Data Science
Autoregessive models

Estimating linear temporal latent factors

Stéphane Marchand-Maillet

Department of Computer Science

UNIVERSITÉ DE GENÈVE

FACULTÉ DES SCIENCES

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What is the lecture about?

- ⋆ To understand temporal data modeling
- ⋆ To understand temporal data analysis
- ⋆ To understand the concept of auto-regression

→ linear temporal latent model

- ⋆ To present the basic of autoregressive models (AR(p))

Reading: [3] and [1] (chap 13)

Definition

- $\star$ A temporal series is a sequence of observations depending on time

$$\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_t, \ldots, \boldsymbol{y}_T \qquad \boldsymbol{y}_t \in \Omega \subseteq \mathbb{R}^D$$

- $\star$ In general, we assume a constant time interval
- $\star$ For statistical temporal data analysis, we study causality

$\Rightarrow$ $\boldsymbol{y}_t$ depends on the p preceding values $\boldsymbol{y}_{t-1}, \ldots, \boldsymbol{y}_{t-p}$

- $\star$ Causality is the information we want to model (give/measures parameters)

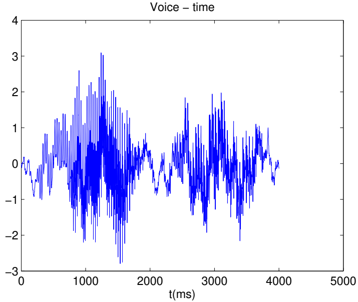
Applications

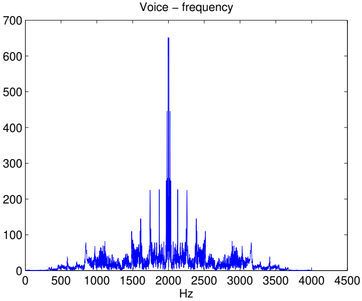
- ⋆ Économetry (trend prediction, risk analysis...)
- ⋆ Social (demography, migrations ...)
- ⋆ Physical measures (explanation of physical constants/values)
- ⋆ Communication and information → telecom, network, coding, speech, video ...

Examples

- ⋆ Temporal data analysis (mean, variance, correlation)
- ⋆ Frequency analysis (periodicity)
- ⋆ Combination time/frequency, time/scale

→ Non-stationary series





Temporal series

- ⋆ Definition

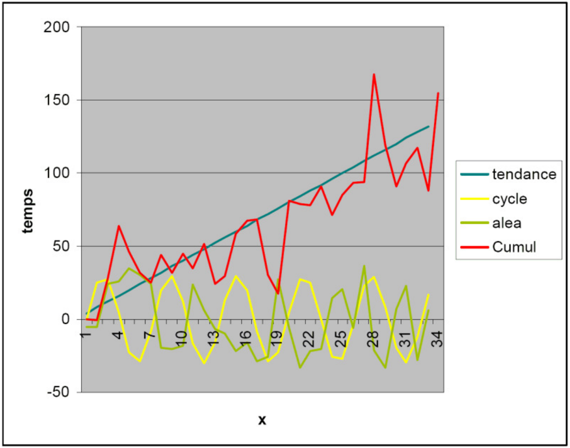
$$\boldsymbol{y}_t = \boldsymbol{g}(t) + \epsilon_t, \quad t \in [\![T]\!]$$

- ⋆ $\boldsymbol{y}_t$ scalar or vector
- ⋆ $\boldsymbol{g}(t)$ structure of the series
- ⋆ ϵ_t noise centered, uncorrelated (white noise)

$\Rightarrow$ ϵ_t random component of $\boldsymbol{y}_t$

Components

- ⋆ Long-term: the trend
- ⋆ Periodic: the seasonality
- ⋆ Random: the residual



Components

- ⋆ Additive scheme $\boldsymbol{y}_t = \boldsymbol{f}_t + \boldsymbol{s}_t + \boldsymbol{\epsilon}_t$
- ⋆ Multiplicative scheme $\boldsymbol{y}_t = \boldsymbol{f}_t \odot \boldsymbol{s}_t \odot (1 + \boldsymbol{\epsilon}_t)$
  (and variations)

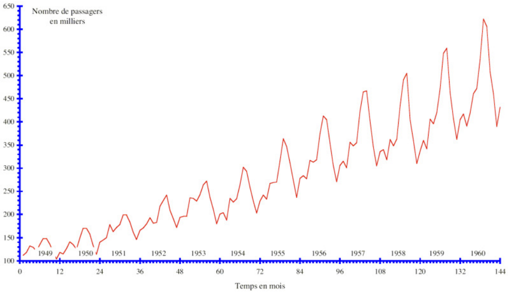


FIG. 4 – Nombre mensuel de passagers internationaux aux États Unis de 1949 à 1960

Note: Multiplicative is equivalent to the additive scheme when considering the log of the series

Component estimation

A rigorous analysis requires the estimation of the trend, independent from seasonality and noise

Smoothing

By definition, $\boldsymbol{s}_t$ and ϵ_t are 0-mean:

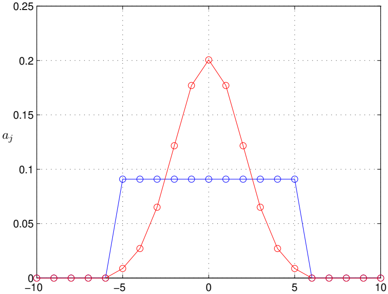
$$\mathbb{E}\,S = \mathbb{E}\,E = 0$$

A moving average of $\boldsymbol{y}_t$ preserves the trend only

Moving average Ψ_p

$$\Psi_p(\boldsymbol{y}_z) = \bar{\boldsymbol{y}}_t = \sum_{j=-p}^{p} a_j \boldsymbol{y}_{t-j}$$

with $a_{-j} = a_j$ (symmetric filter) with $\sum_j a_j = 1$.



Note: This is equivalent to a convolution

Correcting the seasonal variations

⋆ Apply moving average to y_t

$$\Psi_p(\boldsymbol{y}_t) = \Psi_p(\boldsymbol{f}_t) + \Psi_p(\boldsymbol{s}_t) + \Psi_p(\epsilon_t)$$

⋆ if $\boldsymbol{s}_t$ and ϵ_t centered, $\Psi_p(\boldsymbol{s}_t)$, $\Psi_p(\epsilon_t) \approx 0$

$$\hat{\boldsymbol{f}}_t = \Psi_p(\boldsymbol{y}_t)$$

$\Rightarrow$ To "erase" a seasonality with period p, apply moving average Ψ_p

Prediction

- ⋆ Assume we know $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_T$ stationary

⇒ We wish to infer step $t+p$ ($\boldsymbol{y}_{t+p}$) from steps until t ($\boldsymbol{y}_t$) $\rightarrow \hat{\boldsymbol{y}}_t(p)$

- ⋆ k is the horizon
- ⋆ In general $\hat{\boldsymbol{y}}_t(p) \neq \boldsymbol{y}_{t+p}$ (prediction error)$\rightarrow$ we seek minimum error

No prediction is possible if there is no dependence/causality between values $\boldsymbol{y}_t$, $t \in [\![T]\!]$

For the rest of the lecture: $D = 1$ (scalar series):

- ⋆ Can be generalized per component
- ⋆ i.e we consider independent components

⇒ simplify notation ($y_t \in \mathbb{R}$)

Linear prediction

We assume that y_t depends linearly on the previous values

$$y_t = \sum_{k=1}^{n} \alpha_k y_{t-k}$$

$$y_t = \boldsymbol{\alpha}^\mathsf{T} \boldsymbol{y}_{t-1}$$

with $\boldsymbol{\alpha} = [\alpha_1, \ldots \alpha_n]^\mathsf{T}$ and $\boldsymbol{y}_{t-1} = [y_{t-1}, \ldots, y_{t-n}]^\mathsf{T}$

$\Rightarrow$ Linear relationship $\rightarrow$ correlation between values y_t

Stationary process

- ⋆ The series is seen as a stochastic process
  - i.e y_t is a realization (draw/instance) of random variable Y_t
  - In general we know only one realization of $\{Y_t\}_{t \in [\![T]\!]}$

$\Rightarrow$ impossible to use correlation between $y_{t-1}, \dots, y_{t-p}$, to predict y_t

Stationary hypothesis (constant mean):

Covariance $\mathrm{cov}(y_t, y_{t-p}) = \gamma_p$ does not depend on t

$$\gamma_p = \mathrm{cov}(y_t, y_{t-p}) = \frac{1}{T-p} \sum_{t=p}^{T} (y_t - \mu)(y_{t-p} - \mu)$$

Prediction by Exponential smoothing

$$\hat{y}_t(p) = (1 - \tau) \sum_{j=0}^{p} \tau^j y_{t-1-j}$$

with $\tau \in [0, 1]$

- ⋆ Most basic method
- ⋆ The predicted value is the mean of past values
- ⋆ forget effect (exponential decay) making most recent values important

Exponential smoothing

φ Show:

$$\hat{y}_{t+1}(p) = (1-\tau)y_t + \tau\hat{y}_t(p-1)$$

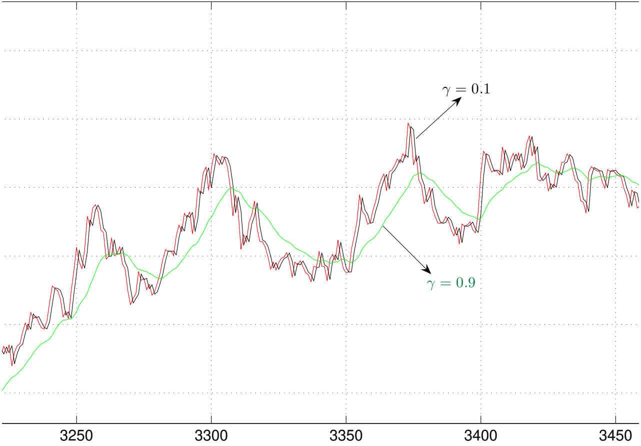
or:

$$\hat{y}_{t+1}(p) = \hat{y}_t(p-1) + (1-\tau)(y_t - \hat{y}_t(p-1))$$

- ⋆ $\tau \to 1$ prediction accounts for far past (smooth)
- ⋆ $\tau \to 0$ prediction only depends on immediate past (less smooth)

Exponential smoothing

Cours de cloture de l'action Microsoft, journalier du 13 mars 1986 au 20 décembre 1999



Auto-regressive models (AR)

- ⋆ order-p AR model: AR(p):

$$y_t = \sum_{j=1}^{p} a_j y_{t-j} + \epsilon_t$$

- ⋆ Values at t linearly depends the on the p preceding values
- ⋆ ϵ_t is a white noise with variance σ_ϵ^2 ($\epsilon_t \sim \mathcal{N}(0, \sigma)$)

Parameter estimation

- Mean-squared error regression:

$$\boldsymbol{a}^* = \underset{\boldsymbol{a}}{\operatorname{argmin}}\, \mathbf{E}\left[e_t^2\right] = \underset{\boldsymbol{a}}{\operatorname{argmin}} \sum_{t=0}^{T} \left(y_t - \sum_{j=1}^{p} a_j y_{t-j} \right)^2$$

- Solution

$$\frac{\partial \mathbf{E}\left[e_t^2\right]}{\partial a_j} = 0 \qquad j \in [\![p]\!]$$

Correlation and regression coefficient

⋆ We get for j:

$$\frac{\partial E[e_t^2]}{\partial a_j} = 2\mathbf{E}_t\left[\left(y_t - \sum_{i=1}^{p} a_i y_{t-i}\right) y_{t-j}\right]$$
$$= \gamma_j - \sum_{i=1}^{p} a_i \gamma_{i-j} = 0$$

⋆ Recall $\gamma_j = \mathrm{cov}(y_t, y_{t-j}) = \gamma_{-j}$ and y_t stationary

Yule-Walker Equations

Let $\rho_k = \frac{\gamma_k}{\gamma_0}$ be the correlation function
$\Rightarrow$ the minimization of mean squared error leads to

$$\rho_k = \sum_{i=1}^{p} a_i \rho_{k-i} \qquad k > 0$$

$$\begin{cases} \rho_1 = a_1\rho_0 + a_2\rho_1 + a_3\rho_2 + \cdots + a_p\rho_{p-1} & k = 1 \\ \rho_2 = a_1\rho_1 + a_2\rho_0 + a_3\rho_1 + \cdots + a_p\rho_{p-2} & k = 2 \\ \vdots & \vdots \\ \rho_p = a_1\rho_{p-1} + a_2\rho_{p-2} + a_3\rho_{p-3} \cdots + a_p\rho_0 & k = p \end{cases}$$

Auto correlation matrix

In matrix form, Yule-Walker equations are:

$$\boldsymbol{R}_p \boldsymbol{a} = \boldsymbol{\rho}$$

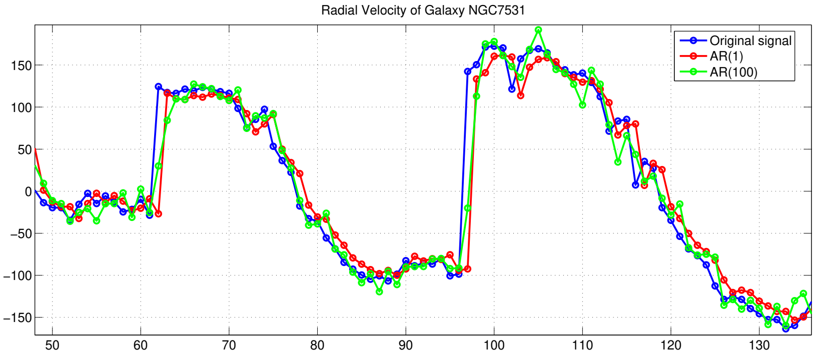
with $\boldsymbol{a} = [a_1, \ldots, a_p]^\mathsf{T}$, $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_p]^\mathsf{T}$ and

$$\boldsymbol{R}_p = \begin{pmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \ldots & \rho_{p-1} \\ \rho_1 & 1 & \rho_1 & \rho_2 & \ldots & \rho_{p-2} \\ & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{p-1} & \rho_{p-2} & \rho_{p-3} & \ldots & \ldots & 1 \end{pmatrix}$$

If $\boldsymbol{R}_p$ is invertible (positive definite)

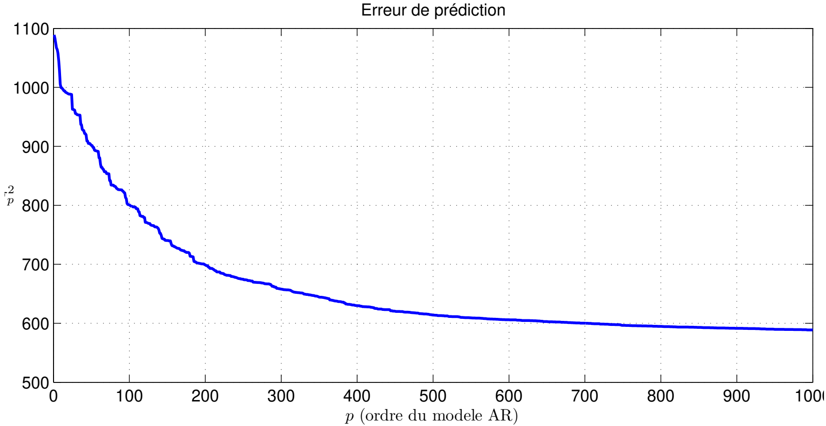
$$\boldsymbol{a} = \boldsymbol{R}_p^{-1} \boldsymbol{\rho}$$

Example



⇒ Choice of optimal order p ?

Example



⇒ Choice of optimal order p ?

Order selection

The higher p the better, but :

- ⋆ Requires the estimation of p correlation coefficients $\rightarrow$ Confidence over a finite series?
- $\Rightarrow$ Rule of thumb for length N: $p < N/3$
- ⋆ Model parsimony : in applications, it is generally better to keep a simple model
- $\Rightarrow$ Define parsimony (low p?)

Order selection

Whitening test (AR model)

$$\epsilon_t = y_t - \sum_{i=1}^{p} a_i y_{t-i} \qquad \text{white noise? (i.i.d} \sim \mathcal{N}(0, \sigma))$$

$\Rightarrow$ Tests: Durbin-Watson, Fisher,...

- $\star$ Check whether values ϵ_t ($t \in [\![T]\!]$) are significantly decorrelated

Final Prediction Error (FPE)

Definition of a global error, combining the variance of the prediction error σ_p and a variance over the imprecision of parameter estimation:

$$\mathrm{FPE}(p) = \sigma_p \cdot \frac{T + p + 1}{T - p - 1}$$

decrease with p $\longleftrightarrow$ increase with p

Criteria for parsimony

Generally from Information Theory :

- ⋆ Minimum Description Length (MDL)

$$\mathrm{MDL}(p) = N \log(\sigma_p^2) + p \log(T)$$

- ⋆ Akaike Information Criterion (AIC)

$$\mathrm{AIC}(p) = N \log(\sigma_p^2) + 2(p+1)$$

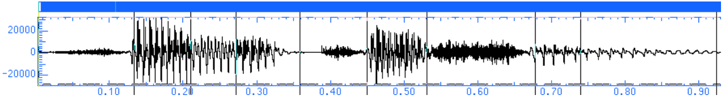
$\Rightarrow$ MDL is a good criterion with low number of samples

Limitations

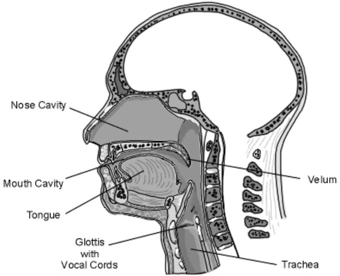
- ⋆ Stationarity is a strong assumption
- ⋆ Econometric data is rarely stationary (trend, multiplicative seasonality)
- → Trend removal, log model, ...
- → More complex models (ARMA, ARIMA, ARCH...) at the price of a more complex estimation
- ⋆ Example of stationary signal: speech (at least locally)

Speech analysis and compression using AR models

Hypothesis: local stationarity



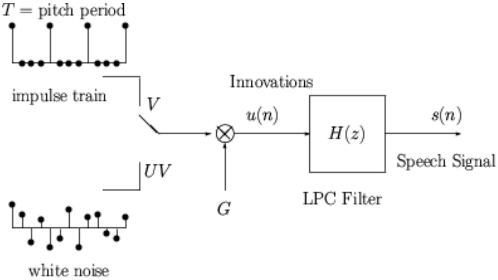
Linear model for speech production



Air flow :

- ⋆ Excited by vocal chords
- ⋆ Echo effect on vocal cavity (linear system)

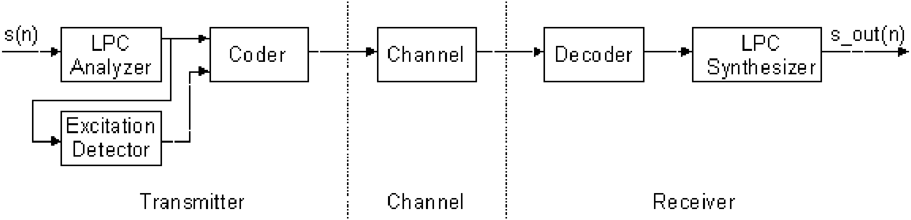
Modeling



- ⋆ Can show that coefficients $H(z)$ are the parameters of the AR model
- ⋆ Characterize the parameters of speech production
- ⋆ Important for speech analysis and compression

Compression by Linear Predictive Coding (LPC)

Vocoder



$\Rightarrow$ used in GSM

Summary

- ⋆ As opposed to static data, temporal data includes causality
- ⋆ Temporal data present a unique sample
- ⋆ Correlation is found by considering stationarity

$\Rightarrow$ need to remove trend ans seasonality

- ⋆ AR(p) models are the base for temporal signal modeling
- ⋆ They are linear (latent) models (coefficient are used for interpretation, generation, ...)

- ⋆ Extension to Neural Nets for sequential data: e.g [2] (chap 15)

Example questions [mostly require formal – mathematical – answers]

- ⋆ What are trend, seasonality and how to remove them? What are the assumptions made?
- ⋆ What is stationarity? What does it imply and allow?
- ⋆ What is exponential smoothing (and exponential decay)?
- ⋆ Provide the base models for linear auto-regression
- ⋆ What is the horizon (and the lag)?

φ It is strongly advised to develop the algebra contained in this chapter
(inc relation to auto-correlation and convolution)

References I

[1] Christopher M. Bishop. *Pattern Recognition and Machine Learning (Information Science and Statistics)*. Springer-Verlag, Berlin, Heidelberg, 2006. (available online).

[2] Kevin P. Murphy. *Probabilistic Machine Learning: an Introduction*. MIT Press, 2022. (available online).

[3] Robert H. Shumway. *Time Series Analysis and its Applications, with R Examples*. Springer, 2017. (available online).