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Theory of Simulation Output Processes Satisfying the Geometric-Moment Contraction Condition

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We study stationary stochastic processes such as those arising from steady-state simulations. For processes satisfying the Geometric-Moment Contraction condition (GMCC), we derive basic properties underlying recent methods to estimate key characteristics of the process's marginal distribution—e.g., the mean, selected ordinates of the distribution function, or selected quantiles. For a GMCC process $\{Y_k : k \geq 0\}$ satisfying certain auxiliary conditions on its marginal distribution, we also derive properties of its autocovariance function. For instance, if $\mathrm{E}\big[|Y_k|^u\big] < \infty$ for some $u > 2$, then we establish geometrically decreasing bounds on the autocovariance function at increasing lags; and thus a GMCC process is short-range dependent. Moreover, for the associated indicator process $\big\{I_k(y) \equiv \mathbf{1}_{\{Y_k \leq y\}} : k \geq 0\big\}$ with given cutoff level y for each observation, we show that the corresponding autocovariance function of the indicator process exhibits similar behavior. Hence the autocovariance functions for $\{Y_k\}$ and $\{I_k(y)\}$ are summable, yielding the respective variance parameters σ^2_Y and $\sigma^2_{I(y)}$. For a selected p-quantile y_p of $\{Y_k\}$ having the usual empirical quantile estimator $\widetilde{y}_p(n)$, we derive the associated variance parameter $\sigma^2_{\widetilde{y}_p}$, provided that the process's marginal density is bounded. We also show that the GMCC allows one to apply a functional central limit theorem and hence is useful in formulating asymptotic confidence intervals for parameters such as the mean and selected quantiles.

Key words: Stationary stochastic process; Geometric-Moment Contraction condition; quantile estimation; variance parameter; simulation analysis; batch-means method.

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1. Introduction

In the analysis of a stationary stochastic process $\{Y_k : k \geq 0\}$ that has marginal cumulative distribution function (c.d.f.) $F_Y(y) \equiv \Pr(Y_0 \leq y)$ for $y \in \mathbb{R} \equiv (-\infty, \infty)$ and finite marginal mean $\mu_Y \equiv \mathrm{E}[Y_0]$, we typically seek to estimate μ_Y or $F_Y(\cdot)$ based on an observed time series $\{Y_k : k = 1, \ldots, n\}$ of size $n \geq 1$ (for ease of exposition, Y_0 is merely reserved for initialization). Thus we compute the sample mean $\overline{Y}_n \equiv n^{-1} \sum_{k=1}^{n} Y_k$ and the empirical c.d.f. $F_{Y,n}(y) \equiv \overline{I}_n(y) \equiv n^{-1} \sum_{k=1}^{n} \mathbf{1}_{\{Y_k \leq y\}}$ for selected values of y, where the indicator function $\mathbf{1}_{\mathscr{E}} \equiv 1$ if event $\mathscr{E}$ occurs, and $\mathbf{1}_{\mathscr{E}} \equiv 0$ otherwise. In this situation, interest often focuses on deriving the asymptotic distributions of the quantities $n^{1/2}(\overline{Y}_n - \mu_Y)$ and $n^{1/2}[F_{Y,n}(y) - F_Y(y)]$ as $n \to \infty$. As detailed below, these asymptotic distributions depend on the *variance parameters* σ_Y^2 and $\sigma_{I(y)}^2$, respectively, when the underlying process $\{Y_k : k \geq 0\}$ is short-range dependent (SRD); see Beran (1994, pp. 3–10). A conventional approach to deriving such asymptotic results requires verifying that $\{Y_k : k \geq 0\}$ satisfies appropriate strong mixing conditions (Bradley 2005), which are usually difficult to check either theoretically or empirically; see Wu (2005, p. 1941), Alexopoulos et al. (2019, p. 1165, 3rd para.), and Alexopoulos et al. (2024, Remark 2, pp. 8–9).

An alternative approach to deriving and verifying such asymptotic results can be used if the process satisfies the following condition, which is more intuitive and considerably easier to verify than strong mixing conditions, especially in the context of an output process generated by a computer simulation that is driven by an input stream consisting of independent and identically distributed (i.i.d.) random variables (r.v.'s). In what follows, we use the standard notations $\mathbb{R} \equiv (-\infty, \infty)$, $\mathbb{Z} \equiv \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{Z}^+ \equiv \{1, 2, \ldots\}$, blah blah. [Need to give at least one ref for the GMCC!]

Geometric-Moment Contraction Condition (GMCC) Consider two independent sequences $\{\varepsilon_j : j \in \mathbb{Z}\}$ and $\{\varepsilon'_j : j \in \mathbb{Z}\}$, each consisting of i.i.d. r.v.'s distributed like ε_0. Suppose $\{Y_k : k \geq 0\}$ and $\{Y'_k : k \geq 0\}$ are stationary processes respectively defined by $Y_k \equiv \xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$ and $Y'_k \equiv \xi(\ldots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k)$ for some function $\xi(\cdot)$. Then $\{Y_k : k \geq 0\}$ satisfies the GMCC if there exist constants $\psi > 0$, $C_\psi > 0$, and $r_\psi \in (0, 1)$ such that

$$\mathrm{E}\big[|Y_k - Y'_k|^\psi\big] \leq C_\psi r_\psi^k \text{ for } k \geq 0. \tag{1}$$

Thus we see that, under the GMCC (1), the r.v.'s Y_k and Y'_k have c.d.f. $F_Y(\cdot)$ for each $k \geq 0$, and the initial r.v.'s Y_0 and Y'_0 are independent. However, for $k \geq 1$ the r.v.'s Y_k and Y'_k are not independent; instead, their trajectories converge at a geometric rate on average as k increases. In the context of a steady-state simulation, the GMCC describes a situation in which two runs of the simulation are initialized independently at time 0, after which those runs share an input stream $\{\varepsilon_1, \ldots, \varepsilon_k\}$ of common random numbers up to time $k \geq 1$, when those runs generate the paired outputs Y_k and Y'_k such that the difference $Y_k - Y'_k$ converges to 0 in the mean of order ψ geometrically fast as $k \to \infty$ (Alexopoulos et al. 2019, p. 1165, next-to-last para.).

The GMCC applies to a large variety of stationary stochastic processes of interest to the simulation community, e.g., finite-order autoregressive–moving average (ARMA) processes (Shao and Wu 2007, Theorem

5.2); generalized autoregressive conditional heteroscedastic (GARCH) processes (Wu and Shao 2004, Theorem 2); the G/G/1 queue-waiting-time process when the service-time distribution is light-tailed (Dingeç et al. 2022); and autoregressive-to-continuous (ARTOC) processes having various marginal distributions (Dingeç et al. 2024b). In addition, linear combinations, mixtures, and the associated indicator processes of GMCC processes are themselves GMCC. Moreover, Alexopoulos et al. (2012) and Alexopoulos et al. (2024, §2.3.1, pp. 7–8) discuss practical methods for empirically checking the GMCC.

In this article we establish geometrically decreasing bounds on the magnitudes of the autocovariance functions at increasing lags for the following processes: (i) a (stationary) stochastic process $\{Y_k : k \geq 0\}$ that satisfies the GMCC (1) and the marginal absolute moment condition,

$$\mathrm{E}\big[|Y_0|^u\big] < \infty \ \text{ for some } \ u > 2; \tag{2}$$

and (ii) the associated indicator process, $\big\{I_k(y) \equiv \mathbf{1}_{\{Y_k \leq y\}} : k \geq 0\big\}$ with given y, where the underlying process $\{Y_k : k \geq 0\}$ satisfies the GMCC and has marginal probability density function (p.d.f.) $f_Y(y)$, $y \in \mathbb{R}$, that satisfies the density-boundedness condition (DBC),

$$f^\star \equiv \sup_{y \in \mathbb{R}} f_Y(y) < \infty. \tag{3}$$

Our results on geometrically decreasing autocovariance term bounds imply that the autocovariance functions for processes (i) and (ii) both have finite sums, which are, respectively, the variance parameters σ_Y^2 and $\sigma_{I(y)}^2$ for given y. See Equation (5) in §2.1 below for the formal definition of σ_Y^2, and then Theorem 1(c) in §3.1 for a proof of its finiteness; and similarly, see Equation (7) in §2.2 for the definition of $\sigma_{I(y)}^2$, and then Corollary 2 in §4 for a proof of its finiteness.

For given $p \in (0, 1)$, the p-quantile of $F_Y(\cdot)$ is defined by $y_p \equiv F_Y^{-1}(p) \equiv \inf\{y : F_Y(y) \geq p\}$ and is typically estimated by the sample quantile $\widetilde{y}_p(n)$ based on the time series $\{Y_1, \ldots, Y_n\}$ of size n. As detailed in §2.3, $\widetilde{y}_p(n)$ has the associated variance parameter $\sigma_{\widetilde{y}_p}^2 = \sigma_{I(y_p)}^2 / f_Y^2(y_p)$ under the GMCC and the following density-regularity condition (DRC) on the marginal p.d.f. (which is more-restrictive than the DBC),

$$\left.\begin{array}{l}\textbf{Density-Regularity Condition} \quad \text{The c.d.f. } F_Y(y) \text{ has a p.d.f. } f_Y(y) \text{ with derivative } f_Y'(y) \text{ for} \\ y \in \mathbb{R}, \text{ both } f_Y(\cdot) \text{ and } f_Y'(\cdot) \text{ are bounded on } \mathbb{R}, \text{ and } f_Y(y_p) > 0 \text{ for selected } p \in (0,1).\end{array}\right\} \tag{4}$$

The rest of this article is organized as follows. Section 2 summarizes additional notation and basic concepts used in the paper, including definitions and result previews involving the variance parameters σ_Y^2, $\sigma_{I(y)}^2$, and $\sigma_{\widetilde{y}_p}^2$. Section 3 gives a derivation of σ_Y^2 when $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) and the moment condition (2). We also show that the GMCC allows one to apply a functional central limit theorem (FCLT) that can be used to obtain asymptotically valid confidence intervals (CIs) for the mean μ_Y, all without necessitating the use of mixing conditions. Further, §3 proves that the batch means computed from

$\{Y_k : k \geq 0\}$—of great interest in the context of simulation output analysis—are asymptotically uncorrelated as the batch size increases when $\{Y_k : k \geq 0\}$ satisfies (1) and (2). Section 4 provides derivations of $\sigma^2_{I(y)}$ [and $\sigma^2_{\widetilde{y}_p}$] when $\{Y_k : k \geq 0\}$ satisfies (1) and the DBC (3) [the DRC (4)]. The companion paper Dingeç et al. (2024a) presents complementary examples in which we derive explicit expressions (often closed-form) for the autocovariance functions and the variance parameters σ^2_Y, $\sigma^2_{I(y)}$, and $\sigma^2_{\widetilde{y}_p}$ for several stationary stochastic processes of interest, including the first-order autoregressive process, the autoregressive-to-Pareto process, and the waiting-time process arising from an M/M/1 queueing system. Section 5 of the current paper summarizes our main findings.

2. Preliminaries

Here we build on the setup described in §1. If the stationary process $\{Y_k : k \geq 0\}$ satisfies the moment condition (2), then it is covariance stationary and has a finite mean $\mu_Y \equiv \mathrm{E}[Y_0]$ and autocovariance function

$$R_Y(\ell) \equiv \mathrm{Cov}[Y_0, Y_\ell] \equiv \mathrm{E}[(Y_0 - \mu_Y)(Y_\ell - \mu_Y)] \quad \text{at lag } \ell \in \mathbb{Z}.$$

In what follows, we formally define and describe the variance parameters for the underlying Y_k's (§2.1), the corresponding $I_k(y)$'s (§2.2), and the empirical p-quantile $\widetilde{y}_p(n)$ (§2.3). The finiteness of these quantities is established in §§3 and 4.

2.1. The Variance Parameter of the Process $\{Y_k : k \geq 0\}$

Theorem 1(c) from §3.1 below proves that the GMCC (1) and the moment condition (2) imply that $\{Y_k : k \geq 0\}$ is SRD, i.e., $\sum_{\ell \in \mathbb{Z}} |R_Y(\ell)| < \infty$; and in this case, the *variance parameter* of the process $\{Y_k : k \geq 0\}$ exists and is defined as follows:

$$\sigma^2_Y \equiv \sum_{\ell \in \mathbb{Z}} R_Y(\ell) \in [0, \infty). \tag{5}$$

In addition, Corollary 1 from §3.1 yields

$$\mathrm{Var}\big[\overline{Y}_n\big] = \frac{\sigma^2_Y}{n} + O\big(n^{-2}\big) \;\text{ as }\; n \to \infty, \;\text{ and }\; \lim_{n\to\infty} n\mathrm{Var}\big[\overline{Y}_n\big] = \sigma^2_Y, \tag{6}$$

where the notation $a(n) = O(b(n))$ indicates that there exist $M \in \mathbb{R}$ and $n_0 \in \mathbb{Z}^+$ such that $|a(n)| \leq Mb(n)$ for every integer $n > n_0$; hence the variance parameter of the process $\{Y_k : k \geq 0\}$ provides a well-defined measure of dispersion of the sample mean $\overline{Y}_n$ as n increases in the sense that $\mathrm{Var}\big[\overline{Y}_n\big] \sim \sigma^2_Y / n$ as $n \to \infty$.

2.2. The Variance Parameter of the Indicator Process $\{I_k(y) : k \geq 0\}$

Next we develop results similar to Equations (5) and (6), but now adapted to the indicator process, $\big\{I_k(y) : k \geq 0\big\}$, for given $y \in \mathbb{R}$. Note that $I_k(y)$ is Bernoulli, i.e., $I_k(y) \overset{\mathrm{d}}{=} \mathrm{Bern}(\Pr(Y_k \leq y)) \overset{\mathrm{d}}{=} \mathrm{Bern}(F_Y(y))$, so that $\mathrm{E}[I_k(y)] = F_Y(y)$ and $\mathrm{Var}[I_k(y)] = F_Y(y)(1 - F_Y(y))$ for all k and y. Moreover, the indicator process $\{I_k(y) : k \geq 0\}$ with given y is covariance stationary, and its autocovariance function is defined as follows:

$$R_{I(y)}(\ell) \equiv \mathrm{Cov}[I_0(y), I_\ell(y)] \equiv \mathrm{E}\big\{[I_0(y) - F_Y(y)][I_\ell(y) - F_Y(y)]\big\} \quad \text{at lag } \ell \in \mathbb{Z}.$$

If both the GMCC (1) and DBC (3) hold, then Corollary 2(b) from §4.1 below establishes that $\{I_k(y) : k \geq 0\}$ is SRD; and thus the variance parameter of the indicator process is well-defined,

$$\sigma^2_{I(y)} \equiv \sum_{\ell \in \mathbb{Z}} R_{I(y)}(\ell) \in [0, \infty) \ \text{ for every } \ y \in \mathbb{R}. \tag{7}$$

It follows from Equation (51) in Corollary 2(d) below that for $\overline{I}_n(y) = n^{-1} \sum_{k=1}^{n} I_k(y)$ with given y, we have

$$\mathrm{Var}\big[\overline{I}_n(y)\big] = \frac{\sigma^2_{I(y)}}{n} + O\big(n^{-2}\big) \ \text{ as } \ n \to \infty, \ \text{ and } \ \lim_{n\to\infty} n\mathrm{Var}\big[\overline{I}_n(y)\big] = \sigma^2_{I(y)}. \tag{8}$$

2.3. The Variance Parameter of the Empirical p-quantile Estimator $\widetilde{y}_p(n)$

Quantile estimation is another application area in which the GMCC (1) together with an appropriate condition on the marginal density $f_Y(\cdot)$ can be used to derive basic asymptotic properties of a relevant statistic defined on a large class of stationary stochastic processes (Wu 2005, §4). We often seek to estimate the p-quantile y_p for selected values of $p \in (0,1)$ based on a time series $\{Y_1, \ldots, Y_n\}$ of length n. Sorting the time series, we obtain the order statistics $Y_{(1)} \leq \cdots \leq Y_{(n)}$; and then the usual point estimator of y_p is the empirical p-quantile estimator $\widetilde{y}_p(n) \equiv Y_{(\lceil np \rceil)}$, where $\lceil \cdot \rceil$ denotes the ceiling function.

In order to provide more perspective on the developments presented in this article, we now derive an expression for the variance parameter for the quantile estimator $\widetilde{y}_p(n)$.

Lemma 1. *If $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) and DRC (4), then the variance parameter for the empirical p-quantile estimator $\widetilde{y}_p(n)$ is given by*

$$\sigma^2_{\widetilde{y}_p} \equiv \lim_{n\to\infty} n\mathrm{Var}\big[\widetilde{y}_p(n)\big] = \frac{\sigma^2_{I(y_p)}}{f_Y^2(y_p)} \in [0, \infty) \ \textit{for} \ \ p \in (0,1). \tag{9}$$

Proof: Under Conditions (1) and (4), Wu (2005, Theorem 4) shows that the *Bahadur representation* of $\widetilde{y}_p(n)$ has the form

$$\widetilde{y}_p(n) = y_p - \frac{\overline{I}_n(y_p) - p}{f_Y(y_p)} + O_{\text{a.s.}}\left[\frac{(\log n)^{3/2}}{n^{3/4}}\right] \text{ as } n \to \infty, \tag{10}$$

where $Q_n \equiv O_{\text{a.s.}}\big[(\log n)^{3/2}/n^{3/4}\big]$ denotes the remainder term in Equation (10) with the following property: there is a nonnegative r.v. $\mathfrak{U}$ that is bounded almost surely (a.s., with probability 1), i.e.,

$$|Q_n| = \left|O_{\text{a.s.}}\left[\frac{(\log n)^{3/2}}{n^{3/4}}\right]\right| \leq \mathfrak{U}\left[\frac{(\log n)^{3/2}}{n^{3/4}}\right] \ \text{ for } \ n \geq 1 \ \text{ a.s.} \tag{11}$$

It follows from Equation (11) (cf. Dingeç et al. 2024c, Lemma 1) that

$$\mathrm{Var}[Q_n] = O\left[\frac{(\log n)^3}{n^{3/2}}\right] \quad \text{as} \quad n \to \infty; \tag{12}$$

and by Equation (51) in Corollary 2(b) from §4.1 below, we have

$$U_n \equiv -\frac{\overline{I}_n(y_p) - p}{f_Y(y_p)} \quad \text{has} \quad \mathrm{Var}[U_n] = \frac{\sigma^2_{I(y_p)}}{n f_Y^2(y_p)} + O\left(n^{-2}\right) \quad \text{as} \quad n \to \infty. \tag{13}$$

It follows from Equations (10) and (13) that $\widetilde{y}_p(n) - y_p = U_n + Q_n$ for $n \geq 1$ a.s.; and hence we have

$$n\mathrm{Var}[\widetilde{y}_p(n) - y_p] = n\mathrm{Var}[U_n] + 2n\,\mathrm{Cov}[U_n, Q_n] + n\mathrm{Var}[Q_n] \tag{14}$$

$$= \left[\frac{\sigma^2_{I(y_p)}}{f_Y^2(y_p)} + O\left(n^{-1}\right)\right] + O\left[\frac{(\log n)^{3/2}}{n^{1/4}}\right] + O\left[\frac{(\log n)^3}{n^{1/2}}\right] \tag{15}$$

$$= \frac{\sigma^2_{I(y_p)}}{f_Y^2(y_p)} + o(1) \quad \text{as} \quad n \to \infty, \tag{16}$$

where Equation (15) follows from Equation (12), the second part of Equation (13), Equation (14), and the Cauchy–Schwarz inequality, and where $a(n) = o(b(n))$ indicates that $a(n)/b(n) \to 0$ as $n \to \infty$. ∎

Table 1, abstracted from Dingeç et al. (2024a, §2), summarizes variance parameters σ^2_Y, $\sigma^2_{I(y)}$, and $\sigma^2_{\widetilde{y}_p}$.

Table 1 Various variance parameters

variance parameter	description	expressions
σ^2_Y	for the underlying Y_k's	$\lim_{n\to\infty} n\mathrm{Var}\left[\overline{Y}_n\right] = \sum_{\ell\in\mathbb{Z}} R_Y(\ell)$
$\sigma^2_{I(y)}$	for the underlying $I_k(y)$'s	$\lim_{n\to\infty} n\mathrm{Var}\left[\overline{I}_n(y)\right] = \sum_{\ell\in\mathbb{Z}} R_{I(y)}(\ell)$
$\sigma^2_{\widetilde{y}_p}$	for the quantile estimator $\widetilde{y}_p(n)$	$\lim_{n\to\infty} n\mathrm{Var}\left[\widetilde{y}_p(n)\right] = \sigma^2_{I(y_p)}/f_Y^2(y_p)$

3. Results for the Underlying Process $\{Y_k : k \geq 0\}$

We derive several results that follow from the GMCC. In particular, §3.1 starts out with a simple motivational result (Lemma 2) giving bounds on the covariance function of the underlying stochastic process $\{Y_k : k \geq 0\}$. Theorem 1 substantially generalizes the conditions under which the bounds hold, and further establishes the existence of the variance parameter σ^2_Y under those conditions. Theorem 2 in §3.2 shows that the GMCC allows us to apply an FCLT to the process $\{Y_k : k \geq 0\}$; this result provides a way to obtain CIs for the mean μ_Y that avoids mixing assumptions. §3.3 is concerned with an interesting consequence of the work in the previous subsections. Namely, we show that under the GMCC, the sample means of nonoverlapping batches of observations from a stationary stochastic process are asymptotically uncorrelated as the batch size becomes large. [Maybe put the CI result right after this one?]

3.1. Bounds on $|R_Y(\ell)|$ and the Finiteness of σ_Y^2

To motivate what is to come in Theorem 1, we begin with a basic result giving exponentially decreasing bounds on the magnitude of the autocovariance function of the Y_k's at increasing lags.

Lemma 2. *If $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) with $\psi = 2$ and $\mathrm{E}\big[Y_0^2\big] < \infty$, then*

$$|R_Y(\ell)| \leq \sqrt{\mathrm{E}\big[Y_0^2\big] C_2 r_2^{|\ell|}} \leq C_2 \sqrt{r_2^{|\ell|}/2} \quad \textit{for all } \ell \in \mathbb{Z}.$$

Proof: Without loss of generality, we can assume $\mathrm{E}[Y_0] = 0$; moreover, it is sufficient to demonstrate the desired conclusion for $\ell \geq 1$. By the stationarity assumption embedded in the GMCC, for each $\ell \geq 1$ we have

$$\begin{aligned} |R_Y(\ell)| &= |\operatorname{Cov}[Y_0, Y_\ell]| \\ &= \big|\operatorname{Cov}[Y_0, Y_\ell - Y'_\ell] + \operatorname{Cov}[Y_0, Y'_\ell]\big| \\ &= |\operatorname{Cov}[Y_0, Y_\ell - Y'_\ell]| \ \ \big(Y_0 \text{ and } Y'_\ell \text{ are independent r.v.'s because they are functions of the} \\ &\qquad \text{independent sequences } \{\varepsilon_j : j \leq 0\} \text{ and } \{\varepsilon'_j : j \leq 0\} \cup \{\varepsilon_j : 1 \leq j \leq \ell\}, \text{ respectively}\big) \qquad (17) \\ &\leq \sqrt{\mathrm{E}\big[Y_0^2\big]\mathrm{E}\big[(Y_\ell - Y'_\ell)^2\big]} \quad \text{(Cauchy–Schwarz inequality)}, \end{aligned}$$

and so the first inequality, $|R_Y(\ell)| \leq \sqrt{\mathrm{E}\big[Y_0^2\big] C_2 r_2^{|\ell|}}$, follows by the GMCC. Since Y_0 and Y'_0 are i.i.d. with $\mathrm{E}[Y_0] = 0$ and $\mathrm{E}\big[Y_0^2\big] < \infty$, the GMCC implies that

$$2\,\mathrm{E}[Y_0^2] = \operatorname{Var}(Y_0) + \operatorname{Var}(Y'_0) = \operatorname{Var}(Y_0) + \operatorname{Var}(Y'_0) - 2\operatorname{Cov}(Y_0, Y'_0) = \operatorname{Var}(Y_0 - Y'_0) = \mathrm{E}\big[(Y_0 - Y'_0)^2\big] \leq C_2,$$

yielding the second inequality, $|R_Y(\ell)| \leq C_2 \sqrt{r_2^{|\ell|}/2}$. ∎

Remark 1. Lemma 2's result can be achieved under different sets of assumptions not involving the GMCC. However, the GMCC is "flexible" and arguably easy to check in practice. Although the lemma requires $\psi = 2$, the following more-general result shows that if $\{Y_k\}$ is GMCC for *some* value of $\psi \in (0, \infty)$ and the moment condition (2) is satisfied for some $u > 2$, then $\{Y_k\}$ is GMCC for *all* values of $\psi \in (0, u)$ (a pretty trick that often proves to be a handy tool in the sequel). □

Theorem 1. *Suppose that (i) $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) for some $\psi > 0$, and (ii) $\mathrm{E}[|Y_0|^u] < \infty$ for some $u > 2$. Then:*

(a) $\{Y_k : k \geq 0\}$ *$\{Y_k\}$ is GMCC for* all $\psi \in (0, u)$.

(b) In terms of the constants

$$v \equiv u/(u-1) \in (1, u), \ \ r_{\psi,u,v} \equiv r_\psi^{(u-v)/(2u)} = r_\psi^{(u-2)/[2(u-1)]} \in (0,1), \ \ \textit{and} \ \ r_{\psi,v} \equiv r_\psi^{v/(2\psi)} \in (0,1), \tag{18}$$

we have

$$\big(\max\{r_{\psi,u,v}, r_{\psi,v}\}\big)^{1/v} \in (0,1) \ \ \textit{and} \ \ |R_Y(\ell)| = O\Big[\big(\max\{r_{\psi,u,v}, r_{\psi,v}\}\big)^{\ell/v}\Big] \ \ \textit{for} \ \ \ell \geq 0. \tag{19}$$

(c) *$\{Y_k : k \geq 0\}$ is SRD with variance parameter σ_Y^2 given by Equation* (5)*.*

Proof. We have $\mathrm{E}[|Y_0|] \leq \{\mathrm{E}[|Y_0|^2]\}^{1/2} \leq \{\mathrm{E}[|Y_0|^u]\}^{1/u} < \infty$ by Lyapounov's inequality (with $u > 2$) and assumption (ii); thus, by Cauchy–Schwarz, $R_Y(\ell)$ is finite for every $\ell \in \mathbb{Z}$, i.e.,

$$|R_Y(\ell)| = \left|\mathrm{Cov}[Y_0, Y_\ell]\right| \leq \sqrt{\mathrm{Var}[Y_0]\mathrm{Var}[Y_\ell]} = \mathrm{Var}[Y_0] \leq \mathrm{E}[Y_0^2] < \infty.$$

Without loss of generality we can assume that $\mathrm{E}[Y_0] = 0$. First we prove conclusions (b) and (c). We know Y_0 is independent of Y_0' because the r.v. sequences $\{\varepsilon_k : k \leq 0\}$ and $\{\varepsilon_k' : k \leq 0\}$ are independent; and for $\ell \geq 1$, the argument given in Equation (17) shows that Y_0 and Y_ℓ' are independent. Hence by (17) we have

$$|R_Y(\ell)| = \left|\mathrm{Cov}\left[Y_0,\, Y_\ell - Y_\ell'\right]\right| = \left|\mathrm{E}\left[Y_0\left(Y_\ell - Y_\ell'\right)\right]\right| \quad \text{for } \ell \geq 0. \tag{20}$$

Starting from Equation (20), we seek to use Hölder's inequality to show that $|R_Y(\ell)|$ declines at least exponentially fast as ℓ increases for $\ell \geq 0$. By definition of v, we have $(1/u) + (1/v) = 1$ so that u, v are conjugate exponents. From stationarity and the observation that

$$\left|Y_\ell - Y_\ell'\right|^v \leq \left[2\max\left\{|Y_\ell|, |Y_\ell'|\right\}\right]^v \leq 2^v\left(|Y_\ell|^v + |Y_\ell'|^v\right) \quad \text{for } \ell \geq 0, \tag{21}$$

we see that

$$\mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v\right] \leq 2^{v+1}\mathrm{E}\left[|Y_0|^v\right] \leq 2^{v+1}\left\{\mathrm{E}\left[|Y_0|^u\right]\right\}^{v/u} < \infty \quad \text{for } \ell \geq 0, \tag{22}$$

where the penultimate inequality follows by Lyapounov's inequality; and thus [since $\mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v\right]$ is finite?] we may apply Hölder's inequality to the expression in Equation (20) as follows:

$$|R_Y(\ell)| = \left|\mathrm{E}\left[Y_0\left(Y_\ell - Y_\ell'\right)\right]\right| \leq \left\{\mathrm{E}\left[|Y_0|^u\right]\right\}^{1/u}\left\{\mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v\right]\right\}^{1/v} \quad \text{for } \ell \geq 0. \tag{23}$$

Since $\{\mathrm{E}[|Y_0|^u]\}^{1/u}$ does not depend on ℓ, we make a second application of Hölder's inequality with the same conjugate exponents in order to derive a suitable upper bound on $\mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v\right]$ for $\ell \geq 0$ that will itself be a version of the GMCC. Let $\vartheta_\ell \equiv r_\psi^{\ell/(2\psi)}$ for $\ell \geq 0$. By Markov's inequality and the GMCC (1), we have

$$\Pr\left\{\left|Y_\ell - Y_\ell'\right| \geq \vartheta_\ell\right\} \leq \vartheta_\ell^{-\psi}\mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^\psi\right] \leq \vartheta_\ell^{-\psi}C_\psi r_\psi^\ell = \vartheta_\ell^{-\psi}C_\psi\vartheta_\ell^{2\psi} = C_\psi\vartheta_\ell^\psi \quad \text{for } \ell \geq 0. \tag{24}$$

We let $s \equiv u/v > 1$ and $t \equiv u/(u-v) = (u-1)/(u-2) > 1$ so that $(1/s) + (1/t) = 1$ and s, t are conjugate exponents. In the underlying probability space, we define the complementary events

$$\mathscr{E}_\ell \equiv \left\{\left|Y_\ell - Y_\ell'\right| \geq \vartheta_\ell\right\} \quad \text{and} \quad \mathscr{E}_\ell^c \equiv \left\{\left|Y_\ell - Y_\ell'\right| < \vartheta_\ell\right\} \quad \text{for } \ell \geq 0. \tag{25}$$

Using the decomposition

$$\mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v\right] = \mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v \mathbf{1}_{\mathscr{E}_\ell}\right] + \mathrm{E}\left[\left|Y_\ell - Y_\ell'\right|^v \mathbf{1}_{\mathscr{E}_\ell^c}\right] \quad \text{for } \ell \geq 0, \tag{26}$$

we seek to apply the Hölder inequality with the conjugate exponents s, t to the first term on the right-hand side (RHS) of Equation (26). Since $vs = u$, $1/s = v/u$, and $1/t = (u-v)/u$, we see that

$$\left\{\mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^{vs}\right]\right\}^{1/s} = \left\{\mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^{u}\right]\right\}^{v/u} < \infty \tag{27}$$

by an argument similar to Equations (21)–(22). Hence we can apply Hölder's inequality as follows:

$$\mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^{v}\mathbf{1}_{\mathscr{E}_\ell}\right] \le \left\{\mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^{vs}\right]\right\}^{1/s}\left\{\mathrm{E}\left[\mathbf{1}_{\mathscr{E}_\ell}\right]\right\}^{1/t} \quad \text{for} \quad \ell \ge 0. \tag{28}$$

Next we consider the second term on the RHS of Equation (28). By Equation (24), we have

$$\left\{\mathrm{E}\left[\mathbf{1}_{\mathscr{E}_\ell}\right]\right\}^{1/t} = \left[\Pr\left\{\left|Y_\ell - Y'_\ell\right| \ge \vartheta_\ell\right\}\right]^{(u-v)/u} \le \left(C_\psi \vartheta_\ell^{\psi}\right)^{(u-v)/u} = C_\psi^{(u-v)/u} r_{\psi,u,v}^{\ell} \quad \text{for} \quad \ell \ge 0. \tag{29}$$

Finally we derive an exponentially decreasing upper bound for the second term on the RHS of Equation (26) as $\ell \to \infty$. We have

$$\mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^{v}\mathbf{1}_{\mathscr{E}_\ell^c}\right] \le \vartheta_\ell^{v}\Pr\left\{\left|Y_\ell - Y'_\ell\right| < \vartheta_\ell\right\} \le \left[r_\psi^{\ell/(2\psi)}\right]^{v} \cdot 1 = r_{\psi,v}^{\ell} \quad \text{for} \quad \ell \ge 0. \tag{30}$$

Combining Equations (26)–(30), we have

$$\max\left\{r_{\psi,u,v}, r_{\psi,v}\right\} \in (0,1) \quad \text{and} \quad \mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^{v}\right] = O\Big[\left(\max\left\{r_{\psi,u,v}, r_{\psi,v}\right\}\right)^{\ell}\Big] \quad \text{for} \quad \ell \ge 0. \tag{31}$$

Combining Equations (20), (23), and (31), we obtain Equations (18) and (19) to complete part (b) of the theorem. The geometric decay of the covariance terms immediately yields the SRD result and Equation (5) required by part (c).

It remains to prove conclusion (a). For the sake of clarity and notational simplicity in the following argument, we let ψ_0, u_0, v_0 denote the relevant fixed quantities that are assumed to satisfy assumptions (i) and (ii) in the statement of Theorem 1 so that in the verification of (a), we are assuming that

$$\mathrm{E}\left[\left|Y_\ell\right|^{u_0}\right] < \infty \quad \text{for some fixed} \quad u_0 > 2; \tag{32}$$

and there are constants $C_{\psi_0} > 0$ and $r_{\psi_0} \in (0,1)$ such that

$$\mathrm{E}\Big[\left|Y_\ell - Y'_\ell\right|^{\psi_0}\Big] \le C_{\psi_0} r_{\psi_0}^{\ell} \quad \text{for} \quad \ell \ge 0. \tag{33}$$

The proof of conclusion (a) is a variant of Equations (21)–(31) in which:

- The real variable v is regarded as an arbitrary element of $(0, u_0)$, and *not* the conjugate exponent of u_0;
- Every instance of ψ is replaced by ψ_0;
- Every instance of u is replaced by u_0;
- Every instance of the conjugate exponent $s \equiv u_0/v$; and
- Every instance of the conjugate exponent $t \equiv u_0/(u_0 - v)$.

With these substitutions, the revised Equation (31) has the form

$$\max\left\{r_{\psi_0,u_0,\nu}, r_{\psi_0,\nu}\right\} \in (0,1) \text{ and } \mathrm{E}\left[\left|Y_\ell - Y'_\ell\right|^\nu\right] = O\left[\left(\max\left\{r_{\psi_0,u_0,\nu}, r_{\psi_0,\nu}\right\}\right)^\ell\right] \text{ for } \ell \geq 0, \tag{34}$$

which is a GMCC in which we can take $\psi = \nu$ for any $\nu \in (0, u_0)$. ∎

Corollary 1. *Under the conditions of Theorem 1, σ_Y^2 is well-defined and satisfies Equation* (6).

Proof: Equation (19) ensures that $|R_Y(\ell)| = O(s^{|\ell|})$ for some $s \in (0,1)$. We define $\gamma_{Y,a} \equiv 2\sum_{\ell=1}^{\infty} \ell^a R_Y(\ell)$ for $a \in \mathbb{N}$ (Aktaran-Kalaycı et al. 2007, §2.1). Thus, for $a \in \mathbb{N}$,

$$\left|\gamma_{Y,a}\right| \leq 2\sum_{\ell=1}^{\infty} \ell^a |R_Y(\ell)| = O\left(\sum_{\ell=1}^{\infty} \ell^a s^\ell\right) < \infty, \tag{35}$$

since exponentials dominate the polynomial terms. By Corollary 2 of Aktaran-Kalaycı et al. (2007) followed by Equation (35) with $a = 1$, we have

$$\operatorname{Var}\left(\overline{Y}_n\right) = \frac{\sigma_Y^2}{n} - \frac{2}{n^2}\sum_{\ell=1}^{\infty} \ell R_Y(\ell) + O\left(\frac{s^n}{n}\right) = \frac{\sigma_Y^2}{n} + O\left(n^{-2}\right). \quad ∎$$

3.2. Functional Central Limit Theorem for $\{Y_k : k \geq 0\}$

It is natural to use the sample mean $\overline{Y}_n$ to construct an asymptotically valid CI estimator for the steady-state mean μ_Y as the length n of the observed time series $\{Y_1, \ldots, Y_n\}$ tends to infinity. An important first step is to show that the pivot $n^{1/2}(\overline{Y}_n - \mu_Y)/\sigma_Y$ converges in distribution to a standard normal r.v. This will prove to be true if the following FCLT holds, in particular, for the case $t = 1$.

FCLT for the $\{Y_k : k \geq 0\}$ Process Define the sequence of random functions $\{\mathcal{Y}_n : n \geq 1\}$ in D, the space of real functions on $[0,1]$ that are left-continuous with right-hand limits, by

$$\mathcal{Y}_n(t) \equiv \frac{\lfloor nt \rfloor}{\sigma_Y \sqrt{n}}\left(\overline{Y}_{\lfloor nt \rfloor} - \mu_Y\right) \text{ for } t \in [0,1] \text{ and } n \geq 1,$$

where $\lfloor\cdot\rfloor$ denotes the floor function. We say that $\{\mathcal{Y}_n : n \geq 1\}$ (or, equivalently, $\{Y_k : k \geq 0\}$) satisfies the FCLT if

$$\mathcal{Y}_n \underset{n\to\infty}{\Longrightarrow} \mathscr{W},$$

where $\mathscr{W}$ is a standard Brownian motion process on $[0,1]$; and $\underset{n\to\infty}{\Longrightarrow}$ denotes weak convergence as $n \to \infty$ (Billingsley 1999, pp. 1–6, §§2–3, §8; Whitt 2002, §3.2, §§4.2–4.4, §11.3). (36)

Fortuitously, the next theorem shows that the GMCC implies the FCLT.

Theorem 2. *If $\{Y_k : k \geq 0\}$ satisfies the GMCC* (1) *and moment condition* (2)*, then the FCLT (36) holds.*

Proof: Suppose $\psi \in (2, \min\{u, 4\})$ and define $Y_{k,\{0\}} \equiv \xi\left(\ldots, \varepsilon_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k\right)$ for $k \geq 1$. Berkes et al. (2014, Theorem 2.1 and Corollary 2.1) find that if

$$\sum_{k=m}^{\infty} \left(\mathrm{E}\left[\left|Y_k - Y_{k,\{0\}}\right|^\psi\right]\right)^{1/\psi} = O\left(m^{-1}(\log m)^{-1/\psi}\right), \tag{37}$$

then there exists a probability space (Ω_c, A_c, P_c) on which we can define r.v.'s $\{Y_k^c : k \geq 1\}$ with the partial sum process $S_n^c \equiv \sum_{k=1}^n Y_k^c$ and a standard Brownian motion $\mathscr{W}(\cdot)$ such that $\{Y_k^c : k \geq 1\} \stackrel{\mathrm{d}}{=} \{Y_k : k \geq 1\}$ and

$$S_n^c - \mu_Y n - \sigma_Y \mathscr{W}(n) = o_{\mathrm{a.s.}}\big(n^{1/\psi}\big) \quad \text{in } (\Omega_c, A_c, P_c). \tag{38}$$

Some quick items for discussion:

- In Equation (37) (and elsewhere) why don't we write something like "for $m \geq 1$" after the order term (though in the case of (37), maybe start at $m \geq 2$ because of the log term)? Whatever, let's be consistent.
- It might be more natural/consistent to write $Y'_{k,\{0\}}$ instead of $Y_{k,\{0\}}$.
- In the discussion leading to Equation (38), there is no explicit mention of $Y_{k,\{0\}}$. That is probably fine, but just curious... are the notations $Y_{k,\{0\}}$ and Y_k^c at all related to each other?
- We haven't defined the $o_{\mathrm{a.s.}}$ notation. Kemal writes that: Regarding your comment on $o_{a.s.}$ in the proof of Theorem 2, I looked for the precise definition of $o_{a.s.}$ in Berkes et al. (2014), but I could not find it. The definition that I found in Wu (2005) is "For a sequence of [r.v.'s] Z_n, we say that $Z_n = o_{a.s.}(r_n)$ if Z_n/r_n converges to 0 almost surely." (see the second paragraph on page 1936 of Wu 2005).

Since the sequences $\left\{S_n^c : n \geq 1\right\}$ and $\left\{S_n \equiv \sum_{k=1}^n Y_k : n \geq 1\right\}$ have the same distribution, then [for all practical purposes,] S_n^c can be replaced by S_n in Expression (38). Moreover, the fact that $o_{\mathrm{a.s.}}$ implies $O_{\mathrm{a.s.}}$ in turn implies the assumption of strong approximation (ASA) of Damerdji (1994):

$$S_n - \mu_Y n - \sigma_Y \mathscr{W}(n) = O_{\mathrm{a.s.}}\big(n^{\frac{1}{2}-\lambda}\big) \quad \text{for some } 0 < \lambda \leq 1/2.$$

And then by Proposition 2.1 of Damerdji (1994), the ASA implies the FCLT.

Thus, to complete our proof, we need only establish Condition (37). To do so, first note that for $k \geq 1$,

$$\begin{aligned}
\mathrm{E}\big[\big|Y_{k,\{0\}} - Y'_k\big|^\psi\big] &= \mathrm{E}\big[\big|\xi(\ldots, \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k) - \xi(\ldots, \varepsilon'_{-2}, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k)\big|^\psi\big] \\
&\stackrel{\mathrm{d}}{=} \mathrm{E}\big[\big|\xi(\ldots, \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k) - \xi(\ldots, \varepsilon'_{-2}, \varepsilon'_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)\big|^\psi\big] \\
&\qquad \text{(since } \varepsilon'_0 \text{ is common to both terms and } \varepsilon'_0 \stackrel{\mathrm{d}}{=} \varepsilon_0) \\
&\stackrel{\mathrm{d}}{=} \mathrm{E}\big[\big|\xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}) - \xi(\ldots, \varepsilon'_{-1}, \varepsilon'_0, \varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1})\big|^\psi\big] \\
&\qquad \text{(by stationarity and an index shift)} \\
&= \mathrm{E}\big[\big|Y_{k+1} - Y'_{k+1}\big|^\psi\big].
\end{aligned} \tag{39}$$

So

$$\begin{aligned}
\big(\mathrm{E}\big[\big|Y_k - Y_{k,\{0\}}\big|^\psi\big]\big)^{1/\psi} &= \big(\mathrm{E}\big[\big|Y_k - Y'_k + Y'_k - Y_{k,\{0\}}\big|^\psi\big]\big)^{1/\psi} \\
&\leq \big(\mathrm{E}\big[\big|Y_k - Y'_k\big|^\psi\big]\big)^{1/\psi} + \big(\mathrm{E}\big[\big|Y_{k,\{0\}} - Y'_k\big|^\psi\big]\big)^{1/\psi} \quad \text{(by the triangle inequality)} \\
&= \big(\mathrm{E}\big[\big|Y_k - Y'_k\big|^\psi\big]\big)^{1/\psi} + \big(\mathrm{E}\big[\big|Y_{k+1} - Y'_{k+1}\big|^\psi\big]\big)^{1/\psi} \quad \text{(by Equation (39))} \\
&\leq \big(C_\psi r_\psi^k\big)^{1/\psi} + \big(C_\psi r_\psi^{k+1}\big)^{1/\psi} \quad \text{(by the GMCC)} \\
&< \widetilde{C} s^k,
\end{aligned}$$

where $\widetilde{C} \equiv 2C_\psi^{1/\psi}$ and $s \equiv r_\psi^{1/\psi} \in (0,1)$. Then Condition (37) is satisfied since

$$\sum_{k=m}^{\infty} \left(\mathrm{E}\left[\left|Y_k - Y_{k,\{0\}}\right|^{\psi}\right]\right)^{1/\psi} < \widetilde{C} \sum_{k=m}^{\infty} s^k = \frac{\widetilde{C} s^m}{1-s}. \quad \blacksquare$$

REMARK 2. A consequence of Theorem 2 is the ability to obtain asymptotically valid CIs for μ_Y, typically of the form $\mu_Y \in \overline{Y}_n \pm t_{\alpha/2,\nu}\, \widehat{\sigma}_Y / \sqrt{n}$, where $t_{\alpha/2,\nu}$ is the $(1-\frac{\alpha}{2})$-quantile of the Student t distribution with ν degrees of freedom (d.f.) and $\widehat{\sigma}_Y^2$ is an estimator for σ_Y^2 based on ν d.f. This is a well-studied research stream, though in previous papers the underlying FCLT requires mixing assumptions to hold, whereas the current paper only requires the GMCC. [Say anything more? Maybe cite the MS/OR paper, if appropriate? Might also explicitly do a batch-means CI after next (BM) section?] $\square$

3.3. Covariance of Nonoverlapping Batch Means

Theorem 1 allows us to do all sorts of interesting things. Notably, in the context of simulation output analysis, we show that the covariance of adjacent batch means vanishes as the batch size $m \to \infty$. We do so while avoiding any mixing conditions. To illustrate, we consider nonoverlapping batches of observations, $\{Y_{(d-1)m+1}, Y_{(d-1)m+2}, \ldots, Y_{dm}\}$, and the resulting batch means, $\overline{Y}_{d,m} \equiv \frac{1}{m}\sum_{i=1}^{m} Y_{(d-1)m+i}$, for batches $d \geq 1$.

REMARK 3. Under the assumptions of Theorem 1, the variance parameter σ_Y^2 is finite; and then it is easy to show that for $d_1 \neq d_2$, $\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right] = O(1/m)$ as $m \to \infty$ using the Cauchy–Schwarz inequality. But we will do a little better in the next theorem.

THEOREM 3. *Under the assumptions of Theorem 1, for $d_1 \neq d_2$, the covariances and correlations of the batch means are*

$$\left|\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]\right| = O\left(\frac{r_2^{(|d_1-d_2|-1)m/2}}{m^{3/2}}\right) \quad \textit{and} \quad \left|\mathrm{Corr}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]\right| = O\left(\frac{r_2^{(|d_1-d_2|-1)m/2}}{m^{1/2}}\right). \tag{40}$$

Proof: We associate with $Y_k = \xi(\ldots, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k)$ the corresponding GMC "coupled" version $Y_k' = \xi(\ldots, \varepsilon_{-1}', \varepsilon_0', \varepsilon_1, \ldots, \varepsilon_k)$, for each $k \geq 1$. We also have the analogous coupled batches and batch means, $\{Y'_{(d-1)m+1}, Y'_{(d-1)m+2}, \ldots, Y'_{dm}\}$ and $\overline{Y}'_{d,m} \equiv \frac{1}{m}\sum_{i=1}^{m} Y'_{(d-1)m+i}$ for batches $d \geq 1$.

First, we assume with no loss of generality that $\mathrm{E}\left[\overline{Y}_{d,m}\right] = \mathrm{E}\left[\overline{Y}'_{d,m}\right] = 0$. By Theorem 1(c), the variance parameter σ_Y^2 exists; and by Corollary 1, we have for $d \geq 1$ that

$$\mathrm{E}\left[\overline{Y}_{d,m}^2\right] = \mathrm{E}\left[\left(\overline{Y}'_{d,m}\right)^2\right] = \mathrm{Var}\left[\overline{Y}_{d,m}\right] = \frac{\sigma_Y^2}{m} + O\left(m^{-2}\right). \tag{41}$$

Second, for batches $d \geq 1$,

$$\begin{aligned}
\mathrm{E}\left[\left(\overline{Y}_{d,m} - \overline{Y}'_{d,m}\right)^2\right] &= \mathrm{Var}\left[\overline{Y}_{d,m} - \overline{Y}'_{d,m}\right] \quad \left(\text{since } \mathrm{E}\left[\overline{Y}_{d,m} - \overline{Y}'_{d,m}\right] = 0\right) \\
&= \mathrm{Cov}\left[\overline{Y}_{d,m} - \overline{Y}'_{d,m},\, \overline{Y}_{d,m} - \overline{Y}'_{d,m}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\mathrm{Cov}\big(Y_{(d-1)m+i}-Y'_{(d-1)m+i},\, Y_{(d-1)m+j}-Y'_{(d-1)m+j}\big)\\
&\le \frac{1}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sqrt{\mathrm{E}\Big[\big(Y_{(d-1)m+i}-Y'_{(d-1)m+i}\big)^2\Big]\mathrm{E}\Big[\big(Y_{(d-1)m+j}-Y'_{(d-1)m+j}\big)^2\Big]}\\
&\qquad \text{(by the Cauchy–Schwarz inequality)}\\
&= \frac{1}{m^2}\left\{\sum_{i=1}^{m}\sqrt{\mathrm{E}\Big[\big(Y_{(d-1)m+i}-Y'_{(d-1)m+i}\big)^2\Big]}\right\}^2\\
&\le \frac{1}{m^2}\left\{\sum_{i=1}^{m}\sqrt{C_2 r_2^{(d-1)m+i}}\right\}^2 \quad \text{(by Theorem 1(a) with } \psi=2\text{, noting that } u>2\text{)}\\
&= \frac{C_2 r_2^{(d-1)m}}{m^2}\left\{\sum_{i=1}^{m}\sqrt{r_2^i}\right\}^2\\
&= \frac{C_2 r_2^{(d-1)m+1}\big(1-r_2^{m/2}\big)^2}{\big(1-\sqrt{r_2}\big)^2 m^2} \quad \text{(geometric series)}\\
&= O\left(\frac{r_2^{(d-1)m}}{m^2}\right).
\end{aligned} \tag{42}
$$

Third, we associate with $\{Y_k\}$ and $\{Y'_k\}$ another related GMC coupled process,

$$
Y''_k = \begin{cases} \xi\big(\ldots,\varepsilon''_{-1},\varepsilon''_0,\varepsilon''_1,\ldots,\varepsilon''_k\big), & \text{for } 0<k\le m\\ \xi\big(\ldots,\varepsilon''_{-1},\varepsilon''_0,\varepsilon''_1,\ldots,\varepsilon''_m,\varepsilon_{m+1},\ldots,\varepsilon_k\big) & \text{for } k>m,\end{cases}
$$

where the ε''_j's are i.i.d. with the same distribution as ε_0 and independent of all of the ε_j's. The analogous batches and batch means are

$$
\big\{Y''_{(d-1)m+1},Y''_{(d-1)m+2},\ldots,Y''_{dm}\big\} \quad\text{and}\quad \overline{Y}''_{d,m}\equiv\frac{1}{m}\sum_{i=1}^{m}Y''_{(d-1)m+i}, \quad \text{for } d\ge 1.
$$

Because of the choices of the noise terms (the ε_j's and ε''_j's), the first batch $\{Y_1,Y_2,\ldots,Y_m\}$ of the original $\{Y_k\}$ process is independent of the first and subsequent batches $\{Y''_{(d-1)m+1},Y''_{(d-1)m+2},\ldots,Y''_{dm}\}$, $d\ge1$, of the $\{Y''_k\}$ process. This is because the first batch of the $\{Y_k\}$ process is composed of elementary noise particles $\{\ldots,\varepsilon_{-1},\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_m\}$, while batch $d\ge1$ of the $\{Y''_k\}$ process is a function of $\{\ldots,\varepsilon''_{-1},\varepsilon''_0,\varepsilon''_1,\ldots,\varepsilon''_m\}$ and (for $d\ge2$) $\{\varepsilon_{m+1},\ldots,\varepsilon_{dm}\}$. The upshot is that the first batch mean $\overline{Y}_{1,m}$ of the $\{Y_k\}$ process is independent of the first and subsequent batch means $\overline{Y}''_{d,m}$, $d\ge1$, of the $\{Y''_k\}$ process. Thus,

$$
\mathrm{Cov}\big(\overline{Y}_{1,m},\overline{Y}''_{d,m}\big) = 0, \quad \text{for } d\ge 1. \tag{43}
$$

By construction, the relationship between $\{Y_{m+k}:k\ge1\}$ and $\{Y''_{m+k}:k\ge1\}$ is the same as that between $\{Y_k:k\ge1\}$ and $\{Y'_k:k\ge1\}$ (i.e., things get shifted by m time units). This and the work leading to Result (42) immediately imply that for $d\ge2$,

$$
\mathrm{E}\Big[\big(\overline{Y}_{d,m}-\overline{Y}''_{d,m}\big)^2\Big] = \mathrm{E}\Big[\big(\overline{Y}_{d-1,m}-\overline{Y}'_{d-1,m}\big)^2\Big] = O\left(\frac{r_2^{(d-2)m}}{m^2}\right). \tag{44}
$$

We are now ready obtain the covariances of the various batch means. For $d_1 \neq d_2$, we have

$$\begin{aligned}
&\left|\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]\right| && (45)\\
&= \left|\mathrm{Cov}\left[\overline{Y}_{1,m}, \overline{Y}_{|d_1-d_2|+1,m}\right]\right| \quad \text{(by stationarity)}\\
&= \left|\mathrm{Cov}\left[\overline{Y}_{1,m},\, \overline{Y}_{|d_1-d_2|+1,m} - \overline{Y}''_{|d_1-d_2|+1,m}\right] + \mathrm{Cov}\left[\overline{Y}_{1,m}, \overline{Y}''_{|d_1-d_2|+1,m}\right]\right|\\
&\le \sqrt{\mathrm{E}\left[\overline{Y}^2_{1,m}\right]\mathrm{E}\left[\left(\overline{Y}_{|d_1-d_2|+1,m} - \overline{Y}''_{|d_1-d_2|+1,m}\right)^2\right]} + 0 \quad \text{(by Cauchy–Schwarz and Equation (43))}\\
&= O\left(\frac{r_2^{(|d_1-d_2|-1)m/2}}{m^{3/2}}\right), && (46)
\end{aligned}$$

where we have used the fact that all of the expected values are finite, along with Results (41) and (44). This is a somewhat better convergence rate than the elementary result given by Remark 3; and this completes the proof of the covariance portion of the theorem.

Finally, by Equations (41) and (46), we have

$$\left|\mathrm{Corr}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]\right| = \frac{\left|\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]\right|}{\sqrt{\mathrm{Var}\left[\overline{Y}_{d_1,m}\right]\mathrm{Var}\left[\overline{Y}_{d_2,m}\right]}} = \frac{O\left(\frac{r_2^{(|d_1-d_2|-1)m/2}}{m^{3/2}}\right)}{\frac{\sigma_Y^2}{m} + o(1/m)} = O\left(\frac{r_2^{(|d_1-d_2|-1)m/2}}{m^{1/2}}\right). \quad \blacksquare$$

Remark 4. [Made this a Remark since it's referenced in GMC1B.] We can squeeze out some more-fine-tuned results on $\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]$ that indirectly use the GMCC via Theorem 1. To begin, some routine algebra reveals that the covariance between batch means is given by a weighted average of autocovariances

$$\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right] = \frac{1}{m}\sum_{q=-m+1}^{m-1}\left(1 - \frac{|q|}{m}\right) R_Y(m|d_2-d_1|+q); \tag{47}$$

see, e.g., Equation (17) in Steiger and Wilson (2001). Moreover, by Theorem 1(b), there exist a positive and finite constant $C \in (0,\infty)$ and a real number $s \in (0,1)$ such that $|R_Y(\ell)| \le Cs^{\ell}$ for $\ell \ge 0$. Therefore, from Equation (47) and a visit with Mathematica, we have

$$\left|\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right]\right| \le \frac{C}{m}\sum_{q=-m+1}^{m-1}\left(1 - \frac{|q|}{m}\right) s^{m|d_2-d_1|+q} = \frac{Cs^{m(|d_2-d_1|-1)+1}(1-s^m)^2}{(1-s)^2m^2}, \tag{48}$$

where, for the last step, we assume $d_1 \neq d_2$. Then the fact that $\mathrm{Var}\left[\overline{Y}_{1,m}\right] = \sigma_Y^2/m + o(1/m)$ together with $\mathrm{Cov}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right] = O(s^{m(|d_2-d_1|-1)}/m^2)$ implies $\mathrm{Corr}\left[\overline{Y}_{d_1,m}, \overline{Y}_{d_2,m}\right] = O(s^{m(|d_2-d_1|-1)}/m)$ as $m \to \infty$. This is a bit of an improvement from Equation (40) of Theorem 3.

In any case, we have established (twice!) that a mild moment requirement plus the GMCC are all that it takes to guarantee that the correlations of the batch means will converge to zero as the batch size increases. The next remarks address related results involving the convergence of the correlations to zero.

REMARK 5. The general conclusions of Theorem 3 are also immediate consequences of Tafazzoli and Wilson (2011, Theorem 1) because finiteness of the variance parameter σ_Y^2 is alone sufficient to ensure that the correlations between batch means at all lags tend to zero as the batch size increases. In fact, for $|d_2 - d_1| \ge 2$, Tafazzoli and Wilson (2011) show that the correlation between batch means is bounded by what is presumably a rapidly decreasing function of the associated ϕ-mixing coefficients of the underlying process as the batch size m grows—a result that complements our Equations (40) and (48). □

REMARK 6. We can obtain an unanticipated fine-tuned result for the special case in which $|d_2 - d_1| = 1$, i.e.,

$$\begin{aligned} m^2 \operatorname{Cov}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big] &= \sum_{q=-m+1}^{m-1} (m-|q|) R_Y(m+q) \\ &= \sum_{q=-m+1}^{0} (m-|q|) R_Y(m+q) + \sum_{q=1}^{m-1} (m-q) R_Y(m+q) \\ &= \sum_{\ell=1}^{m} \ell R_Y(\ell) + O\left(\sum_{q=1}^{m-1} (m-q) s^{m+q}\right) \quad \text{(rearrange terms and note that } |R_Y(\ell)| \le C s^{\ell}) \\ &= \frac{\gamma_{Y,1}}{2} + O(m s^m), \end{aligned}$$

where the first term follows from Corollary 1 of Aktaran-Kalaycı et al. (2007) (cf. our Equation (35)); and the second follows since $\sum_{q=1}^{m-1} (m-q) s^{m+q} \le m s^m \sum_{q=0}^{\infty} s^q = m s^m/(1-s)$. Thus,

$$\begin{aligned} \operatorname{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big] &= \frac{\operatorname{Cov}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]}{\operatorname{Var}\big[\overline{Y}_{1,m}\big]} \\ &= \frac{\gamma_{Y,1}/(2m^2) + O(s^m/m)}{\sigma_Y^2/m - \gamma_{Y,1}/m^2 + O(s^m/m)} \quad \text{(by Corollary 2 of Aktaran-Kalaycı et al. 2007)} \\ &= \frac{\gamma_{Y,1}}{2m\sigma_Y^2} \left[\frac{1}{1 - \gamma_{Y,1}/(m\sigma_Y^2) + O(s^m)}\right] + O(s^m) \\ &= \frac{\gamma_{Y,1}}{2m\sigma_Y^2} \left[1 + \frac{\gamma_{Y,1}}{m\sigma_Y^2} + O(s^m) + O\big(m^{-2}\big)\right] + O(s^m) \quad \text{(by } 1/(1-x) = 1 + x + O(x^2)) \\ &= \frac{\gamma_{Y,1}}{2m\sigma_Y^2} + O\big(m^{-2}\big). \end{aligned} \tag{49}$$

We now show that the correlation result in Equation (49) turns up in an interesting way when studying the well-known nonoverlapping batch means estimator for σ_Y^2 based on $b \ge 2$ contiguous size-m batches,

$$\mathcal{N}(b,m) \equiv \frac{m}{b-1} \sum_{d=1}^{b} \big(\overline{Y}_{d,m} - \overline{Y}_n\big)^2,$$

where $n = bm$ is the total sample size and $\overline{Y}_n = \sum_{k=1}^{n} Y_k/n = \sum_{d=1}^{b} \overline{Y}_{d,m}/b$ is the grand sample mean. By Corollary 3 of Aktaran-Kalaycı et al. (2007), the expected value of $\mathcal{N}(b,m)$ is

$$\mathrm{E}[\mathcal{N}(b,m)] = \sigma_Y^2 - \frac{(b+1)\gamma_{Y,1}}{bm} + O(s^m);$$

and we see that $\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big]$ appears in the following expression for the relative bias of $\mathcal{N}(b,m)$,

$$\frac{\mathrm{E}[\mathcal{N}(b,m)] - \sigma_Y^2}{\sigma_Y^2} = -\frac{(b+1)\gamma_{Y,1}}{bm\sigma_Y^2} + O(s^m) \sim -\frac{2(b+1)}{b}\mathrm{Corr}\big[\overline{Y}_{1,m}, \overline{Y}_{2,m}\big], \quad \text{as } m \to \infty. \quad \text{[Define "}\sim\text{"?]} \qquad \square$$

4. Results for the Indicator Process $\{I_k(y) : k \geq 0\}$

Section 3 studied the GMCC as it was applied to the original stationary process $\{Y_k : k \geq 0\}$. We now obtain analogous results for the associated indicator process $\{I_k(y) : k \geq 0\}$, namely, bounds on $|R_{I(y)}(\ell)|$ and the finiteness of $\sigma^2_{I(y)}$ and $\sigma^2_{\tilde{y}_p}$ in §4.1 and an FCLT for $\{I_k(y_p) : k \geq 0\}$ in §4.2.

4.1. Bounds on $|R_{I(y)}(\ell)|$ and the Finiteness of $\sigma^2_{I(y)}$ and $\sigma^2_{\tilde{y}_p}$

Theorem 4 establishes that if $\{Y_k : k \geq 0\}$ satisfies the GMCC, then so does $\{I_k(y) : k \geq 0\}$; and a number of other findings follow in short order. See Doukhan (2018, Lemma 7.4.2) and Xu (2021, Lemma B.2) for related results.

THEOREM 4. *If $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) and DBC (3), then the associated indicator process, $\big\{I_k(y) : k \geq 0\big\}$ with given y, is GMCC.*

Proof: Using the notation of the GMCC (1), we choose $\delta > 0$ such that

$$r_\psi^{1/\psi} < \delta < 1 \ \text{ so that } \ 0 < r_\psi < \delta^\psi < 1. \tag{50}$$

Moreover, let $\delta_k \equiv \delta^k$ for $k \geq 1$ and let $\{I'_k(y) \equiv \mathbf{1}_{\{Y'_k \leq y\}} : k \geq 1\}$ denote the coupled indicator process. Then

$$\begin{aligned}
\mathrm{E}\big[|I_k(y) - I'_k(y)|^\psi\big] &= \mathrm{E}\big[|I_k(y) - I'_k(y)|\big] \quad \text{(indicators can only be 0 or 1)} \\
&= 0 \cdot \Pr\big(|I_k(y) - I'_k(y)| = 0\big) + 1 \cdot \Pr\big(|I_k(y) - I'_k(y)| = 1\big) \\
&= \Pr\big(\{I_k(y) = 1\} \cap \{I'_k(y) = 0\}\big) + \Pr\big(\{I_k(y) = 0\} \cap \{I'_k(y) = 1\}\big) \\
&= 2\Pr(Y_k \leq y, Y'_k > y) \quad \text{(by symmetry)} \\
&= 2\big[\Pr(Y_k \leq y - \delta_k,\, Y'_k > y) + \Pr(y - \delta_k < Y_k \leq y,\, Y'_k > y)\big] \quad (\text{for } \delta_k > 0) \\
&\leq 2\big[\Pr(Y_k \leq y - \delta_k,\, Y'_k > y) + \Pr(y - \delta_k < Y_k \leq y)\big] \\
&\leq 2\big[\Pr\big(|Y'_k - Y_k| > \delta_k\big) + f^\star \delta_k\big] \quad \text{(by the DBC (3))} \\
&= 2\big[\Pr\big(|Y'_k - Y_k|^\psi > \delta_k^\psi\big) + f^\star \delta_k\big] \\
&\leq \frac{2\mathrm{E}\big[|Y_k - Y'_k|^\psi\big]}{\delta_k^\psi} + 2f^\star\delta_k \quad \text{(by Markov's inequality)} \\
&\leq \frac{2C_\psi r_\psi^k}{\delta^{k\psi}} + 2f^\star\delta^k \quad \text{(by the GMCC (1))} \\
&\leq 2\widetilde{C}\zeta_\psi^k,
\end{aligned}$$

where, by Expression (50),

$$\zeta_\psi \equiv \max\left\{\frac{r_\psi}{\delta^\psi}, \delta\right\} \in (0,1) \ \text{ and } \ \widetilde{C} \equiv \left(C_\psi + f^\star\right). \ \blacksquare$$

We establish exponentially decreasing bounds on the magnitude of the covariance function of the indicator process at increasing lags, and then several useful extensions.

Corollary 2. *If $\{Y_k : k \ge 0\}$ satisfies the GMCC (1) and DBC (3), then*

(a) $|R_{I(y)}(\ell)| = O(s^\ell)$ *as* $\ell \to \infty$ *for some* $s \in (0,1)$.

(b) $\{I_k(y) : k \ge 0\}$ *is SRD, i.e.,* $\sum_{\ell\in\mathbb{Z}} |R_{I(y)}(\ell)| < \infty$.

(c) Equation (7) holds, i.e., for each real y, $\{I_k(y) : k \ge 0\}$ *has a finite variance parameter* $\sigma^2_{I(y)} = \lim_{n\to\infty} n\mathrm{Var}\left[\overline{I}_n(y)\right] = \sum_{\ell\in\mathbb{Z}} R_{I(y)}(\ell) \in [0,\infty)$.

(d) Define $\gamma_{I(y),a} \equiv 2\sum_{\ell=1}^{\infty} \ell^a R_{I(y)}(\ell)$ *for* $a \in \mathbb{N}$ *analogously to* $\gamma_{Y,a}$ *in §3.1. Paralleling Corollary 1, we have* $|\gamma_{I(y),a}| < \infty$ *for all* $a \in \mathbb{N}$ *and the following asymptotic result as* $n \to \infty$:

$$\mathrm{Var}\left[\overline{I}_n(y)\right] = \frac{\sigma^2_{I(y)}}{n} - \frac{\gamma_{I(y),1}}{n^2} + O\left(\frac{\zeta_\psi^n}{n}\right) = \frac{\sigma^2_{I(y)}}{n} + O\left(n^{-2}\right) \quad \textit{for every } y \in \mathbb{R}. \tag{51}$$

(e) If, in addition, the DRC (4) holds, then for $p \in (0,1)$, *the variance parameter for the* p*-quantile estimator* $\widetilde{y}_p(n)$ *is given by* $\sigma^2_{\widetilde{y}_p} = \sigma^2_{I(y_p)} \big/ f_Y^2(y_p) \in [0,\infty)$.

Proof: (a) By Theorem 4, $\{I_k(y) : k \ge 0\}$ satisfies the GMCC. The result is then immediate from Theorem 1(b), with the $I_k(y)$'s replacing the Y_k's. (b) and (c) follow from (a) and the finiteness of a geometric series. [Hmmm. . . Since (a) is only anorder reuslt, might we also need to invoke the boundedness of $|R_{I(y)}(\ell)|$ to assert that the first few terms are finite before the order result kicks in? See the proof of Theorem 5 below.] (d) Since the exponential decay of the generic autocovariance term $R_{I(y)}(\ell)$ dominates the polynomial growth of the corresponding term ℓ^a, it follows that $|\gamma_{I,a}(y)| < \infty$ for every $a \in \mathbb{N}$ and $y \in \mathbb{R}$. (e) This result is proven in Lemma 1, which uses Equation (51) from (d). $\blacksquare$

4.2. Functional Central Limit Theorem for $\{I_k(y_p) : k \ge 0\}$

Analogous to §3.2's discussion, one might seek an asymptotically valid CI estimator of the p-quantile y_p for given $p \in (0,1)$ as $n \to \infty$; and to start this process, we first need an FCLT for the indicator process.

FCLT for $\{I_k(y_p) : k \ge 0\}$ Define the sequence of random functions $\{\mathcal{I}_n : n \ge 1\}$ in D, where

$$\mathcal{I}_n(t;p) \equiv \frac{\lfloor nt \rfloor}{\sigma_{I(y_p)}\sqrt{n}}\left[\overline{I}_{\lfloor nt\rfloor}(y_p) - p\right] \text{ for } t \in [0,1] \text{ and } n \ge 1.$$

We say that $\{\mathcal{I}_n : n \ge 1\}$ (or, equivalently, $\{I_k(y_p) : k \ge 0\}$) satisfies the FCLT if $\mathcal{I}_n \underset{n\to\infty}{\Longrightarrow} \widetilde{\mathscr{W}}$, where $\widetilde{\mathscr{W}}$ is a standard Brownian motion process on $[0,1]$. (52)

THEOREM 5. *If $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) and DBC (3), then $\{I_k(y_p) : k \geq 0\}$ satisfies FCLT* (52).

Proof: By Theorem 4, $\{I_k(y) : k \geq 0\}$ is GMCC. Moreover, all of the moments of $I_k(y)$ exist, so an application of Theorem 2 with the I_k's in place of the Y_k's completes the proof. ∎

REMARK 7. If we assume $\{Y_k : k \geq 0\}$ satisfies the GMCC (1) and DRC (4) (the latter of which implies the DBC (3)), then by Lemma 1's Equations (9) and (10), along with some algebra, we obtain

$$\frac{\sqrt{n}}{\sigma_{I(y_p)}}\left[\bar{I}_n(y_p) - p\right] = \frac{\sqrt{n}}{\sigma_{\widetilde{y}_p}}\left[\widetilde{y}_p(n) - y_p\right] + O_{\text{a.s.}}\left[\frac{(\log n)^{3/2}}{n^{1/4}}\right]. \tag{53}$$

Thus, by Theorem 5 with $t = 1$ and Equation (53), we have

$$\frac{\sqrt{n}}{\sigma_{\widetilde{y}_p}}\left[\widetilde{y}_p(n) - y_p\right] + O_{\text{a.s.}}\left[\frac{(\log n)^{3/2}}{n^{1/4}}\right] \underset{n\to\infty}{\Longrightarrow} \text{Nor}(0,1), \tag{54}$$

Since convergence a.s. implies convergence in probability, we apply Slutsky's theorem to (54) to obtain

$$\frac{\sqrt{n}}{\sigma_{\widetilde{y}_p}}\left[\widetilde{y}_p(n) - y_p\right] \underset{n\to\infty}{\Longrightarrow} \text{Nor}(0,1). \tag{55}$$

Result (55) only requires the GMCC and DRC to hold—which is interesting in light of the fact that, in their work, Alexopoulos et al. (2019, Corollary 1) (building on Wu 2005, Theorem 4) explicitly required what we now know to be the redundant SRD and FCLT assumptions as well.

In any case. we are now in a position to obtain asymptotically valid CIs for the p-quantile y_p as the sample size $n \to \infty$. Such CIs are typically of the form $y_p \in \widetilde{y}_p(n) \pm t_{\alpha/2,\nu}\,\widehat{\sigma}_{\widetilde{y}_p}/\sqrt{n}$, where $\widehat{\sigma}^2_{\widetilde{y}_p}$ is an estimator for $\sigma^2_{\widetilde{y}_p}$ based on ν d.f. [Need to cite and maybe add a little dicussion about the other papers at this point.] □

4.3. Checking the Assumptions

In the context of simulation-based quantile estimation, Alexopoulos et al. (2024, §2.3) detail some useful empirical methods for checking that an output process $\{Y_k : k \geq 0\}$ satisfies the GMCC, DRC, SRD condition, and FCLT (though, as detailed in Remark 7, we really only need to check the GMCC and DRC). On the other hand, there are relatively few theoretical methods for definitively verifying those conditions. We believe that our main results can be used to develop hybrid theoretical/empirical methods for such verification. For example, if the selected output process $\{Y_k : k \geq 0\}$ generated by a queueing simulation can be proved to satisfy the GMCC (1) along the lines of Dingeç et al. (2022), then for selected $p \in (0,1)$, Corollary 2(b) ensures that $\sigma^2_{I(y_p)} \in [0,\infty)$; and to finish verifying the SRD condition, we must merely check that $\sigma^2_{I(y_p)} > 0$. [← Hmm. . . aren't we allowing the variance parameter to = 0 for the SRD? If so, aren't the next few lines on spectral estimation an unnecessary tangent?] We recognize that for the indicator process $\{I_k(y_p) : k \geq 1\}$, the spectral density $s_{I(y_p)}(\omega)$ at frequency $\omega \in [-\pi,\pi]$ has the following key property:

$$\sigma^2_{I(y_p)} = s_{I(y_p)}(0) \quad \text{all} \quad p \in (0,1). \tag{56}$$

Thus we can at least test the hypothesis that $\sigma^2_{I(y_p)} > 0$ by using the spectral method of Heidelberger and Welch (1981) to construct a CI estimator of $s_{I(y_p)}(0)$. Moreover, if the SRD condition holds, then for all practical purposes we may regard the FCLT condition (52) as verified; see Whitt (2002, p. 107, last para.). For more on verifying less-restrictive versions of the DRC (4) that are adapted to specific application contexts, see Alexopoulos et al. (2024, §2.2.2). Although the foregoing discussion requires substantial extension and elaboration, we believe that it provides good evidence of the connections among the GMCC, DRC, SRD, and FCLT conditions. [← this material probably needs a minor re-write.]

5. Conclusions

The Geometric-Moment Contraction condition applies to a large class of stationary stochastic processes and can substitute for hard-to-check mixing conditions in order to carry out mean and quantile estimation.

The main contributions of the current paper concern implications of the GMCC; these results reside in §3 for the underlying stationary stochastic process $\{Y_k : k \geq 0\}$ in the forms of Theorem 1 and Corollary 1, and in §4 for the corresponding quantile process $\{I_k(y) : k \geq 0\}$ via Theorem 4 and Corollary 2. Remarkably, Theorem 1(a) finds that if the process $\{Y_k : k \geq 0\}$ having $\mathrm{E}[|Y_0|^u] < \infty$ for some $u > 2$ satisfies the GMCC (1) for *some* exponent $\psi > 0$, then $\{Y_k : k \geq 0\}$ satisfies the GMCC for *all* $\psi \in (0, u)$. In addition, we find that the GMCC can be used to guarantee: (i) the exponential decay of the autocovariance functions associated with $\{Y_k : k \geq 0\}$ and $\{I_k(y) : k \geq 0\}$, as well as (ii) the existence of the corresponding variance parameters for those processes—σ^2_Y for the Y_k's; $\sigma^2_{I(y)}$ for the $I_k(y)$'s; and, $\sigma^2_{\widetilde{y}_p}$ for the empirical quantile estimator $\widetilde{y}_p(n)$.

In spite of the theory-heavy nature of the current paper, our motivations are all practical. As mentioned previously, the GMCC is more intuitive and easier to justify than mixing conditions for use in applications—particularly those involving steady-state simulation analysis. In fact, with an eye towards simulation analysis, we showed in §3.3 that the GMCC can be used to prove certain (classic) simulation output analysis results, e.g., the fact that nonoverlapping batch means are asymptotically uncorrelated as the batch size grows.

In order to complement and study the robustness of the results in the current paper, our companion paper Dingeç et al. (2024a) finds, for several stochastic processes of interest, exact and asymptotic expressions for: (i) the autocovariance functions $\{R_Y(\ell) : \ell \in \mathbb{Z}\}$ and $\{R_{I(y)}(\ell) : \ell \in \mathbb{Z}\}$; (ii) the variance parameters σ^2_Y, $\sigma^2_{I(y)}$, and $\sigma^2_{\widetilde{y}_p}$; and (iii) the correlation between adjacent batch means. Specifically, we derive results for the AR(1), ARTOP, and M/M/1 waiting-time processes, each presenting interesting challenges and insights.

Future research will encompass more-sophisticated analyses, for instance [The following material still needs work.]:

- Calculation of the expected value and variance of nonoverlapping batch means and standardized time series area variance parameter estimators in quantile estimation problems (Dingeç et al. 2024c).

- Determining how to use the GMCC to obviate the need for obscure ϕ-mixing conditions in estimation problems; see, e.g., the restrictive mixing conditions required in Chien et al. (1997).
- Using the GMCC in conjunction with FCLTs in fixed-sample-size and sequential CI procedures. [← This issue may go away depending on our final comments in this and the MS/OR papers.]
- Robustness analyses when the GMCC fails to hold.
- **[Need to insert additional appropriate Doukhan (2018) references here and there.]**

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