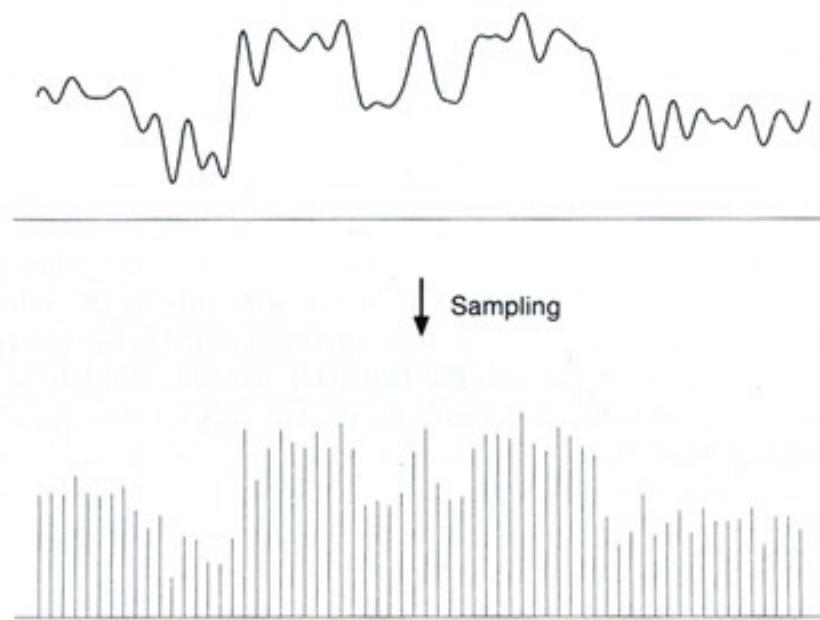


Sampling and reconstruction

CS 4620 Lecture 3

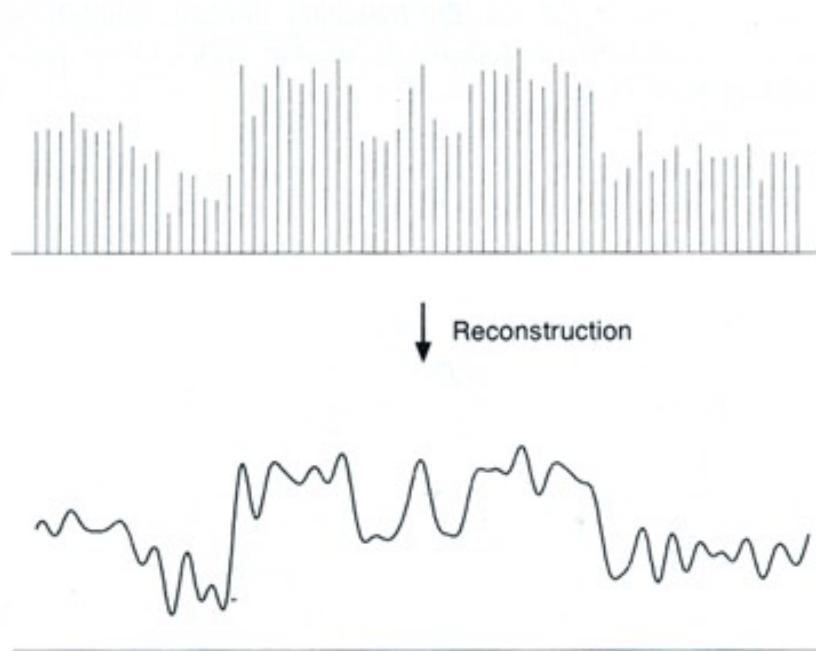
Sampled representations

- How to store and compute with continuous functions?
- Common scheme for representation: samples
 - write down the function's values at many points



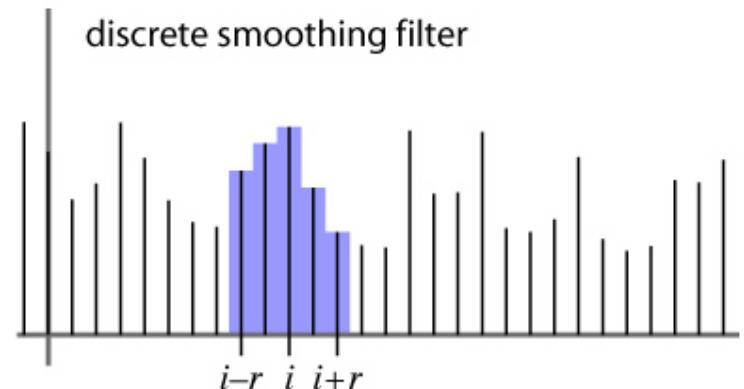
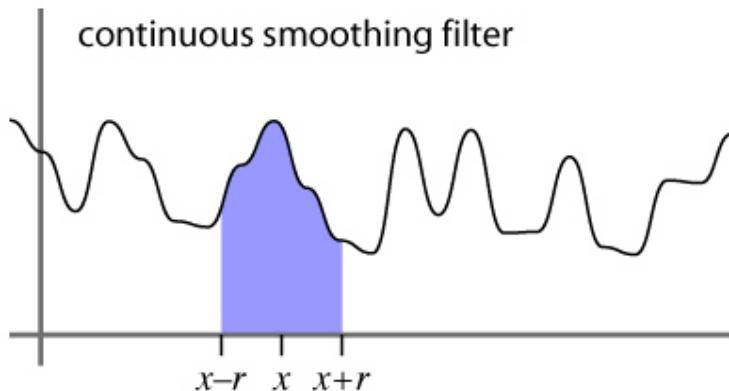
Reconstruction

- Making samples back into a continuous function
 - for output (need realizable method)
 - for analysis or processing (need mathematical method)
 - amounts to “guessing” what the function did in between



Filtering

- Processing done on a function
 - can be executed in continuous form (e.g. analog circuit)
 - but can also be executed using sampled representation
- Simple example: smoothing by averaging

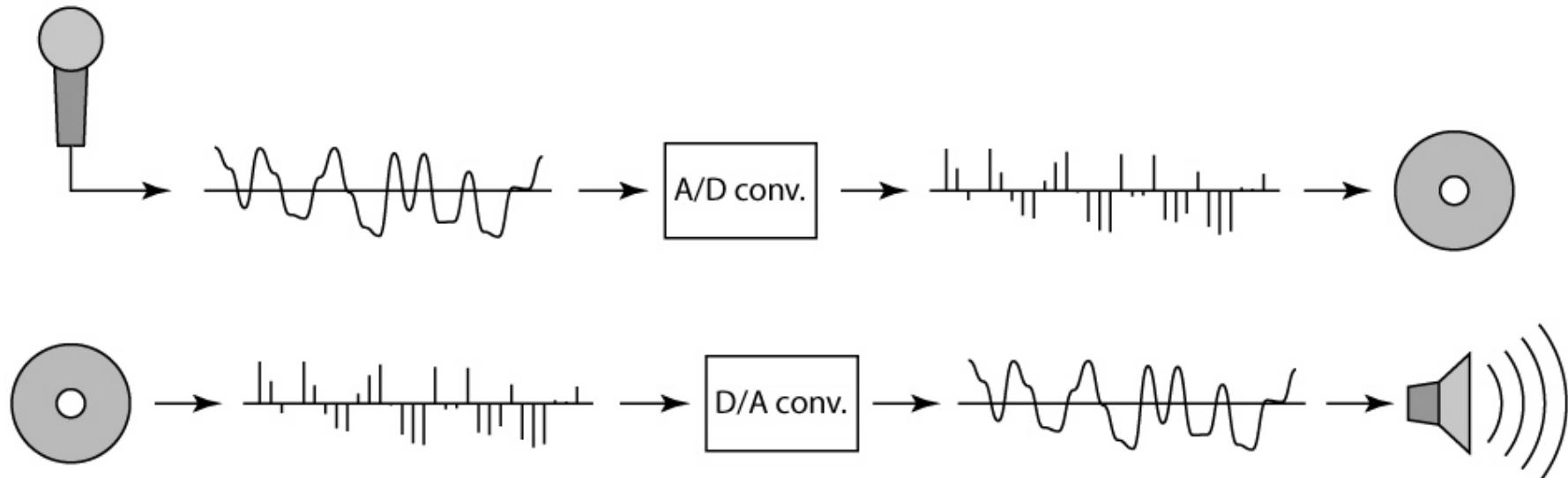


Roots of sampling

- Nyquist 1928; Shannon 1949
 - famous results in information theory
- 1940s: first practical uses in telecommunications
- 1960s: first digital audio systems
- 1970s: commercialization of digital audio
- 1982: introduction of the Compact Disc
 - the first high-profile consumer application
- This is why all the terminology has a communications or audio “flavor”
 - early applications are 1D; for us 2D (images) is important

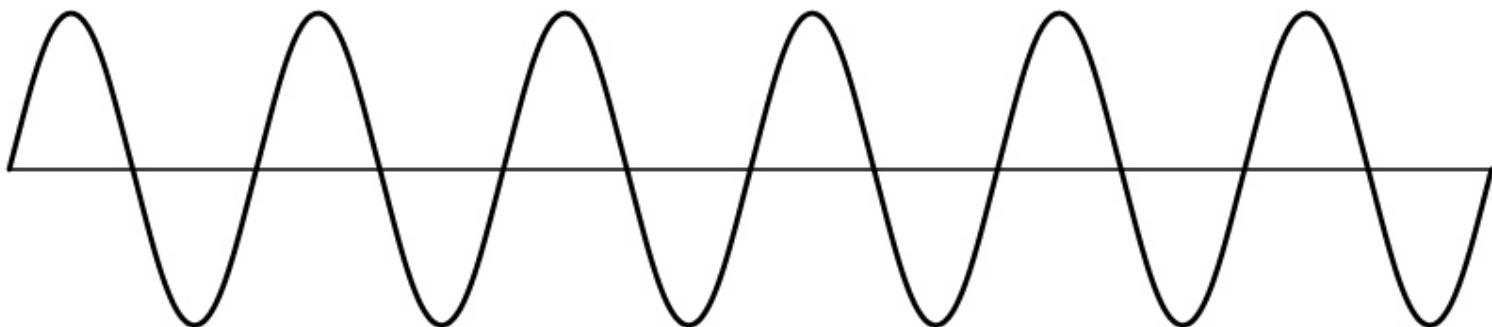
Sampling in digital audio

- Recording: sound to analog to samples to disc
- Playback: disc to samples to analog to sound again
 - how can we be sure we are filling in the gaps correctly?



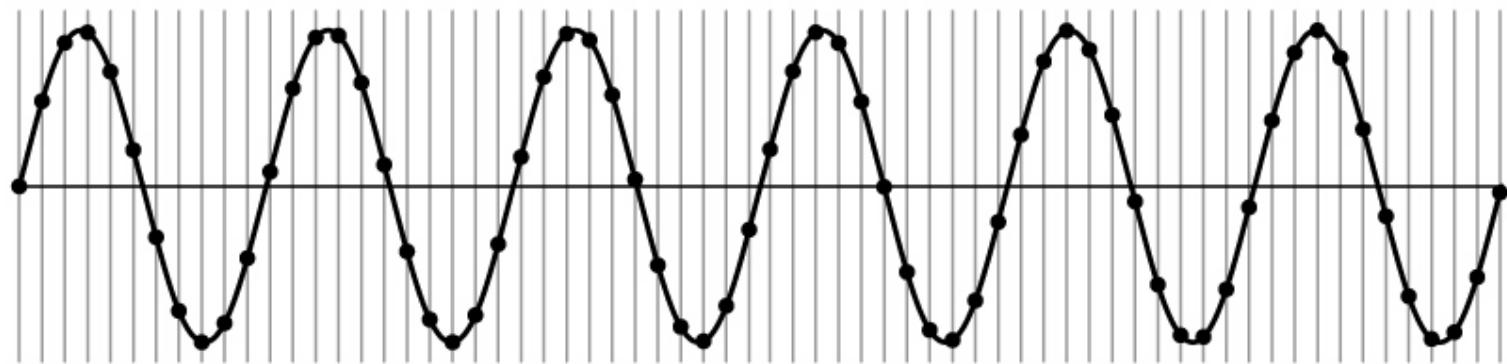
Undersampling

- What if we “missed” things between the samples?
- Simple example: undersampling a sine wave
 - unsurprising result: information is lost
 - surprising result: indistinguishable from lower frequency
 - also was always indistinguishable from higher frequencies
 - aliasing*: signals “traveling in disguise” as other frequencies



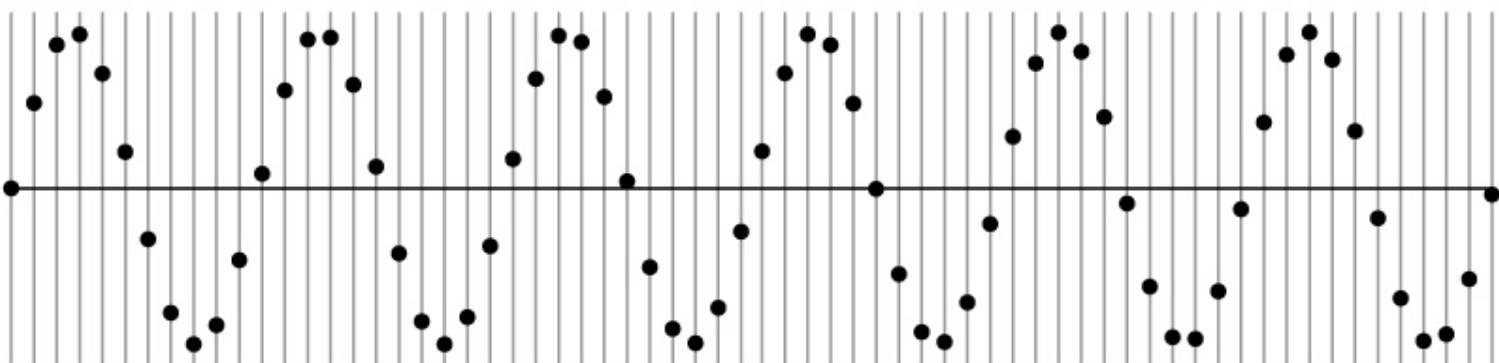
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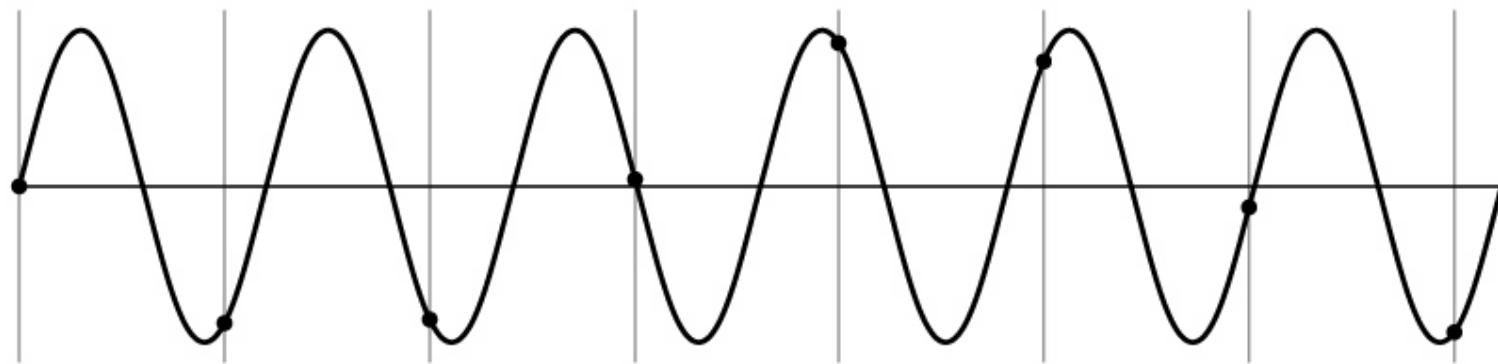
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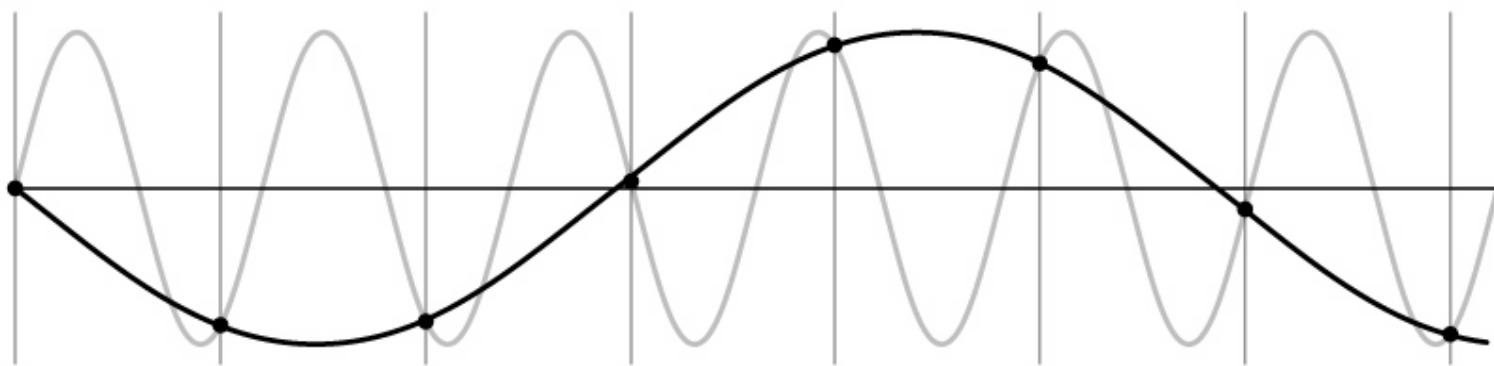
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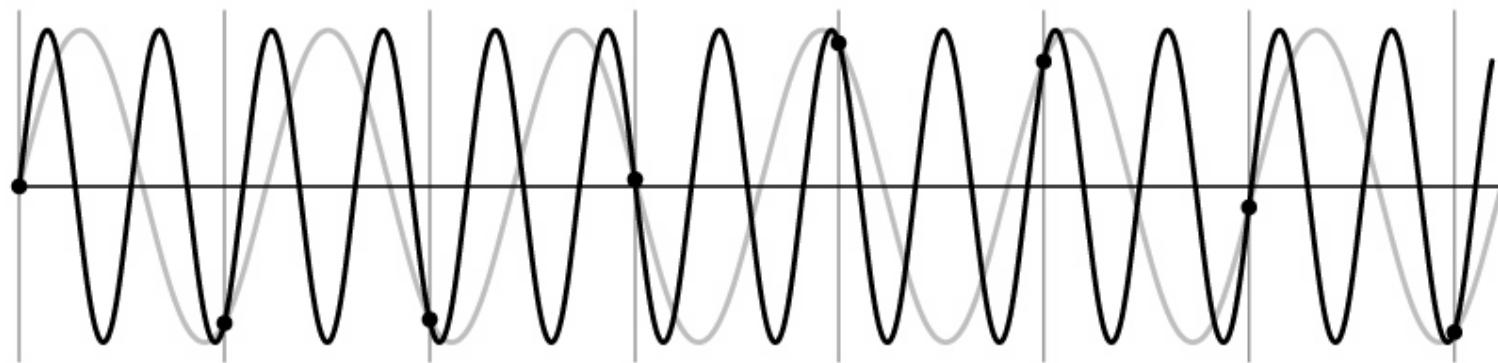
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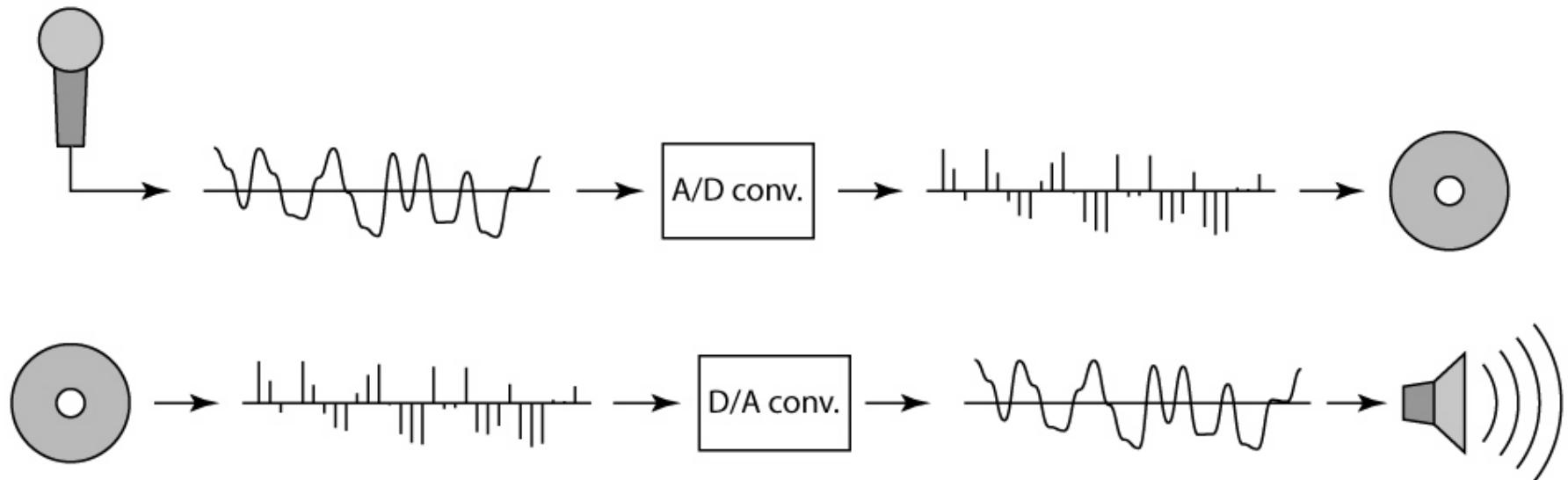
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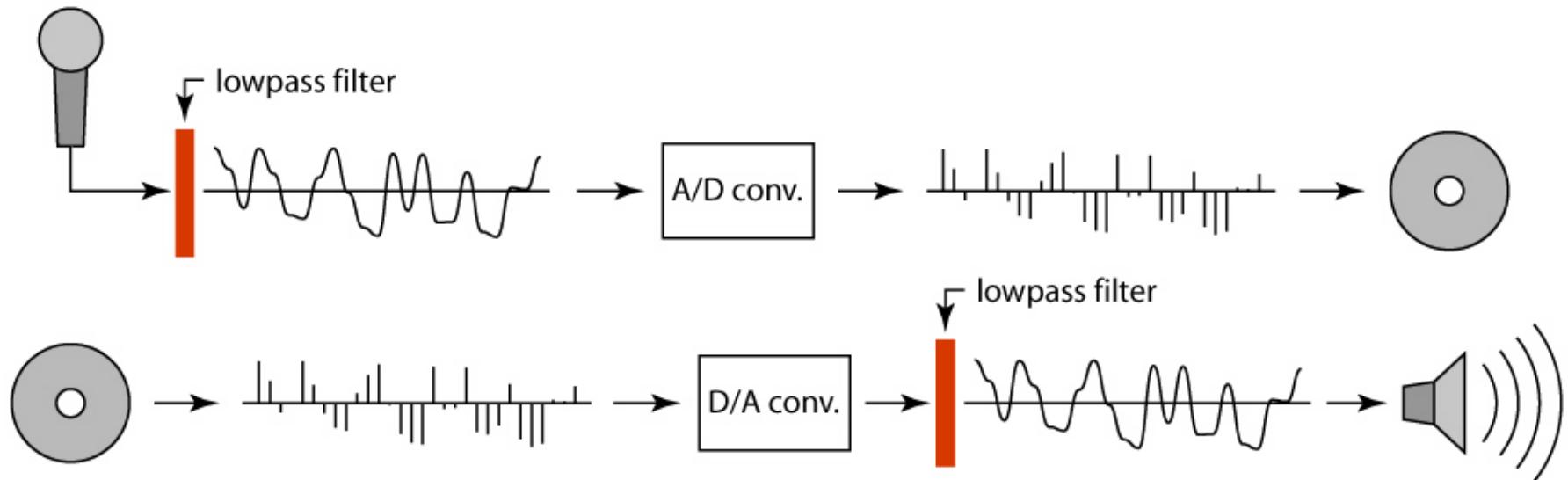
Preventing aliasing

- Introduce lowpass filters:
 - remove high frequencies leaving only safe, low frequencies
 - choose lowest frequency in reconstruction (disambiguate)



Preventing aliasing

- Introduce lowpass filters:
 - remove high frequencies leaving only safe, low frequencies
 - choose lowest frequency in reconstruction (disambiguate)

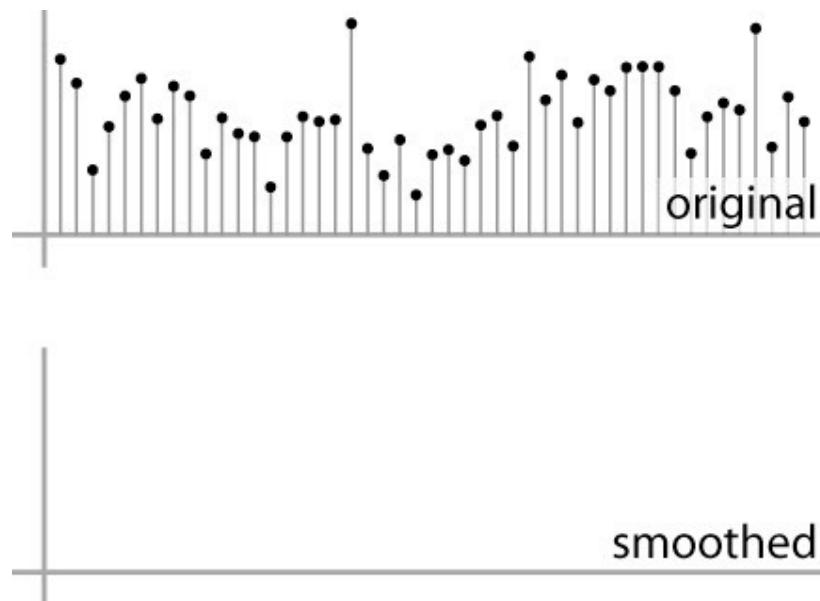


Linear filtering: a key idea

- Transformations on signals; e.g.:
 - bass/treble controls on stereo
 - blurring/sharpening operations in image editing
 - smoothing/noise reduction in tracking
- Key properties
 - linearity: $\text{filter}(f + g) = \text{filter}(f) + \text{filter}(g)$
 - shift invariance: behavior invariant to shifting the input
 - delaying an audio signal
 - sliding an image around
- Can be modeled mathematically by *convolution*

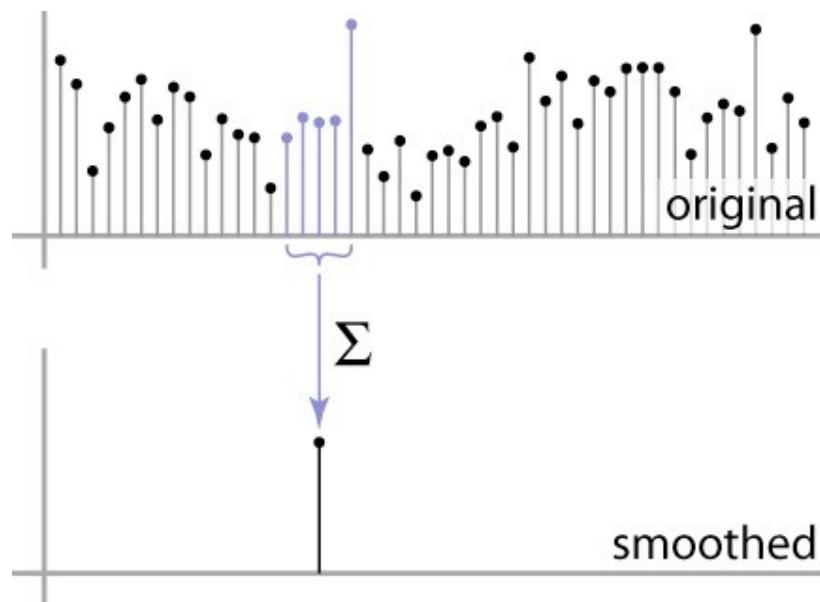
Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



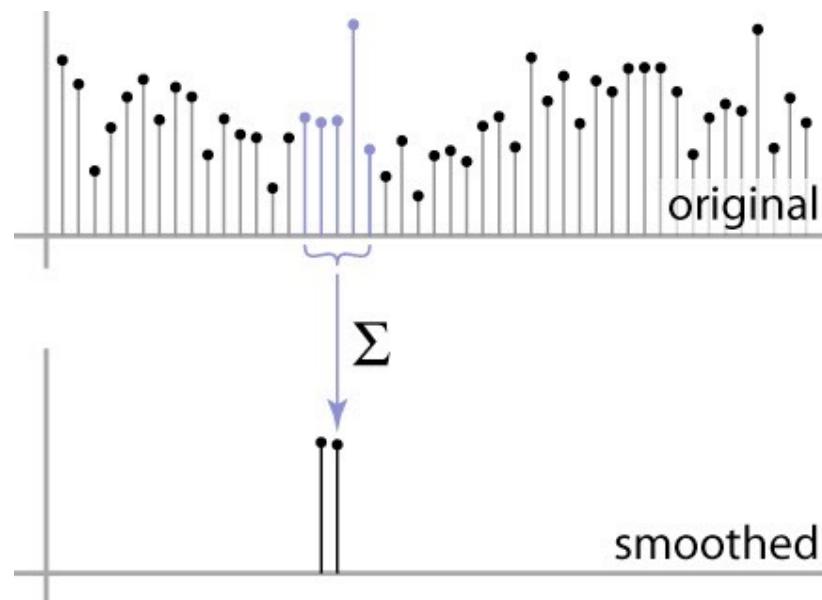
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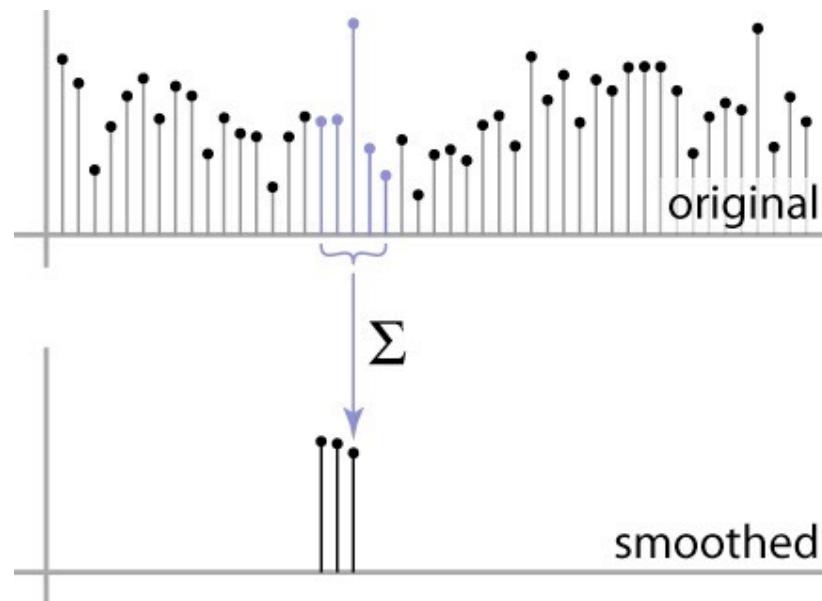
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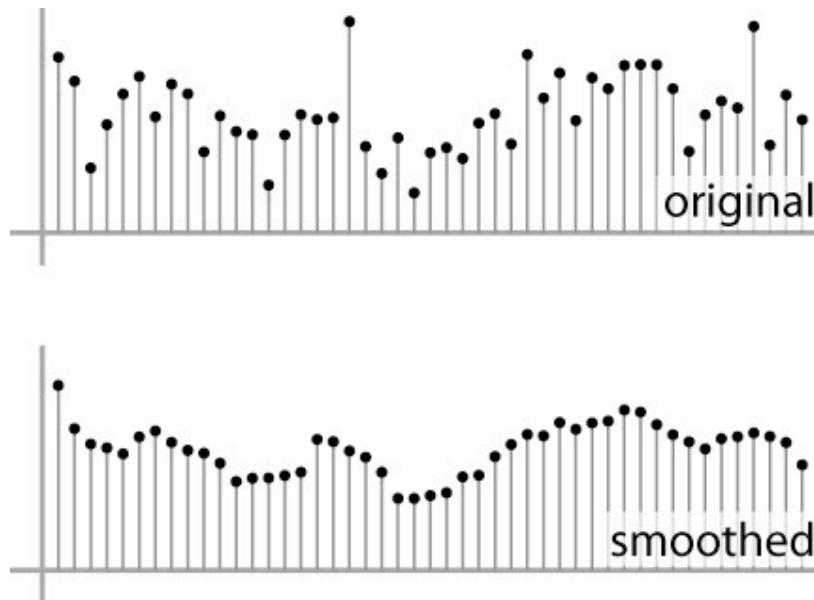
Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



Convolution warm-up

- basic idea: define a new function by averaging over a sliding window
- a simple example to start off: smoothing



Convolution warm-up

- Same moving average operation, expressed mathematically:

$$c[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} a[j].$$

Discrete convolution

- Simple averaging:

$$c[i] = \frac{1}{2r+1} \sum_{j=i-r}^{i+r} a[j].$$

every sample gets the same weight

- Convolution: same idea but with *weighted* average

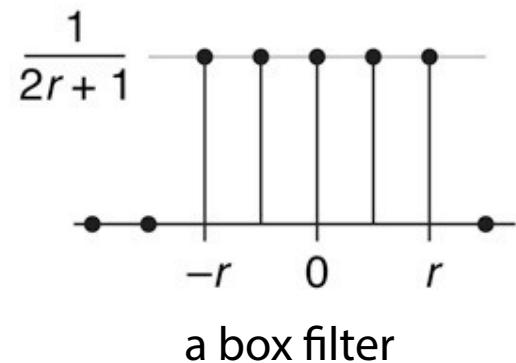
$$(a \star b)[i] = \sum_j a[j]b[i-j].$$

each sample gets its own weight (normally zero far away)

- This is all convolution is: it is a **moving weighted average**

Filters

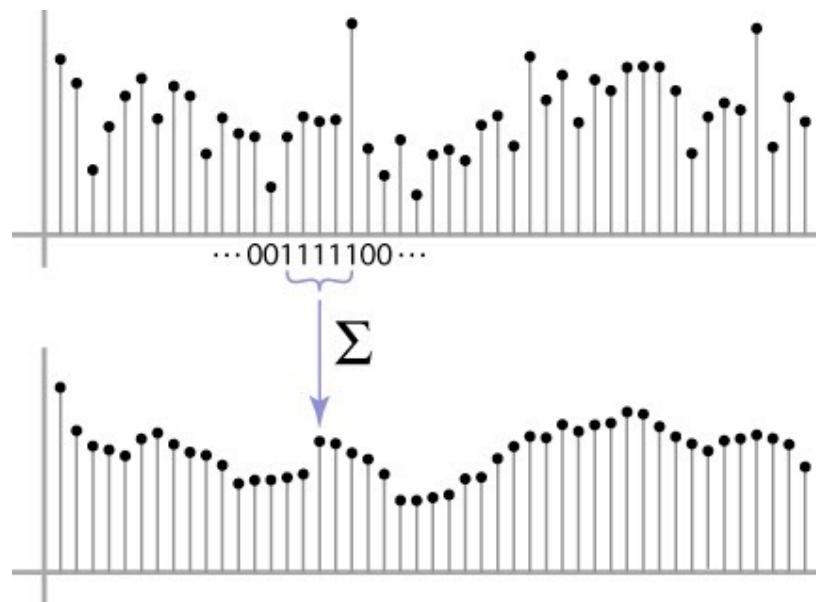
- Sequence of weights b is called a *filter*
- Filter is nonzero over its *region of support*
usually centered on zero: support radius r
- Filter is *normalized* so that it sums to 1.0
this makes for a weighted average, not just any old weighted sum
- Most filters are symmetric about 0
since for images we usually want to treat left and right the same



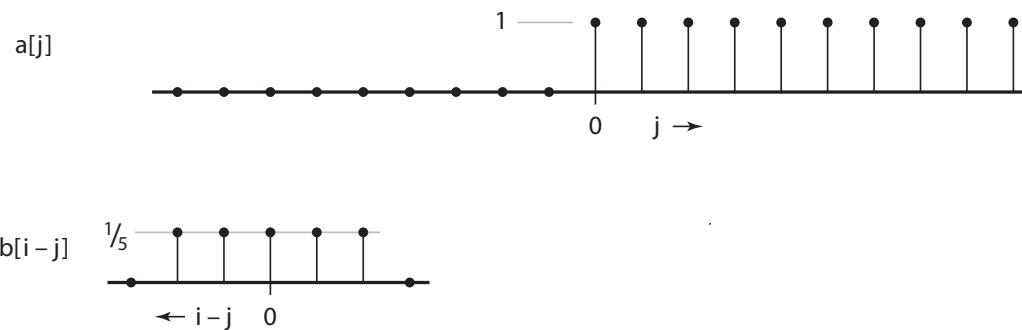
a box filter

Convolution and filtering

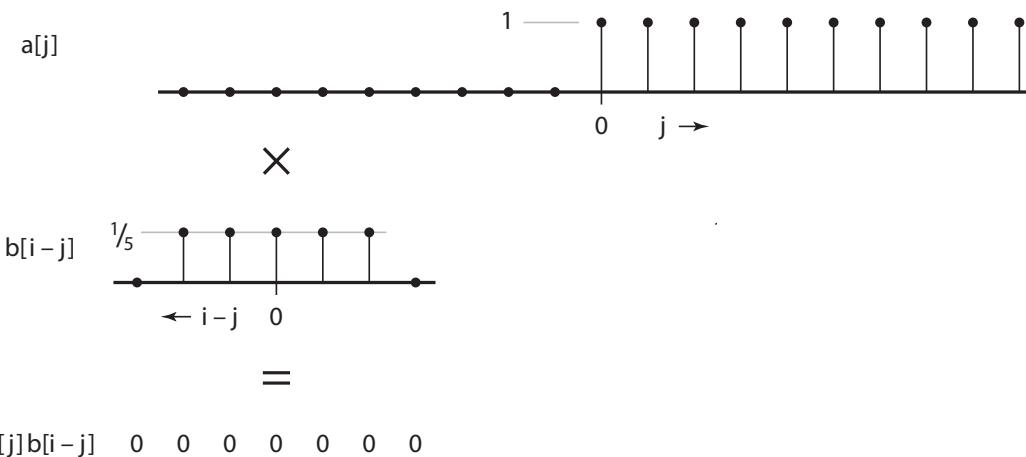
- Can express sliding average as convolution with a *box filter*
- $b_{\text{box}} = [\dots, 0, 1, 1, 1, 1, 1, 0, \dots]/5$



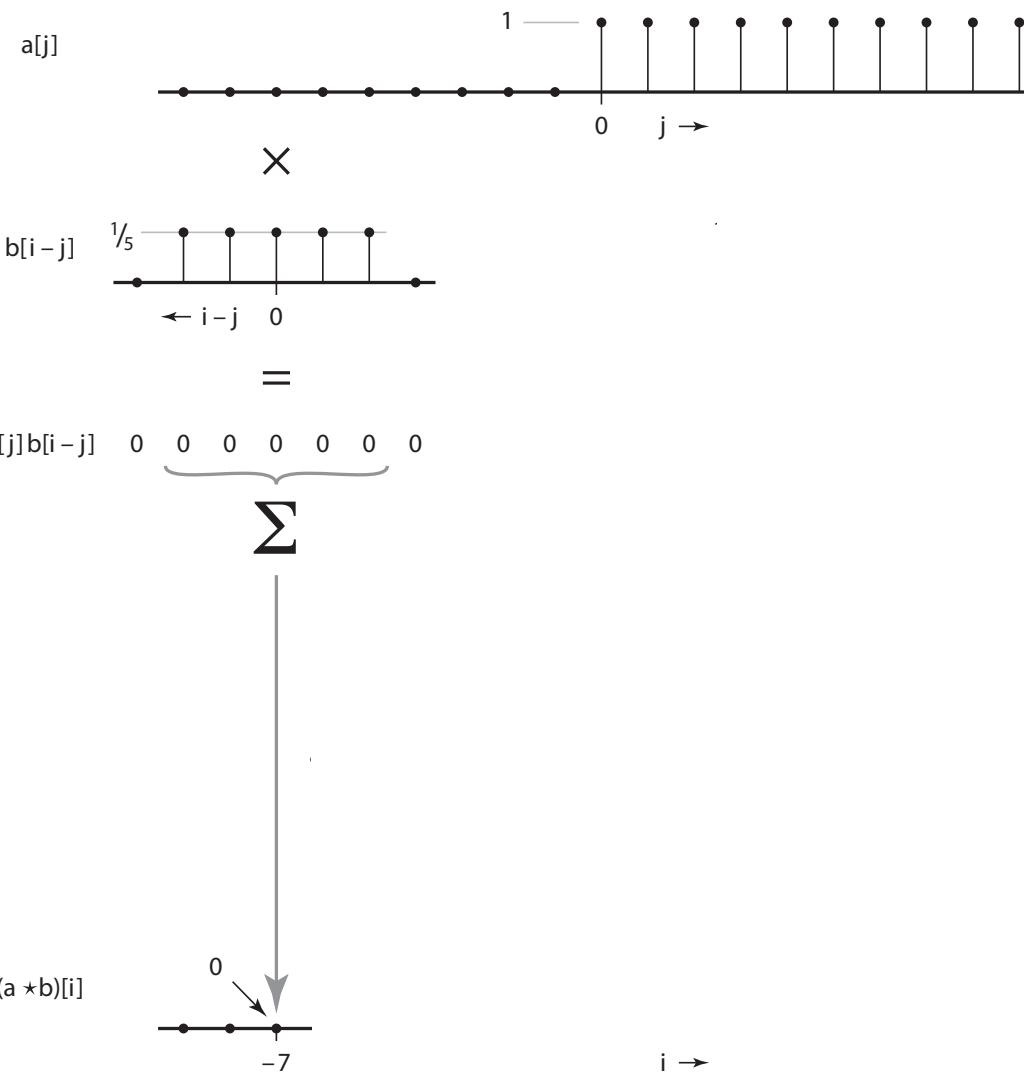
Example: box and step



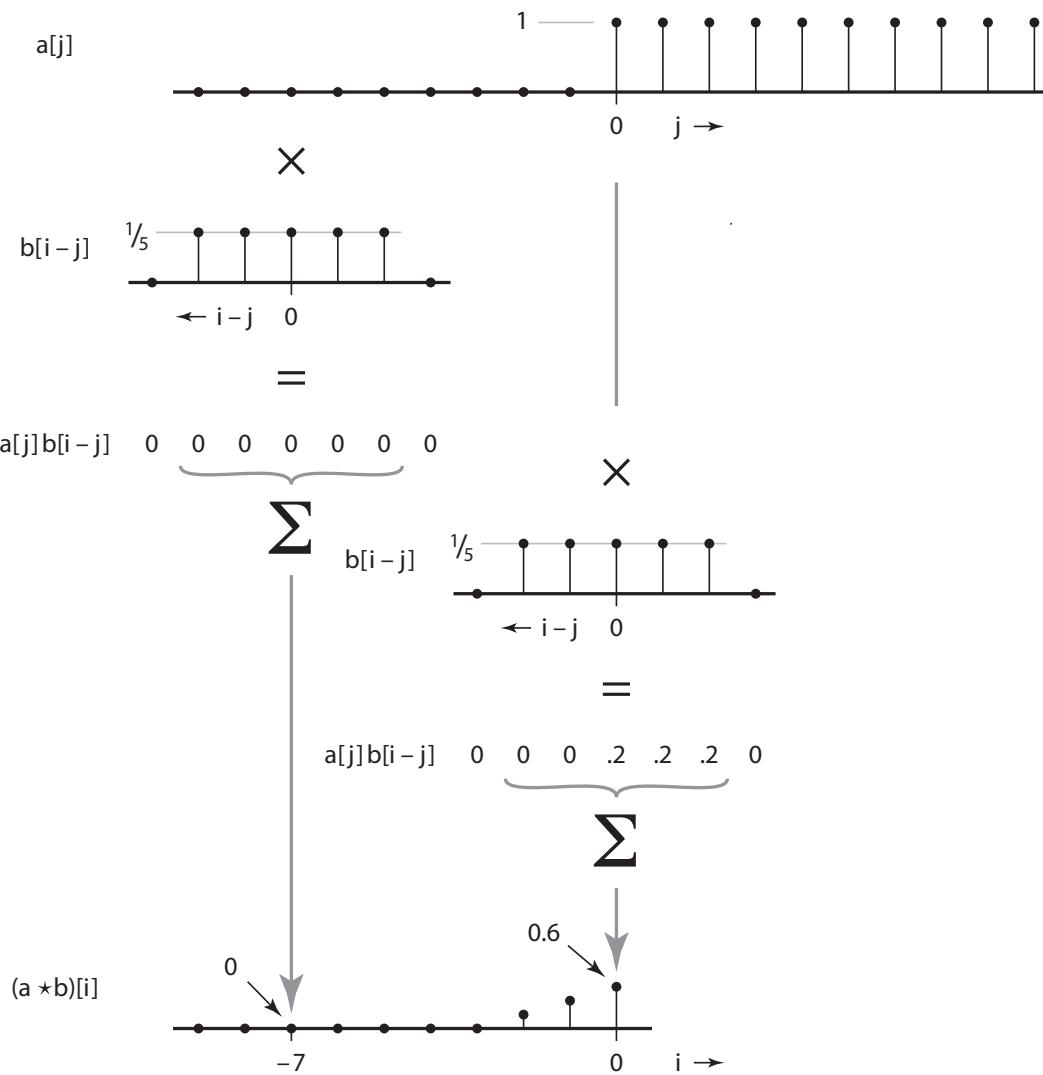
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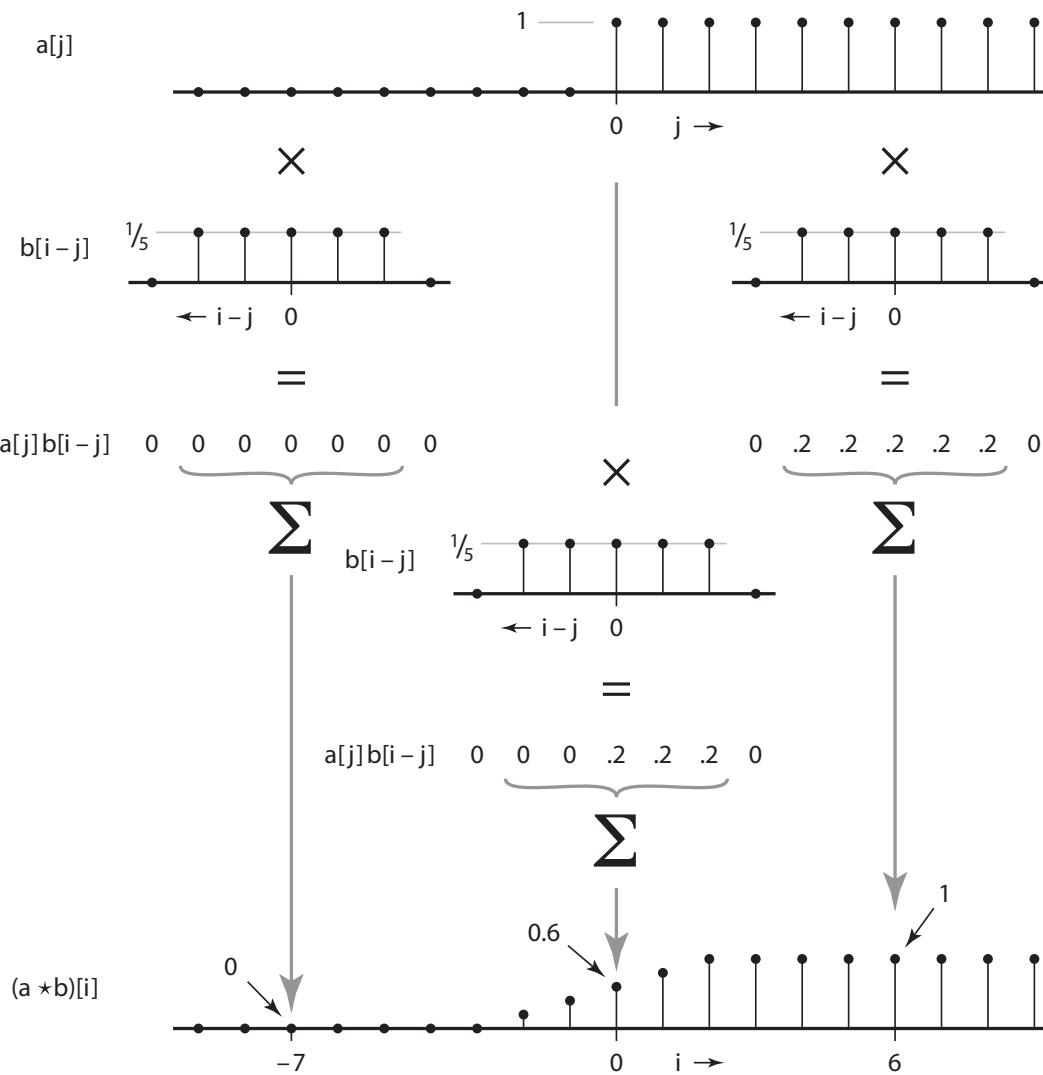
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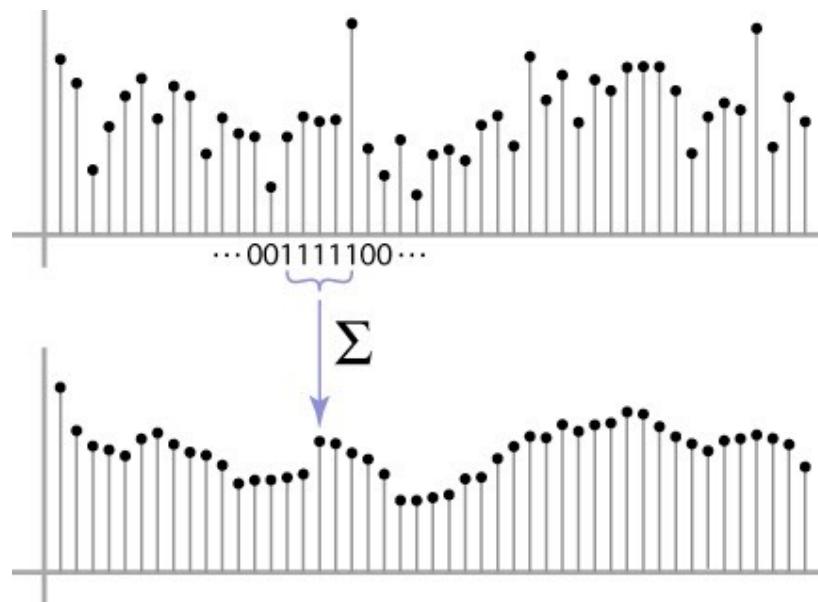


Example: box and step



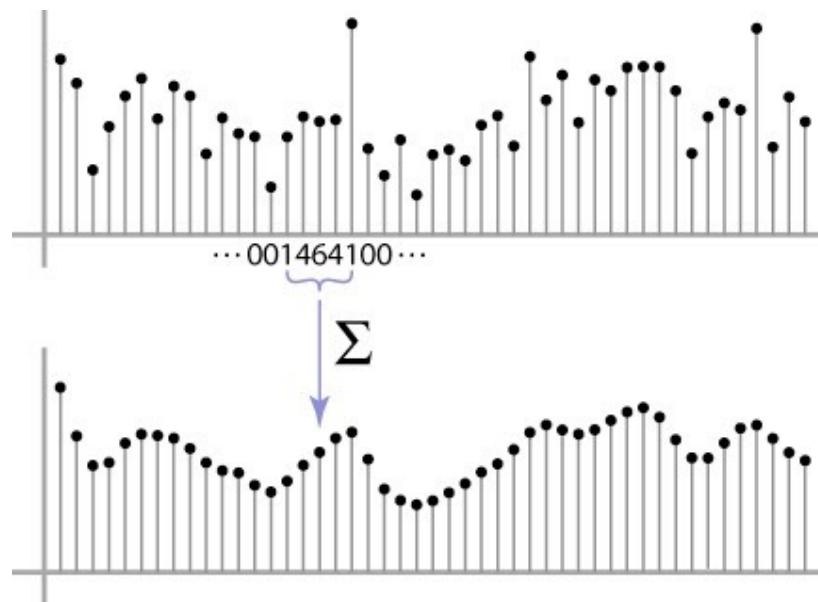
Convolution and filtering

- Convolution applies with any sequence of weights
- Example: bell curve (gaussian-like) $[..., 1, 4, 6, 4, 1, ...]/16$



Convolution and filtering

- Convolution applies with any sequence of weights
- Example: bell curve (gaussian-like) $[..., 1, 4, 6, 4, 1, ...]/16$



Discrete convolution

- Notation: $b = c \star a$
- Convolution is a multiplication-like operation
 - commutative $a \star b = b \star a$
 - associative $a \star (b \star c) = (a \star b) \star c$
 - distributes over addition $a \star (b + c) = a \star b + a \star c$
 - scalars factor out $\alpha a \star b = a \star \alpha b = \alpha(a \star b)$
 - identity: unit impulse $e = [\dots, 0, 0, 1, 0, 0, \dots]$
$$a \star e = a$$
- Conceptually no distinction between filter and signal

Discrete filtering in 2D

- Same equation, one more index

$$(a \star b)[i, j] = \sum_{i', j'} a[i', j']b[i - i', j - j']$$

now the filter is a rectangle you slide around over a grid of numbers

- Commonly applied to images

blurring (using box, using gaussian, ...)

sharpening (impulse minus blur)

- Usefulness of associativity

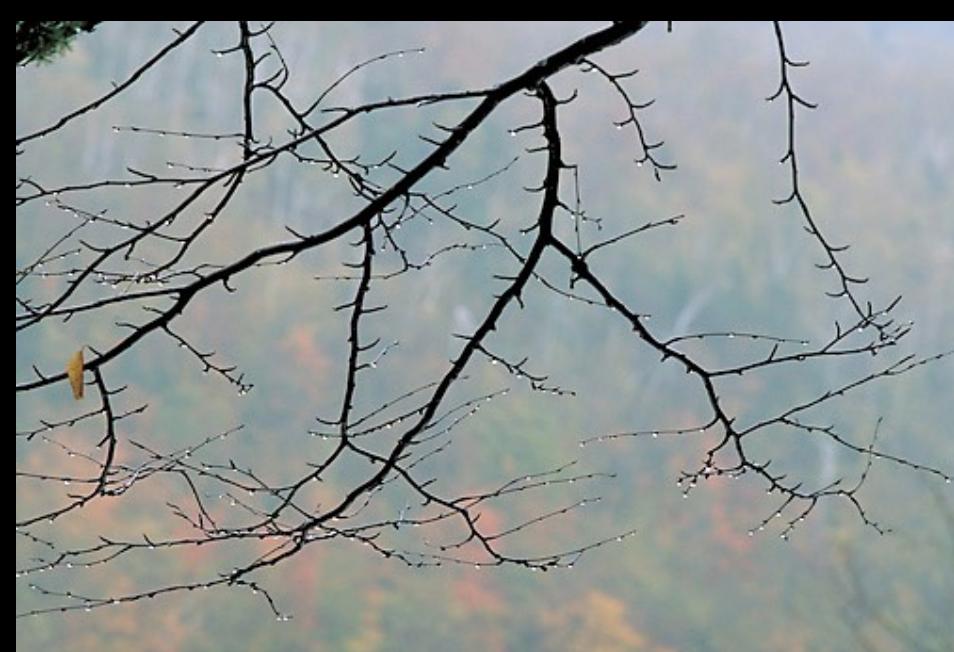
often apply several filters one after another: $((a * b_1) * b_2) * b_3$

this is equivalent to applying one filter: $a * (b_1 * b_2 * b_3)$



[Philip Greenspun]

original ▲ | ▼ box blur



sharpened ▲ | ▼ gaussian blur





[Philip Greenspun]

original ▲ | ▼ box blur

sharpened ▲ | ▼ gaussian blur



Optimization: separable filters

- basic alg. is $O(r^2)$: large filters get expensive fast!
- definition: $a_2(x,y)$ is *separable* if it can be written as:

$$a_2[i, j] = a_1[i]a_1[j]$$

this is a useful property for filters because it allows factoring:

$$\begin{aligned}(a_2 \star b)[i, j] &= \sum_{i'} \sum_{j'} a_2[i', j']b[i - i', j - j'] \\&= \sum_{i'} \sum_{j'} a_1[i']a_1[j']b[i - i', j - j'] \\&= \sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j']b[i - i', j - j'] \right)\end{aligned}$$

Separable filtering

$$a_2[i, j] = a_1[i]a_1[j]$$

| | | | | |
|---|----|----|----|---|
| 1 | 4 | 6 | 4 | 1 |
| 4 | 16 | 24 | 16 | 4 |
| 6 | 24 | 36 | 24 | 6 |
| 4 | 16 | 24 | 16 | 4 |
| 1 | 4 | 6 | 4 | 1 |

=

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 4 | 6 | 4 | 1 |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

*

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 4 | 0 | 0 |
| 0 | 0 | 6 | 0 | 0 |
| 0 | 0 | 4 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |

first, convolve with this

$$\sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j'] b[i - i', j - j'] \right)$$



Separable filtering

$$a_2[i, j] = a_1[i]a_1[j]$$

| | | | | |
|---|----|----|----|---|
| 1 | 4 | 6 | 4 | 1 |
| 4 | 16 | 24 | 16 | 4 |
| 6 | 24 | 36 | 24 | 6 |
| 4 | 16 | 24 | 16 | 4 |
| 1 | 4 | 6 | 4 | 1 |

=

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |
| 1 | 4 | 6 | 4 | 1 |
| 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 |

*

| | | | | |
|---|---|---|---|---|
| 0 | 0 | 1 | 0 | 0 |
| 0 | 0 | 4 | 0 | 0 |
| 0 | 0 | 6 | 0 | 0 |
| 0 | 0 | 4 | 0 | 0 |
| 0 | 0 | 1 | 0 | 0 |

second, convolve with this

first, convolve with this

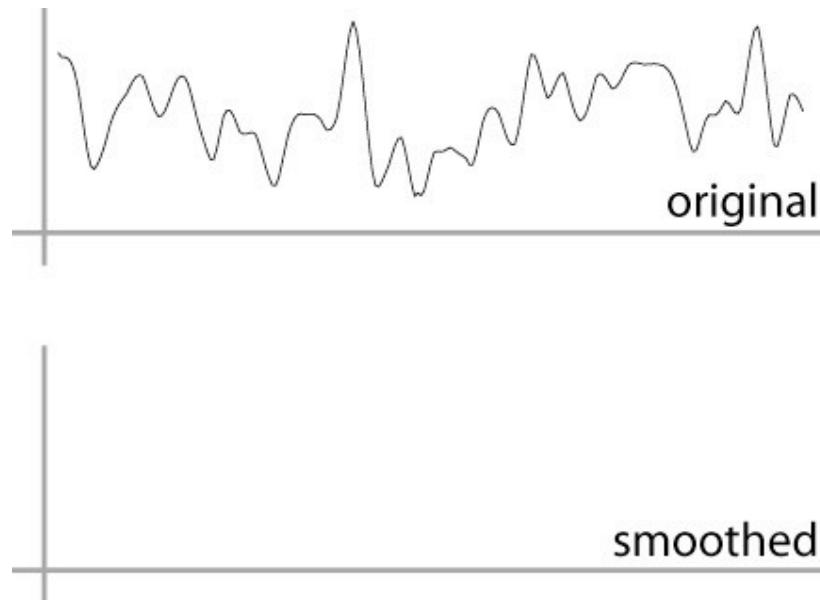
$$\sum_{i'} a_1[i'] \left(\sum_{j'} a_1[j'] b[i - i', j - j'] \right)$$

Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well

output is continuous

integration replaces summation

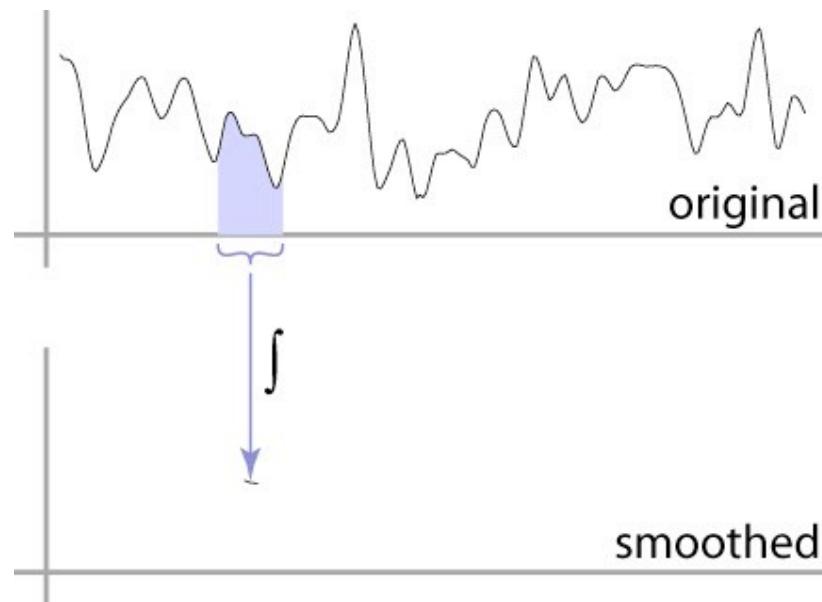


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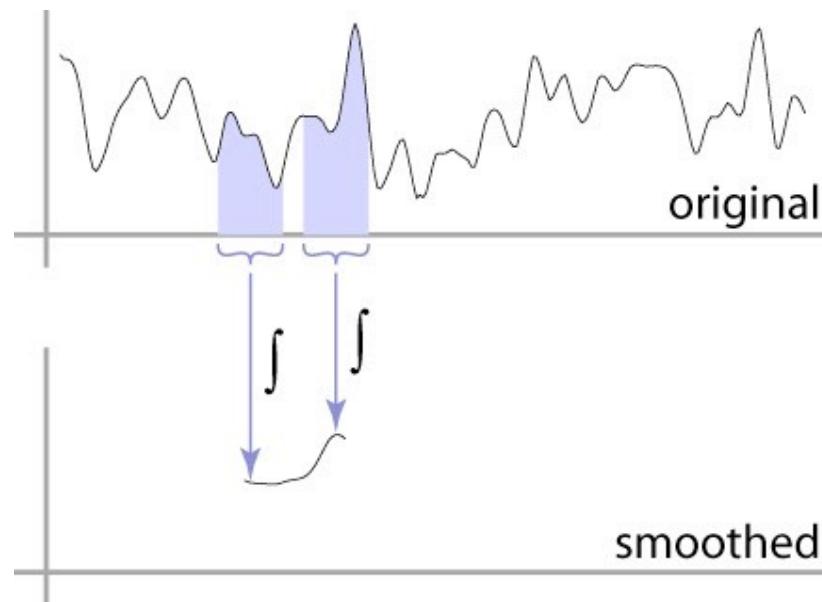


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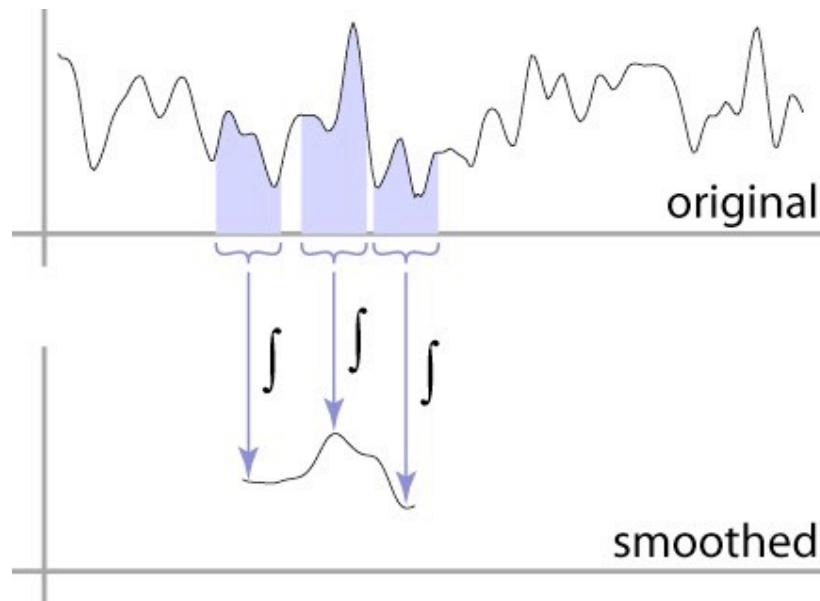


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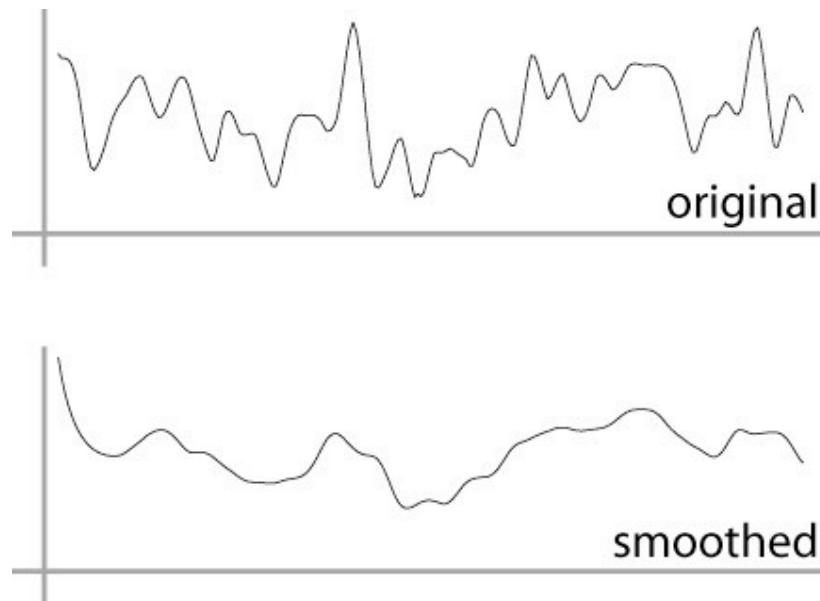


Continuous convolution: warm-up

- Can apply sliding-window average to a continuous function just as well

output is continuous

integration replaces summation



Continuous convolution

- Sliding average expressed mathematically:

$$g_{\text{smooth}}(x) = \frac{1}{2r} \int_{x-r}^{x+r} g(t) dt$$

note difference in normalization (only for box)

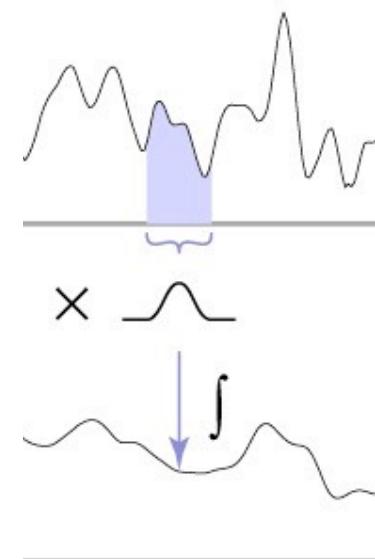
- Convolution just introduces weights

$$(f * g)(x) = \int_{-\infty}^{\infty} f(t)g(x-t) dt$$

weighting is now by a function

weighted integral is like weighted average

again bounds are set by support of $f(x)$



One more convolution

- Continuous-discrete convolution

$$(a \star f)(x) = \sum_i a[i]f(x - i)$$

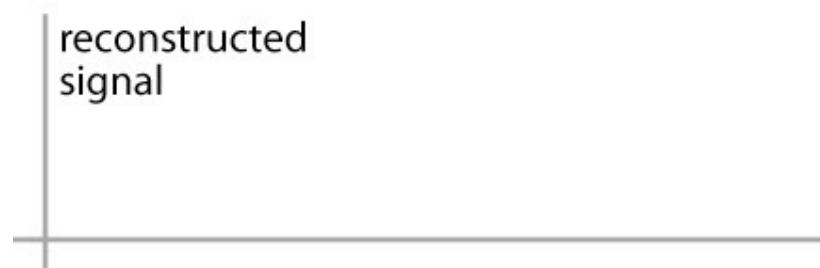
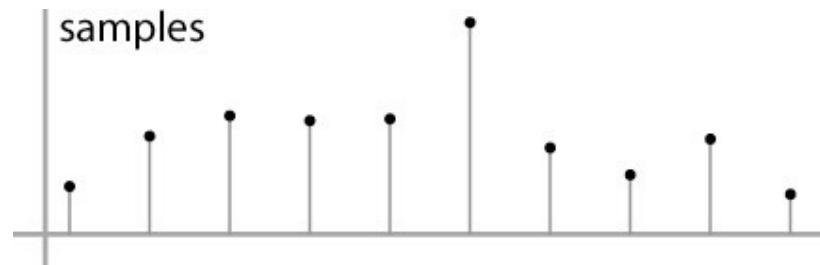
$$(a \star f)(x, y) = \sum_{i,j} a[i, j]f(x - i, y - j)$$

input: a sequence and a continuous function

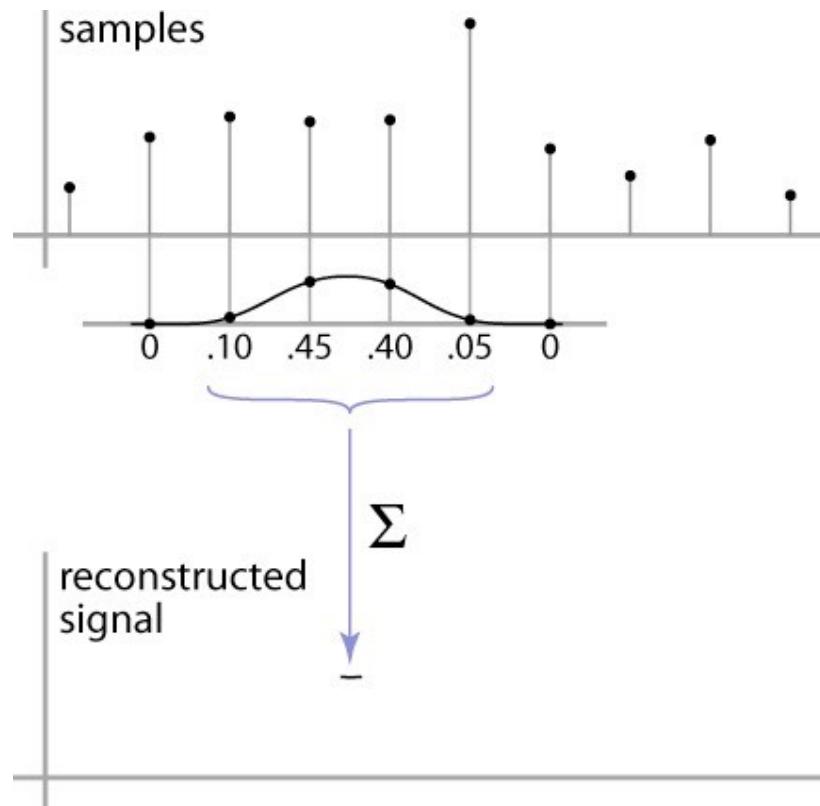
output: a continuous function

used for reconstruction and resampling

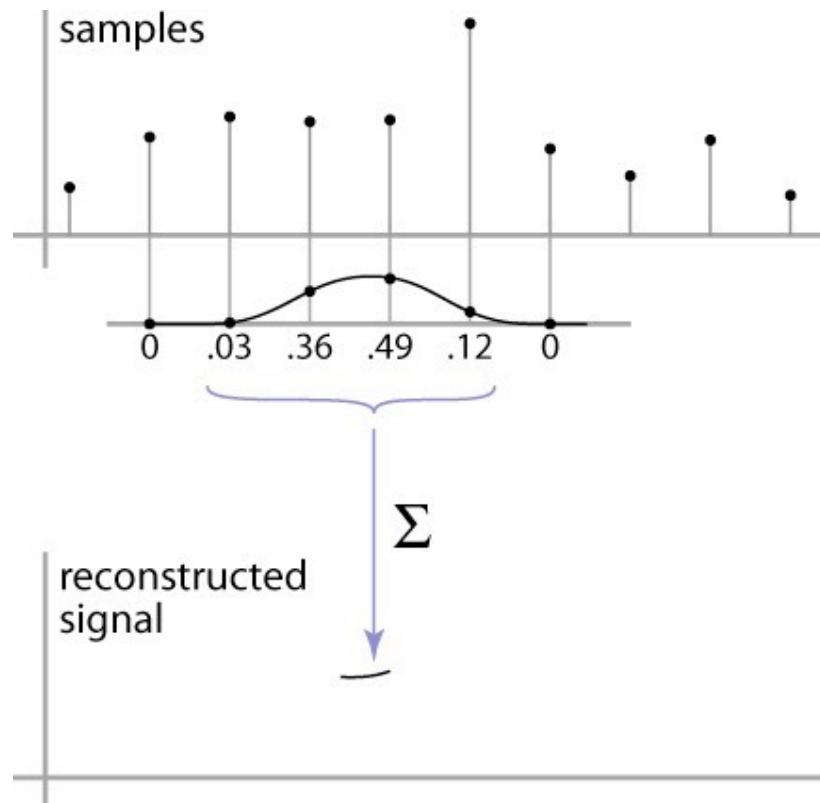
Continuous-discrete convolution



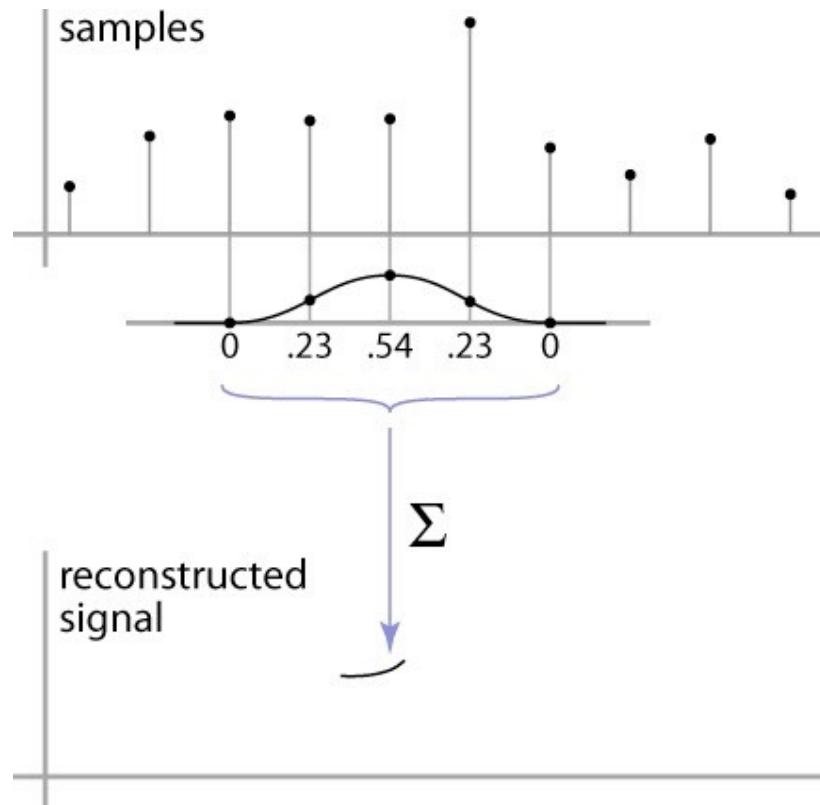
Continuous-discrete convolution



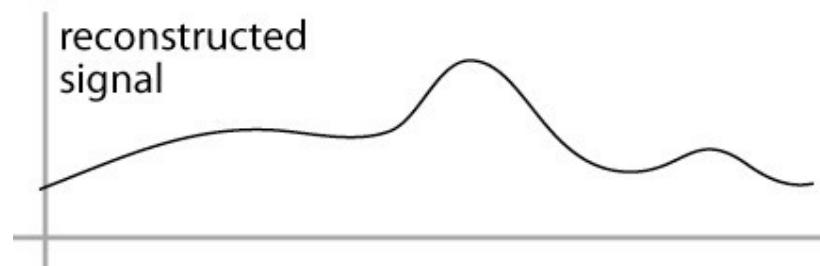
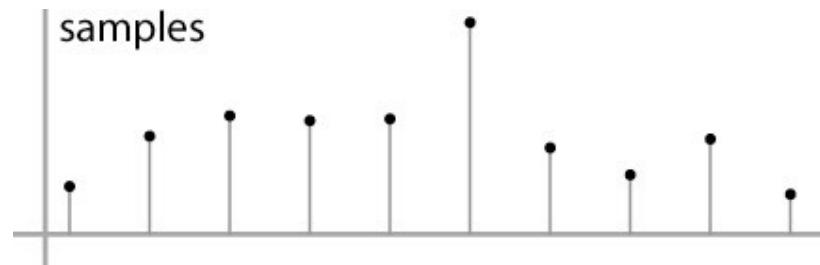
Continuous-discrete convolution



Continuous-discrete convolution



Continuous-discrete convolution

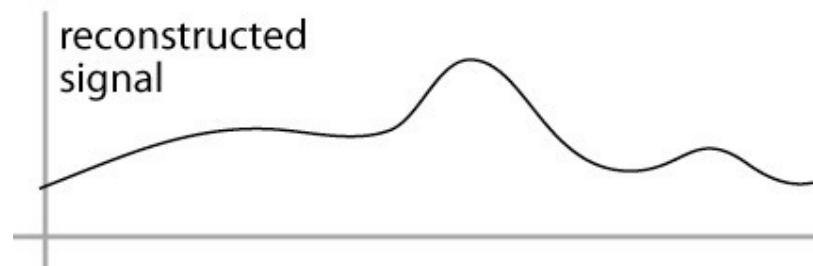
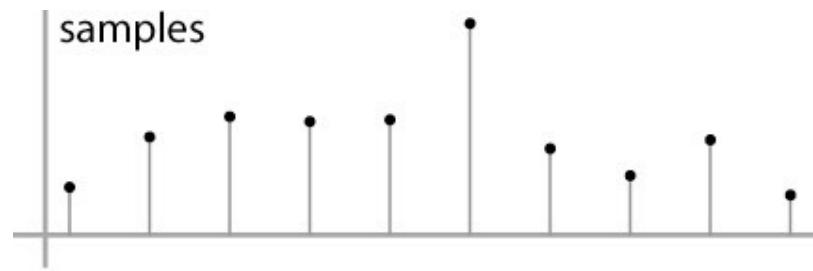


Resampling

- Changing the sample rate
 - in images, this is enlarging and reducing
- Creating more samples:
 - increasing the sample rate
 - “upsampling”
 - “enlarging”
- Ending up with fewer samples:
 - decreasing the sample rate
 - “downsampling”
 - “reducing”

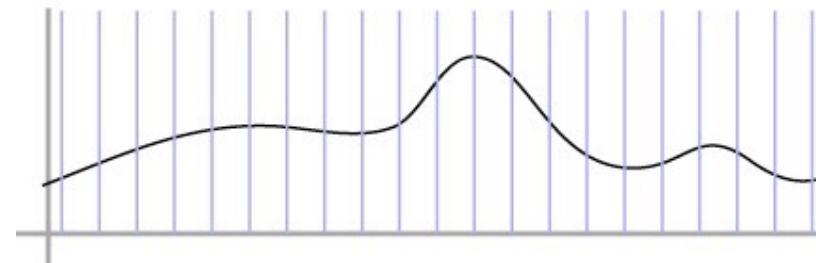
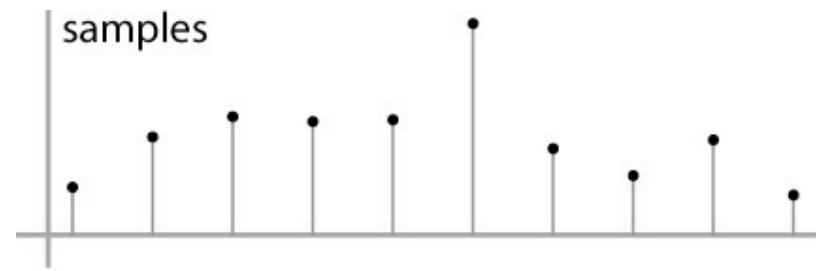
Resampling

- Reconstruction creates a continuous function
forget its origins, go ahead and sample it



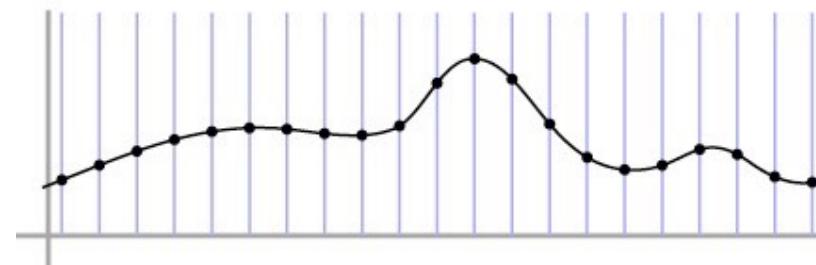
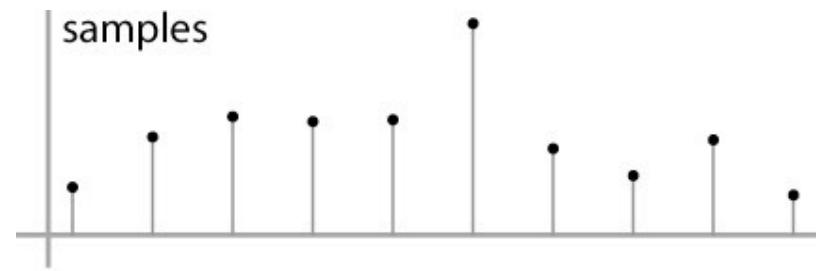
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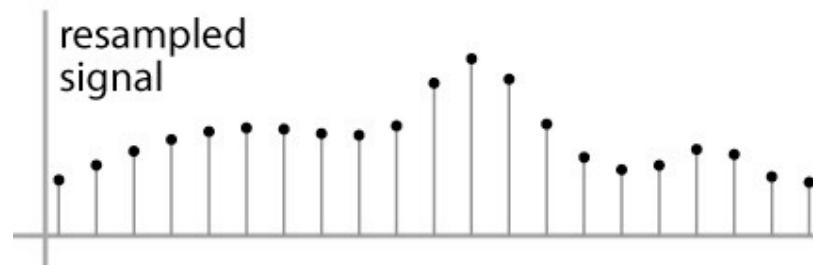
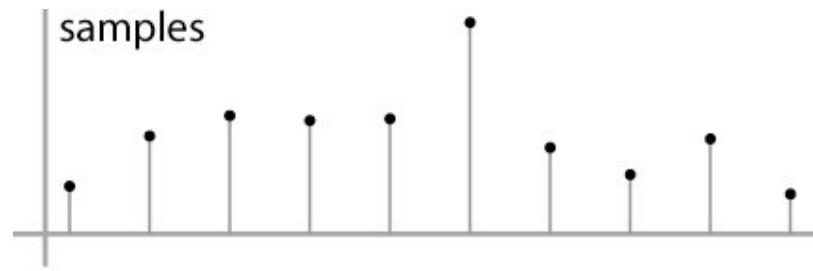
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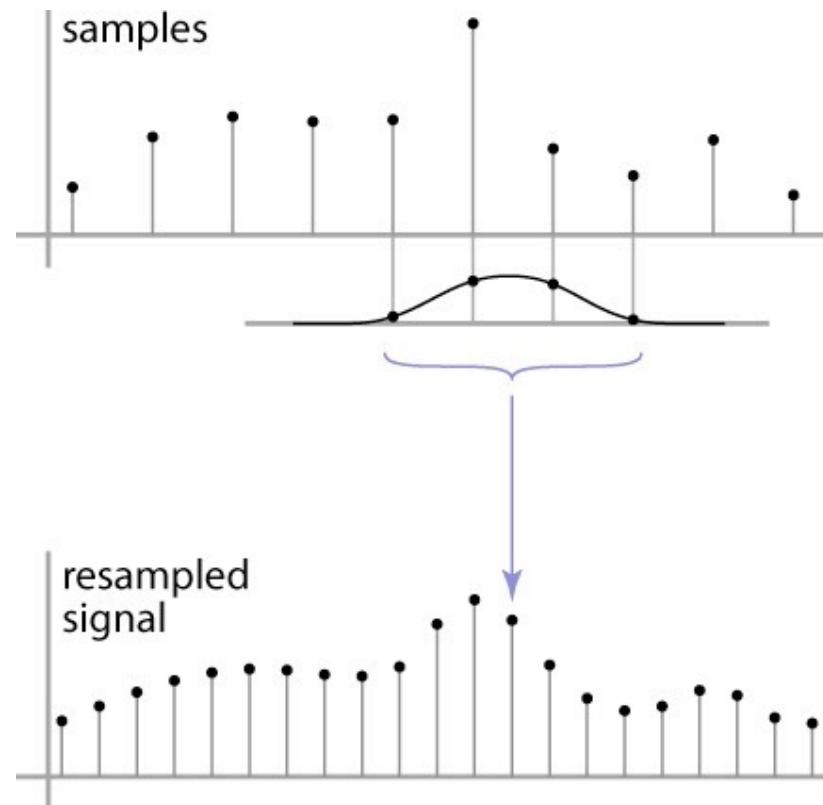
Resampling

- Reconstruction creates a continuous function
forget its origins, go ahead and sample it



Resampling

- Reconstruction creates a continuous function
forget its origins, go ahead and sample it



Cont.-disc. convolution in 2D

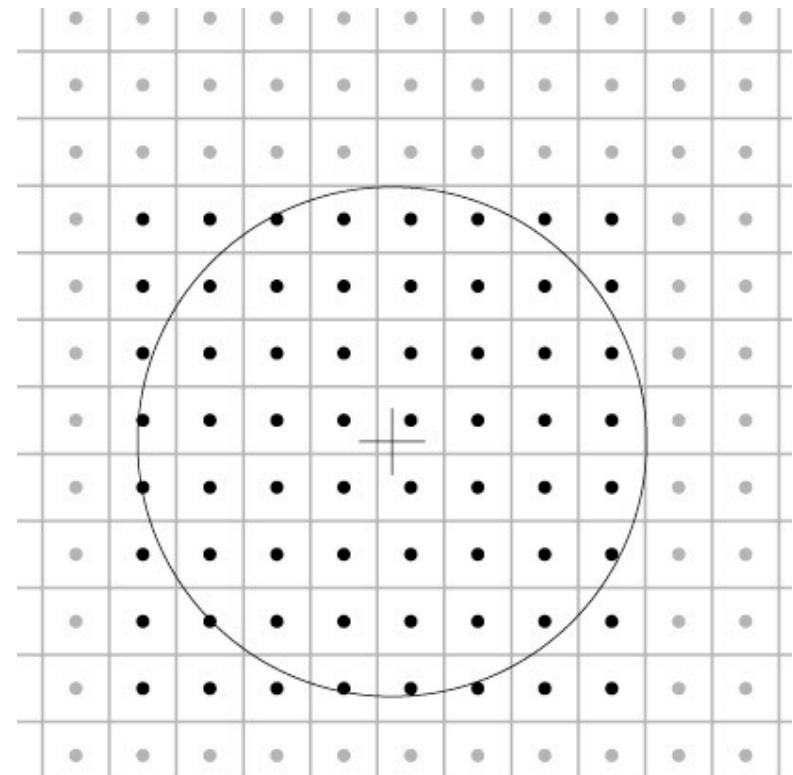
- same convolution—just two variables now

$$(a \star f)(x, y) = \sum_{i,j} a[i, j]f(x - i, y - j)$$

loop over nearby pixels,
average using filter weight

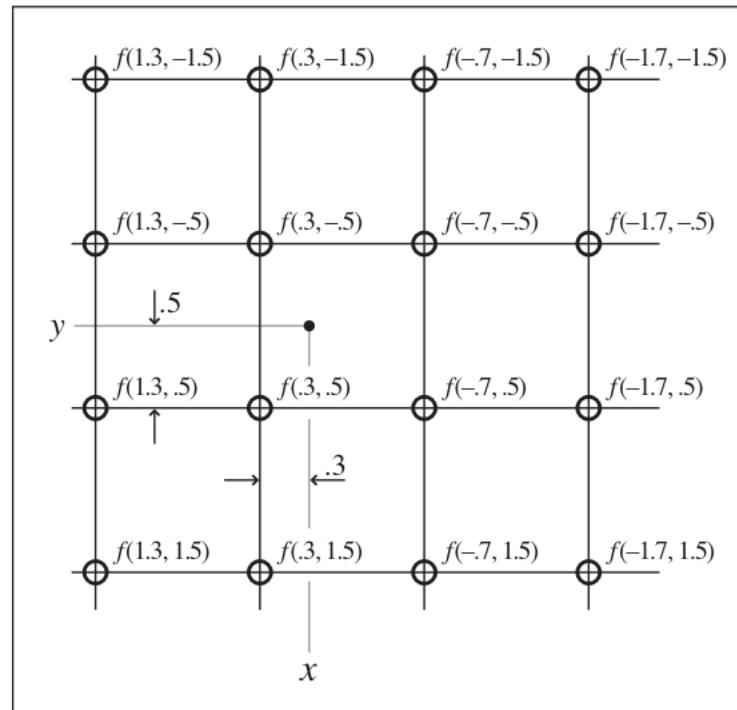
looks like discrete filter,
but offsets are not integers
and filter is continuous

remember placement of filter
relative to grid is variable



Cont.-disc. convolution in 2D

$$(a \star f)(x, y) = \sum_{i,j} a[i, j]f(x - i, y - j)$$



Separable filters for resampling

- just as in filtering, separable filters are useful
separability in this context is a statement about a continuous filter, rather than a discrete one:
$$f_2(x, y) = f_1(x)f_1(y)$$
- resample in two passes, one resampling each row and one resampling each column
- intermediate storage required: product of one dimension of source and the other dimension of destination

[Philip Greenspun]



two-stage resampling using a
separable filter



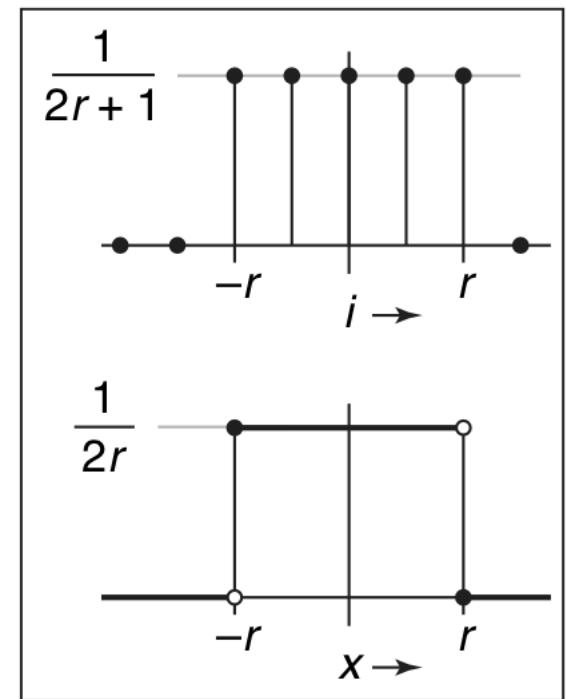
A gallery of filters

- Box filter
 - Simple and cheap
- Tent filter
 - Linear interpolation
- Gaussian filter
 - Very smooth antialiasing filter
- B-spline cubic
 - Very smooth
- Catmull-rom cubic
 - Interpolating
- Mitchell-Netravali cubic
 - Good for image upsampling

Box filter

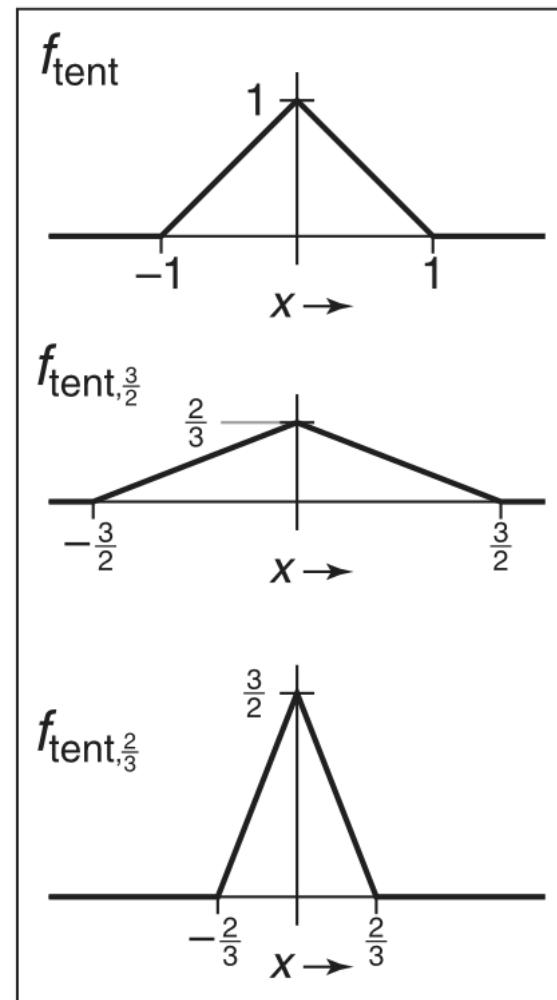
$$a_{\text{box},r}[i] = \begin{cases} 1/(2r+1) & |i| \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_{\text{box},r}(x) = \begin{cases} 1/(2r) & -r \leq x < r, \\ 0 & \text{otherwise.} \end{cases}$$

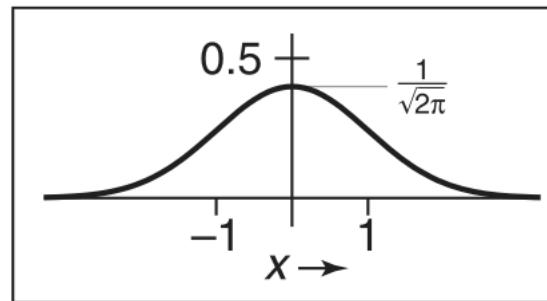


Tent filter

$$f_{\text{tent}}(x) = \begin{cases} 1 - |x| & |x| < 1, \\ 0 & \text{otherwise;} \end{cases}$$
$$f_{\text{tent},r}(x) = \frac{f_{\text{tent}}(x/r)}{r}.$$

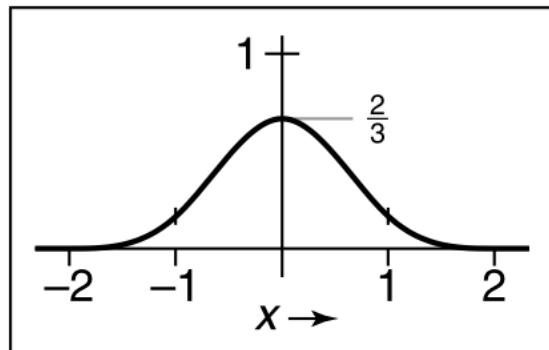


Gaussian filter



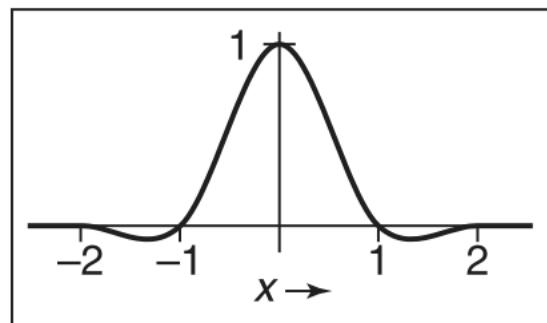
$$f_g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

B-Spline cubic



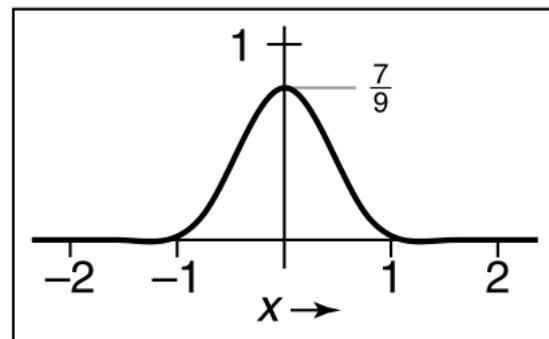
$$f_B(x) = \frac{1}{6} \begin{cases} -3(1 - |x|)^3 + 3(1 - |x|)^2 + 3(1 - |x|) + 1 & -1 \leq x \leq 1, \\ (2 - |x|)^3 & 1 \leq |x| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Catmull-Rom cubic



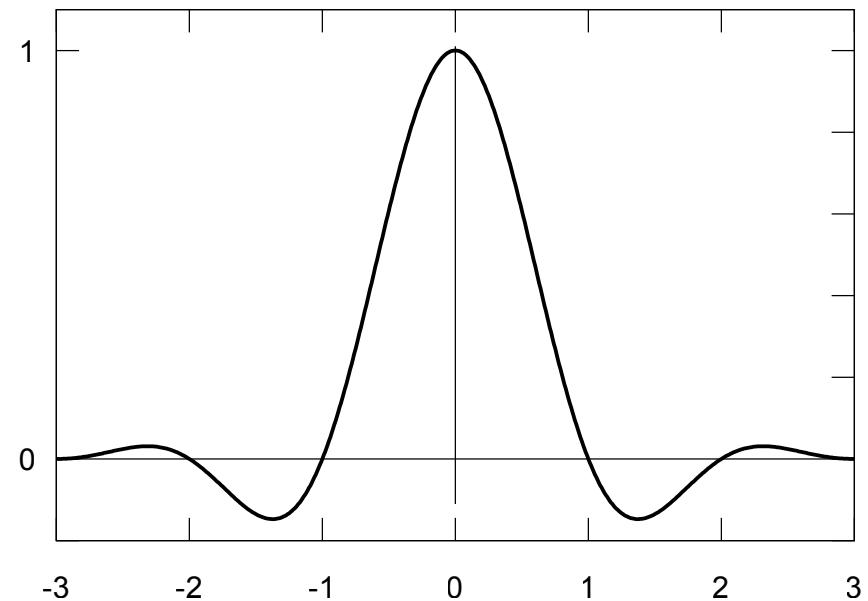
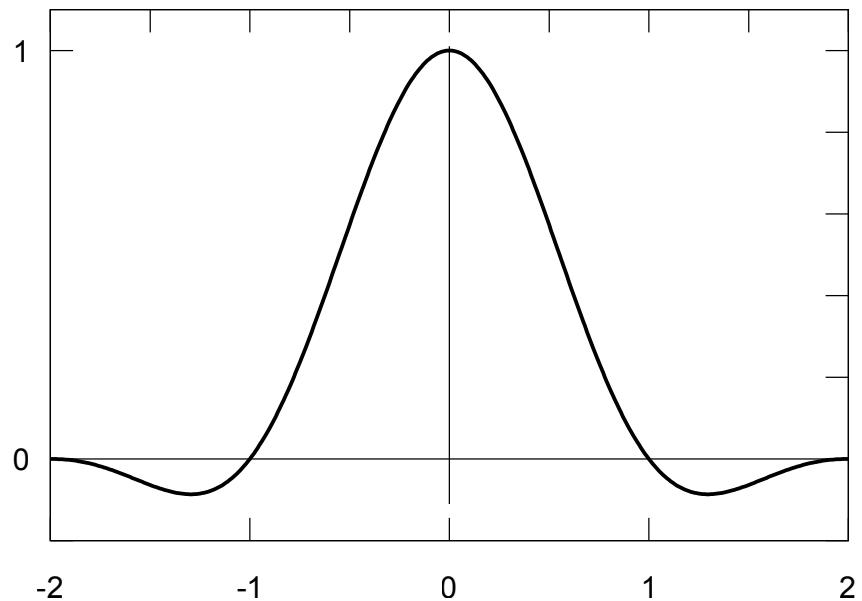
$$f_C(x) = \frac{1}{2} \begin{cases} -3(1 - |x|)^3 + 4(1 - |x|)^2 + (1 - |x|) & -1 \leq x \leq 1, \\ (2 - |x|)^3 - (2 - |x|)^2 & 1 \leq |x| \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Michell-Netravali cubic



$$\begin{aligned}f_M(x) &= \frac{1}{3}f_B(x) + \frac{2}{3}f_C(x) \\&= \frac{1}{18} \begin{cases} -21(1 - |x|)^3 + 27(1 - |x|)^2 + 9(1 - |x|) + 1 & -1 \leq x \leq 1, \\ 7(2 - |x|)^3 - 6(2 - |x|)^2 & 1 \leq |x| \leq 2, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Lanczos



$$f_{L2}(x) = \begin{cases} \text{sinc}(x) \text{sinc}(x/2) & |x| < 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

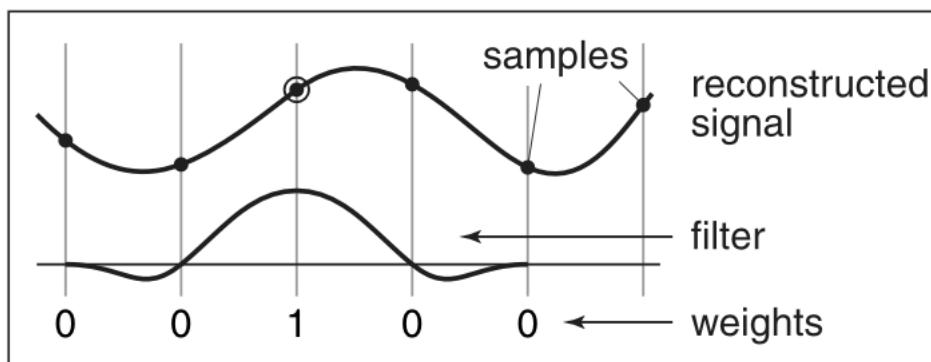
$$f_{L3}(x) = \begin{cases} \text{sinc}(x) \text{sinc}(x/3) & |x| < 3 \\ 0 & \text{otherwise} \end{cases}$$

Effects of reconstruction filters

- For some filters, the reconstruction process winds up implementing a simple algorithm
- Box filter (radius 0.5): nearest neighbor sampling
 - box always catches exactly one input point
 - it is the input point nearest the output point
 - so $\text{output}[i, j] = \text{input}[\text{round}(x(i)), \text{round}(y(j))]$
 - $x(i)$ computes the position of the output coordinate i on the input grid
- Tent filter (radius 1): linear interpolation
 - tent catches exactly 2 input points
 - weights are a and $(1 - a)$
 - result is straight-line interpolation from one point to the next

Properties of filters

- Degree of continuity
- Impulse response
- Interpolating or no
- Ringing, or overshoot

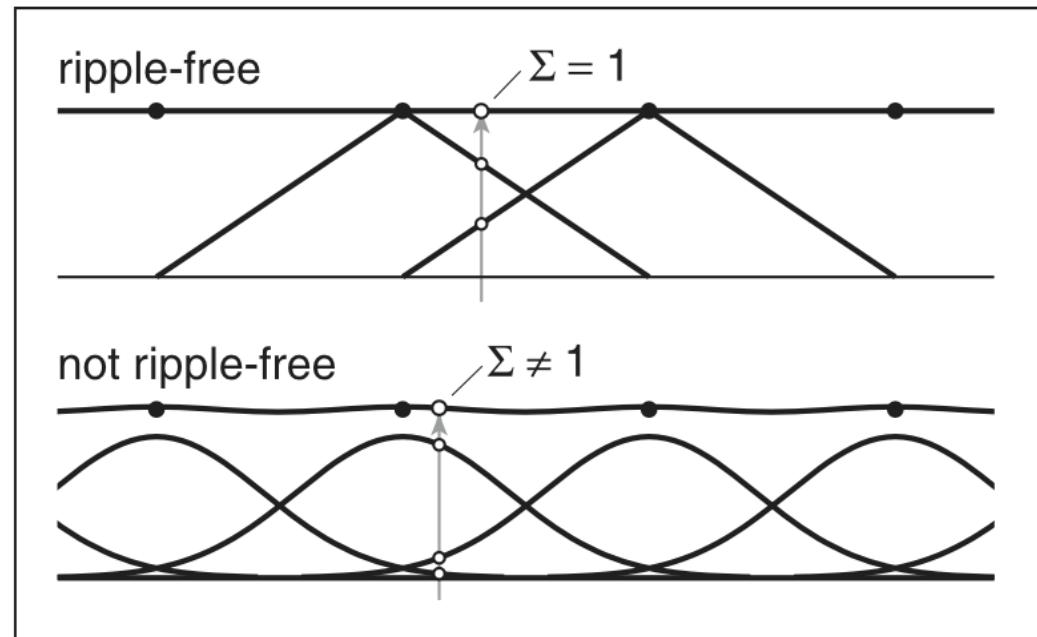
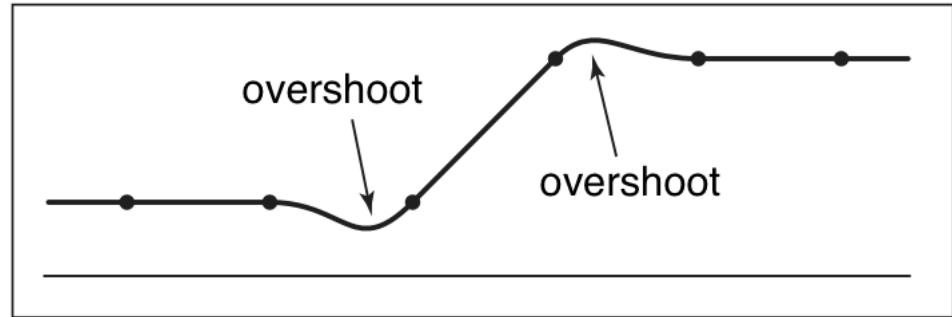


interpolating filter used for reconstruction

Ringing, overshoot, ripples

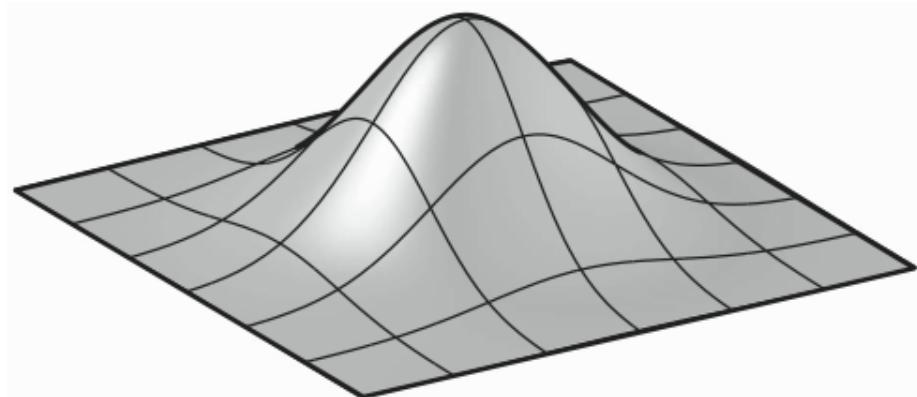
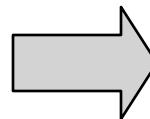
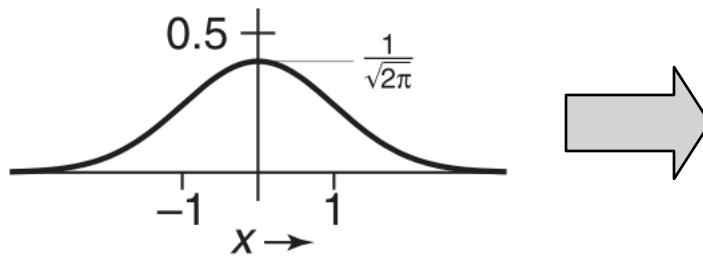
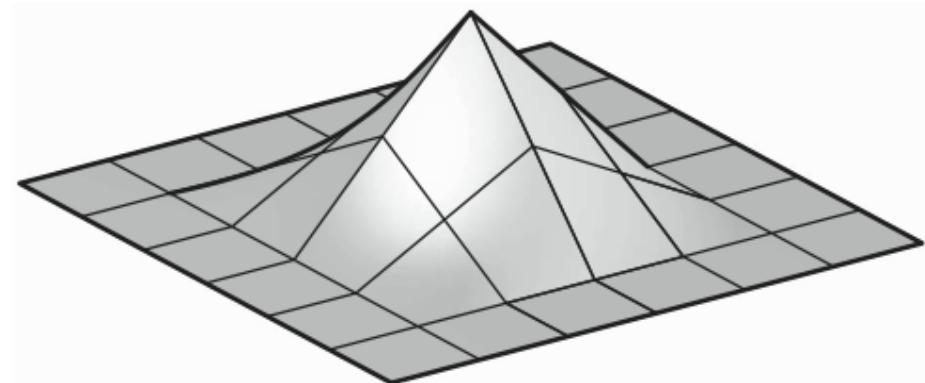
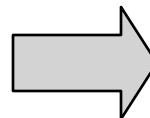
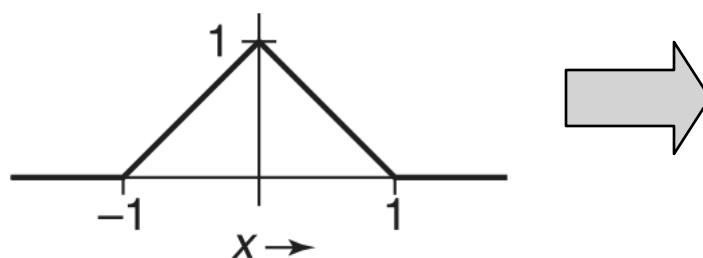
- Overshoot
 - caused by negative filter values
- Ripples
 - constant in, non-const. out
 - ripple free when:

$$\sum_i f(x + i) = 1 \quad \text{for all } x.$$



Constructing 2D filters

- Separable filters (most common approach)



Yucky details

- What about near the edge?
the filter window falls off the edge of the image

need to extrapolate

methods:

- clip filter (black)
- wrap around
- copy edge
- reflect across edge
- vary filter near edge



[Philip Greenspun]

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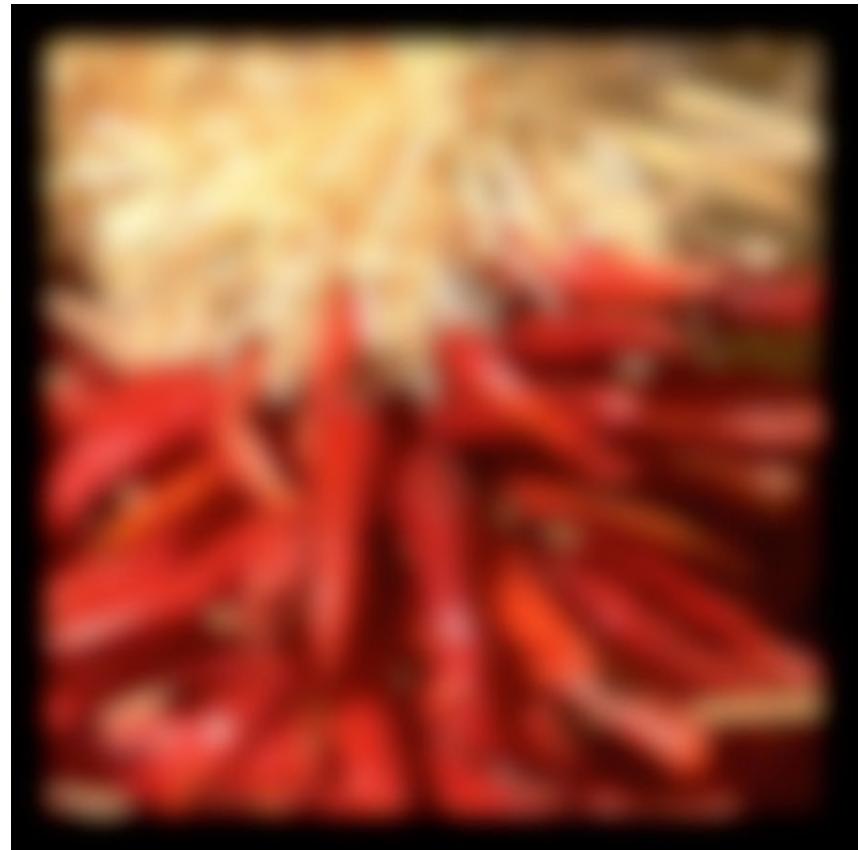
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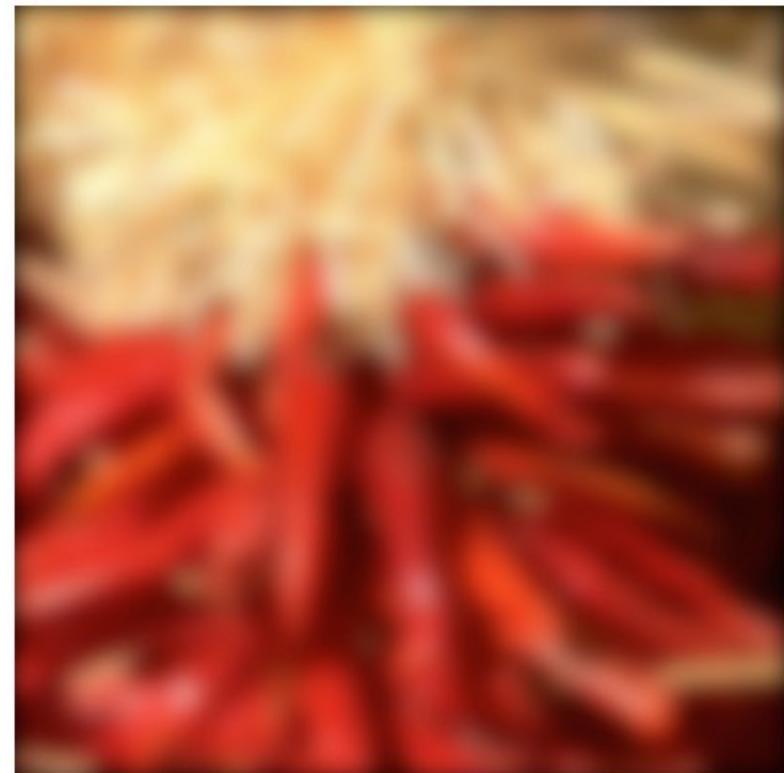
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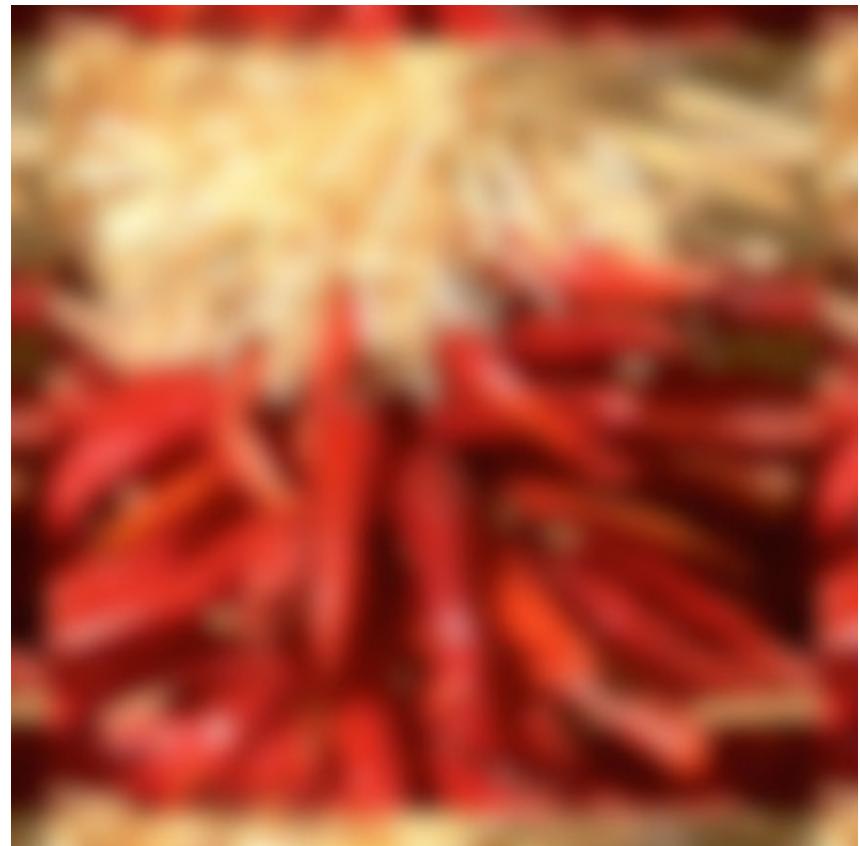
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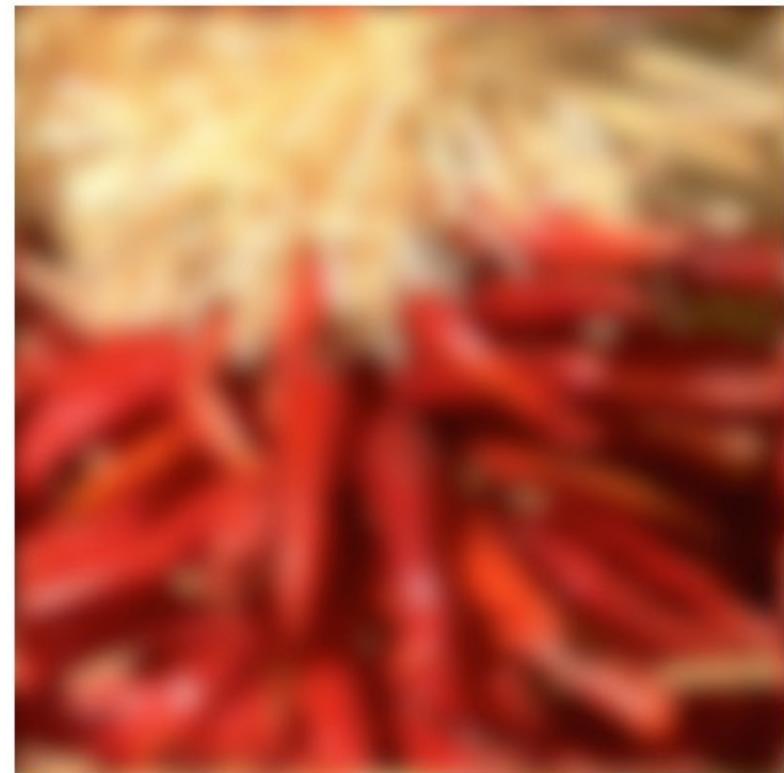
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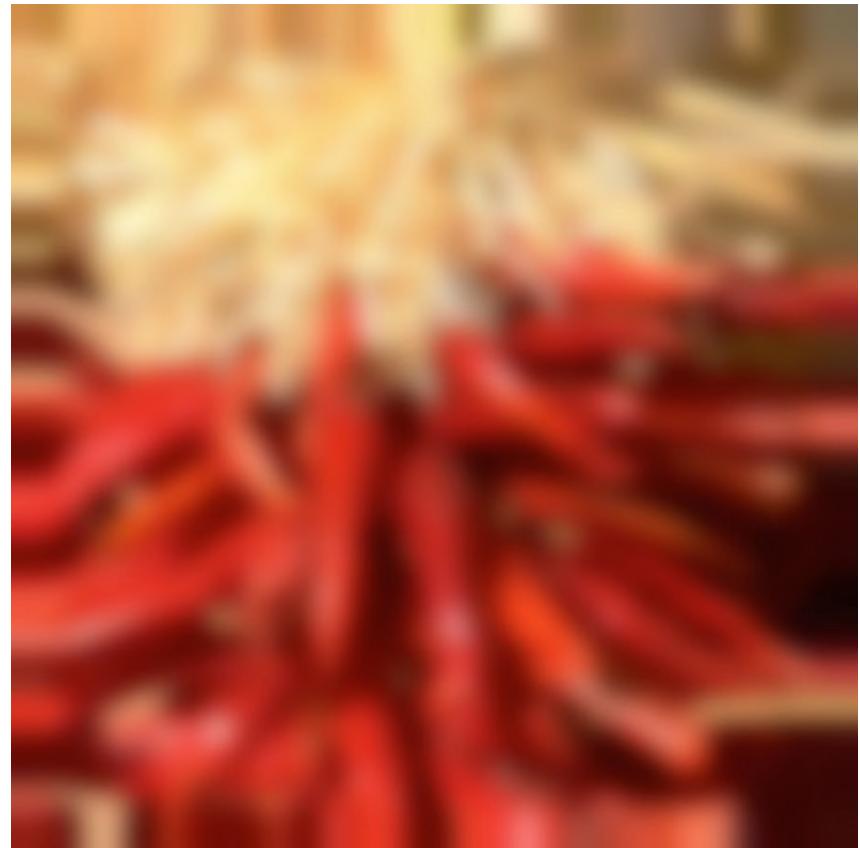
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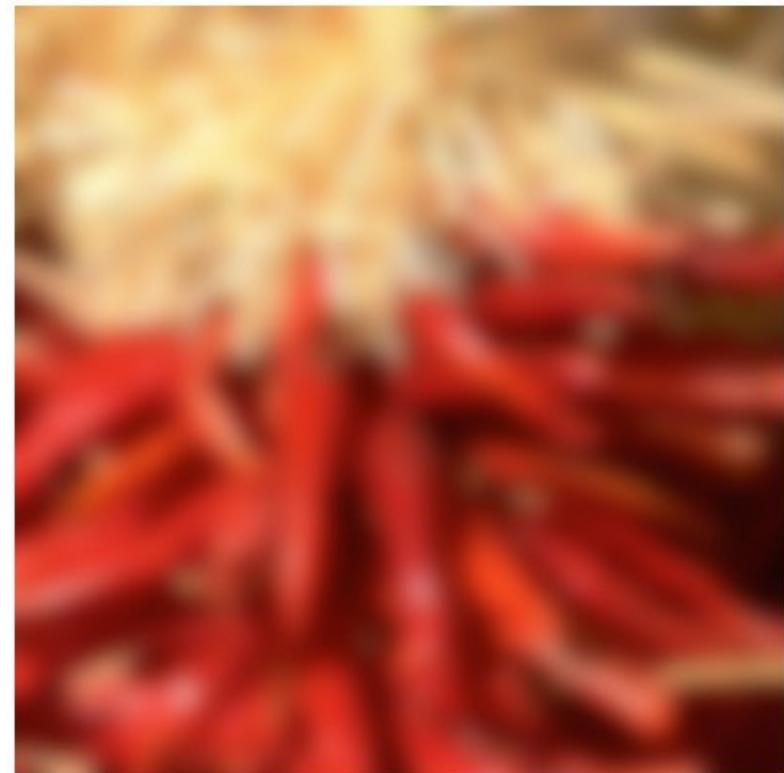
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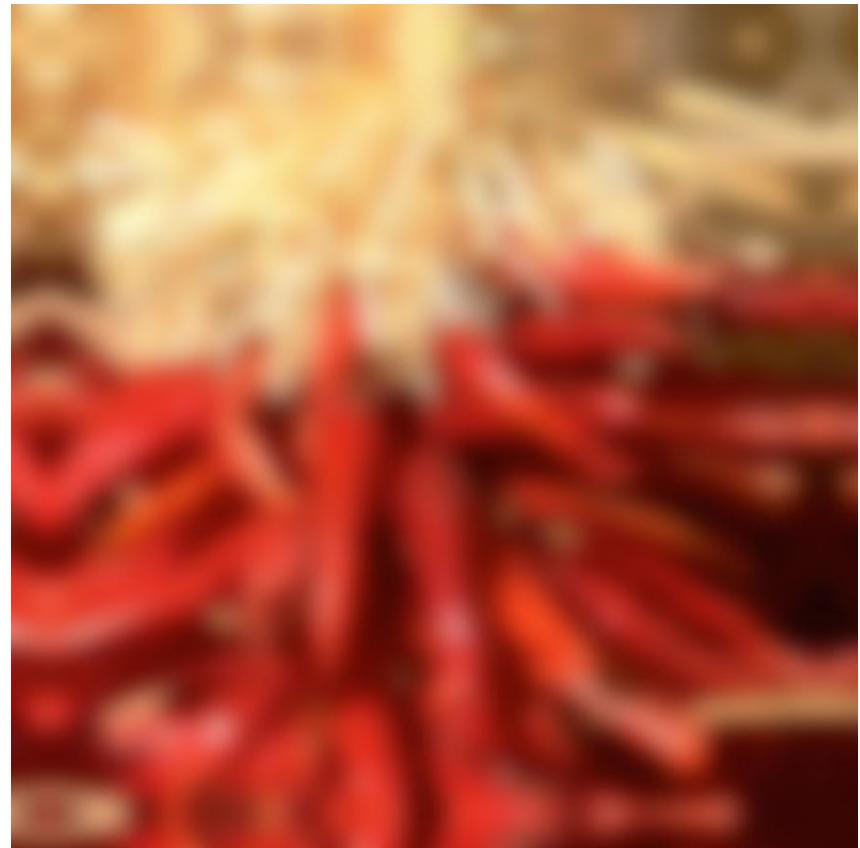
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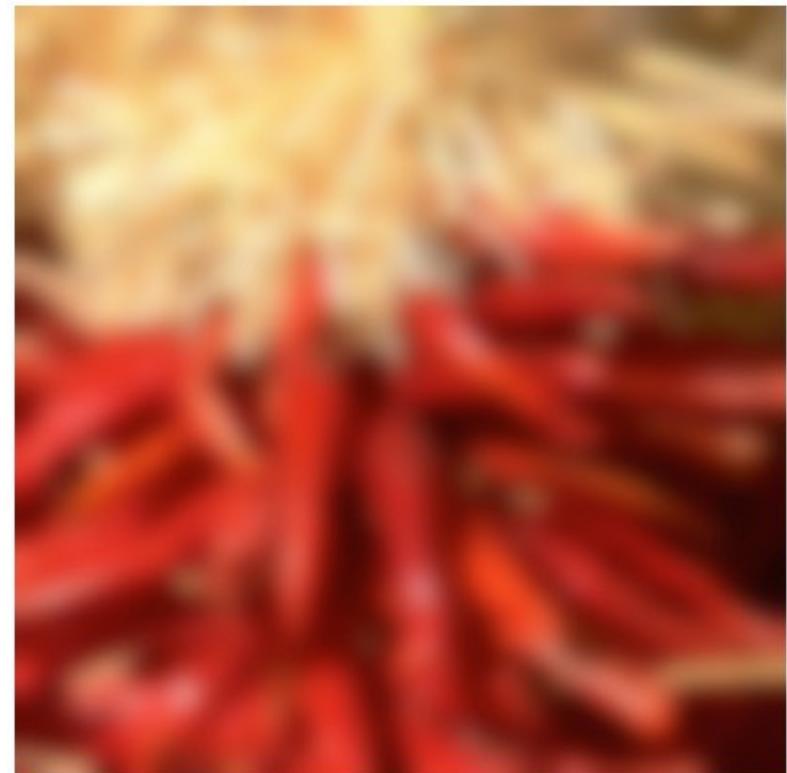
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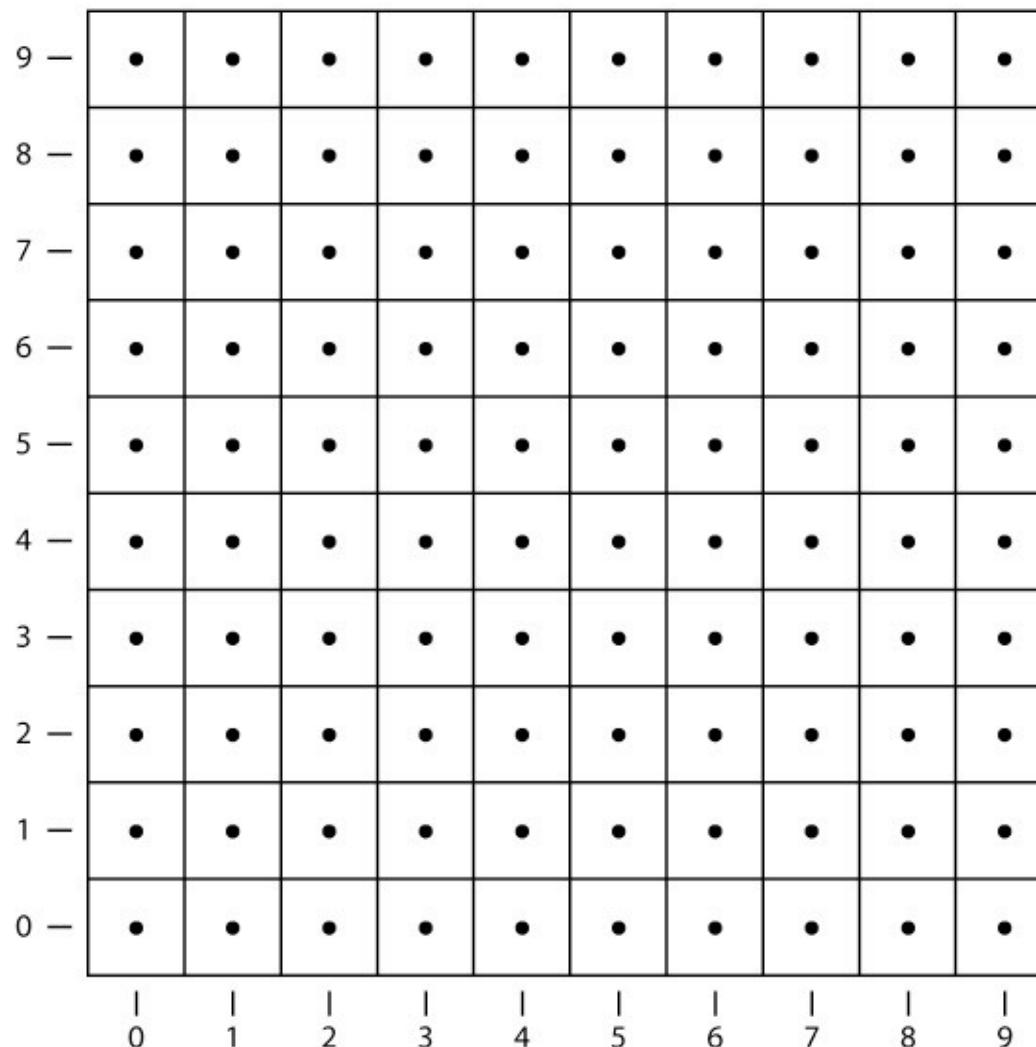


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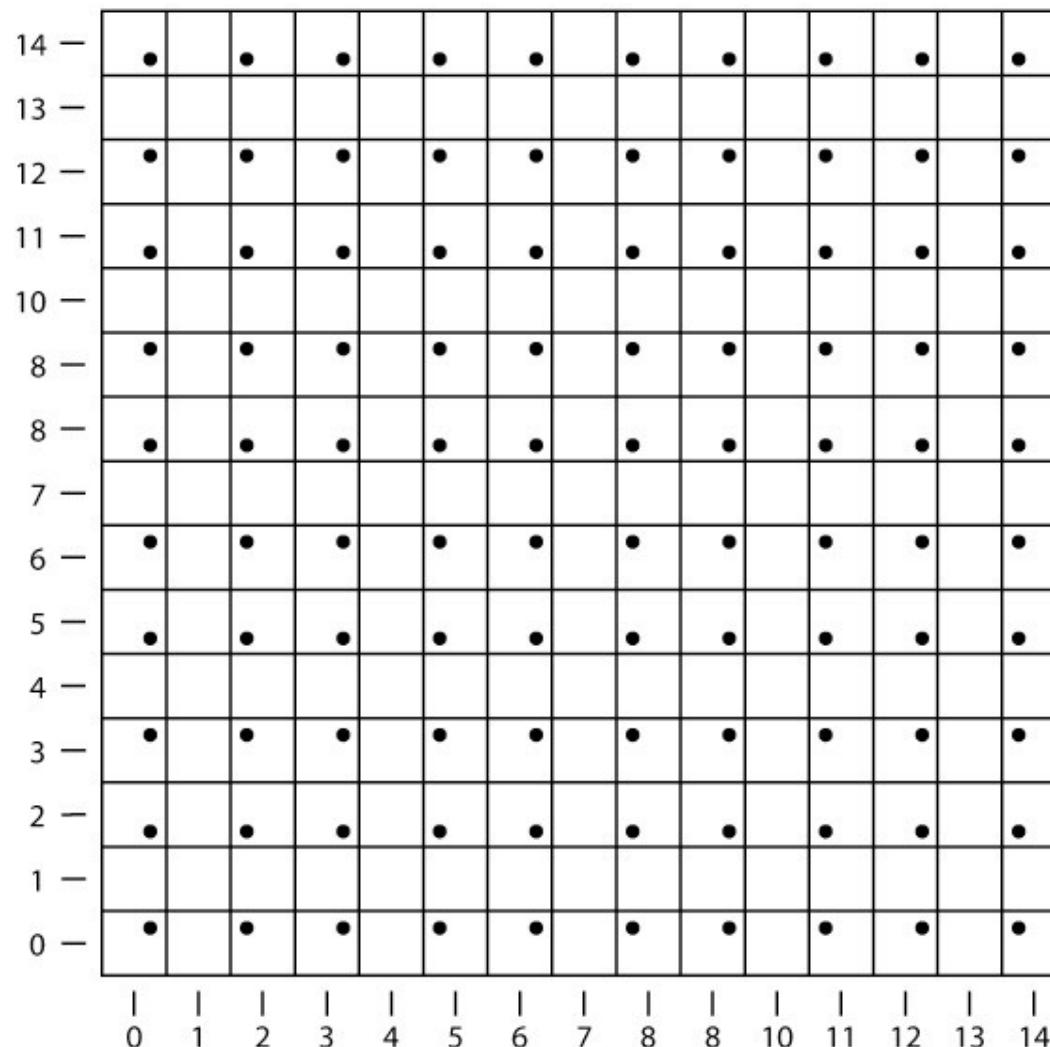
Reducing and enlarging

- Very common operation
 - devices have differing resolutions
 - applications have different memory/quality tradeoffs
- Also very commonly done poorly
- Simple approach: drop/replicate pixels
- Correct approach: use resampling

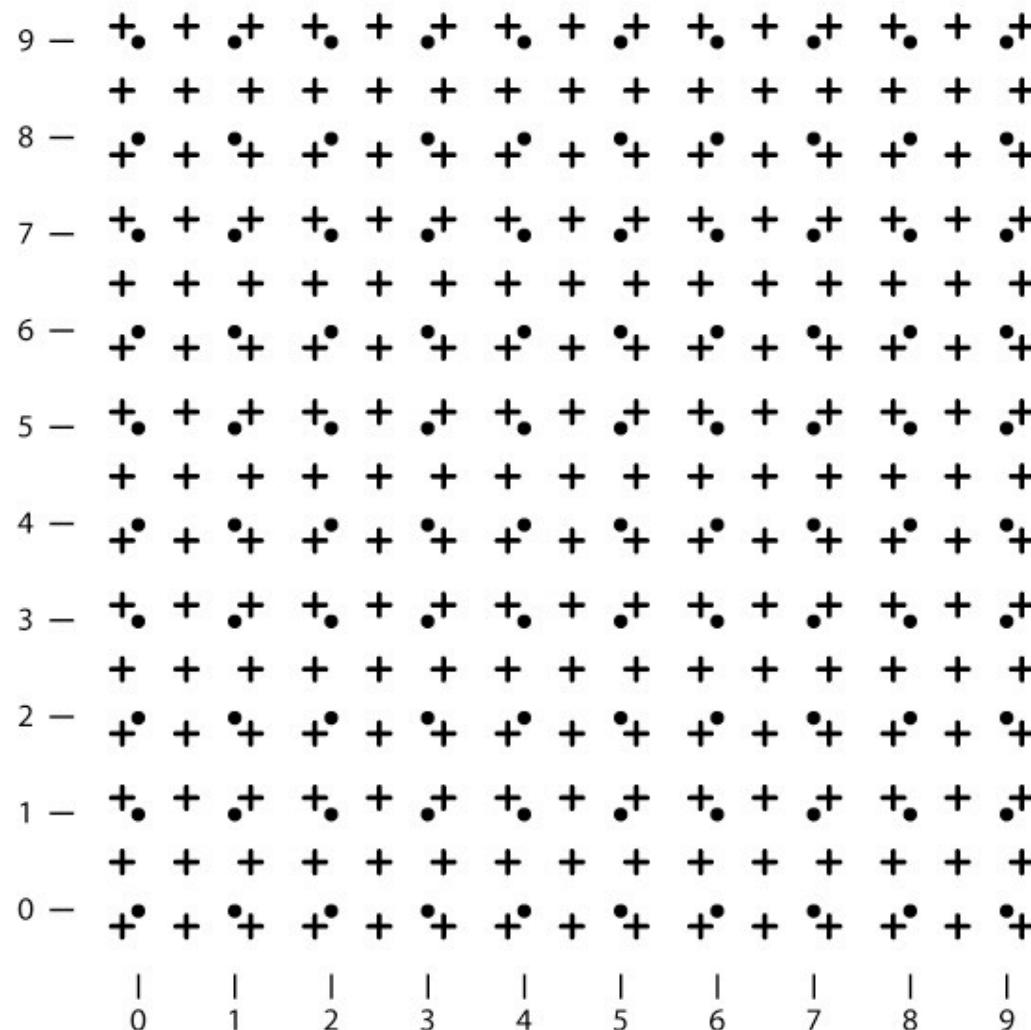
Resampling example



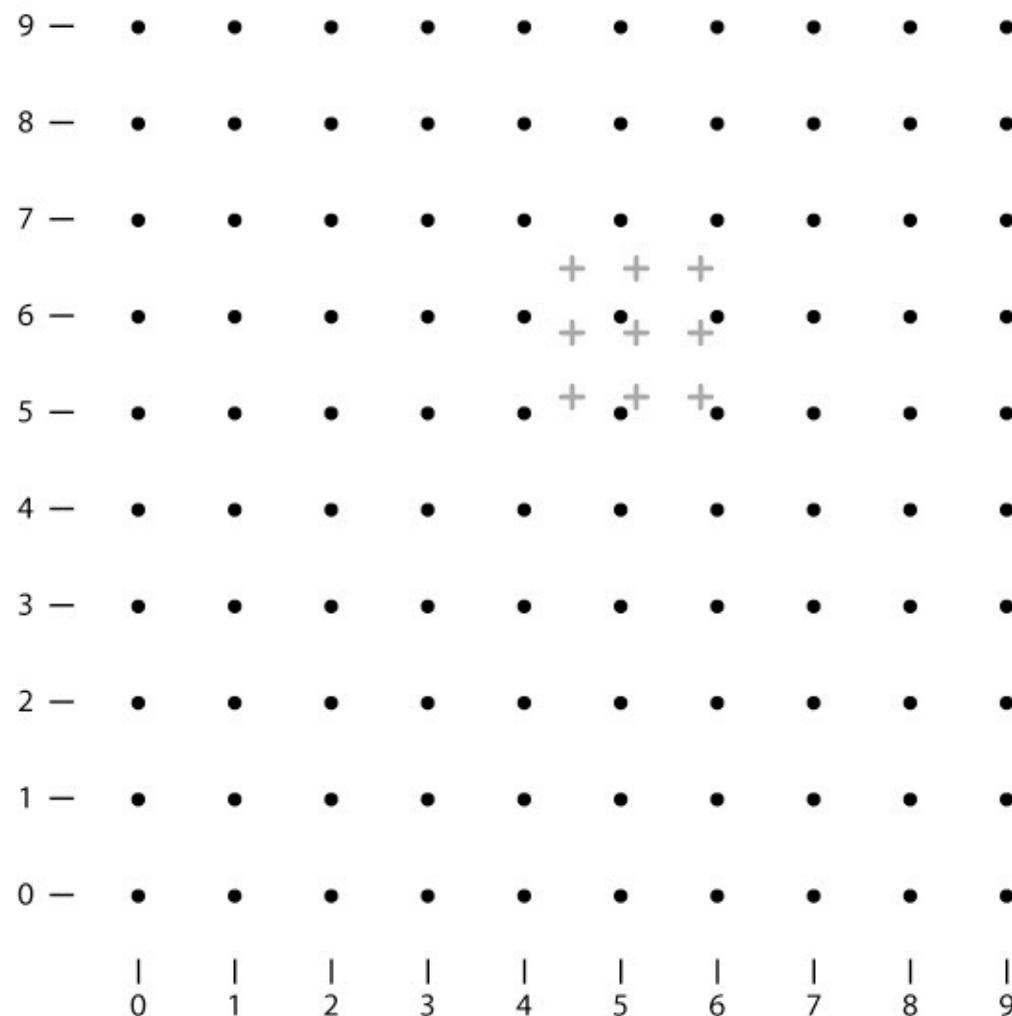
Resampling example



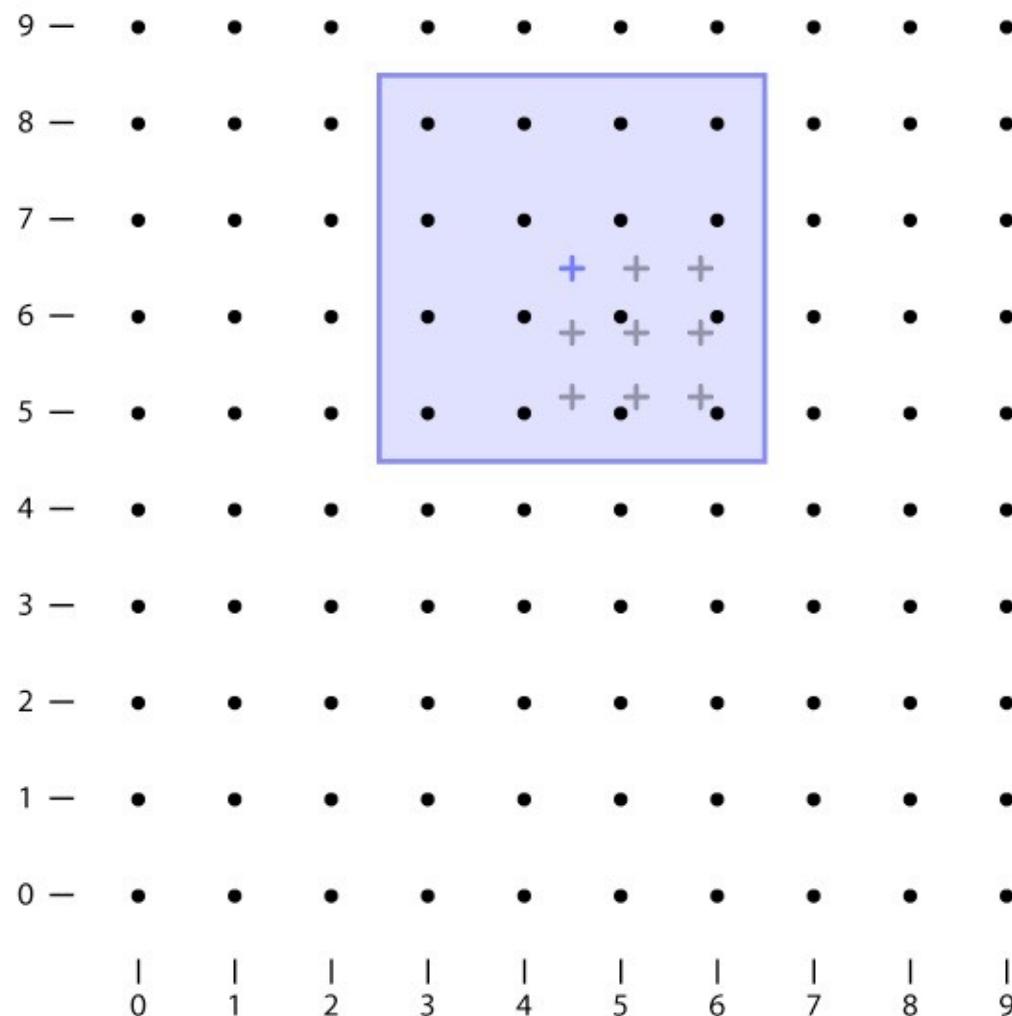
Resampling example



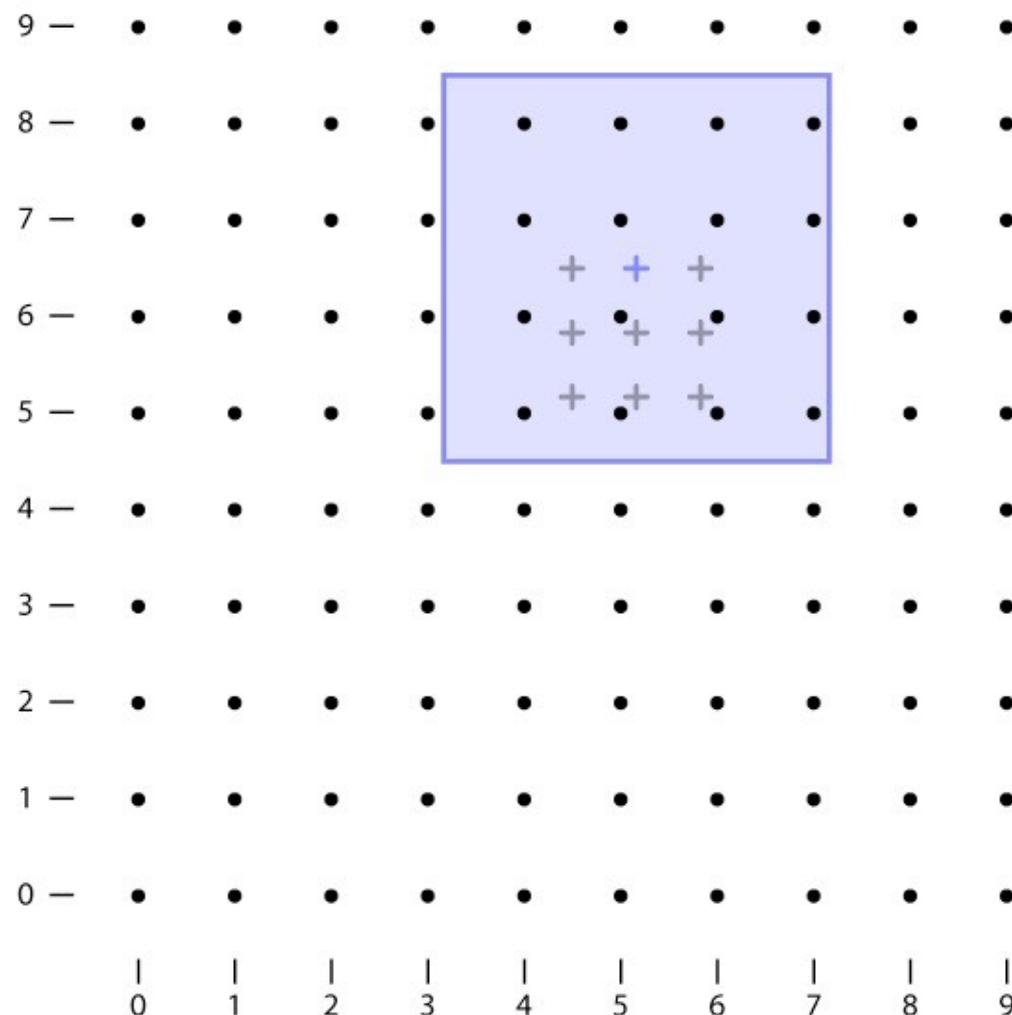
Resampling example



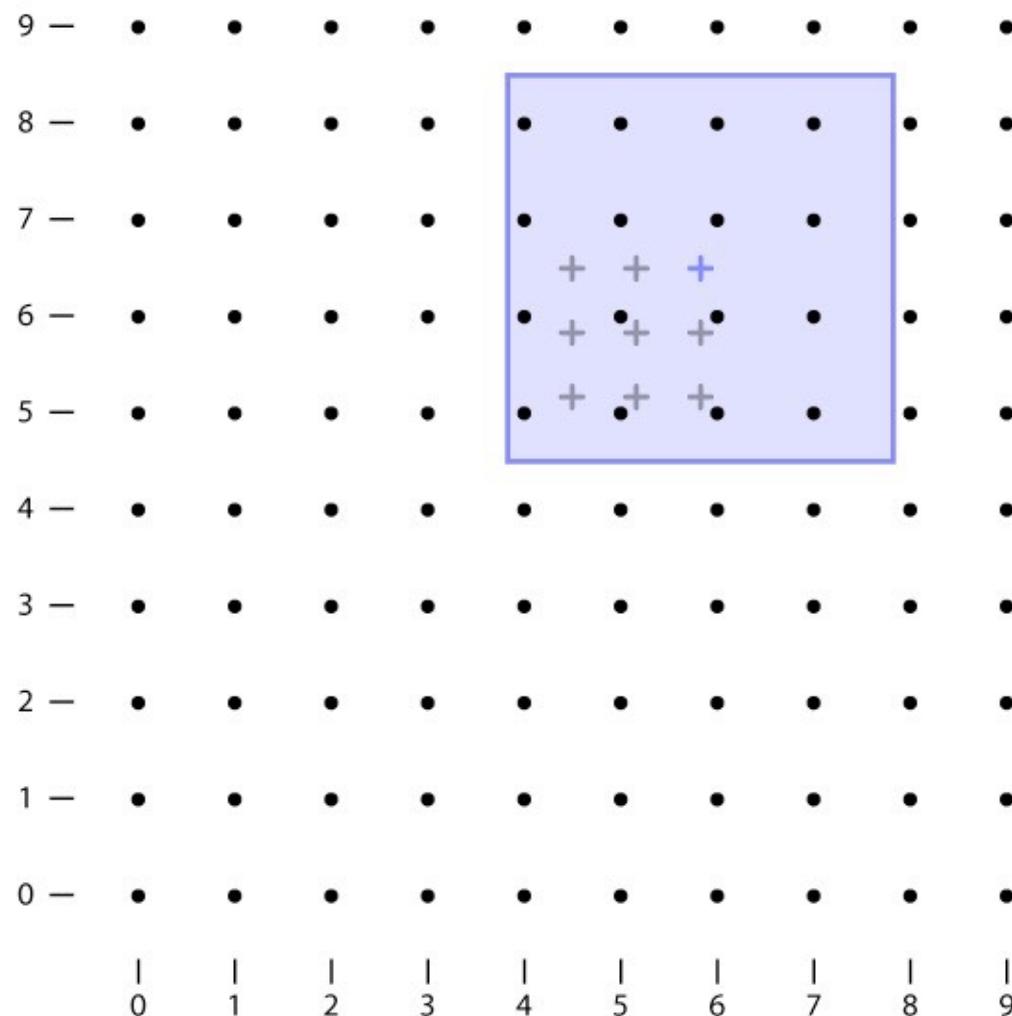
Resampling example



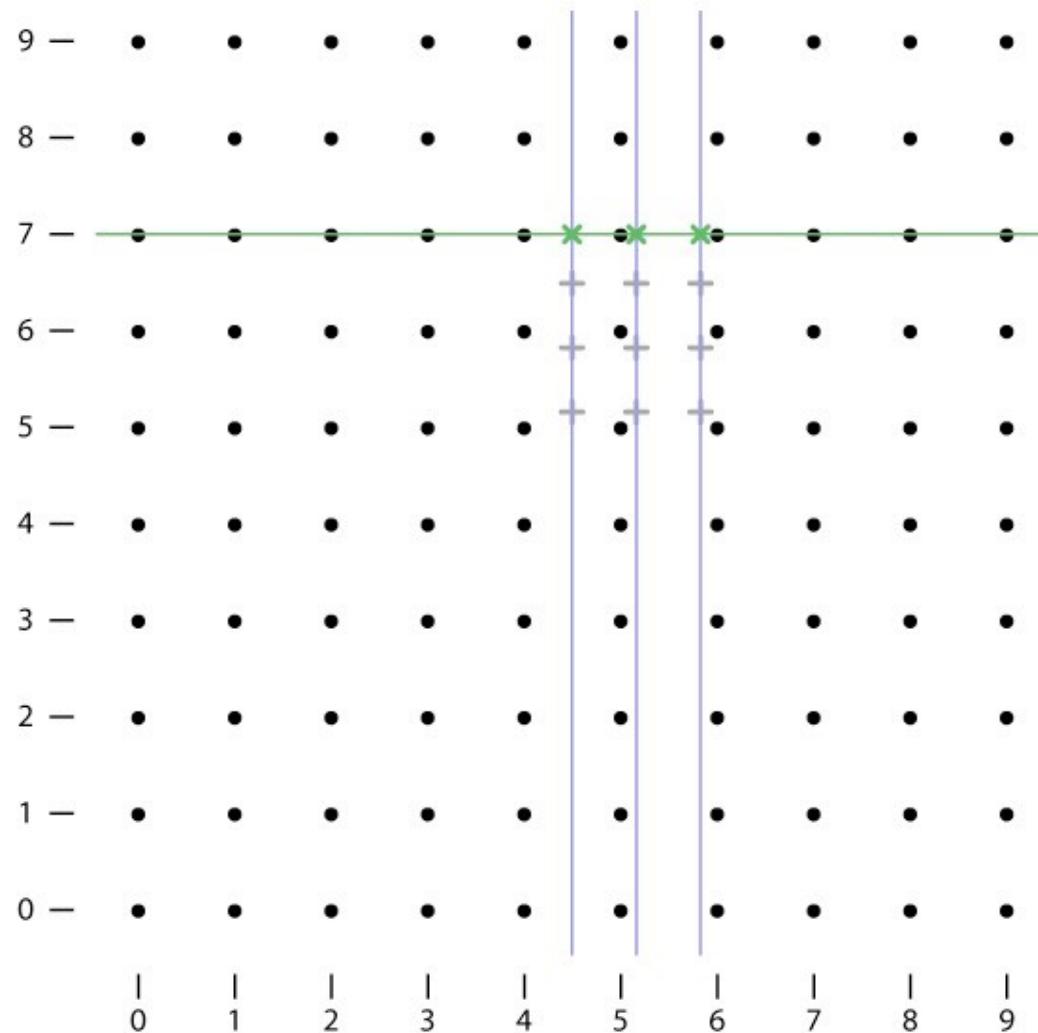
Resampling example



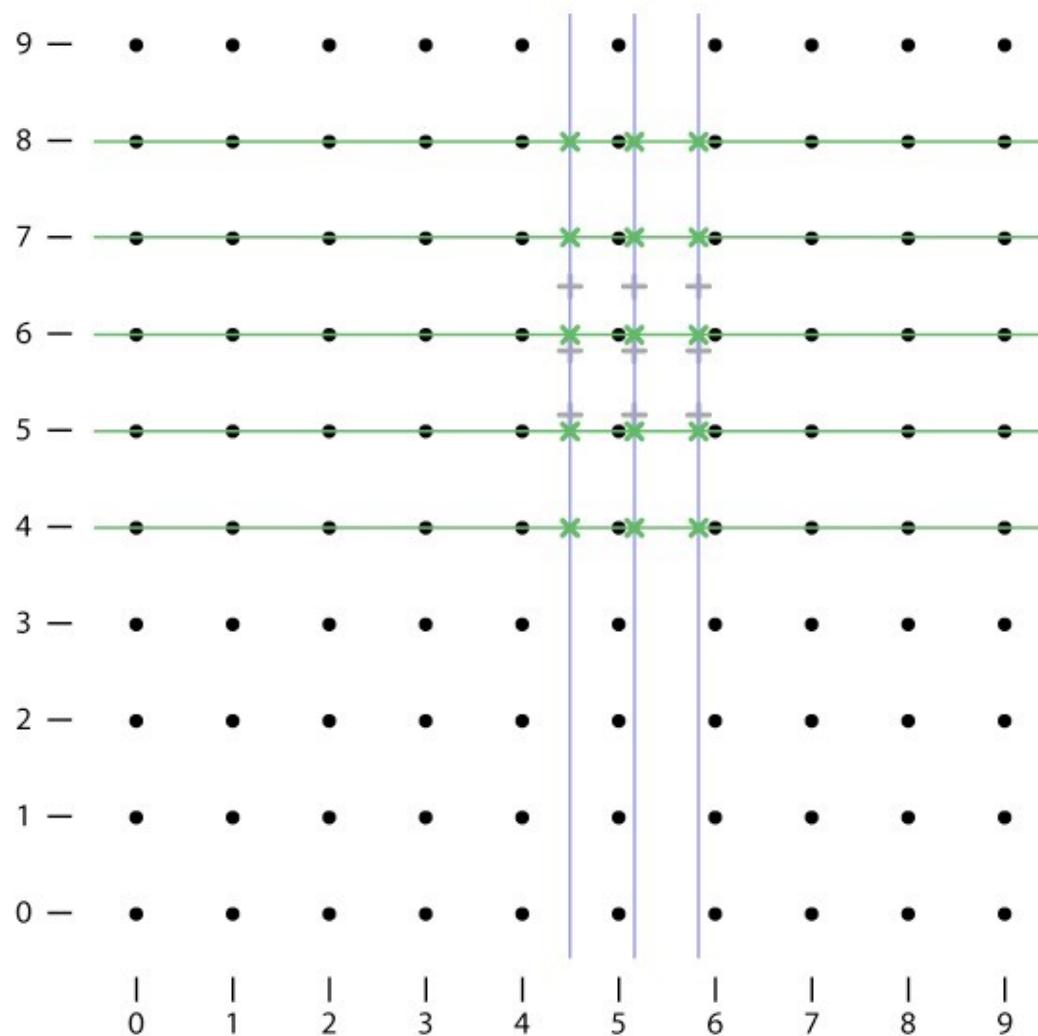
Resampling example



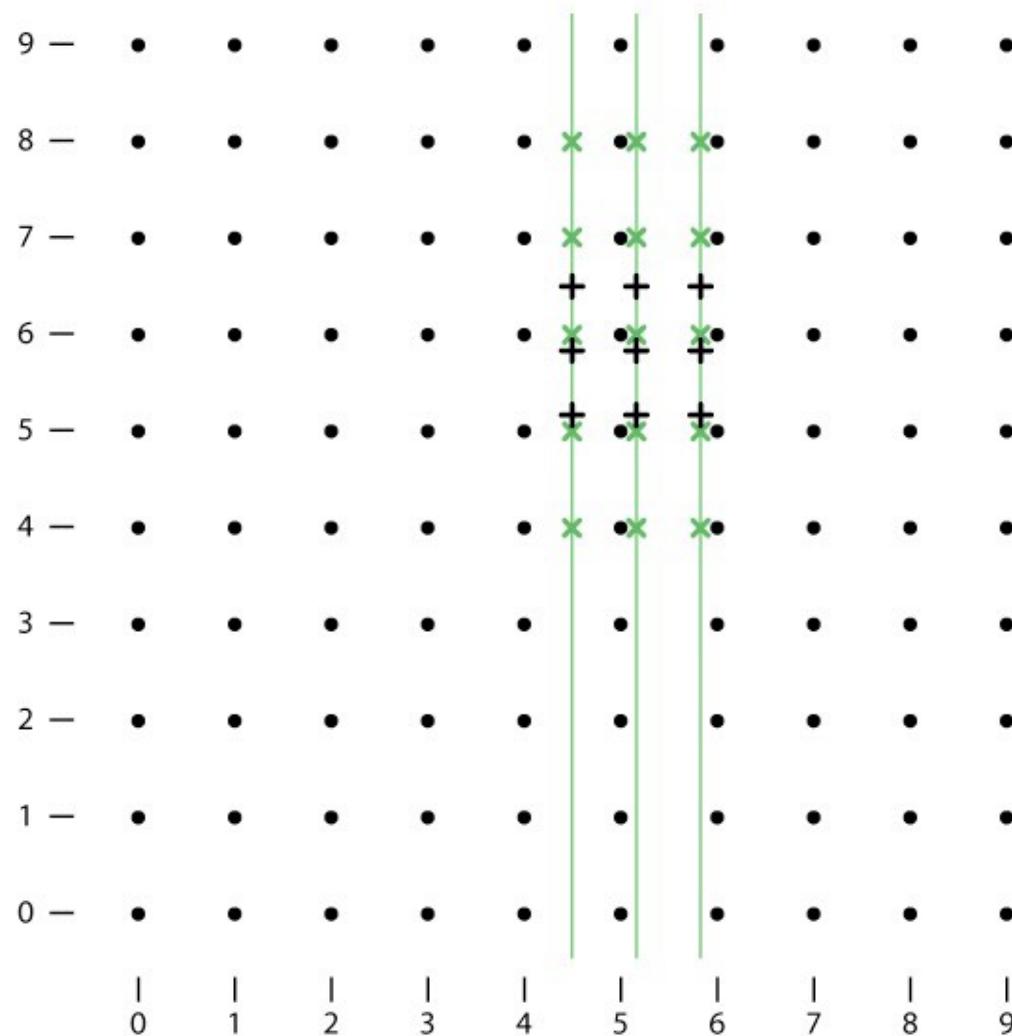
Resampling example



Resampling example

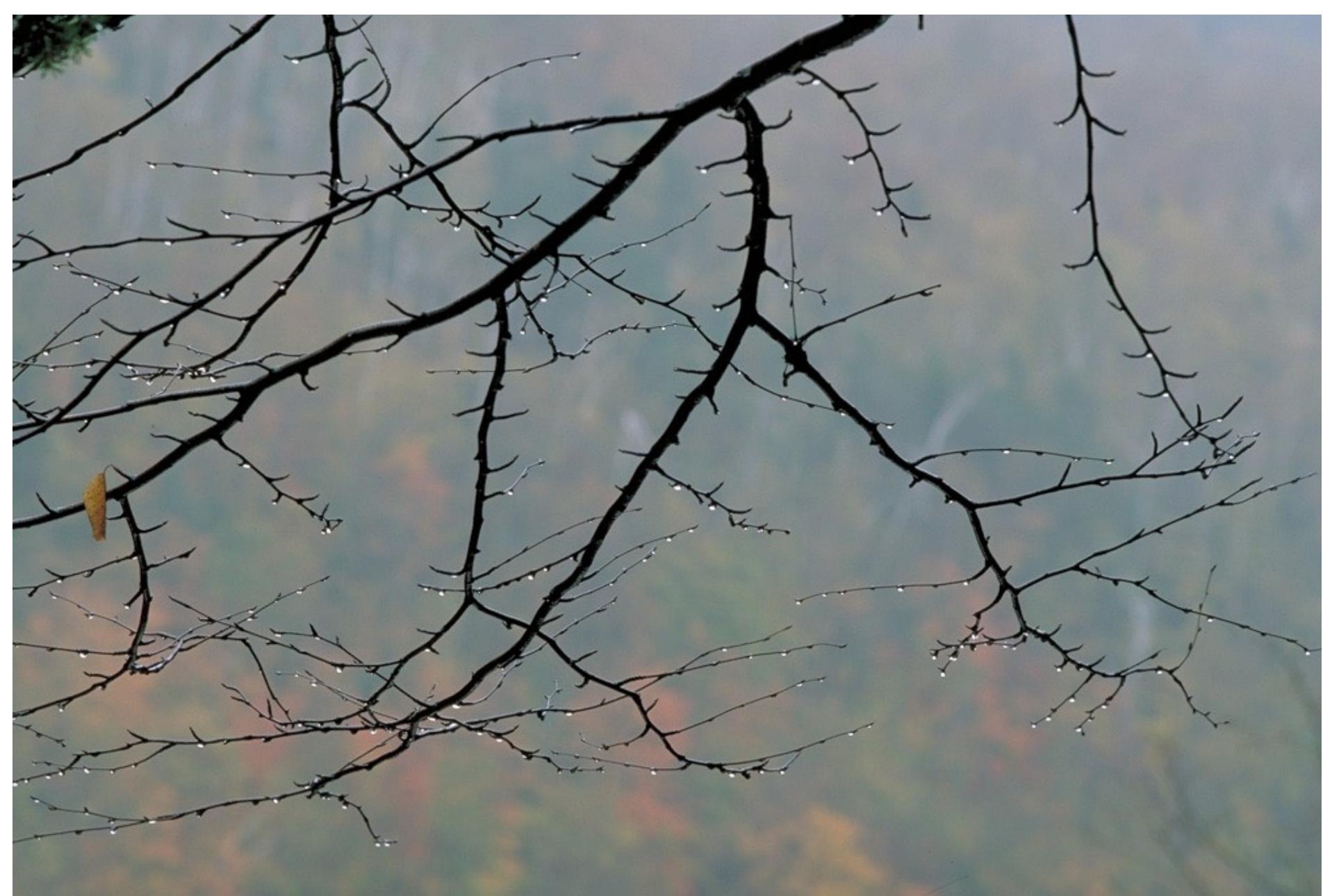


Resampling example



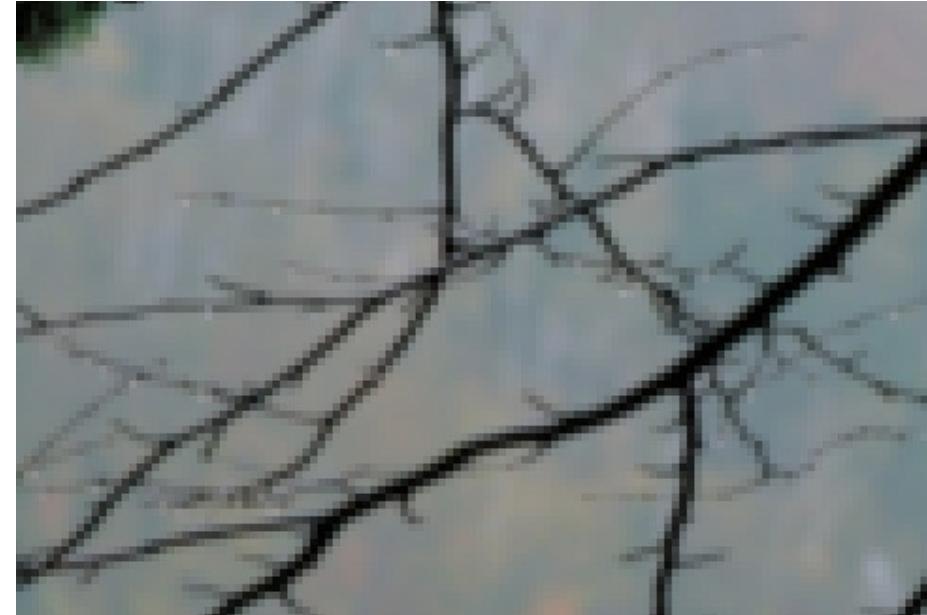
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1000 pixel width

[Philip Greenspun]



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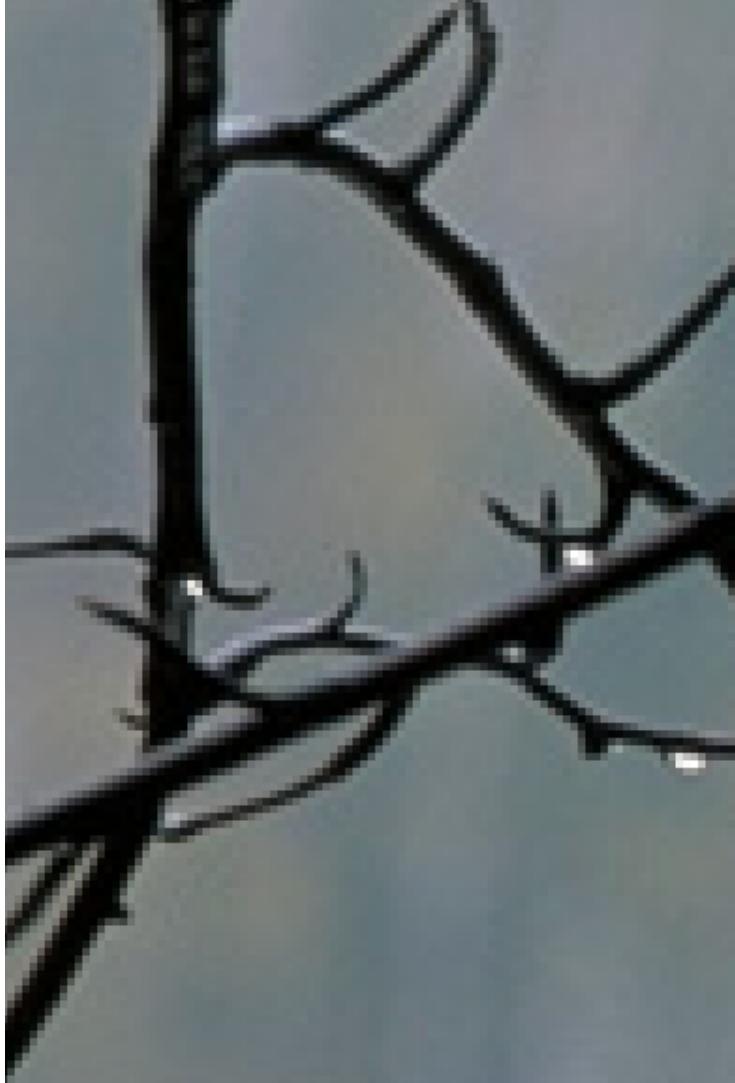


by dropping pixels

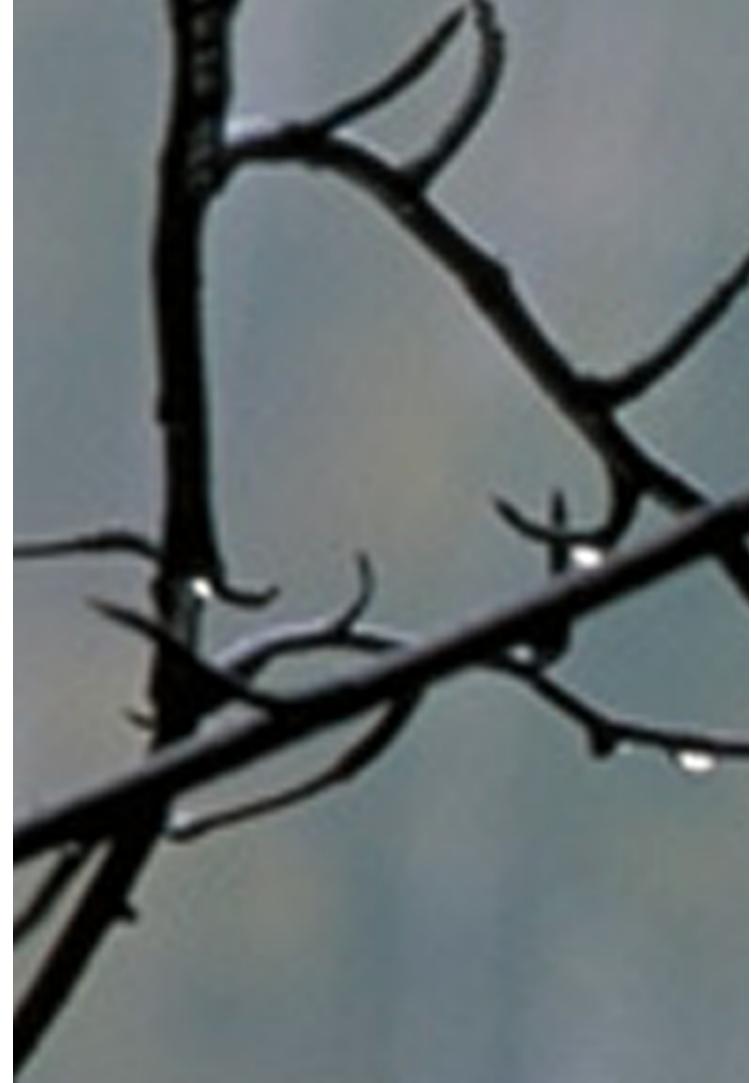


gaussian filter

250 pixel width



box reconstruction filter



bicubic reconstruction filter

4000 pixel width

[Philip Greenspun]

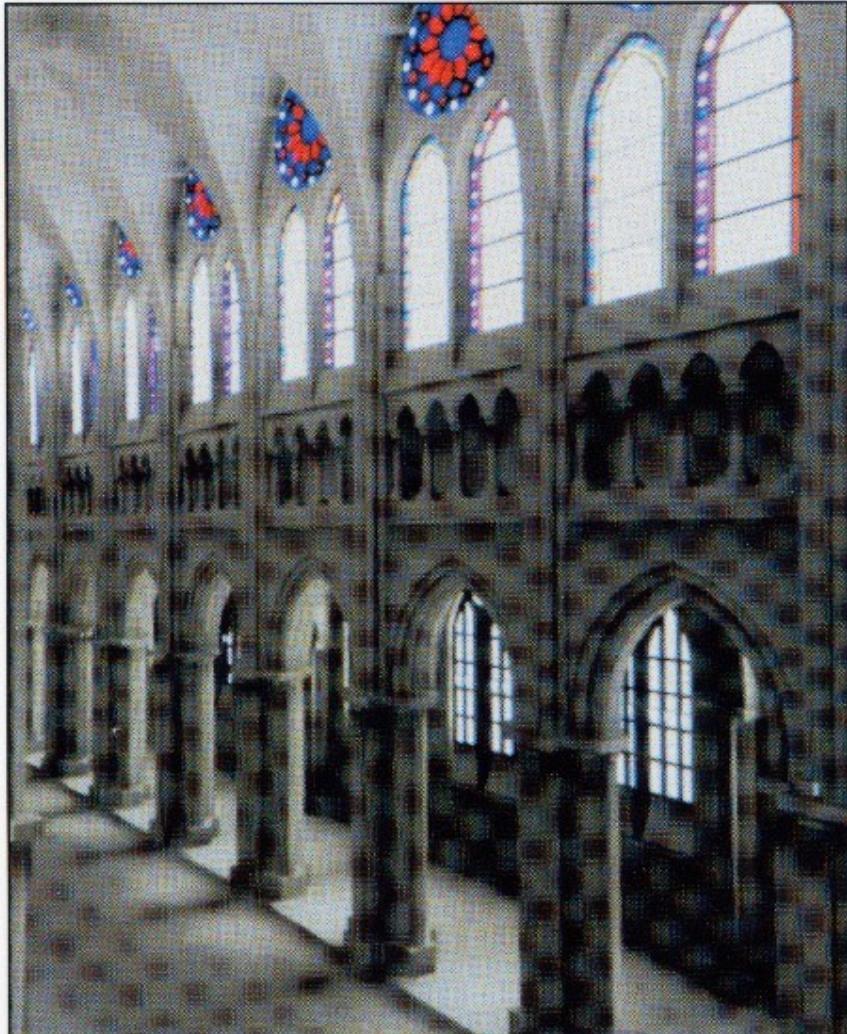
Types of artifacts

- Garden variety
 - what we saw in this natural image
 - fine features become jagged or sparkle
- Moiré patterns

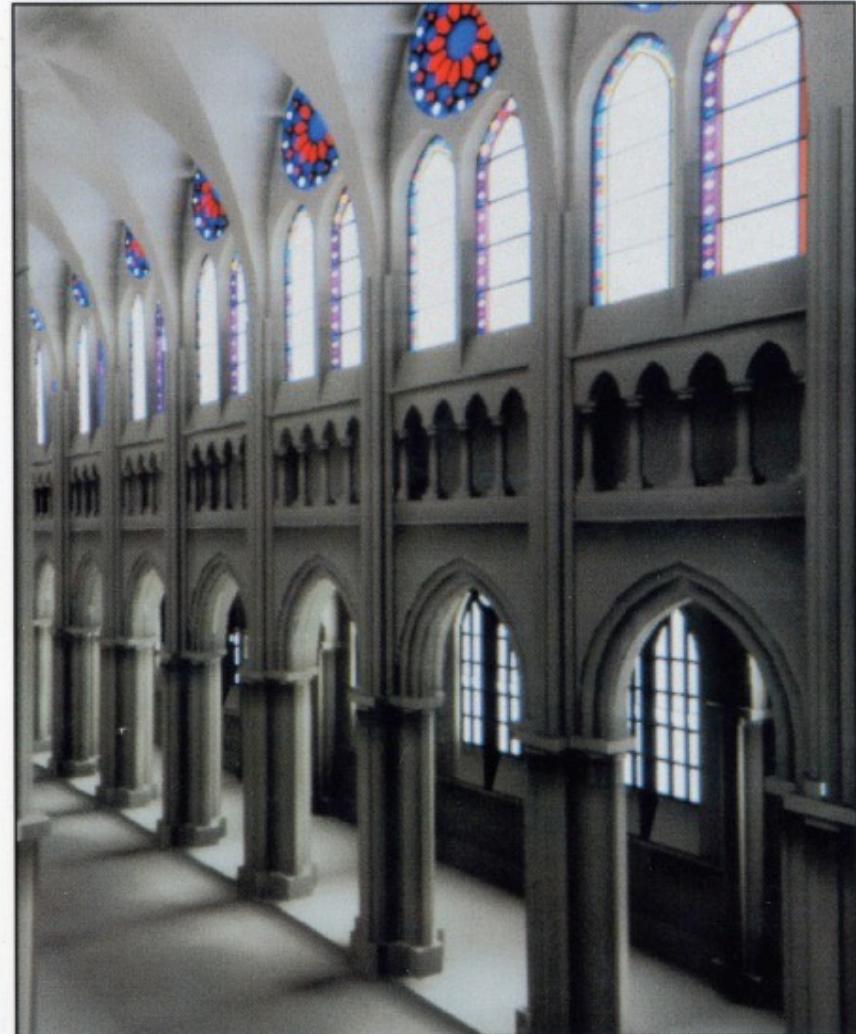


[Hearn & Baker cover]

600ppi scan of a color halftone image



by dropping pixels



gaussian filter

downsampling a high resolution scan

[Hearn & Baker cover]

Types of artifacts

- Garden variety
 - what we saw in this natural image
 - fine features become jagged or sparkle
- Moiré patterns
 - caused by repetitive patterns in input
 - produce large-scale artifacts; highly visible
- These artifacts are *aliasing* just like in the audio example earlier
- How do I know what filter is best at preventing aliasing?
 - practical answer: experience
 - theoretical answer: there is another layer of cool math behind all this
 - based on Fourier transforms
 - provides much insight into aliasing, filtering, sampling, and reconstruction