I.   NORMALIZATION BY ENRICO CAMPOREALE

Here it is explained both the normalization, and how to choose some relevant quantities, such as box size, dx, dt, ecc..

All quantities are given in CGS system (with formulas taken from NRL Plasma Formulary). Let's recall some basic facts about the scaling properties of a plasma. If we take the Newton and Poisson equations:

$$\frac{dv}{dt} = \frac{q}{m}\left(E + \frac{v}{c} \times B\right)$$
$$\nabla \cdot E = 4\pi\rho,$$

and we redefine the following quantities:

$$x = x'L \tag{1}$$
$$v = v'V \tag{2}$$
$$t = t'\frac{L}{V} \tag{3}$$
$$B = B'B_0 \tag{4}$$
$$E = E'B_0 \tag{5}$$
$$\rho = \rho'\rho_0 \tag{6}$$
$$\left(\frac{q}{m}\right)_s = \left(\frac{q}{m}\right)'_s\left(\frac{Q}{M}\right)_i \tag{7}$$

where the quantities with capital letter are typical values of the system that we want to study, while primed quantities are dimensionless, we obtain:

$$\frac{dv'}{dt'} = \frac{QB_0L}{MV^2}\left(\frac{q}{m}\right)'\left(E' + \frac{v'}{c'} \times B'\right) \tag{8}$$
$$\nabla' \cdot E' = \frac{L\rho_0}{B_0}4\pi\rho'. \tag{9}$$

If we have two different physical systems S_0 and S_1, the first denoted by its typical quantites L_0, V_0, B_0, ρ_0, and the second by L_1, V_1, B_1, ρ_1, than what the previous two equations tell us, is that a plasma will behave in the same manner in the two systems, if:

$$\frac{Q_0B_0L_0}{M_0V_0^2} = \frac{Q_1B_1L_1}{M_1V_1^2} \tag{10}$$
$$\frac{L_0\rho_0}{B_0} = \frac{L_1\rho_1}{B_1}. \tag{11}$$

We can then choose to study system S_0 or system S_1 and get the same response if these two relations hold.

Note that if in Eq. (??) we take $V = V_a$ (Alfven velocity), then it becomes identical to Eq.(??). Another property of Eq. (??)-(??) is that if we want the ratio ω_p/Ω_c to be equal in the two systems, than we have $B/n^{1/2} = const$, that if substituted in Eq. (??) gives:

$$\frac{QBL}{MV^2} = \frac{QBL}{T} = \frac{QB^2Ln}{TBn} = const \Rightarrow T = const \tag{12}$$

If we want the ratio $\omega_p/\Omega_c = const$ we need to have the same T in the two systems.

A. How to choose dt and number of cycles

We will consider as the total time of the simulation the time in which a certain linear perturbation saturates. This is fixed as N_t times the (inverse) of the growth rate $\gamma = N_\gamma \omega_{ci}$, so that the total time T is

$$T = \frac{2\pi N_t}{N_\gamma \omega_{ci}}. \tag{13}$$

The time-step dt is chosen in a way that an electron will take N_s steps to perform a complete gyromotion:

$$dt = \frac{2\pi}{N_s \omega_{ce}} = \frac{2\pi m_e c}{N_s q B}; \tag{14}$$

The total number of cycles will be

$$N_{cycles} = \frac{T}{dt} = \frac{N_t N_s}{N_\gamma} \frac{m_i}{m_e}. \tag{15}$$

Considering N_γ as given (and not changeable), N_{cycles} depends only on N_t, N_s, and on the mass ratio m_i/m_e.

To compute the total computing time of the simulation we will use the following formula (rule of thumb), in seconds:

$$TotalTime = 2 * N_{particles} \frac{4 \cdot 10^{-6}}{(N_{procs})^{0.5}}$$

B. How to choose dx and box size

We will say that the box size is N_w times a certain wavelength we are interested in, where the wavevector is given by

$$k = N_k \omega_{pi}/c. \tag{16}$$

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The box size is thus

$$L = \frac{2\pi N_w c}{N_k \omega_{pi}} = \frac{2\pi N_w c}{1.32 \cdot 10^3 n_i^{1/2} N_k} \tag{17}$$

We take $dx = N_\lambda \lambda_D$, where $\lambda_D = 7.43 \cdot 10^2 \left(\frac{T}{n}\right)^{1/2}$ is the Debye length.

The number of cell N_{cell} will then be $L/(N_\lambda \lambda_D)$.

A problem arises when one uses real typical parameters.

For example, chosing $B = 6nT; n = 4cm^{-3}; T = 10^5 kelvin$, and taking $N_k = 1.65; N_w = 2$, one obtains:

$$L = 867.7 km$$

$$\lambda_D = 10.9m$$

that means

$$N_\lambda N_{cell} = 7.95 \cdot 10^4 \tag{18}$$

This could be a problem if one doesn't want to have many cells, or to have a dx/λ_D too big. The idea is now to re-scale the system we are studying by means of Eqs. (??)-(??). We will denote respectively with the subscript 0 and 1 the quantities in the old and in the new system.

Coupling the definitions of L and of λ_D :

$$\lambda_D = \frac{2\pi N_w c}{N_\lambda N_{cell} N_k 1.32 \cdot 10^3 n_1^{1/2}} \tag{19}$$

$$\lambda_D = 7.43 \cdot 10^2 \left(\frac{T_1}{n_1}\right)^{1/2} \tag{20}$$

and eliminating n_1 and λ_D we obtain for T_1:

$$T_1^{1/2} = 1.922 \cdot 10^5 \frac{N_w}{N_\lambda N_{cell} N_k}. \tag{21}$$

If we want to conserve the ratio ω_p/Ω_c in system 1, as we have seen it must be $T_1 = T_0$. This imposes a constrain on the quantities N_λ, N_{cell}, N_w (N_k is considered given and not changeable).

In this case (for $T_1 = T_0$) there is not really any advantage in passing from system 0 to system 1, because we end up with the relation similar to Eq. (??). This is because in the system (??) the term n_1 cancels out.

We have here two possibilities:

- Retain the condition $\omega_p/\Omega_c = const \rightarrow$ allow for big $N_{cell}N_\lambda$

- Relax the condition $\omega_p/\Omega_c = const$ and decrease $N_{cell}N_\lambda$

1. *Relax the condition $\omega_p/\Omega_c = const$*

In this case the ratio T_1/T_0 will tell us "how much" the ratio ω_p/Ω_c is not held constant. Another important constrain is that we do not want for a particle to move more than a cell per cycle. This is expressed as:

$$v_{th,e} < \frac{dx}{dt}$$

with $v_{th,e}$ the electron thermal velocity. Let's re-state the previous relation as

$$\frac{dx}{v_{th,e}dt} = G > 1$$

We than need to fix a value for G that will tell us "how much" the inequality relation is satisfied.

With some easy algebra we have:

$$\frac{dx}{dt} = \frac{N_\lambda \lambda_1}{dt} = N_\lambda \left(\frac{kT}{4\pi n_1 q^2}\right)^{1/2} \frac{qN_sB}{2\pi m_e c} = \frac{N_\lambda N_s B_1}{2\pi c}\left(\frac{kT_1}{m_e}\right)^{1/2}\left(\frac{1}{4\pi n_1 m_e}\right)^{1/2} \tag{22}$$

$$\frac{dx}{v_{th,e}dt} = \frac{N_\lambda N_s B_1}{2\pi c}\left(\frac{1}{4\pi n_1 m_e}\right)^{1/2} = G \tag{23}$$

$$B_1 = \frac{2\pi c G}{N_\lambda N_s}\left(4\pi n_1 m_e\right)^{1/2} \tag{24}$$

This is not a formula to calculate the new B_1, but instead a new constrain on some parameters. If we use the relation (**??**) in the form:

$$\frac{B_0}{\lambda_0 n_0} = \frac{B_1}{\lambda_1 n_1},$$

we have

$$\frac{B_0}{B_1} = \frac{\lambda_0 n_0}{\lambda_1 n_1} = \left(\frac{T_0 n_0}{T_1 n_1}\right)^{1/2}.$$

Substituting Eqs. (**??**) and (**??**) in the previous equation we obtain:

$$\frac{N_s N_w}{N_{cell} N_k} = \frac{4.5 \cdot 10^{-6}(n_0 T_0)^{1/2}G}{B_0(m_i/m_e)^{1/2}}, \tag{25}$$

that gives a condition on N_s, N_k, and N_{cell}, for a given N_k. For example, if we want to double the number of steps for a gyromotion (N_s), we need to double the ratio N_{cell}/N_w. Note that for the case $T_1 = T_0$ (that is when ω_p/Ω_c is held constant passing form system 0 to system 1), we would have:

$$N_s N_\lambda = 0.865 \frac{G n_0^{1/2}}{B_0 (m_i/m_e)^{1/2}}, \tag{26}$$

that is a condition between the values of N_s and N_λ.

At this point, after one has chosen the values of N_s, N_k, and N_{cell} that fullfil Eq. (??), the values for n_1, and B_1 needed to be determined. The following tho choices are equivalent:

$$n_1 = n_0 \implies B_1 = \left(\frac{T_1}{T_0}\right)^{1/2} B_0 \tag{27}$$

or

$$B_1 = B_0 \implies n_1 = \frac{T_0}{T_1} n_0 \implies \lambda_1 = \lambda_0 \tag{28}$$

C. Normalization

Finally we can normalize our quantities.

If in our code, the Newton and Maxwell equaitons are used without any multiplicative factors, then it means that, we can redefine quantities as at the beginning of the section, but with the two following conditions:

$$\frac{Q B_0 L}{M V^2} = 1 \tag{29}$$

$$\frac{L \rho_0}{B_0} = 1. \tag{30}$$

A good choice for Q/M is to fix it either as the physical charge-mass ratio for ions or for electrons. It is trivial that if we choose Q/M as the real ion charge-mass ratio, then $\left(\frac{q_i}{m_i}\right)' = 1$ and $\left(\frac{q_e}{m_e}\right)' = -\frac{m_i}{m_e}$, while if we fix Q/M as the real electron charge-mass ratio, then $\left(\frac{q_e}{m_e}\right)' = 1$ and $\left(\frac{q_i}{m_i}\right)' = -\frac{m_e}{m_i}$. One is allowed to choose freely other two parameters among B_0, L, V, ρ_0, while the remaining two come out from the previous relations.

D. Last annotations

In PARSEK, μ_0 (permeability of free space) and ε_0 (permittivity of free space) are set equal to unity. The only wide used unit systems that allow this choice are the so called Gaussian and Heaviside-Lorentz system. Both of them are CGS systems and the only difference between the two is the 4π factor in the Maxwell equation, where in the Heaviside-Lorentz system it is omitted by a rescaling of charge density (i.e. it is included in the definition of the charge and current density).

It is a misconception that, since $\mu_0 = \varepsilon_0 = 1$, then c (speed of light) must be itself equal to unity, and then chosen as natural normalization factor for velocities. This is of course because the relation $c = 1/\sqrt{(\mu_0 \varepsilon_0)}$ holds in this form only in SI system. Beside, the factor c appears in the Maxwell equations for gaussian-CGS system, and it wouldn't make any sense if it were $c = 1$.

The factor c' can be fixed equal to unity, in the equations and in the code, only if one chooses $V = c$, that is if one chooses the speed of light as normalizing factor for the velocites. This is of course not always the best choice.

As last remark, the user should be aware that the actual value of normalizing factor (B_0, ρ_0, etc..) is not of any concern for the code. Those are only used to link real quantites with code-quantites. Anyway the two relation found previously should be clearly stated in the Collective file. At the same manner it should be clarified if the factor 4π is used or not in the Maxwell equations. If it's not, then it has to be taken in account in the factor β