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Some Steady-State Simulation Processes That Satisfy the Geometric-Moment Contraction Condition

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The Geometric-Moment Contraction (GMC) condition can be used in place of (difficult-to-apply) moment and mixing conditions in the analysis of parameter estimator properties for stationary stochastic processes. We show that several stationary, short-range dependent stochastic processes of interest in discrete-event simulation applications satisfy the GMC condition. In particular, we establish various sets of sufficient conditions for a transformation of a GMC process to yield another GMC process. We also show that linear combinations of GMC processes are GMC; mixtures of GMC processes are GMC; the batch-means process arising from a GMC process is itself GMC; and a certain indicator process associated with quantiles is GMC if the underlying process is GMC. We provide a number of examples that are motivated by their potential use in simulation applications.

Key words: Stationary stochastic process; Geometric-Moment Contraction condition; short-range dependent process; simulation analysis; batch-means method.
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1. Temporary Section with New Result

Consider the following process that looks pretty much like an AR(1) process with pretty arbitrary innovations,

$$X_i = \sum_{\ell=0}^{\infty} \rho^{\ell} \varepsilon_{i-\ell} \quad \text{for } i \in \mathbb{Z},$$

with $\rho \in (-1, 1)$. Let

$$\delta'_{i-\ell} \equiv \begin{cases} \varepsilon_{i-\ell}, & \text{if } \ell < i. \\ \varepsilon'_{i-\ell}, & \text{if } \ell \geq i, \end{cases}$$

and consider the GMC sister process,

$$X_i' = \sum_{\ell=0}^{\infty} \rho^\ell \delta_{i-\ell}' = \sum_{\ell=0}^{i-1} \rho^\ell \varepsilon_{i-\ell} + \sum_{\ell=i}^{\infty} \rho^\ell \varepsilon_{i-\ell}' \quad \text{for } i \in \mathbb{Z}.$$

Thus,

$$\begin{aligned}
\mathrm{E}\left[\left|X_i - X_i'\right|^\psi\right] &= \mathrm{E}\left[\left|X_i - X_i'\right|^\psi\right] \\
&= \mathrm{E}\left[\left|\sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{i-\ell} - \sum_{\ell=0}^{\infty} \rho^\ell \delta_{i-\ell}'\right|^\psi\right] \\
&= \mathrm{E}\left[\left|\sum_{\ell=0}^{\infty} \rho^\ell \varepsilon_{i-\ell} - \sum_{\ell=0}^{i-1} \rho^\ell \varepsilon_{i-\ell} - \sum_{\ell=i}^{\infty} \rho^\ell \varepsilon_{i-\ell}'\right|^\psi\right] \\
&= \mathrm{E}\left[\left|\sum_{\ell=i}^{\infty} \rho^\ell (\varepsilon_{i-\ell} - \varepsilon_{i-\ell}')\right|^\psi\right] \\
&= \mathrm{E}\left[\left|\sum_{\ell=0}^{\infty} \rho^{i+\ell} (\varepsilon_{-\ell} - \varepsilon_{-\ell}')\right|^\psi\right] \\
&= \rho^{i\psi}\, \mathrm{E}\left[\left|\sum_{\ell=0}^{\infty} \rho^{\ell} (\varepsilon_{\ell} - \varepsilon_{\ell}')\right|^\psi\right] \\
&\le (2|\rho|^i)^\psi\, \mathrm{E}\left[\left|\sum_{\ell=0}^{\infty} \rho^{\ell} |\varepsilon_{\ell}|\right|^\psi\right] \qquad (1)
\end{aligned}$$

At this point, we assume that $\mathrm{E}\big[|\varepsilon_i|\big] = M^\star < \infty$ and $\psi \ge 1$. Then Jensen's inequality immediately gives

$$\mathrm{E}\left[\left|X_i - X_i'\right|^\psi\right] \le (2|\rho|^i)^\psi \left(\mathrm{E}\left[\sum_{\ell=0}^{\infty} \rho^\ell |\varepsilon_\ell|\right]\right)^\psi = \left(2|\rho|^i \sum_{\ell=0}^{\infty} \rho^\ell \mathrm{E}\big[|\varepsilon_\ell|\big]\right)^\psi = \left(\frac{2M^\star |\rho|^i}{1-\rho}\right)^\psi.$$

So $\{X_i\}$ is GMC for $\psi \ge 1$. We now use Jim's "old" Theorem 1 to make it GMC for *any* $\psi > 0$. In either case, these results seem to be pretty general. Meanwhile, we know from what's-his-name's paper that for $X_i \sim \mathrm{Bern(p)}$, the process $\{X_i\}$ not strongly mixing. POW! □

2. Introduction

A longstanding, active area of research has been that of analyzing a stationary stochastic process $\{Y_k : k \ge 1\}$, for instance, the output process arising from a steady-state simulation. In particular, it is often of interest to estimate parameters associated with the stationary process such as the marginal mean, $\mu_Y \equiv \mathrm{E}[Y_1]$; the corresponding variance parameter, $\sigma_Y^2 \equiv \lim_{n\to\infty} n\mathrm{Var}(\overline{Y}_n)$, where the sample mean $\overline{Y}_n \equiv n^{-1}\sum_{k=1}^n Y_k$; or a quantile of the form $y_p \equiv F_Y^{-1}(p) \equiv \inf\{y : F_Y(y) \ge p\}$, for $p \in [0,1]$, where the marginal cumulative distribution function (c.d.f.) $F_Y(y) \equiv \Pr(Y_1 \le y)$ for $y \in \mathbb{R}$. There has been a great deal of activity in the simulation, stochastics, and statistics communities focusing on stationary estimation problems, and this work has resulted in various popular estimators, some of which have subsequently been "stress tested" via analytical and Monte Carlo (MC) methods on a battery of challenging stochastic processes in order to evaluate their performance in terms of bias, variance, mean squared error, expected clock time for calculation, etc. For purposes of the current paper, one can most-easily get up to speed on background relevant to estimation by perusing, e.g., Law (2015) for a general introductory discussion of estimation methods for μ and σ_y^2; Aktaran-Kalaycı et al. (2007), which concentrates on establishing convergence properties of certain estimation methods; Alexopoulos et al. (2023), which investigates corresponding methods for quantiles; and Lolos et al. (2023b), which illustrates the use of several test processes to evaluate estimator performance.

[Beware! I sent a lot of this material to the Chien paper 7/11/25 →] A common approach underlying the development and analysis of steady-state performance estimators requires us to assume and verify that the underlying process $\{Y_k : k \geq 1\}$ satisfies appropriate moment and mixing conditions. To this end, we provide some brief background verbiage on mixing. Following Bradley (2005) (also see Billingsley 1995), for sigma-fields $\mathcal{A}$ and $\mathcal{B}$, define $\phi(\mathcal{A}, \mathcal{B}) \equiv \sup \left|\Pr(B|A) - \Pr(B)\right|$, where the supremum is taken over all events $A \in \mathcal{A}$ and $B \in \mathcal{B}$ for which $\Pr(A) > 0$. Specifically, let $\mathcal{M}_i^j$ ($i \leq j \in \mathbb{Z}$) denote the sigma-field generated by the stationary observations $\{Y_k\}_{k=i}^j$. We say that $\{Y_k\}$ is φ-*mixing* if $\varphi(\ell) \equiv \sup_{j \in \mathbb{Z}} \phi\big(\mathcal{M}_{-\infty}^j, \mathcal{M}_{j+\ell}^\infty\big) \to 0$ as $\ell \to \infty$. In other words, for all $j \geq 1$ and $\ell \geq 1$, and any events $A \in \mathcal{M}_{-\infty}^j$ and $B \in \mathcal{M}_{j+\ell}^\infty$, we have $\left|\Pr(B|A) - \Pr(B)\right| \leq \varphi(\ell) \to 0$ as $\ell \to \infty$. Similarly, one can also define *strong*-mixing (aka α-mixing) for the case in which $\left|\Pr(A \cap B) - \Pr(A)\Pr(B)\right| \leq \alpha(\ell) \to 0$ as $\ell \to \infty$. Informally, we can interpret φ-mixing and α-mixing to mean that events in the far future are approximately independent of those in the past.

In practical use of moment and mixing assumptions in the context of the analysis of steady-state performance estimators, one must typically (i) verify that the assumed moments of $\{Y_k\}$ exist (are finite), and (ii) establish the convergence of the coefficients $\varphi(\ell)$ or $\alpha(\ell) \to 0$ as $\ell \to \infty$ as well as the finiteness of certain sums of nontrivial functions of the coefficients. This task can be daunting. For example, Alexopoulos et al. (2001) requires $\mathrm{E}[Y_k^8] < \infty$ and the mixing condition $\varphi(\ell) = O\big(\ell^{-\gamma}\big)$ for some constant $\gamma > 12$: and Chien et al. (1997, Lemma 11) uses the even-more-restrictive assumptions $\mathrm{E}[Y_k^{28}] < \infty$ and $\varphi(\ell) = O(\ell^{-15})$. In particular, mixing conditions are notoriously difficult to check either theoretically or empirically (see, for instance, p. 1941 of Wu, 2005b; p. 1165, 3rd para. of Alexopoulos et al., 2019; and Remark 2, pp. 8–9 of Alexopoulos et al., 2023). So the practical use of mixing assumptions in the context of the analysis of steady-state performance estimators might be somewhat problematic.

Coming to the rescue is the *Geometric-Moment Contraction* (GMC) condition, which is synergistic with an alternative set of assumptions for deriving and verifying such performance results. The GMC condition is more intuitive and easier to check empirically than mixing conditions, especially in the context of a steady-state simulation output process that is driven by an input stream of independent and identically distributed (i.i.d.) random variables (r.v.'s), e.g., i.i.d. customer interarrival times to a station or i.i.d. service times.

Geometric-Moment Contraction Condition Consider two independent sequences of i.i.d. r.v.'s, $\{\varepsilon_j : j \in \mathbb{Z}\}$ and $\{\varepsilon'_j : j \in \mathbb{Z}\}$, with each r.v. distributed like ε_0. The stationary processes $\{Y_k : k \geq 0\}$ and $\{Y'_k : k \geq 0\}$ are defined by a function $\xi(\cdot)$ where $Y_k = \xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$ and $Y'_k = \xi(\ldots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k)$ for $k \geq 0$. In addition, there exist constants $\psi > 0$, $C_\psi > 0$, and $r_\psi \in (0, 1)$ such that

$$\mathrm{E}\left[\left|Y_k - Y'_k\right|^{\psi}\right] \leq C_\psi r_\psi^k \text{ for } k \geq 0. \tag{2}$$

Henceforth, we will use the "prime" notation to distinguish such linked (coupled) processes as $\{Y_k\}$ and $\{Y'_k\}$ in (2).

Clearly, the r.v.'s Y_k and Y'_k are identically distributed for all $k \geq 0$. In addition, the initial pair of r.v.'s, Y_0 and Y'_0, are independent by construction; but any subsequent pair, Y_k and Y'_k, for $k \geq 1$, are (potentially) dependent with the ψth moment of their differences converging to 0 geometrically fast as k increases (Alexopoulos et al. 2019, p. 1165, next-to-last para.). The GMC condition (2) can be likened to the use of common random numbers in simulation, in which two runs of the simulation are initialized independently at time 0 and thereafter share the same inputs $\{\varepsilon_1, \varepsilon_2, \ldots\}$. A result from Dingeç et al. (2023a, Theorem 1(c)) provides additional intuition regarding the GMC condition, where it is established that a GMC process also satisfies the short-range dependence (SRD) condition involving the summability of $\{Y_k\}$'s autocorrelation function (a.c.f.), $\rho_\ell \equiv \mathrm{Corr}(Y_0, Y_\ell)$, for $\ell = 0, \pm1, \pm2, \ldots$; namely,

$$\sum_{\ell=-\infty}^{\infty} |\rho_\ell| = \sum_{\ell=-\infty}^{\infty} \left|\mathrm{Corr}(Y_0, Y_\ell)\right| < \infty$$

(also see Doukhan 2018, Proposition 7.4.2 and Remark 7.4.6). Thus, the far future of GMC process is uncorrelated with the present (similar to our analogous remark on φ-mixing).

A feature of the GMC condition (2) is that it seems to hold for a larger collection of stochastic processes than does mixing. For instance, as is well known, a first-order autoregressive process (AR(1)) is not φ-mixing (Davidson 1994, Example 14.8, p. 218), yet it satisfies (2) (see below). For this reason, results from work based on GMC will tend to be applicable in more generality and in more-practical situations than corresponding work reliant on φ-mixing assumptions. One could also argue that validation of the GMC condition in practice either by analytic methods or by conducting moment-convergence diagnostics is simpler than attempting to validate relatively abstruse φ-mixing conditions (Whitt 2002, pp. 107–108, see Equations (4.3)–(4.8), Theorem 4.4.1, and the immediately following paragraph). We also caution that one encounters additional difficulties when attempting to verify assumptions on the summability of certain functions of the mixing coefficients—for example, $\sum_{\ell=1}^{\infty} \sqrt{\varphi(\ell)}$—let alone estimating the sum.

The main goal of this article is to present several interesting stochastic processes that satisfy the GMC condition (2). We are especially on the lookout for processes that might prove useful in the context of computer simulation applications. In fact, the ultimate objective is that in follow-up work on the design and evaluation of simulation-based procedures for estimating steady-state parameters such as means or quantiles, these GMC processes can be used to stress test those procedures (**?**, Lolos et al. 2022). The results in this paper facilitate the formulation of a test bed of old and new GMC processes that can be rapidly simulated and easily configured to exhibit interesting/problematic characteristics such as a strongly marked autocorrelation function, undefined third- or higher-order marginal moments, or a highly irregular marginal probability density function (p.d.f.) or probability mass function (p.m.f.).

The article is organized as follows. Section 3 provides background in the forms of a representative list of stochastic processes that are already known to be GMC; some assumptions that will help us proceed as we study additional GMC processes of interest to the simulation community; and two instructive, self-contained proofs demonstrating the (known) results that certain finite-state Markov chains and AR(1) processes are GMC processes. Sections 4–6 detail a portmanteau of sufficient conditions and machinations designed to establish that various classes of transformations and/or other tricks involving GMC processes themselves result in GMC processes. Section 4 establishes sufficient conditions based on Lipschitz-continuous and convex/concave transformations that preserve the GMC property of a process. Section 5 gives numerous examples, many of which rely on the transformations from the previous section, with most involving versions of the "autoregressive-to-continuous" (ARTOC) process, i.e., an AR(1) process that has been transformed to have a given (continuous) marginal p.d.f. Section 7 proves that linear combinations of GMC processes are GMC; and, in fact, the component processes of the linear combination do not have to be independent of each other. We apply this result to obtain GMC processes based on Taylor and other series representations of functions. Section 8 establishes that products of GMC processes are GMC, and then uses the previous section's results to prove that mixtures of processes are GMC. Section 9 deals with batch-means processes, which often appear in the context of simulation output analysis. Specifically, we prove that if the underlying process is GMC, then so is the corresponding nonoverlapping batch means process. Section 6 is concerned with quantile indicator processes; and in that section, we show that if a process $\{Y_k : k \geq 1\}$ satisfies the GMC condition (2) and the density-boundedness condition (4) below, then the associated indicator process, $\left\{I_k(y) \equiv \mathbf{1}_{\{Y_k \leq y\}} : k \geq 1\right\}$ with given y, satisfies the GMC condition. This result is key to the work in our companion paper Dingeç et al. (2023a). Section 10 summarizes our main findings and discusses how these will used in future research directions. The Appendix contains the proofs of certain results presented in the main part of the paper as well as complementary material.

3. Background

Section 3.1 serves as a quick literature review by listing various classes of stationary time series that are GMC, and §3.2 establishes some terminology and results that will be used in the sequel. §3.3 gives succinct proofs that two-state Markov chains and the AR(1) process are GMC—(known) results that will (i) provide additional motivation for the GMC condition and (ii) serve as the basis of several examples in the sequel.

3.1. Some Old GMC Friends

In recent years a number of time series models that are widely used in simulation applications have been shown to satisfy the GMC condition (2); and we list some of those models here. More interestingly, the GMC condition holds for the usual finite-order moving average (MA), autoregressive (AR), and autoregressive–moving average (ARMA) processes (Box et al. 2008, Chap. 3; Shao and Wu 2007, Theorem 5.2) Moreover, Wu and Shao (2004, Theorem 2) show that the GMC condition is satisfied by a large class of other linear and nonlinear processes, including: autoregressive conditional heteroscedastic (ARCH) processes (Engle 1982), generalized autoregressive conditional heteroscedastic (GARCH) processes (Bollerslev 1986), ARMA–ARCH and ARMA–GARCH processes (Li et al. 2002), random coefficient autoregressive (RCA) processes (Nicholls and Quinn 1982), threshold autoregressive (TAR) processes (Tong 1990); and all these processes are used in the simulation of economic systems (Law 2015, p. 73). Moreover a wide collection of Markov chains satisfy the GMC condition; and they are widely used in Markov chain Monte Carlo (MCMC) studies. Dingeç et al. (2022) proved that the GMC condition is satisfied by the queue-waiting-time process in a G/G/1 queueing system with a non-heavy-tailed service-time distribution (i.e., the service time's moment generating function exists in a neighborhood of zero). Moreover, practical methods for checking the GMC condition empirically on more-arbitrary stochastic processes are detailed in Alexopoulos et al. (2012) and Alexopoulos et al. (2023, §2.3.1, pp. 7–8). [Make sure this doesn't copy OR Sequest paper verbiage.]

3.2. Some Assumptions and Useful Results

As discussed above, a process satisfying the GMC condition (2) includes stationarity (by definition) and SRD (Dingeç et al. 2023a, Theorem 1(c)). Additionally, along the way, we will adhere to certain reasonable assumptions beyond GMC, namely:

- The marginal absolute moment condition,

$$\mathrm{E}\big[|Y_0|^{u_Y}\big] < \infty \text{ for some } u_Y > 2. \tag{3}$$

- The density-boundedness condition, in which Y_k's marginal p.d.f. $f_Y(y)$ satisfies

$$\sup_{y\in\mathbb{R}} f_Y(y) < \infty. \tag{4}$$

The following is a result from our companion paper Dingeç et al. (2023a, Theorem 1(a)) that we will occasionally put to use—in plain English, if $\{Y_k\}$ is GMC for some $\psi > 0$, then it remains GMC for other values of ψ.

Theorem 1. *Suppose the GMC (2) and moment condition (3) hold for* $\{Y_k : k \ge 0\}$ *for some* $\psi > 0$ *and* $u_Y > 2$. *Then* $\{Y_k : k \ge 0\}$ *satisfies (2) for* all $\psi \in (0, u_Y)$.

Without loss of generality and for ease of exposition, Theorem 1 will typically allow us to use a common $\psi > 0$ when we deal simultaneously with multiple GMC processes in the sequel.

Finally, in the subsequent discussion, we will make occasionally use of certain bounds related to the standard normal p.d.f. $\phi(z) \equiv (2\pi)^{-1/2} \exp\left(-\frac{1}{2}z^2\right)$ and c.d.f. $\Phi(z) \equiv \int_{-\infty}^{z} \phi(x)\,\mathrm{d}x$, for $z \in \mathbb{R}$.

Lemma 1. *We have the following bounds on standard normal tail probabilities and integrals:*

$$\frac{z\phi(z)}{z^2+1} < \Phi(-z) < \frac{\phi(z)}{z} \quad \textit{for } z > 0. \tag{5}$$

Suppose that $c \in [0, 3/2]$. *Then*

$$\int_{\mathbb{R}} [\Phi(-z)]^{-c}\, \phi^2(z)\,\mathrm{d}z < \infty. \tag{6}$$

Proof: Equation (5) is an old result due to Gordon (1941, Eq. (10)). On the way to Equation (6), we use (5) to first obtain

$$\frac{\phi(z)}{2z} < \frac{z\phi(z)}{z^2+1} < \Phi(-z) \quad \text{for} \quad z \geq 1; \tag{7}$$

and it follows that

$$\begin{aligned}
\int_1^\infty \left[\Phi(-z)\right]^{-c} \phi^2(z)\,\mathrm{d}z &< \int_1^\infty \left[\frac{\phi(z)}{2z}\right]^{-c} \phi^2(z)\,\mathrm{d}z = 2^c \int_1^\infty z^c \left[\phi(z)\right]^{2-c} \mathrm{d}z \\
&< 2^{3/2} \int_1^\infty z^2 \phi^{1/2}(z)\,\mathrm{d}z < 2^{3/2} \int_{-\infty}^\infty z^2 \phi^{1/2}(z)\,\mathrm{d}z \qquad (8)\\
&= 2^{13/4} \pi^{1/4} < \infty, \qquad (9)
\end{aligned}$$

where Equation (8) follows from the assumption that $c \leq 3/2$ and the resulting inequalities $2 - c \geq 1/2$ and $\left[\phi(z)\right]^{2-c} \leq \phi^{1/2}(z) < 1$ for $z \in \mathbb{R}$; and Equation (9) is simple calculus.

Next, because $[\Phi(-z)]^{-c}\phi^2(z)$ is continuous on $[-1, 1]$, we have

$$\int_{-1}^{1} \left[\Phi(-z)\right]^{-c} \phi^2(z)\,\mathrm{d}z < \infty. \tag{10}$$

Finally, we have

$$\int_{-\infty}^{-1} [\Phi(-z)]^{-c}\, \phi^2(z)\,\mathrm{d}z = \int_1^\infty [\Phi(z)]^{-c}\, \phi^2(z)\,\mathrm{d}z < \int_1^\infty [\Phi(-z)]^{-c}\, \phi^2(z)\,\mathrm{d}z < \infty, \tag{11}$$

where the equality follows by a change of variables; the first inequality is a consequence of the fact that $[\Phi(z)]^{-c} < [\Phi(-z)]^{-c}$ for $z > 0$; and the finiteness follows from Equation (9).

The proof is completed by adding the expressions in Equations (9)–(11). □

3.3. Some Elementary Processes That Are GMC

First of all, if the i.i.d. sequence $\{Y_k : k \geq 0\}$ satisfies $\mathrm{E}[|Y_0|^{u_Y}] < \infty$ for some $u_Y > 2$, then by Theorem 1 below we will see that for any $\psi \in (0, u_Y)$ and any $r_\psi \in (0,1)$, we have $C_\psi \equiv \mathrm{E}\big[\big|Y_k - Y_k'\big|^\psi\big] = C_\psi r_\psi^k < \infty$ for $k = 0$; and for $k \geq 1$, we have $\mathrm{E}\big[\big|Y_k - Y_k'\big|^\psi\big] = 0 < C_\psi r_\psi^k$ for $k \geq 1$. Thus the i.i.d. process $\{Y_k : k \geq 0\}$ is GMC. We show that two simple yet interesting stochastic processes satisfy the GMC condition. These are well-known results, but we nevertheless provide succinct, motivational proofs in order for the current paper to be self-contained and to establish useful notation. §3.3.1 establishes that a two-state Markov chain is GMC (cf. Wu and Woodroofe 2000) [check for precise reference, if applicable?], while §3.3.2 does so for the AR(1) process (cf. Shao and Wu 2007, Theorem 5.2).

3.3.1. A Two-State Markov Chain Is GMC A good starting place, at least for purposes of motivation, is the two-state Markov chain $\{X_k : k \geq 0\}$ on the state space $\mathcal{S} \equiv \{0,1\}$ with transition probability matrix $\boldsymbol{P}$ defined by the elements

$$P_{i,j} \equiv \Pr(X_{k+1} = j | X_k = i) \quad \text{for } i, j \in \mathcal{S} \text{ and } k \geq 0. \tag{12}$$

To verify the GMC condition for this process, we introduce another two-state Markov chain $\{X_k' : k \geq 0\}$ such that

$$\{X_k : k \geq 0\} \stackrel{\mathrm{d}}{=} \{X_k' : k \geq 0\}, \tag{13}$$

where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution—i.e., the two processes induce the same probability measure on the underlying probability space; and ultimately we will couple those processes using the method of common random numbers to verify the GMC condition. For both processes the one-step transition probability matrix has the form

$$\boldsymbol{P} \equiv \begin{pmatrix} a & 1-a \\ b & 1-b \end{pmatrix}, \tag{14}$$

where $a, b \in (0,1)$; and without loss of generality, we assume that $a < b$. Let $\boldsymbol{\pi} = (\pi_0, \pi_1)$ denote the corresponding stationary p.m.f., which is the solution of the usual equations $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ and $\pi_0 + \pi_1 = 1$ (Ross 2019) and is given by

$$\pi_0 = \frac{b}{1+b-a} \quad \text{and} \quad \pi_1 = 1 - \pi_0 = \frac{1-a}{1+b-a}. \tag{15}$$

Lemma 2. *The two-state Markov chain as defined by Equations (12)–(15) is GMC.*

Proof: Let U_0, U_0', and $\{U_k : k \geq 1\}$ denote mutually independent random numbers. We initialize the Markov chains $\{X_k : k \geq 0\}$ and $\{X_k' : k \geq\}$ using the stationary distribution, so that the two processes are stationary and equivalent in distribution. Thus X_0 and X_0' are respectively generated as follows:

$$X_0 = \begin{cases} 0, & \text{if } U_0 \leq \pi_0, \\ 1 & \text{if } U_0 > \pi_0, \end{cases} \quad \text{and} \quad X_0' = \begin{cases} 0, & \text{if } U_0' \leq \pi_0, \\ 1 & \text{if } U_0' > \pi_0 \end{cases} \tag{16}$$

so that X_0 and X_0' are independent r.v.'s. At time step $k \geq 0$, the r.v. values X_{k+1} and X_{k+1}' at the next time step are respectively generated from the current r.v. values X_k and X_k' using the common random number U_{k+1} as follows:

$$X_{k+1} = \begin{cases} 0, & \text{if } X_k = 0 \text{ and } U_{k+1} \leq P_{0,0} = a, \\ 1, & \text{if } X_k = 0 \text{ and } U_{k+1} > P_{0,0} = a, \\ 0, & \text{if } X_k = 1 \text{ and } U_{k+1} \leq P_{1,0} = b, \\ 1, & \text{if } X_k = 1 \text{ and } U_{k+1} > P_{1,0} = b; \end{cases} \quad \text{and} \quad X_{k+1}' = \begin{cases} 0, & \text{if } X_k' = 0 \text{ and } U_{k+1} \leq P_{0,0} = a, \\ 1, & \text{if } X_k' = 0 \text{ and } U_{k+1} > P_{0,0} = a, \\ 0, & \text{if } X_k' = 1 \text{ and } U_{k+1} \leq P_{1,0} = b, \\ 1, & \text{if } X_k' = 1 \text{ and } U_{k+1} > P_{1,0} = b. \end{cases} \tag{17}$$

From another perspective, we see that Equation (14) immediately yields the following table of transitions as a function of U_{k+1} for $k \geq 0$:

	$X_k = 0$	$X_k = 1$	$X_k' = 0$	$X_k' = 1$
$U_{k+1} \leq a$	$X_{k+1} = 0$	$X_{k+1} = 0$	$X_{k+1}' = 0$	$X_{k+1}' = 0$
$a < U_{k+1} \leq b$	$X_{k+1} = 1$	$X_{k+1} = 0$	$X_{k+1}' = 1$	$X_{k+1}' = 0$
$U_{k+1} > b$	$X_{k+1} = 1$	$X_{k+1} = 1$	$X_{k+1}' = 1$	$X_{k+1}' = 1$

(18)

From the table (18) we observe that the bivariate process

$$\left\{ \boldsymbol{Y}_k \equiv \left(X_k, X_k' \right) : k \geq 0 \right\} \tag{19}$$

is a Markov chain with the state space

$$\widetilde{S} \equiv \{(0,0), (0,1), (1,0), (1,1)\} \tag{20}$$

and the transition probability matrix

$$\widetilde{\boldsymbol{P}} = \begin{array}{c} \\ (0,0) \\ (0,1) \\ (1,0) \\ (1,1) \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} (0,0) & (0,1) & (1,0) & (1,1) \end{array} \\ \begin{pmatrix} a & 0 & 0 & 1-a \\ a & 0 & b-a & 1-b \\ a & b-a & 0 & 1-b \\ b & 0 & 0 & 1-b \end{pmatrix} \end{array}, \tag{21}$$

where the origin and destination states for each transition appear on the border of $\widetilde{\boldsymbol{P}}$. Thus, for example, as indicated by the circled entries in the table (18), we have

$$\widetilde{P}_{(0,1),(1,0)} \equiv \Pr\left\{ (X_{k+1}, X_{k+1}') = (1,0) \,\middle|\, (X_k, X_k') = (0,1) \right\} = b - a \quad \text{for } k \geq 0.$$

It is easy to show that depending on whether $\ell = 2k-1$ or $\ell = 2k$ for $k \geq 1$, the ℓ-step transition matrices for the Markov process $\left\{ \boldsymbol{Y}_k \equiv \left(X_k, X_k' \right) : k \geq 0 \right\}$ are given by

$$\widetilde{\boldsymbol{P}}^{(2k-1)} \equiv \widetilde{\boldsymbol{P}}^{2k-1} = \begin{pmatrix} \star & 0 & 0 & \star \\ \star & 0 & (b-a)^{2k-1} & \star \\ \star & (b-a)^{2k-1} & 0 & \star \\ \star & 0 & 0 & \star \end{pmatrix}, \tag{22}$$

and

$$\widetilde{\boldsymbol{P}}^{(2k)} \equiv \widetilde{\boldsymbol{P}}^{2k} = \begin{pmatrix} \star & 0 & 0 & \star \\ \star & (b-a)^{2k} & 0 & \star \\ \star & 0 & (b-a)^{2k} & \star \\ \star & 0 & 0 & \star \end{pmatrix}, \tag{23}$$

where "$\star$" denotes a generic nonnegative probability $\widetilde{P}_{(g,h),(i,j)}$ whose only relevant property is that it is it is in $[0,1]$. For instance, as indicated by the circled entry in Equation (22), the probability of transitioning from state $(X_2, X_2') = (0,1)$ to $(X_7, X_7') = (1,0)$ in precisely $2k-1 = 5$ steps is $(b-a)^5$.

To verify that the two-state Markov chain $\{X_k : k \geq 0\}$ is GMC for given time step $\ell \geq 0$, we apply the law of total probability for expectations (Karlin and Taylor 1975, p. 8), the chain rule for conditional expectations (Billingsley 1995, p. 51, Eq. (4.2)), and Equations (22)–(23) to compute

$$\mathrm{E}\big[|X_\ell - X_\ell'|^\psi\big] = \sum_{(g,h)\in\widetilde{\mathcal{S}}} \sum_{(i,j)\in\widetilde{\mathcal{S}}} \Pr\{\boldsymbol{Y}_0 = (g,h)\} \widetilde{P}^{(\ell)}_{(g,h),(i,j)} \mathrm{E}\big[|X_\ell - X_\ell'|^\psi \,\big|\, \boldsymbol{Y}_0 = (g,h);\ \boldsymbol{Y}_\ell = (i,j)\big] \tag{24}$$

$$= \sum_{(g,h)\in\widetilde{\mathcal{S}}} \sum_{(i,j)\in\widetilde{\mathcal{S}}} \Pr\{X_0 = g\} \Pr\{X_0' = h\} \widetilde{P}^{(\ell)}_{(g,h),(i,j)} |i-j|^\ell \tag{25}$$

In terms of Equations (22), (23), and (25) as well as the auxiliary vectors

$$\widetilde{\boldsymbol{V}} \equiv \big[\pi_0^2,\, \pi_0(1-\pi_0),\, \pi_0(1-\pi_0),\, (1-\pi_0)^2\big] \quad \text{and} \quad \widetilde{\boldsymbol{W}} \equiv [0,1,1,0]^\mathsf{T}, \tag{26}$$

it is straightforward to verify that

$$\mathrm{E}\big[|X_\ell - X_\ell'|^\psi\big] = \widetilde{\boldsymbol{V}}\widetilde{\boldsymbol{P}}^{(\ell)}\widetilde{\boldsymbol{W}} = 2\pi_0(1-\pi)(b-a)^\ell \ \text{ for } \ \ell \geq 0 \ \text{ and } \ \psi > 0; \tag{27}$$

thus the two-state Markov chain $\{X_k : k \geq 0\}$ satisfies the GMC condition for any $\psi > 0$ with $C_\psi = 2\pi_0(1-\pi_0)$, and $r_\psi = b - a \in (0,1)$. ∎

3.3.2. The AR(1) Process Is GMC The stationary (Gaussian) AR(1) process is defined by the relation

$$Z_k = \beta Z_{k-1} + \varepsilon_k \ \text{ for } \ k \geq 1, \tag{28}$$

where: (i) the autoregressive parameter $\beta \in (-1,1)$; (ii) the innovations $\{\varepsilon_k : k \geq 0\}$ are i.i.d. $\mathrm{Nor}(0, \sigma_\varepsilon^2)$ with $\sigma_\varepsilon^2 = 1 - \beta^2$; and (iii) the initial AR(1) observation $Z_0 \equiv \varepsilon_0/\sigma_\varepsilon$ is $\mathrm{Nor}(0,1)$.

Lemma 3. *The AR(1) process as defined by Equation (28) is GMC.*

This is another known result (Shao and Wu 2007, Theorem 5.2), **[← check!]** but we give a short proof in order for the current paper to be self-contained and to establish useful notation.

Proof: We generate a coupled version of the process $\{Z_k : k \geq 0\}$ using the relations

$$Z_0' = \varepsilon_0'/\sigma_\varepsilon \quad \text{and} \quad Z_k' = \beta Z_{k-1}' + \varepsilon_k \ \text{ for } \ k \geq 1, \tag{29}$$

where $\varepsilon_0' \sim \text{Nor}\big(0, \sigma_\varepsilon^2\big)$ and is independent of all of the ε_k's. From Equations (28) and (29), we can easily derive the explicit representations for Z_k and Z_k' in terms of their respective innovations up to time k,

$$\left.\begin{aligned} Z_k &= \xi_Z(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) \equiv \big(\beta^k/\sigma_\varepsilon\big)\varepsilon_0 + \textstyle\sum_{i=1}^k \beta^{k-i}\varepsilon_i \\ Z_k' &= \xi_Z(\varepsilon_0', \varepsilon_1, \ldots, \varepsilon_k) \equiv \big(\beta^k/\sigma_\varepsilon\big)\varepsilon_0' + \textstyle\sum_{i=1}^k \beta^{k-i}\varepsilon_i \end{aligned}\right\} \text{ for } k \ge 0 \text{ a.s.,} \tag{30}$$

where the notation "a.s." denotes "almost surely"; see Wu (2005b, Eq. (17), p. 1940). **[← What is the precise reason for this ref? Don't we already have (Shao and Wu 2007, Theorem 5.2)? Maybe just pick the earliest appropriate one?]**

To obtain the GMC condition (2) for the AR(1) processes specified by Equations (28)–(30), we select $\psi > 0$ arbitrarily. Then $Z_0 - Z_0' \sim 2^{1/2}Q$, where $Q \sim \text{Nor}(0,1)$; and $\text{E}\big[|Q|^{\lceil\psi\rceil}\big] < \infty$ since a normal distribution has finite moments of all orders (Billingsley 1995, Ex. 30.1, p. 389). By Lyapounov's inequality, we have

$$\text{E}\big[|Q|^\psi\big] \le \Big(\text{E}\big[|Q|^{\lceil\psi\rceil}\big]\Big)^{\psi/\lceil\psi\rceil} < \infty \tag{31}$$

(Billingsley 1995, Eq. (5.37), p. 81). Then Equation (31) (or Theorem 1) ensures that the base AR(1) process $\{Z_k : k \ge 0\}$ satisfies the GMC condition for *every* $\psi > 0$ and the following values of $C_\psi > 0$ and $r_\psi \in (0,1)$:

$$\left.\begin{aligned} &\text{E}\big[\big|Z_k - Z_k'\big|^\psi\big] = \tfrac{\beta^{k\psi}}{\sigma_\varepsilon^\psi}\text{E}\big[\big|\varepsilon_0 - \varepsilon_0'\big|^\psi\big] = C_\psi r_\psi^k \text{ for each } k \ge 0 \\ &\text{and } \psi > 0, \text{ where } C_\psi \equiv 2^{\psi/2}\text{E}\big[|Q|^\psi\big] \text{ and } r_\psi \equiv |\beta|^\psi \in (0,1). \end{aligned}\right\} \quad \blacksquare \tag{32}$$

[Please check: Should σ_ε be somewhere in the C_ψ term above?]

Example 1. We illustrate coupling—the most-direct consequence of the GMC condition—via an example involving an AR(1) process with parameter $\beta = 0.9$. Figure 1 depicts a "basic" Monte Carlo realization $\{Z_k : k \ge 0\}$ as well as the corresponding "coupled" realization $\{Z_k' : k \ge 0\}$ initialized, respectively, with different Z_0 and Z_0' values, but thereafter both incorporating the common stream of innovations $\{\varepsilon_k : k \ge 1\}$. We see that the Z_k and Z_k' streams start out distinctly separated, but quickly couple as the effects of the common random numbers kick in. □

With Lemma 3 in mind, one would think that a first-order autoregression with non-normal innovations or a "nice" transformation applied to an AR(1) process would result in processes that retain the GMC property. This is our goal for the first-order autoregressive process with general innovations (ARG(1)), the first-order exponential autoregressive process (EAR(1)), and the ARTOC process, which are coming up in §§5.2.1, 5.2.2, and 5.2.3, respectively.

4. Sufficient Conditions for GMC

This section presents two sets of sufficient conditions for a transformation $g(x)$ of a GMC process to be GMC. §4.1 discusses conditions involving Lipschitz-continuous transformations of an underlying GMC process, while §4.2 does so for convex or concave transformations. The two sets of conditions are often

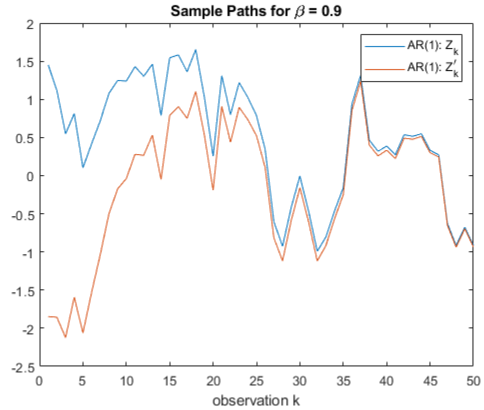


Figure 1 Monte Carlo results for an AR(1) process with $\beta = 0.9$ illustrating coupling of the realizations $\{Z_k\}$ and $\{Z'_k\}$ initialized with different Z_0 and Z'_0 values.

(but not always) complementary in that if one holds, then the other does not; see §5 for relevant evidentiary examples and §5.4, in particular, for more-rigorous justification of this phenomenon in the special case of ARTOC processes.

In order to prepare for what follows, we often start with paired (coupled) versions $\{X_k : k \geq 0\}$ and $\{X'_k : k \geq 0\}$ of a base GMC process. The paired versions of the transformed process are denoted by $\{Y_k = g(X_k) : k \geq 0\}$ and $\{Y'_k = g(X'_k) : k \geq 0\}$; and we invoke the first- or second-order subscript "Y" on the associated constants $\psi_Y > 0$, $C_{\psi_Y} > 0$, and $r_{\psi_Y} \in (0,1)$ in the obvious manner. We make the following assumptions about the base process.

$$\left.\begin{array}{l}\text{A base process } \{X_k : k \geq 0\} \text{ satisfies the GMC condition (2) with constants } \psi_X > 0,\ C_{\psi_X} > 0,\\ \text{and } r_{\psi_X} \in (0,1) \text{ as well as the marginal absolute moment condition}\\ \qquad\qquad \mathrm{E}\big[|X_0|^{u_X}\big] < \infty \ \text{ for some } \ u_X > 2.\\ \text{The process has a given marginal c.d.f. } F_X(x),\ x \in \mathbb{R}, \text{ and a marginal p.d.f. } f_X(x),\ x \in \mathbb{R},\\ \text{whose support } \mathbb{S}_X \equiv \{x \in \mathbb{R} : f_X(x) > 0\} \text{ is an interval in } \mathbb{R} \text{ such that } f_X(x) \text{ is continuous at}\\ \text{every } x \in \mathbb{R} \text{ except possibly at the endpoints (if any) of } \mathbb{S}_X.\end{array}\right\} \quad (33)$$

4.1. Lipschitz-Continuous Transformations

We use Lipschitz-continuous transformations to enhance our collection of GMC processes. A real-valued function $g(x) : \mathbb{R} \to \mathbb{R}$ is said to be Lipschitz continuous if there exists a constant $K_g > 0$ such that, for all $x, y \in \mathbb{R}$, we have $|g(x) - g(y)| \leq K_g |x - y|$. Such functions can be differentiable or nondifferentiable. In fact, a function $g(x)$ that is everywhere differentiable is Lipschitz continuous if and only if it has a bounded first derivative (Wikipedia 2023); [← Thanos suggests that we find a more-standard reference] and in that case, we can take $K_g = \sup_{x \in \mathbb{R}} |g'(x)|$ and use Theorem 2 with more ease than Theorem 3 (which has additional requirements); cf. Doukhan (2018, Lemma 7.4.1).

THEOREM 2. *If $\{X_k : k \geq 0\}$ is a base process satisfying Assumption (33) and $g(x)$ is a Lipschitz-continuous function with the Lipschitz constant $K_g > 0$, then the transformed process $\{Y_k \equiv g(X_k) : k \geq 0\}$ satisfies the GMC condition* (2) *with constants $\psi_Y = \psi_X \in (0, u_X)$, $C_{\psi_Y} = K_g^{\psi_X} C_{\psi_X} > 0$, and $r_{\psi_X}, r_{\psi_Y} \in (0, 1)$.*

Proof: By the definition of Lipschitz continuity,

$$\left|Y_k - Y_k'\right| = \left|g(X_k) - g(X_k')\right| \leq K_g \left|X_k - X_k'\right| \quad \text{for all } k \geq 0 \text{ a.s.}$$

Taking $\psi_Y = \psi_X \in (0, u_X)$ so that the expectations $\mathrm{E}\left[\left|Y_k - Y_k'\right|^{\psi_Y}\right]$ and $\mathrm{E}\left[\left|X_k - X_k'\right|^{\psi_X}\right]$ both exist, we see that $\{Y_k : k \geq 0\}$ satisfies the GMC condition (2) with constants $C_{\psi_Y} = K_g^{\psi_X} C_{\psi_X} > 0$, and $r_{\psi_X}, r_{\psi_Y} \in (0, 1)$. Note that C_{ψ_Y} and r_{ψ_Y}(resp. C_{ψ_X} and r_{ψ_X}) depend on the transformed process (resp., base process); and thus in general $C_{\psi_Y} \neq C_{\psi_X}$ and $r_{\psi_Y} \neq r_{\psi_X}$. ∎

4.2. Convex/Concave Transformations

The main idea in this subsection—formalized in Theorem 3—is to pick a sufficiently smooth, well-behaved transformation $Y_k = g(X_k)$ for $k \geq 0$ such that the transformed process $\{Y_k : k \geq 0\}$ is a GMC process that is readily verified to satisfy Equations (2)–(4).

THEOREM 3. *Suppose that the base process $\{X_k : k \geq 0\}$ satisfies Assumption (33), and the function $g(x)$, $x \in \mathbb{R}$, satisfies the following conditions:*

$$\textit{The derivative } g'(x) \textit{ exists at every } x \in \operatorname{int} \mathbb{S}_X, \tag{34}$$

$$\textit{The function } g(x) \textit{ is convex or concave on } \operatorname{int} \mathbb{S}_X, \tag{35}$$

$$\mathrm{E}\left[\left|g(X_0)\right|^{u_Y}\right] < \infty \textit{ for some } u_Y > 2, \textit{ and} \tag{36}$$

$$\mathrm{E}\left[\left|g'(X_0)\right|^{\mathfrak{v}}\right] < \infty \textit{ for some } \mathfrak{v} > 0. \tag{37}$$

Then the transformed process $\{Y_k = g(X_k) : k \geq 0\}$ satisfies the GMC condition (2) for all $\psi = \psi_Y \in \left(0, u_Y\right)$.

Theorem 3 is proven in Appendix A1.

THEOREM 4. *Suppose that the base process $\{X_k : k \geq 0\}$ satisfies Assumption (33), and the function $g(x)$, $x \in \mathbb{R}$, satisfies the following conditions:*

$$\textit{The derivative } g'(x) \textit{ exists at every } x \in \operatorname{int} \mathbb{S}_X, \tag{38}$$

$$\textit{There exist a quasiconvex function } h(x) \textit{ on } \operatorname{int} \mathbb{S}_X \textit{ and a finite constant } C < \infty \textit{ such that}$$

$$|g'(x)| \leq C h(x) \textit{ for all } x \in \operatorname{int} \mathbb{S}_X, \tag{39}$$

$$\mathrm{E}\left[\left|g(X_0)\right|^{u_Y}\right] < \infty \textit{ for some } u_Y > 2, \textit{ and} \tag{40}$$

$$\mathrm{E}\left[\left|h(X_0)\right|^{\mathfrak{v}}\right] < \infty \textit{ for some } \mathfrak{v} > 0. \tag{41}$$

Then the transformed process $\{Y_k = g(X_k) : k \geq 0\}$ satisfies the GMC condition (2) for all $\psi = \psi_Y \in \left(0, u_Y\right)$.

Proof: Let $\{X_k : k \geq 0\}$ and $\{X'_k : k \geq 0\}$ denote paired versions of the base process. Following in parallel the steps leading to Equation (A.44) in the proof of Corollary 1 in §A8, **[← NB: This step skips around to the Online Appendix. It would be really nice if we could somehow avoid this. Is it possible to shorten the proof a bit using some of the machinery we now have?]** we exploit the continuity of $f_X(x)$ and the differentiability of $g(x)$ on $\operatorname{int}\mathbb{S}_X$ to apply the mean value theorem, concluding that there exists $X^*_k \in \Big(\{X_k \wedge X'_k\}, \{X_k \vee X'_k\}\Big)$ for which we have

$$\begin{aligned} \left|g(X_k) - g(X'_k)\right| &= \left|g'(X^*_k)\right|\left|X_k - X'_k\right| \text{ for every } k \geq 0 \text{ a.s.} \\ &\leq C\left|h(X^*_k)\right|\left|X_k - X'_k\right| \text{ for every } k \geq 0 \text{ a.s.} \end{aligned} \tag{42}$$

Since the function $p(\mathfrak{s}) \equiv \mathfrak{s}^{\mathfrak{v}} : (0,\infty) \mapsto (0,\infty)$ is nondecreasing, we see that the composition $|h(x)|^{\mathfrak{v}}$ of the function $|h(x)|$ with the function $p(\mathfrak{s})$ is quasiconvex on $\operatorname{int}\mathbb{S}_X$ (Boyd and Vandenberghe 2004, §3.4.4, p. 102); therefore by the definition of a quasiconvex function (Boyd and Vandenberghe 2004, Eq. (3.19), p. 98), we have

$$\begin{aligned} \left|h(X^*_k)\right|^{\mathfrak{v}} &\leq \max\left\{\left|h(X_k \wedge X'_k)\right|^{\mathfrak{v}}, \left|h(X_k \vee X'_k)\right|^{\mathfrak{v}}\right\} \\ &= \max\left\{\left|h(X_k)\right|^{\mathfrak{v}}, \left|h(X'_k)\right|^{\mathfrak{v}}\right\} \\ &\leq \left|h(X_k)\right|^{\mathfrak{v}} + \left|h(X'_k)\right|^{\mathfrak{v}} \text{ for all } k \geq 0 \text{ a.s.} \end{aligned} \tag{43}$$

Taking the expected values of both sides of Equation (43), by Assumption (41) we have

$$\mathrm{E}\left[\left|h(X^*_k)\right|^{\mathfrak{v}}\right] \leq 2\mathrm{E}\left[\left|h(X_0)\right|^{\mathfrak{v}}\right] < \infty \text{ for all } k \geq 0. \tag{44}$$

Without loss of generality, we can assume $\mathfrak{v} \in (0, u_X)$. Then by Theorem 1, $\{X_k : k \geq 0\}$ satisfies the GMC condition for $\psi_X = \mathfrak{v}$; i.e., there exist $C_{\psi_X} \in (0,\infty)$ and $r_{\psi_X} \in (0,1)$ such that

$$\mathrm{E}\left[\left|X_k - X'_k\right|^{\psi_X}\right] \leq C_{\psi_X} r^k_{\psi_X} \text{ for all } k \geq 0. \tag{45}$$

Thus, we can prove that $\{Y_k = g(X_k) : k \geq 0\}$ satisfies the GMC condition for $\psi_Y = \mathfrak{v}/2$,

$$\begin{aligned} \mathrm{E}\left[\left|Y_k - Y'_k\right|^{\psi_Y}\right] &= \mathrm{E}\left[\left|g(X_k) - g(X'_k)\right|^{\mathfrak{v}/2}\right] \\ &= C^{\mathfrak{v}/2}\mathrm{E}\left[\left|h(X^*_k)\right|^{\mathfrak{v}/2}\left|X_k - X'_k\right|^{\mathfrak{v}/2}\right] && \text{(by Equation (42))} \\ &\leq C^{\mathfrak{v}/2}\left\{\mathrm{E}\left[\left|h(X^*_k)\right|^{\mathfrak{v}}\right]\mathrm{E}\left[\left|X_k - X'_k\right|^{\mathfrak{v}}\right]\right\}^{1/2} && \text{(by the Cauchy–Schwarz inequality)} \\ &\leq C_{\psi_Y} r^k_{\psi_Y} \text{ for all } k \geq 0, \end{aligned}$$

where, by Equations (44) and (45),

$$C_{\psi_Y} \equiv C^{\mathfrak{v}/2}\left\{2C_{\psi_X}\mathrm{E}\left[\left|h(X_0)\right|^{\mathfrak{v}}\right]\right\}^{1/2} \text{ and } r_{\psi_Y} \equiv r^{1/2}_{\psi_X}.$$

Theorem 1 together with the moment condition (40) ensures that $\{Y_k = g(X_k) : k \geq 0\}$ satisfies the GMC condition for all $\psi = \psi_Y = \mathfrak{v}/2 \in (0, u_Y)$. ∎

5. A Potpourri of Examples Involving Transformations

This section shows that a number of stochastic processes that may be of some interest to the simulation community satisfy the GMC condition. Specifically, we use Theorems 2 and 3 (from §§4.1 and 4.2 above) and Corollary 1 (in §5.2.3 below) to transform a GMC process into other GMC processes. These examples are meant to: (i) illustrate how and when to use the theorems, and (ii) obtain GMC processes that might be interesting and useful for purposes of stress testing simulation-analysis procedures in our concurrent research efforts (see §10).

§5.1 presents several elementary examples to get things going. §5.2 proves that three variants variants of the AR(1) process are GMC, namely, ARG(1), EAR(1), and ARTOC processes. §5.3 shows that various ARTOC processes with specific marginals of interest to the simulation community are GMC. §5.4 summarizes the findings of the section and opines on a surprising complementarity between Theorems 2 and 3.

5.1. Some Starter Examples

We begin with simple examples to demonstrate the applicability (or non-applicability) of Theorems 2 and 3. The next example is a trivial case for which both theorems are satisfied at once.

Example 2. Suppose that $\{X_k : k \geq 0\}$ is a GMC process whose components are continuous r.v.'s with marginal p.d.f. $f_X(x) = 3x^2$, for $0 \leq x \leq 1$. Consider the transformation $g(x) = x^2$, so that $Y_k = g(X_k) = X_k^2$, for $k \geq 0$, with resulting marginal p.d.f. $f_Y(y) = \frac{3}{2}y^{1/2}$ for $0 \leq y \leq 1$. It is clear that $g(x)$ is Lipschitz continuous on $0 \leq x \leq 1$, whence Theorem 2 is applicable. It is also obvious that all of the sufficient conditions listed in Theorem 3 hold. Thus, either theorem could be applied to show that $\{Y_k : k \geq 0\}$ is GMC. □

Example 3. Suppose that $\{X_k : k \geq 0\}$ satisfies the GMC condition (2). Since $g(x) = \cos(\gamma x)$ and $g(x) = \sin(\gamma x)$ have bounded derivatives for any constant γ, we have that these functions are Lipschitz continuous. Thus, Theorem 2 implies that $\{\cos(\gamma X_k) : k \geq 0\}$ and $\{\sin(\gamma X_k) : k \geq 0\}$ are GMC. However, for this example, Theorem 3 does not apply, since, in both cases, $g(x)$ is neither convex nor concave. □

The Lipschitz condition is often useful because it can also be applied to *non-differentiable* functions $g(x)$, where the assumptions of Theorem 3 might not hold.

Example 4. The two non-differentiable functions, $g(x) = (x - y)^+ = \max(x - y, 0)$ for some threshold $y \in \mathbb{R}$ and $g(x) = |x|$, preserve the GMC condition, since both functions are Lipschitz continuous with the constant $K_g = 1$. Thus, if $\{X_k : k \geq 0\}$ is GMC, then for the two choices of $g(x)$ given here, $\{g(X_k) : k \geq 0\}$ is also GMC. □

Although the Lipschitz condition is so far looking very good, it is not without its own issues, as we now demonstrate with an example of interest in the simulation domain.

Example 5. Consider a standard M/M/1 queueing system with exponential interarrivals and services with respective interarrival and service rates α and β, where $\alpha < \beta$, so that the traffic intensity is $\tau \equiv \alpha/\beta < 1$. If we initialize the system in steady-state, then the stream of waiting times $\{W_k : k \geq 0\}$ is stationary with c.d.f.

$$F_W(w) = \begin{cases} 0, & \text{if } w < 0, \\ 1-\tau, & \text{if } w = 0, \\ 1-\tau e^{-\gamma w}, & \text{if } w > 0, \end{cases}$$

where $\gamma \equiv \beta - \alpha$; see, e.g., Asmussen and Glynn (2007, p. 4).

Dingeç et al. (2022, Theorem 1) showed that the base waiting-time process $\{W_k : k \geq 0\}$ is GMC. We will use our Theorem 3 to establish that the process arising from the transformation $g(W_k) = W_k^a$ is also GMC for any $a > 0$. **[I believe that Lipschitz seems to fail for all $a \neq 1$? Please check.]** To begin with, we immediately see that the theorem's Assumptions (34)–(35) are satisfied trivially. In order to establish Assumptions (36) and (37), observe that

$$\mathrm{E}[W^a] = 0 \cdot (1-\tau) + \int_0^\infty w^a \tau\gamma e^{-\gamma w}\, \mathrm{d}w = \frac{\tau\,\Gamma(a+1)}{\gamma^a} < \infty \quad \text{for any } a \geq 0, \tag{46}$$

where $\Gamma(\cdot)$ is the gamma function. Moreover, hypothesis (33) holds since the base process is GMC, Equation (46) implies that all moments are finite, and the p.d.f. of W_k is continuous except for the endpoint $w = 0$. Equation (46) implies that

$$\mathrm{E}\big[|g(W_0)|^{u_Y}\big] = \mathrm{E}\big[W_0^{a u_Y}\big] = \frac{\tau\,\Gamma(a u_Y + 1)}{\gamma^{a u_Y}} < \infty \quad \text{for any } a > 0 \text{ and } u_Y > 2$$

and

$$\mathrm{E}\big[|g'(W_0)|^{\mathfrak{v}}\big] = \mathrm{E}\big[a W_0^{(a-1)\mathfrak{v}}\big] = \frac{a^{\mathfrak{v}}\,\tau\,\Gamma((a-1)\mathfrak{v}+1)}{\gamma^{(a-1)\mathfrak{v}}} < \infty \quad \text{for any } \mathfrak{v} > 0 \text{ and } a > 1 - \tfrac{1}{\mathfrak{v}} > 0.$$

With all of the assumptions in hand, Theorem 3 establishes that $\{W_k^a : k \geq 0\}$ is GMC. Note that a bonus direct proof from the definition of the GMC condition for the $a = 2$ case is given in Appendix A2.

Figure 2 illustrates MC results based on a sample of $n = 1{,}000{,}000$ consecutive autocorrelated waiting times, $W_1, W_2, \ldots, W_n$. We display the sample histogram (a combined discrete-continuous distribution owing to the point mass at $w = 0$) for the case $\alpha = 0.9$, $\beta = 1.0$, and $g(w) = \sqrt{w}$ (i.e., $a = 0.5$), as well as sample a.c.f.'s for the cases $a = 0.5$, 1, and 2, for lags $\ell = 0(1)500$, where the sample a.c.f. is defined as

$$\widehat{\rho}_\ell \equiv \widehat{\mathrm{Corr}}(W_1^a, W_{1+\ell}^a) \equiv \frac{\sum_{i=1}^{n-\ell}(W_i^a - \overline{W}_n^a)(W_{i+\ell}^a - \overline{W}_n^a)}{\sum_{i=1}^{n}(W_i^a - \overline{W}_n^a)^2} \quad \text{for lags } \ell \geq 1,$$

where, of course, $\overline{W}_n^a \equiv \sum_{k=1}^n W_i^a/n$. We see that the a.c.f.'s slowly decrease to 0, with higher values of the exponent a taking longer to do so (see Blomqvist 1967, for the case $a = 1$). □

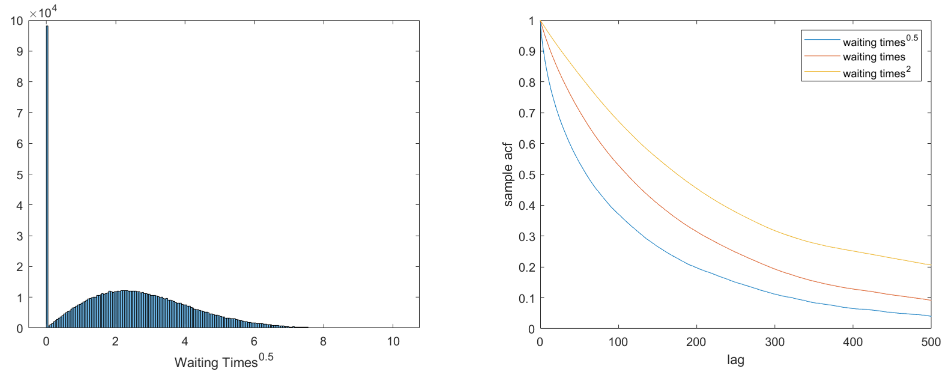


Figure 2 Monte Carlo results for an M/M/1 waiting-time base process with interarrival rate $\alpha = 0.9$ and service rate $\beta = 1.0$. Sample p.d.f. (left) for $g(w) = \sqrt{w}$ and sample a.c.f.'s (right) for $g(w) = \sqrt{w}$, w, and w^2. All MC results are based on a single run of $n = 1{,}000{,}000$ consecutive waiting times, $W_1, W_2, \ldots, W_n$.

5.2. ARG(1), EAR(1), and ARTOC Processes Are GMC

The next three subsections give conditions under which the ARG(1), EAR(1), and ARTOC variants of the AR(1) process are GMC.

5.2.1. ARG(1) Processes Consider a process $\{Y_k : k \geq 0\}$ having the form of an AR(1) as in Equation (28), but instead of i.i.d. zero-mean normal innovations $\{\varepsilon_k : k \geq 1\}$, we use arbitrary i.i.d. zero-mean innovations $\{\eta_k : k \geq 1\}$, again assuming that Y_0 follows the stationary distribution and is independent of $\{\eta_k : k \geq 1\}$. We refer to this AR(1) **g**eneralization by the nomenclature ARG(1). We now give a simple condition under which the GMC condition holds for the ARG(1).

Lemma 4. *If there exists a constant $\psi > 0$ such that $\mathrm{E}[|Y_0|^\psi] < \infty$, then the ARG(1) process is GMC.*

Proof: Similar to the case of normal innovations, we mimic Equation (30) and find that

$$Y_k = \beta^k Y_0 + W_k, \text{ for } k \geq 1, \text{ where } W_k = \sum_{i=1}^{k} \beta^{k-i} \eta_i.$$

By defining the coupled process $Y_k' \equiv \beta^k Y_0' + W_k$, $k \geq 1$, for an independent copy Y_0' of Y_0, we obtain

$$\mathrm{E}\big[\big|Y_k - Y_k'\big|^\psi\big] = \mathrm{E}\big[\big|Y_0 - Y_0'\big|^\psi\big] |\beta|^{k\psi}. \tag{47}$$

By Lemma 1 in Dingeç et al. (2022),

$$|Y_0 - Y_0'|^\psi \leq \max\{1, 2^{\psi-1}\}(|Y_0|^\psi + |Y_0'|^\psi),$$

so that

$$\mathrm{E}\big[\big|Y_0 - Y_0'\big|^\psi\big] \leq \max\{2, 2^\psi\} \mathrm{E}\big[\big|Y_0\big|^\psi\big].$$

By assumption, the first term on the right-hand side of (47) is finite, and so the process $\{Y_k, k \ge 0\}$ satisfies the GMC condition (2) with $C_\psi = \mathrm{E}[|Y_0 - Y_0'|^\psi] < \infty$ and $r_\psi = |\beta|^\psi$. ∎

Lemma 4 is interesting in that its proof is in some sense easier than the task of actually coming up with examples of its use. Of course, Lemma 3 (normal innovations) is an obvious special case. But the roadblock in implementing practical applications of Lemma 4 arises from the necessity to either: (i) select a desired steady-state marginal distribution for the Y_k's and then reverse engineer the resulting distribution of the innovations (Balakrishna 2021, Chaps. 3 and 4), or (ii) select the η_i's and then determine the steady-state distribution of the Y_k's (Balakrishna 2021, Chap. 6). In the companion paper Dingeç et al. (2023b), we illustrate the use of Lemma 4 on several examples; for instance, Cauchy innovations with $\psi < 1$ yield a GMC process.

5.2.2. The EAR(1) Process Is GMC The EAR(1) (**?**) is defined by

$$Y_k = \beta Y_{k-1} + \delta_k, \quad \text{with} \quad \delta_k \equiv \begin{cases} 0, & \text{w.p. } \beta \\ \varepsilon_k, & \text{w.p. } 1-\beta \end{cases} \quad \text{for } k \ge 1, \tag{48}$$

$0 \le \beta < 1$, $Y_0 \sim \mathrm{Exp}(1)$, and the ε_k's are i.i.d. Exp(1) r.v.'s that are independent of Y_0. The EAR(1) has Exp(1) marinals and has the same covariance structure as the AR(1), except that $0 \le \beta < 1$, that is, $\mathrm{Cov}(Y_i, Y_{i+k}) = \beta^{|k|}$ for all $k \in \mathbb{Z}$.

Lemma 5. *The EAR(1) process (48) is GMC.*

Proof: Using the proof of Lemma 3 as our guide, we generate a coupled version of the process $\{Y_k : k \ge 0\}$,

$$Y_0' = \varepsilon_0' \quad \text{and} \quad Y_k' = \beta Y_{k-1}' + \delta_k \ \text{ for } \ k \ge 1, \tag{49}$$

where $\varepsilon_0' \sim \mathrm{Exp}(1)$ and is independent of all of the ε_k's. From Equations (48) and (49), we find that

$$Y_k = \beta^k \varepsilon_0 + \sum_{i=1}^{k} \beta^{k-i} \varepsilon_i \ \text{ and } \ Y_k' = \beta^k \varepsilon_0' + \sum_{i=1}^{k} \beta^{k-i} \varepsilon_i \ \text{ for } \ k \ge 0, \tag{50}$$

resulting in $|Y_k - Y_k'| = \beta^k |\varepsilon_0 - \varepsilon_0'|$. The quantity $\varepsilon_0 - \varepsilon_0'$ is the difference between two independent Exp(1) r.v.'s, which is well known to be a Laplace r.v. It is also true that all moments of the Laplace exist; and so the remainder of the proof proceeds analogously to that of the AR(1). □

5.2.3. ARTOC Processes In this subsection, we provide sufficient conditions for transforming a base AR(1) process that is GMC into an ARTOC process that is also GMC. As we will detail below, an ARTOC process is obtained using the inversion method on the AR(1) such that certain assumptions on the resulting marginal distribution are satisfied.

Definition 1. Suppose that $\{Y_k : k \geq 0\}$ is a stationary process having a given marginal p.d.f. $f_Y(y)$ and c.d.f. $F_Y(y)$, and satisfying Equations (3) and (4). Further suppose that the p.d.f.'s support $\mathbb{S} \equiv \{y \in \mathbb{R} : f_Y(y) > 0\}$ is an interval in $\mathbb{R}$ such that $f_Y(y)$ is continuous at every $y \in \mathbb{R}$ except possibly at the endpoints (if any) of $\mathbb{S}$. The boundary of $\mathbb{S}$ consists of the endpoints of $\mathbb{S}$ and is denoted by $\operatorname{bd}\mathbb{S}$; and the interior of $\mathbb{S}$ is given by $\operatorname{int}\mathbb{S} \equiv \mathbb{S} \setminus \operatorname{bd}\mathbb{S}$. Let $\{Z_k : k \geq 0\}$ be an AR(1) process as per Equation (28). Finally, we say that $\{Y_k : k \geq 0\}$ is an *ARTOC process* if it defined by

$$Y_k = \xi(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) \equiv F_Y^{-1}\left\{\Phi\left[(\beta^k/\sigma_\varepsilon)\varepsilon_0 + \sum_{i=1}^{k}\beta^{k-i}\varepsilon_i\right]\right\} = F_Y^{-1}[\Phi(Z_k)] \quad \text{for} \quad k \geq 0, \tag{51}$$

where the last equality is a direct result of Equation (30). Thus, for any ARTOC process, the appropriate transformation function is $g(z) = F_Y^{-1}[\Phi(z)]$. □

Definition 1 allows us to easily simulate the $\{Y_k : k \geq 0\}$ process. Moreover, notice that $Z_k \sim \text{Nor}(0,1)$; so inversion implies $\Phi(Z_k) \sim \text{Unif}(0,1)$; and then, since $F_Y^{-1}(\cdot)$ is well-defined, $Y_k = F_Y^{-1}[\Phi(Z_k)]$ has the desired marginal. Similarly, we can generate the associated coupled ARTOC process,

$$Y_k' = \xi(\varepsilon_0', \varepsilon_1, \ldots, \varepsilon_k) \equiv F_Y^{-1}\left\{\Phi\left[(\beta^k/\sigma_\varepsilon)\varepsilon_0' + \sum_{i=1}^{k}\beta^{k-i}\varepsilon_i\right]\right\} = F_Y^{-1}[\Phi(Z_k')] \quad \text{for} \quad k \geq 0. \tag{52}$$

The following useful result follows so obviously from Theorem 2 that no proof is necessary.

Corollary 1. *Suppose that the ARTOC process $\{Y_k : k \geq 0\}$ specified by Definition 1 satisfies the condition*

$$\mathfrak{M} \equiv \sup_{z\in\mathbb{R}} |\mathfrak{L}(z)| < \infty, \quad \textit{where} \quad \mathfrak{L}(z) \equiv g'(z) = \frac{\phi(z)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} \quad \textit{for } z \in \mathbb{R}. \tag{53}$$

Then $\{Y_k : k \geq 0\}$ satisfies the GMC condition (2) for every $\psi > 0$ and the corresponding values of $C_\psi > 0$ and $r_\psi \in (0,1)$ given by Equation (32).

Appendix A8 gives an instructive alternative proof that incorporates from first principles a pretty application of a stochastic mean value theorem. Besides being an interesting proof in its own right, we refer to some of its steps when we prove Theorem 3 in Appendix A1. **[← Would really prefer not to skip around appendices when we do that proof in Appendix A1. Can we somehow make it self-contained?]**

5.3. Selected ARTOC Examples

Having established in §5.2.3 conditions under which ARTOC processes are GMC, we now present a number of specific examples, targeted at those of interest to simulation practitioners. Recall that, for the ARTOC case, one can attempt to apply either Theorem 2 or. equivalently, its Corollary 1.

EXAMPLE 6. [Give motivation.] Consider the ARTOC–Uniform process defined by a sequence of serially dependent uniforms $\{U_k = \Phi(Z_k) = F_U^{-1}[\Phi(Z_k)] : k \geq 0\}$, where $\{Z_k : k \geq 0\}$ is an AR(1) process. Since the transformation function $g(z) = \Phi(z)$, so that

$$|\mathfrak{L}(z)| = |g'(z)| = |\phi(z)| \leq K_g = 1/\sqrt{2\pi} \quad \text{for all } z \in \mathbb{R},$$

we have that Theorem 2 and, equivalently, its Corollary 1, are satisfied. Therefore, the ARTOC–Uniform process is GMC. Theorem 3 does not apply, since $g(z) = \Phi(z)$ is neither convex nor concave. □

EXAMPLE 7. [Give motivation.] Consider an ARTOC–Exponential process with exponential marginals $\{E_k \equiv -\ell\text{n}(\Phi(Z_k)) : k \geq 0\}$, where $\{Z_k : k \geq 0\}$ is an AR(1) process. In this case, the transformation function $g(z) = -\ell\text{n}(\Phi(z))$ is *not* Lipschitz continuous since, by Equation (5) and symmetry,

$$\sup_{z\in\mathbb{R}} |\mathfrak{L}(z)| = \sup_{z\in\mathbb{R}} |g'(z)| = \sup_{z\in\mathbb{R}} \frac{\phi(z)}{\Phi(z)} = \sup_{z<0} \frac{\phi(z)}{\Phi(z)} = \sup_{s>0} \frac{\phi(-s)}{\Phi(-s)} = \sup_{s>0} \frac{\phi(s)}{\Phi(-s)} > \sup_{s>0} s = \infty; \tag{54}$$

and thus we cannot use Theorem 2 for this example. Luckily, Assumptions (35)–(37) of Theorem 3 are satisfied. First, we have $g'(z) = -\phi(z)/\Phi(z)$, which exists for all $z \in \mathbb{Z}$, so that Assumption (34) holds. Second, a little algebra reveals that

$$g''(z) = \frac{\phi(z)}{\Phi(z)} [z\Phi(z) + \phi(z)] .$$

If $z \geq 0$, it is obvious that $g''(z) \geq 0$. If $z < 0$, then an argument similar to that in Equation (54) again shows that $g''(z) \geq 0$. This takes care of Assumption (35). Third, note that $E_k = -\ell\text{n}(\Phi(Z_k)) \sim \text{Exp}(1)$, indicating that all moments exist and thereby satisfying Assumption (36). Fourth, to verify Assumption (37), it remains to show that $\mathrm{E}\big[|g'(Z_0)|^{\mathfrak{v}}\big]$ is finite for *some* $\mathfrak{v} > 0$. By the Law of the Unconscious Statistician with $\mathfrak{v} = 1$, we have

$$\mathrm{E}\big[|g'(Z_0)|\big] = \int_{\mathbb{R}} g'(z)\phi(z)\,\mathrm{d}z = \int_{\mathbb{R}} [\Phi(z)]^{-1}\phi^2(z)\,\mathrm{d}z < \infty, \tag{55}$$

where the finiteness follows by Lemma 1's Equation (6) with $c = 1$. Thus, Theorem 3 implies that $\{E_k : k \geq 0\}$ satisfies the GMC condition. □

EXAMPLE 8. We examine an ARTOC–Power$(a, b; \theta)$ process $\{Y_k : k \geq 0\}$, that is, an ARTOC process with the power-function marginal p.d.f. $f_Y(y) = \theta(y-a)^{\theta-1}/(b-a)^{\theta}$ for $y \in [a, b]$, where: $a < b$; the p.d.f.'s support $\mathbb{S} = (a, b]$ has interior $\operatorname{int}\mathbb{S} = (a, b)$ and boundary $\operatorname{bd}\mathbb{S} = \{a, b\}$; and the shape parameter $\theta > 0$. The power-function distribution is often used to model real-world processes encountered in finance, population demographics, physics, and computer simulation. It arises in cases where there are occasional outliers, while the majority of observations have more-moderate values (see Pareto 1896, Mitzenmacher 2000, Clauset et al. 2009, just to cite a few references).

For ease of exposition, we take $a = 0$ and $b = 1$, in which case the marginal c.d.f. is

$$F_Y(y) = \begin{cases} 0 & \text{if } y < 0, \\ y^\theta & \text{if } y \in [0,1], \\ 1 & \text{if } y > 0; \end{cases}$$

and the inverse c.d.f. is $F_Y^{-1}(s) = s^{1/\theta}$ for $s \in (0,1)$. Thus, the transformation function in play here is $g(z) = F_Y^{-1}[\Phi(z)] = [\Phi(z)]^{1/\theta}$. Unfortunately, $g(z)$ is neither convex nor concave, thus violating Condition (35) of Theorem 3.

On the other hand, Theorem 2 / Corollary 1 will work. The sufficient condition (53) for the GMC property requires the finiteness of the supremum over all $z \in \mathbb{R}$ of the auxiliary function

$$\mathfrak{L}(z) = g'(z) = \frac{\phi(z)}{\theta[\Phi(z)]^{(\theta-1)/\theta}}. \tag{56}$$

Because the function $g'(z)$ in Equation (56) is continuous on $\mathbb{R}$, we verify condition (53) by showing that this function tends to 0 as $z \longrightarrow \pm\infty$. The limits $\lim_{z\to\infty} \phi(z) = 0$ and $\lim_{z\to\infty} \Phi(z) = 1$ together with the requirement $\theta > 0$ ensure that

$$\lim_{z\longrightarrow\infty} g'(z) = 0. \tag{57}$$

Since $\phi(z)$ is symmetric about the origin, we also have

$$\begin{aligned}
\lim_{z\to-\infty} \frac{\phi(z)}{\theta[\Phi(z)]^{(\theta-1)/\theta}} &= \lim_{s\to\infty} \frac{\phi(s)}{\theta[\Phi(-s)]^{(\theta-1)/\theta}} \\
&= \frac{1}{\theta} \lim_{s\to\infty} \left[\frac{s\phi(s)/(s^2+1)}{\Phi(-s)}\right] \cdot \left(\frac{s^2+1}{s}\right) \cdot [\Phi(-s)]^{1/\theta} \qquad (58) \\
&< \frac{1}{\theta} \lim_{s\to\infty} [1] \cdot (2s) \cdot [\Phi(-s)]^{1/\theta} \\
&= \frac{2}{\theta} \lim_{s\to\infty} \left[\int_{-\infty}^{-s} s^\theta \phi(z)\,\mathrm{d}z\right]^{1/\theta} \\
&< \frac{2}{\theta} \lim_{s\to\infty} \left[\int_{-\infty}^{-s} |z|^\theta \phi(z)\,\mathrm{d}z\right]^{1/\theta} = 0, \qquad (59)
\end{aligned}$$

where: (i) Equation (5) ensures that the term enclosed in large square brackets on the RHS of Equation (58) is positive and less than 1; (ii) the term enclosed in large parentheses on the RHS of Equation (58) is positive and less than $2s$ when $s > 1$; and (iii) the limit of 0 on the RHS of Equation (59) follows from the dominated convergence theorem and the observation paralleling Equation (31) that

$$\int_{-\infty}^{\infty} |z|^\theta \phi(z)\,\mathrm{d}z \le \left[\int_{-\infty}^{\infty} |z|^{\lceil\theta\rceil} \phi(z)\,\mathrm{d}z\right]^{\theta/\lceil\theta\rceil} < \infty. \tag{60}$$

Thus, by Theorem 2 / Corollary 1 and Equations (56)–(60), the process $\{Y_k : k \ge 0\}$ is GMC for every $\psi > 0$ and the respective values of $C_\psi > 0$ and $r_\psi \in (0,1)$ in Equation (32).

We present some Monte Carlo results for the ARTOC–Power($a = 0, b = 1; \theta = 2$) process. First of all, Figure 3 depicts sample paths based on time series snippets of 100 consecutive observations from AR(1) processes for various β-values, along with their resulting ARTOC–Power(0, 1; 2) realizations. For the purpose of apples-to-apples comparison, the AR(1) processes are actually shifted up by $\mathrm{E}[Y_k] = (b\theta + a)/(\theta + 1) = 2/3$ so as to have the same mean as the ARTOC–Power(0, 1; 2) processes. Figure 3's three panels correspond to the cases $\beta = -0.8$ (left), $\beta = 0$ (middle), and $\beta = 0.8$ (right). We see that the AR(1) and ARTOC–Power(0, 1; 2) realization patterns are generally the same—which makes sense since the AR(1)'s values directly feed into the ARTOC's—though the AR(1)'s exhibits significantly greater variation. Note that the snippets become "smoother" as we move from $\beta = -0.8$ to 0 to 0.8—as anticipated based on well-known AR(1) properties. Figure 4 depicts the sample p.d.f. and sample a.c.f. for the AR(1) and ARTOC–Power(0, 1; 2) processes, both for the case $\beta = 0.8$, and now based on a sample of $n = 100{,}000$ consecutive autocorrelated observations, $Y_1, Y_2, \ldots, Y_n$, for lags $\ell = 0, 1, \ldots, 20$. We observe that the sample p.d.f. of $\{Y_k\}$ is indeed Power(0, 1; 2) (in fact, the right-triangular special case), and that the a.c.f. decreases exponentially almost in lock step with the underlying AR(1)'s a.c.f. $\mathrm{Cov}(Z_1, Z_{1+\ell}) = \beta^\ell = 0.8^\ell$ for $\ell = 0, 1, \ldots, 20$.

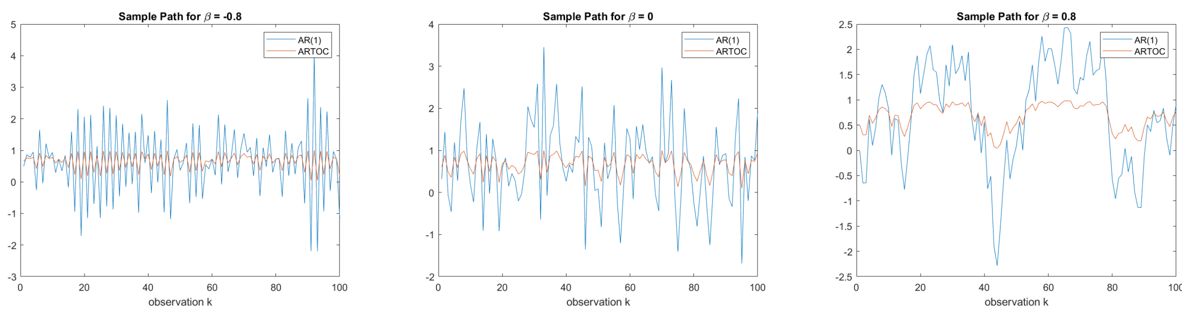


Figure 3 Monte Carlo results for AR(1) and ARTOC–Power(0, 1; 2) processes. Sample paths of 100 consecutive observations for the cases $\beta = -0.8$ (left), $\beta = 0$ (middle), and $\beta = 0.8$ (right).

Example 9. Over the last four decades, there has been remarkable growth in the literature on the Pareto family of distributions (Arnold 2015, pp. 17–18; Johnson et al. 1994, p. 573). Moreover in our experience, the ARTOC–Pareto process $\{Y_k : k \geq 0\}$ with a Pareto marginal distribution (often called the ARTOP process) provides a highly flexible and effective vehicle for stress-testing simulation-analysis procedures (Lada et al. 2007; Tafazzoli et al. 2011; Alexopoulos et al. 2016, 2017, 2019). With the location parameter $\gamma > 0$ and shape parameter $\theta > 0$, the Pareto p.d.f., c.d.f., and inverse c.d.f. are given respectively by

$$f_Y(y) = \begin{cases} \dfrac{\theta\gamma^\theta}{y^{\theta+1}} & \text{if } \; y \geq \gamma, \\ 0 & \text{if } \; y < \gamma; \end{cases}$$

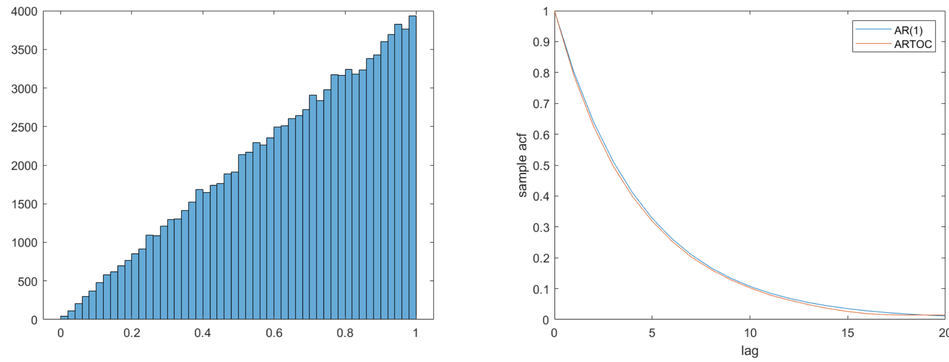


Figure 4 Monte Carlo results for the ARTOC–Power$(0,1;2)$ process with $\beta = 0.8$. Sample p.d.f. histogram (left) and exact AR(1) and sample ARTOC–Power$(0,1;2)$ a.c.f.'s (right), with all MC results based on a single run of $n = 100{,}000$ AR(1) observations, $Z_1, Z_2, \ldots, Z_n$.

$$F_Y(y) = \begin{cases} 1-(\gamma/y)^\theta & \text{if } y \ge \gamma, \\ 0 & \text{if } y < \gamma; \text{ and} \end{cases}$$

$$F_Y^{-1}(s) = \frac{\gamma}{(1-s)^{1/\theta}} \text{ for all } s \in (0,1).$$

For the Pareto r.v. Y_0, the marginal moment of order $\ell \in \mathbb{N}^* \equiv \{1,2,\ldots\}$ and the variance are

$$\mathrm{E}\big[Y_0^\ell\big] = \frac{\theta\gamma^\ell}{\theta-\ell} \text{ if } \theta > \ell, \quad \text{and} \quad \mathrm{Var}(Y_0) = \frac{\gamma^2\theta}{(\theta-1)^2(\theta-2)} \text{ if } \theta > 2. \tag{61}$$

Assuming $\theta > 2$, we will use Theorem 3 to show that the ARTOP process satisfies the GMC condition. Thus we again use the AR(1) process $\{Z_k : k \ge 0\}$ described in Equation (28) as the base process, and we take

$$g(z) = F_Y^{-1}[\Phi(z)] = \frac{\gamma}{[1-\Phi(z)]^{1/\theta}} = \frac{\gamma}{[\Phi(-z)]^{1/\theta}} \quad \text{for } z \in \mathbb{R} \text{ and } \theta > 2.$$

Because $\Phi(-Z_0) \sim \mathrm{Unif}(0,1)$, the inverse-transform method ensures that $Y_k \equiv g(Z_k)$ has the c.d.f. $F_Y(y)$ for $k \ge 0$; and thus we have

$$\mathrm{E}\Big[\big|g(Z_0)\big|^{u_Y}\Big] = \mathrm{E}\big[Y_0^{u_Y}\big] < \infty \text{ for } u_Y \in (2,\theta).$$

It is also easy to show that for $z \in \mathbb{R}$,

$$\mathfrak{L}(z) = g'(z) = \frac{\phi(z)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} = \frac{\gamma\phi(z)}{\theta\,[\Phi(-z)]^{\frac{\theta+1}{\theta}}} > 0 \quad \text{and}$$

$$g''(z) = g'(z)\left[\left(\frac{\theta+1}{\theta}\right)\frac{\phi(z)}{\Phi(-z)} - z\right] > 0,$$

where the last inequality follows easily from Equation (5) and the observation that $z\Phi(-z)/\phi(z) < 1$ for all $z \in \mathbb{R}$. Hence $g(z)$ is differentiable, strictly increasing, and strictly convex on $\mathbb{R}$.

In order to invoke Theorem 3, it remains to show that $\mathrm{E}\big[\big|g'(Z_0)\big|^{\mathfrak{v}}\big]$ is finite for some $\mathfrak{v} > 0$. Similar to the machinations leading to Example 7's Equation (55), we take $\mathfrak{v} = 1$ and obtain

$$\mathrm{E}\big[\big|g'(Z_0)\big|\big] = \int_{\mathbb{R}} g'(z)\phi(z)\,\mathrm{d}z = \frac{\gamma}{\theta}\int_{\mathbb{R}} [\Phi(-z)]^{-(\theta+1)/\theta}\,\phi^2(z)\,\mathrm{d}z < \infty,$$

where the finiteness follows by Lemma 1's Equation (6) with $c = (\theta+1)/\theta \in (0, 3/2)$ for $\theta > 2$.

Thus, Theorem 3 implies that the ARTOP process satisfies the GMC condition for all $\psi_Y \in \big(0, u_Y\big)$ and $u_Y \in (2, \theta)$ such that $\lceil u_Y \rceil < \theta$. (Note that if $2 < u_Y \le \lceil u_Y \rceil < \theta$, then the Assumption (36) of Theorem 3 follows from Equation (61) for the integer-order moments of a Pareto r.v. using Lyapounov's inequality in the same way that inequality was used to justify Equation (31) based on the integer-order moments of a normal r.v.)

Even though we do not provide details here, we remark for completeness that it is straightforward to show that one can not invoke Theorem 2 / Corollary 1 to prove that the ARTOP process is GMC. □

Example 10. Three major obstacles can impede the rapid development of the initial version of a large-scale simulation: 1) the lack of adequate data on the stochastic processes driving the simulation, 2) substantial uncertainty about the values of key deterministic parameters of the target system, and 3) the inability to collect and analyze quickly enough the data needed to overcome obstacles 1) and 2). In those situations, subject-matter experts are often asked to provide three subjective estimates of, for example, the time to perform a given activity in the operation of the target system—namely, the optimistic (minimum feasible) time a, the most likely (modal) time m, and the pessimistic (maximum feasible) time b so that $a < b$ and the duration of that activity is modeled as a triangular r.v., denoted as $\mathrm{Tria}(a, m, b)$, whose support has interior (a, b) and mode $m \in [a, b]$ (Law 2015, §6.11, pp. 375–380). For simplicity in this example, we discuss the ARTOC process having a $\mathrm{Tria}(0, m, 1)$ marginal distribution with location parameters $a = 0$ and $b = 1$; and separate analyses must be carried out for three cases: (i) $m = 1$; (ii) $m = 0$; and (iii) $0 < m < 1$. Note that these special cases easy generalize to an arbitrary $\mathrm{Tria}(a, m, b)$.

In case (i), the paired processes $\{Y_k : k \ge 0\}$ and $\big\{Y'_k : k \ge 0\big\}$ have the marginal p.d.f. of a power-function r.v. with $a = 0$, $b = 1$, and $\theta = 2$ (aka, a right-triangular distribution). Thus by Theorem 1 and the analysis in Example 8, we have $\mathrm{E}\big[\big|Y_k - Y'_k\big|^{\psi}\big] \le C_\psi r_\psi^k$ for all $\psi > 0$ and the associated constants $C_\psi > 0$ and $r_\psi \in (0, 1)$ given by Equation (32).

For case (ii) (a left-triangular distribution), consider the paired processes $\{Y_k : k \ge 0\}$ and $\big\{Y'_k : k \ge 0\big\}$ of case (i). By symmetry, we obtain the paired processes $\big\{\widetilde{Y}_k \equiv 1 - Y_k : k \ge 0\big\}$ and $\big\{\widetilde{Y}'_k \equiv 1 - Y'_k : k \ge 0\big\}$ that conform to case (ii). Moreover, we have

$$\big|\widetilde{Y}_k - \widetilde{Y}'_k\big| = \big|Y'_k - Y_k\big| \ \text{ for } \ k \ge 0;$$

and it follows immediately from case (i) that $\big\{\widetilde{Y}_k : k \ge 0\big\}$ is GMC for all $\psi > 0$, and $C_\psi > 0$ and $r_\psi \in (0, 1)$ given by Equation (32).

In case (iii), the marginal p.d.f., c.d.f., and inverse c.d.f. are respectively given by

$$f_Y(y) = \begin{cases} 0 & \text{if } \; y < 0, \\ 2y/m & \text{if } \; 0 \le y \le m, \\ 2(1-y)/(1-m) & \text{if } \; m \le y \le 1, \\ 0 & \text{if } \; y > 1; \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & \text{if } \; y < 0, \\ y^2/m & \text{if } \; 0 \le y \le m, \\ 1-(1-y)^2/(1-m) & \text{if } \; m \le y \le 1, \\ 1 & y > 1; \;\; \text{and} \end{cases}$$

$$F_Y^{-1}(s) = \begin{cases} (ms)^{1/2} & \text{if } \; 0 < s \le m, \\ 1-[(1-m)(1-s)]^{1/2} & \text{if } \; m \le s < 1. \end{cases}$$

It is easy to establish via graphical means (or via analytical means if we are willing to patiently crank out the second derivatives of $\mathfrak{L}_1(z)$ and $\mathfrak{L}_2(z)$) that the transformation function $g(z) = F_Y^{-1}[\Phi(z)]$ for $z \in \mathbb{R}$ is neither convex nor concave, so that Theorem 3 does not apply.

Instead, with an eye towards using Theorem 2, we define the auxiliary functions analogous to Equation (56) for $z \in \mathbb{R}$,

$$\mathfrak{L}_1(z) \equiv \frac{\phi(z)}{2F_Y^{-1}(\Phi(z))/m} \mathbf{1}_{\{0<F_Y^{-1}(\Phi(z)) \le m\}} = \frac{m^{1/2}\phi(z)}{2[\Phi(z)]^{1/2}} \mathbf{1}_{\{-\infty<z\le\Phi^{-1}[F_Y(m)]\}} \;\; \text{and}$$

$$\mathfrak{L}_2(z) \equiv \frac{\phi(z)}{2\big[1-F_Y^{-1}(\Phi(z))\big]/(1-m)} \mathbf{1}_{\{0<F_Y^{-1}(\Phi(z)) \le m\}} = \frac{(1-m)^{1/2}\phi(z)}{2[\Phi(-z)]^{1/2}} \mathbf{1}_{\{\Phi^{-1}[F_Y(m)]<z<\infty\}},$$

which implies that

$$\mathfrak{L}(z) = \mathfrak{L}_1(z) + \mathfrak{L}_2(z) \;\; \text{for} \;\; z \in \mathbb{R}.$$

Finally, we define $\mathfrak{M}_\nu \equiv \sup_{z\in\mathbb{R}} \mathfrak{L}_\nu(z)$ for $\nu = 1, 2$ and $\mathfrak{M} = \max\{\mathfrak{M}_1, \mathfrak{M}_2\}$ so that by the same argument given in Equations (56)–(60) for $\theta = 2$, we have

$$\mathfrak{M}_\nu < \infty \;\; \text{for} \;\; \nu = 1, 2; \;\; \text{hence} \;\; \mathfrak{M} < \infty.$$

It follows immediately that in all cases the ARTOC–Triangular process $\{Y_k : k \ge 0\}$ is GMC for all $\psi > 0$ and the corresponding values of $C_\psi > 0$ and $r_\psi \in (0, 1)$ given by Equation (32).

Analogous to our work in Example 8, we give Monte Carlo results for the ARTOC–Tria($a = 0, m = 0.5, b = 1$) process. Figure 5 depicts sample paths based on time series of 100 consecutive observations from AR(1) and ARTOC–Tria$(0, 0.5, 1)$ processes with comparable β-values. For comparison purposes, the AR(1)'s observations are shifted up by 0.5, yielding the same mean as the ARTOC-Tria$(0, 0.5, 1)$. Figure 5's three panels correspond to the cases $\beta = -0.8$ (left), $\beta = 0$ (middle), and $\beta = 0.8$ (right). As with Example 8, we see that the AR(1) and ARTOC–Tria$(0, 0.5, 1)$ realization patterns are in general the same, though the AR(1) again exhibits significantly greater variation than the ARTOC. Further, as before, the realizations become "smoother" moving from $\beta = -0.8$ to 0 to 0.8. Figure 6 depicts the sample p.d.f. and sample a.c.f. for the AR(1) and ARTOC–Tria$(0, 0.5, 1)$ processes, both for the case $\beta = 0.8$, and based on a larger sample of $n = 100{,}000$ consecutive autocorrelated observations, $Y_1, Y_2, \ldots, Y_n$. The sample p.d.f. of $\{Y_k\}$ appears to be Tria$(0, 0.5, 1)$, and the a.c.f. decreases exponentially, essentially matching the AR(1)'s $\text{Corr}(Z_1, Z_{1+\ell}) = 0.8^\ell$ for $\ell = 0, 1, \ldots, 20$. See the related Example 20 in §8 where we use mixtures of left- and right-triangular r.v.'s to produce not-quite-ARTOC processes having triangular (and not-quite-triangular) marginals. □

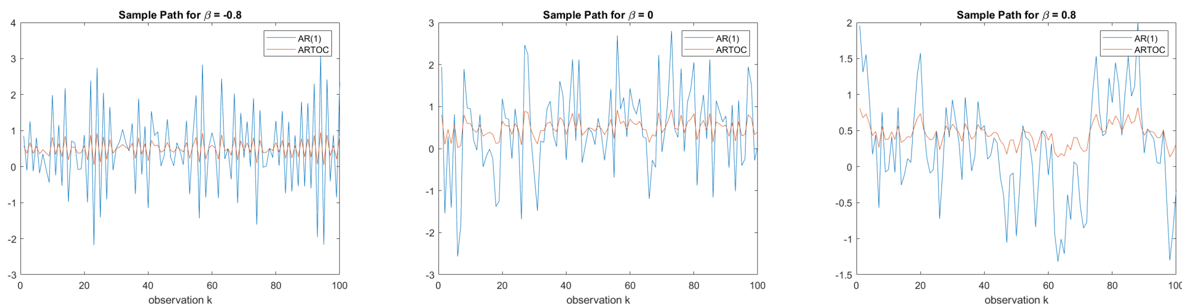


Figure 5 Monte Carlo results for AR(1) and ARTOC–Tria$(0, 0.5, 1)$ processes. Sample paths of 100 consecutive observations for the cases $\beta = -0.8$ (left), $\beta = 0$ (middle), and $\beta = 0.8$ (right).

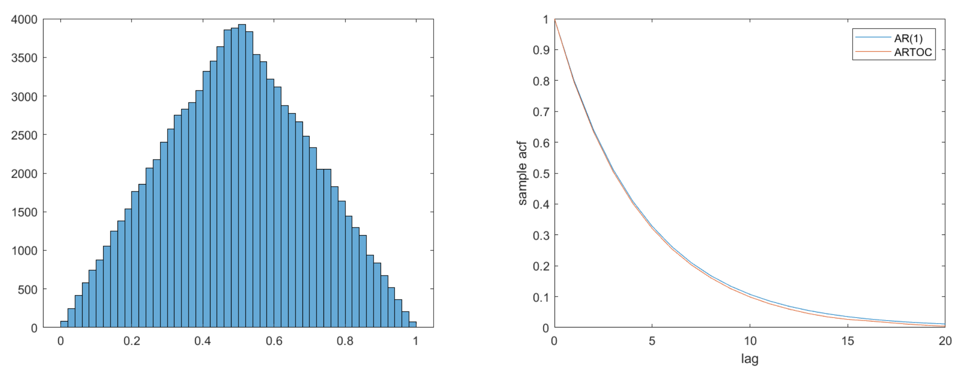


Figure 6 Monte Carlo results for the ARTOC–Tria$(0, 0.5, 1)$ process with $\beta = 0.8$. Sample p.d.f. (left) and exact AR(1) and sample ARTOC–Tria$(0, 0.5, 1)$ a.c.f.'s (right), with all MC results based on a single run of $n = 100{,}000$ AR(1) observations, $Z_1, Z_2, \ldots, Z_n$.

Example 11. In situations where a large dataset of observed responses is available and no standard distribution can adequately represent some of the prominent features of the empirical marginal distribution, the true (but unknown) p.d.f. $f_Y(y)$ can be approximated by a frequency polygon, which is a valid continuous p.d.f. defined by linear interpolation between the midpoints of a *deterministic* histogram with equal-width bins that is used to summarize the given dataset (Scott 2015, Chap. 3, pp. 51–56 and Chap. 4, pp. 101–110). The analysis given in Example 10 using Theorem 2 for the triangular p.d.f. is easily generalized to handle the $\mathfrak{d}$ bins of the frequency polygon (Scott 2015, Fig. 4.1, p. 101) so as to yield the functions $\{\mathfrak{L}_\nu(z) : z \in \mathbb{R};\ \nu = 1, \ldots, \mathfrak{d} - 1\}$ (corresponding to the support of each of the histograms cells) and the associated suprema

$$\mathfrak{M}_\nu < \infty \ \text{ for } \ \nu = 1, \ldots, \mathfrak{d} - 1, \ \text{ and } \ \mathfrak{M} = \max\{\mathfrak{M}_\nu : \nu = 1, \ldots \mathfrak{d} - 1\} < \infty.$$

Thus conditioned on the dataset used to approximate $f_Y(y)$, the associated ARTOC process $\{Y_k : k \geq 0\}$ satisfies the GMC condition for every $\psi > 0$ and the corresponding values of $C_\psi > 0$ and $r_\psi \in (0, 1)$ given by Equation (32). Of course, Theorem 3, in general, does not apply in this example because there is obviously no guarantee that a histogram will satisfy any of the required conditions. □

Example 12. Because the lognormal is used extensively in the engineering, life, medical, social, and natural sciences (Johnson et al. 1994, Chap. 14, §2, pp. 209–211), it is of interest to develop ARTOC processes with lognormal marginals that are GMC. With the "location" parameter $\mu \in \mathbb{R}$ and shape parameter $\sigma > 0$, the lognormal p.d.f., c.d.f., and inverse c.d.f. are respectively given by

$$f_Y(y) = \frac{1}{y\sigma(2\pi)^{1/2}} \exp\left[-\frac{(\log y - \mu)^2}{2\sigma^2}\right] \mathbf{1}_{\{y>0\}}; \tag{62}$$

$$F_Y(y) = \Phi\left(\frac{\log y - \mu}{\sigma}\right) \mathbf{1}_{\{y>0\}}; \ \text{ and} \tag{63}$$

$$F_Y^{-1}(s) = \exp[\mu + \sigma\Phi^{-1}(s)] \ \text{ for } \ 0 < s < 1. \tag{64}$$

It follows from Equations (62)–(64) that the transformation $g(z) = F_Y^{-1}[\Phi(z)] = \exp\left[\mu + \sigma\,\Phi^{-1}\big(\Phi(z)\big)\right] = \exp(\mu + \sigma z)$, and thus

$$\mathfrak{L}(z) = g'(z) = \frac{\phi(z)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} = \sigma\exp(\mu + z\sigma) \ \text{ for all } \ z \in \mathbb{R} \ \text{ so that } \ \mathfrak{M} = \lim_{z\to\infty} \sigma\exp(\mu + z\sigma) = \infty.$$

This precludes the use Equation (53) from Theorem 2 / Corollary 1 to verify that the ARTOC–Lognormal process $\left\{Y_k = F_Y^{-1}[\Phi(Z_k)] = \exp(\mu + \sigma Z_k) : k \geq 0\right\}$ is GMC when $\{Z_k : k \geq 0\}$ is an AR(1) process.

On the other hand, $g(z)$ is clearly positive and infinitely differentiable on $\mathbb{R}$ with $g'(z) = \sigma\exp(\mu + \sigma z) > 0$ and $g''(z) = \sigma^2\exp(\mu + \sigma z) > 0$ for all $z \in \mathbb{R}$. Thus, $g(z)$ is strictly convex, and $g'(z)$ is continuous and strictly increasing on $\mathbb{R}$. For simplicity, we examine the case in which $\mu = 0$ and $\sigma = 1$. It is straightforward to compute $\mathrm{E}\left[\left|g(Z_0)\right|^{u_Y}\right] = \mathrm{E}[\exp(u_Y Z_0)] = \exp\left(u_Y^2/2\right) < \infty$ for every $u_Y \in \mathbb{R}$; and $\mathrm{E}\left[\left|g'(Z_0)\right|^{\mathfrak{v}}\right] = \mathrm{E}\left[\exp\left(\mathfrak{v}Z_0\right)\right] = \exp\left(\mathfrak{v}^2/2\right) < \infty$ for every $\mathfrak{v} \in \mathbb{R}$. Hence Theorem 3 implies that the ARTOC process $\{Y_k = \exp(Z_k) : k \geq 0\}$ satisfies the GMC condition for every $\psi = \psi_Y \in (0, \infty)$. □

Example 13. **[Motivate the Weibull.]** The marginal distribution of the ARTOC–Weibull process has the p.d.f., c.d.f., and inverse c.d.f.

$$f_Y(y) = \frac{a}{b}\left(\frac{y}{b}\right)^{a-1} e^{-(y/b)^a}, \quad \text{for } y > 0;$$

$$F_Y(y) = 1 - e^{-(y/b)^a}, \quad \text{for } y > 0; \quad \text{and}$$

$$F_Y^{-1}(s) = b\big[-\ell\text{n}(1-s)\big]^{1/a}, \quad \text{for } 0 < s < 1,$$

where $a > 0$ and $b > 0$ are the shape and scale parameters, respectively. We immediately have

$$g(z) = F_Y^{-1}[\Phi(z)] = b\big(-\ell\text{n}[\Phi(-z)]\big)^{1/a} \quad \text{for } z \in \mathbb{R},$$

and

$$\mathfrak{L}(z) = g'(z) = \frac{\phi(z)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} = \frac{b\phi(z)}{a\,\Phi(-z)\big(-\ell\text{n}[\Phi(-z)]\big)^{(a-1)/a}} > 0 \quad \text{for } z \in \mathbb{R}. \tag{65}$$

Under this setup, it can be shown that the ARTOC–Weibull process satisfies the GMC condition, though we defer the proof to Appendix A3, owing to the tedious nature of the necessary calculations. In particular, we find that:

- The Lipschitz-continuity assumption of Theorem 2 is satisfied for the case $a \geq 2$; but it is not satisfied for $0 < a < 2$.
- The various conditions of Theorem 3 hold for the case $0 \leq a \leq 2$; but not for $a > 2$. [Actually, we still have to show it doesn't hold for $a > 2$.]

Evidently, the ARTOC–Weibull satisfies *both* theorems for the borderline case, $a = 2$. □

Example 14. **[Motivate the Laplace.] [This seems to be an example in which neither Theorem 2 nor 3 work!]** The marginal distribution of the ARTOC–Laplace process has the following p.d.f., c.d.f., and inverse c.d.f.,

$$f_Y(y) = \frac{1}{2}e^{-|y|}, \quad \text{for } y > \mathbb{R};$$

$$F_Y(y) = \begin{cases} \frac{1}{2}e^{y} & \text{if } y < 0, \\ 1 - \frac{1}{2}e^{-y} & \text{if } y \geq 0; \end{cases} \quad \text{and}$$

$$F_Y^{-1}(s) = \begin{cases} \ell\text{n}(2s) & \text{if } 0 \leq s < 1/2, \\ -\ell\text{n}(2(1-s)) & \text{if } 1/2 \leq s \leq 1. \end{cases} \tag{66}$$

Then we have

$$\mathfrak{L}(z) = g'(z) = \frac{\phi(z)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} = \begin{cases} \frac{\phi(z)}{f_Y\{\ell\text{n}[2\Phi(z)]\}} & \text{if } z < 0, \\ \frac{\phi(z)}{f_Y\{-\ell\text{n}[2\Phi(-z)]\}} & \text{if } z \geq 0 \end{cases} = \frac{\phi(z)}{\Phi(-|z|)} \quad \text{if } z \in \mathbb{R}. \tag{67}$$

Thus, $\mathfrak{L}(z) \to +\infty$ as $z \to \pm\infty$. This violates the assumptions of Theorem 2.

With regard to Theorem 3, note that, after a bit of pedestrian algebra and an invocation of Equation (5), we have

$$g''(z) = \begin{cases} \frac{\phi(z)[|z|\Phi(-|z|)-\phi(z)]}{\Phi^2(-|z|)} < 0 & \text{if } z<0, \\ \frac{\phi(z)[-z\Phi(-z)+\phi(z)]}{\Phi^2(-z)} > 0 & \text{if } z \ge 0. \end{cases} \tag{68}$$

Therefore, $g(z) = F_Y^{-1}[\Phi(z)]$ is neither convex nor concave, and so we cannot invoke Theorem 3 to establish whether or not the ARTOC–Laplace process is GMC. [I guess you could figure this out by graphing $g(z)$. In any case, will add an allusion to upcoming proof that it's actually GMC w/o use of Thms 2 or 3.] **[In an email, Kemal mentions that $|g'(z)|$ is quasi-convex as required by the proof of Theorem 3. What are the consequences of this observation?]**

[For purposes of maybe applying the quasi-convex generalization of Thm 3 that Jim/Kemal are working on, Jim asked me to see if the Laplace satisfies Equation (36), i.e., is $\mathrm{E}\big[|g(Z_k)|^{u_Y}\big] < \infty$ for some $u_Y > 2$? Here we go....] By Equation (66), we have

$$g(z) = F_Y^{-1}(\Phi(z)) = \begin{cases} \ell\mathrm{n}(2\Phi(z)) & \text{if } z<0, \\ -\ell\mathrm{n}(2\Phi(-z)) & \text{if } z \ge 0. \end{cases}$$

Since we only need to show that result holds for *some* $u_Y > 2$, we will (arbitrarily) take $u_Y = 3$. In that case, by symmetry and LOTUS,

$$\begin{aligned}
\mathrm{E}\big[|g(Z_k)|^{u_Y}\big] &= 2\int_0^\infty \big(-\ell\mathrm{n}(2\Phi(-z))\big)^3 \phi(z)\,\mathrm{d}z \\
&= 2\int_0^\infty \left[-\ell\mathrm{n}\left(\frac{2z\phi(z)}{1+z^2}\right)\right]^3 \phi(z)\,\mathrm{d}z \quad \text{(by the right-hand side of (5))} \\
&= 2\int_0^\infty \left[\frac{\ell\mathrm{n}(\pi/2)}{2} - \ell\mathrm{n}(z) + \frac{z^2}{2} + \ell\mathrm{n}\big(1+z^2\big)\right]^3 \phi(z)\,\mathrm{d}z \\
&\le 2\int_0^\infty \left[\frac{1}{4} - \ell\mathrm{n}(z) + \frac{3z^2}{2}\right]^3 \phi(z)\,\mathrm{d}z \quad \text{(since } z^2 \ge \ell\mathrm{n}\big(1+z^2\big) \text{ for all } z) \\
&\le 54\int_0^\infty \left[\frac{1}{64} + |\ell\mathrm{n}^3(z)| + \frac{27z^6}{8}\right] \phi(z)\,\mathrm{d}z \\
&\qquad \text{(since } (a+b+c)^3 \le \big(3\max\{a,b,c\}\big)^3 \le 27(a^3+b^3+c^3) \text{ for } a,b,c>0).
\end{aligned}$$

The first and third terms in the above integral are clearly finite because all moments of the standard normal are finite. The middle term is shown to be finite by paying a visit to Mathematica (Wolfram Research, Inc. 2022),

$$\int_0^\infty |\ell\mathrm{n}^3(z)|\phi(z)\,\mathrm{d}z = \frac{28\zeta(3) + 2\gamma^3 + 2\,\ell\mathrm{n}^3(2) + 3\gamma\big(\pi^2 + 2\,\ell\mathrm{n}^2(2)\big) + \pi^2\,\ell\mathrm{n}(8) + \gamma^2\ell\mathrm{n}(64)}{32} \doteq 2.355,$$

where $\zeta(\cdot)$ is the zeta function and γ is Euler's constant. Thus, Requirement (36) holds.

Finally, we show that Requirement (37) is satisfied for *some* $\mathfrak{v} > 0$, which we (arbitrarily) choose to be $\mathfrak{v} = 1$. In that case, LOTUS, symmetry, and the left-hand side of Inequality (7) imply that

$$\mathrm{E}\big[|g'(Z_k)|^{\mathfrak{v}}\big] = 2\int_0^\infty \frac{\phi(z)}{\Phi(-z)}\,\phi(z)\,\mathrm{d}z \le \int_0^\infty \frac{2z\phi(z)}{\phi(z)}\,\phi(z)\,\mathrm{d}z \le 4\int_0^\infty z\,\phi(z)\,\mathrm{d}z = 2\sqrt{\frac{2}{\pi}}. \quad \square$$

5.4. Summary of Examples

We have provided a number of examples of potential interest to the simulation community, establishing that certain transformations of GMC process remain GMC. Our main tools have been Theorem 2 (focusing on Lipschitz-continuous transformations) and Theorem 3 (which comes into play for convex/concave transformations). Table 1 summarizes some of our findings—in particular, which Theorem(s) work for which GMC processes?

One immediately notices that the two theorems seem to be complementary to each other. When one theorem works for a GMC process, the other likely can not be applied. **[Might we have a theorem here?]** For instance, in the ARTOC examples we considered, when the convexity/concavity condition of Theorem 3 in Equation (35) fails to hold, this means that there exists a point $x^\star$ such that $g''(x^\star)=0$, with $g''(x)>0$ for $x<x^\star$, and $g''(x)<0$ for $x>x^\star$. **[Do we still have to say something about the finiteness of $|g'(x)|$ as $x\to\pm\infty$? Also, what about the "opposite" case in which $g''(x)<0$ for $x<x^\star$, and $g''(x)>0$ for $x>x^\star$?]** This makes the first derivative a unimodal function with the maximum $g'(x^\star)<\infty$ and so makes the function $g(x)$ Lipschitz continuous with a bounded derivative $g'(x)\le g'(x^\star)<\infty$, indicating that Theorem 2 holds. Therefore, in our ARTOC examples, we observed that two theorems are complementary. **[←Except for the very special case of Weibull, eh?]** [Also, how about some comments for cases when Theorem 2 fails but Theorem 3 holds? Maybe mention that we'll be seeing some of this in upcoming Table 1.] [And dg is still thinking about the following: Because of the $\Phi(\cdot)$ that's inside of $g(z)$, it is unlikely that such a transformation will be convex/concave, thus suggesting that Theorem 3 doesn't have much of a chance.]

Lemma 6. [This result is only half-baked. Jim is working on it, maybe with Kemal's help. This has the potential to be an important finding that will simplify some of our previous findings — so it's worth a little effort.] *Suppose that a function $g(x)$ is continuous and differentiable on $\mathbb{R}$. Then it is not possible for the following two conditions to hold simultaneously: (i) the Lipschitz constant $K_g=\sup_{x\in\mathbb{R}}|g'(x)|<\infty$ and (ii) $g(x)$ is convex for all $x\in\mathbb{R}$ or concave for all $x\in\mathbb{R}$.*

Proof: Assume without loss of generality that $g''(x)\ge\epsilon$ for all $x\in\mathbb{R}$ and some $\epsilon>0$. Then for $a<b$, the second fundamental theorem of calculus implies that

$$(b-a)\epsilon \;\le\; \int_a^b g''(x)\,\mathrm{d}x \;=\; g'(b)-g'(a),$$

which implies

$$g'(b)\;\ge\; g'(a)+(b-a)\epsilon.$$

If we take $b > a + (K_g - g'(a))/\epsilon$, we obtain

$$g'(b) > g'(a) + \left[a + \frac{K_g - g'(a)}{\epsilon} - a\right]\epsilon = K_g,$$

which is a contraction. ※

Thus, under the conditions of Lemma 6, it follows that Theorem 2 and Theorem 3 cannot be applied simultaneously.

[I'm pretty sure that we can also provide a (trivial) example of a non-ARTOC transformation on a distribution having finite support that will satisfy both theorems.]

Process	Source	Theorem 2	Theorem 3
X_k^2	Example 2	✓	✓
$\cos(\gamma X_k)$ or $\sin(\gamma X_k)$	Example 3	✓	×
M/M/1 waiting times	Example 5	×	✓
ARTOC–Uniform	Example 6	✓	×
ARTOC–Exponential	Example 7	×	✓
ARTOC–Power	Example 8	✓	×
ARTOC–Pareto	Example 9	×	✓
ARTOC–Triangular	Example 10	✓	×
ARTOC freq. polygon	Example 11	✓	×
ARTOC–Lognormal	Example 12	×	✓
ARTOC–Weibull	Example 13	✓ $(a \ge 2)$ × $(0 < a < 2)$	× $(a > 2)$ ✓ "$(0 < a \le 2)$"
ARTOC–Laplace	Example 14	×	×

Table 1 How did we prove that certain processes are GMC? Theorem 2 (Lipschitz-continuous transformations) or Theorem 3 (convex/concave transformations)? The symbol "✓" indicates that the cited result was used to prove GMC, while "×" indicates that the result was not useful in proving GMC. **[Still need to completely resolve the Weibull "$(0 < a \le 2)$" case for Theorem 3.]**

5.5. Some new ideas on ARTOC: Quasiconcave densities

Due to the following upper bound

$$|g'(z)| = g'(z) = \frac{\phi(z)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} \le \frac{\phi(0)}{f_Y\{F_Y^{-1}[\Phi(z)]\}} \text{ for } z \in \mathbb{R},$$

the condition (39) of Theorem 4 can be satisfied with the constant $C = \phi(0) = 1/\sqrt{2\pi} < \infty$, if $h(z) \equiv (f_Y\{F_Y^{-1}[\Phi(z)]\})^{-1}$ is a quasiconvex function.

If $f_Y(y)$ is quasiconcave on int $\mathbb{S}$, then $h(z) = (f_Y\{F_Y^{-1}[\Phi(z)]\})^{-1}$ is quasiconvex on $\mathbb{R}$.

[In Definition 1, the notation for the support is $\mathbb{S}$. Should we use $\mathbb{S}$ or $\mathbb{S}_Y$?]

Proof:

If $f_Y(y)$ is quasiconcave on $\operatorname{int}\mathbb{S}$, then it is true that for any $y_1, y_2 \in \operatorname{int}\mathbb{S}$ and $\lambda \in [0,1]$

$$f_Y(\lambda y_1 + (1-\lambda)y_2) \;\geq\; \min\{f_Y(y_1), f_Y(y_2)\}$$

and

$$\frac{1}{f_Y(\lambda y_1 + (1-\lambda)y_2)} \;\leq\; \frac{1}{\min\{f_Y(y_1), f_Y(y_2)\}} \;=\; \max\left\{\frac{1}{f_Y(y_1)}, \frac{1}{f_Y(y_2)}\right\}.$$

The monotonicity of $g(z) = F_Y^{-1}[\Phi(z)]$ implies that for any $z_1, z_2 \in \mathbb{R}$ and $\lambda \in [0,1]$, there exists a $\gamma \in [0,1]$ such that

$$g(\lambda z_1 + (1-\lambda)z_2) \;=\; \gamma g(z_1) + (1-\gamma)g(z_2)$$

and

$$\frac{1}{f_Y(g(\lambda z_1 + (1-\lambda)z_2))} \;=\; \frac{1}{f_Y(\gamma g(z_1) + (1-\gamma)g(z_2))} \;\leq\; \max\left\{\frac{1}{f_Y(g(z_1))}, \frac{1}{f_Y(g(z_2))}\right\}$$

establishing that $(f_Y\{g(z)\})^{-1} = (f_Y\{F_Y^{-1}[\Phi(z)]\})^{-1}$ is quasiconvex on $\mathbb{R}$. ∎

The condition (41) is satisfied if there exists a $\mathfrak{v} > 0$ such that $\mathrm{E}[|h(Z_0)|^{\mathfrak{v}}] < \infty$. By the new definition $h(z) = f_Y\{F_Y^{-1}[\Phi(z)]\})^{-1}$,

$$\mathrm{E}[|h(Z_0)|^{\mathfrak{v}}] \;=\; \mathrm{E}[(f_Y\{F_Y^{-1}[\Phi(Z_0)]\})^{-\mathfrak{v}}] \;=\; \mathrm{E}[(f_Y(Y_0))^{-\mathfrak{v}}] \;=\; \int_{\mathbb{S}} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y.$$

Let $\Omega(v) \equiv \mathrm{E}[(f_Y(Y_0))^{-v}] = \int_{\mathbb{S}} [f_Y(y)]^{1-v}\,\mathrm{d}y$ for $v \in [0,1]$. Its values at the endpoints are $\Omega(0) = \int_{\mathbb{S}} f_Y(y)\mathrm{d}y = 1$ and $\Omega(1) = \mathrm{E}[(f_Y(Y_0))^{-1}] = \int_{\mathbb{S}} \mathrm{d}y$. If $\mathbb{S}$ is finite, then the condition (41) is satisfied for $\mathfrak{v} = 1$, since $\mathrm{E}[|h(Z_0)|] = \Omega(1) = \int_{\mathbb{S}} \mathrm{d}y < \infty$. On the other hand, if $\mathbb{S}$ is infinite, it is clear that the condition (41) is not satisfied for $\mathfrak{v} = 1$, since $\mathrm{E}[|h(Z_0)|] = \Omega(1) = \int_{\mathbb{S}} \mathrm{d}y = \infty$. But it is still possible that the condition (41) holds for some $\mathfrak{v} \in (0,1)$, if the derivative of $\Omega(v)$ at $v = 0$ is finite. By the monotone convergence theorem, the derivative is

$$\Omega'(v) \;=\; \frac{\mathrm{d}}{\mathrm{d}\mathfrak{v}}\mathrm{E}[(f_Y(Y_0))^{-v}] \;=\; \mathrm{E}\left[\frac{\mathrm{d}}{\mathrm{d}v}(f_Y(Y_0))^{-v}\right] \;=\; -\mathrm{E}\left[f_Y(y)^{-v}\log f_Y(y)\right] \;=\; -\int_{\mathbb{S}} f_Y(y)^{1-v}\log f_Y(y)\mathrm{d}y,$$

for $v \in [0,1]$, and the derivative at $v = 0$ is

$$\Omega'(0) = -\int_{\mathbb{S}} f_Y(y)\log f_Y(y)\mathrm{d}y,$$

which corresponds to the *differential entropy* of Y_0. [A sufficient condition for a finite differential entropy, $\Omega'(0) < +\infty$, is the finiteness of mean: See, for instance, Rioul (2011).

https://stats.stackexchange.com/questions/155939/is-differential-entropy-always-less-than-infinity].

If $\Omega'(0) < +\infty$ and $\Omega'(0) > -\infty$, then by Taylor's theorem,

$$\Omega(v) = \Omega(0) + \Omega'(0)v + v\mathcal{H}_1(v) = 1 + \Omega'(0)v + v\mathcal{H}_1(v), \quad \text{where } \lim_{v\to 0}\mathcal{H}_1(v) = 0.$$

So, for any $K < \infty$, there exists a $\mathfrak{v} \in (0,1)$ such that $\Omega(\mathfrak{v}) < K$. If $\Omega'(0) = -\infty$, then since $\Omega(1) = +\infty$ and by continuity, there exists a $\mathfrak{v} \in (0,1)$ such that $\Omega(\mathfrak{v}) < +\infty$.

[Some arguments based on tail behavior]

Assume that $f_Y(y) = O(|y|^{-\gamma})$ as $|y| \to \infty$, for some $\gamma > 1$, that is, there exist $a, C \in (0,\infty)$ such that $f_Y(y) \le C|y|^{-\gamma}$ for $|y| \ge a$. Then for $\mathfrak{v} \in (0,1)$,

$$\begin{aligned}\int_{\mathbb{S}} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y &= \int_{\mathbb{R}} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y \\ &= \left\{\int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{+\infty}\right\} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y \\ &\le \int_{-a}^{a} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y + 2C^{1-\mathfrak{v}} \int_{a}^{\infty} y^{-(1-\mathfrak{v})\gamma}\,\mathrm{d}y. \end{aligned} \tag{69}$$

To bound the first term, we note that if $f_Y(y) > 1$, then $[f_Y(y)]^{1-\mathfrak{v}} < f_Y(y)$, and if $f_Y(y) \le 1$, then $[f_Y(y)]^{1-\mathfrak{v}} \le 1$ for $\mathfrak{v} \in (0,1)$. By splitting the interval $(-a,a)$ into two pieces $(-a,a) = A \cup B$, where $A \equiv \{y \in (-a,a) : f_Y(y) > 1\}$ and $B \equiv \{y \in (-a,a) : f_Y(y) \le 1\}$, we get

$$\int_{-a}^{a} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y \;=\; \int_{A} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y + \int_{B} [f_Y(y)]^{1-\mathfrak{v}}\,\mathrm{d}y \;\le\; \int_{A} f_Y(y)\,\mathrm{d}y + \int_{B}\mathrm{d}y \;<\; 1 + 2a \;<\; \infty.$$

On the other hand, the integral in the second term of (69) is

$$\int_{a}^{\infty} y^{-(1-\mathfrak{v})\gamma}\mathrm{d}y \;=\; \frac{a^{1-(1-\mathfrak{v})\gamma}}{(1-\mathfrak{v})\gamma - 1} \;<\; \infty, \quad \text{if } (1-\mathfrak{v})\gamma > 1.$$

So, for any $\mathfrak{v} \in (0, 1-1/\gamma)$, the condition (41) holds. Moreover, the moment in the condition (40) is

$$\begin{aligned}\mathrm{E}[|g(Z_0)|^u] &= \mathrm{E}[|Y_0|^u] \\ &= \int_{\mathbb{R}} |y|^u f_Y(y)\,\mathrm{d}y \\ &= \left\{\int_{-\infty}^{-a} + \int_{-a}^{a} + \int_{a}^{+\infty}\right\} |y|^u f_Y(y)\,\mathrm{d}y \\ &\le \int_{-a}^{a} |y|^u f_Y(y)\,\mathrm{d}y + 2C \int_{a}^{\infty} y^{u-\gamma}\,\mathrm{d}y \\ &\le a^u + 2C\frac{a^{1+u-\gamma}}{\gamma - u - 1} < \infty, \quad \text{if } \gamma - u > 1.\end{aligned}$$

So, if $\gamma > 3$, then for any $u \in (2, \gamma - 1)$, the moment condition (40) holds.

6. Indicator Functions of GMC Processes Are GMC

[Need a transition sentence or two either here or at end of last section stating that now we're going from simple transforms to what might be more-interesting general tools. I dunno.] Suppose we are interested in whether or not a machine breakdown time X is at most y, an event characterized by the indicator function $\mathbf{1}_{\{X \le y\}}$. Besides being of interest in and of themselves, indicator functions have applications in quantile estimation, as discussed in our companion papers Alexopoulos et al. (2023) and Dingeç et al. (2023a,b,d). To this end, we note that the indicator function $g(x) = \mathbf{1}_{\{x \le y\}}$, for some threshold $y \in \mathbb{R}$, does not satisfy some of the assumptions of Theorems 2 and 3, e.g., Theorem 2's requirement that $\left|\mathbf{1}_{\{u \le y\}} - \mathbf{1}_{\{v \le y\}}\right| \le K_g |u - v|$ for all $u, v \in \mathbb{R}$ (think of $u < y < v < u + \epsilon$ for some vanishingly small $\epsilon > 0$) and Theorem 3's derivative requirement (34) . Nevertheless, Theorem 5 establishes that if the underlying process $\{X_k : k \ge 0\}$ satisfies the GMC condition (2) and the density-boundedness condition (4), then the sequence of indicators $\{\mathbf{1}_{\{X_k \le y\}} : k \ge 0\}$ is GMC.

Theorem 5. *If a stationary stochastic process $\{X_k : k \ge 1\}$ satisfies the GMC condition (2) and the density-boundedness condition (4), then the associated indicator process, $\left\{I_k(y) \equiv \mathbf{1}_{\{X_k \le y\}} : k \ge 1\right\}$ with given y, satisfies the GMC condition.*

A proof of Theorem 5 is given by Dingeç et al. (2023a, Theorem 4). Also see the related result of Doukhan (2018, Lemma 7.4.2). In addition, Xu (2021, Lemma B.2) proved a version of Theorem 5 using a slightly different definition of the coupled process, namely, $Y_k' = \xi\left(\ldots, \varepsilon_{-1}, \varepsilon_0', \varepsilon_1, \ldots, \varepsilon_k\right)$, for $k \ge 1$.

The next example examines the behavior of the indicator process arising from a GMC ARTOC process.

Example 15. This is a continuation of Example 10 where we studied the ARTOC process with triangular marginals $\{Y_k : k \ge 0\}$, which was determined to be GMC. We now consider the corresponding indicator process $\left\{I_k(y) \equiv \mathbf{1}_{\{Y_k \le y\}} : k \ge 0\right\}$ with given y. Since the triangular distribution clearly satisfies the density-boundedness condition (4), Theorem 5 tells us that the indicator process itself is GMC.

Figure 7 depicts the Monte Carlo sample a.c.f. of the ARTOC–Tria$(0, 0.5, 1)$'s indicator function process $\{I_k(y) : k \ge 0\}$ with $\beta = 0.8$ and $y = 0.75$. The MC results are based on a single run of $n = 1{,}000{,}000$ AR(1) observations, $Z_1, Z_2, \ldots, Z_n$. The a.c.f. of $\{I_k(y) : k \ge 0\}$ appears to converge to 0 more quickly than the a.c.f. of the AR(1) process $\{Z_k : k \ge 0\}$—an outcome we observed for all β- and y-values that we studied, and one that makes sense because the indicator function contains "less information" than the underlying $\{Z_k : k \ge 0\}$ process. □

Lemma 7 is a complementary result that applies to discrete base r.v.'s $\{X_k : k \ge 0\}$; it is proven in Appendix A4.

Lemma 7. *Let $\{V_k : k \ge 0\}$ denote a discrete GMC process having marginal p.m.f.* $\Pr(V_k = i) = p_i$ *for $i = 1, 2, \ldots, d$ and $k \ge 0$, where $p_1, p_2, \ldots, p_d > 0$ and $\sum_{i=1}^{d} p_i = 1$. Let $W_{ik} \equiv \mathbf{1}_{\{V_k = i\}}$ for $i = 1, 2, \ldots, d$ and $k \ge 0$. If $\{V_k : k \ge 0\}$ is a GMC process, then $\{W_{ik} : k \ge 0\}$, for $i = 1, 2, \ldots, d$, are GMC.*

7. Linear and Other Such Combinations of GMC Processes Are GMC

In §7.1, we present results establishing that linear combinations and products of GMC processes are themselves GMC. These findings will have several interesting applications as we proceed. For instance,

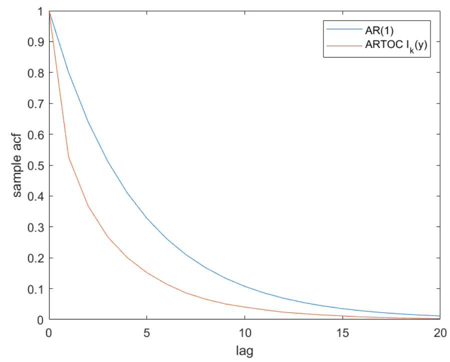


Figure 7 Monte Carlo sample a.c.f. for the ARTOC-Tria$(0,0.5,1)$'s indicator function process $\{I_k(y): k \ge 0\}$ with $\beta = 0.8$ and $y = 0.75$. MC results are based on a single run of $n = 1{,}000{,}000$ AR(1) observations, $Z_1, Z_2, \ldots, Z_n$.

Example 17 in the current section solves a burning mystery; and §7.2 shows that a truncated Taylor series consisting of GMC components is GMC, with the same remark extending to such useful tools as Fourier series and Walsh functions. The work in this section will also be referred to elsewhere in the sequel.

7.1. Two Jake Lemmata and Two Simple Examples

Lemma 8 gives the main tool of the section.

Lemma 8. *Suppose that* $\{Y_{ik}: k \ge 0\}$, $i = 1, 2, \ldots, d$, *are GMC processes, not necessarily independent of each other. Then their linear combination* $\{X_{dk} \equiv a_0 + \sum_{i=1}^{d} a_i Y_{ik} : k \ge 0\}$ *is GMC.*

Proof: We carry out induction on d. For the $d = 2$ base case, Lemma 1 in Dingeç et al. (2022) implies that

$$\begin{aligned} \mathrm{E}[|X_{2k} - X'_{2k}|^{\psi}] &= \mathrm{E}\big[\big|a_1(Y_{1k} - Y'_{1k}) + a_2(Y_{2k} - Y'_{2k})\big|^{\psi}\big] \\ &\le \max\{1, 2^{\psi-1}\}\Big(|a_1|^{\psi}\mathrm{E}\big[\big|Y_{1k} - Y'_{1k}\big|^{\psi}\big] + |a_2|^{\psi}\mathrm{E}\big[\big|Y_{2k} - Y'_{2k}\big|^{\psi}\big]\Big). \end{aligned}$$

Thus, $\{X_{2k}: k \ge 0\}$ is a GMC process for $d = 2$. For the inductive step, we assume that $\{X_{dk}: k \ge 0\}$ is GMC. Assuming that $\{Y_{d+1,k}: k \ge 0\}$ is GMC, we see that $X_{d+1,k} = X_{dk} + a_{d+1}Y_{d+1,k}$. $k \ge 0$, so that $\{X_{d+1,k}: k \ge 0\}$ is a linear combination of two GMC processes and therefore is itself GMC by the $d = 2$ case. ∎

Example 16. Consider the two independent replications $\{Z_{1k}: k \ge 0\}$ and $\{Z_{2k}: k \ge 0\}$ of the serially correlated AR(1) process satisfying Equation (28) with $\beta = 0.8$. Note that $Y_{ik} \equiv \Phi(Z_{ik}) \sim \mathrm{Unif}(0,1)$ for $i = 1, 2$ and $k \ge 0$; and $\Phi(Z_{1j})$ is independent of $\Phi(Z_{2k})$ for $j, k \ge 0$.

We first consider the case in which we sum two independent ARTOC processes with autocorrelated Unif$(0,1)$ marginals to obtain a not-quite-ARTOC triangular stream $X_k = Y_{1k} + Y_{2k} \sim \mathrm{Tria}(0,1,2)$ for $k \ge 0$, where Lemma 8 assures us that the serially correlated process $\{X_k: k \ge 0\}$ is GMC. Figure 8(a) displays the corresponding sample histogram and sample a.c.f. based on a run of $n = 100{,}000$ observations. These depictions are similar to those of the ARTOC-triangular process given in Figure 6 from Example 10. However, the new sample a.c.f. lives slightly below the AR(1) and ARTOC-triangular a.c.f.'s from Example 10; see Table 2 for specific numerics for lags $\ell = 1(1)5$.

lag ℓ	1	2	3	4	5
$\rho(Z_k, Z_{k+\ell}) = \beta^\ell$	0.800	0.640	0.512	0.410	0.328
$\widehat{\rho}(X_k, X_{k+\ell})$	0.786	0.619	0.489	0.384	0.303
$\widehat{\rho}(\widetilde{X}_k, \widetilde{X}_{k+\ell})$	0.893	0.703	0.557	0.444	0.354

Table 2 Exact a.c.f. for AR(1) process with $\beta = 0.8$ and sample a.c.f.'s for sums of two Unif$(0,1)$ for lags $\ell = 1(1)5$.

An alternative formulation is to work with the single stream $\{Z_{1k} : k \ge 0\}$ and set $\widetilde{Y}_{1k} \equiv \Phi(Z_{1k})$, $\widetilde{Y}_{2k} \equiv \Phi(Z_{1,k-1})$, and $\widetilde{X}_k \equiv \widetilde{Y}_{1k} + \widetilde{Y}_{2k}$ for $k \ge 1$. Lemma 8 guarantees that this sister process $\{\widetilde{X}_k : k \ge 0\}$ is also GMC. Figure 8(b) shows that, perhaps surprisingly, $\{\widetilde{X}_k : k \ge 0\}$ has non-triangular marginals, together with an a.c.f. that lives above the AR(1) and ARTOC-triangular a.c.f.'s from Example 10. Some of the low-lag covariances for $\{\widetilde{X}_k : k \ge 0\}$ are quite large, indicating that successive values of $\widetilde{X}_k$ might tend to "hang out" near the p.d.f.'s endpoints 0 and 2—at least more so than observations from a Tria$(0,1,2)$ marginal. □

Simlar to linear combinations of GMC processes, Lemma 9 states that the product of GMC processes is GMC; the proof can be found in §A5.

Lemma 9. *If $\{A_k : k \ge 0\}$ and $\{B_k : k \ge 0\}$ are both GMC processes, not necessarily independent of each other, then $\{A_k B_k : k \ge 0\}$ is GMC.* [Make sure that $\mathrm{E}[|A_0|^{2\psi}]$ and $\mathrm{E}[|B_0|^{2\psi}] < \infty$.]

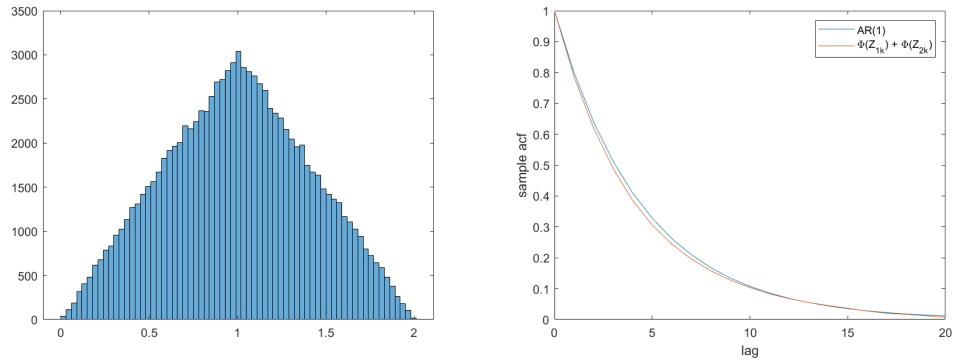
Example 17. A mystery solved! In Example 14, we found that the ARTOC–Laplace process fails to satisfy the sufficient conditions of either Theorems 2 or 3; and we were not able to conclude at that point whether or not the ARTOC–Laplace is GMC. We put that issue to rest now by writing the inverse c.d.f. from Equation (66) as

$$F_Y^{-1}(s) \;=\; \ell\mathrm{n}(2s)\,\mathbf{1}_{\{0\le s<1/2\}} - \ell\mathrm{n}(2(1-s))\,\mathbf{1}_{\{1/2\le s\le 1\}}.$$

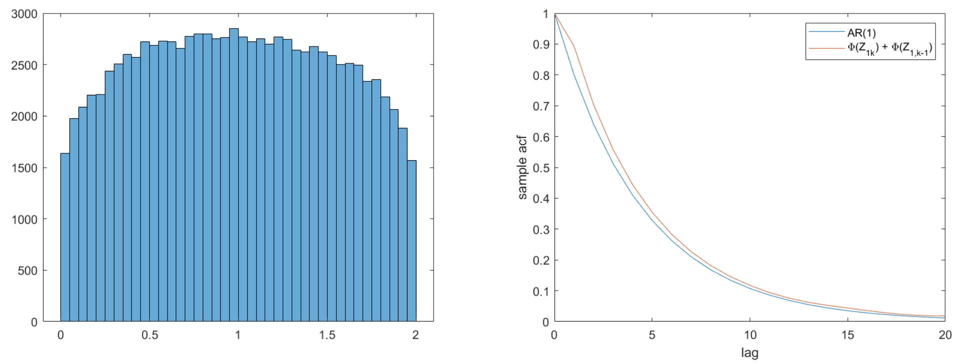
The inverse transform method with $\Phi(Z_k) \stackrel{\mathrm{d}}{=} \mathrm{Unif}(0,1)$ gives us ARTOC–Laplace realizations,

$$\begin{aligned} Y_k &= \ell\mathrm{n}\big(2\Phi(Z_k)\big)\,\mathbf{1}_{\{0\le\Phi(Z_k)<1/2\}} - \ell\mathrm{n}\big(2(1-\Phi(Z_k))\big)\,\mathbf{1}_{\{1/2\le\Phi(Z_k)\le 1\}} \\ &= \ell\mathrm{n}\big(2\Phi(Z_k)\big)\,\mathbf{1}_{\{Z_k<0\}} - \ell\mathrm{n}\big(2\Phi(-Z_k)\big)\,\mathbf{1}_{\{Z_k\ge 0\}}. \end{aligned}$$

We have that $\{-\ell\mathrm{n}\big(2\Phi(-Z_k)\big) \sim \mathrm{Exp}(1) - \ell\mathrm{n}(2) : k \ge 0\}$ is GMC by Example 7 and Lemma 8; the indicator process $\{\mathbf{1}_{\{Z_k<0\}} : k \ge 0\}$ is GMC by Theorem 5; and $\{\ell\mathrm{n}\big(2\Phi(Z_k)\big) : k \ge 0\}$ and $\{\mathbf{1}_{\{Z_k\ge 0\}} : k \ge 0\}$ are GMC by symmetry. By Lemma 9, both products are GMC; and then by Lemma 8, $\{Y_k : k \ge 0\}$ is GMC. □



(a) $X_k = \Phi(Z_{1k}) + \Phi(Z_{2k}) \sim \text{Tria}(0,1,2)$, for $k \geq 0$.



(b) $\widetilde{X}_k = \Phi(Z_{1k}) + \Phi(Z_{1,k-1})$, for $k \geq 1$.

Figure 8 Monte Carlo results for the sum of two Unif$(0,1)$ r.v.'s. (a) From two independent ARTOC processes with $\beta = 0.8$ and Unif$(0,1)$ marginals: $X_k = \Phi(Z_{1k}) + \Phi(Z_{2k}) \sim \text{Tria}(0,1,2)$ for $k \geq 0$. (b) From a single ARTOC process with $\beta = 0.8$ and Unif$(0,1)$ marginals: $\widetilde{X}_k = \Phi(Z_{1k}) + \Phi(Z_{1,k-1})$ for $k \geq 1$. Sample p.d.f. (left) and sample a.c.f. (right). The MC results are based on two runs of $n = 100{,}000$ AR(1) observations, $Z_{11}, Z_{12}, \ldots, Z_{1n}$ and $Z_{21}, Z_{22}, \ldots, Z_{2n}$.

7.2. Taylor Series and Beyond

Suppose that a function $g(x)$ has derivatives of all orders in an open interval containing a point a, Then the Taylor series for a function $g(x)$ is given by

$$g(x) \equiv \sum_{i=0}^{\infty} \frac{g^{(i)}(a)}{i!}(x-a)^i, \quad x \in \mathbb{R}.$$

with the case $a = 0$ being the Maclaurin series. We consider the partial sum associated with the Maclaurin series,

$$g_d(x) \equiv \sum_{i=0}^{d} \frac{g^{(i)}(0)}{i!} x^i, \quad x \in \mathbb{R},\ d \geq 1. \tag{70}$$

If $\{X_k : k \geq 0\}$ is a GMC process, then for fixed d, Lemma 8 implies that $\{g_d(X_k) : k \geq 0\}$ is GMC.

EXAMPLE 18. It is a straightforward exercise to show that the Maclaurin series for $g(z) = \Phi(z)$ is given by (Abramowitz and Stegun 1964, Result 26.2.10, p. 932)

$$\Phi(x) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)\cdot 2^i \cdot i!}, \quad x \in \mathbb{R}.$$

Let $\Phi_d(x)$ denote the corresponding partial sum encompassing $d \geq 1$ terms as in Equation (70). If $\{Z_k : k \geq 0\}$ is our usual AR(1) process from Equation (28), then $\{\Phi_d(Z_k) : k \geq 0\}$ is GMC. Of course, we know that $\Phi(Z_k) \sim \text{Unif}(0,1)$ for all k. But it is of interest to examine how quickly $\Phi_d(x)$ approaches the Unif$(0,1)$ distribution as d becomes larger; and so we conduct some Monte Carlo work to look into this query. Figure 9 illustrates the corresponding sample histogram and sample a.c.f. based on a run of $n = 100{,}000$ observations. Subfigures (a), (b), and (c) correspond, respectively, to the cases $d = 4$, 10, and 20 of the Maclaurin expansion $\Phi_d(x)$. The case $d = 4$ shows that the sample p.d.f. primarily lives near $[0,1]$ (good!) but exhibits tremendous variability (bad!) due to the fact that we are using relatively few expansion terms; and the a.c.f. function tails off more quickly than the AR(1)'s. As we progress to $d = 10$ and then to $d = 20$, we see that as we increase the number of terms in the expansion, the sample p.d.f. seems to approach that of the limiting Unif$(0,1)$, and the a.c.f. seems to match up nicely with the AR(1)'s. □

With the simple Maclaurin Example 18 in mind, we can begin to consider several extensions.

EXAMPLE 19. Suppose that $g(x)$ is a function having a *Fourier series* representation,

$$g(x) = A_0 + \sum_{i=1}^{\infty} \left[A_i \cos\left(\frac{2\pi i x}{T}\right) + B_i \sin\left(\frac{2\pi i x}{T}\right)\right], \quad k \geq 0,$$

where the Fourier coefficients are, of course, defined by

$$A_0 \equiv \frac{1}{T}\int_{-T/2}^{T/2} g(x)\,\mathrm{d}x, \quad A_i \equiv \frac{1}{T}\int_{-T/2}^{T/2} g(x)\cos\left(\frac{2\pi i x}{T}\right)\mathrm{d}x, \quad B_i \equiv \frac{1}{T}\int_{-T/2}^{T/2} g(x)\sin\left(\frac{2\pi i x}{T}\right)\mathrm{d}x, \quad i \geq 1.$$

Further suppose that $\{X_k : k \geq 0\}$ is a GMC process. Then by Example 3 and Lemma 8, the truncated ($d \geq 1$) Fourier series

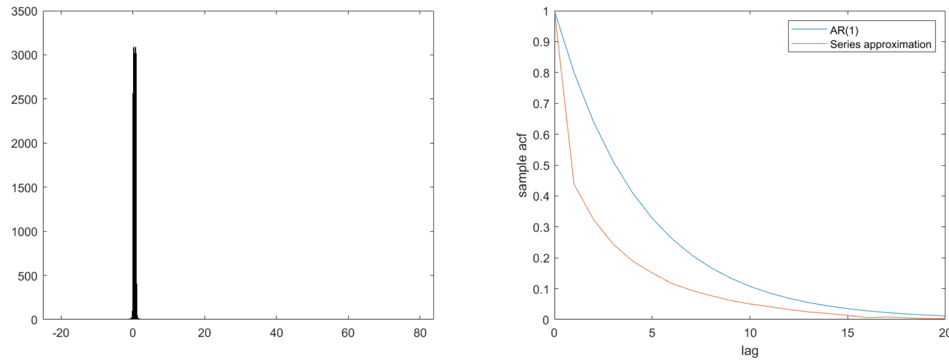
$$g_d(X_k) \equiv A_0 + \sum_{i=1}^{d} \left[A_i \cos\left(\frac{2\pi i X_k}{T}\right) + B_i \sin\left(\frac{2\pi i X_k}{T}\right)\right], \quad k \geq 0,$$

is also a GMC process—which is certainly a rich class of processes to add to our library. □

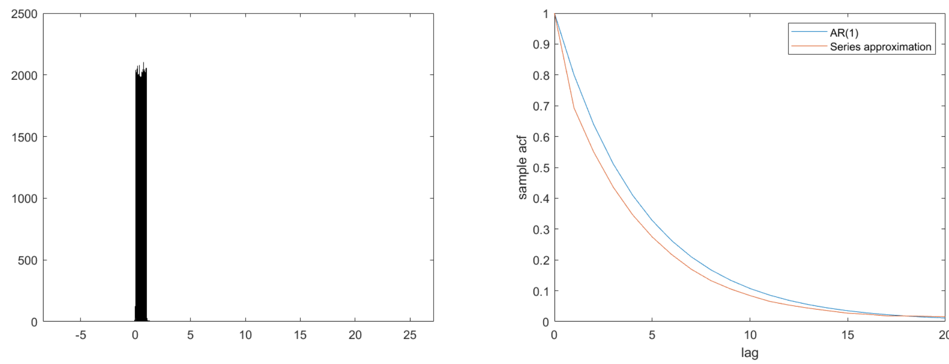
REMARK 1. We defer to a future paper discussion involving additional examples, other classes of infinite series, and convergence issues as the number of terms d gets large. □

8. Mixtures of GMC Processes Are GMC

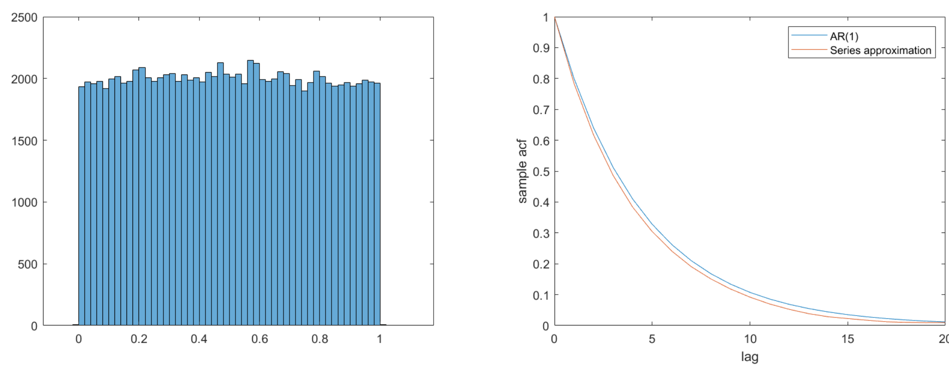
In many simulation and statistical applications encountered in practice, one works with a *mixture* of r.v.'s (Robert and Casella 2004). (Some authors, e.g., Law 2015, use the appellation *composition* instead of



(a) Maclaurin series with $d = 4$ terms for $\Phi(Z_k)$.



(b) Maclaurin series with $d = 10$ terms for $\Phi(Z_k)$.



(c) Maclaurin series with $d = 20$ terms for $\Phi(Z_k)$.

Figure 9 Monte Carlo results on obtaining Unif$(0,1)$'s the hard way—via Maclaurin series for $\Phi(Z_k)$, where $\{Z_k : k \geq 0\}$ is an AR(1) process with $\beta = 0.8$. (a) $d = 4$ expansion terms; (b) $d = 10$; and (c) $d = 20$. Sample p.d.f. (left) and sample a.c.f. (right), with MC results for Cases (a), (b), and (c), each based on a single run of $n = 100{,}000$ AR(1) observations, $Z_1, Z_2, \ldots, Z_n$.

mixture.) We start off with a motivational example. A coffee shop has a very long line when the store opens up in the morning. Any customer is Type A w.p. p_a and Type B w.p. $p_b = 1 - p_a$, though the successive customer types may be autocorrelated. Types A and B customers have different service-time distributions, and those times may be autocorrelated. In order to simulate this store until the line clears out, we need to sample each customer's type and then its service time from the distribution corresponding to its type.

The next theorem shows that a mixture of GMC processes is itself GMC.

THEOREM 6. *Suppose that $\{Y_{ik} : k \geq 0\}$, $i = 1, 2, \ldots, d$, are d processes satisfying the GMC condition, where the d processes need not be independent of each other. Let $\{V_k : k \geq 0\}$ denote a discrete GMC process having marginal p.m.f.* $\Pr(V_k = i) = p_i$ *for $i = 1, 2, \ldots, d$ and $k \geq 0$, where $p_1, p_2, \ldots, p_d > 0$ and $\sum_{i=1}^{d} p_i = 1$. Let $W_{ik} \equiv \mathbf{1}_{\{V_k = i\}}$ for $i = 1, 2, \ldots, d$ and $k \geq 0$. Then the mixture*

$$X_{dk} \equiv \begin{cases} Y_{1k} & w.p.\ p_1 \\ Y_{2k} & w.p.\ p_2 \\ \vdots & \\ Y_{dk} & w.p.\ p_d \end{cases} = \sum_{i=1}^{d} Y_{ik} \mathbf{1}_{\{V_k = i\}} = \sum_{i=1}^{d} Y_{ik} W_{ik}, \quad \textit{for } k \geq 0, \tag{71}$$

satisfies the GMC condition.

The proof of Theorem 6 is given in §A6.

EXAMPLE 20. In Example 10, we showed that a Tria$(0, m, 1)$ ARTOC process $\{Y_k : k \geq 1\}$ is GMC and remarked that this result readily generalizes to a Tria(a, m, b) ARTOC process. It is easy via composition to generate other autocorrelated GMC processes with triangular marginals. For instance, we can use composition on the left-Tria$(0, m, m)$ and right-Tria$(m, m, 1)$ ARTOC processes to obtain other (non-ARTOC) GMC processes with Tria$(0, m, 1)$ marginals. To this end, the c.d.f.'s of $Y_{1k} \sim$ Tria$(0, m, m)$ and $Y_{2k} \sim$ Tria$(m, m, 1)$ are given by, respectively,

$$F_{Y_1}(y) = (y/m)^2, \text{ for } 0 \leq y \leq m \quad \text{and} \quad F_{Y_2}(y) = 1 - [(1-y)/(1-m)]^2, \text{ for } m \leq y \leq 1,$$

which immediately yields the inverse functions

$$F_{Y_1}^{-1}(\mathfrak{w}) = m\mathfrak{w}^{1/2}, \text{ for } 0 \leq \mathfrak{w} \leq 1 \quad \text{and} \quad F_{Y_2}^{-1}(\mathfrak{w}) = 1 - (1-m)(1-\mathfrak{w})^{1/2}, \text{ for } 0 \leq \mathfrak{w} \leq 1.$$

These results give rise to the following composition scheme to generate a process with Tria$(0, m, 1)$ marginals from a base stationary AR(1) process $\{Z_k : k \geq 1\}$, where $\{U_k : k \geq 1\}$ is a stream of Unif(0,1) r.v.'s (not necessarily independent),

$$\begin{aligned} X_k &= \begin{cases} Y_{1k}, & \text{w.p. } m \\ Y_{2k}, & \text{w.p. } 1-m \end{cases} \\ &= \begin{cases} m[\Phi(Z_k)]^{1/2}, & \text{if } U_k \leq m \\ 1-(1-m)[1-\Phi(Z_k)]^{1/2}, & \text{if } U_k > m \end{cases} \\ &= m[\Phi(Z_k)]^{1/2}\, \mathbf{1}_{\{V_k = 1\}} + \Big\{1-(1-m)[\Phi(-Z_k)]^{1/2}\Big\} \mathbf{1}_{\{V_k = 2\}} \quad \text{for } k \geq 1, \end{aligned} \tag{72}$$

where the events $\{V_k = 1\} \equiv \{U_k \leq m\}$ and $\{V_k = 2\} \equiv \{U_k > m\}$. What remains is to propose different choices for the stream of uniforms $\{U_k : k \geq 1\}$ driving the sequence of composition components $\{V_k : k \geq 1\}$, and then to show for each choice of stream that the respective indicator processes $W_{ik} = \mathbf{1}_{\{V_k = i\}}$, for $i = 1, 2$ and $k \geq 0$, are GMC. Some obvious examples that come to mind are:

(i) $U_0, U_1, U_2, \ldots$ i.i.d. $\mathrm{Unif}(0,1)$.

(ii) $U_0 \sim \mathrm{Unif}(0,1)$, and then

$$U_{k+1} = \begin{cases} U_k, & \text{w.p. } p_1 = p \\ \text{an independent Unif}(0,1), & \text{w.p. } p_2 = 1-p \end{cases}, \quad \text{for } p \in [0,1] \text{ and } k \geq 0.$$

(iii) $U_k = \Phi(Z_k)$, for $k \geq 0$.

For each of the three example Cases (i)–(iii), we can easily show that these choices of $\{U_k : k \geq 1\}$ are all GMC; and then $\{V_k : k \geq 1\}$; and then $\{W_{ik} : k \geq 1\}$ for $i = 1$ and 2. First of all, the GMC property for Case (i) follows trivially from the i.i.d. discussion at the beginning of §3.1. With regard to Case (ii), define D as the value of $k \geq 1$ such that $U_k \neq U_0$ for the first time, so that:

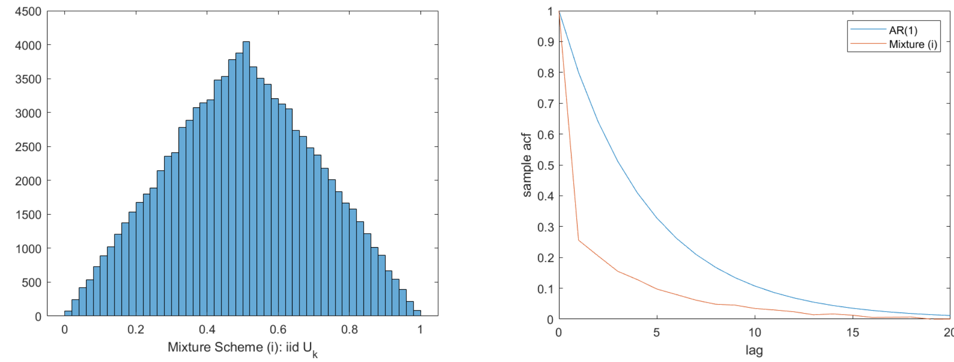
$$D \sim \mathrm{Geom}(1-p), \quad U_k \text{ is independent of } U_0 \text{ for } k \geq D, \quad \text{and} \quad U_k = U_k' \text{ for } k \geq D.$$

These facts imply that

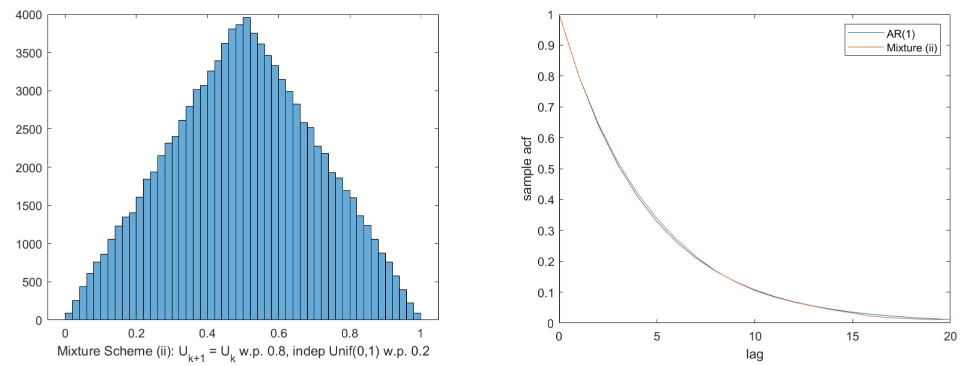
$$\begin{aligned} \mathrm{E}\big[|U_k - U_k'|^\psi\big] &= \mathrm{E}\big[|U_k - U_k'|^\psi \big| D \leq k\big] \Pr(D \leq k) + \mathrm{E}\big[|U_k - U_k'|^\psi \big| D > k\big] \Pr(D > k) \\ &= (0)(1-p^k) + \mathrm{E}\big[|U_k - U_k'|^\psi \big| D > k\big] p^k \\ &\leq p^k, \end{aligned}$$

which completes (ii). Case (iii) follows from the Lipschitz Example 6. Thus, in all of Cases (i)–(iii), the $\{U_k : k \geq 1\}$ processes are GMC, so equivalently, the respective $\{V_k : k \geq 1\}$ and $\{W_{ik} : k \geq 1\}$ processes are GMC. Then Theorem 6 immediately implies that $\{X_k : k \geq 0\}$ is GMC for each of the Cases (i)–(iii).

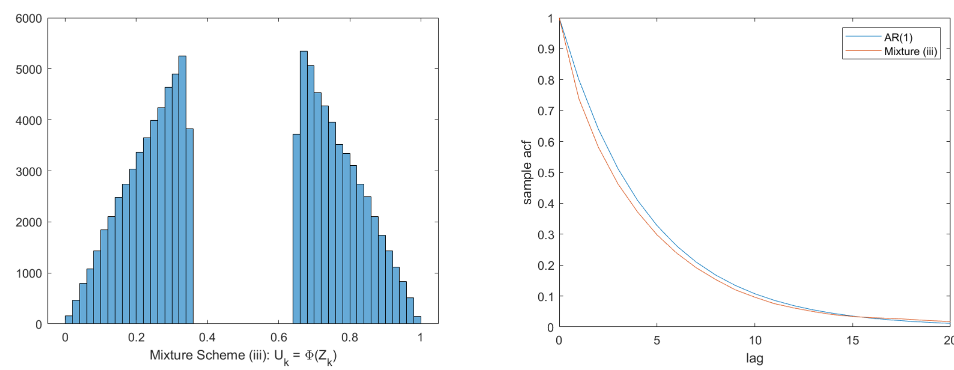
Figure 6 back in §5.3 depicted the sample histogram and sample a.c.f. of an ARTOC process with a $\mathrm{Tria}(0,0.5,1)$ marginal and AR(1) parameter $\beta = 0.8$. Figures 10(a)–(c) extend the analysis for the respective mixture Cases (i)–(iii) in which—for purposes of apples-to-apples comparison—the marginals are "targeted" to be $\mathrm{Tria}(0,0.5,1)$ and the underlying base process is still AR(1) with $\beta = 0.8$. We see from Figures 10(a)–(c) that Cases (i) and (ii) indeed produce marginal $\mathrm{Tria}(0,0.5,1)$ distributions, but that Case (iii) yields a sample p.d.f. having a significant gap from $0.5^{1.5} \doteq 0.3536$ to $1 - 0.5^{1.5} \doteq 0.6464$; the gap arises as a consequence of $U_k = \Phi(Z_k)$ being used to determine both V_k (used to decide which of the left- or right-triangles to sample from) and Y_{ik} (the sample generated via inversion from the chosen triangle). In terms of the a.c.f., recall that Figure 6's sample a.c.f. tracked that of the underlying AR(1) almost exactly. However, Figure 10(a)'s sample a.c.f. falls well under that of the AR(1), no doubt due to the added randomness of the i.i.d. $\{U_k : k \geq 0\}$ process used to choose which triangle to invert for each X_k. Figure 10(b)'s sample a.c.f. is for Case (ii) with $p = p_1 = 0.8$, indicating that there is strong positive correlation between the sequential choice of triangles as sampling proceeds. We find that for this choice of p, the a.c.f. of $\{X_k\}$ tracks the AR(1)'s. However, for $p > 0.8$ [$p < 0.8$], it turns out (not illustrated here) that the a.c.f. of $\{X_k\}$ is generally larger [smaller] than the AR(1)'s across all lags. The sample a.c.f. displayed in Figure 10(c) tracks the AR(1)'s modestly well, even though the marginal sample p.d.f. is significantly "gap-toothed." □



(a) Mixture (i): $U_1, U_2, \ldots$ i.i.d. $\mathrm{Unif}(0,1)$.



(b) Mixture (ii): $U_{k+1} = U_k$ w.p. $p = 0.8$, or else an independent $\mathrm{Unif}(0,1)$ w.p. $1-p=0.2$, for $k \geq 1$.



(c) Mixture (iii): $U_k = \Phi(Z_k)$ for $k \geq 1$.

Figure 10 Monte Carlo results for mixtures "targeting" $\mathrm{Tria}(0, 0.5, 1)$ distributions. As presented in Equation (73), the MC results are based on single runs of $n = 100{,}000$ observations from an AR(1) process $Z_1, Z_2, \ldots, Z_n$ with $\beta = 0.8$, and a set of $\mathrm{Unif}(0,1)$ observations $U_1, U_2, \ldots, U_k$ generated under each of the following schemes: (a) Case (i), in which the base $U_1, U_2, \ldots$ are i.i.d. $\mathrm{Unif}(0,1)$; (b) Case (ii), in which $U_{k+1} = U_k$ w.p. $p = 0.8$, or else an independent $\mathrm{Unif}(0,1)$ w.p. $1-p=0.2$, for $k \geq 1$; and (c) Case (iii), in which $U_k = \Phi(Z_k)$ for $k \geq 1$. Sample p.d.f. (left) and sample a.c.f. (right).

Example 21. [This eg still a work in progress and not even sure if needed. Might be useful to establish mixture notation and to see if we can actually get an ARTOC-Tria out of it.]

As described in the original example, the ARTOC–Laplace process has marginal p.d.f. $f_Y(y)$, c.d.f. $F_Y(y)$, and inverse c.d.f. $F_Y^{-1}(s)$ respectively given by Equations (**??**), (**??**), and (66). It is then is obvious by symmetry that we can obtain a Laplace r.v. X_k via a 50–50% mixture of a "negative" exponential r.v. Y_{1k} and an exponential r.v. Y_{2k} and a having respective c.d.f.'s

$$F_{Y_1}(y) \equiv \begin{cases} e^y & \text{if } y < 0 \\ 1 & \text{if } y > 0 \end{cases} \quad \text{and} \quad F_{Y_2}(y) \equiv \begin{cases} 0 & \text{if } y < 0 \\ 1 - e^{-y} & \text{if } y > 0 \end{cases},$$

from which it is easily verified that the mixture's c.d.f., $F_Y(y) = \frac{1}{2}F_{Y_1}(y) + \frac{1}{2}F_{Y_2}(y)$, indeed matches the expression given by Equation (**??**). Then by Equation (66), we generate the mixture r.v. via

$$\begin{aligned} X_k &= \begin{cases} Y_{1k}, & \text{w.p. } 1/2 \\ Y_{2k}, & \text{w.p. } 1/2 \end{cases} \\ &= \begin{cases} \ell\text{n}(\Phi(Z_k)) & \text{w.p. } 1/2 \\ -\ell\text{n}(1 - \Phi(Z_k)) & \text{w.p. } 1/2 \end{cases} \\ &= \text{TBD } \mathbf{1}_{\{V_k = 1\}} + \text{TBD } \mathbf{1}_{\{V_k = 2\}} \quad \text{for } k \geq 1 \end{aligned} \tag{73}$$

blah blah □

9. Nonoverlapping Batch Means and Related Processes are GMCC

The use of batch means in simulation output analysis has been ubiquitous for over 60 years (Conway 1963, Law 2015) and has a great deal of theoretical backing. But as related in §2, much of the theory relies on ϕ-mixing assumptions that are sometimes abstruse (Chien et al. 1997), hard to verify in practice, and perhaps not applicable to a broad class of well-known stationary stochastic processes—not even the AR(1) process! For this reason, it is of interest to provide conditions under which the process of successive nonoverlapping batch means is GMC, a question that Theorem 10 in §**??** definitively addresses. We then present an extension in §9.4 in which we show that processes involving more-general functions of the observations from nonoverlapping batches are GMC.

[8/11/24: Moved the proof to the Chien paper. **But on 8/16/25, I moved it back from Chien, and added OBM and STS. This follows in magenta]**

This section reports on findings involving various incarnations of batch means when the underlying observations $\{Y_k : k \geq 0\}$ are GMCC. In particular, §9.1 gives conditions for which the nonoverlapping batch means process is itself GMCC; §9.2 does the same for the overlapping batch means (OBM) process; and §9.3 studies the related standardized time series (STS) area functional, finding that it, too, is GMCC.

9.1. Nonoverlapping Batch Means are GMCC

Consider a GMCC process $\{Y_k : k \geq 0\}$ which we subdivide into nonoverlapping batches of length $m \geq 1$ so that $\{Y_{(d-1)m+1}, Y_{(d-1)m+2}, \ldots, Y_{dm}\}$ constitutes nonoverlapping batch d having batch mean $\overline{Y}_{d,m} \equiv$

$m^{-1}\sum_{j=1}^{m} Y_{(d-1)m+j}$, for $d \geq 1$. In §??, we will show how NBMs can be used to estimate the variance parameter σ^2 when the GMCC is in play. For now, we will show that if the underlying $\{Y_k : k \geq 0\}$ process is GMCC, then so is the resulting NBM process $\{\overline{Y}_{d,m} : d \geq 1\}$.

To be self-contained, we start with a result on the finiteness of the batch means' moments.

Lemma 10. *If $\{Y_k : k \geq 0\}$ is stationary with $\mathrm{E}\big[|Y_k|^u\big] < \infty$ for some $u > 0$, then $\mathrm{E}\big[|\overline{Y}_{d,m}|^v\big] < \infty$, for $0 < v \leq u$ and $d, m \geq 1$.*

Proof. Suppose that $u \geq 1$ and pick a $v \in [1, u]$. Then by Minkowski's inequality and Lyapunov's inequality,

$$\left(\mathrm{E}\big[|\overline{Y}_{1,m}|^v\big]\right)^{1/v} = \frac{1}{m}\left(\mathrm{E}\left[\left|\sum_{k=1}^{m} Y_k\right|^v\right]\right)^{1/v} \leq \frac{1}{m}\sum_{k=1}^{m}\left(\mathrm{E}\big[|Y_k|^v\big]\right)^{1/v} = \left(\mathrm{E}\big[|Y_1|^v\big]\right)^{1/v} \leq \left(\mathrm{E}\big[|Y_1|^u\big]\right)^{1/u} < \infty. \tag{74}$$

For $0 < v < 1$, Lyapunov's inequality and then Equation (74) imply that [How? Probably easy.]

$$\left(\mathrm{E}\big[|\overline{Y}_{1,m}|^v\big]\right)^{1/v} \leq \mathrm{E}\big[|\overline{Y}_{1,m}|\big] < \infty.$$

On the other hand, if $u < 1$, then for $0 < v \leq u < 1$,

$$\mathrm{E}\big[|\overline{Y}_{1,m}|^v\big] = \frac{1}{m^v}\mathrm{E}\left[\left|\sum_{k=1}^{m} Y_k\right|^v\right] \leq \frac{1}{m^v}\sum_{k=1}^{m}\mathrm{E}\big[|Y_k|^v\big] = \frac{1}{m^{v-1}}\mathrm{E}\big[|Y_1|^v\big] \leq \frac{1}{m^{v-1}}\left(\mathrm{E}\big[|Y_1|^u\big]\right)^{v/u} < \infty. \quad \square$$

So if $\mathrm{E}\big[|Y_k|^u\big] < \infty$, then we do not need to worry about the corresponding moments of the batch means. [Need to say exactly why the lemma is important; note that it's not mentioned again.]

Our next result shows that if the underlying $\{Y_k : k \geq 0\}$ process is GMCC, then so is the NBM process.

Theorem 7. *Suppose that $\{Y_k : k \geq 0\}$ satisfies the GMCC (2) for some $\psi > 0$, $C_\psi \geq 1$, and $r_\psi \in (0,1)$. Then the associated NBM process $\{\overline{Y}_{d,m} : d \geq 1\}$ is GMCC, i.e., for each $d, m \geq 1$, we have*

$$\mathrm{E}\big[|\overline{Y}_{d,m} - \overline{Y}'_{d,m}|^\psi\big] \leq C_{\text{NBM}}\, r_{\text{NBM}}^d,$$

where $\{\overline{Y}'_{d,m} : d \geq 1\}$ is the coupled NBM process,

$$C_{\text{NBM}} \equiv C_{\text{NBM}}(m,\psi) \equiv \frac{C'_\psi}{m^\psi r_\psi^{m-1}}, \quad C'_\psi \equiv \frac{C_\psi}{\min\left[1 - r_\psi,\ \left(1 - r_\psi^{1/\psi}\right)^\psi\right]} > 0, \quad \text{and} \quad r_{\text{NBM}} \equiv r_{\text{NBM}}(m,\psi) \equiv r_\psi^m. \tag{75}$$

Proof. We first put in our pockets a result from Dingeç et al. (2022, Lemma 1) which states that for $a, b \in \mathbb{R}$,

$$|a+b|^\psi \leq \begin{cases} |a|^\psi + |b|^\psi & \text{if } 0 < \psi \leq 1 \\ 2^{\psi-1}\left(|a|^\psi + |b|^\psi\right) & \text{if } \psi > 1. \end{cases} \tag{76}$$

Then we have

$$
\begin{aligned}
m^{\psi}\mathrm{E}\Big[\big|\overline{Y}_{d,m}-\overline{Y}'_{d,m}\big|^{\psi}\Big] &= \mathrm{E}\left[\left|\sum_{k=(d-1)m+1}^{dm}(Y_k-Y'_k)\right|^{\psi}\right]\\
&\le \begin{cases}\displaystyle\sum_{k=(d-1)m+1}^{dm}\mathrm{E}\Big[\big|Y_k-Y'_k\big|^{\psi}\Big] & \text{if } 0<\psi\le 1 \quad \text{(by, e.g., Equation (76))}\\ \displaystyle\left[\sum_{k=(d-1)m+1}^{dm}\Big(\mathrm{E}\Big[\big|Y_k-Y'_k\big|^{\psi}\Big]\Big)^{1/\psi}\right]^{\psi} & \text{if } \psi>1 \quad \text{(by an iterated version of Minkowski's inequality)}\end{cases}\\
&\le \begin{cases}\displaystyle\sum_{k=(d-1)m+1}^{dm} C_{\psi}r_{\psi}^{k} & \text{if } 0<\psi\le 1\\ \displaystyle\left[\sum_{k=(d-1)m+1}^{dm}\big(C_{\psi}r_{\psi}^{k}\big)^{1/\psi}\right]^{\psi} & \text{if } \psi>1\end{cases} \quad \text{(by the GMCC)}\\
&= C_{\psi}r_{\psi}^{(d-1)m+1}\times\begin{cases}\dfrac{1-r_{\psi}^{m}}{1-r_{\psi}} & \text{if } 0<\psi\le 1\\ \left(\dfrac{1-r_{\psi}^{m/\psi}}{1-r_{\psi}^{1/\psi}}\right)^{\psi} & \text{if } \psi>1.\end{cases}
\end{aligned}
$$

The proof is completed after noting that both $1-r_{\psi}^{m}$ and $1-r_{\psi}^{m/\psi}<1$. □

An obvious interpretation of this result is that the expected value decreases *extremely* quickly in terms of both the batch size m and the nonoverlapping batch number $d\ge 1$; and this result will manifest later on when we find that correlations related to non-adjacent batch means effectively disappear for $d>1$.

Corollary 2. *Suppose that the conditions of Lemma 1 hold, including the requirement that* $\mathrm{E}\big[|Y_k|^{u}\big]<\infty$ *for some* $u>2$. *Then the associated NBM process* $\{\overline{Y}_{d,m}: d\ge 1\}$ *satisfies the GMCC for* all $\psi\in(0,u)$.

Proof. By Theorem 7, we know that the NBM process $\{\overline{Y}_{d,m}: d\ge 1\}$ satisfies the GMCC for *some* $\psi>0$. The result then follows immediately from Lemma 1(a). □

[I'm in favor of deleting/vastly reducing Rmk 2 since not ever used (not even in §?? anymore). . .]

Remark 2. By Proposition 1 in Appendix A2 of Dingeç et al. (2023c), if $\{Y_k, k\ge 0\}$ is GMCC and $\mathrm{E}[|Y_1|^{u}]<\infty$, then the squared NBMs are GMCC for $\psi<u/2$, i.e.,

$$\mathrm{E}\Big[\big|\overline{Y}_{d,m}^{2}-(\overline{Y}'_{d,m})^{2}\big|^{\psi}\Big] \le C_{\text{SNBM}}(m,\psi)\, r_{\text{SNBM}}^{d}(m,\psi),$$

where

$$C_{\text{SNBM}}(m,\psi)\equiv\max\{\sqrt{2},2^{\psi}\}\sqrt{C_{\text{NBM}}(m,2\psi)\,\mathrm{E}\big[\overline{Y}_{m}^{2\psi}\big]} \quad\text{and}\quad r_{\text{SNBM}}(m,\psi)\equiv\sqrt{r_{\text{NBM}}(m,2\psi)}=r_{2\psi}^{m/2}. \quad \triangle$$

9.2. Overlapping Batch Means are GMCC

We can subdivide the process $\{Y_k : k \geq 1\}$ into *overlapping* batches of length $m \geq 1$ so that $\{Y_j, Y_{j+1}, \ldots, Y_{j+m-1}\}$ constitutes overlapping batch j having batch mean $\overline{Y}^{\text{O}}_{j,m} \equiv m^{-1} \sum_{k=j}^{j+m-1} Y_k$, for $j \geq 1$. The OBM approach leads to an estimator for σ^2 that has lower variance than the analogous NBM-based estimator (Aktaran-Kalaycı et al. 2007); however, the next theorem merely establishes that if the underlying $\{Y_k : k \geq 1\}$ process satisfies the GMCC, then so does the OBM process.

Theorem 8. *Suppose that $\{Y_k : k \geq 1\}$ satisfies the GMCC (2) for some $\psi > 0$, $C_\psi \geq 1$, and $r_\psi \in (0,1)$. Then the associated OBM process $\{\overline{Y}^{\text{O}}_{j,m} : j \geq 1\}$ is GMCC, i.e., for each $j, m \geq 1$, we have*

$$\mathrm{E}\big[\big|\overline{Y}^{\text{O}}_{j,m} - \overline{Y}'^{\text{O}}_{j,m}\big|^\psi\big] \leq C_{\text{OBM}}\, r^j_{\text{OBM}},$$

where $\{\overline{Y}'^{\text{O}}_{j,m} : d \geq 1\}$ is the coupled OBM process,

$$C_{\text{OBM}} \equiv C_{\text{OBM}}(m,\psi) \equiv \frac{C_\psi}{m^\psi \min\big[1 - r_\psi,\ (1 - r_\psi^{1/\psi})^\psi\big]} > 0, \quad \textit{and} \quad r_{\text{OBM}} \equiv r_{\text{OBM}}(m,\psi) \equiv r_\psi.$$

Proof. Closely following the proof of Theorem 7, we have

$$m^\psi \mathrm{E}\big[\big|\overline{Y}^{\text{O}}_{j,m} - \overline{Y}'^{\text{O}}_{j,m}\big|^\psi\big] = \mathrm{E}\left[\left|\sum_{k=j}^{j+m-1} (Y_k - Y'_k)\right|^\psi\right] = C_\psi r^j_\psi \times \begin{cases} \dfrac{1 - r^m_\psi}{1 - r_\psi} & \text{if } 0 < \psi \leq 1 \\[2ex] \left(\dfrac{1 - r^{m/\psi}_\psi}{1 - r^{1/\psi}_\psi}\right)^\psi & \text{if } \psi > 1. \end{cases} \qquad \square$$

This expected value for the OBM case decreases exponentially at rate r^j_ψ, $j \geq 1$; but not nearly as quickly as the rate r^{dm}_ψ, $d \geq 1$, for the NBM case. No surprise here, since consecutive OBMs are clearly more highly correlated than are consecutive NBMs — and so the comparison comes with a scent of apples-to-oranges.

Corollary 3. *Suppose that the conditions of Lemma 1 hold, including the requirement that $\mathrm{E}\big[|Y_0|^u\big] < \infty$ for some $u > 2$. Then the associated OBM process $\{\overline{Y}^{\text{O}}_{j,m} : j \geq 1\}$ satisfies the GMCC for* all $\psi \in (0,u)$.

Proof. Immediate from Theorem 8 and Lemma 1(a). $\square$

9.3. Standardized Time Series Batch Weighted Area Functionals are GMCC

We discuss a slightly more-exotic example that comes up in the simulation output analysis literature, again in the context of variance estimation. Consider a GMCC process $\{Y_k : k \geq 0\}$ that we present in nonoverlapping batches of size m as in §9.1, i.e., the observations $\{Y_{(d-1)m+1}, Y_{(d-1)m+2}, \ldots, Y_{dm}\}$ constitute nonoverlapping batch $d \geq 1$. As in Foley and Goldsman (1999), define the STS *weighted area* functional from nonoverlapping batch d and its coupled pair from nonoverlapping batch d by

$$A_d(m) \equiv \frac{1}{m^{3/2}} \sum_{k=(d-1)m+1}^{dm} g(k-(d-1)m) Y_k \quad \text{and} \quad A'_d(m) \equiv \frac{1}{m^{3/2}} \sum_{k=(d-1)m+1}^{dm} g(k-(d-1)m) Y'_k,$$

respectively, where

$$g(j) \equiv \sum_{k=1}^{m}(k/m)w(k/m) - \sum_{k=j}^{m} w(k/m), \quad \text{for } 1 \le j \le m,$$

where $w(t)$ is itself continuous and $\sup_{t\in[0,1]}|w(t)| \le W$.

Theorem 9. *Suppose that $\{Y_k : k \ge 1\}$ satisfies the GMCC (2) for some $\psi > 0$, $C_\psi \ge 1$, and $r_\psi \in (0,1)$. Then the associated STS nonoverlapping batch area process $\{A_d(m) : d \ge 1\}$ is GMCC.*

Proof. First of all, we have

$$|g(j)| \le \sum_{k=1}^{m}\frac{Wk}{m} + \sum_{k=j}^{m} W \le \frac{3W(m+1)}{2}, \quad \text{for } 1 \le j \le m.$$

This bound and work from the previous two proofs imply that

$$\begin{aligned}
m^{3\psi/2}\mathrm{E}\Big[\big|A_d(m) - A'_d(m)\big|^\psi\Big] &= \mathrm{E}\left[\left|\sum_{k=(d-1)m+1}^{dm} g(k-(d-1)m)(Y_k - Y'_k)\right|^\psi\right] \\
&\le \begin{cases} \displaystyle\sum_{k=(d-1)m+1}^{dm} \mathrm{E}\Big[\big|g(k-(d-1)m)(Y_k - Y'_k)\big|^\psi\Big] & \text{if } 0 < \psi \le 1 \\ \displaystyle\left[\sum_{k=(d-1)m+1}^{dm}\left(\mathrm{E}\Big[\big|g(k-(d-1)m)(Y_k - Y'_k)\big|^\psi\Big]\right)^{1/\psi}\right]^\psi & \text{if } \psi > 1 \end{cases} \\
&\le (3Wm)^\psi \times \begin{cases} \displaystyle\sum_{k=(d-1)m+1}^{dm} \mathrm{E}\Big[\big|(Y_k - Y'_k)\big|^\psi\Big] & \text{if } 0 < \psi \le 1 \\ \displaystyle\left[\sum_{k=(d-1)m+1}^{dm}\left(\mathrm{E}\Big[\big|(Y_k - Y'_k)\big|^\psi\Big]\right)^{1/\psi}\right]^\psi & \text{if } \psi > 1. \end{cases}
\end{aligned}$$

And then previous work from §9.1 just as immediately yields

$$\mathrm{E}\Big[\big|A_d(m) - A'_d(m)\big|^\psi\Big] \le \frac{(3W)^\psi C_\psi r_\psi^{(d-1)m+1}}{m^{\psi/2}\min\big[1-r_\psi,\, (1-r_\psi^{1/\psi})^\psi\big]}. \quad \square$$

Thus the $\{A_d(m) : d \ge 1\}$ process satisfies the GMCC (almost "as quickly" as the NBM process does). By machinations similar to those in §9.2, we can easily show that the overlapping batch version of the area function is also GMCC.

Theorem 10. *If the process $\{Y_k : k \ge 0\}$ satisfies the GMC condition (2), then for each batch size $m \ge 1$, the batch-means process*

$$\left\{\overline{Y}_j(m) \equiv \frac{1}{m}\sum_{k=jm}^{(j+1)m-1} Y_k : j \ge 0\right\} \tag{77}$$

also satisfies the GMC condition.

See Appendix A7 for the proof. More discussion on batch-means processes in the context of the GMC problem can be found in our companion papers Alexopoulos et al. (2023) and Dingeç et al. (2023a,b,d).

9.4. An Extension to More-General Functions of Batched Observations

[Add a motivational sentence.] Let $g : \mathbb{R}^m \to \mathbb{R}$ be an α-Hölder continuous function, i.e., a function for which there exist $A_g \in (0,\infty)$ and $\alpha \in (0,1]$ such that for any $x, y \in \mathbb{R}^m$, we have

$$|g(x) - g(y)| \;\le\; A_g ||x - y||^\alpha \;\equiv\; A_g \left(\sum_{i=1}^m |x_i - y_i|^2\right)^{\alpha/2}.$$

Of course, if the exponent is $\alpha = 1$, the function becomes Lipschitz continuous.

THEOREM 11. *Suppose that $\{Y_k : k \ge 0\}$ is GMC and define the following functionals of the batch processes $\{(Y_{jm}, \dots, Y_{(j+1)m-1}) : j \ge 0\}$ and $\{(Y'_{jm}, \dots, Y'_{(j+1)m-1}) : j \ge 0\}$,*

$$\left\{B_j(m) \equiv g(Y_{jm}, \dots, Y_{(j+1)m-1}) : j \ge 0\right\} \quad \textit{and} \quad \left\{B'_j(m) \equiv g(Y'_{jm}, \dots, Y'_{(j+1)m-1}) : j \ge 0\right\}.$$

If $g(x)$ is an α-Hölder-continuous function with $\mathrm{E}\left[|B_j(m)|^{\frac{2}{\alpha}+\epsilon}\right] < \infty$ for some $\epsilon > 0$ (the $\epsilon > 0$ is needed to guarantee uniform integrability when $\alpha = 1$), then $\{B_j(m) : j \ge 1\}$ is a GMC process for any $\psi \in (0, \frac{2}{\alpha} + \epsilon)$.

Again, see Appendix A7 for the proof.

EXAMPLE 22. Any linear function defined by $g(x_1, \dots, x_m) = \sum_{i=1}^m \alpha_i x_i$ for $\alpha_i \in \mathbb{R}$, $i = 1, \dots, m$ is Lipschitz continuous. This implies, for example, batch means $\{\overline{Y}_j(m) = g(Y_{jm}, \dots, Y_{(j+1)m-1}), j \ge 0\}$ obtained by setting $\alpha_i = 1/m$, $i = 1, \dots, m$, are GMC by Theorem 11, if $\mathrm{E}[|\overline{Y}_j(m)|^u] < \infty$ for some $u > 2$. Moreover, by Dingeç et al. (2023a, Theorem 1(b)), $\mathrm{Cov}[\overline{Y}_1(m), \overline{Y}_{1+\ell}(m)] = O(s^\ell)$ for some $s \in (0,1)$. □

EXAMPLE 23. It can be shown that the order statistics $g(x_1, \dots, x_m) = x_{(k)}$ for $1 \le k \le m$ are Lipschitz continuous. Let $x_{(k)}$ and $y_{(k)}$ denote the kth order statistics of two samples $x_1, \dots, x_m$ and $y_1, \dots, y_m$, respectively. Without loss of generality, we can assume $x_{(k)} \ge y_{(k)}$. By definition, there are k elements in the second sample, which are less than equal to $y_{(k)}$, and at most $k-1$ elements in the first sample which are strictly smaller than $x_{(k)}$. So there is at least one index ℓ such that $y_\ell \le y_{(k)}$ and $x_\ell \ge x_{(k)}$. This implies

$$|x_{(k)} - y_{(k)}| \;=\; x_{(k)} - y_{(k)} \;\le\; x_\ell - y_\ell \;=\; |x_\ell - y_\ell| \;\le\; ||x - y||.$$

That is, $g(x_1, \dots, x_m) = x_{(k)}$ is Lipschitz continuous with a Lipschitz constant 1.

Let $Y_{j,(1)} \le Y_{j,(2)} \le \cdots \le Y_{j,(m)}$ be the order statistics of $\{Y_{jm}, \dots, Y_{(j+1)m-1}\}$. The jth batch quantile estimator (BQE) of the quantile $y_p = F^{-1}(p)$ is defined by $\widehat{y}_p(j,m) \equiv Y_{j,(\lceil mp \rceil)}$. If $\mathrm{E}[|\widehat{y}_p(j,m)|^u] < \infty$ for some $u > 2$, then by Theorem 11 and the Lipschitz continuity of order statistics, the BQEs $\{\widehat{y}_p(j,m), j \ge 0\}$ are GMC. Moreover, Dingeç et al. (2023a, Theorem 1(b)) implies $\mathrm{Cov}[\widehat{y}_p(1,m), \widehat{y}_p(1+\ell,m)] = O(s^\ell)$ for some $s \in (0,1)$. □

10. Conclusions and Future Research Directions

The Geometric-Moment Contraction condition applies to a large class of stationary stochastic processes and can substitute for hard-to-check mixing conditions in order to carry out mean and quantile estimation. The main contribution of the current paper is the development and experimental evaluation of "competing" sets of sufficient conditions that can be used to prove that various useful stochastic processes are GMC. We also provided numerous tricks to show how to transform GMC processes into other GMC processes. We placed special emphasis on processes that might be of interest to the computer simulation community.

Our first findings of the paper—Theorems 2 and 3 in §4—extend the class of stochastic processes that satisfy the GMC condition by providing sufficient conditions for a transformation of a GMC process to remain GMC. Theorem 2 presents a Lipschitz-based sufficient consition, while Theorem 3 mandates transformation functions satisfying certain derivative and moment assumptions. An extensive selection of examples is presented in §5, many of which establish that a variety of ARTOC processes are GMC. Remarkably, the various sets of sufficient conditions for Theorems 2 and 3 do not seem to directly subsume each other—some stochastic processes pass muster on one theorem but may fail the sufficient conditions on the other. For this reason, Table 1 in §5.4 and the subsequent discussion conveniently summarize how some of our examples adhere to the different sufficient conditions—if only to illustrate that a single, grand theorem is not yet available.

The remaining sections in the paper discussed various manipulations on GMC processes that preserve the GMC property. Section 7 showed that a linear combination of GMC processes is GMC; and it is noteworthy that the constituent GMC processes do not have to be independent of each other. We illustrated this result with a simple example involving the sum of two $\mathrm{Unif}(0,1)$ GMC process, and then a more-substantive example involving a Taylor series. Our work in §8 established that mixtures of GMC processes are GMC, a result that we use to generate several serially autocorrelated processes with triangular (or almost-triangular) marginals. We proved in §9 that if the underlying process is GMC, then the resulting nonoverlapping batch-means process is also GMC, a finding of direct interest for purposes of simulation output analysis. Finally, §6 studied the quantile indicator processes, where we showed that if $\{Y_k : k \geq 0\}$ is GMC, then the associated indicator process, $\left\{I_k(y) \equiv \mathbf{1}_{\{Y_k \leq y\}} : k \geq 0\right\}$ with given y, is GMC.

The work presented in this paper has motivated a great deal of our current research. In fact, the various GMC results studied herein are being used in in a rich set of complementary undertakings. For instance:

- Alexopoulos et al. (2023) exploits the assumption of GMC to derive asymptotically valid confidence intervals (CIs) based on the method of standardized time series for the quantiles $F^{-1}(p)$ of a stationary stochastic process.
- Dingeç et al. (2023a) studies implications of the GMC condition on the autocovariance functions $\{\mathrm{Cov}(Y_1, Y_{1+\ell}) : \ell \in \mathbb{Z}\}$ and $\{\mathrm{Cov}(I_1(y), I_{1+\ell}(y)) : \ell \in \mathbb{Z}\}$, as well as the associated variance parameters (i.e., sums of autovariances over all ℓ); and analogous results are reported for quantile estimators.

- Dingeç et al. (2023b) provides exact and asymptotic results in support of the theoretical findings in Dingeç et al. (2023a). Specifically, we derive results for the AR(1), ARTOP, and M/M/1 waiting time processes, each of which presents interesting challenges and insights.
- Dingeç et al. (2023d) derives expressions for the expected value and variance of nonoverlapping batch means and standardized time series area variance parameter estimators in quantile estimation problems involving GMC processes.
- We seek to establish GMC conditions that completely obviate the need for obscure ϕ-mixing conditions in estimation problems; see, e.g., the restrictive mixing conditions required in Chien et al. (1997).
- Lolos et al. (2023a,b) have formulated and extensively tested automated, easy-to-use sequential procedures that return CIs for steady-state marginal quantiles.
- We are tying together results establishing that GMC conditions that allow for Functional Central Limit Theorems (FCLTs) to hold; cf. Wu (2005a, 2009) and Berkes et al. (2014). We will also show how to apply such FCLTs in fixed-sample-size and sequential CI procedures. **[Also mention (Doukhan 2018)?]**
- We are currently studying convergence properties of GMC estimators that are incorporated in infinite sums such as Taylor series, Fourier series, or Walsh functions.
- We are carrying out robustness analyses to determine the consequences in cases when the GMC condition fails to hold.

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APPENDIX

This appendix contains several proofs of results presented in the main part of the article, as well as various additional discussions of interest. §A1 proves Theorem 3, which gives conditions under which a function of a GMC process retains the GMC property. §A2 uses the definition of the GMC condition and an easy moment condition to directly establish that if $\{X_k : k \geq 0\}$ is GMC, then so is $\{X_k^2 : k \geq 0\}$. The result is then applied to the $a = 2$ special case of Example 5's M/M/1 waiting-time process (which required a couple more conditions to hold). [Add discussion of other appendices.] §A6 proves Theorem 6, which establishes that mixtures of GMC processes are themselves GMC. §A7 proves Theorem 10: If a process satisfies the GMC condition, then so does the corresponding nonoverlapping batch-means process; we also prove Theorem 11, a generalization of the former result. §A4 proves Theorem 5, i.e., if $\{X_k : k \geq 0\}$ is GMC, then so is the associated indicator process $\{I_k(y) : k \geq 0\}$. Online Appendix A8 provides an alternative proof-from-scratch of Corollary 1 (of Theorem 2) for ARTOC processes.

A1. Proof of Theorem 3

Let $\{X_k : k \geq 0\}$ and $\left\{X_k' : k \geq 0\right\}$ denote paired versions of the base process. Following in parallel the steps leading to Equation (A.44) in the proof of Corollary 1 in §A8, **[← NB: This step skips around to the Online Appendix. It would be really nice if we could somehow avoid this. Is it possible to shorten the proof a bit using some of the machinery we now have?]** we exploit the continuity of $f_X(x)$ and the differentiability of $g(x)$ on $\operatorname{int}\mathbb{S}_X$ to apply the mean value theorem, concluding that there exists $X_k^* \in \left(\left\{X_k \wedge X_k'\right\}, \left\{X_k \vee X_k'\right\}\right)$ for which we have

$$\left|g(X_k) - g(X_k')\right| = \left|g'(X_k^*)\right| \left|X_k - X_k'\right| \text{ for every } k \geq 0 \text{ a.s.} \tag{A.1}$$

Assumptions (34)–(35), and Avriel (1976, Theorem 4.26, p. 90) ensure that

$$\left.\begin{array}{l}\text{Because } g(x) \text{ is convex (resp., concave) and differentiable on } \operatorname{int}\mathbb{S}_X\text{, the derivative } g'(x)\\ \text{is nondecreasing (resp., nonincreasing) on } \operatorname{int}\mathbb{S}_X.\end{array}\right\} \tag{A.2}$$

Equation (A.2) and Avriel (1976, Theorem 4.28, p. 91) ensure that

$$\text{The derivative } g'(x) \text{ is continuous on } \operatorname{int}\mathbb{S}_X. \tag{A.3}$$

Equations (A.2)–(A.3) and Boyd and Vandenberghe (2004, Eq. (3.19), p. 98 and 2nd para., p. 99) together yield the following result:

$$\text{The derivative } g'(x) \text{ is } \textit{quasiconcave} \text{ on } \operatorname{int}\mathbb{S}_X. \tag{A.4}$$

Equation (A.4) involves two subcases:

$$\left.\begin{array}{l}\text{If } g'(x) \text{ is nondecreasing (or nonincreasing) and is nonnegative (or nonpositive) on } \operatorname{int}\mathbb{S}_X,\\ \text{then the function } |g'(x)| \text{ is nondecreasing (or nonincreasing) on } \operatorname{int}\mathbb{S}_X\text{; and}\end{array}\right\} \tag{A.5}$$

$$\left.\begin{array}{l}\text{If } g'(z) \text{ is nondecreasing (or nonincreasing) and takes negative, zero, and positive values}\\ \text{on } \operatorname{int}\mathbb{S}_X\text{, then the graph of } |g'(x)| \text{ is U-shaped on } \operatorname{int}\mathbb{S}_X\text{; i.e., there is a point } x_0 \in \operatorname{int}\mathbb{S}_X\\ \text{such that for } x \in \operatorname{int}\mathbb{S}_X \text{ and } x \le x_0\text{, the function } |g'(x)| \text{ is nonincreasing, while for } x \in \operatorname{int}\mathbb{S}_X\\ \text{and } x \ge x_0\text{, the function } |g'(x)| \text{ is nondecreasing.}\end{array}\right\} \tag{A.6}$$

In both subcases (A.5) and (A.6), we see that

$$\text{The function } |g'(x)| \text{ is quasiconvex on } \operatorname{int}\mathbb{S}_X.$$

Since the function $h(\mathfrak{s}) \equiv \mathfrak{s}^{\mathfrak{v}} : (0,\infty) \mapsto (0,\infty)$ is nondecreasing, we see that the composition $|g'(x)|^{\mathfrak{v}}$ of the function $|g'(x)|$ with the function $h(\mathfrak{s})$ is quasiconvex on $\operatorname{int}\mathbb{S}_X$ (Boyd and Vandenberghe 2004, §3.4.4, p. 102); therefore by the definition of a quasiconvex function (Boyd and Vandenberghe 2004, Eq. (3.19), p. 98), we have

$$\begin{aligned}\left|g'(X_k^*)\right|^{\mathfrak{v}} &\le \max\left\{\left|g'(X_k \wedge X_k')\right|^{\mathfrak{v}}, \left|g'(X_k \vee X_k')\right|^{\mathfrak{v}}\right\}\\ &= \max\left\{\left|g'(X_k)\right|^{\mathfrak{v}}, \left|g'(X_k')\right|^{\mathfrak{v}}\right\}\\ &\le \left|g'(X_k)\right|^{\mathfrak{v}} + \left|g'(X_k')\right|^{\mathfrak{v}} \text{ for all } k \ge 0 \text{ a.s.}\end{aligned} \tag{A.7}$$

Taking the expected values of both sides of Equation (A.7), by Assumption (37) we have

$$\mathrm{E}\left[\left|g'(X_k^*)\right|^{\mathfrak{v}}\right] \le 2\mathrm{E}\left[\left|g'(X_0)\right|^{\mathfrak{v}}\right] < \infty \text{ for all } k \ge 0. \tag{A.8}$$

Without loss of generality, we can assume $\mathfrak{v} \in (0, u_X)$. Then by Theorem 1, $\{X_k : k \ge 0\}$ satisfies the GMC condition for $\psi_X = \mathfrak{v}$; i.e., there exist $C_{\psi_X} \in (0,\infty)$ and $r_{\psi_X} \in (0,1)$ such that

$$\mathrm{E}\left[\left|X_k - X_k'\right|^{\psi_X}\right] \le C_{\psi_X} r_{\psi_X}^k \text{ for all } k \ge 0. \tag{A.9}$$

Thus, we can prove that $\{Y_k = g(X_k) : k \ge 0\}$ satisfies the GMC condition for $\psi_Y = \mathfrak{v}/2$,

$$\begin{aligned}\mathrm{E}\left[\left|Y_k - Y_k'\right|^{\psi_Y}\right] &= \mathrm{E}\left[\left|g(X_k) - g(X_k')\right|^{\mathfrak{v}/2}\right]\\ &= \mathrm{E}\left[\left|g'(X_k^*)\right|^{\mathfrak{v}/2}\left|X_k - X_k'\right|^{\mathfrak{v}/2}\right] && \text{(by Equation (A.1))}\\ &\le \left\{\mathrm{E}\left[\left|g'(X_k^*)\right|^{\mathfrak{v}}\right]\mathrm{E}\left[\left|X_k - X_k'\right|^{\mathfrak{v}}\right]\right\}^{1/2} && \text{(by the Cauchy–Schwarz inequality)}\\ &\le C_{\psi_Y} r_{\psi_Y}^k \quad \text{for all } k \ge 0,\end{aligned}$$

where, by Equations (A.8) and (A.9),

$$C_{\psi_Y} \equiv \left\{2C_{\psi_X}\mathrm{E}\left[\left|g'(X_0)\right|^{\mathfrak{v}}\right]\right\}^{1/2} \text{ and } r_{\psi_Y} \equiv r_{\psi_X}^{1/2}.$$

Theorem 1 together with the moment condition (36) ensures that $\{Y_k = g(X_k) : k \ge 0\}$ satisfies the GMC condition for all $\psi = \psi_Y = \mathfrak{v}/2 \in (0, u_Y)$. ∎

A2. A Direct Proof That $\{X_k\}$ Is GMC implies $\{X_k^2\}$ Is GMC

We start with a general result from which the $a = 2$ case from Example 5 follows quickly.

PROPOSITION 1. *If the process $\{X_k : k \geq 0\}$ is GMC and $\mathrm{E}\big[X_k^{2\psi}\big] < \infty$, then the process $\{X_k^2 : k \geq 0\}$ is GMC.*

Proof: We have

$$
\begin{aligned}
&\mathrm{E}\big[\big|X_k^2 - (X_k')^2\big|^{\psi}\big] \\
&= \mathrm{E}\big[\big|X_k - X_k'\big|^{\psi}\big|X_k + X_k'\big|^{\psi}\big] \\
&\leq \sqrt{\mathrm{E}\big[\big|X_k - X_k'\big|^{2\psi}\big] \cdot \mathrm{E}\big[\big|X_k + X_k'\big|^{2\psi}\big]} \quad \text{(Cauchy–Schwarz)} \\
&\leq \sqrt{\mathrm{E}\big[\big|X_k - X_k'\big|^{2\psi}\big] \cdot \max\{1, 2^{2\psi-1}\}\Big(\mathrm{E}\big[X_k^{2\psi}\big] + \mathrm{E}\big[(X_k')^{2\psi}\big]\Big)} \quad \text{(Dingeç et al. 2022, Lemma 1)} \\
&= \max\{\sqrt{2}, 2^{\psi}\}\sqrt{\mathrm{E}\big[\big|X_k - X_k'\big|^{2\psi}\big] \cdot \mathrm{E}\big[X_k^{2\psi}\big]} \quad (X_k \text{ and } X_k' \text{ are identically distributed}) \\
&\leq C r^{k/2} \ \text{ for } \ k \geq 0 \quad \text{(by the GMC and finite-moment assumptions).} \ \blacksquare
\end{aligned}
$$

In particular, for Example 5's M/M/1 waiting-time process $\{W_k : k \geq 0\}$, Equation (46) reminds us that $\mathrm{E}\big[W_k^{2\psi}\big] = \tau\,\Gamma(2\psi + 1)/\gamma^{2\psi} < \infty$.

A3. Proofs of Example 13 Results (ARTOC–Weibull)

In this appendix, we show that:

- The Lipschitz-continuity assumption of Theorem 2 is satisfied for the ARTOC–Weibull process when $a \geq 2$; but it is not satisfied for $0 < a < 2$ (see §A3.1).
- The conditions of Theorem 3 hold for the case $0 \leq a \leq 2$; but not for $a > 2$ (see §A3.2).

A3.1. Applying Theorem 2

In order to check the viability of Theorem 2, continuity tells us that it is enough to check the boundedness of $\mathfrak{L}(z)$ from Equation (65) only for the limiting cases as $z \to -\infty$ and $z \to +\infty$. It is clear that $\lim_{z\to-\infty} \mathfrak{L}(z) = 0$. On the other hand, the limiting value of $\mathfrak{L}(z)$ for $z \to +\infty$ depends on the shape parameter a. Consider the case $0 < a < 2$. By squeezing both sides of the Equation (5) inequalities so that $\Phi(-z) \sim \phi(z)/z$ as $z \to \infty$, we find that Equation (65) yields

$$
\mathfrak{L}(z) = O\left(\frac{z}{\big(-\ell\mathrm{n}\big[\frac{\phi(z)}{z}\big]\big)^{(a-1)/a}}\right) = O\left(\frac{z}{\big(\ell\mathrm{n}(z) + \frac{\ell\mathrm{n}(2\pi)}{2} + \frac{z^2}{2}\big)^{(a-1)/a}}\right) = O\big(z^{\frac{2}{a}-1}\big) \to \infty \quad \text{as } z \to +\infty. \tag{A.10}
$$

Thus, we conclude that the Lipschitz-continuity assumption of Theorem 2 is not satisfied if $0 < a < 2$.

Now consider the case $a \geq 2$, where Equations (5) and (65) result in the following upper bound when $z > 0$,

$$
g'(z) = \mathfrak{L}(z) < \frac{b\,(z^2+1)}{az\,(-\ell\mathrm{n}[\Phi(-z)]\,)^{(a-1)/a}} \sim \frac{b\left(z + \frac{1}{z}\right)}{a\big(\ell\mathrm{n}(z) + \frac{\ell\mathrm{n}(2\pi)}{2} + \frac{z^2}{2}\big)^{(a-1)/a}},
$$

which converges to $b/\sqrt{2}$ as $z \to +\infty$ if $a = 2$, and converges to zero if $a > 2$. So, by Theorem 2, the ARTOC–Weibull process with shape parameter $a \geq 2$ is GMC.

A3.2. Applying Theorem 3

We now show that the sufficient conditions of Theorem 3 are satisfied for $0 < a \leq 2$. The derivative $g'(z)$ is explicitly given in Equation (65), thereby satisfying Assumption (34). After a great deal of algebra, one can calculate the second derivative,

$$g''(z) = g'(z)\left(-z + \frac{\phi(z)}{\Phi(-z)}\left[1 + \frac{(a-1)/a}{\ell\text{n}[\Phi(-z)]}\right]\right). \tag{A.11}$$

Since $g'(z) > 0$ for all $z \in \mathbb{R}$ by Equation (65), we merely need to establish the positivity of Equation (A.11)'s second term in order to obtain the convexity of $g(z)$. For the case $0 < a \leq 1$ with $z \leq 0$, this is easily shown to be true from first principles and the fact that $\ell\text{n}[\Phi(-z)] < 0$; and for $z > 0$ via Equation (5). This leaves the case $1 < a \leq 2$, for which

$$-z + \frac{\phi(z)}{\Phi(-z)}\left[1 + \frac{(a-1)/a}{\ell\text{n}[\Phi(-z)]}\right] \geq -z + \frac{\phi(z)}{\Phi(-z)}\left[1 + \frac{1}{2\,\ell\text{n}[\Phi(-z)]}\right]. \tag{A.12}$$

Figure A1 gives numerical evidence that this last quantity is positive, at least for all $z \in [-6, 6]$. In order to proceed more rigorously, we further divide our toils into the subcases $|z| < 1$ and $|z| \geq 1$.

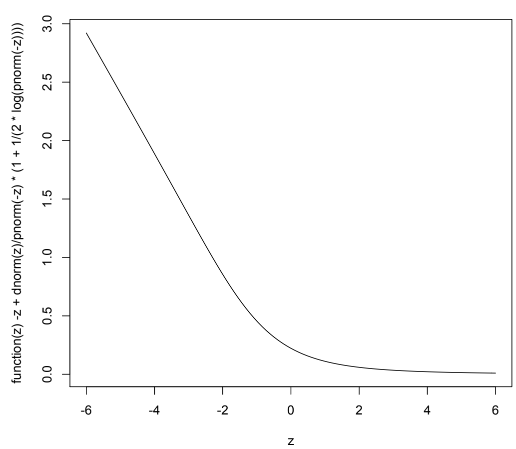

Figure A1 Plot of $h(z) \equiv -z + \frac{\phi(z)}{\Phi(-z)}\left[1 + \frac{1}{2\,\ell\text{n}[\Phi(-z)]}\right]$ for $z \in (-6, 6)$ from Equation (A.11). [Kemal, can you change the messy expression on the y-axis to plain old $h(z)$?]

Subcase $1 < a \leq 2$*, for* $z < 0$: [This *should* be easier than $0 \leq z < 1$ case that follows immediately below — especially after looking at Kemal's figure — but I've really, really tried for the last few days and haven't been able to do it yet! Maybe there's an easy trick that I'm just missing. But I'm getting carpal tunnel from this.]

Subcase $1 < a \le 2$*, for* $0 \le z < 1$: We begin with well-known expansions of $\phi(z)$ and $\Phi(-z)$,

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\,2^k} = \frac{1}{\sqrt{2\pi}} \left(1 - \frac{z^2}{2} + \frac{z^4}{8} - \frac{z^6}{48} + \cdots \right) \tag{A.13}$$

and

$$\Phi(-z) = \frac{1}{2} - \big(\Phi(z) - \Phi(0)\big) = \frac{1}{2} - \int_0^z \phi(t)\,\mathrm{d}t = \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,2^k\,(2k+1)}. \tag{A.14}$$

Then

$$\begin{aligned}
\ell\mathrm{n}\big(\Phi(-z)\big) &= \ell\mathrm{n}\big(2\Phi(-z)\big) - \ell\mathrm{n}(2) \\
&= \ell\mathrm{n}\left(1 - \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,2^k\,(2k+1)}\right) - \ell\mathrm{n}(2) \\
&= -\sum_{\ell=1}^{\infty} \frac{1}{\ell} \left[\sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,2^k\,(2k+1)}\right]^{\ell} - \ell\mathrm{n}(2) && \text{(A.15)} \\
&= -\sum_{\ell=1}^{\infty} \frac{1}{\ell} \left[\sqrt{\frac{2}{\pi}} \left(z - \frac{z^3}{6} + \frac{z^5}{40} - \frac{z^7}{336} + \cdots \right)\right]^{\ell} - \ell\mathrm{n}(2) && \text{(A.16)} \\
&\le -\sum_{\ell=1}^{\infty} \frac{1}{\ell} \left[\sqrt{\frac{2}{\pi}} \left(z - \frac{z^3}{6}\right)\right]^{\ell} - \ell\mathrm{n}(2) && \text{(A.17)} \\
&\le -2 \sum_{\ell=1}^{\infty} \left[\sqrt{\frac{1}{2\pi}} \left(z - \frac{z^3}{6}\right)\right]^{\ell} - \ell\mathrm{n}(2) \\
&= \frac{-2\big(z - \frac{z^3}{6}\big)}{\sqrt{2\pi} - \big(z - \frac{z^3}{6}\big)} - \ell\mathrm{n}(2), && \text{(A.18)}
\end{aligned}$$

where Equation (A.15) follows from the facts that $0 \le 2\big(\Phi(z) - \Phi(0)\big) < 1$ for all $z \ge 0$ and $\ell\mathrm{n}(1-x) = -\sum_{\ell=1}^{\infty} \frac{x^\ell}{\ell}$ for $-1 \le x < 1$; Equation (A.16) simply lists a few of the terms; Equation (A.17) follows from telescoping of the power series in their radii of convergence along with the fact that $\frac{1}{2^{\ell-1}} < \frac{1}{\ell}$ for $\ell \in \mathbb{Z}^+$; and Equation (A.18) is the sum of a geometric series.

In addition, by Equations (A.13) and (A.14),

$$\begin{aligned}
-z\Phi(-z) + \phi(z) &= -z \left[\frac{1}{2} - \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{k!\,2^k\,(2k+1)}\right] + \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!\,2^k} \\
&= \frac{1}{\sqrt{2\pi}} \left[1 - z\sqrt{\frac{\pi}{2}} + \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+2}}{(k+1)!\,2^{k+1}\,(2k+1)}\right] \\
&= \frac{1}{\sqrt{2\pi}} \left[1 - z\sqrt{\frac{\pi}{2}} + \frac{z^2}{2} - \frac{z^4}{24} + \frac{z^6}{240} - \frac{z^8}{2688} + \cdots \right]. && \text{(A.19)}
\end{aligned}$$

To establish $g''(z) > 0$ for this case, it is sufficient by Equation (A.12) to show that

$$\phi(z) \le -2\,\ell\mathrm{n}[\Phi(-z)]\big(-z\Phi(-z) + \phi(z)\big) \quad \text{for } 0 \le z < 1,$$

or, equivalently via Equations (A.13), (A.16), and (A.19),

$$1-\frac{z^2}{2}+\frac{z^4}{8}-\frac{z^6}{48}+\cdots \;\le\; 2\left\{\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left[\sqrt{\frac{2}{\pi}}\left(z-\frac{z^3}{6}+\frac{z^5}{40}-\cdots\right)\right]^{\ell}+\ell\mathrm{n}(2)\right\}\left[1-z\sqrt{\frac{\pi}{2}}+\frac{z^2}{2}-\frac{z^4}{24}+\frac{z^6}{240}-\cdots\right].$$

With telescoping of each of the series in the above expressions as well as Equation (A.18) in mind, it is sufficient to establish that

$$1-\frac{z^2}{2}+\frac{z^4}{8} \;\le\; \left\{\frac{2\left(z-\frac{z^3}{6}\right)}{\sqrt{2\pi}-\left(z-\frac{z^3}{6}\right)}+\ell\mathrm{n}(2)\right\}\left[2-\sqrt{2\pi}\,z+z^2-\frac{z^4}{12}\right].$$

This is equivalent to showing that

$$\begin{aligned}0 \;\le\; &\sqrt{2\pi}\left(2\ell\mathrm{n}(2)-1\right)+\left(5-2\ell\mathrm{n}(2)-2\pi\ell\mathrm{n}(2)\right)z+\sqrt{2\pi}\left(\frac{3}{2}+2\ell\mathrm{n}(2)\right)z^2+\frac{2}{3}\left(1-\ell\mathrm{n}(2)\right)z^3\\ &+\sqrt{2\pi}\left(\frac{-\ell\mathrm{n}(2)}{4}+\frac{5}{24}\right)z^4+\left(\frac{\ell\mathrm{n}(2)}{4}-\frac{7}{24}\right)z^5+\left(\frac{-\ell\mathrm{n}(2)}{72}+\frac{1}{144}\right)z^7\\ \doteq\; &0.96830-0.74147\,z-0.28592\,z^2+0.20457\,z^3+0.08785\,z^4-0.11838\,z^5-0.00268\,z^7. \qquad \text{(A.20)}\end{aligned}$$

Since $0\le z<1$, the right-hand side of Equation (A.20) can be bounded from below,

$$\begin{aligned}a(z) \;&\equiv\; \text{RHS of (A.20)}\\ &\ge\; 0.96830-0.74147\,z-0.28502\,z^2+0.20457\,z^3-0.03321\,z^4\\ &\ge\; 0.96830-0.74147\,z-0.28502\,z^2+0.17136\,z^3.\end{aligned}$$

Note that $a(0)=0.96839$, $a(1)=0.11317$, and $a'(z)=-0.74147-0.57004\,z+0.51341\,z^2<0$ for all $0\le z<1$, which implies that $a(z)>0$ for all $0\le z<1$. This finally establishes that $g''(z)>0$ for the subcase $1<a\le 2$, for $0\le z<1$.

Subcase $1<a\le 2$, for $z\ge 1$: [Unfortunately, the (many) tricks / bounds I used for the $0<z\le 1$ case run out of steam when you get much beyond $z=1$. I strongly suspect that the result is true, judging from the numerics that Kemal and I have done.]

Subcase $a>2$, for all $z\in\mathbb{R}$: We have already established in §A3.1 that $\lim_{z\to-\infty}g'(z)=\lim_{z\to+\infty}g'(z)=0$. Taken together with the strict positivity of $g(z)$ at some z (e.g., $z=0$) indicates that $g'(z)$ is not monotone (first increasing from zero to a positive number, then decreasing to zero); and so $g(z)$ is neither convex nor concave for this subcase.

Let $Y_0=g(Z_0)\sim\mathrm{Weibull}(a,b)$. It is well known that the moments of Weibull distribution are finite,

$$\mathrm{E}[|g(Z_0)|^{u_Y}] \;=\; \mathrm{E}[Y_0^{u_Y}] \;=\; b^{u_Y}\Gamma\left(1+\frac{u_Y}{a}\right)<\infty, \quad \text{for } u_Y>0.$$

In addition, by Equation (65), $g'(z) > 0$ for $z \in \mathbb{R}$, and so the expected value of the absolute derivative is

$$\mathrm{E}[|g'(Z_0|] = \mathrm{E}[g'(Z_0)] = \frac{b}{a}\int_{\mathbb{R}} \frac{\big(-\ell\mathrm{n}[\Phi(-z)]\big)^{(1-a)/a}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z. \tag{A.21}$$

To establish the finiteness of $\mathrm{E}[g'(Z_0)]$, we first consider the case $0 < a \le 1$. The integral in Equation (A.21) can be written as

$$\int_{\mathbb{R}} \frac{\big(-\ell\mathrm{n}[\Phi(-z)]\big)^{\frac{1-a}{a}}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z = \left(\int_{-\infty}^{1} + \int_{1}^{\infty}\right)\frac{\big(-\ell\mathrm{n}[\Phi(-z)]\big)^{\frac{1-a}{a}}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z. \tag{A.22}$$

Since $1/\Phi(-z)$ and $-\ell\mathrm{n}[\Phi(-z)]$ are increasing functions of z, and since $\phi^2(z) < \phi(z)$ for all $z \in \mathbb{R}$, the first integral in Equation (A.22) is

$$\int_{-\infty}^{1} \frac{\big(-\ell\mathrm{n}[\Phi(-z)]\big)^{\frac{1-a}{a}}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z < \frac{\big(-\ell\mathrm{n}[\Phi(-1)]\big)^{\frac{1-a}{a}}}{\Phi(-1)}\int_{-\infty}^{1}\phi^2(z)\,\mathrm{d}z < \frac{\big(-\ell\mathrm{n}[\Phi(-1)]\big)^{\frac{1-a}{a}}\Phi(1)}{\Phi(-1)} < \infty.$$

Meanwhile, Equation (7) asserts that $\Phi(-z) > \phi(z)/(2z)$ for $z \ge 1$. Then

$$-\ell\mathrm{n}[\Phi(-z)] < -\ell\mathrm{n}\left[\frac{\phi(z)}{2z}\right] = \frac{\ell\mathrm{n}(2\pi)}{2} + \frac{z^2}{2} + \ell\mathrm{n}(2z) < \frac{\ell\mathrm{n}(2\pi)}{2} + \frac{z^2}{2} + 2z - 1 < \frac{z^2}{2} + 2z, \tag{A.23}$$

where the penultimate inequality reflects the fact that $\ell\mathrm{n}(2z) < 2z - 1$ for $z \ge 1$. So, by Equations (7) and (A.23), the second integral in Equation (A.22) is

$$\begin{aligned}
\int_{1}^{\infty} \frac{\big(-\ell\mathrm{n}[\Phi(-z)]\big)^{\frac{1-a}{a}}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z &< 2\int_{1}^{\infty} z\left(\frac{z^2}{2} + 2z\right)^{\frac{1-a}{a}}\phi(z)\,\mathrm{d}z \\
&= 2^{\frac{2a-1}{a}}\int_{1}^{\infty} z^{1/a}\,(z+4)^{\frac{1-a}{a}}\,\phi(z)\,\mathrm{d}z \\
&< 4\int_{1}^{\infty}(z+4)^{2/a}\,\phi(z)\,\mathrm{d}z \\
&< 4\int_{\mathbb{R}}(z+4)^{2/a}\,\phi(z)\,\mathrm{d}z \\
&< \infty,
\end{aligned}$$

where the final inequality follows from the finiteness of normal moments.

Now we consider the case in which $1 < a \le 2$, by using the inequality $\ell\mathrm{n}[\Phi(-z)] < \Phi(-z) - 1 = \Phi(z)$ and the fact that $1 - a < 0$, we obtain

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\big(-\ell\mathrm{n}[\Phi(-z)]\big)^{\frac{1-a}{a}}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z &< \int_{\mathbb{R}} \frac{[\Phi(z)]^{\frac{1-a}{a}}}{\Phi(-z)}\phi^2(z)\,\mathrm{d}z \\
&< \int_{\mathbb{R}} \frac{\phi^2(z)}{\Phi(z)\Phi(-z)}\,\mathrm{d}z \\
&= 2\int_{0}^{\infty} \frac{\phi^2(z)}{\Phi(z)\Phi(-z)}\,\mathrm{d}z \quad \text{(by symmetry)} \\
&< 4\int_{0}^{\infty} \frac{\phi^2(z)}{\Phi(-z)}\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&< 4\left[\int_0^1 \frac{\phi(z)}{\Phi(-z)}\,\mathrm{d}z + \int_1^\infty \frac{\phi^2(z)}{\Phi(-z)}\,\mathrm{d}z\right] \quad (\text{since } \phi^2(z) < \phi(z)) \\
&< 4\left[-\ell\mathrm{n}[\Phi(-z)]\Big|_0^1 + 2\int_1^\infty z\phi(z)\,\mathrm{d}z\right] \quad (\text{by Equation (7)}) \\
&= 4\left[-\ell\mathrm{n}[2\Phi(-1)] + \sqrt{\frac{2}{e\pi}}\right] < \infty,
\end{aligned}
$$

where the last equality is a known result (see, e.g., Wolfram Research, Inc. 2022) .

Therefore, $\mathrm{E}[|g'(z)|] < \infty$ and by Theorem 3, the ARTOC–Weibull process with a shape parameter $0 < a \le 2$ is GMC. Note that both of Theorems 2 and 3 work when $a = 2$, since $g'(z)$ is bounded and monotone with $g''(z) > 0$ in that case. □

A4. Proof Lemma 7 from §6

Proof of Lemma 7: We have

$$
\begin{aligned}
\mathrm{E}\left[|W_{ik} - W'_{ik}|^{\psi}\right] &= \mathrm{E}\left[|\mathbf{1}_{\{V_k=i\}} - \mathbf{1}_{\{V'_k=i\}}|^{\psi}\right] \\
&= \mathrm{E}\left[|\mathbf{1}_{\{V_k=i\}} - \mathbf{1}_{\{V'_k=i\}}|\right] \quad (\text{since the inside of } |\cdot| \text{ can only be 0 or 1}) \\
&= \Pr\left(\mathbf{1}_{\{V_k=i\}} \ne \mathbf{1}_{\{V'_k=i\}}\right) \\
&= \Pr\left(\{V_k = i, V'_k \ne i\} \cup \{V_k \ne i, V'_k = i\}\right) \\
&= 2\Pr\left(V_k = i, V'_k \ne i\right) \\
&= 2\sum_{j=1, j\ne i}^{d} \Pr\left(V_k = i, V'_k = j\right) \\
&\le 2\sum_{j=1}^{d} |i-j| \Pr\left(V_k = i, V'_k = j\right) \\
&\le 2\sum_{i=1}^{d}\sum_{j=1}^{d} |i-j| \Pr\left(V_k = i, V'_k = j\right) \\
&= \mathrm{E}\left[|V_k - V'_k|\right].
\end{aligned} \tag{A.24}
$$

Meanwhile, since all moments of V_k exist and $\{V_k : k \ge 0\}$ is GMC, we have from Theorem 1 that $\mathrm{E}\left[|V_k - V'_k|^{\psi}\right] < \infty$ for all $\psi > 0$, in particular, for $\psi = 1$. Then Equation (A.24) immediately implies that

$$
\mathrm{E}\left[|W_{ik} - W'_{ik}|^{\psi}\right] \le \mathrm{E}\left[|V_k - V'_k|\right] \le C_1 r_1^k, \quad \text{for } k \ge 0 \text{ and all } i,
$$

for some $C_1 > 0$ and $r_1 \in (0, 1)$; and thus, $\{W_k : k \ge 0\}$ is GMC. ∎

A5. Proof of Lemma 9 from §7.1

Proof of Lemma 9: We have

$$
\mathrm{E}\left[|A_k B_k - A'_k B'_k|^{\psi}\right] = \mathrm{E}\left[\left|(A_k - A'_k)B_k + A'_k(B_k - B'_k)\right|^{\psi}\right]
$$

$$\begin{aligned}
&\le \mathrm{E}\big[\big(|(A_k - A'_k)B_k| + |A'_k(B_k - B'_k)|\big)^{\psi}\big] \\
&\le \max\{1, 2^{\psi-1}\}\Big(\mathrm{E}\big[|(A_k - A'_k)B_k|^{\psi}\big] + \mathrm{E}\big[|A'_k(B_k - B'_k)|^{\psi}\big]\Big) \\
&\qquad \text{(Dingeç et al. 2022, Lemma 1)} \\
&\le K_{\psi}\Big(\sqrt{\mathrm{E}\big[|A_k - A'_k|^{2\psi}\big]\mathrm{E}\big[|B_k|^{2\psi}\big]} + \sqrt{\mathrm{E}\big[|A'_k|^{2\psi}\big]\mathrm{E}\big[|B_k - B'_k|^{2\psi}\big]}\Big) \\
&\le K^{\star}_{\psi}\Big(\sqrt{\mathrm{E}\big[|A_k - A'_k|^{2\psi}\big]} + \sqrt{\mathrm{E}\big[|B_k - B'_k|^{2\psi}\big]}\Big), \qquad \text{(A.25)}
\end{aligned}$$

where the last steps incorporate large-enough constants K_{ψ} and $K^{\star}_{\psi} > 0$, the Cauchy–Schwarz inequality, and the facts that $\mathrm{E}\big[|A'_k|^{2\psi}\big] = \mathrm{E}\big[|A_k|^{2\psi}\big] < \infty$ and $\mathrm{E}\big[|B_k|^{2\psi}\big] < \infty$. Then since $\{A_k : k \ge 0\}$ and $\{B_k : k \ge 0\}$ are both GMC, we have that the RHS of Equation (A.25) converges to 0 exponentially fast. [May also have to finesse that $2\psi <$ our old friend u.] ∎

A6. Proof of Theorem 6 from §[composition]

We can now prove the main mixture result.

Proof of Theorem 6: By Lemma 7, each of the processes $\{W_{ik} : k \ge 0\}$, for $i = 1, 2, \ldots, d$, are GMC. Then by Lemma 9, each of the (product) processes $\{Y_{ik}W_{ik} : k \ge 0\}$, for $i = 1, 2, \ldots, d$, are GMC. We now apply Lemma 8 to Equation (71) (with $Y_{ik}W_{ik}$ in the equation serving in place of Y_{ik} in the lemma) to complete the proof. ∎

A7. Proofs of Theorems 10 and 11:

[Moved to Chien paper.]

Proof of Theorem 11: We have

$$\begin{aligned}
\mathrm{E}\big[|B_j(m) - B'_j(m)|^{2/\alpha}\big] &\le A_g^{2/\alpha}\,\mathrm{E}\left[\sum_{k=jm}^{(j+1)m-1} |Y_k - Y'_k|^2\right] \\
&= A_g^{2/\alpha} \sum_{k=jm}^{(j+1)m-1} \mathrm{E}[|Y_k - Y'_k|^2] \\
&\le A_g^{2/\alpha} C_2 \sum_{k=jm}^{(j+1)m-1} r_2^k \\
&\le A_g^{2/\alpha} C_2\, m\, r_2^{jm},
\end{aligned}$$

where $C_2 > 0$ and $r_2 > 0$ are appropriate constants as per Theorem 1. Then Theorem 1 implies $\{B_j(m) : j \ge 0\}$ is GMC for any $\psi \in \big(0, \frac{2}{\alpha} + \epsilon\big)$. ∎

ONLINE APPENDIX

A8. Alternative Proof of Corollary 1 from First Principles

Before undertaking the formal proof that carefully uses a stochastic form of the mean value theorem, we first present an informal high-level sketch of the proof.

Sketch of the Alternative Proof: Let $\mathfrak{s} \wedge \mathfrak{t} \equiv \min\{\mathfrak{s}, \mathfrak{t}\}$ and $\mathfrak{s} \vee \mathfrak{t} \equiv \max\{\mathfrak{s}, \mathfrak{t}\}$ for $\mathfrak{s}, \mathfrak{t} \in \mathbb{R}$. By Definition 1 and Equations (51)–(53), the mean value theorem ensures that there exists $Z_k^* \in \Big(\big\{Z_k \wedge Z_k'\big\}, \big\{Z_k \vee Z_k'\big\}\Big)$ a.s. such that

$$\begin{aligned}
\big|Y_k - Y_k'\big| &= F^{-1}\big[\Phi\big(Z_k \vee Z_k'\big)\big] - F^{-1}\big[\Phi\big(Z_k \wedge Z_k'\big)\big] \\
&= \left\{\frac{\mathrm{d}}{\mathrm{d}z} F^{-1}\big[\Phi(z)\big]\bigg|_{z=Z_k^*}\right\}\Big(\big\{Z_k \vee Z_k'\big\} - \big\{Z_k \wedge Z_k'\big\}\Big) \\
&= \frac{\phi\big(Z_k^*\big)}{f\big\{F^{-1}\big[\Phi\big(Z_k^*\big)\big]\big\}}\big|Z_k - Z_k'\big| \ \le \mathfrak{M}\big|Z_k - Z_k'\big| \ \text{ for all } k \ge 0 \text{ a.s.};
\end{aligned} \tag{A.26}$$

and thus by Equations (32) and (A.26), we have $\mathrm{E}\big[\big|Y_k - Y_k'\big|^{\psi}\big] \le (\mathfrak{M}^{\psi} C_{\psi}) r_{\psi}^k$ for each $k \ge 0$ and $\psi > 0$, and $C_{\psi} > 0$ and $r_{\psi} \in (0, 1)$ from Equation (32). ∎

Formal Alternative Proof: Defnition 1 ensures that $\operatorname{int}\mathbb{S}$ is an open interval in $\mathbb{R}$. The proof of Theorem 1 requires verifying and applying certain intermediate results to the following types of random intervals:

$$\left.\begin{array}{c}
\Big(\big\{Z_k \wedge Z_k'\big\}, \big\{Z_k \vee Z_k'\big\}\Big) \subset \mathbb{R}, \\
\Big(\Phi\big\{Z_k \wedge Z_k'\big\}, \Phi\big\{Z_k \vee Z_k'\big\}\Big) \subset (0, 1), \text{ and} \\
\Big[\big\{Y_k \wedge Y_k'\big\}, \big\{Y_k \vee Y_k'\big\}\Big] = \Big[F^{-1}\big(\Phi\big\{Z_k \wedge Z_k'\big\}\big), F^{-1}\big(\Phi\big\{Z_k \vee Z_k'\big\}\big)\Big] \subset \operatorname{int}\mathbb{S}
\end{array}\right\} \text{ for all } k \ge 0 \text{ a.s.}$$

Enumerated below are the intermediate results required to complete the proof of Corollary 1.

IR1. The c.d.f. $F : \mathbb{R} \mapsto [0, 1]$ is continuous on $\mathbb{R}$. The restriction of F to $\operatorname{int}\mathbb{S}$ is differentiable and strictly increasing so that

$$\frac{\mathrm{d}}{\mathrm{d}y} F(y) = f(y) > 0 \text{ for all } y \in \operatorname{int}\mathbb{S}; \text{ and the image } F(\operatorname{int}\mathbb{S}) = (0, 1), \tag{A.27}$$

where the image of $\operatorname{int}\mathbb{S}$ under the restricted function F is defined by

$$F(\operatorname{int}\mathbb{S}) \equiv \{\mathfrak{w} : \mathfrak{w} = F(y) \text{ for some } y \in \operatorname{int}\mathbb{S}\}.$$

For simplicity, we use the symbol F to denote both the c.d.f. and its restriction to $\operatorname{int}\mathbb{S}$ since no confusion can result from this widely used convention.

IR2. The inverse c.d.f. $F^{-1} : (0, 1) \mapsto \operatorname{int}\mathbb{S}$ is differentiable and strictly increasing with

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{w}} F^{-1}(\mathfrak{w}) = \frac{1}{f[F^{-1}(\mathfrak{w})]} > 0 \text{ for all } \mathfrak{w} \in (0, 1); \text{ and } F[F^{-1}(\mathfrak{w})] = \mathfrak{w} \text{ for all } \mathfrak{w} \in (0, 1). \tag{A.28}$$

IR3. The composite function $F^{-1}[\Phi(\cdot)] : \mathbb{R} \mapsto \operatorname{int}\mathbb{S}$ is differentiable and strictly increasing with

$$\frac{\mathrm{d}}{\mathrm{d}z}F^{-1}[\Phi(z)] = \frac{\phi(z)}{f\{F^{-1}[\Phi(z)]\}} > 0 \text{ for all } z \in \mathbb{R}; \text{ and } F^{-1}[\Phi(\mathbb{R})] = \operatorname{int}\mathbb{S}. \tag{A.29}$$

IR4. We have the key property

$$\left|Y_k - Y_k'\right| = \left\{Y_k \vee Y_k'\right\} - \left\{Y_k \wedge Y_k'\right\} = F^{-1}\left[\Phi\left(Z_k \vee Z_k'\right)\right] - F^{-1}\left[\Phi\left(Z_k \wedge Z_k'\right)\right] > 0 \text{ for all } k \geq 0 \text{ a.s.} \tag{A.30}$$

IR5. For each $k \geq 0$ there exists $Z_k^* \in \left(\left\{Z_k \wedge Z_k'\right\}, \left\{Z_k \vee Z_k'\right\}\right)$ such that

$$\left|Y_k - Y_k'\right| = \frac{\phi\left(Z_k^*\right)}{f\left\{F^{-1}\left[\Phi\left(Z_k^*\right)\right]\right\}}\left|Z_k - Z_k'\right| \leq \mathfrak{M}\left|Z_k - Z_k'\right| \text{ for all } k \geq 0 \text{ a.s.} \tag{A.31}$$

The first step in verifying **IR1** exploits the fundamental relation

$$F(y) = \int_{-\infty}^{y} f(\mathfrak{s})\,\mathrm{d}\mathfrak{s} \text{ for all } y \in \mathbb{R}, \tag{A.32}$$

which ensures that F is continuous on $\mathbb{R}$ (Billingsley 1995, 2nd para., p. 401). The next step is to select arbitrarily a bounded interval $[\mathfrak{a}_0, \mathfrak{b}_0] \subset \operatorname{int}\mathbb{S}$; and since f is continuous on $[\mathfrak{a}_0, \mathfrak{b}_0]$, we have $\frac{\mathrm{d}}{\mathrm{d}y}F(y) = f(y) > 0$ for all $y \in (\mathfrak{a}_0, \mathfrak{b}_0)$ by the fundamental theorem of integral calculus (Fitzpatrick 2006, Theorem 6.29, p. 168). Since $[\mathfrak{a}_0, \mathfrak{b}_0]$ is an arbitrary closed interval in $\operatorname{int}\mathbb{S}$, we must have $\frac{\mathrm{d}}{\mathrm{d}y}F(y) = f(y) > 0$ for all $y \in \operatorname{int}\mathbb{S}$ so that F is strictly increasing on $\operatorname{int}\mathbb{S}$.

To finish verifying **IR1**, we must show that $F(\operatorname{int}\mathbb{S}) = (0, 1)$. Since F is continuous on $\mathbb{R}$ and $\operatorname{int}\mathbb{S}$ is an interval in $\mathbb{R}$, we have the image $F(\operatorname{int}\mathbb{S}) = I$ for some interval $I \subset [0, 1]$ (Fitzpatrick 2006, Theorem 3.14, p. 65). Let the notation nbd $\mathfrak{c}$ (resp., nbd $\mathfrak{w}$) denote a neighborhood of $\mathfrak{c}$ (resp., $\mathfrak{w}$) in the space $\operatorname{int}\mathbb{S}$ (resp., $[0, 1]$). We use proof by contradiction to rule out the following tentative assumptions about the interval I: (i) $0 \in I$; (ii) $1 \in I$; or (iii) I is closed, half closed, or open with lower endpoint $\mathfrak{u}_1 > 0$ or upper endpoint $\mathfrak{v}_1 < 1$. In case (i), there must exist $\mathfrak{a}_1 \in \operatorname{int}\mathbb{S}$ such that $\mathfrak{u}_1 \equiv F(\mathfrak{a}_1) = 0$. Starting from the Euclidean topology on the space $\mathbb{R}$, we see that in the relative (induced) topology on the subspace $[0, 1]$, for any given $\mathfrak{v}_1 \in (0, 1]$ we have the following neighborhood of $\mathfrak{u}_1$,

$$\text{nbd}\,\mathfrak{u}_1 = [\mathfrak{u}_1, \mathfrak{v}_1) = [0, \mathfrak{v}_1); \tag{A.33}$$

see Dugundji (1966, Ex. 1(c), p. 77). Because F is continuous on $\mathbb{R}$, the restriction of F to $\operatorname{int}\mathbb{S}$ is also continuous (Dugundji 1966, Property 8.2(2), p. 79). The latter result implies that there exist a neighborhood nbd $\mathfrak{a}_1 \subset \operatorname{int}\mathbb{S}$ and points $\mathfrak{c}_1, \mathfrak{d}_1 \in \operatorname{int}\mathbb{S}$ such that

$$\mathfrak{a}_1 \in (\mathfrak{c}_1, \mathfrak{d}_1), \quad [\mathfrak{c}_1, \mathfrak{d}_1] \subset \text{nbd}\,\mathfrak{a}_1, \quad \text{and} \quad F\{[\mathfrak{c}_1, \mathfrak{d}_1]\} \subset F(\text{nbd}\,\mathfrak{a}_1) \subset \text{nbd}\,\mathfrak{u}_1;$$

see Dugundji (1966, Theorem 8.3(4), pp. 79–80). However, we also have

$$\mathfrak{c}_1 < \mathfrak{a}_1 \text{ implies } F(\mathfrak{c}_1) < F(\mathfrak{a}_1) = 0 \tag{A.34}$$

so that Equation (A.34) contradicts Equation (A8) as well as the basic requirement that $F(\mathfrak{c}_1) \in [0,1]$; and thus the tentative assumption (i) is false. A similar argument shows that the tentative assumption (ii) is false.

To show that the tentative assumption (iii) is false, we must consider separately the subcases in which I is closed, half closed, or open with lower endpoint $\mathfrak{u}_1 > 0$, or upper endpoint $\mathfrak{v}_1 < 1$. For example, under the tentative assumption that I is the open interval $(\mathfrak{u}_2, \mathfrak{v}_2)$ with $0 < \mathfrak{u}_2 < \mathfrak{v}_2 < 1$, we see that the only possible configuration of some amount of probability mass on $\mathbb{R}$ whose associated c.d.f. F and p.d.f. f can satisfy Equations (1) and (A.32) while yielding $F(\operatorname{int}\mathbb{S}) = (\mathfrak{u}_2, \mathfrak{v}_2)$ must have the following characteristics: (i) the support $\mathbb{S}$ of the p.d.f. f has some lower and upper endpoints $\mathfrak{a}_2$ and $\mathfrak{b}_2$ such that $-\infty < \mathfrak{a}_2 < \mathfrak{b}_2 < \infty$ and $\mathbb{S}$ has boundary $\operatorname{bd}\mathbb{S} = \{\mathfrak{a}_2, \mathfrak{b}_2\}$; and (ii) the c.d.f. has the functional form

$$F(y) = \begin{cases} \int_{(-\infty,y]} f(\mathfrak{s})\,\mathrm{d}\mathfrak{s} = 0 & \text{if } y \in (-\infty, \mathfrak{a}_2), \\ \mathfrak{u}_2 & \text{if } y = \mathfrak{a}_2, \\ \mathfrak{u}_2 + \int_{(\mathfrak{a}_2,y]} f(\mathfrak{s})\,\mathrm{d}\mathfrak{s} & \text{if } y \in (\mathfrak{a}_2, \mathfrak{b}_2), \\ \mathfrak{v}_2 + \int_{(\mathfrak{b}_2,y]} f(\mathfrak{s})\,\mathrm{d}\mathfrak{s} = \mathfrak{v}_2 & \text{if } y \in [\mathfrak{b}_2, \infty), \end{cases} \tag{A.35}$$

because $f(\mathfrak{s}) = 0$ for all $\mathfrak{s} \in (-\infty, \mathfrak{a}_2) \cup (\mathfrak{b}_2, \infty)$. Equation (A.35) implies that F has the following properties, which contradict two of the basic requirements for a continuous c.d.f.:

$$\left.\begin{aligned} &\lim_{\substack{y\to\mathfrak{a}_2\\ y<\mathfrak{a}_2}} F(y) = 0 \\ &\lim_{\substack{y\to\mathfrak{a}_2\\ y>\mathfrak{a}_2}} F(y) = \mathfrak{u}_2 \end{aligned}\right\} \text{ so } F \text{ has an atom (jump) at } \mathfrak{a}_2 \text{ with } \Pr(Y_0 = \mathfrak{a}_2) = \mathfrak{u}_2 > 0, \text{ and} \tag{A.36}$$

$$\lim_{y\to\infty} F(y) = \mathfrak{v}_2 < 1 \text{ so } F \text{ specifies a defective probability distribution with defect } 1 - \mathfrak{v}_2 > 0. \tag{A.37}$$

The contradictions (A.36) and (A.37) show that the tentative assumption $F(\operatorname{int}\mathbb{S}) = (\mathfrak{u}_2, \mathfrak{v}_2)$ with $0 < \mathfrak{u}_2 < \mathfrak{v}_2 < 1$ is false. Moreover, the tentative assumption that $F(\operatorname{int}\mathbb{S}) = (\mathfrak{u}_2, \mathfrak{v}_2]$ with $0 < \mathfrak{u}_2 < \mathfrak{v}_2 < 1$ is easily seen to be false because we can select points $\mathfrak{a}_2, \mathfrak{d}_2 \in \operatorname{int}\mathbb{S}$ such that

$$\mathfrak{a}_2 < \mathfrak{d}_2 \ \text{ and } \ F(\mathfrak{a}_2) = \mathfrak{v}_2 < F(\mathfrak{d}_2), \tag{A.38}$$

which contradicts the tentative assumption that $F(\mathfrak{d}) \in (\mathfrak{u}_2, \mathfrak{v}_2]$ for every $\mathfrak{d} \in \operatorname{int}\mathbb{S}$. Similar arguments apply to all other subcases of tentative assumption (iii). The only remaining possibility for the layout of I is that $F(\operatorname{int}\mathbb{S}) = I = (0,1)$, which completes the verification of **IR1**.

We verify **IR2** by applying both parts of Equation (A.27) and the inverse-function rule (Fitzpatrick 2006, Theorem 4.11, p. 97 and Corollary 4.12, p. 98).

We verify **IR3** by applying **IR2** and the chain rule (Fitzpatrick 2006, Theorem 4.14, p. 99).

To verify **IR4**, we observe that the first part of Equation (A.30) follows immediately from the definitions of $\{Y_k \vee Y_k'\}$ and $\{Y_k \wedge Y_k'\}$. To show the second part of Equation (A.30), we note that Equation (30) implies

$$\left.\begin{aligned} &Z_k - Z_k' = \beta^k (Z_0 - Z_0') \sim \operatorname{Nor}(0, 2\beta^{2k}); \\ &\text{so } Z_k - Z_k' \neq 0 \text{ and } Z_k \wedge Z_k' < Z_k \vee Z_k' \end{aligned}\right\} \text{ for all } k \geq 0 \text{ a.s.} \tag{A.39}$$

since the $\mathrm{Nor}(0, 2\beta^{2k})$ distribution is continuous so that $\Pr\left(Z_k - Z'_k = 0 \text{ for some } k \geq 0\right) = 0$. Applying **IR3** (monotonicity) and Equation (A.39), we have

$$\left\{Y_k \vee Y'_k\right\} = F^{-1}\left[\Phi\left(Z_k \vee Z'_k\right)\right] \text{ and } \left\{Y_k \wedge Y'_k\right\} = F^{-1}\left[\Phi\left(Z_k \wedge Z'_k\right)\right] \text{ for all } k \geq 0 \text{ a.s.} \tag{A.40}$$

To verify **IR5**, we apply **IR3** and Equations (A.39)–(A.40) to conclude that the open interval

$$\left(\left\{Y_k \wedge Y'_k\right\}, \left\{Y_k \vee Y'_k\right\}\right) \subset \operatorname{int}\mathbb{S} \text{ for all } k \geq 0 \text{ a.s.}, \tag{A.41}$$

so that we have the following version of the first part of Equation (A.27) based on the restriction of $F(\cdot)$ to the random interval $\left(\left\{Y_k \wedge Y'_k\right\}, \left\{Y_k \vee Y'_k\right\}\right)$:

$$\frac{\mathrm{d}}{\mathrm{d}y}F(y) = f(y) > 0 \text{ at each } y \in \left(\left\{Y_k \wedge Y'_k\right\}, \left\{Y_k \vee Y'_k\right\}\right) \text{ for all } k \geq 0 \text{ a.s.}$$

Since $\Phi(z)$ is strictly increasing on $\mathbb{R}$, Equation (A.39) ensures that

$$0 < \Phi\left(Z_k \wedge Z'_k\right) < \Phi\left(Z_k \vee Z'_k\right) < 1 \text{ for all } k \geq 0 \text{ a.s.}; \tag{A.42}$$

and from Equation (A.42) we obtain the following version of Equation (A.28) based on the restriction of $F^{-1}(\cdot)$ to the random interval $\left[\Phi\left\{Z_k \wedge Z'_k\right\}, \Phi\left\{Z_k \vee Z'_k\right\}\right]$:

$$\frac{\mathrm{d}}{\mathrm{d}\mathfrak{w}}F^{-1}(\mathfrak{w}) = \frac{1}{f[F^{-1}(\mathfrak{w})]} > 0 \text{ for all } \mathfrak{w} \in \left[\Phi\left\{Z_k \wedge Z'_k\right\}, \Phi\left\{Z_k \vee Z'_k\right\}\right] \text{ and all } k \geq 0 \text{ a.s.}$$

Equation (A.39) also yields the following version of Equation (A.29) based on the restriction of $F^{-1}[\Phi(\cdot)]$ to the random interval $\left[\left\{Z_k \wedge Z'_k\right\}, \left\{Z_k \vee Z'_k\right\}\right]$:

$$\frac{\mathrm{d}}{\mathrm{d}z}F^{-1}\left[\Phi(z)\right] = \frac{\phi(z)}{f\left\{F^{-1}\left[\Phi(z)\right]\right\}} \text{ for all } z \in \left[\left\{Z_k \wedge Z'_k\right\}, \left\{Z_k \vee Z'_k\right\}\right] \text{ for all } k \geq 0 \text{ a.s.} \tag{A.43}$$

Thus, by Equation (A.43) and the mean value theorem for derivatives (Fitzpatrick 2006, Theorem 4.18, p. 103), there exists $Z^*_k \in \left(\left\{Z_k \wedge Z'_k\right\}, \left\{Z_k \vee Z'_k\right\}\right)$ such that

$$\begin{aligned} &F^{-1}\left[\Phi\left(Z_k \vee Z'_k\right)\right] - F^{-1}\left[\Phi\left(Z_k \wedge Z'_k\right)\right] \\ &\qquad = \left\{\frac{\mathrm{d}}{\mathrm{d}z}F^{-1}\left[\Phi(z)\right]\bigg|_{z=Z^*_k}\right\}\left(\left\{Z_k \vee Z'_k\right\} - \left\{Z_k \wedge Z'_k\right\}\right) \text{ for all } k \geq 0 \text{ a.s.} \end{aligned} \tag{A.44}$$

Finally, Equation (A.31) of **IR5** follows from Equations (53), (A.30), and (A.44); and then by Equation (32), we obtain the GMC condition,

$$\mathrm{E}\left[\left|Y_k - Y'_k\right|^{\psi}\right] \leq \mathfrak{C}_{\psi} r^k_{\psi} \text{ for all } \psi > 0 \text{ and } k \geq 0, \text{ where } \mathfrak{C}_{\psi} \equiv 2^{\psi/2}\mathfrak{M}^{\psi}\mathrm{E}\left[|Q|^{\psi}\right] \text{ and } r_{\psi} \equiv |\beta|^{\psi}. \blacksquare$$