover, there exist constants $\psi > 0$, $C_\psi > 0$, and $r_\psi \in (0,1)$ such that for two independent sequences $\{\varepsilon_j : j \in \mathbb{Z}\}$ and $\{\varepsilon'_j : j \in \mathbb{Z}\}$, each consisting of i.i.d. r.v.'s distributed like ε_0, we have

$$\mathrm{E}\Big[\big|\xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) - \xi(\ldots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k)\big|^{\psi}\Big] \leq C_\psi r_\psi^k \quad \text{for } k \geq 0. \qquad (1)$$

Let

$$Y_k' \equiv \begin{cases} \xi\left(\ldots, \varepsilon_{-1}', \varepsilon_0'\right) & \text{for } k = 0, \\ \xi\left(\ldots, \varepsilon_{-1}', \varepsilon_0', \varepsilon_1, \ldots, \varepsilon_k\right) & \text{for } k \geq 1 \end{cases} \tag{2}$$

denote the simulation response that is paired with Y_k for $k \geq 0$. For $k = 0$, we see that Y_k and Y_k' are independent r.v.'s because their respective input sequences $\left(\ldots, \varepsilon_{-1}, \varepsilon_0\right)$ and $\left(\ldots, \varepsilon_{-1}', \varepsilon_0'\right)$ are independent. However for $k \geq 1$, we see that Y_k and Y_k' are dependent (coupled) r.v.'s since they share the common i.i.d. inputs $\varepsilon_1, \ldots, \varepsilon_k$. Thus the GMC condition (1) can be simply expressed as $\mathrm{E}\left[\left|Y_k - Y_k'\right|^{\psi}\right] \leq C_\psi r_\psi^k$ for $k \geq 0$.

Dingeç et al. (2024a,b) give additional motivation regarding the GMC condition (1), noting that it applies to many processes related to the autoregressive–moving average (ARMA) family (cf. Shao and Wu 2007, Theorem 5.2) as well as the waiting-time process in a G/G/1 queueing system with a non-heavy-tailed service-time distribution (Dingeç et al. 2022). Dingeç et al. (2024a) then go on to detail how their alternative set of assumptions can be used to evaluate the point estimators $\overline{Y}_n$, $\overline{I}_n(y)$, and $\widetilde{y}_p(n)$ for the respective parameters μ, $F(y)$, and y_p, in particular, by deriving properties of the associated variance parameters σ_Y^2, $\sigma_{I(y)}^2$, and $\sigma_{\widetilde{y}_p}^2$ as well as certain related autocovariance functions. Dingeç et al. (2024b) also show that many processes in the ARMA family satisfy the GMC condition but do not satisfy certain mixing conditions, which is good evidence of the theoretical and practical significance of the GMC condition.

The current paper provides a number of examples to complement the theory-based companion paper Dingeç et al. (2024a), and it proceeds as follows. Section 2 summarizes most of the remaining notation and basic concepts used in this article, including a brief synopsis of the salient results from Dingeç et al. (2024a,b). Section 3 discusses sample-size considerations in comparing the effort required to estimate μ, $F(y)$, and y_p to a certain precision in the setting where $\{Y_k : k \geq 0\}$ is a dependent stationary process versus the setting where $\{Y_k : k \geq 0\}$ consists of i.i.d. r.v.'s having the same c.d.f. $F(y)$. Section 4 houses the main contributions of the paper, where we present the results of analytical calculations for the following stationary GMC processes: (a) the (Gaussian) first-order autoregressive (AR(1)) process; (b) the AR(1) process with Cauchy (heavy-tailed) innovations (instead of normal noise terms); (c) the Autoregressive-to-Pareto (ARTOP) process; and (d) the waiting-time process arising from an M/M/1 queueing system. For each of these processes, we calculate and discuss the following characteristics:

- the autocovariance functions $\left\{R_Y(\ell) \equiv \mathrm{Cov}\left[Y_1, Y_{1+|\ell|}\right] : \ell \in \mathbb{Z}\right\}$ and $\left\{R_{I(y)}(\ell) \equiv \mathrm{Cov}\left[I_1(y), I_{1+|\ell|}(y)\right] : \ell \in \mathbb{Z}\right\}$ respectively corresponding to the original process $\{Y_k : k \geq 0\}$ and the associated indicator process $\{I_k(y) : k \geq 0\}$;
- the variance parameters σ_Y^2, $\sigma_{I(y)}^2$, and $\sigma_{\widetilde{y}_p}^2$;
- the corresponding "information" metrics that we call the Bayley–Hammersley (BH) dependent sampling factors; and
- the autocovariance function for the sequence of nonoverlapping batch means with batch size m that is computed from the original simulation-generated output process $\{Y_k : k \geq 1\}$, thus providing a core result for applications of simulation output analysis.

All of our findings validate the theory from Dingeç et al. (2024a); and, to our knowledge, most are completely new. Section 5 summarizes our main findings and gives suggestions for future research.

2. Preliminaries

This section reprises several of the key findings of the companion papers Dingeç et al. (2024a,b), all the while establishing additional notation that will be needed as we move forward.

First of all, we present a slightly weakened composite version of Theorem 1 and its corollary from Dingeç et al. (2024a) (henceforth called Theorem C1 in this article) that: (a) gives sufficient conditions for the GMC condition to hold; (b) establishes that the covariance function $R_Y(\ell)$ decays exponentially as $\ell \to \infty$; and (c) proves that the variance parameter σ_Y^2 exists and can be written in the "classic" form as $\sum_{\ell \in \mathbb{Z}} R_Y(\ell)$, as well as a first-order formula relating σ_Y^2 and $n\,\mathrm{Var}\big[\overline{Y}_n\big]$.

THEOREM C1. *If (i) $\{Y_k : k \geq 0\}$ is stationary; (ii) $\mathrm{E}[|Y_0|^u] < \infty$ for some $u > 2$; and (iii) for some $\psi > 0$, there exist constants $C_\psi > 0$ and $r_\psi \in (0,1)$ such that $\mathrm{E}\big[\big|Y_k - Y_k'\big|^\psi\big] \leq C_\psi r_\psi^k$ for $k \geq 0$, where $\{Y_k' : k \geq 0\}$ is the associated process as defined in Equation (2), then*

- *(a) $\{Y_k : k \geq 0\}$ satisfies the GMC condition (1) for all $\psi \in (0,u)$;*
- *(b) $|R_Y(\ell)| = O(s^{|\ell|})$ as $|\ell| \to \infty$ for some $s \in (0,1)$, where the notation $a(\ell) = O(b(\ell))$ means that there exist $M \in \mathbb{R}^+$ and $\ell_0 \in \mathbb{Z}^+$ such that $|a(\ell)| \leq M b(\ell)$ for every integer $\ell > \ell_0$; and*
- *(c) The variance parameter σ_Y^2 is given by*

$$\sigma_Y^2 = \sum_{\ell \in \mathbb{Z}} R_Y(\ell) = n\,\mathrm{Var}\big[\overline{Y}_n\big] + O(1/n) \in [0,\infty) \text{ as } n \to \infty. \tag{3}$$

Moving to the quantile parameters, we state a slightly weakened version of Theorem 4 and its immediate corollary from Dingeç et al. (2024a) (henceforth called Theorem C2 herein), establishing the exponential decay of the covariance function $R_{I(y)}(\ell)$ for given y as $\ell \to \infty$, as well as the existence (and hence the boundedness) of the associated variance parameters.

THEOREM C2. *If $\{Y_k : k \geq 0\}$ satisfies the GMC condition (1) and the marginal c.d.f. $F(y)$ has a probability density function (p.d.f.) $f(y)$ with derivative $f'(y)$, where both $f(y)$ and $f'(y)$ are bounded for $y \in \mathbb{R}$, then*

- *(a) $R_{I(y)}(\ell) = O(s^{|\ell|})$ as $|\ell| \to \infty$ for some $s \in (0,1)$.*
- *(b) For each y, the indicator process $\big\{I_k(y) : k \geq 0\big\}$ has a finite variance parameter*

$$\sigma_{I(y)}^2 = \sum_{\ell \in \mathbb{Z}} R_{I(y)}(\ell) = n\,\mathrm{Var}\big[\overline{I}_n(y)\big] + O(1/n) \in [0,\infty) \text{ for every } y \in \mathbb{R} \text{ as } n \to \infty. \tag{4}$$

- *(c) If, in addition, for given $p \in (0,1)$, we have $f(y_p) > 0$, then the variance parameter for $\widetilde{y}_p(n)$ is*

$$\sigma_{\widetilde{y}_p}^2 = \frac{\sigma_{I(y_p)}^2}{f^2(y_p)} \in [0,\infty).$$

Last, we state an amalgamation of Remarks 4 and 6 from Dingeç et al. (2024a) (herein Remark C1) giving order results on the covariance and correlation between the sample means arising from nonoverlapping size-m batches of observations, $\{Y_{(d-1)m+1}, Y_{(d-1)m+2}, \ldots, Y_{dm}\}$, $d \geq 1$, a.k.a. the batch means, $\overline{Y}_{d,m} \equiv m^{-1}\sum_{i=1}^{m} Y_{(d-1)m+i}$, $d \geq 1$.

Remark C1. *Under the assumptions of Theorem C1, for $d_1 \neq d_2$,*

$$m^2 \operatorname{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right] = \sum_{j=-m+1}^{m-1} \left(m - |j|\right) R_Y\big(m|d_2 - d_1| + j\big) = O\left(s^{m(|d_2-d_1|-1)}\right) \tag{5}$$

and

$$m \operatorname{Corr}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right] = O\left(s^{m(|d_2-d_1|-1)}\right),$$

for some $s \in (0,1)$ as $m \to \infty$; and in particular,

$$m^2 \operatorname{Cov}\left[\overline{Y}_{1,m}, \overline{Y}_{2,m}\right] = \frac{c_Y}{2} + O(ms^m) \quad \text{and} \quad m \operatorname{Corr}\left[\overline{Y}_{1,m}, \overline{Y}_{2,m}\right] = \frac{c_Y}{2\sigma_Y^2} + O(1/m), \tag{6}$$

where $c_Y \equiv 2\sum_{\ell=1}^{\infty} \ell R_Y(\ell)$ is finite since $\mathrm{E}[|Y_0|^u] < \infty$ for some $u > 2$ and $|R_Y(\ell)| = O\left(s^{|\ell|}\right)$ as $|\ell| \to \infty$ (Dingeç et al. 2024a, Theorem 3). ◄

Table 1 summarizes the three distinct variance parameters under study here. [dg — 6/6/25 — (6) not mentioned elsewhere.]

Table 1 Various variance parameters

variance parameter	description	expressions
σ_Y^2	for underlying Y_k's	$\lim_{n\to\infty} n\operatorname{Var}\left[\overline{Y}_n\right] = \sum_{\ell\in\mathbb{Z}} R_Y(\ell)$
$\sigma_{I(y)}^2$	for underlying $I_k(y)$'s	$\lim_{n\to\infty} n\operatorname{Var}\left[\overline{I}_n(y)\right] = \sum_{\ell\in\mathbb{Z}} R_{I(y)}(\ell)$
$\sigma_{\widetilde{y}_p}^2$	for p-quantile estimator $\widetilde{y}_p(n)$	$\lim_{n\to\infty} n\operatorname{Var}\left[\widetilde{y}_p(n)\right] = \sigma_{I(y_p)}^2 / f^2(y_p)$

3. Impact of Stochastic Dependence on Required Dataset Size

This section is a high-level overview of the challenges arising in estimating a given characteristic (e.g., $\mu = \mathrm{E}[Y_1]$) of the marginal distribution of a dependent process $\{Y_k : k \geq 0\}$ that satisfies the assumptions of Theorems C1 and C2. In the context of the current article, these challenges are especially significant when $\{Y_k : k \geq 0\}$ is used as a benchmark (test) process in a large-scale simulation-based performance evaluation of user-selected estimators of the given characteristic. To quantify the impact of stochastic dependence in the benchmark process on the precision (variance) of a user-selected estimator $\left(\text{e.g., } \overline{Y}_n\right)$ of the given characteristic, we seek to identify an appropriate functional relationship $n = \nu(\mathfrak{n})$ between (i) the size n of a sample (time series) taken from the dependent process $\{Y_k : k \geq 0\}$ with marginal c.d.f. $F(\cdot)$; and (ii) the size $\mathfrak{n}$ of the "gold-standard" sample taken from the i.i.d. process $\{\mathfrak{X}_k : k \geq 0\}$ having the same marginal c.d.f. $F(\cdot)$ such that the variances of the respective estimators (e.g., $\mathrm{Var}\big[\overline{Y}_n\big]$ and $\mathrm{Var}\big[\overline{\mathfrak{X}}_\mathfrak{n}\big]$) computed from samples (i) and (ii) are *asymptotically equivalent* as $\mathfrak{n} \to \infty$ and $n = \nu(\mathfrak{n}) \to \infty$. Such asymptotic equality is denoted by the expression $\mathrm{Var}\big[\overline{Y}_{\nu(\mathfrak{n})}\big] \sim \mathrm{Var}\big[\overline{\mathfrak{X}}_\mathfrak{n}\big]$ as $\mathfrak{n} \to \infty$, meaning that $\lim_{\mathfrak{n}\to\infty} \mathrm{Var}\big[\overline{Y}_{\nu(\mathfrak{n})}\big] / \mathrm{Var}\big[\overline{\mathfrak{X}}_\mathfrak{n}\big] = 1$ (Scheinerman 2019, Def. 1, p. 9). As detailed in the rest of this section, the limiting ratio $\lim_{\mathfrak{n}\to\infty} \nu(\mathfrak{n})/\mathfrak{n}$ quantifies the long-run impact of dependence in the benchmark process $\{Y_k : k \geq 0\}$ compared to the corresponding i.i.d. process $\{\mathfrak{X}_k : k \geq 0\}$.

If the marginal distribution has high variance or fat tails, then such anomalous conditions can significantly impede precise estimation of the marginal mean $\mu = \mathrm{E}[Y_1]$, a probability $F(y) = \mathrm{E}[I_1(y)]$ for a given $y \in \mathbb{R}$, or a quantile $y_p = F^{-1}(p)$ for a given $p \in (0,1)$. To address the issue of estimator precision for these characteristics in the sequel, here we consider the variance parameters σ^2_Y, $\sigma^2_{I(y)}$, and $\sigma^2_{\widetilde{y}_p}$ in Table 1 that are respectively used to formulate the *asymptotic precision* as $n \to \infty$ of the statistics $\overline{Y}_n$, $\overline{I}_n(y)$, and $\widetilde{y}_p(n)$ computed from a dependent sequence $\{Y_1, \ldots, Y_n\}$. Our precision analysis uses the following key results:

$$\mathrm{Var}\big[\overline{Y}_n\big] = \frac{\sigma^2_Y}{n} + O\big(n^{-2}\big) \sim \frac{\sigma^2_Y}{n} \quad \text{as } n \to \infty, \tag{7}$$

$$\mathrm{Var}\big[\overline{I}_n(y)\big] = \frac{\sigma^2_{I(y)}}{n} + O\big(n^{-2}\big) \sim \frac{\sigma^2_{I(y)}}{n} \quad \text{as } n \to \infty, \text{ and} \tag{8}$$

$$\mathrm{Var}\big[\widetilde{y}_p(n)\big] = \frac{\sigma^2_{I(y_p)}}{nf^2(y_p)} + O\left[\frac{(\log n)^3}{n^{5/4}}\right] \sim \frac{\sigma^2_{I(y_p)}}{nf^2(y_p)} \quad \text{as } n \to \infty; \tag{9}$$

Equations (7) and (8) are immediate from Equations (3) and (4); and the middle term in Equation (9) is given in Dingeç et al. (2024c, §3.5). [← Jim, I suggest that we do NOT include the derivation of this result in the current paper because: (i) it would take 3 or 4 pages, and (ii) for the current paper, we really only need the RHS, not the cited result in the middle.]

To assess the impact of stochastic dependence in the simulation-generated sequence $\{Y_1, \ldots, Y_n\}$ of length n on the precision with which μ, $\mathrm{E}[I_1(y)]$, and y_p are estimated, first we examine the gold-standard situation in which the i.i.d. sequence $\big\{\mathfrak{X}_1, \ldots, \mathfrak{X}_\mathfrak{n}\big\}$ of length $\mathfrak{n}$ has the given marginal c.d.f. $F(\cdot)$ and p.d.f. $f(\cdot)$. For the latter i.i.d. sample with $y \in \mathbb{R}$, $1 \leq k \leq \mathfrak{n}$, and $p \in (0,1)$, we define the statistics

$$\mathcal{I}_k(y) \equiv \begin{cases} 1 & \text{if } \mathfrak{X}_k \leq y, \\ 0 & \text{otherwise,} \end{cases} \qquad \overline{\mathcal{I}}_\mathfrak{n}(y) \equiv \mathfrak{n}^{-1} \sum_{k=1}^{\mathfrak{n}} \mathcal{I}_k(y), \qquad \text{and} \qquad \widetilde{\mathfrak{x}}_p(\mathfrak{n}) \equiv \mathfrak{X}_{(\lceil \mathfrak{n}p \rceil)}.$$

Because the $\mathfrak{X}_i$'s are i.i.d., we have

$$\mathrm{Var}\big[\overline{\mathfrak{X}}_\mathfrak{n}\big] = \frac{\mathrm{Var}[\mathfrak{X}_1]}{\mathfrak{n}} = \frac{\mathrm{Var}[Y_1]}{\mathfrak{n}} = \frac{R_Y(0)}{\mathfrak{n}} \quad \text{for all } \mathfrak{n} \geq 1, \tag{10}$$

$$\mathrm{Var}\big[\overline{\mathcal{I}}_\mathfrak{n}(y)\big] = \frac{\mathrm{Var}[\mathcal{I}_1(y)]}{\mathfrak{n}} = \frac{\mathrm{Var}[I_1(y)]}{\mathfrak{n}} = \frac{R_{I(y)}(0)}{\mathfrak{n}} = \frac{F(y)[1-F(y)]}{\mathfrak{n}} \quad \text{for all } \mathfrak{n} \geq 1, \text{ and} \tag{11}$$

$$\mathrm{Var}\big[\widetilde{\mathfrak{x}}_p(\mathfrak{n})\big] \sim \frac{R_{I(y_p)}(0)}{\mathfrak{n}f^2(y_p)} = \frac{F(y_p)[1-F(y_p)]}{\mathfrak{n}f^2(y_p)} = \frac{p(1-p)}{\mathfrak{n}f^2(y_p)} \quad \text{as } \mathfrak{n} \to \infty; \tag{12}$$

cf. Equations (7), (8), and (9), respectively.

Next we compare the statistics $\overline{Y}_n$, $\overline{I}_n(y)$, and $\widetilde{y}_p(n)$ vs. the respective statistics $\overline{\mathfrak{X}}_{\mathfrak{n}}$, $\overline{\mathcal{I}}_{\mathfrak{n}}(y)$, and $\widetilde{\mathfrak{x}}_p(\mathfrak{n})$ in terms of their precision as estimators of the respective marginal process characteristics μ, $\mathrm{E}[I_1(y)]$, and y_p. To begin, we compare $\overline{Y}_n$ vs. $\overline{\mathfrak{X}}_{\mathfrak{n}}$ as (unbiased) estimators of μ. For a user-selected value of $\mathfrak{n}$, we regard $\mathrm{Var}\big[\overline{\mathfrak{X}}_{\mathfrak{n}}\big] = \mathrm{Var}[Y_1]/\mathfrak{n}$ in Equation (10) as the gold-standard level of precision for estimating μ. For a dependent sequence $\{Y_1, \ldots, Y_n\}$, we seek to specify the sequence length $n = \nu_Y(\mathfrak{n})$ required to achieve precision $\mathrm{Var}\big[\overline{\mathfrak{X}}_{\mathfrak{n}}\big]$ asymptotically as $\mathfrak{n} \to \infty$ and $\nu_Y(\mathfrak{n}) \to \infty$. So, $\nu_Y(\cdot)$ must satisfy the asymptotic relation

$$\mathrm{Var}\big[\overline{Y}_{\nu_Y(\mathfrak{n})}\big] \sim \mathrm{Var}\big[\overline{\mathfrak{X}}_{\mathfrak{n}}\big] \text{ as } \mathfrak{n} \to \infty \text{ and } \nu_Y(\mathfrak{n}) \to \infty, \tag{13}$$

which ensures that for practical purposes $\mathrm{Var}\big[\overline{Y}_{\nu_Y(\mathfrak{n})}\big] \approx \mathrm{Var}\big[\overline{\mathfrak{X}}_{\mathfrak{n}}\big]$, *provided that* $\mathfrak{n}$ *is sufficiently large*. Using Equations (7) and (10), we characterize $\nu_Y(\cdot)$ by the asymptotic relations

$$\frac{\sigma_Y^2}{\nu_Y(\mathfrak{n})} \sim \mathrm{Var}\big[\overline{Y}_{\nu_Y(\mathfrak{n})}\big] \sim \mathrm{Var}\big[\overline{\mathfrak{X}}_{\mathfrak{n}}\big] = \frac{\mathrm{Var}[Y_1]}{\mathfrak{n}} \text{ as } \mathfrak{n} \to \infty \text{ and } \nu_Y(\mathfrak{n}) \to \infty.$$

This implies that

$$\frac{\nu_Y(\mathfrak{n})}{\mathfrak{n}} \sim h_Y \equiv \frac{\sigma_Y^2}{\mathrm{Var}[Y_1]} = \frac{\sum_{\ell \in \mathbb{Z}} R_Y(\ell)}{R_Y(0)} = \sum_{\ell \in \mathbb{Z}} \rho_Y(\ell) \text{ as } \mathfrak{n} \to \infty \text{ and } \nu_Y(\mathfrak{n}) \to \infty, \tag{14}$$

where $\rho_Y(\ell) \equiv \mathrm{Corr}(Y_0, Y_\ell)$ for $\ell \in \mathbb{Z}$ is the autocorrelation function of the Y_k's at lag ℓ; and we call h_Y the *BH dependent sampling factor* (see Bayley and Hammersley 1946).

Similarly, we can derive an appropriate function $n = \nu_{I(y)}(\mathfrak{n})$ representing the length of a dependent sequence $\{Y_1, \ldots, Y_n\}$ that is required to achieve precision $\mathrm{Var}[\overline{\mathfrak{X}}_{\mathfrak{n}}] = \mathrm{Var}[I_1(y)]/\mathfrak{n}$ asymptotically as $\mathfrak{n} \to \infty$ and $\nu_{I(y)}(\mathfrak{n}) \to \infty$. Using Equations (8) and (11), we seek a function $\nu_{I(y)}(\cdot)$ such that

$$\frac{\sigma_{I(y)}^2}{\nu_{I(y)}(\mathfrak{n})} \sim \frac{\mathrm{Var}[I_1(y)]}{\mathfrak{n}} \text{ as } \mathfrak{n} \to \infty \text{ and } \nu_{I(y)}(\mathfrak{n}) \to \infty.$$

Thus, we obtain the following BH factor $h_{I(y)}$ for estimation of $\mathrm{E}[I_1(y)]$, this time via the relations

$$\frac{\nu_{I(y)}(\mathfrak{n})}{\mathfrak{n}} \sim h_{I(y)} \equiv \frac{\sigma_{I(y)}^2}{\mathrm{Var}[I_1(y)]} = \frac{\sum_{\ell \in \mathbb{Z}} R_{I(y)}(\ell)}{F(y)[1 - F(y)]} = \sum_{\ell \in \mathbb{Z}} \rho_{I(y)}(\ell) \text{ as } \mathfrak{n} \to \infty \text{ and } \nu_{I(y)}(\mathfrak{n}) \to \infty,$$

where $\rho_{I(y)}(\ell) \equiv \mathrm{Corr}\big(I_0(y), I_\ell(y)\big) = R_{I(y)}(\ell)/\big\{F(y)[1 - F(y)]\big\}$ for $\ell \in \mathbb{Z}$.

Finally, in the context of estimating the p-quantile y_p for $p \in (0, 1)$ using Equations (9) and (12), it is straightforward to derive the BH factor $h_{\widetilde{y}_p}$:

$$\frac{\nu_{\widetilde{y}_p}(\mathfrak{n})}{\mathfrak{n}} \sim h_{\widetilde{y}_p} \equiv \frac{\sigma_{\widetilde{y}_p}^2 f^2(y_p)}{p(1-p)} = \frac{\sigma_{I(y_p)}^2}{p(1-p)} = h_{I(y_p)} \text{ as } \mathfrak{n} \to \infty \text{ and } \nu_{\widetilde{y}_p}(\mathfrak{n}) \to \infty.$$

To summarize the relationships, we have

$$\sigma_Y^2 = h_Y R_Y(0), \quad \sigma_{I(y)}^2 = h_{I(y)} R_{I(y)}(0), \quad \text{and} \quad h_{\widetilde{y}_p} = h_{I(y_p)}.$$

Any such BH factor can be regarded informally as a multiplicative adjustment to $\mathfrak{n}$ that yields the length of a simulation-generated dependent sequence that is required to obtain the same amount of "information" about a selected marginal characteristic that would ideally be obtained from an i.i.d. sequence of length $\mathfrak{n}$, *provided that* $\mathfrak{n}$ *is sufficiently large*. The quantity h_Y, in particular, has appeared in the simulation folklore over the years; typically $h_Y > 1$ for processes $\{Y_k : k \geq 1\}$ exhibiting positive serial correlation. In any case, either a highly variable marginal distribution or a large BH factor could imply a "more-difficult" process.

4. Examples Featuring Variance Parameters and Related Quantities

For certain stationary processes, we can obtain analytical expressions for the autocovariance functions associated with the underlying observations and the indicator process, i.e., $R_Y(\ell)$ and $R_{I(y)}(\ell)$, respectively. These in turn yield expressions for the variance parameter σ_Y^2 associated with the underlying observations; the correlation $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$ between adjacent batch means; the variance parameter $\sigma_{I(y)}^2$ corresponding to the indicator process; and then (often after an easy additional calculation) the variance parameter $\sigma_{\widetilde{y}_p}^2$ for the associated p-quantile process. Some calculations are approximate (but still accurate), possibly involving truncation of an infinite sum of terms. We will detail such calculations for:

- (§4.1) The Gaussian AR(1) process.
- (§4.2) A generalized AR(1) process with non-normal innovations (noise terms). An example is an AR(1) process with heavy-tailed innovations such as those arising from a Cauchy distribution, where it turns out that the autocovariance function of the underlying observations, $\{R_Y(\ell) : \ell \in \mathbb{Z}\}$, does not even exist—yet that of the indicator process, $\{R_{I(y)}(\ell) : \ell \in \mathbb{Z}\}$, does!
- (§4.3) The ARTOP process for which the covariances $\{R_{I(y)}(\ell) : \ell \in \mathbb{Z}\}$ are the same as those of the underlying AR(1) process even though the marginal distribution is the Pareto. Thus, to calculate $\sigma_{\widetilde{y}_p}^2$ we use in the numerator the same $\sigma_{I(y_p)}^2$ as for the AR(1) process, whereas the denominator, $f(y_p)$, is calculated from the marginal distribution.
- (§4.4) The M/G/1 waiting-time process, for which we provide an analytical expression containing integrals for the infinite sum $\sum_{\ell=1}^{\infty} R_{I(y)}(\ell)$ (see Equations (54)–(56) below). In addition, for the special case of the M/M/1 system, we obtain a closed form for $\sigma_{\widetilde{y}_p}^2$ by simplification of the analytical expression for $\sum_{\ell=1}^{\infty} R_{I(y)}(\ell)$ and using the closed forms of the steady-state quantile y_p and $f(y_p)$.

Generally speaking, each of the three subsections §§4.1, 4.3, and 4.4 are themselves divided into (at least) three parts: the first involving preliminary analysis; the second focusing on the calculation of the non-quantile-related quantities σ_Y^2 and $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$; and the third dedicated to the quantile-related variance parameters $\sigma^2_{I(y)}$ and $\sigma^2_{\widetilde{y}_p}$. The extra §4.4.4 is devoted to elucidating examples on sample-size requirements for confidence intervals of the variance parameters under study for the special case of the M/M/1. §4.2 is organized a bit differently, where we instead divide the content into preliminaries; a discussion of σ^2_Y, $\sigma^2_{I(y)}$, and $\sigma^2_{\widetilde{y}_p}$ all in one pass; and finally an extended example involving Cauchy innovations. For reasons that will become apparent, §4.2 does not concern itself with batch means.

4.1. AR(1) Process

4.1.1. AR(1) Preliminaries. The purpose of this subsection is to define the basic notation and present results that will facilitate our subsequent work. Consider the (Gaussian) AR(1) process defined by

$$Y_k \;=\; \beta Y_{k-1} + \eta_k, \quad \text{for } k \ge 1, \tag{15}$$

where $-1 < \beta < 1$ and the η_k's are i.i.d. $\text{Nor}(0, 1-\beta^2)$ random variables that are independent of Y_0. It is well known that $Y_k \stackrel{\text{d}}{=} \text{Nor}(0,1)$, $k \ge 1$; and, further, it is easy to show that $Y_k = \beta^k Y_0 + W_k$, where

$$W_k \;\equiv\; \sum_{i=1}^{k} \beta^{k-i}\eta_i \;\stackrel{\text{d}}{=}\; \text{Nor}\left(0,\; (1-\beta^2)\sum_{i=1}^{k}\beta^{2(k-i)}\right) \;\stackrel{\text{d}}{=}\; \text{Nor}\big(0,\; 1-\beta^{2k}\big) \;\text{ for }\; k \ge 1,$$

and where Y_0 and W_k are independent (since they consist of different η_k's). It is also well known that $R_Y(\ell) = \text{Cov}[Y_0, Y_\ell] = \beta^{|\ell|}$. Hereupon, let $\phi(\cdot)$ and $\Phi(\cdot)$ denote the $\text{Nor}(0,1)$ p.d.f. and c.d.f. Our forthcoming derivations will use the "probabilist's" version of Hermite polynomials (Cramér 1946, §12.6, p. 133),

$$\text{He}_j(z) \;\equiv\; (-1)^j[\phi(z)]^{-1}\frac{\text{d}^j}{\text{d}z^j}\phi(z) \;=\; (-1)^j e^{z^2/2}\frac{\text{d}^j}{\text{d}z^j}e^{-z^2/2}, \quad \text{for } j \ge 0 \text{ and } z \in \mathbb{R}, \tag{16}$$

where $\frac{\text{d}^0}{\text{d}z^0}\phi(z) \equiv \phi(z)$ for all $z \in \mathbb{R}$. Thus the Hermite polynomials have the following orthogonality property:

$$\int_{\mathbb{R}} \text{He}_i(z)\text{He}_j(z)\phi(z)\,\text{d}z = \begin{cases} 0, & \text{if } i \ne j, \\ j!, & \text{if } i = j. \end{cases} \tag{17}$$

The next lemma is useful for obtaining certain results for the AR(1) and ARTOP processes.

Lemma 1. *If $\{Y_k : k \ge 0\}$ is an AR(1) process and $g(z)$ is a function such that $\text{E}[g^2(Y_0)] \in (0,\infty)$, then*

$$\text{Cov}[g(Y_0), g(Y_\ell)] \;=\; \sum_{j=1}^{\infty} a_j \beta^{j\ell} \;=\; O(|\beta|^\ell) \quad \textit{for } \ell \ge 0, \tag{18}$$

where

$$a_j \;\equiv\; \frac{1}{j!}\left(\int_{\mathbb{R}} g(z)\text{He}_j(z)\phi(z)\,\text{d}z\right)^2 \;\le\; \text{E}[g^2(Y_0)] \;<\; \infty \quad \textit{for } j \ge 1. \tag{19}$$

Proof: See Appendix S1 for a self-contained derivation.

Remark 1. Gebelein (1941), Lancaster (1957), Rozanov (1967), and Yu (2008) (among others) discuss related results (obtained via similar Hermite polynomial-based arguments) on the maximal correlation coefficient for functions of bivariate normal components. In fact, for instance, one can apply Rozanov (1967, Lemma 10.2) to an AR(1) process along with some algebra to obtain

$$|\text{Corr}[g(Y_0), g(Y_\ell)]| \;\le\; |\text{Corr}[Y_0, Y_\ell]| \;=\; O(|\beta|^\ell) \quad \text{for } \ell \ge 0. \quad ◀$$

Remark 2. Under the assumption that $\{Y_k : k \geq 1\}$ is a strongly mixing process, Gastwirth and Rubin (GR) (1975) analyze the asymptotic variance parameters of estimators such as the sample mean and median that are based on linear combinations of order statistics; see their Theorem 4.4 for an AR(1) process for the case $\beta \geq 0$. Moreover, Lemma 2.1 and Proposition 2.1 of GR (1975) adopt the same Hermite polynomial expansion approach that we use in our Lemma 1. In addition, the numerical results in our Table 2 below confirm a limiting variance ratio result found in GR (1975, Theorem 4.4). In particular, in our Table 2, the ratio of the variance parameters of the mean and median for $\beta = 0.995$ is $\sigma_Y^2/\sigma_{\widetilde{y}_{0.5}}^2 = 399/434.5 = 0.9183$, which is very close to the asymptotic ratio 0.9184 given in their Equation (4.9). ◄

4.1.2. Calculation of σ_Y^2 and the Correlation Between Batch Means for the AR(1) Process.

Some of the following results for the AR(1) process are known from the literature, but we state then here anyway to facilitate the subsequent exposition. As mentioned in §1, the AR(1) satisfies the GMC condition. The variance parameter is $\sigma_Y^2 = \sum_{\ell \in \mathbb{Z}}^{\infty} R_Y(\ell) = 1 + 2\sum_{\ell=1}^{\infty} \beta^\ell = (1+\beta)/(1-\beta)$; and since $\mathrm{Var}[Y_1] = 1$, we have $h_Y = \sigma_Y^2/\mathrm{Var}[Y_1] = \sigma_Y^2$ for the AR(1). By Fishman (1978, pp. 232–233) and Aktaran-Kalaycı et al. (2007, Lemma 2), the variance of the sample mean can be written as

$$\mathrm{Var}\big[\,\overline{Y}_n\big] = \frac{1}{n}\left[R_Y(0) + 2\sum_{\ell=1}^{n-1}\left(1 - \frac{\ell}{n}\right)R_Y(\ell)\right] \tag{20}$$

$$= \frac{1}{n}\left[1 + 2\sum_{\ell=1}^{n-1}\left(1 - \frac{\ell}{n}\right)\beta^\ell\right]$$

$$= \frac{(1-\beta^2)n - 2\beta(1-\beta^n)}{(1-\beta)^2 n^2} = \frac{\sigma_Y^2}{n} - \frac{2\beta(1-\beta^n)}{(1-\beta)^2 n^2}, \tag{21}$$

confirming Theorem C1(c). On the other hand, by Equation (5), the covariance of batch means $d_1 \neq d_2$ is

$$\mathrm{Cov}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big] = \frac{1}{m}\sum_{q=-m+1}^{m-1}\left(1 - \frac{|q|}{m}\right)\beta^{m|d_1-d_2|+q} = \frac{\beta^{m(|d_1-d_2|-1)+1}(1-\beta^m)^2}{(1-\beta)^2 m^2}. \tag{22}$$

So, by Equations (21) and (22), the correlation of batch means $d_1 \neq d_2$ is

$$\mathrm{Corr}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big] = \frac{\beta^{m(|d_1-d_2|-1)+1}(1-\beta^m)^2}{(1-\beta^2)m - 2\beta(1-\beta^m)} \sim \frac{\beta^{m(|d_1-d_2|-1)+1}}{(1-\beta^2)m}, \quad \text{as } m \to \infty. \tag{23}$$

Figure 1 depicts the correlation of two adjacent batch means, $\mathrm{Corr}[\overline{Y}_{1,m}, \overline{Y}_{2,m}]$, as a decreasing function of the batch size m for each of various values of β.

4.1.3. Calculation of $\sigma_{I(y)}^2$ and $\sigma_{\widetilde{y}_p}^2$ for the AR(1) Process.

We begin by calculating $\sigma_{I(y)}^2$ for the AR(1) process. By Lemma 1 with $g(z) \equiv \mathbf{1}_{\{z \leq y\}}$, we have

$$R_{I(y)}(\ell) = \mathrm{Cov}[\mathbf{1}_{\{Y_0 \leq y\}}, \mathbf{1}_{\{Y_\ell \leq y\}}] = \sum_{j=1}^{\infty} a_j \beta^{j\ell} \quad \text{for all } \ell \geq 1 \text{ and } y \in \mathbb{R}, \tag{24}$$

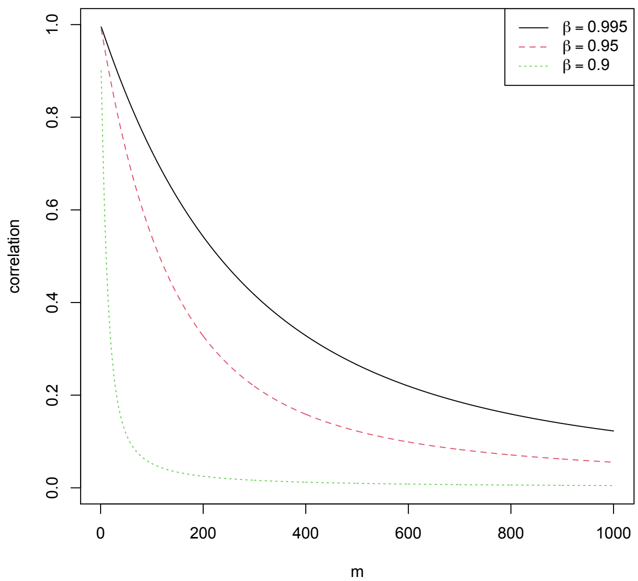

Figure 1 **Correlation $\mathrm{Corr}[\overline{Y}_{1,m}, \overline{Y}_{2.m}]$ of adjacent batch means of size $m = 1, 2, \ldots, 1000$, for an AR(1) process. [← Jim, does this read OK?]**

where the definition of the indicator function $g(z)$ and the first relation in Equation (19) imply that

$$\begin{aligned} a_j &= \frac{1}{j!}\left(\int_{-\infty}^{y} \mathrm{He}_j(z)\phi(z)\,\mathrm{d}z\right)^2 \\ &= \frac{1}{j!}\left((-1)^j \int_{-\infty}^{y} \frac{\mathrm{d}^j}{\mathrm{d}z^j}\phi(z)\,\mathrm{d}z\right)^2 \quad \text{(by Equation (16))} \\ &= \frac{1}{j!}\left(\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\phi(y)\right)^2 \quad \text{(by the “swap” discussion in Appendix §S2.1)} \qquad (25) \\ &= \frac{1}{j!}\phi^2(y)\mathrm{He}_{j-1}^2(y) \quad \text{(again by Equation (16)),} \qquad (26) \end{aligned}$$

which matches Gastwirth and Rubin (1975, Equation (2.11)).

Equations (24) and (26) imply that the autocovariance at lag ℓ for the indicator process $\{I_k(y) : k \geq 0\}$ is

$$R_{I(y)}(\ell) = \phi^2(y) \sum_{j=1}^{\infty} \frac{1}{j!}\mathrm{He}_{j-1}^2(y)\beta^{j\ell} \quad \text{for all } \ell \geq 1 \text{ and } y \in \mathbb{R}.$$

By Equations (16) and (26), we have $a_1 = \phi^2(y)\mathrm{He}_0^2(y) = \phi^2(y)$; and by Equation (24), it follows that

$$R_{I(y)}(\ell) = \phi^2(y)\beta^\ell + O(\beta^{2\ell}) \quad \text{for all } \ell \geq 1 \text{ and } y \in \mathbb{R},$$

in agreement with Theorem C2(a). [← Had (b) there before. Check?]

Theorem C2(b) [← Had (a) there before. Check?] establishes that the variance parameter $\sigma^2_{I(y)}$ is finite; and in Appendix S2.2 we show that it can be expressed as an absolutely convergent double series,

$$\sigma^2_{I(y)} = \Phi(y)[1-\Phi(y)] + 2\sum_{\ell=1}^{\infty}\sum_{j=1}^{\infty} a_j \beta^{j\ell} \in [0,\infty) \text{ for all } y \in \mathbb{R}. \tag{27}$$

Remark 3. Appendix S3 distills several other ways to calculate $R_{I(y)}(\ell)$ and then $\sigma^2_{I(y)}$. The motivations behind the alternative formulations are: (i) they provide checks when we undertake numerical calculations later on; (ii) they give rise to additional expressions that are used elsewhere in the paper; and (iii) they are interesting as stand-alone results. In particular, in Appendix S3.1, we give an expression for $R_{I(y)}(\ell)$ (Equation (S.30)) that is based on a simple conditioning argument. Appendix S3.2 provides another expression for $R_{I(y)}(\ell)$ (Equation (S.41)) that uses Owen's T function—which is (for the most part) what we implement in our upcoming numerical calculations. Appendix S3.3 applies a result from Drezner and Wesolowsky (1990), yielding yet another incarnation of $R_{I(y)}(\ell)$ (Equation (S.43)), and then two additional expressions for $\sigma^2_{I(y)}$ (Equations (S.44) and (S.45)); these latter results can be used as checks on previous work. ◄

Implementation Considerations: We can use standard quadrature methods to numerically calculate $R_{I(y)}(\ell)$ for $\ell \geq 1$ via Equations (24), (S.30), (S.41), or (S.43); and then utilize the latter approximate values to obtain a suitably truncated version of $\sigma^2_{I(y)} = R_{I(y)}(0) + 2\sum_{\ell=1}^{\infty} R_{I(y)}(\ell)$. All of our subsequent calculations for Figure 2 and Table 2 in this subsection are based on Equation (S.41); but all have been checked via Equations (24) and (S.30). In any case, as alluded to above, the implementations of all of these methods rely on obtaining good truncation points for their respective infinite sums. That process is described in great detail in Appendix S4 for Equation (S.30) (but can also be applied to the other methods). We remark that Wolff et al. (1967, Eqns. (12)–(14)) analyze $\sigma^2_{I(y)}$ for the special case $y = 0$ for the AR(1) process; we have checked via Mathematica (Wolfram Research, Inc. 2022) that the coefficients of their Equation (14) match those from the expression for $\sigma^2_{I(0)}$ given by our Equation (S.46) in Appendix S3.3.

Figure 2 depicts the behavior of the autocovariance and autocorrelation functions, $R_{I(y_p)}(\ell)$ and $\rho_{I(y_p)}(\ell) = R_{I(y_p)}(\ell)/[p(1-p)]$, of the AR(1)'s indicator process for coefficient $\beta = 0.9$ (so that $\sigma^2_Y = 19$) and $p = 0.5$, 0.95, and 0.99. Table 2 provides the quantiles $y_p = \Phi^{-1}(p)$ along with the associated variance parameters and BH ratios $\sigma^2_{I(y_p)}$, $h_{I(y_p)} = \sigma^2_{I(y_p)}/[p(1-p)]$, and $\sigma^2_{\tilde{y}_p}$, for $\beta = 0.9$ and 0.995 (for which $\sigma^2_Y = h_Y = 19$ and 399, respectively), and several values of p. Many of the choices for p are "extreme" (close to 1) and are used to illustrate the asymptotic behavior as $p \to 1-$ meaning that we take the left-hand limit of the relevant function of p $\left(\text{such as } \sigma^2_{I(y_p)}\right)$ through values of p less than 1 in the interval $(0,1]$; see Apostol (1974, p. 93), Bartle and Sherbert (2011, p. 117, Definition 4.3.1), or Rudin (1976, p. 94, Definition 4.25). The figure and table yield some interesting results that we have organized into several synergistic remarks.

Remark 4. First of all, Figure 2 illustrates the anticipated result that for fixed p, the autocovariances and autocorrelations decrease (presumably geometrically) as the lag ℓ increases. ◄

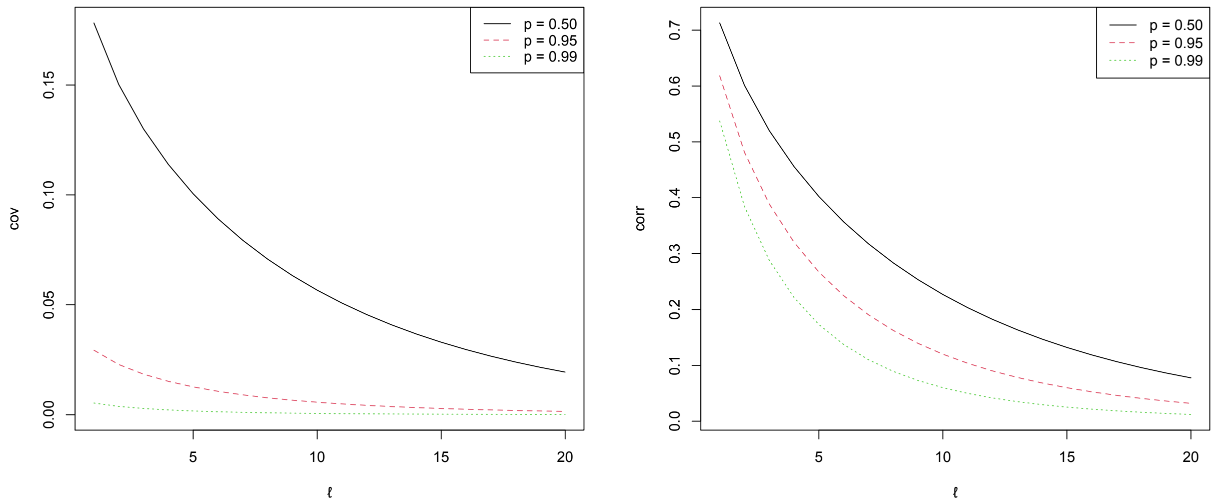

Figure 2 **Autocovariances (left) $R_{I(y_p)}(\ell)$, $\ell = 1, 2, \ldots, 20$, and autocorrelations (right) $\rho_{I(y_p)}(\ell)$, $\ell = 1, 2, \ldots, 20$, of the indicator process for an AR(1) process with $\beta = 0.9$ and various values of p.**

Table 2 **Variance parameters for the quantiles of an AR(1) process with $\beta = 0.9$ and 0.995.**

		$\beta = 0.9$ and $\sigma_Y^2 = h_Y = 19$			$\beta = 0.995$ and $\sigma_Y^2 = h_Y = 399$		
p	y_p	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$
0.3	−0.524	2.664	12.68	22.0	55.441	264.0	458.6
0.5	0	3.320	13.28	20.9	69.148	276.6	434.5
0.7	0.524	2.664	12.68	22.0	55.441	264.0	458.6
0.9	1.282	0.912	10.13	29.6	18.891	209.9	613.3
0.95	1.645	0.407	8.57	38.3	8.393	176.7	789.0
0.99	2.326	0.058	5.86	81.6	1.177	118.9	1,657
0.995	2.576	0.025	5.03	120.3	0.506	101.7	2,420
0.999	3.090	0.00378	3.79	333.7	0.07427	74.3	6,551
0.9999	3.719	0.000280	2.80	1,786	0.005268	52.7	33,617
0.99999	4.265	2.26e−05	2.26	11,286	0.0004066	40.7	202,687
0.999999	4.753	1.94e−06	1.94	79,116	3.31e−05	33.1	1,351,953

REMARK 5. As might be inferred from Figure 2, we claim that for $\beta > 0$ and fixed $\ell \geq 1$, these curves decrease as $p \geq 0.5$ increases. To prove this, note that, in general, for $p \in (0,1)$, $\beta \in (-1,1)$, and $\ell \geq 1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}p} R_{I(y_p)}(\ell) &= \frac{\mathrm{d}}{\mathrm{d}p}\left[\frac{1}{2\pi}\int_0^{\beta^\ell} \frac{1}{\sqrt{1-r^2}} \exp\left(\frac{-y_p^2}{1+r}\right) \mathrm{d}r\right] \quad \text{(by Equation (S.43))} \\
&= \frac{1}{2\pi}\int_0^{\beta^\ell} \frac{1}{\sqrt{1-r^2}} \frac{\mathrm{d}}{\mathrm{d}p}\exp\left(\frac{-y_p^2}{1+r}\right) \mathrm{d}r \quad \text{(by the swap discussion in Appendix S2.3)} \qquad (28) \\
&= -\frac{1}{\pi}\int_0^{\beta^\ell} \frac{y_p}{\sqrt{1-r^2}\,(1+r)} \exp\left(\frac{-y_p^2}{1+r}\right) \frac{\mathrm{d}y_p}{\mathrm{d}p} \, \mathrm{d}r
\end{aligned}$$

$$= -\frac{1}{\pi}\int_0^{\beta^\ell} \frac{y_p}{\phi(y_p)\sqrt{1-r^2}\,(1+r)} \exp\left(\frac{-y_p^2}{1+r}\right) \mathrm{d}r, \tag{29}$$

where the last step follows after we use the usual derivative of an inverse function to obtain

$$\frac{\mathrm{d}y_p}{\mathrm{d}p} = \frac{\mathrm{d}\Phi^{-1}(p)}{\mathrm{d}p} = \frac{1}{\phi(\Phi^{-1}(p))} = \frac{1}{\phi(y_p)}. \tag{30}$$

When $\beta > 0$ and $p > 0.5$, Equation (29) clearly establishes that $\frac{\mathrm{d}}{\mathrm{d}p}R_{I(y_p)}(\ell) < 0$, as claimed. ◀

Remark 6. Next we apply Theorem C2(b) to derive the key property,

$$\frac{\mathrm{d}}{\mathrm{d}p}\sigma^2_{I(y_p)} = \frac{\mathrm{d}}{\mathrm{d}p}\left[R_{I(y_p)}(0) + 2\sum_{\ell\ge 1} R_{I(y_p)}(\ell)\right] = \frac{\mathrm{d}}{\mathrm{d}p}R_{I(y_p)}(0) + 2\sum_{\ell\ge 1}\frac{\mathrm{d}}{\mathrm{d}p}R_{I(y_p)}(\ell) \;\text{ for all }\; p \in (0,1), \tag{31}$$

where Appendix S2.4 formally justifies Equation (31)'s derivative/infinite summation swap. In any case, Equation (31) and Remark 5 imply that, for fixed $\beta \in (0,1)$, the quantity $\sigma^2_{I(y_p)}$ decreases as $p \ge 0.5$ increases—a modestly surprising finding that is substantiated by Table 2. ◀

Remark 7. In fact, for $0 \le \beta < 1$, we have $\sigma^2_{I(y_p)} \sim p(1-p)$, as $p \to 0+$ or $p \to 1-$. where in the former case we take the right-hand limit of the function $\sigma^2_{I(y_p)}/[p(1-p)]$ through values of p greater than 0 in the interval $[0,1)$. where $g(p) \sim h(p)$ indicates that $g(p)/h(p) \to 1-$ in the applicable limit. See Appendix S5 for a proof of this result. ◀

Remark 8. The next potpourri of results from Table 2 is perhaps obvious for the AR(1) model: (i) All performance measures are symmetric around $p = 0.5$. (ii) As β increases from 0.9 to 0.995, the estimation of y_p becomes "harder," and all of the measures increase commensurately by factors a bit above 20. (iii) The quantity $\sigma^2_{\tilde{y}_p} = \sigma^2_{I(y_p)}/f^2(y_p) = 2\pi\exp(y_p^2)\sigma^2_{I(y_p)}$ increases as $p \ge 0.5$ increases, primarily because the quantiles y_p increase rapidly. Appendix S6 proves that $\frac{\mathrm{d}}{\mathrm{d}p}\sigma^2_{\tilde{y}_p} > 0$, at least for large-enough p. ◀

Remark 9. Table 2's results (along with additional numerical evidence not presented here) suggest that, for the AR(1), $h_{I(y_p)} = \sigma^2_{I(y_p)}/[p(1-p)]$ is a decreasing function of $p \ge 0.5$. (We have been unable to formally prove that $\frac{\mathrm{d}}{\mathrm{d}p}h_{I(y_p)} < 0$.) Nevertheless, Remark 7 implies that $\lim_{p\to 1-} h_{I(y_p)} = 1$, indicating that for $p \to 1-$ and sufficiently large n, the BH information about y_p from the dependent sequence $\{Y_1, \ldots, Y_n\}$ is about the same as what would be obtained from an i.i.d. sequence of length n. ◀

4.2. AR(1) Process with General Innovations

4.2.1. ARG(1) Preliminaries. Consider an AR(1) process $\{Y_k : k \ge 0\}$ as in Equation (15), but now with i.i.d. innovations $\{\eta_k : k \ge 1\}$ from an *arbitrary* zero-mean distribution constructed in such a way so that $\{Y_k : k \ge 0\}$ is stationary, with Y_0 initialized in the stationary distribution independently of the η_k's. We refer to this AR(1) generalization by the nomenclature ARG(1). In Dingeç et al. (2024b, §4.2), we show that if there exists a $\psi > 0$ such that $\mathrm{E}[|Y_0|^\psi] < \infty$, then $\{Y_k : k \ge 0\}$ satisfies the GMC condition (1) with $C_\psi = \mathrm{E}[|Y_0 - Y_0'|^\psi] < \infty$ and $r_\psi = |\beta|^\psi$.

4.2.2. Calculation of σ_Y^2, $\sigma_{I(y)}^2$, and $\sigma_{\widetilde{y}_p}^2$ for the ARG(1) Process. As in the normal innovations case in §4.1.2, we have $Y_k = \beta^k Y_0 + W_k$; the autocovariance function of $\{Y_k : k \geq 0\}$ is $R_Y(\ell) = \beta^{|\ell|}\mathrm{Var}[Y_0]$; the autocorrelation function is $\rho_Y(\ell) = \beta^{|\ell|}$; and $\sigma_Y^2 = h_Y \mathrm{Var}(Y_0)$ for $h_Y = (1+\beta)/(1-\beta)$. The autocorrelations and h_Y are independent of the distribution of the innovations. On the other hand, the autocovariance function of the indicator process depends on the joint distribution of the underlying Y_k's, one representation being

$$\begin{aligned}
R_{I(y)}(\ell) &= \mathrm{Cov}\big[\mathbf{1}_{\{Y_0 \leq y\}}, \mathbf{1}_{\{Y_\ell \leq y\}}\big] \\
&= \mathrm{Cov}\big[\mathbf{1}_{\{Y_0 > y\}}, \mathbf{1}_{\{Y_\ell > y\}}\big] \\
&= \Pr(Y_0 > y, Y_\ell > y) - [\Pr(Y_0 > y)]^2 \quad (\text{since } \mathrm{E}[\mathbf{1}_{\mathscr{E}}] = \Pr(\mathscr{E}) \text{ for any event } \mathscr{E}) \\
&= \Pr(Y_0 > y)\big[\Pr(Y_\ell > y \,|\, Y_0 > y) - \Pr(Y_0 > y)\big]. \qquad (32)
\end{aligned}$$

Thus, if one can calculate $R_{I(y)}(\ell)$ for each ℓ exactly or numerically, then $\sigma_{I(y)}^2$ and $\sigma_{\widetilde{y}_p}^2$ will be just around the corner via the usual infinite sums. We perform such calculations for Cauchy innovations in §4.2.3. For now, we use Equation (32) to study the autocorrelation $\rho_{I(y_p)}(\ell) = R_{I(y_p)}(\ell)/R_{I(y_p)}(0)$ as $p \to 1-$, i.e., For the p-quantile y_p, the autocovariance and autocorrelation functions become

$$R_{I(y_p)}(\ell) = (1-p)\big[\Pr(Y_\ell > y_p \,|\, Y_0 > y_p) - (1-p)\big]$$

and

$$\lim_{p\to 1-} \rho_{I(y_p)}(\ell) = \lim_{p\to 1-} \frac{(1-p)\big[\Pr(Y_\ell > y_p \,|\, Y_0 > y_p) - (1-p)\big]}{p(1-p)} = \lim_{p\to 1-} \Pr(Y_\ell > y_p \,|\, Y_0 > y_p),$$

Therefore, the limiting autocorrelation is given by

$$\lim_{p\to 1-} \rho_{I(y_p)}(\ell) = \lim_{p\to 1-} \Pr(Y_\ell > y_p \,|\, Y_0 > y_p),$$

which corresponds to the *coefficient of upper-tail dependence* of two r.v.'s Y_0 and Y_ℓ (see McNeil et al. 2005, p. 209). In Davis and Mikosch (2009, §2.5), it is shown that if the η_k innovations follow a symmetric and regularly varying distribution with tail index $\alpha > 0$, then the limiting autocorrelation function of the indicators (a.k.a. the *extremogram*) of the ARG(1) process is

$$\lim_{p\to 1-} \rho_{I(y_p)}(\ell) = \beta^{\alpha\ell}, \quad \text{for } \ell \geq 1. \qquad (33)$$

A regularly varying distribution is a heavy-tailed distribution with a right-tail probability given by $\overline{F}(y) = 1 - F(y) = y^{-\alpha}L(y)$ for a slowly varying function $L(y)$ (see McNeil et al. 2005, p. 268). The index α measures the heaviness of the tail, with the tail being heavier for a smaller α. For example, it can be verified that the Student's t-distribution with ν degrees of freedom is regularly varying with a tail index equal to ν. Proposition 1 uses Equation (33) to derive a result on the limiting sum of the autocorrelation function.

Proposition 1. *Let $\{Y_k : k \ge 0\}$ denote an ARG(1) process whose innovations $\{\eta_k : k \ge 0\}$ have a symmetric and regularly varying distribution with tail index $\alpha > 0$. If the series of functions $\sum_{\ell=1}^{\infty} \rho_{I(y_p)}(\ell)$ converges uniformly on $\mathbb{S} \equiv (0,1)$ and the function $\rho_{I(y_p)}(\ell)$ is left continuous at $p = 1$ for every $\ell \ge 1$, then*

$$\lim_{p \to 1-} \left[\sum_{\ell=1}^{\infty} \rho_{I(y_p)}(\ell) \right] = \sum_{\ell=1}^{\infty} \left[\lim_{p \to 1-} \rho_{I(y_p)}(\ell) \right] = \frac{\beta^\alpha}{1-\beta^\alpha}. \tag{34}$$

Proof: Let $\Psi(p) \equiv \sum_{\ell=1}^{\infty} \rho_{I(y_p)}(\ell)$ for $p \in \mathbb{S}$ denote the function implicitly defined by the condition that $\sum_{\ell=1}^{\infty} \rho_{I(y_p)}(\ell)$ converges uniformly on $\mathbb{S}$. The analysis given by Davis and Mikosch (2009, §2.5) to justify Equation (33) also implies that $\rho_{I(y_p)}(\ell)$ is left continuous at $p = 1$ for each $\ell \ge 1$; hence Apostol (1974, Thm. 9.7, p. 223) ensures that $\Psi(p)$ is also left continuous at $p = 1$. Moreover, Equation (33) ensures that $p = 1$ must be an accumulation point of $\mathbb{S}$. Thus Equation (34) holds by Apostol (1974, note on p. 224). □

Under the assumptions of Proposition 1, it immediately follows that the limit of the BH ratio as $p \to 1-$ is

$$\lim_{p \to 1-} h_{I(y_p)} = \lim_{p \to 1-} \left[1 + 2 \sum_{\ell=1}^{\infty} \rho_{I(y_p)}(\ell) \right] = 1 + 2 \sum_{\ell=1}^{\infty} \beta^{\alpha\ell} = \frac{1+\beta^\alpha}{1-\beta^\alpha}. \tag{35}$$

These results show that the tail behavior of the innovations has some effect on the autocorrelations of the indicator process and the BH ratio for extreme quantiles. In fact, for innovations with heavier tails, i.e., smaller α, the limiting autocorrelations and $\lim_{p \to 1-} h_{I(y_p)}$ are larger.

4.2.3. Cauchy Innovations. In order to illustrate our work with non-normal innovations, consider the well-known Cauchy distribution having p.d.f. and c.d.f.

$$f_{\mathrm{C}}(y;\gamma) \equiv \frac{\gamma}{\pi(\gamma^2 + y^2)} \quad \text{and} \quad F_{\mathrm{C}}(y;\gamma) \equiv \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{y}{\gamma}\right), \quad \text{for } y \in \mathbb{R},$$

where $\gamma > 0$ is the scale parameter. We denote by ARC(1) the ARG(1) process having Cauchy innovations with $\gamma = 1 - \beta$. Then the marginal distribution of the ARC(1) is Cauchy with $\gamma = 1$ (see e.g., Balakrishna 2021, §6.2.3), so that the steady-state quantile is given by $y_p = F_{\mathrm{C}}^{-1}(p;1) = \tan\left[\pi\left(p - \frac{1}{2}\right)\right]$ for $p \in (0,1)$.

Because the marginal Cauchy distribution has an infinite second moment, it is clear that the autocovariance function $R_Y(\ell)$ does not exist—hence, for the ARC(1), it is meaningless to discuss the resulting variance parameter σ_Y^2 or the covariance structure of the batch means. But the good news is that the ARC(1) satisfies the GMC condition for $\psi \in (0,1)$, since $\mathrm{E}[|Y_0|^\psi] = \gamma^\psi \sec(\pi\psi/2) < \infty$ for $\psi \in (0,1)$. Moreover, since the marginal p.d.f. is bounded, i.e., $\sup_{y \in \mathbb{R}} f_{\mathrm{C}}(y;1) = f_{\mathrm{C}}(0;1) = 1/\pi < \infty$, Theorem C2(a)–(b) implies that $R_{I(y)}(\ell) = O(s^\ell)$ as $\ell \to \infty$ for some $s \in (0,1)$ and $\sigma_{I(y)}^2 \in [0,\infty)$. Since the Cauchy distribution is regularly varying with $\alpha = 1$, we also obtain from Equations (33) and (35) that $R_{I(y_p)}(\ell) \sim p(1-p)\beta^\ell$ and $\sigma_{I(y_p)}^2 = F_{\mathrm{C}}(y_p;1)(1 - F_{\mathrm{C}}(y_p;1)) h_{I(y_p)} \sim p(1-p)(1+\beta)/(1-\beta)$ as $p \to 1-$.

To numerically calculate $R_{I(y)}(\ell)$, note that $W_k = \sum_{i=1}^k \beta^{k-i}\eta_i$ follows a Cauchy distribution with parameter $\gamma = (1-\beta)\sum_{i=1}^k \beta^{k-i} = 1-\beta^k$ due to the fact that the Cauchy distribution is closed under linear transformation and convolution. Therefore, the joint c.d.f. of Y_0 and Y_ℓ is

$$\begin{aligned}\Pr\left(Y_0 \le y, Y_\ell \le y\right) &= \Pr\left(Y_0 \le y, \beta^\ell Y_0 + W_\ell \le y\right)\\ &= \int_{\mathbb{R}} \Pr\left(Y_0 \le y, \beta^\ell Y_0 + W_\ell \le y \,|\, Y_0 = z\right) f_{\mathrm{C}}(z;1)\,\mathrm{d}z\\ &= \int_{-\infty}^{y} \Pr(W_\ell \le y - \beta^\ell z) f_{\mathrm{C}}(z;1)\,\mathrm{d}z \quad \text{(by independence of the } W_\ell\text{'s and } Y_0)\\ &= \int_{-\infty}^{y} F_{\mathrm{C}}(y-\beta^\ell z; 1-\beta^\ell) f_{\mathrm{C}}(z;1)\,\mathrm{d}z\\ &= \frac{1}{\pi}\int_{-\infty}^{y}\left[\frac{1}{2} + \frac{1}{\pi}\arctan\left(\frac{y-\beta^\ell z}{1-\beta^\ell}\right)\right]\left(\frac{1}{1+z^2}\right)\mathrm{d}z. \end{aligned} \tag{36}$$

The autocovariance function of the indicator process, $R_{I(y)}(\ell) = \Pr(Y_0 \le y, Y_\ell \le y) - [F_{\mathrm{C}}(y;1)]^2$, can be calculated via the marginal and joint c.d.f.'s. Specifically, for the p-quantile, we have $R_{I(y_p)}(\ell) = \Pr(Y_0 \le y_p, Y_\ell \le y_p) - p^2$, which then allows us to calculate the variance parameter $\sigma^2_{I(y_p)}$. Since the derivative of the marginal p.d.f. is bounded, i.e., $\sup_{y\in\mathbb{R}} |f'_{\mathrm{C}}(y;1)| = \sup_{y\in\mathbb{R}}[2y/(1+y^2)^2] = 3\sqrt{3}/8$, and $f_{\mathrm{C}}(y_p;1) > 0$ for all $p \in (0,1)$, Theorem C2(c) implies that the variance parameter for the quantile process is well-defined, i.e., $\sigma^2_{\tilde{y}_p} = \sigma^2_{I(y_p)}/f^2_{\mathrm{C}}(y_p;1) < \infty$ for $p \in (0,1)$. So the ARC(1) has well-defined expressions for $\sigma^2_{I(y_p)}$ and $\sigma^2_{\tilde{y}_p}$, yet not for σ^2_Y.

Table 3 reports numerical values for y_p, $\sigma^2_{I(y_p)}$, $h_{I(y_p)}$, and $\sigma^2_{\tilde{y}_p}$. In order to carry out the calculation of $\sigma^2_{I(y_p)}$, we numerically evaluated the integral in Equation (36) using R's `integrate` function (R Core Team 2022), with the infinite sum being truncated at 1,000 terms for $\beta = 0.9$ and 10,000 terms for $\beta = 0.995$. Table 3's entries are larger than the analogous values for the original AR(1) process depicted in Table 2—sometimes by several orders of magnitude—no doubt due to the comparatively high variability of the innovations for the ARC(1). Of note is that the ARC(1)'s $h_{I(y_p)}$ values are converging to $(1+\beta)/(1-\beta)$ as $p \to 1-$, in line with Equation (35), while the AR(1)'s $h_{I(y_p)}$ values from Table 2 are converging to 1 as $p \to 1-$, in line with Remark 7.

Table 3 **Variance parameters for the quantiles of an ARC(1) process with Cauchy innovations and $\beta = 0.9$ and 0.995.**

		$\beta = 0.9$			$\beta = 0.995$		
p	y_p	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$
0.3	−0.727	4.396	20.93	101.3	92.255	439.31	2,126
0.5	0	5.322	21.29	52.5	111.663	446.65	1,102
0.7	0.727	4.396	20.93	101.3	92.255	439.31	2,126
0.9	3.078	1.786	19.85	1,933	37.499	416.65	40,587
0.95	6.314	0.924	19.45	15,226	19.399	408.39	319,698
0.99	31.821	0.189	19.09	1,916,591	3.970	400.97	40,247,068
0.995	63.657	0.094	19.05	15,364,671	1.990	399.99	322,652,665

4.3. ARTOP Process

4.3.1. ARTOP Preliminaries. The c.d.f. of the Pareto distribution with location parameter $\gamma > 0$ and shape parameter $\theta > 0$ is $F(y) = 1 - (\gamma/y)^\theta$ for $y \geq \gamma$. The ARTOP process $\{Y_k : k \geq 1\}$ is defined by

$$Y_k = F^{-1}[\Phi(Z_k)] = \frac{\gamma}{[\Phi(-Z_k)]^{1/\theta}}, \quad \text{for } k \geq 1,$$

where $\{Z_k : k \geq 1\}$ is the AR(1) process defined in §4.1 with the symbol "Z" in place of "Y." It is well known that the marginal mean and variance of the ARTOP process are $\mu = \mathrm{E}[Y_0] = \gamma\theta/(\theta - 1)$ for $\theta > 1$ and $R_Y(0) = \mathrm{Var}[Y_0] = \gamma^2\theta/[(\theta-1)^2(\theta-2)]$ for $\theta > 2$, respectively. Henceforth, we consider the finite-variance case by assuming $\theta > 2$. The ARTOP is a flexible bellwether process that enables us to perform rigorous stress testing because (i) if $\theta \in (2,3)$, then the marginal skewness of the process is infinite; (ii) if $\theta \in [3,4)$, then the marginal skewness of the process is finite but the marginal kurtosis is infinite; and (iii) if $\theta \in [4,\infty)$, then both the marginal skewness and kurtosis are finite. In our experience, the ARTOP process is particularly effective in stress-testing candidate point and confidence-interval estimators for the mean and given quantiles of a stationary simulation output process.

4.3.2. Calculation of σ_Y^2 and the Correlation Between Batch Means for the ARTOP Process. Dingeç et al. (2024b) prove that the ARTOP process satisfies the GMC condition; and thus its autocovariance structure decays exponentially. In fact, we can provide an explicit expression for the covariance function.

THEOREM 1. *For the ARTOP process $\{Y_k : k \geq 1\}$, we have $|R_Y(\ell)| = O(|\beta|^\ell)$ as $\ell \to \infty$, where $R_Y(\ell)$ is given by Equation (38) below.*

Proof: We define the continuous function $g(z) \equiv F^{-1}[\Phi(z)] = \gamma/[\Phi(-z)]^{1/\theta}$ for $z \in \mathbb{R}$. Thus, for $\theta > 2$, the Law of the Unconscious Statistician (LOTUS) (see, e.g., Ross 2019) implies

$$\begin{aligned}
\mathrm{E}\big[g^2(Z_0)\big] &= \int_{\mathbb{R}} g^2(z)\phi(z)\,\mathrm{d}z \\
&= \gamma^2 \int_{\mathbb{R}} [\Phi(-z)]^{-2/\theta}\,\phi(z)\,\mathrm{d}z \\
&= \gamma^2 \int_{\mathbb{R}} [\Phi(z)]^{-2/\theta}\,\phi(z)\,\mathrm{d}z \quad \text{(by symmetry)} \\
&= \frac{\gamma^2[\Phi(z)]^{1-\frac{2}{\theta}}}{1-\frac{2}{\theta}}\bigg|_{z=-\infty}^{\infty} = \frac{\gamma^2}{1-\frac{2}{\theta}} \equiv C_\theta < \infty.
\end{aligned} \tag{37}$$

Since the conditions of Lemma 1 are satisfied, Equations (18) (with "Z" in place of "Y") and (19) yield

$$R_Y(\ell) = \mathrm{Cov}[Y_0, Y_\ell] = \mathrm{Cov}[g(Z_0), g(Z_\ell)] = \sum_{j=1}^{\infty} a_j \beta^{j\ell} = \sum_{j=1}^{\infty} \frac{\gamma^2}{j!}\left(\int_{\mathbb{R}} \frac{\mathrm{He}_j(z)\phi(z)}{[\Phi(-z)]^{1/\theta}}\,\mathrm{d}z\right)^2 \beta^{j\ell}, \quad \text{for } \ell \geq 0; \tag{38}$$

and, moreover, Equations (19) and (37) give

$$|R_Y(\ell)| = \left|\sum_{j=1}^{\infty} a_j \beta^{j\ell}\right| \leq \mathrm{E}\big[g^2(Z_0)\big]\left|\sum_{j=1}^{\infty} \beta^{j\ell}\right| = \frac{C_\theta |\beta|^\ell}{1-\beta^\ell} = O(|\beta|^\ell), \quad \text{for } \ell \geq 0. \quad \square$$

REMARK 10. With the Hermite a_j's in hand from Equation (19), we have from Equation (38) that

$$\sigma_Y^2 = R_Y(0) + 2\sum_{\ell=1}^{\infty} R_Y(\ell) = \frac{\gamma^2\theta}{(\theta-1)^2(\theta-2)} + 2\sum_{\ell=1}^{\infty}\sum_{j=1}^{\infty} a_j \beta^{j\ell} = \frac{\gamma^2\theta}{(\theta-1)^2(\theta-2)} + 2\sum_{j=1}^{\infty} \frac{a_j\beta^j}{1-\beta^j}.$$

Although this formula is elegant, it runs into some possible issues. Namely, (i) we could not obtain closed-form expressions for the ARTOP's a_j's; and (ii) the numerical calculation of the integral in Equation (38) is challenging for large j, since calculating $\mathrm{He}_j(z)$ requires arithmetic involving huge numbers, giving rise to potential rounding errors. Therefore, in what follows, we will obtain expressions for $R_Y(\ell)$ and σ_Y^2 that are more numerically amenable. ◄

A better way to calculate $R_Y(\ell)$ and σ_Y^2 uses the representation $Z_\ell = \beta^\ell Z_0 + \sqrt{1-\beta^{2\ell}}Z'$, where $Z' \stackrel{\mathrm{d}}{=} \mathrm{Nor}(0,1)$ and is independent of Z_0, to obtain the exact integral formula (39) below for the ARTOP's autocovariance function. Specifically, for $\ell \ge 0$, we have

$$\begin{aligned}
R_Y(\ell) &= \mathrm{E}[Y_0 Y_\ell] - \mu^2 = \mathrm{E}\Big[F^{-1}[\Phi(Z_0)]F^{-1}[\Phi(Z_\ell)]\Big] - \mu^2 \\
&= \gamma^2\,\mathrm{E}\Big[\big(\Phi(Z_0)\Phi(Z_\ell)\big)^{-1/\theta}\Big] - \frac{\gamma^2\theta^2}{(\theta-1)^2} \\
&= \gamma^2\,\mathrm{E}\left[\left(\Phi(Z_0)\,\Phi\left(\beta^\ell Z_0 + \sqrt{1-\beta^{2\ell}}Z'\right)\right)^{-1/\theta}\right] - \gamma^2\int_0^1\int_0^1 (uv)^{-1/\theta}\,\mathrm{d}u\,\mathrm{d}v \\
&= \gamma^2\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\Phi(x)\,\Phi\left(\beta^\ell x + \sqrt{1-\beta^{2\ell}}y\right)\right)^{-1/\theta}\phi(x)\phi(y)\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad -\gamma^2\int_{\mathbb{R}}\int_{\mathbb{R}}(\Phi(x)\Phi(y))^{-1/\theta}\phi(x)\phi(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \gamma^2\int_{\mathbb{R}}\int_{\mathbb{R}}[\Phi(x)]^{-1/\theta}\left\{\left[\Phi\left(\beta^\ell x + \sqrt{1-\beta^{2\ell}}y\right)\right]^{-1/\theta} - [\Phi(y)]^{-1/\theta}\right\}\phi(x)\phi(y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \gamma^2\int_0^1\int_0^1 u^{-1/\theta}G_\ell\big(\Phi^{-1}(u),\Phi^{-1}(v)\big)\,\mathrm{d}u\,\mathrm{d}v,
\end{aligned} \tag{39}$$

where

$$G_\ell(x,y) \equiv \left[\Phi\left(\beta^\ell x + \sqrt{1-\beta^{2\ell}}y\right)\right]^{-1/\theta} - [\Phi(y)]^{-1/\theta}, \quad u \equiv \Phi(x), \quad \text{and} \quad v \equiv \Phi(y). \tag{40}$$

With $R_Y(0)$ available from §4.3.1, the autocorrelation function is given by

$$\rho_Y(\ell) = \frac{R_Y(\ell)}{R_Y(0)} = \frac{(\theta-1)^2(\theta-2)}{\theta}\int_0^1\int_0^1 u^{-1/\theta}G_\ell\big(\Phi^{-1}(u),\Phi^{-1}(v)\big)\,\mathrm{d}u\,\mathrm{d}v, \tag{41}$$

which is independent of γ. The plots in Figure 3 are based on Equation (41) and compare the autocorrelation function of the AR(1) process, β^ℓ, for $\ell \ge 1$, with that of the ARTOP process with the same β values as for the AR(1) and different values of the ARTOP's θ parameter. The figure indicates that for the values of $\theta > 2$ under study, the autocorrelations of the ARTOP process are smaller than those of the corresponding AR(1) process,

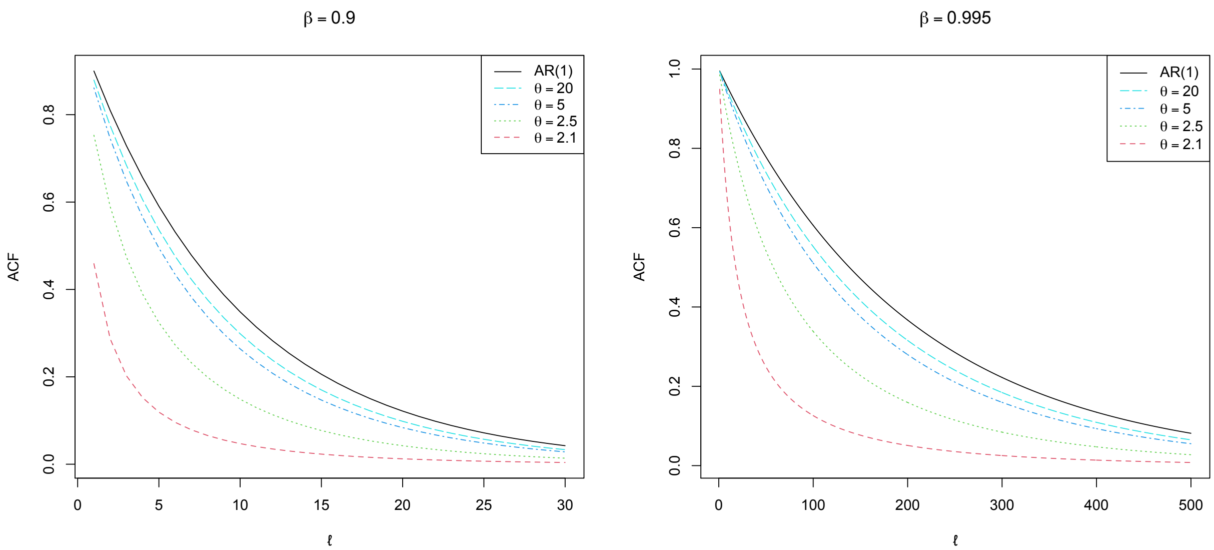

Figure 3 **Autocorrelation functions $\rho_Y(\ell)$ of AR(1) and various ($\theta > 2$) ARTOP processes with $\beta = 0.9$ (left) and $\beta = 0.995$ (right).**

especially for values of θ barely greater than 2; and this adheres to the inequality $|\mathrm{Corr}(g(Z_0), g(Z_\ell))| \le |\mathrm{Corr}(Z_0, Z_\ell)| = |\beta|^\ell$, with $g(z) = F^{-1}[\Phi(z)] = \gamma/[\Phi(-z)]^{1/\theta}$ for $z \in \mathbb{R}$, as per Remark 1.

The variance parameter for the ARTOP process becomes

$$\sigma_Y^2 = R_Y(0) + 2\sum_{\ell=1}^{\infty} R_Y(\ell) = \frac{\gamma^2\theta}{(\theta-1)^2(\theta-2)} + 2\gamma^2 \sum_{\ell=1}^{\infty} \int_0^1 \int_0^1 u^{-1/\theta} G_\ell\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)\,\mathrm{d}u\,\mathrm{d}v. \quad (42)$$

The right-hand side (RHS) of Equation (42) can be approximated by truncation of the infinite sum and numerical quadrature (similar to that undertaken for the AR(1) process in §4.1.3).

Further, the variance, autocovariance, and autocorrelation of the batch means can be calculated via Equations (5), (20), and (39)–(41). Figure 4 plots the correlation of two adjacent batch means, $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$, as a function of the batch size m, for $\gamma = 1$, $\theta = 2.1$, and three different parameter values $\beta = 0.9$, 0.95, and 0.995. Compared to the AR(1) process in Figure 1, we observe smaller correlations—due to the nonlinear transformation $Y_k = F^{-1}[\Phi(Z_k)]$ applied to the AR(1) process (Remark 1).

4.3.3. Calculation of $\sigma^2_{I(y)}$ and $\sigma^2_{\tilde{y}_p}$ for the ARTOP Process. Recall that the steady-state quantile of the ARTOP process is $y_p = F^{-1}(p) = \gamma/(1-p)^{1/\theta}$ for $p \in (0,1)$. Since $F^{-1}[\Phi(\cdot)]$ is monotone, the associated quantile indicator function with $y = y_p$ is

$$\mathbf{1}_{\{Y_k \le y_p\}} = \mathbf{1}_{\{F^{-1}[\Phi(Z_k)] \le y_p\}} = \mathbf{1}_{\{Z_k \le \Phi^{-1}[F(y_p)]\}} = \mathbf{1}_{\{Z_k \le \Phi^{-1}(p)\}}\,.$$

Thus, the autocovariance function of the indicator process evaluated at the p-quantile is the same as that of the base AR(1) process $\{Z_k : k \ge 1\}$, i.e.,

$$R_{I(y_p)}(\ell) = \mathrm{Cov}\big[\mathbf{1}_{\{Y_0 \le y_p\}}, \mathbf{1}_{\{Y_\ell \le y_p\}}\big]$$

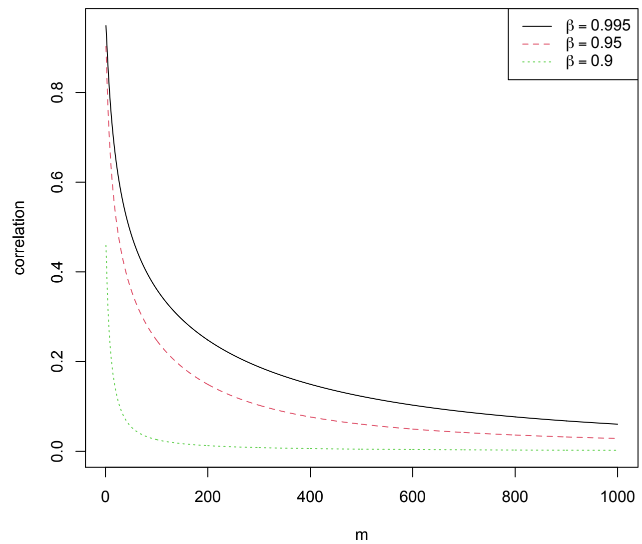

Figure 4 **Correlation of adjacent batch means,** $\mathrm{Corr}\left[\overline{Y}_{1,m}, \overline{Y}_{2,m}\right]$**,** $m = 1, 2, \ldots, 1000$**, for an ARTOP process with** $\gamma = 1$**,** $\theta = 2.1$**, and various** β **values.**

$$\begin{aligned}
&= \mathrm{Cov}\left[\mathbf{1}_{\{Z_0 \le \Phi^{-1}(p)\}}, \mathbf{1}_{\{Z_\ell \le \Phi^{-1}(p)\}}\right] \quad \text{(by stationarity)} \\
&= \Pr\left(Z_0 \le \Phi^{-1}(p),\, Z_\ell \le \Phi^{-1}(p)\right) - [\Pr(Z_0 \le \Phi^{-1}(p))]^2 \\
&= \Pr\left(Z_0 \le \Phi^{-1}(p),\, Z_\ell \le \Phi^{-1}(p)\right) - p^2, \qquad (43)
\end{aligned}$$

which can be calculated numerically via any of Equations (24), (S.30), (S.41), or (S.43), each with $y = \Phi^{-1}(p)$. Therefore, we see that $\sigma^2_{I(y_p)}$ (the sum of the indicator covariances) is independent of the marginal distribution and depends only on the correlation coefficient $\beta \in (-1, 1)$ (through the underlying AR(1) process $\{Z_k\}$). Moreover, by using the p.d.f. of the Pareto distribution, $f(y) = \theta\gamma^\theta / y^{\theta+1}$ for $y \ge \gamma$, we obtain

$$f(y_p) = \frac{\theta\gamma^\theta}{y_p^{\theta+1}} = \frac{\theta\gamma^\theta}{\left[\gamma/(1-p)^{1/\theta}\right]^{\theta+1}} = \frac{\theta}{\gamma}(1-p)^{\frac{(\theta+1)}{\theta}}.$$

With $\sigma^2_{I(y_p)}$ and $f(y_p)$ at our disposal, it is then straightforward to calculate $\sigma^2_{\widetilde{y}_p}$. Note that, for a fixed $p \in (0, 1)$, the quantity $1/f^2(y_p)$ (and so $\sigma^2_{\widetilde{y}_p} = \sigma^2_{I(y_p)}/f^2(y_p)$) increases exponentially as the shape parameter $\theta \to 0$. [Don't we really only care about $\theta > 2$?] Also, for a fixed $\theta > 0$, the quantity $1/f^2(y_p)$ increases as $p \to 1$. [I deleted a paragraph because it created confusion and we only consider $\theta > 2$.]

Table 4 lists steady-state quantiles y_p and the associated variance parameters and related quantities σ^2_Y, h_Y, $\sigma^2_{I(y_p)}$, $h_{I(y_p)}$, and $\sigma^2_{\widetilde{y}_p}$ for an ARTOP process with parameters $\gamma = 1$, $\theta = 2.1$, and $\beta = 0.9$ and 0.995. The variance parameter σ^2_Y of the underlying ARTOP process is calculated by truncation of the infinite sum in Equation (42) at 200 and 2000 terms, for $\beta = 0.9$ and 0.995, respectively. Of course, the ARTOP's Pareto marginal distribution is bounded by 0 from below and skewed right, so we do not observe quantile symmetry around $p = 0.5$ as for the AR(1). In addition, owing to the Pareto's fat tails (and relatively high marginal variance $\mathrm{Var}[Y_1]$), the σ_Y^2 variance parameter values are greater than the analogous AR(1)'s, and its quantiles increase more rapidly than do the underlying AR(1)'s. On the other hand, in light of Figure 3 (showing that the AR(1) autocorrelations $\rho_Y(\ell)$ are greater than those of the corresponding ARTOP, particularly for θ just above 2), it is of interest to calculate the BH information values $h_Y = \sigma_Y^2/\mathrm{Var}[Y_1]$ for both processes,

$$\text{AR(1)}:\ h_Y^{\text{AR(1)}} \equiv \frac{\frac{1+\beta}{1-\beta}}{1} = 1 + \frac{2\beta}{1-\beta} \quad \text{and} \quad \text{ARTOP}:\ h_Y^{\text{ARTOP}} \equiv 1 + \frac{2(\theta-1)^2(\theta-2)}{\gamma^2\theta} \sum_{j=1}^{\infty} \frac{a_j \beta^j}{1-\beta^j}.$$

By definition of the BH ratio in Equation (14) and the maximal correlation property of bivariate normals (Remark 1 with "Z" now in place of "Y"), we have that for $\beta \in (0,1)$ (whence $\mathrm{Corr}(Z_0, Z_\ell) > 0$),

$$h_Y^{\text{ARTOP}} = \sum_{\ell \in \mathbb{Z}} \mathrm{Corr}\big(Y_0, Y_{|\ell|}\big) = \sum_{\ell \in \mathbb{Z}} \mathrm{Corr}\big(g(Z_0), g(Z_{|\ell|})\big) \le \sum_{\ell \in \mathbb{Z}} \mathrm{Corr}\big(Z_0, Z_{|\ell|}\big) = h_Y^{\text{AR(1)}}.$$

Figure 5 depicts $h_Y^{\text{AR(1)}}$ and h_Y^{ARTOP} for $\beta = 0.9$, the latter as a function of $\theta \in [2.1, 15.1]$. For θ in that range, we have $h_Y^{\text{ARTOP}} < h_Y^{\text{AR(1)}} = 19$, indicating (perhaps surprisingly) that the estimation of the ARTOP's mean is not necessarily more "difficult" than for the AR(1).

Table 4 **Variance parameters for the quantiles of an ARTOP process with $\gamma = 1$, $\theta = 2.1$, and $\beta = 0.9$ and 0.995.**

		$\beta = 0.9$			$\beta = 0.995$		
		$\sigma_Y^2 = 83.36$ and $h_Y = 4.80$			$\sigma_Y^2 = 1{,}690$ and $h_Y = 97.38$		
p	y_p	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$
0.3	1.185	2.664	12.68	1.73	55.441	264.0	36.0
0.5	1.391	3.320	13.28	5.83	69.148	276.6	121.4
0.7	1.774	2.664	12.68	21.13	55.441	264.0	439.7
0.9	2.994	0.912	10.13	185.3	18.891	209.9	3,838
0.95	4.164	0.407	8.57	640.2	8.393	176.7	13,201
0.99	8.962	0.058	5.86	10,562	1.177	118.9	214,277
0.995	12.466	0.025	5.03	35,238	0.506	101.7	713,292

In terms of the variance parameters related to quantiles, the ARTOP's $\sigma^2_{\tilde{y}_p}$ values significantly surpass those of the AR(1) process as p approaches 1 from the left—also, presumably, a consequence of the Pareto's tails. Other than that, qualitatively speaking, we observe the same general behavior here as for the AR(1) from Table 2. Indeed, as noted just after Equation (43), the values for $\sigma^2_{I(y_p)}$ and $\sigma^2_{I(y_p)}/[p(1-p)] = h_{I(y_p)} = h_{\tilde{y}_p}$ are precisely the same for the AR(1) process (Table 2) and the ARTOP process (Table 4)—indicating that, in some sense, both processes yield the same "information" per observation with respect to quantile estimation.

4.4. M/M/1 Waiting-Time Process

This subsection follows the usual recipe, but with one bonus. §4.4.1 establishes basic notation and results for the M/M/1 waiting-time process; §4.4.2 calculates σ_Y^2 and the correlation between batch means for the waiting process; §4.4.3 deals with $\sigma^2_{I(y)}$ and $\sigma^2_{\tilde{y}_p}$; and §4.4.4 discusses some sample-size results related to confidence intervals involving the various variance parameters.

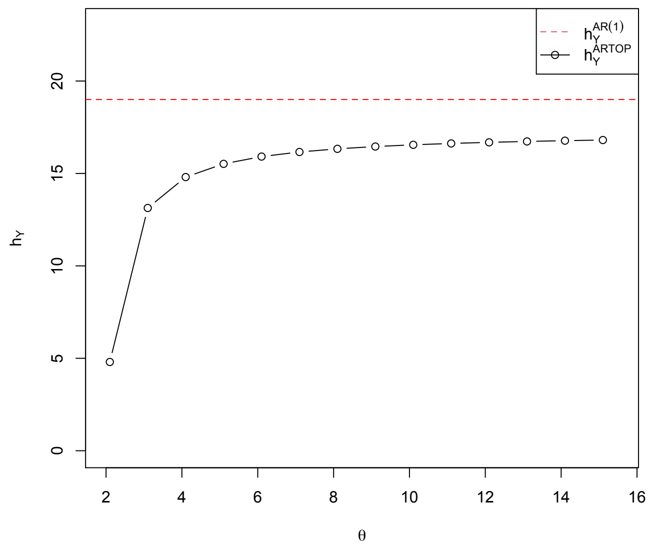

Figure 5 **BH information values: $h_Y^{\text{AR(1)}}$ for the AR(1) process with $\beta = 0.9$; and h_Y^{ARTOP} for the ARTOP process with $\beta = 0.9$, $\gamma = 1$, and $\theta = 2.1, 3.1, \ldots, 15.1$.**

4.4.1. M/M/1 Preliminaries. Consider the waiting-time process in a steady-state M/G/1 queueing system; i.e., customers arrive in an i.i.d. exponential fashion with rate $\lambda > 0$, and they are served first-in-first-out with i.i.d. service times having rate $\omega = \lambda/\rho > 0$, where $\rho \in (0, 1)$ is the server utilization (traffic intensity). Let Y_k denote the waiting time of customer $k \geq 0$, where Y_0 is initialized in steady state from the c.d.f.

$$F(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1 - \rho, & \text{if } y = 0, \\ 1 - \rho e^{-\omega(1-\rho)y}, & \text{if } y > 0. \end{cases} \tag{44}$$

Dingeç et al. (2022) proved that the GMC condition is satisfied by the waiting-time process in a G/G/1 queueing system with a non-heavy-tailed service-time distribution (i.e., the service time's moment generating function exists in a neighborhood of zero). For an M/M/1 queue (with i.i.d. exponential service times), the marginal steady-state mean and variance of the waiting-time process are, respectively,

$$\mu = \mathrm{E}[Y_0] = \frac{\rho}{\omega(1-\rho)} \quad \text{and} \quad R_Y(0) = \mathrm{Var}[Y_0] = \frac{\rho^3(2-\rho)}{\lambda^2(1-\rho)^2}. \tag{45}$$

4.4.2. Calculation of σ_Y^2 and the Correlation Between Batch Means for the M/M/1 Waiting-Time Process. We validate and study the consequences of Theorems C1 and C2 for M/M/1 waiting times. Let $r \equiv 4\rho/(1+\rho)^2 \in (0, 1)$, since $\rho \in (0, 1)$, and $\kappa(t) \equiv t^{3/2}(r-t)^{1/2}/(1-t)^3$ for $t \in (0, r)$. Then Equation (34) of Daley (1968), as corrected by Song and Schmeiser (1995, p. 117), yields the autocovariance function

$$R_Y(\ell) = c\, R_Y(0) \int_0^r t^\ell \kappa(t)\, \mathrm{d}t, \quad \text{for } \ell \geq 0, \tag{46}$$

where

$$c \equiv \frac{1-\rho^2}{2\pi\lambda^2 R_Y(0)} = \frac{(1-\rho)^3(1+\rho)}{2\pi\rho^3(2-\rho)}.$$

As indicated by Theorem C1, the autocovariance function is $O(r^\ell)$ as $\ell \to \infty$. Also, from Daley (1968, Equation (32), p. 696), we obtain the variance parameter,

$$\sigma_Y^2 = \frac{\rho^3(2+5\rho-4\rho^2+\rho^3)}{(1-\rho)^4\lambda^2}, \tag{47}$$

and then we have the standardized BH variance parameter,

$$h_Y = \frac{\sigma_Y^2}{R_Y(0)} = \frac{1+\rho}{1-\rho} + \frac{2\rho(3-\rho)}{(2-\rho)(1-\rho)^2},$$

which is a sharply increasing function as $\rho \to 1-$.

In addition, we can derive an exact expression for $\mathrm{Var}\big[\overline{Y}_{1,m}\big]$ using Equations (20) and (46) to obtain

$$\begin{aligned}
\mathrm{Var}\big[\overline{Y}_{1,m}\big] &= \frac{1}{m}\left[R_Y(0) + 2\sum_{\ell=1}^{m-1}\left(1-\frac{\ell}{m}\right) c\, R_Y(0)\int_0^r t^\ell \kappa(t)\,\mathrm{d}t\right] \\
&= \frac{R_Y(0)}{m}\left(1 + 2c\int_0^r \left[\sum_{\ell=1}^{m-1}\left(1-\frac{\ell}{m}\right)t^\ell\right]\kappa(t)\,\mathrm{d}t\right) \\
&= \frac{R_Y(0)}{m}\left(1 + 2c\int_0^r \frac{t\big(t^m + m(1-t) - 1\big)\kappa(t)}{m(1-t)^2}\,\mathrm{d}t\right).
\end{aligned} \tag{48}$$

Similarly, by Equations (5) and (46), the covariance of batch means $d_1 \neq d_2$ is

$$\begin{aligned}
\mathrm{Cov}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big] &= \frac{1}{m}\sum_{\ell=-m+1}^{m-1}\left(1-\frac{|\ell|}{m}\right) c\, R_Y(0)\int_0^r t^{m|d_1-d_2|+\ell}\kappa(t)\,\mathrm{d}t, \\
&= \frac{c\,R_Y(0)}{m}\int_0^r \left[\sum_{\ell=-m+1}^{m-1}\left(1-\frac{|\ell|}{m}\right)t^{m|d_1-d_2|+\ell}\right]\kappa(t)\,\mathrm{d}t \\
&= \frac{c\,R_Y(0)}{m^2}\int_0^r t^{m(|d_1-d_2|-1)+1}\left(\frac{1-t^m}{1-t}\right)^2\kappa(t)\,\mathrm{d}t \quad \text{(via Mathematica)} \qquad (49) \\
&= O\big(r^{m(|d_1-d_2|-1)}m^{-2}\big), \quad \text{as } m \to \infty. \qquad (50)
\end{aligned}$$

Then (7) and (50) immediately imply that for $d_1 \neq d_2$,

$$m\,\mathrm{Corr}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big] = \frac{m^2\,\mathrm{Cov}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big]}{m\,\mathrm{Var}\big[\overline{Y}_{1,m}\big]} = \frac{O\big(r^{m(|d_1-d_2|-1)}\big)}{\sigma_Y^2 + O(1/m)} = O\big(r^{m(|d_1-d_2|-1)}\big),$$

which matches the order of the analogous result for the AR(1) process given by Equation (23).

More precisely, we can apply Equations (48) and (49) to obtain the exact correlation of batch means,

$$\mathrm{Corr}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big] = \frac{\mathrm{Cov}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big]}{\mathrm{Var}\big[\overline{Y}_{1,m}\big]} = \frac{\int_0^r t^{m(|d_1-d_2|-1)+1}\big(\frac{1-t^m}{1-t}\big)^2\kappa(t)\,\mathrm{d}t}{\frac{m}{c} + 2\int_0^r \frac{t[t^m+m(1-t)-1]}{(1-t)^2}\kappa(t)\,\mathrm{d}t}. \tag{51}$$

We can also provide a closed-form asymptotic expression for $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$, i.e., the special case of adjacent batches, as $m \to \infty$. Given $r \in (0,1)$, the increasing sequence of nonnegative functions on $(0,r)$,

$$\mathfrak{w}_m(t) \equiv t\left(\frac{1-t^m}{1-t}\right)^2 \kappa(t) = \frac{t^{5/2}(r-t)^{1/2}(1-t^m)^2}{(1-t)^5} \quad \text{for} \quad m \ge 1,$$

converges at every point $t \in (0,r)$ to the function

$$\mathfrak{w}(t) \equiv \frac{t^{5/2}(r-t)^{1/2}}{(1-t)^5}.$$

Thus, the Monotone Convergence Theorem (Royden and Fitzpatrick 2010, p. 83) ensures that

$$m^2\,\mathrm{Cov}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big] = c\,R_Y(0)\int_0^r \mathfrak{w}_m(t)\,\mathrm{d}t \to c\,R_Y(0)\int_0^r \mathfrak{w}(t)\,\mathrm{d}t = \frac{5\rho^4}{(1-\rho)^6\lambda^2}, \text{ as } m \to \infty, \tag{52}$$

where the last term follows from Mathematica. Then Equations (47) and (52) imply that

$$m\,\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big] = \frac{m^2\,\mathrm{Cov}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]}{m\,\mathrm{Var}\big[\overline{Y}_{1,m}\big]} \to \frac{\frac{5\rho^4}{(1-\rho)^6\lambda^2}}{\sigma_Y^2} = \frac{5\rho}{(1-\rho)^2(2+5\rho-4\rho^2+\rho^3)}, \text{ as } m \to \infty. \tag{53}$$

Figure 6 illustrates the correlation between two adjacent batches, $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$, with respect to batch size m, for three different traffic intensity values $\rho = 0.8$, 0.85, and 0.9. These calculations applied R's numerical `integrate` function (R Core Team 2022) on the exact expression in Equation (51) [and were checked via the asymptotic expression given in Equation (53)].

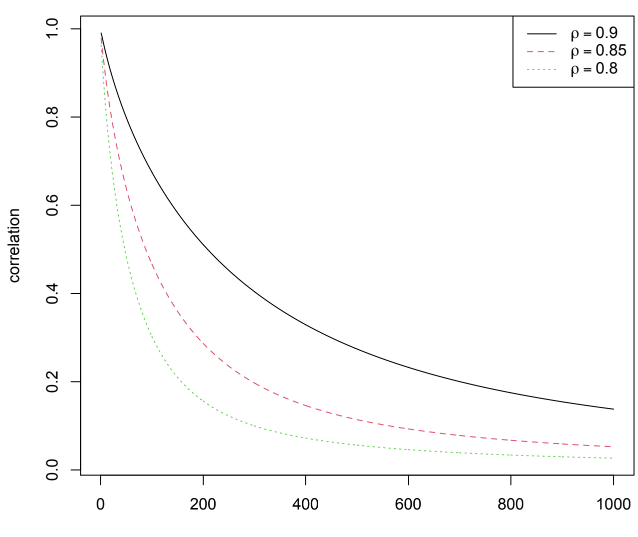

Figure 6 **Correlation of adjacent batch means,** $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$, $m = 1, 2, \ldots, 1000$, **for the M/M/1 waiting-time process with** $\omega = 1$ **and various** ρ **values. [← Do we need to say anything about** ω **here? is it ever used???]**

4.4.3. Calculation of $\sigma^2_{I(y)}$ and $\sigma^2_{\bar{y}_p}$ for the M/M/1 Waiting-Time Process. This process does not satisfy Assumption (c) of Theorem C2 because the marginal c.d.f. in Equation (44) is discontinuous; hence the marginal p.d.f. does not exist. Fortunately, the consequences of the theorem hold. In order to calculate the variance parameters $\sigma^2_{I(y)}$ and $\sigma^2_{\bar{y}_p}$, we first define the generator $\pi(s;y)$ of the autocovariance function $R_{I(y)}(\ell)$ of the indicators,

$$\pi(s;y) \equiv \sum_{\ell=0}^{\infty} s^{\ell} R_{I(y)}(\ell), \quad \text{for } |s| \le 1 \text{ and } y \ge 0.$$

We obtain the variance parameter of the indicator process as

$$\sigma^2_{I(y)} = R_{I(y)}(0) + 2\sum_{\ell=1}^{\infty} R_{I(y)}(\ell) = 2\pi(1;y) - F(y)[1-F(y)], \quad \text{for } y \ge 0. \tag{54}$$

Equation (19) in Blomqvist (1967) gives a formula for $\pi(1;y)$ for the M/G/1 waiting-time process,

$$\pi(1;y) = \frac{1}{1-\rho}\left\{[1-F(y)]\left[G(y) - \lambda\int_0^y G(x)\,\mathrm{d}x\right] + \lambda F(y)\left(\mu F(y) - \int_0^y x\,\mathrm{d}F(x)\right)\right\}, \tag{55}$$

where $G(y)$ is the convolution of $F(x)$ with itself, i.e.,

$$G(y) \equiv \int_0^y F(y-x)\,\mathrm{d}F(x). \tag{56}$$

Equations (54)–(56) can be used to calculate $\sigma^2_{I(y)}$ for any waiting-time c.d.f. $F(y)$.

For certain special cases, closed-form results are possible. For instance, for the M/M/1, we can obtain the p-quantile of the waiting-time process by inverting the c.d.f. in Equation (44),

$$y_p = F^{-1}(p) = \begin{cases} 0, & \text{if } 0 < p \le 1-\rho, \\ \frac{1}{\omega(1-\rho)}\,\ell\mathrm{n}\left(\frac{\rho}{1-p}\right), & \text{if } 1-\rho < p < 1. \end{cases} \tag{57}$$

Simplification of $\pi(1;y_p)$ via Mathematica using the formulas for $F(y)$, μ, and y_p yields the following closed form,

$$\sigma^2_{I(y_p)} = \frac{(1-p)}{(1-\rho)^2}\left(\left[-2+p(3-\rho)+2\rho\right](1+\rho) - 4(1-p)\rho\,\ell\mathrm{n}\left(\frac{\rho}{1-p}\right)\right), \quad \text{for } p \in (1-\rho,1). \tag{58}$$

Furthermore, by using $f(y_p) \equiv F'(y_p) = \omega(1-p)(1-\rho)$ for $p \in (1-\rho,1)$, we obtain

$$\sigma^2_{\bar{y}_p} = \frac{\sigma^2_{I(y_p)}}{f^2(y_p)} = \frac{1}{\omega^2(1-\rho)^4}\left(\frac{\left[-2+p(3-\rho)+2\rho\right](1+\rho)}{1-p} - 4\rho\,\ell\mathrm{n}\left(\frac{\rho}{1-p}\right)\right), \quad \text{for } p \in (1-\rho,1). \tag{59}$$

Here $f(y)$ stands for the derivative of the steady-state c.d.f. $F(y)$, as the p.d.f. of M/M/1 queue waiting times does not exist.

Table 5 lists steady-state quantiles y_p and the associated variance parameters and related quantities σ^2_Y, $\sigma^2_{I(y_p)}$, $h_{I(y_p)}$, and $\sigma^2_{\bar{y}_p}$, for the M/M/1 waiting-time process with $\omega = 1$ [← do we need ω?] and $\rho = 0.8$ and 0.9. The values of the quantiles and associated variance parameters are quite high. What is notable is that—unlike the cases of the AR(1) and ARTOP processes reported earlier—the values of $h_{I(y_p)} = \sigma^2_{I(y_p)}/[p(1-p)]$ are *increasing* as p becomes larger. This can be shown analytically by taking the derivative of $h_{I(y_p)}$ with respect to p and using the inequality $\ell\text{n}(x) \geq 1 - \frac{1}{x}$,

$$\frac{\mathrm{d}h_{I(y_p)}}{\mathrm{d}p} = \frac{2\big(1-2p\rho-\rho^2+2\rho\,\ell\text{n}[\rho/(1-p)]\big)}{p^2(1-\rho^2)} \geq \frac{2(-1-2p\rho-\rho^2+2\rho+2p)}{p^2(1-\rho^2)} > 0, \quad \text{for } p \in (1-\rho, 1). \tag{60}$$

Table 5 **Variance parameters for the quantiles of an M/M/1 waiting-time process with $\omega = 1$ and $\rho = 0.8$ and 0.9.**

	$\rho = 0.8$, $\sigma^2_Y = 1{,}976$ and $h_Y = 82.33$				$\rho = 0.9$, $\sigma^2_Y = 35{,}901$ and $h_Y = 362.64$			
p	y_p	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$	y_p	$\sigma^2_{I(y_p)}$	$h_{I(y_p)}$	$\sigma^2_{\tilde{y}_p}$
0.300	0.668	2.956	14.1	150.8	2.513	12.858	61.2	2,624
0.500	2.350	6.350	25.4	634.5	5.878	27.849	111.4	11,140
0.700	4.904	8.328	39.7	2,313	10.986	36.795	175.2	40,883
0.900	10.397	5.446	60.5	13,616	21.972	24.200	268.9	242,000
0.950	13.863	3.248	68.4	32,480	28.904	14.451	304.2	578,047
0.990	21.910	0.765	77.3	191,261	44.998	3.408	344.2	3.41e+06
0.995	25.376	0.392	78.9	392,375	51.930	1.748	351.4	6.99e+06

In addition, we can use Equation (58) to calculate the limiting value $\lim_{p\to 1-} h_{I(y_p)} = \lim_{p\to 1-}\left\{\sigma^2_{I(y_p)}/[p(1-p)]\right\}$. A darg of algebra (i.e., more than you can shake a stick at) eventually results in

$$\sigma^2_{I(y_p)} \sim p(1-p)\left[\frac{1+\rho}{1-\rho}\right]^2, \quad \text{as } p \to 1-,$$

which is greater than the analogous quantity for the AR(1) process with $0 \leq \beta < 1$, i.e., $\sigma^2_{I(y_p)} \sim p(1-p)$, as $p \to 1-$ (Remark 7 and Appendix S5).

Furthermore, since by Equation (60), $\mathrm{d}h_{I(y_p)}/\mathrm{d}p > 0$ for $p \in (1-\rho, 1)$, we have

$$\frac{h_Y}{h_{I(y_p)}} = \frac{\sigma^2_Y/\mathrm{Var}[Y_0]}{\sigma^2_{I(y_p)}/[p(1-p)]} \geq \frac{\sigma^2_Y/\mathrm{Var}[Y_0]}{\lim_{p\to 1-}\left\{\sigma^2_{I(y_p)}/[p(1-p)]\right\}} = \frac{\rho^3-4\rho^2+5\rho+2}{(2-\rho)(1+\rho)^2} > 1, \quad \text{for } \rho \in (0,1),$$

and we conclude that the “difficulty” caused by autocorrelation is always more pronounced in the estimation of the mean than that of quantiles—at least for the M/M/1 case.

For completeness, the lag-ℓ covariances can be calculated via the derivatives of the generator function,

$$R_{I(y)}(\ell) = \frac{1}{\ell!}\left.\frac{\mathrm{d}^\ell \pi(s;y)}{\mathrm{d}s^\ell}\right|_{s=0}, \quad \text{for } \ell \geq 1.$$

Equation (22) of Blomqvist (1967) gives a formula for the generator function $\pi(s;y)$ in the M/M/1 case,

$$\pi(s;y) = \overline{F}(y)\frac{\omega\big(\omega+v(s)+\rho(\lambda-v(s))\big)}{(v(s)+\omega-\lambda)(2v(s)+\omega-\lambda)} - \frac{\omega^2\overline{F}^2(y)}{v(s)(v(s)+\omega-\lambda)}\left[\rho - \frac{(1-\rho)(\lambda-v(s))^2}{\lambda(2v(s)+\omega-\lambda)}e^{-2v(s)y}\right]$$

where $\overline{F}(y) \equiv 1 - F(y) = \rho e^{-\omega(1-\rho)y}$, for $y \geq 0$, and

$$v(s) \equiv \frac{1}{2}\left(\lambda - \omega + \sqrt{(\omega+\lambda)^2 - 4\lambda\omega s}\right).$$

After a corvée of algebra, the generating function at the p-quantile y_p simplifies to

$$\pi(s; y_p) = (1-p)\frac{\frac{1+\rho}{1-\rho} + \sqrt{1-rs}}{\frac{1+\rho}{1-\rho}(1-rs) + \sqrt{1-rs}} - \frac{(1-p)^2}{1-s} + \frac{(1-p)^2(1-\rho^2)\left(1-\sqrt{1-rs}\right)^2}{4\rho^2(1-s)\sqrt{1-rs}}\left[\frac{\rho}{1-p}\right]^{1-\frac{1+\rho}{1-\rho}\sqrt{1-rs}},$$

for $|s| < 1$ and (as defined previously) $r = 4\rho/[(1+\rho)^2] \in (0,1)$. The autocovariances $R_{I(y_p)}(\ell)$ and autocorrelations $\rho_{I(y_p)}(\ell)$ can be computed by calculating the derivatives of the generator function $\pi(s; y_p)$ at $s = 0$. In particular, the lag-1 autocorrelation has a simple formula,

$$\rho_{I(y_p)}(1) = 1 - \frac{(1-\rho)}{p(1+\rho)},$$

and is an increasing function of p, in contrast to the AR(1) example (cf. Figure 2). In addition, its limiting value as $p \to 1-$ is $\lim_{p\to 1-} \rho_{I(y_p)}(1) = \frac{2\rho}{1+\rho} < 1$.

4.4.4. Comparison of Sample Sizes for Computing $100(1-\alpha)$% Confidence Intervals. If we assume that the variance parameters σ_Y^2 and $\sigma_{\widetilde{y}_p}^2$ are somehow *known*, then for sufficiently large n, approximately valid $100(1-\alpha)$% confidence intervals (CIs) for the marginal mean μ and p-quantile y_p are:

$$\mu \in \overline{Y}_n \pm z_{\alpha/2}\frac{\sigma_Y}{\sqrt{n}} \quad \text{and} \quad y_p \in \widetilde{y}_p(n) \pm z_{\alpha/2}\frac{\sigma_{\widetilde{y}_p}}{\sqrt{n}},$$

respectively, where $z_\delta \equiv \Phi^{-1}(1-\delta)$ for $\delta \in (0,1)$. Then estimates for the smallest sample sizes n required to compute CIs with relative precision (RP) (ratio of the CI half-length divided by the absolute value of the metric of interest) $\epsilon \in (0,1)$ can be computed from the respective constraints,

$$z_{\alpha/2}\frac{\sigma_Y}{|\mu|\sqrt{n}} \leq \epsilon \quad \text{and} \quad z_{\alpha/2}\frac{\sigma_{\widetilde{y}_p}}{|y_p|\sqrt{n}} \leq \epsilon.$$

The resulting sample sizes are

$$n_\mu \equiv \left\lceil \left(\frac{z_{\alpha/2}\,\sigma_Y}{\mu\,\epsilon}\right)^2 \right\rceil \quad \text{and} \quad n_p \equiv \left\lceil \left(\frac{z_{\alpha/2}\,\sigma_{\widetilde{y}_p}}{y_p\,\epsilon}\right)^2 \right\rceil \tag{61}$$

for the mean and the p-quantile, respectively, where "$\lceil\cdot\rceil$" denotes the greatest integer function.

With Equation (61) in hand, we add Equations (45), (47), (57), and (59) to the mix in order to compare the sample sizes required to achieve an apples-to-apples RP for estimating μ and y_p for the M/M/1 waiting-time process, namely,

$$n_\mu = \left\lceil \frac{z_{\alpha/2}^2}{\epsilon^2}\frac{(2+5\rho-4\rho^2+\rho^3)}{(1-\rho)^2\rho} \right\rceil, \quad \text{for } \rho \in (0,1) \tag{62}$$

and

$$n_p = \left\lceil \frac{z_{\alpha/2}^2}{\epsilon^2} \frac{\left[-2+p(3-\rho)+2\rho\right](1+\rho)-4(1-p)\rho\,\ell\mathrm{n}[\rho/(1-p)]}{(1-p)(1-\rho)^2\big(\ell\mathrm{n}[\rho/(1-p)]\big)^2} \right\rceil, \quad \text{for } p \in (1-\rho, 1) \text{ and } \rho \in (0,1). \tag{63}$$

Figure 7 plots the sample sizes n_μ and n_p required for the cases $\omega = 1$, [$\leftarrow$ is ω necessary?] $\epsilon = 0.02$, $\alpha = 0.05$, and $\rho \in [0.5, 0.995]$. For a reference point, the traffic intensity $\rho = 0.9$ yields $n_\mu = 4{,}256{,}550$ for estimation of the mean μ, and $n_{p=0.5} = 3{,}096{,}488$, $n_{p=0.9} = 4{,}813{,}960$, and $n_{p=0.95} = 6{,}644{,}949$ for estimation of the quantiles y_p at $p = 0.5$, 0.9, and 0.95, respectively. Notice that the values of n_p are substantially larger than the respective values of n_μ only for extreme quantiles ($p \geq 0.95$).

On the other hand, we see that setting the traffic intensity $\rho = 0.51$ yields $n_\mu = 285{,}656$ for the mean and $n_{p=0.5} = 77{,}509{,}754$, $n_{p=0.9} = 236{,}840$, and $n_{p=0.95} = 275{,}170$ for the quantiles. In this case, the sample size for the median is much larger than that for the mean and other quantiles. This latter phenomenon is borne out in Figure 7, where the graph of n_p for $p = 0.5$ (as a function of ρ) differs greatly from the other curves, diverging as $\rho \to 0.5 = 1 - p$. In fact, for a fixed $\rho \in (0,1)$, we have that $n_p \to \infty$ as $p \to 1 - \rho$. This is due to the fact that the quantile y_p to be estimated approaches zero while the respective variance parameter $\sigma^2_{\tilde{y}_p}$ approaches a strictly positive constant as $p \to 1 - \rho$, i.e., by Equation (59),

$$\lim_{p \to 1-\rho} \sigma^2_{\tilde{y}_p} = \frac{1+\rho}{\omega^2 \rho (1-\rho)^2} > 0 \quad \text{for } \rho \in (0,1).$$

The rapid inflation of n_p as $p \to 1 - \rho$ is manifested by the term $\big(\ell\mathrm{n}[\rho/(1-p)]\big)^2$ in Equation (63)'s denominator.

There is also interest in determining which of mean or quantile estimation requires more observations in the heavy traffic limit as $\rho \to 1-$. By Equations (62) and (63), the limiting ratio of the two sample sizes is

$$\lim_{\rho \to 1-} \frac{n_p}{n_\mu} = \frac{1}{\ell\mathrm{n}(1-p)} \left[1 + \frac{p/(1-p)}{\ell\mathrm{n}(1-p)}\right] \in (1/2, \infty),$$

which is an increasing function of p that, moreover, goes to 1/2 as $p \to 0+$ and goes to ∞ as $p \to 1-$. That is, in the heavy traffic limit, the required sample size for quantile estimation is at least 50% of that required for mean estimation. For estimation of the quartiles for the cases $p = 0.25$, $p = 0.5$, and $p = 0.75$, the respective limiting ratios are 0.55, 0.64, and 0.84. In addition, the above ratio is at least 1 if $p \geq 0.834$, and at least 2 if $p \geq 0.96$. For extreme quantiles with $p \geq 0.995$, the limiting ratio is at least 6.9. Therefore, in the heavy traffic limit, quantile estimation requires significantly more observations than mean estimation only for extreme and near-extreme upper quantiles.

Table 6 compares the "theoretical" sample sizes n_p from Equation (61) against the average sample sizes $\bar{n}$ reported as part of a Monte Carlo (MC) evaluation of the SQSTS sequential quantile-estimation method of Lolos et al. (2023b, Table 3). SQSTS is fully automated, includes steps to alleviate the effects of initialization

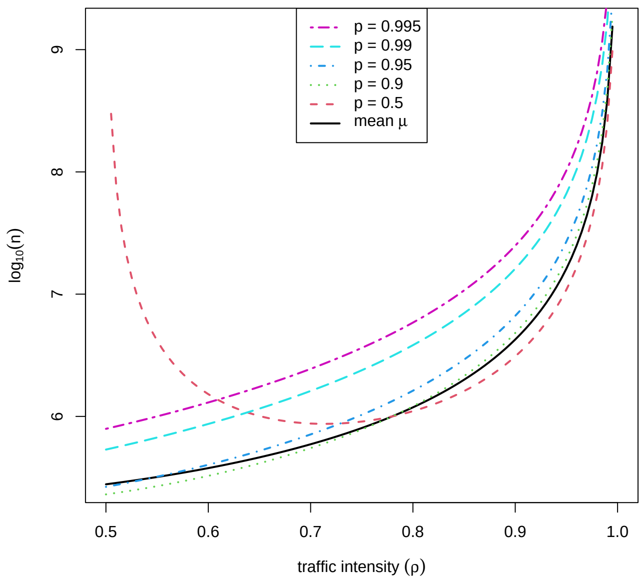

Figure 7 **Estimated sample sizes n_μ and n_p for computing 95% CIs for μ and y_p with RP $\epsilon = 0.02$ for the stationary M/M/1 waiting-time process with service rate $\omega = 1$ and traffic intensity $\rho \in (0.5, 1)$.**

bias, and estimates the variance parameter $\sigma^2_{\tilde{y}_p}$ by a linear combination of estimators constructed from the sample variance of the empirical quantiles based on a set of b nonoverlapping batches having length m and standardized times series "area" estimators obtained from the same set of nonoverlapping batches. The limiting distribution of the associated CI pivot ratio for fixed b and $m \to \infty$ is a scaled Student's t-distribution with $(2b-1)$ degrees of freedom. Column 3 of Table 6 displays average sample sizes $\bar{n}$ obtained from 1,000 independent MC replications of an M/M/1 waiting-time process with $\rho = 0.9$ and $\epsilon = 0.02$ (i.e., 2% RP), where herein we ignore an initial transient phase that comprises only a negligible proportion of the sample before entering approximate steady state. Column 4 contains estimates of coverage probabilities for 95% CIs for y_p; the standard errors of these point estimates are roughly 0.007.

The close proximities of the MC averages $\bar{n}$ in column 3 to the "theoretical" values n_p computed from Equation (61) using the actual values of $\sigma^2_{\tilde{y}_p}$ in this challenging experimental setting are indicative of the sampling effectiveness of the SQSTS method in spite of the presence in each sample path of an initial transient and the need to estimate the unknown variance parameter $\sigma^2_{\tilde{y}_p}$. We believe that the discrepancies between the theoretical values n_p and the estimates $\overline{n}$ are attributable to the gap between the targeted and final attained relative precisions of the CI delivered by SQSTS: in its final stages the sequential method updates the batch size and batch count until the RP of the CI for y_p drops below ϵ. For example, when $p = 0.95$, the average 95% CI RP reported in Table 3 of Lolos et al. (2023b) was 1.892% (vs. the desired 2% RP). If one uses the latter value in Equation (63), the resulting theoretical value of n_p becomes 7,425,221. Adding the average length of the initial truncated portion of the sample path, which hovered near 5,800, yields the total (approximate) average sample size of 7,431,021, which is very close to the average MC sample size $\overline{n} = 7{,}500{,}116$ reported in column 3 of Table 6.

Table 6 "Theoretical" sample sizes n_p from Equation (61), average MC sample sizes $\bar{n}$, and MC coverages reported by SQSTS for computing 95% CIs for selected quantiles y_p with RP $\epsilon = 0.02$ for the stationary M/M/1 waiting-time process with service rate $\omega = 1$ [$\leftarrow$ is ω needed?] and traffic intensity $\rho = 0.9$. All MC results are based on 1,000 replications.

p	n_p	$\bar{n}$	Estimated CI coverage
0.300	3,990,095	4,528,399	0.951
0.500	3,096,488	3,576,460	0.946
0.700	3,253,068	3,731,135	0.946
0.900	4,813,960	5,461,971	0.946
0.950	6,644,949	7,500,116	0.941
0.990	16,164,456	18,479,751	0.930
0.995	24,904,615	28,290,323	0.936

Table 7 concerns the performance of the fixed-sample-size quantile-estimation procedure FQUEST of Lolos et al. (2023a) for the same M/M/1 queue waiting-time process. In particular, we report on how the procedure's pre-specified sample size n affects the CI's RP and coverage probability. The asymptotically exact RPs, $\left(z_{\alpha/2}\,\sigma_{\widetilde{y}_p}\right)/\left(y_p\sqrt{n}\right)$, are displayed in column 4. Columns 5 and 6, respectively, contain the estimated average RPs and coverages obtained by FQUEST; these estimates are based on truncated sample paths after an initial portion of about 700 observations (on average) was removed. The most-important observation is that FQUEST's exact and estimated RPs tend to approach each other for large sample sizes, while the estimated coverage probabilities reported by FQUEST are often greater than the nominal 95% level. We surmise that the primary reason for the excessive estimated coverage is the use of heuristic CIs that are amalgams of asymptotically valid CIs when the sample size is deemed to be insufficient.

5. Conclusions

This article examined in detail key properties of four bellwether stochastic processes that can ultimately be used for evaluating and stress-testing proposed statistical estimators for the mean and quantiles of any stationary stochastic process $\{Y_k : k \geq 0\}$ that satisfies the Geometric-Moment Contraction condition and has a bounded marginal density with a bounded derivative. In particular, we examined the following test processes: (a) an AR(1) process with normal (light-tailed) innovations and various values for the autoregressive parameter; (b) an AR(1) process with Cauchy (heavy-tailed) innovations; (c) an ARTOP process with Pareto marginals having various shape and autoregressive parameters; and (d) the waiting-time process in an M/M/1 queueing system for several traffic intensities.

Table 7 Asymptotically exact RPs, estimated average RPs, and estimated coverages reported by FQUEST for computing 95% CIs for selected quantiles y_p for the stationary M/M/1 waiting-time process with service rate $\omega = 1$ [← ω??] and traffic intensity $\rho = 0.9$. All MC results are based on 1,000 replications.

p	y_p	n	"Exact" RP (%)	FQUEST RP (%)	FQUEST CI coverage
0.7	10.986	100,000	11.41	21.41	0.973
		200,000	8.07	12.52	0.984
		500,000	5.10	6.77	0.967
		1,000,000	3.61	4.54	0.970
0.95	28.904	100,000	16.30	36.54	0.965
		200,000	11.53	22.65	0.963
		500,000	7.29	11.31	0.963
		1,000,000	5.16	7.20	0.967
0.99	44.998	100,000	25.43	40.03	0.939
		200,000	17.98	31.68	0.949
		500,000	11.37	22.54	0.958
		1,000,000	8.04	15.13	0.952

For each of the test processes (a)–(d), we obtained closed (or nearly closed) forms for the autocovariance functions associated with the underlying observations and the indicator process, that is, $\{R_Y(\ell) : \ell \in \mathbb{Z}\}$ (with the exception of the AR(1)–Cauchy case) and $\{R_{I(y)}(\ell) : \ell \in \mathbb{Z}\}$, respectively. These derivations led to analogous results for the variance parameters associated with the underlying observations, the indicator process, and the p-quantile, i.e., σ_Y^2, $\sigma_{I(y)}^2$, and $\sigma_{\tilde{y}_p}^2$, respectively. Some calculations were approximate (but still accurate), possibly involving carefully controlled truncation of an infinite series. Moreover, in §§4.1.2, 4.3.2, and 4.4.2, we derived expressions for the covariance $\mathrm{Cov}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big]$ and the correlation $\mathrm{Corr}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big]$ between nonoverlapping batch means with batch size m that are indexed by d_1 and d_2 respectively and are computed from the underlying process $\{Y_k : k \ge 1\}$ at all lags $d_2 - d_1 \ne 0$ between successive batches. In particular, to the best of our knowledge, the integral formula given in Equation (51) for $\mathrm{Corr}\big[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\big]$ for the M/M/1 is new.

Our calculations exemplified several results from Dingeç et al. (2024a,b), namely: under the GMC condition, the autocovariance functions $\{R_Y(\ell) : \ell \in \mathbb{Z}\}$ and $\{R_{I(y)}(\ell) : \ell \in \mathbb{Z}\}$ both decrease exponentially in ℓ, the three variance parameters are all well-defined and finite, and the correlation of adjacent batch means decreases in order $O(1/m)$. Based on the magnitude of the corresponding variance parameters, the experimental results showed that estimation of the marginal steady-state quantiles is highly dependent on p and can be (much) harder than estimation of the mean (cf. Tafazzoli et al. 2011a,b, Tafazzoli and Wilson 2011). For the AR(1) and ARTOP test processes (a)–(c) specified in the previous paragraph, as $p \to 1-$ the estimation of the p-quantile y_p became progressively harder as measured by the Bayley–Hammersley dependent sampling factor. This behavior is not solely attributable to the variability of the associated indicator process $\{I_k(y_p)\}$ because the BH factor $h_{I(y_p)} = \sigma_{I(y_p)}^2 / [p(1-p)]$ decreases as $p \to 1-$ for both the AR(1) and ARTOP processes but increases for the M/M/1 waiting-time process. Clearly, the "difficulty" of the quantile estimation problem depends on the ratio $\sigma^2_{I(y_p)}/f^2(y_p)$, whose behavior as a function of p is problem dependent.

A promising direction for future research on quantile estimation is the formulation of new estimators of the variance parameter $\sigma^2_{\tilde{y}_p}$ based on the method of standardized time series (STS) such that those STS-based variance estimators can be suitably combined to yield asymptotically exact rectangular confidence-region (CR) estimators for a given vector of marginal quantiles $(y_{p_1}, \ldots, y_{p_c})$ as an appropriate precision requirement on that CR tends to zero. [← There are really TWO different problems alluded to in the previous, extremely long sentence.] Other research avenues will encompass more-sophisticated analyses, e.g., calculating the expected value and variance of nonoverlapping batch means and STS area variance parameter estimators in quantile estimation problems (Dingeç et al. 2024c); establishing GMC conditions that completely obviate the need for mixing conditions in estimation problems; robustness analyses when the GMC fails to hold; and use of the GMC condition in fixed-sample-size and sequential CI procedures.

Comments on References

- Not necessary to supply THREE different schools for our tech reports. Also need to update dates, titles, etc. (if necessary).
- URL not necessary on Dingec (2022) Pierre paper.
- Update Lolos (2023a) WSC paper, which has now appeared.
- Update Lolos (2023b) JoS paper, which has now appeared.

Acknowledgments

The authors gratefully thank our colleague Anup Mokashi of the Memorial Sloan Kettering Cancer Center for his enlightening comments and suggestions.

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Online Supplement

This Online Supplement contains detailed proofs of new theoretical results, complete specifications of new computational methods, and illustrative ancillary results. In §S1 we prove Lemma 1, which is stated in §4.1.1 and applied in §4.1.3 and §4.3.2.

In §§S1–S2 we verify (under appropriate conditions) that for each given pair of mathematical operations to be performed on a given sequence or series of functions on $\mathbb{R}^d$ for $d \geq 1$, the result is the same when we interchange ("swap") the order of performing those operations. For example, given a sequence $\{f_k(x) : k \geq 1\}$ of Lebesgue-integrable functions on $\mathbb{R}^d$ that converges almost everywhere to a limit function $f(x)$, often we need to verify that the Lebesgue integral of $f(x)$ equals the limiting value of the corresponding sequence $\left\{\int_{\mathbb{R}^d} f_k(x)\,\mathrm{d}x : k \geq 1\right\}$ of termwise Lebesgue integrals derived from the given function sequence $\{f_k(x) : k \geq 1\}$,

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x \equiv \int_{\mathbb{R}^d} \Big[\lim_{k\to\infty} f_k(x)\Big]\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^d} f_k(x)\,\mathrm{d}x. \tag{S.1}$$

In particular, Lebesgue's Dominated Convergence Theorem (LDCT) (Jones 2001, p. 136; Royden and Fitzpatrick 2010, p. 376) provides conditions sufficient to justify interchanging the limit and integration operations in Equation (S.1). More-complex swaps are described and justified in §§S1–S2.

In §S3 we derive computationally efficient expressions for $R_{I(y)}(\ell)$ and $\sigma^2_{I(y)}$ in the AR(1) process. In §S4 we provide a full explanation of the method used to compute $\sigma^2_{I(y)}$ for the AR(1) process. Finally, in §§S5–S6 we detail some technicalities that arise in our work on the AR(1) process.

S1. Proof of Lemma 1 for $\mathrm{Cov}[g(Y_0), g(Y_\ell)]$

Lemma 1 in §4.1.1 gives a succinct expression for $\mathrm{Cov}[g(Y_0), g(Y_\ell)]$, where: $\{Y_k : k \geq 0\}$ is an AR(1) process with lag-1 correlation $\beta \in (-1, 1)$; and $g(\cdot)$ is a real function with $\mathrm{E}\big[g^2(Y_0)\big] < \infty$. We follow an approach similar to that of Carpena et al. (2020, Eqns. (10)–(17), p. 5), which yields an expansion of the correlation function of transformed normal random variables. In the sequel, $\phi_2(x, y; \rho)$ denotes the bivariate standard normal density with correlation $\rho \in (-1, 1)$. The proof of Lemma 1 involves three steps.

<u>Step 1</u>: We formulate an expansion of the following quantity in terms of Hermite polynomials:

$$J(\rho) \equiv \int_{\mathbb{R}}\int_{\mathbb{R}} g(x)g(y)\phi_2(x, y; \rho)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} g(x)g(y)\left[\phi(x)\phi(y)\sum_{j=0}^{\infty}\frac{\rho^j}{j!}\mathrm{He}_j(x)\mathrm{He}_j(y)\right]\mathrm{d}x\,\mathrm{d}y, \tag{S.2}$$

which follows from Mehler's (1866) Hermite polynomial formula (cf. Cramér 1946, Eqn. (12.6.8), p. 133),

$$\sum_{j=0}^{\infty}\frac{\rho^j}{j!}\mathrm{He}_j(x)\mathrm{He}_j(y) = \frac{1}{\sqrt{1-\rho^2}}\exp\left[-\frac{\rho^2(x^2+y^2) - 2\rho xy}{2(1-\rho^2)}\right] \text{ for all } (x, y) \in \mathbb{R}^2.$$

<u>Step 2</u>: We use Equations (15)–(17) and (S.2) to show that

$$\sum_{j=0}^{\infty}\left\{\int_{\mathbb{R}}\int_{\mathbb{R}} \big|g(x)g(y)\big|\,\phi(x)\phi(y)\,\frac{|\rho|^j}{j!}\,\big|\mathrm{He}_j(x)\mathrm{He}_j(y)\big|\,\mathrm{d}x\,\mathrm{d}y\right\} < \infty.$$

We have

$$\int_{\mathbb{R}} \left|g(x)\right| \phi(x) \left|\mathrm{He}_j(x)\right| \mathrm{d}x = \int_{\mathbb{R}} \left|g(x)\right| [\phi(x)]^{1/2} \left|\mathrm{He}_j(x)\right| [\phi(x)]^{1/2} \, \mathrm{d}x \tag{S.3}$$

$$\leq \left[\int_{\mathbb{R}} g^2(x)\phi(x)\,\mathrm{d}x \right]^{1/2} \left[\int_{\mathbb{R}} \mathrm{He}_j^2(x)\phi(x)\,\mathrm{d}x \right]^{1/2} = \left\{ \mathrm{E}\left[g^2(Y_0) \right] j! \right\}^{1/2} < \infty \quad \text{for} \quad j \geq 0, \tag{S.4}$$

where Equation (S.3), the Cauchy-Schwarz inequality, and Equation (17) imply Equation (S.4) so that the function $g(x)\phi(x)\mathrm{He}_j(x)$ is Lebesgue integrable for $j \geq 0$. Tonelli's Theorem for Nonnegative Functions on $\mathbb{R}^n$ (Jones 2001, pp. 183–184; Royden and Fitzpatrick 2010, pp. 420–421) and Equation (S.4) imply that

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}} \left|g(x)g(y)\right| \phi(x)\phi(y) \, \frac{|\rho|^j}{j!} \left|\mathrm{He}_j(x)\mathrm{He}_j(y)\right| \mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \left(\frac{|\rho|^j}{j!} \left|g(x)\right| \phi(x) \left|\mathrm{He}_j(x)\right| \right) \left(\left|g(y)\right| \phi(y) \left|\mathrm{He}_j(y)\right| \right) \mathrm{d}y \right] \mathrm{d}x \\
&= \int_{\mathbb{R}} \frac{|\rho|^j}{j!} \left|g(x)\right| \phi(x) \left|\mathrm{He}_j(x)\right| \left[\int_{\mathbb{R}} \left|g(y)\right| \phi(y) \left|\mathrm{He}_j(y)\right| \mathrm{d}y \right] \mathrm{d}x \\
&= \frac{|\rho|^j}{j!} \left[\int_{\mathbb{R}} \left|g(y)\right| \phi(y) \left|\mathrm{He}_j(y)\right| \mathrm{d}y \right] \left[\int_{\mathbb{R}} \left|g(x)\right| \phi(x) \left|\mathrm{He}_j(x)\right| \mathrm{d}x \right] \\
&= \frac{|\rho|^j}{j!} \left[\int_{\mathbb{R}} \left|g(y)\right| \phi(y) \left|\mathrm{He}_j(y)\right| \mathrm{d}y \right]^2 \leq |\rho|^j \mathrm{E}\left[g^2(Y_0) \right] < \infty \quad \text{for} \quad j \geq 0;
\end{aligned} \tag{S.5}$$

thus the function $g(x)g(y)\phi(x)\phi(y)\left(\rho^j / j!\right)\mathrm{He}_j(x)\mathrm{He}_j(y)$ is Lebesgue integrable on $\mathbb{R}^2$ for every $j \geq 0$.

Step 3: Finally to derive Equations (18) and (19), we justify (i) interchanging a bivariate Lebesgue-integration operation performed on $\mathbb{R}^2$ and a summation operation performed on all $j \geq 0$, and (ii) expressing each bivariate integral in (i) as an iterated (interchanged) integral. Equation (S.5) ensures that

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \left| g(x)g(y)\phi(x)\phi(y)\left(\rho^j / j!\right)\mathrm{He}_j(x)\mathrm{He}_j(y) \right| \mathrm{d}x\,\mathrm{d}y \leq \mathrm{E}\left[g^2(Y_0) \right] / \left(1 - |\rho| \right) < \infty; \tag{S.6}$$

and thus we have

$$J(\rho) = \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \int_{\mathbb{R}} \int_{\mathbb{R}} g(x)g(y)\phi(x)\phi(y)\mathrm{He}_j(x)\mathrm{He}_j(y)\,\mathrm{d}x\,\mathrm{d}y \tag{S.7}$$

$$= \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \int_{\mathbb{R}} g(x)\mathrm{He}_j(x)\phi(x) \left[\int_{\mathbb{R}} g(y)\mathrm{He}_j(y)\phi(y)\,\mathrm{d}y \right] \mathrm{d}x \tag{S.8}$$

$$\begin{aligned}
&= \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \left[\int_{\mathbb{R}} g(y)\mathrm{He}_j(y)\phi(y)\,\mathrm{d}y \right] \left[\int_{\mathbb{R}} g(x)\mathrm{He}_j(x)\phi(x)\,\mathrm{d}x \right] \\
&= \sum_{j=0} \frac{\rho^j}{j!} \left[\int_{\mathbb{R}} g(y)\mathrm{He}_j(y)\phi(y)\,\mathrm{d}y \right]^2 \\
&= \sum_{j=0}^{\infty} a_j \rho^j,
\end{aligned}$$

where Equation (S.7) follows from Equation (S.6) by interchanging Lebesgue integration over $\mathbb{R}^2$ of the function $g(x)g(y)\phi(x)\phi(y)\big(\rho^j/j!\big)\mathrm{He}_j(x)\mathrm{He}_j(y)$ and summation over all $j \geq 0$ based on Levi's Theorem for Integrable Functions on $\mathbb{R}^n$ (Jones 2001, bot. of p. 136–top of p. 137; Apostol 1974, Theor. 10.26, p. 269) Then Equation (S.8) follows by expressing the required integration over $\mathbb{R}^2$ in Equation (S.7) as an iterated (interchanged) integral based on Fubini's Theorem for Integrable Functions on $\mathbb{R}^n$ (Jones 2001, p. 189). Since $\mathrm{Cov}[g(Y_0), g(Y_\ell)] = J(\beta^\ell) - J(0) = J(\beta^\ell) - a_0$, we have

$$\big|\mathrm{Cov}[g(Y_0), g(Y_\ell)]\big| \leq \sum_{j=1}^{\infty}\big|a_j\beta^{j\ell}\big| \leq \mathrm{E}\big[g^2(Y_0)\big]\sum_{j=1}^{\infty}|\beta|^{j\ell} = \mathrm{E}\big[g^2(Y_0)\big]\,\frac{|\beta|^\ell}{1-|\beta|^\ell} = O(|\beta|^\ell). \quad \square$$

S2. Mathematical Details Involving Various Swaps

This section provides miscellaneous mathematical details—often involving the formal swaps of various combinations of infinite summations, integrals, derivatives, and limits—that might be considered as tangential to the flow of the main text.

S2.1. Proof of Equation (25)

We use induction to show that for each $j \geq 1$, there exist coefficients $\{c_{j-1,k} : k = 0, 1, \ldots, j-1\}$ such that

$$\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\phi(z) = \sum_{k=0}^{j-1} c_{j-1,k}\, z^k \phi(z) \quad \text{for all} \quad z \in \mathbb{R} \quad \text{and} \quad j \geq 1. \tag{S.9}$$

By the definition of an improper integral, we have

$$\begin{aligned}
\int_{-\infty}^{y}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\phi(z)\,\mathrm{d}z &\equiv \lim_{\mathscr{L}\to-\infty}\int_{\mathscr{L}}^{y}\frac{\mathrm{d}^j}{\mathrm{d}z^j}\phi(z)\,\mathrm{d}z && \text{(S.10)}\\
&= \lim_{\mathscr{L}\to-\infty}\left[\frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\phi(y) - \frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\phi(\mathscr{L})\right] && \text{(S.11)}\\
&= \frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\phi(y) - \sum_{k=0}^{j-1}\Big[\lim_{\mathscr{L}\to-\infty} c_{j-1,k}\mathscr{L}^k\phi(\mathscr{L})\Big] && \text{(S.12)}\\
&= \frac{\mathrm{d}^{j-1}}{\mathrm{d}z^{j-1}}\phi(y) \quad \text{for all} \quad y \in \mathbb{R} \quad \text{and} \quad j \geq 1, && \text{(S.13)}
\end{aligned}$$

where Equation (S.11) follows immediately by the Second Fundamental Theorem of Calculus (Apostol 2006, Theorem 5.3, p. 205); and Equation (S.12) follows from Equations (S.9) and (S.11). Finally, Equation (S.13) follows from the result $\lim_{|z|\to\infty} z^k\phi(z) = 0$ for all $z \in \mathbb{R}$ and $k \geq 0$ (Apostol 2006, Theor. 7.11, p. 301). $\square$

S2.2. Proof of Equation (27)

We show that the iterated series $\sum_{\ell=1}^{\infty}\big(\sum_{j=1}^{\infty} a_j\beta^{j\ell}\big)$ is absolutely convergent. To do so, we have

$$\sum_{\ell=1}^{\infty}\left(\sum_{j=1}^{\infty}\big|a_j\beta^{j\ell}\big|\right) = \sum_{\ell=1}^{\infty}\left[\sum_{j=1}^{\infty} a_j\big(|\beta|^\ell\big)^j\right] \tag{S.14}$$

$$\le \mathrm{E}\big[g^2(Y_0)\big] \sum_{\ell=1}^{\infty} |\beta|^{\ell} \left[\sum_{j=1}^{\infty} \left(|\beta|^{\ell}\right)^{j-1} \right] \quad \text{(by Eq. (19))} \tag{S.15}$$

$$= \mathrm{E}\big[g^2(Y_0)\big] \sum_{\ell=1}^{\infty} \frac{|\beta|^{\ell}}{1-|\beta|^{\ell}} \tag{S.16}$$

$$\le \frac{\mathrm{E}\big[g^2(Y_0)\big]}{1-|\beta|} \sum_{\ell=1}^{\infty} |\beta|^{\ell} \quad (\text{since } \beta \in (-1,1)) \tag{S.17}$$

$$= \frac{\mathrm{E}[g^2(Y_0)]|\beta|}{(1-|\beta|)^2} < \infty. \tag{S.18}$$

Equation (4) and Equations (S.14)–(S.18) together with Theorem 8.43 of Apostol (1974, pp. 202–203) ensure that Equation (27) holds. □

S2.3. Proof of Equation (28)

To justify interchanging the order of differentiation and integration on the RHS of Equation (28), we apply Theorem 7.40 of Apostol (1974, p. 167). To verify the hypotheses of that theorem, we pick $p \in (0,1)$ and $\beta \in (-1,1)$ arbitrarily. For simplicity, here we assume that $\beta \in [0,1)$; a similar argument applies when $\beta \in (-1,0)$. Let

$$\delta_p \equiv \min\{p, 1-p\}/2. \tag{S.19}$$

On the following rectangular subregions of $\mathbb{R}^2$,

$$\mathscr{G}_p(\ell) \equiv \left[0, \beta^{\ell}\right] \times \left[p-\delta_p, p+\delta_p\right] \ \text{ for } \ \ell \ge 1, \tag{S.20}$$

we define that the corresponding bivariate functions

$$\theta(r,q;\ell) \equiv \frac{\exp\left[-y_q^2/(1+r)\right]}{\sqrt{1-r^2}} \ \text{ for all } \ (r,q) \in \mathscr{G}_p(\ell) \ \text{ and } \ \ell \ge 1;$$

and $\theta(r,q;\ell)$ is bounded and continuous on $\mathscr{G}_p(\ell)$ because $\beta^{\ell} \in [0,1)$ for $\ell \ge 1$ and $\left[p-\delta_p, p+\delta_p\right] \subset (0,1)$ so that the Riemann (and Lebesgue) integral

$$\Theta_q(\ell) \equiv \int_0^{\beta^{\ell}} \theta(r,q;\ell)\,\mathrm{d}r \ \text{ exists for each } \ q \in \left[p-\delta_p, p+\delta_p\right] \ \text{ and } \ \ell \ge 1.$$

Moreover, the partial derivative

$$\frac{\partial}{\partial q}\theta(r,q;\ell) \ = \ -\frac{2\theta(r,q;\ell)y_q}{\phi(y_q)(1+r)}$$

is continuous on $\mathscr{G}_p(\ell)$ for $\ell \ge 1$; and thus Theorem 7.40 of Apostol (1974, , p. 167) ensures that

$$\frac{\mathrm{d}}{\mathrm{d}q}\Theta_q(\ell) = \int_0^{\beta^{\ell}} \frac{\partial}{\partial q}\theta(r,q;\ell)\,\mathrm{d}r \ \text{ for each } \ q \in \left[p-\delta_p, p+\delta_p\right] \ \text{ and } \ \ell \ge 1. \tag{S.21}$$

Since $p \in (0,1)$ was chosen arbitrarily, Equation (S.21) implies that Equations (28) and (29) hold for every $p \in (0,1)$ and $\ell \ge 1$. □

S2.4. Proof of Equation (31)

To justify Equation (31), we apply Theorem 9.14 of Apostol (1974, p. 230). As in Remark 5, we pick $p \in (0,1)$ and $\beta \in (-1,1)$ arbitrarily. For simplicity, here we assume that $\beta \in [0,1)$; a similar argument applies when $\beta \in (-1,0)$. Here δ_p is again defined by Equation (S.19). On the following rectangular subregion of $\mathbb{R}^2$,

$$\mathscr{H}_p \equiv [0,\beta] \times \left[p - \delta_p,\; p + \delta_p\right],$$

we define the function

$$\omega_p(r,q) \equiv \left| -\frac{1}{\pi}\left[\frac{y_q}{\phi(y_q)\sqrt{1-r^2}\,(1+r)}\exp\left(\frac{-y_q^2}{1+r}\right)\right]\right| \quad \text{for } (r,q) \in \mathscr{H}_p,$$

which is continuous on $\mathscr{H}_p$. Moreover, because $\mathscr{H}_p$ is closed and bounded in $\mathbb{R}^2$, the function $\omega_p(r,q)$ attains a finite maximum value on $\mathscr{H}_p$ (Dugundji 1966, Theorem 4.3, p. 233 and Theorem 2.3, p. 227) so that we have

$$\omega_p^* \equiv \sup\left\{\omega_p(r,q) : (r,q) \in \mathscr{H}_p\right\} = \omega_p(r^*,q^*) < \infty \;\text{ for some } (r^*,q^*) \in \mathscr{H}_p. \tag{S.22}$$

It follows immediately from Equations (29) and (S.22) that we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(\ell)\right| \le \int_0^{\beta^\ell} \omega_p^*\,\mathrm{d}r = \omega_p^*\beta^\ell \;\text{ for all } \ell \ge 1 \tag{S.23}$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(0)\right| = \left|\frac{\mathrm{d}}{\mathrm{d}q}\Phi(y_q)[1-\Phi(y_q)]\right| = \left|\frac{\mathrm{d}}{\mathrm{d}q}(q-q^2)\right| = |1-2q| \le 1. \tag{S.24}$$

Equations (S.23) and (S.24) imply that for $\mathfrak{L} \ge 1$,

$$\left|2\sum_{\ell \ge \mathfrak{L}} \frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(\ell)\right| \le 2\sum_{\ell \ge \mathfrak{L}}\left|\frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(\ell)\right| \le 2\omega_p^* \sum_{\ell \ge \mathfrak{L}} \beta^\ell = \frac{2\omega_p^*\beta^{\mathfrak{L}}}{1-\beta}. \tag{S.25}$$

Pick $\zeta > 0$ arbitrarily. From Equations (S.24)–(S.25), we see that

$$\mathfrak{L} = \left\lceil \log\left[\zeta(1-\beta)\big/\left(2\omega_p^*\right)\right] \big/ \log(\beta)\right\rceil \quad \text{implies} \quad \left|2\sum_{\ell \ge \mathfrak{L}} \frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(\ell)\right| < \zeta \;\text{ for all } q \in \mathscr{H}_p; \tag{S.26}$$

and thus by the Cauchy Criterion for proving uniform convergence (Apostol 1974, Theor. 9.5, p. 223), we see that the series

$$\sum_{\ell \in \mathbb{Z}} \frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(\ell) = \frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(0) + 2\sum_{\ell \ge 1} \frac{\mathrm{d}}{\mathrm{d}q} R_{I(y_q)}(\ell) \;\text{ is uniformly convergent on } \left[p - \delta_p\; p + \delta_p\right]. \tag{S.27}$$

Moreover, Equation (27) ensures that the series

$$\sum_{\ell \in \mathbb{Z}} R_{I(y_q)}(\ell) \;\text{ is convergent for every } q \in \left[p - \delta_p,\; p + \delta_p\right]. \tag{S.28}$$

Since p was selected arbitrarily in $(0,1)$, Equations (S.27) and (S.28) show that the respective assumptions of Theorem 9.14 of Apostol (1974, p. 230) are satisfied; and thus Equation (31) holds. □

S3. Alternative Expressions for $R_{I(y)}(\ell)$ and $\sigma^2_{I(y)}$ for the AR(1) Process

We provide several additional expressions that can be used to calculate the covariance function $R_{I(y)}(\ell)$ for the AR(1) process, which can then be used to calculate the corresponding variance parameter $\sigma^2_{I(y)} = R_{I(y)}(0) + 2\sum_{\ell=1}^{\infty} R_{I(y)}(\ell)$. §S3.1 applies a simple conditioning argument to obtain an exact integral expression for $R_{I(y)}(\ell)$ (Equation (S.30)), after which we derive an asymptotic expression as $\ell \to \infty$ (Equation (S.31)). §S3.2 gives another expression for $R_{I(y)}(\ell)$ (Equation (S.41)) based on Owen's T function. §S3.3 uses a result from Drezner and Wesolowsky (1990) to obtain still another expression for $R_{I(y)}(\ell)$ (Equation (S.43)), that we have used to check our previous calculations.

S3.1. A Simple Conditioning Argument

By stationarity and the fact that $\mathrm{E}[\mathbf{1}_{\mathscr{E}}] = \Pr(\mathscr{E})$ for any event $\mathscr{E}$, we have that for any $y \in \mathbb{R}$ and $\ell \geq 0$,

$$\begin{aligned}
R_{I(y)}(\ell) &= \mathrm{E}[I_0(y)I_\ell(y)] - \mathrm{E}[I_0(y)]\mathrm{E}[I_\ell(y)] \\
&= \Pr(Y_0 \leq y, Y_\ell \leq y) - [\Pr(Y_0 \leq y)]^2 \quad (= \Phi_{2,\ell}(y,y) - \Phi^2(y)) && \text{(S.29)} \\
&= \Pr\left(Y_0 \leq y, \beta^\ell Y_0 + W_\ell \leq y\right) - \Phi^2(y) \quad (W_\ell \stackrel{\text{d}}{=} \mathrm{Nor}\left(0, 1-\beta^{2\ell}\right) \text{ is defined in §4.1.1)} \\
&= \int_{\mathbb{R}} \Pr\left(Y_0 \leq y, \beta^\ell Y_0 + W_\ell \leq y \,|\, Y_0 = z\right)\phi(z)\,\mathrm{d}z - \Phi^2(y) \\
&= \int_{-\infty}^{y} \Pr(W_\ell \leq y - \beta^\ell z)\phi(z)\,\mathrm{d}z - \Phi^2(y) \\
&= \int_{-\infty}^{y} \left[\Phi\left(\frac{y-\beta^\ell z}{\sqrt{1-\beta^{2\ell}}}\right) - \Phi(y)\right]\phi(z)\,\mathrm{d}z \quad \text{for all } y \in \mathbb{R} \text{ and } \ell \geq 1. \quad \square && \text{(S.30)}
\end{aligned}$$

Having the exact integral expression (S.30) for $R_{I(y)}(\ell)$ in hand, we now seek to prove that

$$\frac{R_{I(y)}(\ell)}{\phi^2(y)\beta^\ell} \to 1, \quad \text{as } \ell \to \infty. \tag{S.31}$$

Towards the goal of proving Equation (S.31), we first invoke the generalized mean-value theorem (Rudin 1976, Theorem 5.10), which ensures that for each $\ell \geq 1$, there is a point $\mathfrak{w}_\ell \equiv \mathfrak{w}_\ell(y,z) \in (0,1)$ for which

$$\Phi\left(\frac{y-\beta^\ell z}{\sqrt{1-\beta^{2\ell}}}\right) - \Phi(y) = \phi\left(\mathfrak{w}_\ell \frac{y-\beta^\ell z}{\sqrt{1-\beta^{2\ell}}} + (1-\mathfrak{w}_\ell)y\right)\left(\frac{y-\beta^\ell z}{\sqrt{1-\beta^{2\ell}}} - y\right) \tag{S.32}$$

since $\phi(\zeta) \leq 1$ for every $\zeta \in \mathbb{R}$. We now define the following terms for $\ell \geq 1$:

$$\left.\begin{aligned}
\mathfrak{z}_\ell(y,z) &\equiv \mathfrak{w}_\ell(y,z)\left(\frac{y-\beta^\ell z}{\sqrt{1-\beta^{2\ell}}}\right) + [1-\mathfrak{w}_\ell(y,z)]y, \\
\mathscr{S}_\ell(y) &\equiv \int_{-\infty}^{y} \phi[\mathfrak{z}_\ell(y,z)]y\left(\frac{1-\sqrt{1-\beta^{2\ell}}}{\sqrt{1-\beta^{2\ell}}}\right)\phi(z)\,\mathrm{d}z, \\
\mathscr{T}_\ell(y) &\equiv -\int_{-\infty}^{y} \phi[\mathfrak{z}_\ell(y,z)]z\left(\frac{\beta^\ell}{\sqrt{1-\beta^{2\ell}}}\right)\phi(z)\,\mathrm{d}z.
\end{aligned}\right\} \tag{S.33}$$

Note that Equations (S.30), (S.32), and (S.33) imply that

$$R_{I(y)}(\ell) = \mathscr{S}_\ell(y) + \mathscr{T}_\ell(y), \quad \text{for } \ell \geq 1. \tag{S.34}$$

At this point we will prove the key results

$$\mathscr{S}_\ell(y) \sim \tfrac{1}{2} y\phi(y)\Phi(y)\beta^{2\ell} \quad \text{and} \quad \mathscr{T}_\ell(y) \sim \phi^2(y)\beta^\ell, \quad \text{as } \ell \to \infty.$$

Since $\mathfrak{w}_\ell \in (0,1)$ for each $\ell \geq 1$, we have

$$\liminf_{\ell\to\infty} \mathfrak{w}_\ell \geq 0 \quad \text{and} \quad \limsup_{\ell\to\infty} \mathfrak{w}_\ell \leq 1;$$

and it follows that

$$\liminf_{\ell\to\infty} \left(\frac{1-\sqrt{1-\beta^{2\ell}}}{\sqrt{1-\beta^{2\ell}}} \right) \mathfrak{w}_\ell = \left(\lim_{\ell\to\infty} \frac{1-\sqrt{1-\beta^{2\ell}}}{\sqrt{1-\beta^{2\ell}}} \right) \liminf_{\ell\to\infty} \mathfrak{w}_\ell = 0, \tag{S.35}$$

$$\limsup_{\ell\to\infty} \left(\frac{1-\sqrt{1-\beta^{2\ell}}}{\sqrt{1-\beta^{2\ell}}} \right) \mathfrak{w}_\ell = \left(\lim_{\ell\to\infty} \frac{1-\sqrt{1-\beta^{2\ell}}}{\sqrt{1-\beta^{2\ell}}} \right) \limsup_{\ell\to\infty} \mathfrak{w}_\ell = 0. \tag{S.36}$$

Equations (S.35) and (S.36) ensure that

$$\lim_{\ell\to\infty} \left(\frac{1-\sqrt{1-\beta^{2\ell}}}{\sqrt{1-\beta^{2\ell}}} \right) \mathfrak{w}_\ell = 0;$$

and by a similar analysis we have

$$\lim_{\ell\to\infty} \left(\frac{\beta^\ell z}{\sqrt{1-\beta^{2\ell}}} \right) \mathfrak{w}_\ell = 0,$$

so that by the definition (S.33) of $\mathfrak{z}_\ell(y,z)$, we have

$$\lim_{\ell\to\infty} \mathfrak{z}_\ell(y,z) = y \quad \text{and} \quad \lim_{\ell\to\infty} \phi[\mathfrak{z}_\ell(y,z)] = \phi(y), \quad \text{for all } y, z \in \mathbb{R}. \tag{S.37}$$

Equations (S.33) and (S.37) imply that

$$\begin{aligned} \lim_{\ell\to\infty} \frac{\mathscr{T}_\ell(y)}{\beta^\ell} &= -\int_{-\infty}^{y} \left\{ \lim_{\ell\to\infty} \phi[\mathfrak{z}_\ell(y,z)] \frac{z}{\sqrt{1-\beta^{2\ell}}} \right\} \phi(z)\,\mathrm{d}z \\ &= -\phi(y) \int_{-\infty}^{y} z\phi(z)\,\mathrm{d}z \\ &= \phi(y) \int_{-\infty}^{y} \frac{d}{dz}\phi(z)\,\mathrm{d}z \\ &= \phi^2(y). \end{aligned} \tag{S.38}$$

Similarly, Equations (S.33) and (S.37) imply that

$$\begin{aligned} \lim_{\ell\to\infty} \frac{\mathscr{S}_\ell(y)}{\beta^{2\ell}} &= y \int_{-\infty}^{y} \left\{ \lim_{\ell\to\infty} \phi[\mathfrak{z}_\ell(y,z)] \left(\frac{1-\sqrt{1-\beta^{2\ell}}}{\beta^{2\ell}\sqrt{1-\beta^{2\ell}}} \right) \right\} \phi(z)\,\mathrm{d}z \\ &= y \int_{-\infty}^{y} \left\{ \lim_{\ell\to\infty} \phi[\mathfrak{z}_\ell(y,z)] \left(\frac{\frac{1}{2} + \frac{\beta^{2\ell}}{8} + \frac{\beta^{4\ell}}{16} + \cdots}{\sqrt{1-\beta^{2\ell}}} \right) \right\} \phi(z)\,\mathrm{d}z \quad \text{(Maclaurin series for } \sqrt{1-x}\text{)} \\ &= \frac{y\phi(y)\Phi(y)}{2}. \end{aligned} \tag{S.39}$$

Equation (S.31) immediately follows from Equations (S.34), (S.38), and (S.39). □

S3.2. Using Owen's T Function

Another way to calculate $R_{I(y)}(\ell)$ arises from Equation (3.13) on p. 2407 of Meyer (2013) with $\varrho = \mathrm{Corr}[Y_0, Y_\ell] = \beta^\ell$, which reveals that

$$\Pr(Y_0 \le y,\, Y_\ell \le y) \;=\; \Phi(y) - 2\,\mathcal{T}\!\left(y\,,\, \sqrt{\frac{1-\beta^\ell}{1+\beta^\ell}}\right),$$

where

$$\mathcal{T}(h,a) \;\equiv\; \frac{1}{2\pi}\int_0^a \frac{e^{-\frac{1}{2}h^2(1+x^2)}}{1+x^2}\,\mathrm{d}x, \quad \text{for } h, a \in \mathbb{R}, \tag{S.40}$$

is Owen's T function. Therefore, by Equation (S.29), we have

$$R_{I(y)}(\ell) \;=\; \Phi(y)(1-\Phi(y)) - 2\,\mathcal{T}\!\left(y\,,\, \sqrt{\frac{1-\beta^\ell}{1+\beta^\ell}}\right), \quad \text{for } \ell \ge 1. \tag{S.41}$$

Notably, for $p = 1/2$ (whence $y_p = 0$ for the AR(1) process), we have a remarkably simpler formula, $R_{I(y_{1/2})}(\ell) \;=\; R_{I(0)}(\ell) \;=\; \frac{1}{2\pi}\arcsin\!\left(\beta^\ell\right), \quad \text{for } \ell \ge 1.$ $\square$

S3.3. Using Drezner and Wesolowsky (1990)

Equation (6) of Drezner and Wesolowsky (1990) gives a formula for the probability $\Pr(Z_1 \ge h, Z_2 \ge k)$ for a bivariate standard normal vector (Z_1, Z_2) having correlation $\rho \in (-1,1)$. By using the fact that $\Pr(Z_1 \ge h, Z_2 \ge k) = \Pr(Z_1 \le -h, Z_2 \le -k)$ and setting $h = k = -y$ and $\rho = \beta^\ell$ for $\ell \ge 1$ in their Equation (6), we obtain

$$\Pr(Y_0 \le y,\, Y_\ell \le y) = \left\{\begin{array}{ll} \Phi(y) & \text{for } \ \ell = 0, \\ \Phi^2(y) \;+\; \dfrac{1}{2\pi}\displaystyle\int_0^{\beta^\ell} \frac{\exp\left[-y^2/(1+r)\right]}{\sqrt{1-r^2}}\,\mathrm{d}r & \text{for } \ell \ge 1 \end{array}\right\} \tag{S.42}$$

so that, by Equation (S.29), we have

$$R_{I(y)}(\ell) = \left\{\begin{array}{ll} \Phi(y)[1-\Phi(y)] & \text{for } \ \ell = 0, \\ \dfrac{1}{2\pi}\displaystyle\int_0^{\beta^\ell} \frac{\exp\left[-y^2/(1+r)\right]}{\sqrt{1-r^2}}\,\mathrm{d}r & \text{for } \ell \ge 1. \end{array}\right\} \tag{S.43}$$

Thus, the corresponding variance parameter is given by

$$\sigma^2_{I(y)} \;=\; R_{I(y)}(0) + 2\sum_{\ell=1}^{\infty} R_{I(y)}(\ell) \;=\; \Phi(y)[1-\Phi(y)] \;+\; \frac{1}{\pi}\sum_{\ell=1}^{\infty}\int_0^{\beta^\ell} \frac{\exp\left[-y^2/(1+r)\right]}{\sqrt{1-r^2}}\,\mathrm{d}r, \tag{S.44}$$

which is a convergent series of Lebesgue integrals of nonnegative functions.

Define

$$A_i(y) \;\equiv\; \frac{(2i-2)!}{(2i-1)}\left[\sum_{k=0}^{i-1} \frac{(-1)^k y^{2k}}{2^{i-k-1}(2k)!(i-k-1)!}\right]^2 \quad \text{for } i \ge 1$$

and

$$B_i(y) \;\equiv\; \frac{y^2(2i-1)!}{2i}\left[\sum_{k=0}^{i-1} \frac{(-1)^k y^{2k}}{2^{i-k-1}(2k+1)!(i-k-1)!}\right]^2 \quad \text{for } i \ge 1.$$

[In another paper because this is already too long,] we show that the variance parameter can be written as

$$\sigma^2_{I(y)} = \Phi(y)[1-\Phi(y)] + \frac{e^{-y^2}}{\pi}\sum_{i=1}^{\infty}\left\{\frac{A_i(y)\,\beta^{2i-1}}{1-\beta^{2i-1}} + \frac{B_i(y)\,\beta^{2i}}{1-\beta^{2i}}\right\}. \tag{S.45}$$

The last formula holds for all $\beta \in (-1,1)$. Notice that $\sigma^2_{I(y)}$ is an increasing function of $\beta \in (0,1)$. Furthermore, for $y=0$, we obtain a simpler expression,

$$\sigma^2_{I(0)} = \frac{1}{4} + \frac{1}{\pi}\sum_{i=1}^{\infty}\frac{(2i-2)!}{2^{2i-2}(2i-1)[(i-1)!]^2}\left(\frac{\beta^{2i-1}}{1-\beta^{2i-1}}\right). \tag{S.46}$$

These expressions can be used to check previous work.

S4. Calculation Details for $\sigma^2_{I(y)}$ for the AR(1) Process

In order to calculate $\sigma^2_{I(y)}$ for the AR(1) process, we use truncation of the infinite sum in Equation (4), where the individual $R_{I(y)}(\ell)$ terms for $\ell \geq 1$ are based on Equations (S.30) or (S.41). For calculations that involve Owen's T function $\mathcal{T}(h,a)$ given by Equation (S.40), we used the R implementation of Azzalini (2020) as well as the built-in function `OwenT` of Mathematica.

The numerical approximation of the variance parameter's infinite sum $\sigma^2_{I(y)} = R_{I(y)}(0) + 2\sum_{\ell=1}^{\infty} R_{I(y)}(\ell)$ was typically based on the truncation point of $N = 10{,}000$ terms (after informally noticing that the sum stabilizes well before that). But we are still interested in formally finding an appropriate truncation point that guarantees reasonable precision. To this end, we will now conduct a more-precise analysis on how the terms decay as ℓ becomes large, and then propose an algorithm to calculate $\sigma^2_{I(y)}$.

Recall that Equation (S.31) establishes that $R_{I(y)}(\ell) \sim \phi^2(y)\beta^{\ell}$ as $\ell \to \infty$, which forms the basis for the following proposed method to compute $\sigma^2_{I(y)}$ to an arbitrary level of relative precision (RP) $\delta_{\text{rerr}} \in (0,1)$. We seek to estimate $\sigma^2_{I(y)}$ from Equation (4) in terms of the partial sum of covariances and the associated remainder that are respectively given by

$$\mathcal{Q}_N \equiv \sum_{\ell=1}^{N} R_{I(y)}(\ell) \quad \text{and} \quad \mathcal{R}_N \equiv \sum_{\ell=N+1}^{\infty} R_{I(y)}(\ell), \quad \text{for } N \geq 1.$$

Our algorithm to approximate $\sigma^2_{I(y)}$ stops at step N if the latest approximation $\widetilde{\sigma}^2_{I(y)} \equiv R_{I(y)}(0) + 2\mathcal{Q}_N$ satisfies the RP requirement

$$\left|\frac{\sigma^2_{I(y)} - \widetilde{\sigma}^2_{I(y)}}{\sigma^2_{I(y)}}\right| = \frac{2|\mathcal{R}_N|}{R_{I(y)}(0) + 2(\mathcal{Q}_N + \mathcal{R}_N)} \leq \delta_{\text{rerr}}, \quad \text{for sufficiently large } N.$$

If N is large enough so that we have $R_{I(y)}(\ell) \doteq \phi^2(y)\beta^{\ell}$ for $\ell > N$ as well as the conditions

$$\mathcal{Q}_N \geq 0, \quad R_{I(y)}(N+1) \geq 0, \quad \text{and} \quad \beta^{N+1} > 0,$$

then we have the following approximation for $\mathscr{R}_N$:

$$\sum_{\ell=N+1}^{\infty} R_{I(y)}(\ell) \doteq \phi^2(y) \sum_{k=0}^{\infty} \beta^{N+1+k} = \frac{\phi^2(y)\beta^{N+1}}{1-\beta} \geq 0.$$

Thus, computations stop at step N if

$$\frac{\phi^2(y)\beta^{N+1}}{1-\beta} \leq \frac{\delta_{\text{rerr}}\left(\mathcal{Q}_N + \frac{1}{2}R_{I(y)}(0)\right)}{1-\delta_{\text{rerr}}}.$$

Algorithm 1 presents a formal procedural method for estimating $\sigma^2_{I(y)}$.

Algorithm 1 Computing $\sigma^2_{I(y_p)}$ for the AR(1) Process

Initialization: Set autoregressive parameter, e.g., $\beta \leftarrow 0.9$; selected quantile order, e.g., $p \leftarrow 0.95$; indicator variance $R_{I(y_p)}(0) = p(1-p)$; current lag, $N \leftarrow 0$; accumulated covariances, $\mathcal{Q}_N \leftarrow 0$; relative precision for approximating indicator variance parameter, e.g., $\delta_{\text{rerr}} \leftarrow 10^{-4}$; absolute precision for achieving asymptotic behavior of covariance function, e.g., $\gamma_{\text{aerr}} \leftarrow 5 \times 10^{-3}$; and stopping indicator $I_{\text{stop}} \leftarrow 0$.
if $\beta = 0$ **then**
    $\sigma^2_{I(y_p)} \leftarrow R_{I,0} \equiv R_{I(y_p)}(0)$
    return $\sigma^2_{I(y_p)}$
    $I_{\text{stop}} \leftarrow 1$
end if
repeat
    $N \leftarrow N+1$, compute $R_{I,N} \equiv R_{I(y_p)}(N)$ from (S.30), $\mathcal{Q}_N \leftarrow \mathcal{Q}_{N-1} + R_{I,N}$
until $\left|R_{I,N} / \left[\phi^2(y)\beta^N\right] - 1\right| \leq \gamma_{\text{aerr}}$
while $I_{\text{stop}} = 0$ **do**
    repeat
        $N \leftarrow N+1$, compute $R_{I,N}$ from (S.30), $\mathcal{Q}_N \leftarrow \mathcal{Q}_{N-1} + R_{I,N}$
    until $\mathcal{Q}_{N-1} \geq 0$, $R_{I,N} \geq 0$, and $\beta^N > 0$
    if $\left[\phi^2(y)\beta^N\right] / (1-\beta) \leq \left[\delta_{\text{rerr}}\left(\mathcal{Q}_{N-1} + \frac{1}{2}R_{I,0}\right)\right] / (1-\delta_{\text{rerr}})$ **then**
        $\sigma^2_{I(y_p)} \leftarrow R_{I,0} + 2\mathcal{Q}_{N-1}$
        return $\sigma^2_{I(y_p)}$
        $I_{\text{stop}} \leftarrow 1$
    end if
end while

S5. Proof of Remark 7

The goal is to first establish an upper bound (Expression (S.47) below) for $\sigma^2_{I(y)}$, and then to use this bound to complete the proof for the case $0 \le \beta < 1$. With Equations (S.43) and (S.44) from §S3.3 in mind, we define

$$q(r) \equiv \frac{e^{-\frac{y^2}{1+r}}}{\sqrt{1-r^2}},$$

which is an increasing function of $r \in (0,1)$. Then for $\beta \in [0,1)$,

$$\begin{aligned}
\sum_{\ell=1}^{\infty} \int_0^{\beta^\ell} q(r)\,\mathrm{d}r &= \sum_{\ell=1}^{\infty} \ell \int_{\beta^{\ell+1}}^{\beta^\ell} q(r)\,\mathrm{d}r \\
&\le \sum_{\ell=1}^{\infty} \ell(\beta^\ell - \beta^{\ell+1}) q(\beta^\ell) \\
&\le q(\beta) \sum_{\ell=1}^{\infty} \ell\beta^\ell (1-\beta) \\
&= \frac{e^{-\frac{y^2}{1+\beta}}}{\sqrt{1-\beta^2}} \left(\frac{\beta}{1-\beta}\right) \\
&= \frac{\beta e^{-\frac{y^2}{1+\beta}}}{(1-\beta)^{3/2}\sqrt{1+\beta}}.
\end{aligned}$$

Therefore an upper bound for $\sigma^2_{I(y)}$ is

$$\sigma^2_{I(y)} \le \Phi(y)[1-\Phi(y)] + \frac{1}{\pi} \frac{\beta e^{-\frac{y^2}{1+\beta}}}{(1-\beta)^{3/2}\sqrt{1+\beta}}, \tag{S.47}$$

which is finite if $0 \le \beta < 1$. By using the upper bound in Equation (S.47) and the following asymptotic relations,

$$\begin{aligned}
\Phi(y)[1-\Phi(y)] &\sim 1-\Phi(y), \quad \text{as } y \to \infty \quad (\text{since } \Phi(y) \to 1) \\
&\sim y^{-1}\phi(y), \quad \text{as } y \to \infty \quad (\text{Feller 1968, Lemma 2, p. 175}),
\end{aligned} \tag{S.48}$$

we obtain

$$\begin{aligned}
\frac{\sigma^2_{I(y)}}{\Phi(y)[1-\Phi(y)]} - 1 &\le \left[\frac{1}{\pi} \frac{\beta}{(1-\beta)^{3/2}\sqrt{1+\beta}}\right] \frac{e^{-\frac{y^2}{1+\beta}}}{\Phi(y)[1-\Phi(y)]} \quad (\text{by Equation (S.47)}) \\
&\sim \left[\frac{1}{\sqrt{2\pi^3}} \frac{\beta}{(1-\beta)^{3/2}\sqrt{1+\beta}}\right] \frac{y e^{-\frac{y^2}{1+\beta}}}{e^{-y^2/2}}, \quad \text{as } y \to \infty \quad (\text{by Equation (S.48)}) \\
&= \frac{1}{\sqrt{2\pi^3}} \frac{\beta}{(1-\beta)^{3/2}\sqrt{1+\beta}} \exp\left\{-y^2\left[\frac{1-\beta}{1+\beta}\right] + \ell\mathrm{n}(y)\right\} \\
&\to 0, \quad \text{as } y \to \infty.
\end{aligned}$$

On the other hand, by Equation (S.44), it is clear that $\sigma^2_{I(y)} \geq \Phi(y)[1-\Phi(y)]$. These latter two facts immediately imply that

$$\sigma^2_{I(y)} \sim \Phi(y)[1-\Phi(y)], \quad \text{as } y \to \infty,$$

so that, in particular,

$$\sigma^2_{I(y_p)} \sim \Phi(y_p)[1-\Phi(y_p)] = p(1-p), \quad \text{as } p \to 1.$$

By symmetry, the same result can be obtained for $p \to 0$. $\square$ [Add Jim's ± comments.]

S6. Proof that $\frac{\mathrm{d}}{\mathrm{d}p}\sigma^2_{\tilde{y}_p} > 0$ for the AR(1) Process for Sufficiently Large p

Since $\sigma^2_{\tilde{y}_p} = \sigma^2_{I(y_p)}/f^2(y_p) = 2\pi e^{y_p^2}\sigma^2_{I(y_p)}$, we have

$$\begin{aligned}
\frac{1}{2\pi e^{y_p^2}}\frac{\mathrm{d}}{\mathrm{d}p}\sigma^2_{\tilde{y}_p} &= \frac{1}{e^{y_p^2}}\left[\sigma^2_{I(y_p)}\frac{\mathrm{d}}{\mathrm{d}p}e^{y_p^2} + e^{y_p^2}\frac{\mathrm{d}}{\mathrm{d}p}\sigma^2_{I(y_p)}\right] \\
&= \left(\sum_{\ell\in\mathbb{Z}} R_{I(y_p)}(\ell)\right)2y_p\frac{\mathrm{d}y_p}{\mathrm{d}p} + \frac{\mathrm{d}}{\mathrm{d}p}\sum_{\ell\in\mathbb{Z}} R_{I(y_p)}(\ell) \quad \text{(by Equation (4))} \\
&= 2\sqrt{2\pi}\, y_p\, e^{y_p^2/2}\sum_{\ell\in\mathbb{Z}} R_{I(y_p)}(\ell) + \sum_{\ell\in\mathbb{Z}}\frac{\mathrm{d}}{\mathrm{d}p}R_{I(y_p)}(\ell) \\
&\qquad \text{(by Equation (30), and a legal sum/derivative swap as per Equation (31) of Remark 6)} \\
&= 2\sqrt{2\pi}\, y_p\, e^{y_p^2/2}\sum_{\ell\in\mathbb{Z}}\frac{1}{2\pi}\int_0^{\beta^{|\ell|}}\frac{1}{\sqrt{1-r^2}}\exp\left[\frac{-y_p^2}{1+r}\right]\mathrm{d}r \\
&\qquad -\sum_{\ell\in\mathbb{Z}}\sqrt{\frac{2}{\pi}}\int_0^{\beta^{|\ell|}}\frac{1}{\sqrt{1-r^2}\,(1+r)}\exp\left[\frac{y_p^2(-1+r)}{2(1+r)}\right]\mathrm{d}r \quad \text{(by Equations (S.43) and (29))} \\
&= \sqrt{\frac{2}{\pi}}\, e^{y_p^2/2}\sum_{\ell\in\mathbb{Z}}\int_0^{\beta^{|\ell|}}\left\{\frac{1}{\sqrt{1-r^2}}\left(y_p - \frac{1}{1+r}\right)\exp\left[\frac{-y_p^2}{1+r}\right]\right\}\mathrm{d}r.
\end{aligned}$$

This quantity is guaranteed to be greater than 0 as long as $y_p > 1$, i.e., if $p > \Phi(1) \doteq 0.8413$. $\square$