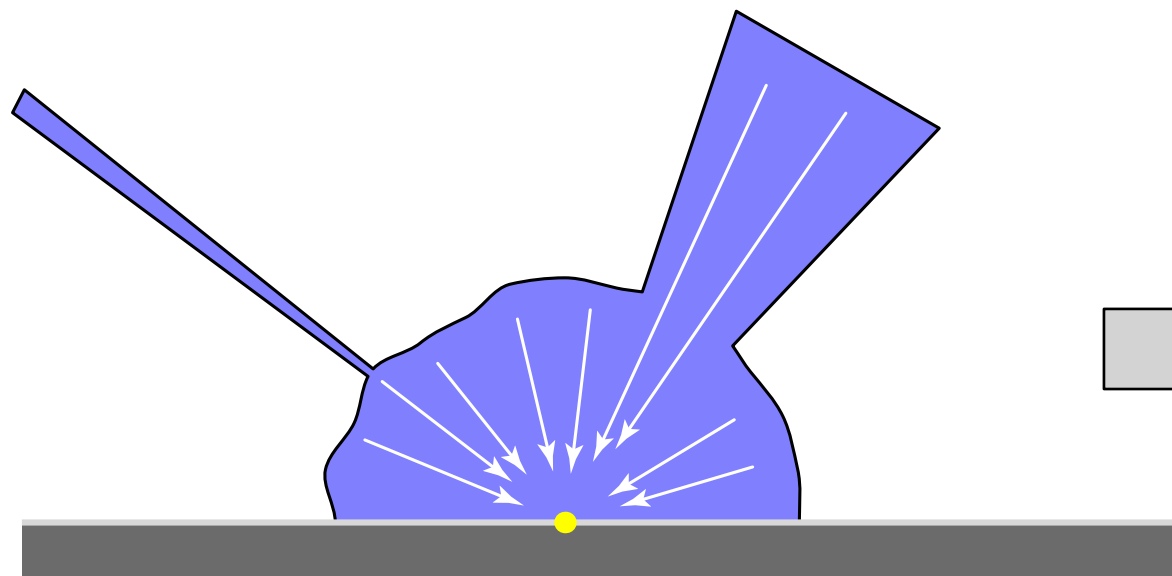


Monte Carlo Illumination

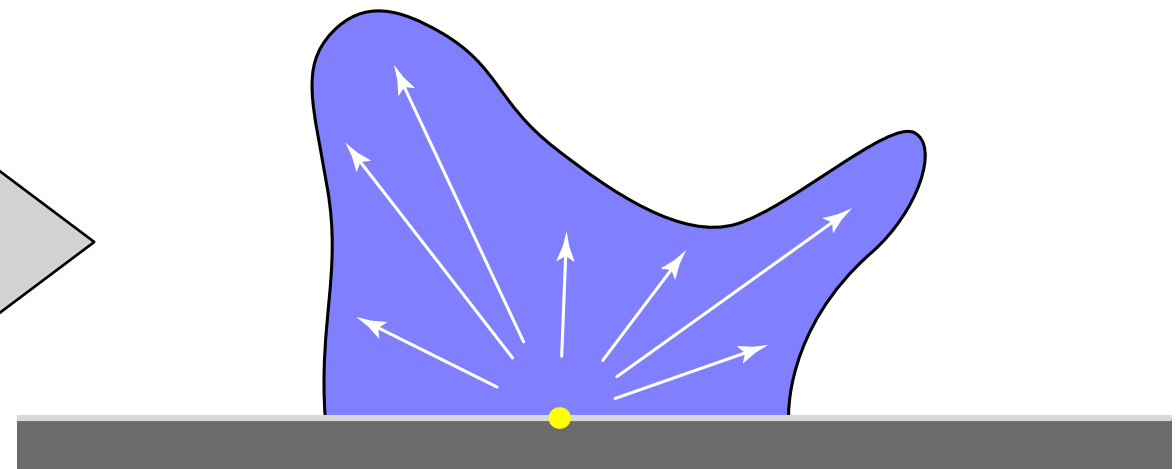
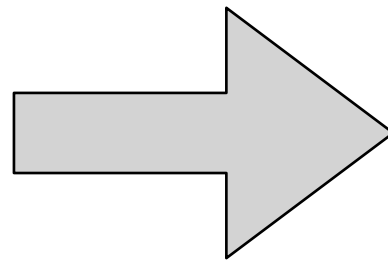
CS 4620 Lecture 19

Light reflection: full picture

- **when writing a shader, think like a bug standing on the surface**
 - bug sees an incident distribution of light arriving at the surface
 - physics question: what is the outgoing distribution of light?

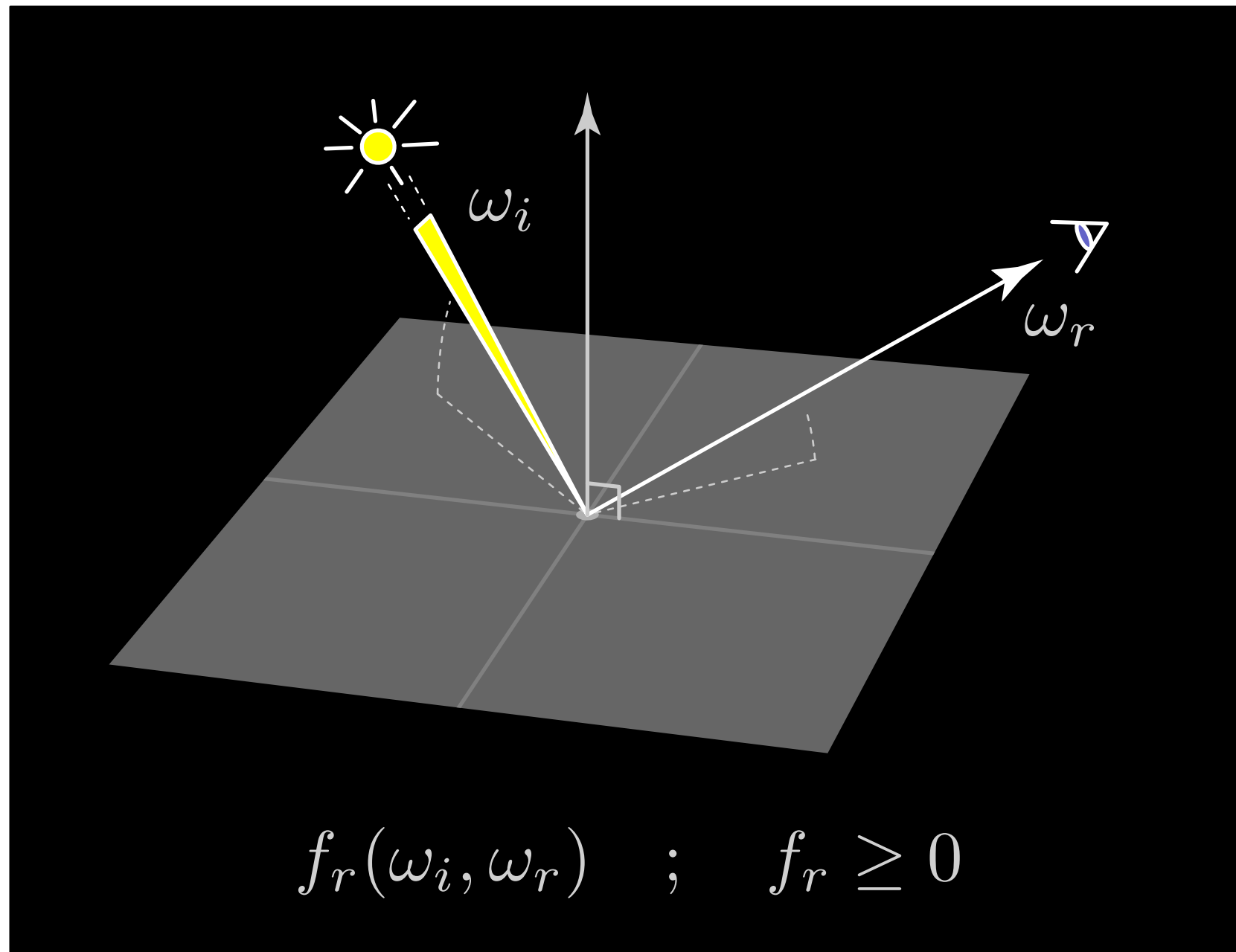


incident distribution
(function of direction)



reflected distribution
(function of direction)

BRDF



Bidirectional Reflectance Distribution Function

Surface illumination integral (as sum)

- **BRDF tells you how light from a single direction is reflected**
- **Light coming from a small source behaves similarly**
- **What about light coming from everywhere?**
 - approximate incoming light with many small sources on a sphere (the little bug can't tell the difference...)
 - reflected light is sum of reflected light due to each source (each source has its size Ω_k , brightness L_k , and direction ω_k)

$$L_r(\omega_r) = \sum_k \Omega_k L_k f_r(\omega_k, \omega_r) |\omega_k \cdot \mathbf{n}|$$

reflected light
in direction ω_r

“intensity” of
light source k

BRDF

cosine factor

Surface illumination integral

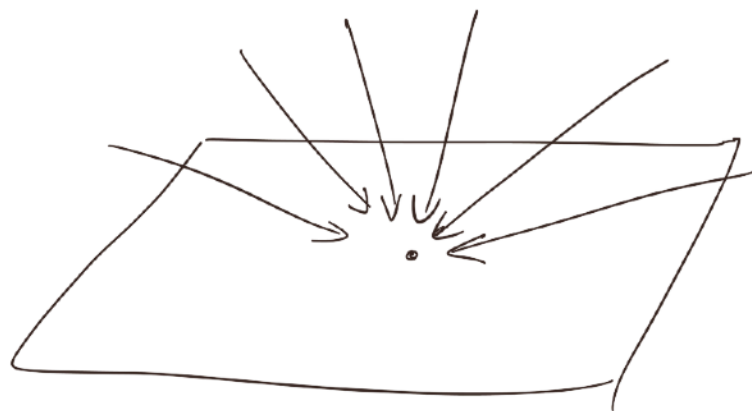
- **Take the limit as the little area sources get smaller**
 - collection of separate brightnesses L_k becomes a function $L_i(\boldsymbol{\omega}_i)$
 - size of sources turns into an integration measure $d\boldsymbol{\sigma}$

$$L_r(\omega_r) = \int_{S_+^2} L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}| d\sigma(\omega_i)$$

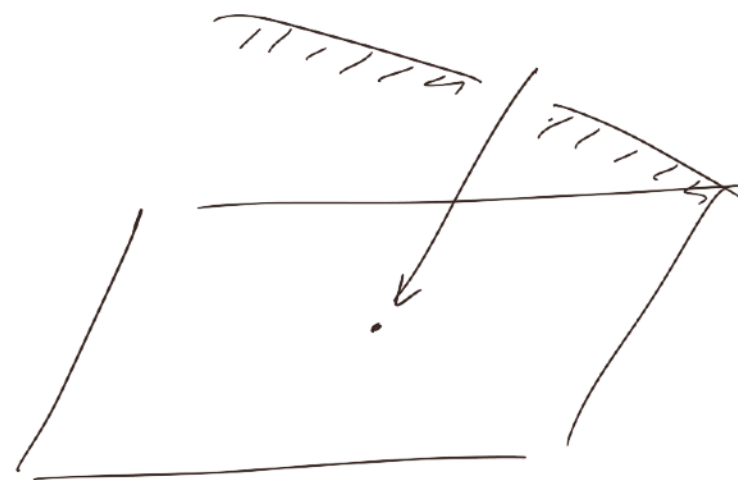
“The light reflected to direction $\boldsymbol{\omega}_r$ is the integral, over the positive unit hemisphere, of the incoming light times the BRDF times the incoming cosine factor, with respect to surface area.”

A word on radiometric units

- **Power**
 - energy per unit time, Watts
- **Irradiance**
 - energy per unit area, W/m^2
- **Radiance**
 - energy per unit area and per unit solid angle, $\text{W}/(\text{m}^2 \text{ sr})$



irradiance



radiance

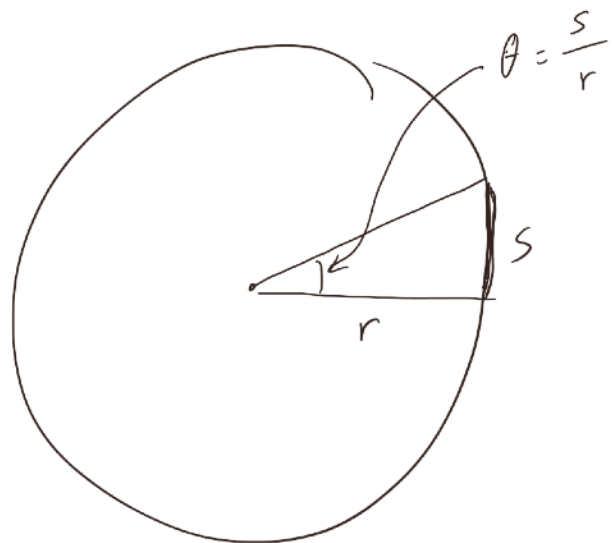
Angle and solid angle

- **Angle**

- size of a set of 2D directions (subset of unit circle)
- length / distance; whole circle has angle 2π radians

- **Solid angle**

- the size of a set of 3D directions (subset of unit sphere)
- area / distance²; whole sphere has solid angle 4π steradians



Monte Carlo Integration

- **Monte Carlo idea: design a random experiment whose average outcome is the answer we want**
- **Integration:**

$$I = \int_a^b f(x) dx$$

- **want to define an “estimator” $g(x)$ such that**

$$E\{g(x)\} = I \quad \text{for random values of } x$$

- **that is, the expected value of g is the answer we seek when x is chosen randomly.**

Uniform sampling

- **If x is chosen uniformly at random from $[a, b]$:**

$$E\{f(x)\} = \frac{1}{b-a} \int_a^b f(x) dx$$

- **so, to get the desired answer, set**

$$g(x) = (b-a)f(x)$$

- **then**

$$E\{g(x)\} = \int_a^b f(x) dx = I \quad \text{for } x \text{ uniform in } [a, b]$$

Aside: probability density functions

- **Probability distribution: familiar notion in the discrete case**

- a distribution divides up one unit of probability among the elements of a *probability space*.
- e.g. roll two dice; probability space is $\Omega = \{1, \dots, 6\}^2$
- each possible roll is equally likely: $p((i, j)) = \frac{1}{36}$
- probability distribution p has to be normalized: $\sum_{x \in \Omega} p(x) = 1$
- a random variable is a function on Ω
- e.g. sum of the two dice: $S((i, j)) = i + j$
- values of S are distributed over $\{2, \dots, 12\}$
- $S \sim p_S$ where $p_S(n) = \Pr\{S(x) = n\}$

Aside: probability density functions

- **Probability distribution can also be over a continuous set**
 - e.g. spin a spinner from 0 to 6; probability space is $\Omega = [0, 6)$
 - each possible spin is equally likely: $p(x_0) = \frac{1}{6} = \frac{\Pr\{x_0 < x < x_0 + dx\}}{dx}$
 - probability density p has to be normalized: $\int_{\Omega} p(x) dx = 1$
 - a random variable is a function on Ω
 - e.g. sum of two spins: $S : \Omega^2 \rightarrow \mathbb{R} : S(x, y) = x + y$
 - values of S are distributed over $[0, 12)$
 - $S \sim p_S$ where $p_S(z) dz = \Pr\{z < S(x, y) < z + dz\}$
 $p(0) = 0; \quad p(1) = \frac{1}{36}; \quad p(6) = \frac{1}{6}; \quad p(12) = 0$

Expectation

- **Discrete case**

$$\text{when } x \sim p(x), \quad E\{f(x)\} = \sum_{x \in \Omega} f(x)p(x)$$

- **Continuous case**

$$\text{when } x \sim p(x), \quad E\{f(x)\} = \int_{\Omega} f(x)p(x) \, dx$$

Uniform sampling revisited

- **Choosing points uniformly from $[a, b]$ is sampling from a pdf that has density $1 / (b - a)$.**

– if we use an estimator g with uniformly sampled x :

$$E\{g(x)\} = \int_a^b g(x)p(x) dx = \frac{1}{b-a} \int_a^b g(x)dx$$

– so if f is the desired integrand, the correct estimator is

$$g(x) = (b-a)f(x)$$

Nonuniform sampling

- **Choosing points instead from some other distribution over the interval $[a, b]$ also works just as well**
 - if we use an estimator g with $x \sim p(x)$

$$E\{g(x)\} = \int_a^b g(x)p(x) dx$$

- so if f is the desired integrand, the correct estimator is

$$g(x) = \frac{f(x)}{p(x)}$$

$$E\{g(x)\} = \int_a^b \frac{f(x)}{p(x)} p(x) dx = \int_a^b f(x) dx \quad \text{as long as } p(x) \text{ is not zero!}$$

Monte Carlo illumination

- **Monte Carlo integration is widely used to compute illumination integrals**

- integrand: product of illumination and BRDF and cosine factor

$$L_r(\omega_r) = \int_{S_+^2} L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}| d\sigma(\omega_i)$$

- if we choose:

$$\omega_i \sim p(\omega_i) \quad \text{and set: } g(\omega_i) = \frac{L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|}{p(\omega_i)}$$

- then: $E\{g(\omega_i)\} = L_r(\omega_r)$ (as long as $p > 0$ over the whole hemisphere)
- this is an algorithm for computing L_r !

Example: cosine-proportional sampling

- **If we select directions proportional to $|\omega_i \cdot \mathbf{n}|$**

- then:

$$p(\omega_i) \sim |\omega_i \cdot \mathbf{n}|/\pi$$

- factor of π needed so that probability integrates to 1

- the correct estimator is:

$$\begin{aligned} g(\omega_i) &= \frac{L_i(\omega_i) f_r(\omega_i, \omega_r) |\omega_i \cdot \mathbf{n}|}{p(\omega_i)} \\ &= L_i(\omega_i) f_r(\omega_i, \omega_r) \end{aligned}$$