Exploring fix-orders on Groups

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Definitions

A **quasi-order** (also known as a pre-order) is a binary relation which is both reflexive and transitive.

Suppose G is a group of permutations acting on a set X. Then the **fix-set** for an element $g \in G$ is the set of all elements in X which are not moved when acted on by g. Formally:

$$fix(g) = \{x \in X \mid g(x) = x\}$$

A **fix-set quasi-order** is a quasi-order $\Subset$, where $g_1 \Subset g_2$ if and only if $fix(g_1) \subseteq fix(g_2)$. All fix-set quasi-orders satisfy the following axioms:

$$1 \Subset g \Rightarrow g = 1 \tag{1}$$

$$g \Subset g^{-1} \tag{2}$$

$$g \Subset h \wedge g \Subset k \Rightarrow g \Subset hk \tag{3}$$

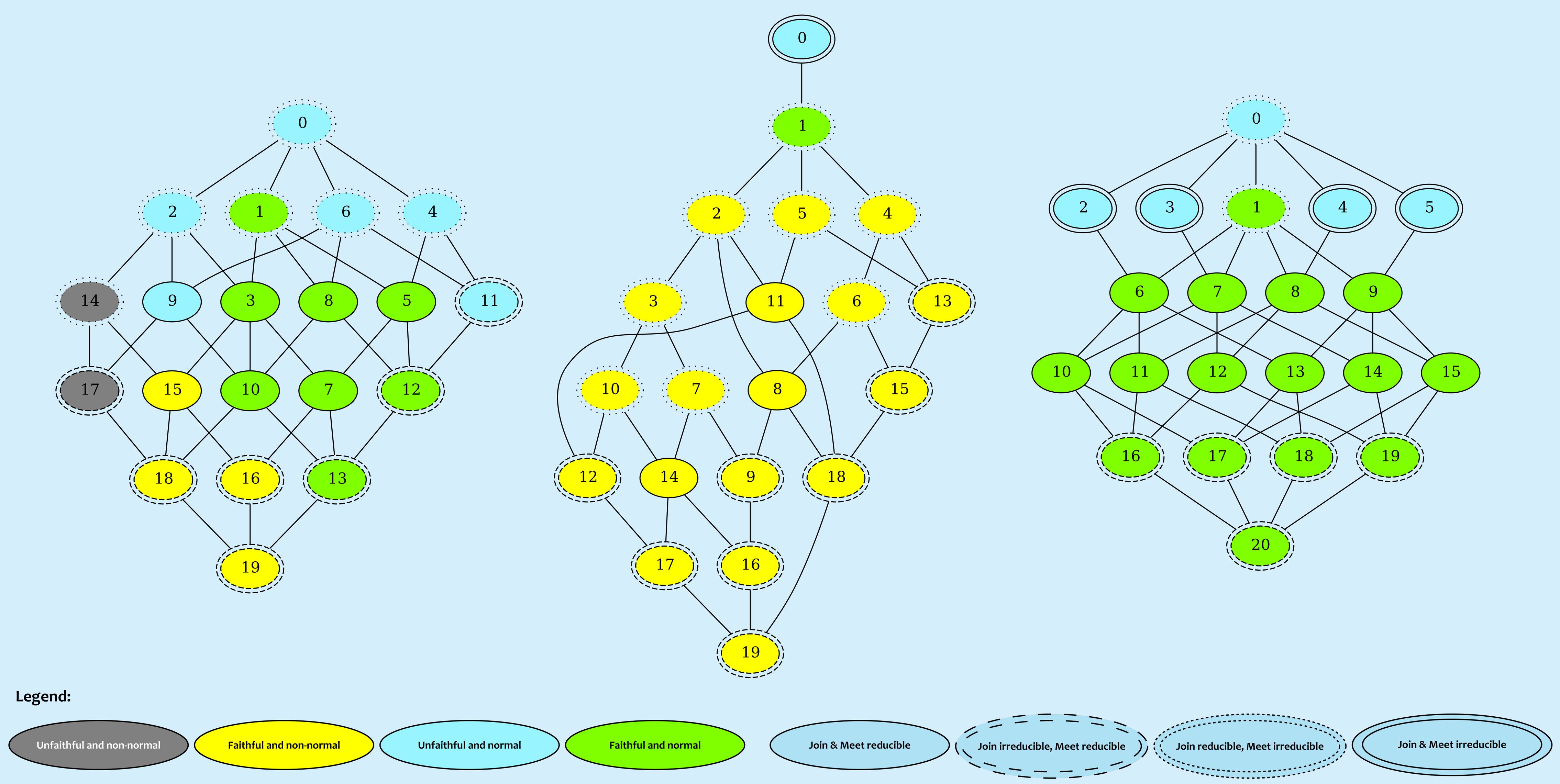
$$g \Subset h \Rightarrow g^k \Subset h^k \tag{4}$$

In fact, every group G with a quasi-order $\Subset$ satisfying the above axioms is isomorphic to a group of permutations where $\Subset$ corresponds to the fix-set quasi-order. Therefore any quasi-order which satisfies the above axioms is a **fix-order**.

Aim

The aim of this project is to determine the set of all unique fix-orders on as many groups as possible. Since most groups have a large number of fix-orders, the fix-orders of each group are to be visualised as a lattice, ordered by set inclusion. For example:

Dicyclic group of order 12 Alternating group of order 5 Elementary abelian group of order 9



References

[1] I. Hawthorn, T. Stokes, Groups with fix-set quasi-order, in preparation.
[2] T. Stokes, Axioms for function semigroups with agreement quasi-order, Algebra Univ 66 (2011), 85-98.

Acknowledgements

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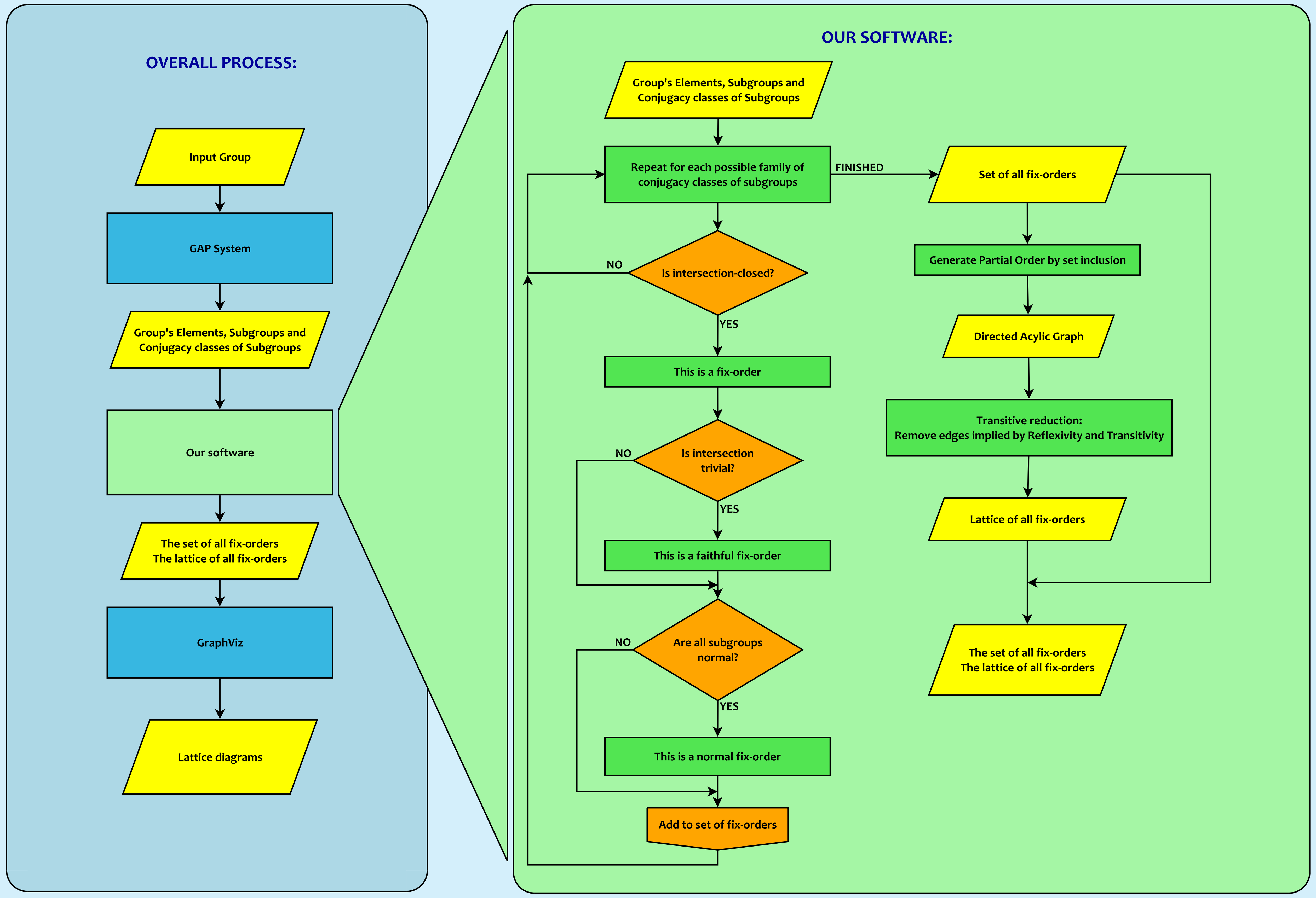
Process



Outcomes

The outcomes of this project are:

- Complete description of the lattice of fix-orders of all groups of order 15 or less, including:
  - Faithfulness and normality of the fix-orders.
  - Modularity and Distributivity of the lattice of fix-orders.
  - The join irreducible and meet irreducible fix-orders of the lattice.
  - The fix-orders related by an automorphism transformation in the lattice.
- Description of the fix orders of some selected Dihedral, Symmetric and Alternating Groups as well as some groups of order 31 or less. Approximately 120 groups have been processed.
- A software package which is able to determine the lattice of fix-orders for any input group, in $O(2^N)$, where N is the number of conjugacy classes of subgroups.

Open Problems

In general, lattice distributivity is a stronger condition than lattice modularity. However, in all of the lattices of fix-orders we examined, distributivity and modularity were equivalent. It is not known whether this holds for all lattices of fix-orders.