Theorems for Polynomial Interpolation

Fundamental Theorem of Algebra

Every non-zero, single-variable, degree n polynomial has, counted with multiplicity, at most n roots.

Existence and Uniqueness of Polynomial Interpolation

Given $(x_i, y_i)_{i=0}^n$, with x_i's distinct. There exists one and only polynomial $P_n(x)$ of degree $\leq n$ such that $P_n\left(x_i\right) = y_i$ for all $i = 0, 1, \cdots, n$.

Proof: Existence: by construction.

Uniqueness: Assume we have two polynomials $p(x), q(x) \in \mathcal{P}_n$, such that

$$p\left(x_i\right) = y_i, \quad q\left(x_i\right) = y_i, \quad i = 0, 1, \cdots, n$$

Now, let $g(x) = p(x) - q(x)$, a polynomial of degree $\leq n$.

$$g\left(x_i\right) = p\left(x_i\right) - q\left(x_i\right) = y_i - y_i = 0, \quad i = 0, 1, \cdots, n$$

So $g(x)$ has $n+1$ zeros. By the Fundamental Theorem of Algebra, we must have $g(x) \equiv 0$, therefore $p(x) \equiv q(x)$.

Error Theorem for Polynomial Interpolation

Given a function $f(x)$ on $x \in [a, b]$, and a set of distinct points $x_i \in [a, b]$, $i = 0, 1, \cdots, n$. Let $P_n(x) \in \mathcal{P}_n$ s.t.,

$$P_n\left(x_i\right) = f\left(x_i\right), \quad i = 0, 1, \cdots, n$$

error function : $\quad e(x) = f(x) - P_n(x), \quad x \in [a, b]$.

Theorem. There exists some value $\xi \in [a, b]$, such that

$$e(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^{n}\left(x - x_i\right), \quad \text{for all } x \in [a, b]$$

Proof. If $f \in \mathcal{P}_n$, then by Uniqueness Theorem of polynomial interpolation we must have $f(x) = P_n(x)$. Then $e(x) \equiv 0$ and the proof is trivial.

Now assume $f \notin \mathcal{P}_n$. If $x = x_i$ for some i, we have $e\left(x_i\right) = f\left(x_i\right) - P_n\left(x_i\right) = 0$, arid the result holds.

Now consider $x \neq x_i$ for any i.

$$W(x) = \prod_{i=0}^{n}\left(x - x_i\right) \quad \in \mathcal{P}_{n+1},$$

it holds

$$W\left(x_i\right)=0, \quad W(x)=x^{n+1}+\cdots, \quad W^{(n+1)}=(n+1)!$$

Fix an y such that $a \leq y \leq b$ and $y \neq x_i$ for any i. We define a constant

$$c=\frac{f(y)-P_n(y)}{W(y)}$$

and another function

$$\varphi(x)=f(x)-P_n(x)-c W(x).$$

We find all the zeros for $\varphi(x)$. We see that x_i 's are zeros since

$$\varphi\left(x_i\right)=f\left(x_i\right)-P_n\left(x_i\right)-c W\left(x_i\right)=0, \quad i=0,1, \cdots, n$$

and also y is a zero because

$$\varphi(y)=f(y)-P_n(y)-c W(y)=0$$

$$\begin{array}{rccl}
\multicolumn{4}{l}{\text{Here goes our deduction:}} \\
\varphi(x) \text{ has at least} & n+2 & & \text{zeros on } [a, b]. \\
\varphi'(x) \text{ has at least} & n+1 & & \text{zeros on } [a, b]. \\
\varphi''(x) \text{ has at least} & n & & \text{zeros on } [a, b]. \\
 & \vdots & & \\
\varphi^{(n+1)}(x) \text{ has at least} & 1 & \text{zero on } [a, b]. & \text{Call it } \xi \text{ s.t. } \varphi^{(n+1)}(\xi)=0.
\end{array}$$

So we have

$$\varphi^{(n+1)}(\xi)=f^{(n+1)}(\xi)-0-c W^{(n+1)}(\xi)=0.$$

Recall $W^{(n+1)}=(n+1)$!, we have, for every y,

$$f^{(n+1)}(\xi)=c W^{(n+1)}(\xi)=\frac{f(y)-P_n(y)}{W(y)}(n+1)!.$$

Writing y into x, we get

$$e(x)=f(x)-P_n(x)=\frac{1}{(n+1)!} f^{(n+1)}(\xi) W(x)=\frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^n\left(x-x_i\right),$$

for some $\xi \in[a, b]$.

Example of Error Formula

Recall the error formula: $\quad e(x)=\frac{1}{(n+1)!} f^{(n+1)}(\xi) \prod_{i=0}^n\left(x-x_i\right)$

Example 1. If $n=1, x_0=a, x_1=b, b>a$, find an upper bound for error.

Answer. Let

$$M=\max _{a \leq x \leq b}\left|f^{\prime \prime}(x)\right|=\left\|f^{\prime \prime}\right\|_{\infty}$$

and observe

$$\max_{a \le x \le b} |(x-a)(x-b)| = \cdots = \frac{(b-a)^2}{4}.$$

For $x \in [a, b]$, we have

$$|e(x)| = \frac{1}{2}|f''(\xi)| \cdot |(x-a)(x-b)| \le \frac{1}{2}\|f''\|_\infty \frac{(b-a)^2}{4} = \frac{1}{8}\|f''\|_\infty (b-a)^2.$$

Error depends on the distribution of nodes x_i.

Uniform Grid.

Equally distribute the nodes (x_i) : on $[a, b]$, with $n+1$ nodes.

$$x_i = a + ih, \quad h = \frac{b-a}{n}, \quad i = 0, 1, \cdots, n.$$

One can show that for $x \in [a, b]$, it holds

$$\prod_{i=0}^{n} |x - x_i| \le \frac{1}{4} h^{n+1} \cdot n!$$

Proof. If $x = x_i$ for some i, then $x - x_i = 0$ and the product is 0 , so it trivially holds. Now assume $x_i < x < x_{i+1}$ for some i. We have

$$\max_{x_i < x < x_{i+1}} \left|(x - x_i)\left(x - x_{i+1}\right)\right| = \frac{1}{4}(x_{i+1} - x_i)^2 = \frac{h^2}{4}.$$

Now consider the other terms in the product, say $x - x_j$, for either $j > i+1$ or $j < i$. Then $|x - x_j| \le h(j-i)$ for $j > i+1$ and $|x - x_j| \le h(i+1-j)$ for $j < i$. In all cases, the product of these terms are bounded by $h^{n-1} n!$, proving the result.

We have the error estimate

$$|e(x)| \le \frac{1}{4(n+1)}\left|f^{(n+1)}(x)\right| h^{n+1} \le \frac{M_{n+1}}{4(n+1)} h^{n+1}$$

where

$$M_{n+1} = \max_{x \in [a,b]} \left|f^{(n+1)}(x)\right| = \left\|f^{(n+1)}\right\|_\infty$$

Chebychev nodes: equally distributing the error

Type I: including the end points. For interval $[-1, 1] \quad : \quad \bar{x}_i = \cos\left(\frac{i}{n}\pi\right), \quad i = 0, 1, \cdots, n$ For interval $[a, b] \quad : \quad \bar{x}_i = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)\cos\left(\frac{i}{n}\pi\right), \quad i = 0, 1, \cdots,$ One can show that

$$\max_{a \le x \le b}\left\{\prod_{k=0}^{n} |x - \bar{x}_k|\right\} = 2^{-n} \le \max_{a \le x \le b}\left\{\prod_{k=0}^{n} |x - x_k|\right\}$$

where x_k is any other choice of nodes.

Error bound: $|e(x)| \leq \frac{1}{(n+1)!} ||f^{(n+1)}(x)||_\infty 2^{-n}$.

Splines (Optional)

Intro

General Idea: Using iecewise polynomials to approximate the function.

Given a set of data

x	t_0	t_1	$\cdots$	t_n
y	y_0	y_1	$\cdots$	y_n

Find a function $\mathcal{S}(x)$ which interpolates the points $(t_i, y_i)_{i=0}^n$. The set $t_0 < t_1 < \cdots < t_n$ are called knots. Note that they need to be ordered. $\mathcal{S}(x)$ consists of piecewise polynomials

$$\mathcal{S}(x) \doteq \begin{cases} \mathcal{S}_0(x), & t_0 \leq x \leq t_1 \\ \mathcal{S}_1(x), & t_1 \leq x \leq t_2 \\ \vdots & \\ \mathcal{S}_{n-1}(x), & t_{n-1} \leq x \leq t_n \end{cases}$$

$\mathcal{S}(x)$ is called a spline of degree k, if

- $\mathcal{S}_i(x)$ is a polynomial of degree k;
- $\mathcal{S}(x)$ is $(k-1)$ times continuous differentiable, i.e., for $i = 1, 2, \cdots, k-1$ we have

$$\begin{gathered} \mathcal{S}_{i-1}(t_i) = \mathcal{S}_i(t_i), \\ \mathcal{S}'_{i-1}(t_i) = \mathcal{S}'_i(t_i), \\ \vdots \\ \mathcal{S}_{i-1}^{(k-1)}(t_i) = \mathcal{S}_i^{(k-1)}(t_i), \end{gathered}$$

Commonly used splines:

- $n = 1$: linear spline (simplest)
- $n = 2$: quadratic spline (less popular)
- $n = 3$: cubic spline (most used)

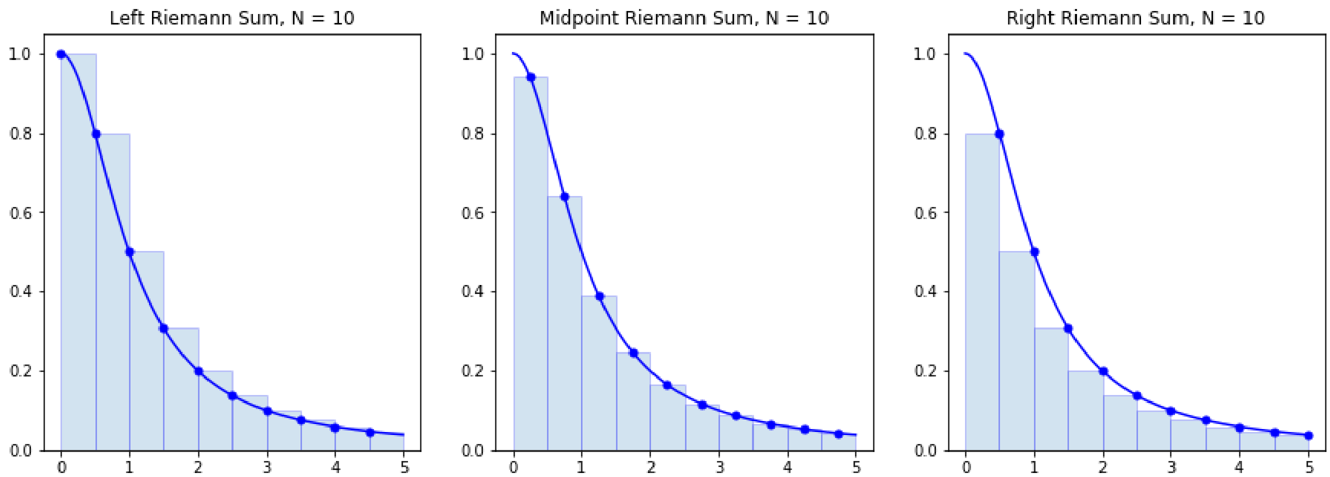
Numerical Integration

Problem Description: Given a function $f(x)$, defined on an interval $[a, b]$, we want to find an approximation to the integral

$$I(f) = \int_a^b f(x)dx.$$

Main ideas:

- Cut up $[a, b]$ into smaller sub-intervals;
- In each sub-interval, find a polynomial $p_i(x) \approx f(x)$;
- Integrate $p_i(x)$ on each sub-interval, and sum them up.



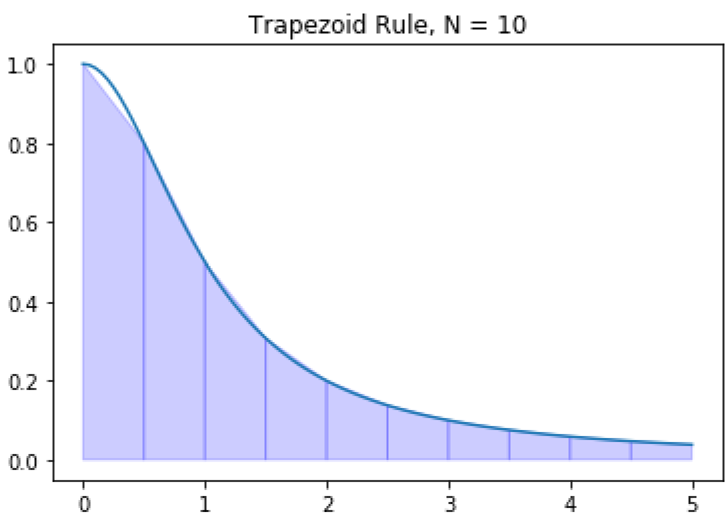


Image Source: Mathematical Python

Trapezoid Rule on Uniform Grid: $\int_a^b f(x)dx \approx h\left[\frac{1}{2}f\left(x_0\right) + \sum_{i=1}^{n-1} f\left(x_i\right) + \frac{1}{2}f\left(x_n\right)\right]$

Error estimate for trapezoid rule

Intermediate Value Theorem : In mathematical analysis, the intermediate value theorem states that if f is a continuous function whose domain contains the interval $[a, b]$, then it takes on any given value between $f(a)$ and $f(b)$ at some point within the interval.

We define the error:

$$E(f;h) \doteq I(f) - T(f;h) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} [f(x) - p_i(x)]\, dx = \sum_{i=0}^{n-1} E_i(f;h),$$

where $E_i(f;h)$ is the error on each sub-interval

$$E_i(f;h) = \int_{x_i}^{x_{i+1}} [f(x) - p_i(x)]\, dx, \quad (i = 0, 1, \cdots, n-1)$$

Error bound with polynomial interpolation:

$$f(x) - p_i(x) = \frac{1}{2} f''(\xi_i)(x - x_i)(x - x_{i+1}), \quad (x_i < \xi_i < x_{i+1})$$

Error estimate on each sub-interval:

$$E_i(f;h) = \frac{1}{2} f''(\xi_i) \int_{x_i}^{x_{i+1}} (x - x_i)(x - x_{i+1})\, dx = -\frac{1}{12} h^3 f''(\xi_i).$$

The total error is:

$$\begin{aligned} E(f;h) &= \sum_{i=0}^{n-1} E_i(f;h) = \sum_{i=0}^{n-1} -\frac{1}{12} h^3 f''(\xi_i) \\ &= -\frac{1}{12} h^3 \underbrace{\left[\sum_{i=0}^{n-1} f''(\xi_i)\right] \cdot \frac{1}{n}}_{=f''(\xi)} \cdot \underbrace{\frac{b-a}{h}}_{} = n \end{aligned}$$

which gives

$$E(f;h) = -\frac{b-a}{12} h^2 f''(\xi), \quad \xi \in (a,b).$$

Error bound

$$|E(f;h)| \le \frac{b-a}{12} h^2 \max_{x \in (a,b)} |f''(x)|$$

Simpson's rule

We now explorer possibility of using higher order polynomials. We cut up $[a,b]$ into $2n$ equal sub-intervals

$$x_0 = a, \quad x_{2n} = b, \quad h = \frac{b-a}{2n}, \quad x_{i+1} - x_i = h$$

On $[x_{2i}, x_{2i+2}]$, interpolates $f(x)$ at the points $x_{2i}, x_{2i+1}, x_{2i+2}$ with a quadratic polynomial $p_i(x)$.

Lagrange form for $p_i(x)$:

$$\begin{aligned} p_i(x) =& f(x_{2i}) \frac{(x - x_{2i+1})(x - x_{2i+2})}{(x_{2i} - x_{2i+1})(x_{2i} - x_{2i+2})} + f(x_{2i+1}) \frac{(x - x_{2i})(x - x_{2i+2})}{(x_{2i+1} - x_{2i})(x_{2i+1} - x_{2i+2})} \\ &+ f(x_{2i+2}) \frac{(x - x_{2i})(x - x_{2i+1})}{(x_{2i+2} - x_{2i})(x_{2i+2} - x_{2i+1})} \end{aligned}$$

With uniform nodes, this becomes

$$
\begin{aligned}
p_i(x) = &\frac{1}{2h^2} f\left(x_{2i}\right)\left(x - x_{2i+1}\right)\left(x - x_{2i+2}\right) - \frac{1}{h^2} f\left(x_{2i+1}\right)\left(x - x_{2i}\right)\left(x - x_{2i+2}\right) \\
&+ \frac{1}{2h^2} f\left(x_{2i+2}\right)\left(x - x_{2i}\right)\left(x - x_{2i+1}\right)
\end{aligned}
$$

We work out the integrals (try to fill in the details yourself!)

$$
\begin{aligned}
I_1 &= \int_{x_{2i}}^{x_{2i+2}} \left(x - x_{2i+1}\right)\left(x - x_{2i+2}\right) dx = \frac{2}{3}h^3 \\
I_2 &= \int_{x_{2i}}^{x_{2i+2}} -\left(x - x_{2i}\right)\left(x - x_{2i+2}\right) dx = \frac{4}{3}h^3 \\
I_3 &= \int_{x_{2i}}^{x_{2i+2}} \left(x - x_{2i}\right)\left(x - x_{2i+1}\right) dx = \frac{2}{3}h^3
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{x_{2i}}^{x_{2i+2}} p_i(x)dx &= \frac{1}{2h^2} f\left(x_{2i}\right) \cdot I_1 + \frac{1}{h^2} f\left(x_{2i+1}\right) \cdot I_2 + \frac{1}{2h^2} f\left(x_{2i+2}\right) \cdot I_3 \\
&= \frac{1}{2h^2} f\left(x_{2i}\right) \frac{2}{3}h^3 + \frac{1}{h^2} f\left(x_{2i+1}\right) \frac{4}{3}h^3 + \frac{1}{2h^2} f\left(x_{2i+2}\right) \frac{2}{3}h^3 \\
&= \frac{h}{3}\left[f\left(x_{2i}\right) + 4f\left(x_{2i+1}\right) + f\left(x_{2i+2}\right)\right].
\end{aligned}
$$

We now sum them up

$$
\begin{aligned}
\int_a^b f(x)dx \approx S(f;h) &= \sum_{i=0}^{n-1} \int_{x_{2i}}^{x_{2i+2}} p_i(x)dx \\
&= \frac{h}{3} \sum_{i=0}^{n-1} \left[f\left(x_{2i}\right) + 4f\left(x_{2i+1}\right) + f\left(x_{2i+2}\right)\right]
\end{aligned}
$$

Simpson's Rule:

$$
S(f;h) = \frac{h}{3}\left[f\left(x_0\right) + 4\sum_{i=1}^{n} f\left(x_{2i-1}\right) + 2\sum_{i=1}^{n-1} f\left(x_{2i}\right) + f\left(x_{2n}\right)\right]
$$

Simpson's Rule: $S(f;h) = \frac{h}{3}\left[f\left(x_0\right) + 4\sum_{i=1}^{n} f\left(x_{2i-1}\right) + 2\sum_{i=1}^{n-1} f\left(x_{2i}\right) + f\left(x_{2n}\right)\right]$

In [ ]:

```
import numpy as np

def f(x):
    return np.sqrt(x**2 + 1)

a = -2
b = 2
n = 10

h = (b - a) / (2 * n)
x = np.linspace(a, b, 2*n + 1)

S = (h / 3) * (f(x[0]) + 4 * np.sum(f(x[1:2*n:2])) + 2 * np.sum(f(x[2:2*n-1:2])) + f(x[2
```

```
S
```

Out[ ]: 5.915769549490477

In [ ]:
```
import sympy as sp

# Define the symbol and the function
x = sp.symbols('x')
f = sp.sqrt(x**2 + 1)

# Calculate the integral
I = sp.integrate(f, (x, -2, 2))

I
```

Out[ ]: $\operatorname{asinh}(2) + 2\sqrt{5}$

In [ ]:
```
I.evalf()
```

Out[ ]: 5.91577143017839

Error estimate for Simpson's Rule

The basic error on each sub-interval is

$$E_{S,i}(f;h) = -\frac{1}{90}h^5 f^{(4)}(\xi_i), \quad \xi_i \in (x_{2i}, x_{2i+2}).$$

(See lecture notes or textbook for the proof.) Then, the total error is

$$E_S(f;h) = I(f) - S(f;h) = -\frac{1}{90}h^5 \sum_{i=0}^{n-1} f^{(4)}(\xi_i)\frac{1}{n} \cdot \frac{b-a}{2h} = -\frac{b-a}{180}h^4 f^{(4)}(\xi),$$

This gives us the error bound

$$|E_S(f;h)| \leq \frac{b-a}{180}h^4 \max_{x\in(a,b)} \left|f^{(4)}(x)\right|.$$

Gaussian Quadrature

All the numerical integration rules follow the form

$$\int_a^b f(x)dx \approx A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_n).$$

Here $x_i \in [a,b]$ are called the nodes, and A_i 's are the weights. For example, the trapezoid rule is:

$$x_0 = a, x_1 = b, A_0 = A_1 = \frac{b-a}{2}$$

and the Simpson's rule corresponds to

$$x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b, A_0 = A_2 = (b-a)\frac{1}{6}, A_1 = (b-a)\frac{2}{3}.$$

In these rules, we fix the points x_i, then we adjust the weights A_i.

Idea: allow both the nodes x_i and the weights A_i to be adjusted, to achieve high accuracy.

One chooses the nodes x_i and weight $A_i (i = 0, 1, 2, \cdots, N)$ such that the algorithm gives the exact value for polynomial functions $f(x)$ of highest possible degree m :

$$\int_a^b f(x) dx = A_0 f(x_0) + A_1 f(x_1) + \cdots + A_n f(x_m), \quad \text{if } f \in P^m.$$

Here, m is called the degree of precision.

To fix the idea, we consider now the interval $[-1, 1]$. Start with $N = 1$, i.e., we have 2 nodes x_0, x_1 and 2 weights A_0, A_1. The rule satisfies

$$\int_{-1}^1 f(x) dx = A_0 f(x_0) + A_1 f(x_1), \quad f \in \mathcal{P}_m$$

$\Rightarrow$ the rule must be exact for functions $1, x, x^2, x^3, \cdots, x^m$. We will need 4 equations for 4 unknowns, therefore $m = 3$. Recall that

$$\int_{-1}^1 x^k dx = \begin{cases} \dfrac{2}{k+1}, k \text{ even} \\ 0, k \text{ odd.} \end{cases}$$

$$\begin{aligned} f(x) = 1 &: A_0 + A_1 = 2 \\ f(x) = x &: A_0 x_0 + A_1 x_1 = 0 \\ f(x) = x^2 &: A_0 x_0^2 + A_1 x_1^2 = \frac{2}{3} \\ f(x) = x^3 &: A_0 x_0^3 + A_1 x_1^3 = 0 \end{aligned}$$

One could solve this system and find

$$x_0 = -\frac{1}{\sqrt{3}}, \quad x_1 = \frac{1}{\sqrt{3}}, \quad A_0 = 1, \quad A_1 = 1.$$

Then, we have the Gaussian quadrature rule for $n = 1$,

$$\int_{-1}^1 f(x) dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

This will have degree of precision 3 .

Gaussian quadrature, $N = 2$ Nodes: x_0, x_1, x_2, and weights: A_0, A_1, A_2.

$$\int_{-1}^1 f(x) dx \approx A_0 f(x_0) + A_1 f(A_1) + A_2 f(x_2)$$

The rule should give exact solution for polynomials of degree $m = 5$. symmetry:
$x_0 = -x_2, \quad x_1 = 0, \quad A_0 = A_2,$

Only need to check with $f(x) = 1, x^2, x^4$:

$$\begin{aligned} f(x) = 1: &\quad 2A_0 + A_1 = 2 \\ f(x) = x^2: &\quad 2A_0 x_0^2 = 2/3 \\ f(x) = x^4: &\quad 2A_0 x_0^4 = 2/5 \end{aligned}$$

$$x_0 = \sqrt{3/5}, \quad x_1 = 0, \quad x_2 = -\sqrt{3/5}, \quad A_0 = A_2 = 5/9, \quad A_1 = 8/9.$$