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Reverse Routing to Obtain Upstream Hydrographs

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SUMMARY Methods are presented to calculate upstream hydrographs from downstream hydrographs for a storage reservoir such as a dam with spillway or detention basin with pipe outlet, and for Muskingum river reach routing. These methods avoid the numerical instabilities which occur when conventional routing procedures are used.

1. INTRODUCTION

Flood routing usually involves the use of an upstream hydrograph to estimate a downstream hydrograph. Examples are estimation of the outflow hydrograph from a storage reservoir with spillway or from a detention basin with culvert outlet, and estimating the hydrograph at the downstream end of a river reach. Less common, but still significant, is the case where a downstream hydrograph has been recorded and an estimate of the upstream hydrograph is required. This can be needed for example where flow is recorded downstream of a dam, or recorded by means of the water level in a storage, or to fill in missing records using those at a downstream station.

In Chapter 7 of Australian Rainfall and Runoff (Institution of Engineers, Australia, 1987), referred to herein as ARR87, difficulties in reverse routing due to numerical instabilities have been noted by the second author. This can cause divergence or oscillation in the solution of the routing equation. Figure 1 shows typical results for the hydrograph at Gingera (National Station Number 410730), upstream of Corin dam on the Cotter River near Canberra, estimated from the flow recorded downstream of the dam. As a short reach of river is involved, the problem can be considered essentially as reverse routing through a reservoir. Attempts to estimate the inflow hydrograph resulted in severe oscillations.

This paper presents results of an investigation into methods of reducing or eliminating this numerical instability.

2. DATA USED

Two cases were studied, routing through a concentrated storage reservoir, and routing through distributed storage along a river reach. The study used examples from ARR87. The concentrated reservoir (Section 7.5.7) provides a more severe test of the method than many practical applications, since it consists of a culvert plus spillway and the elevation - discharge relation has two sharp discontinuities, when the culvert inlet submerges, and when the spillway begins discharging. Since the detention basin is small, the data are listed at 5 minute intervals. The river reach data (Section 7.4.4 ARR87) should also provide a good test since there is considerable delay between the upstream and downstream hydrograph peaks. For the long reach length of approximately 110 km between Doctors Point and Corowa, the data are listed at 24 hour intervals.

For the first part of the study, the upstream hydrographs given in ARR87 were adopted, and these were used to calculate downstream hydrographs. Methods used were Puls' routing and Muskingum routing for the reservoir and river reach respectively. For reverse routing, these calculated hydrographs were used in an attempt to reproduce the given upstream hydrograph. Thus these tests are of the numerical calculation procedures given "perfect" data. As a second part of the study, the most satisfactory of the methods tested using perfect data were applied to recorded upstream and downstream hydrographs to test their effectiveness with real data.

3 CONCENTRATED RESERVOIR

The continuity equation for flood routing is

$$I(t) - Q(t) = dS(t)/dt \tag{1}$$

where I(t) is the upstream or inflow discharge, Q(t) is the downstream or outflow discharge and S(t) is the volume of water held in temporary storage above the reservoir outlet, all being instantaneous values at time t. Since the storage is concentrated, storage is a function of the outflow discharge only:

$$S(t) = f[Q(t)] \tag{2}$$

where the function depends on the elevation - discharge relation associated with the hydraulics of the outlet works, and on the elevation - storage relation associated with the site topography.

In conventional forward routing to obtain downstream from upstream hydrographs, and in the reverse application of these procedures suggested in ARR87 and other places for the estimation of upstream hydrographs, equation (1) is expressed in finite difference form in terms of the inflow and outflow discharges averaged over time Δt:

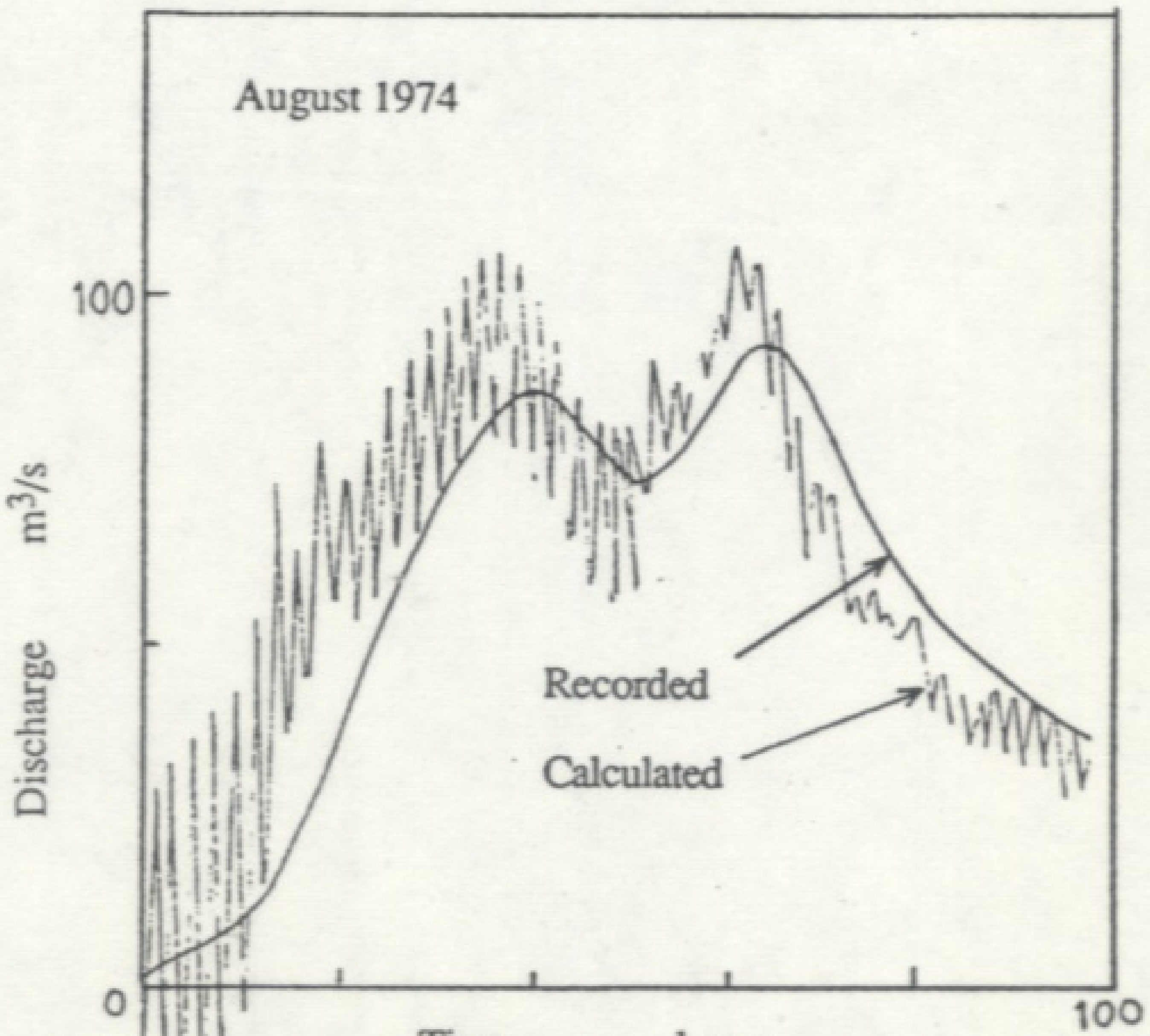

Figure 1 Corin Dam on Cotter River
Recorded and Calculated Upstream Hydrographs

$$(I_{i+1} + I_i)\,\Delta t/2 - (Q_{i+1} + Q_i)\,\Delta t/2 = S_{i+1} - S_i \quad (3)$$

where the subscripts refer to values spaced Δt apart and the solution is sought at time i + 1.

For reverse routing however, this can result in oscillations as illustrated in Figure 1. A better approach for reverse routing is to express equation (1) in finite difference form in terms of the instantaneous inflow and outflow rates occurring at time i. This follows a procedure used by Pilgrim and Watson (3) for a similar problem in estimating the input from a recorded output, for a radiation ratemeter involving an electrical storage system:

$$I_i = Q_i + dS/dt\,|_i \quad (4)$$

The derivative in equation (4) is expressed in finite difference form as discussed for example in Salvadori and Baron (4). The simplest two point scheme (equation 5a) can be used, or higher order formulae can be applied, such as the four point scheme given in equation (5b):

$$dS/dt\,|_i = (S_{i+1} - S_{i-1})/2\Delta t \quad (5a)$$

$$dS/dt\,|_i = (8S_{i+1} - 8S_{i-1} - S_{i+2} + S_{i-2})/12\Delta t \quad (5b)$$

Results using both schemes were very similar, so the simpler equation (5a) was adopted for the derivative. For derivatives at the beginning and end of the period being analysed, forward and backward differences must be used. The forward difference formula used to calculate the derivative at the time t = 0 is:

$$dS/dt\,|_o = (-3S_o + 4S_1 - S_2)\,/\,2\Delta t \quad (5c)$$

and the backward difference formula is similar but with signs reversed.

Figure 2 shows results of reverse routing for the storage reservoir using equations (4) and (5a) and (5c). The results were remarkably good, giving very close reproduction of the upstream hydrograph when a time step of Δt = 5 minutes was used. Although the discontinuities in the storage - discharge relation for this detention basin cause abrupt changes in the downstream hydrograph Q, the variation of storage S (given by equation 2) with time is much smoother, allowing accurate finite difference representation of the derivative dS/dt.

A discontinuity in the storage discharge relation may cause oscillations in some cases in the computed upstream hydrograph. If this should occur, it may be necessary to divide the total period into two separate parts at the time when the discontinuity is reached, and to calculate the last values in the first part and the first values in the second part by backward and forward difference formulae respectively.

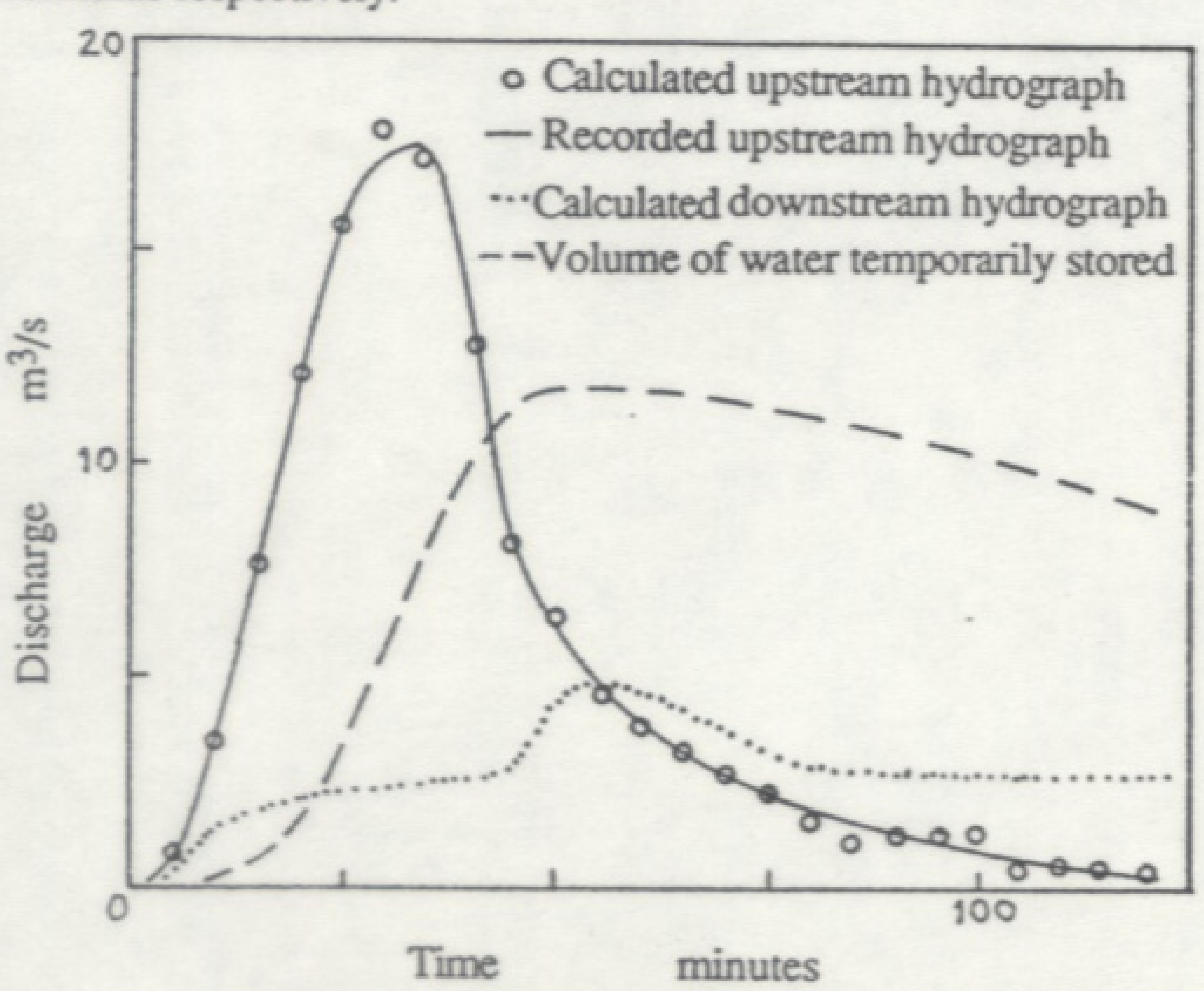

Figure 2 Concentrated Reservoir Routing

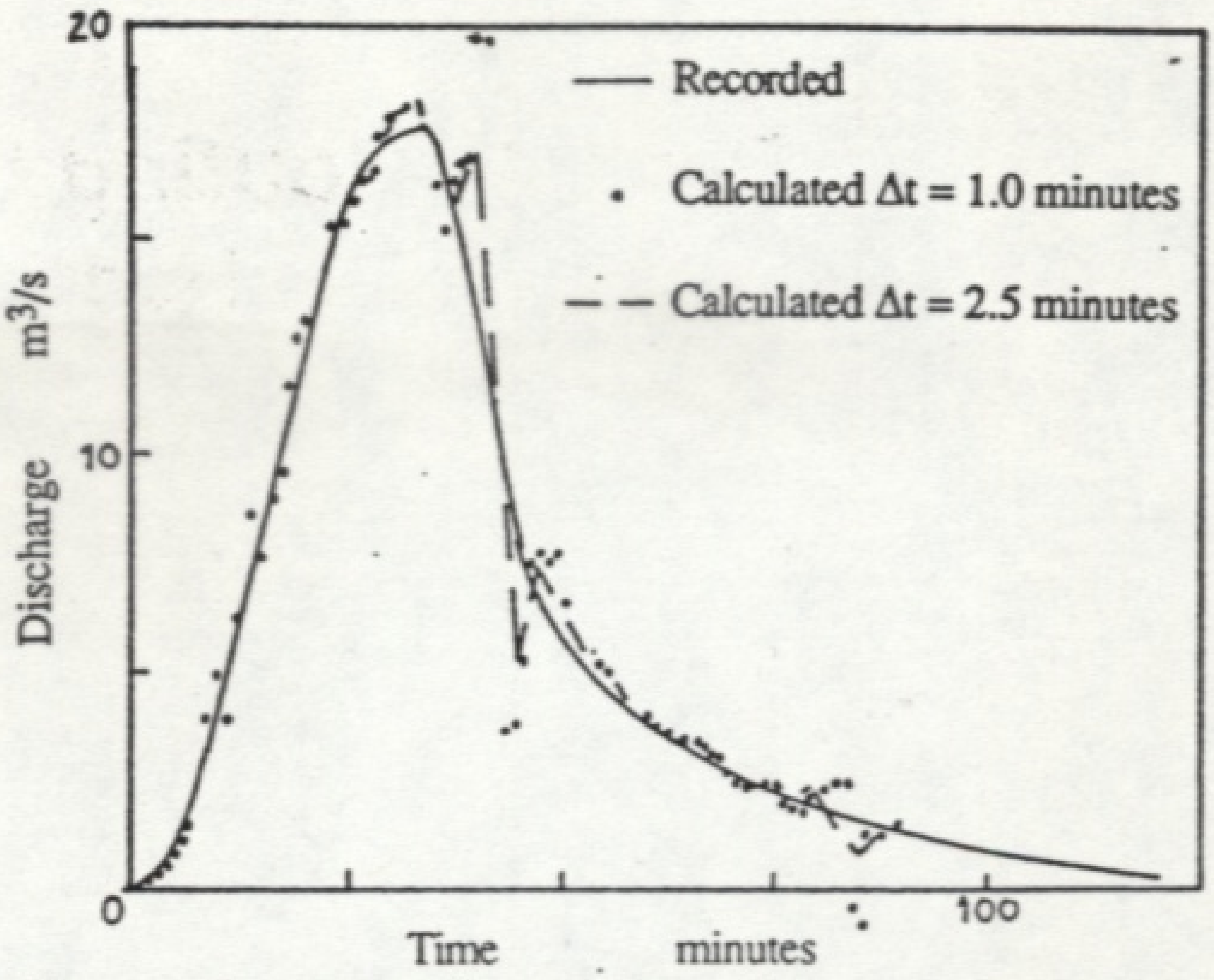

Figure 3 Concentrated Storage Reservoir
Effect of Time Step on Calculated Upstream Hydrograph

While good estimates of the upstream hydrograph are obtained for Δt = 5 minutes, oscillations are introduced for values of Δt less than this (Figure 3). A similar effect has been noted regarding oscillations occurring in derived unit hydrographs (Bree, 1). Thus the time step used in calculation is important. If Δt is too small, oscillations occur. If Δt is too large, not all points on the hydrograph are considered and the peak may be missed. If oscillations do occur, a simple smoothing algorithm such as equation (6) may be used to reduce them.

$$I_i = (I_{i-1} + 2I_i + I_{i+1})/4 \quad (6)$$

The expression of equation (1) in the finite difference form of equations (4) and (5a) and (5c) appears to provide a satisfactory procedure for reverse routing in reservoirs, and gives much better results than the reverse application of normal routing procedures as expressed in equation (3).

4. RIVER REACH

4.1 Reverse Routing Moving Forward in Time

Muskingum routing is normally used for distributed storage in river reaches. For the example given in ARR87, the Murray River from Doctors Point to Corowa, K = 66 hours, x = 0.45 and Δt = 24 hours.

The Muskingum storage-discharge relation is

$$S = K[xI + (1 - x)\,Q] \quad (7)$$

Combining equations (1) and (7) in finite difference form gives the well known Muskingum equation:

$$Q_{i+1} = C_o\, I_{i+1} + C_1\, I_i + C_2\, Q_i \quad (8)$$

where

$$C_o = (-Kx + \frac{\Delta t}{2})\,/\,(K - Kx + \frac{\Delta t}{2}) \quad (9a)$$

$$C_1 = (Kx + \frac{\Delta t}{2})/(K - Kx + \frac{\Delta t}{2}) \quad (9b)$$

$$C_2 = (K - Kx - \frac{\Delta t}{2})\,/\,(K - Kx + \frac{\Delta t}{2}) \quad (9c)$$

For reverse routing moving forward in time, with inflow I_{i+1} estimated at time Δt later than time i, equation (8) is re-arranged to

$$I_{i+1} = \frac{1}{C_o}\cdot Q_{i+1} - \frac{C_2}{C_o}\,Q_i - \frac{C_1}{C_o}\cdot I_i \quad (10)$$

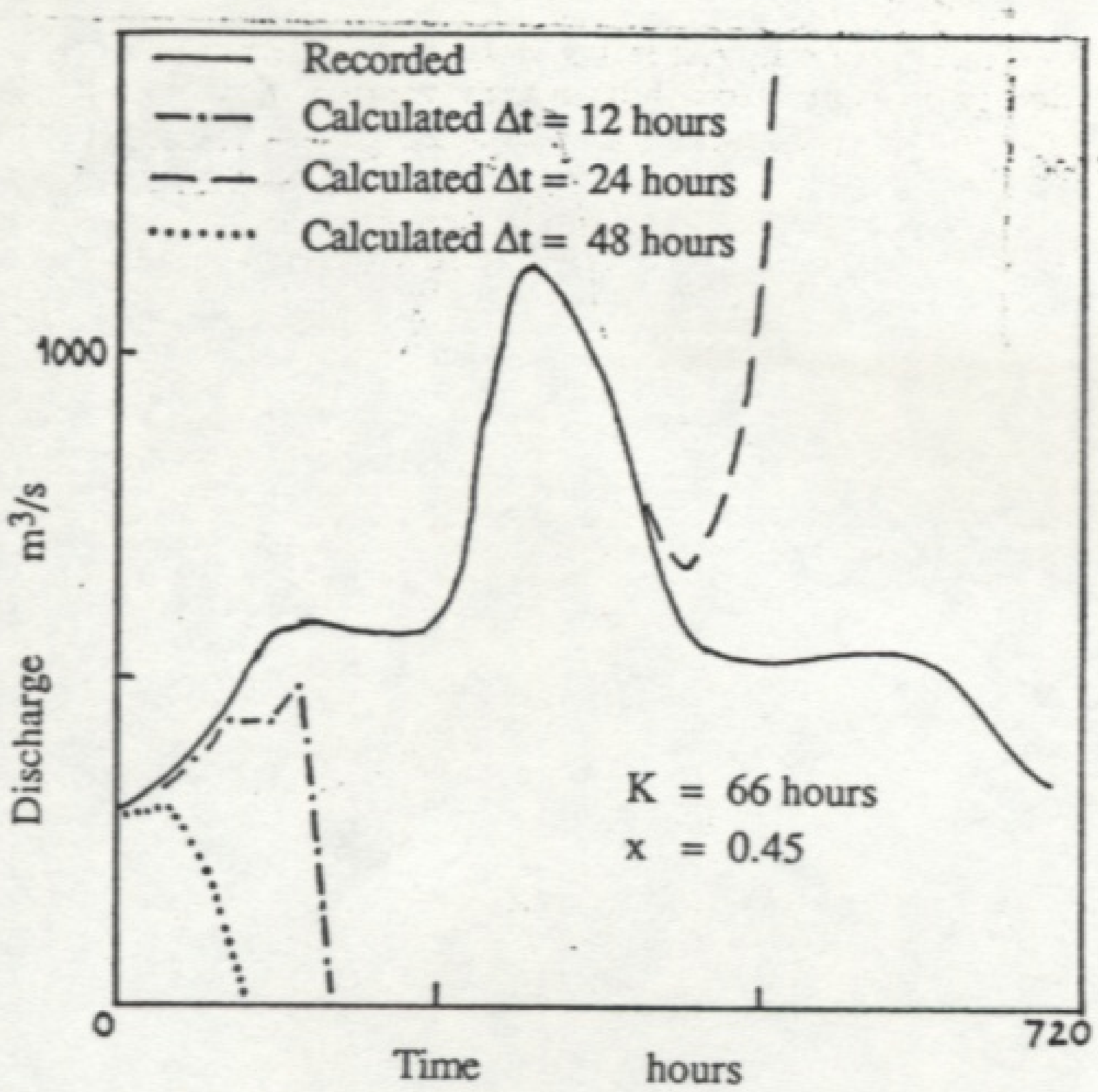

Figure 4 River Reach Routing Moving Forward in Time
Effect of Time Step on Calculated Upstream Hydrograph

Figure 4 shows upstream hydrographs estimated using equation (10) for $K = 66$ hours, $x = 0.45$ and various values of Δt.

The calculated upstream hydrograph is initially good, but soon diverges. This occurs because the multiplying factor for the calculated I_i value, C_1/Co always exceeds 1.0, so that any error entering into I_i is carried forward into the calculation and amplified giving very rapid divergence. By contrast, in the usual Muskingum equation (8), the coefficient C_2 is always less than 1.0, so that any error entering into the calculated Q_i value is carried forward but diminished, so that equation (8) is numerically stable. Table I shows values of C_1/Co calculated for $K = 66$ hours, $x = 0.45$ hours and various values of Δt. The value of C_1/Co always exceeds 1.0, so equation (9) will always diverge. The value of C_1/Co is maximum when $\Delta t = 2Kx = 59.4$ hours for these data. While values of Δt near to 59.4 hours give very rapid divergence, values of Δt greater and less than this are also unsatisfactory, in the first case because the time step is too coarse to adequately define the hydrograph, and in the second case because the increased number of calculation steps introduces more opportunity for errors to amplify.

A similar result occurs when $K = 66$, $\Delta t = 24$ hours and parameter x is varied (Table II). The value of C_1/Co is always greater than 1.0 except for $x = 0$ when it equals 1.0. Equation (10) therefore should be satisfactory for $x = 0$ but divergence occurs for all $x \neq 0$. The maximum value of C_1/Co occurs when $x = \Delta t/2K = 24/2 \times 66 = 0.182$ for the event. Calculations using equation (10) verified this result. The method was

Table I Effect of Time Period on Term C_1/Co

Δt	12	24	48	60	72	96
C_1/Co	-1.51	-2.36	-9.42	199.0	10.43	4.25

Table II Effect of Parameter x onTerm C_1/Co

x	0	0.1	0.2	0.3	0.4	0.45	0.5
C_1/Co	1.0	3.44	-21.00	-4.08	-2.67	-2.36	-2.14

satisfactory for $x = 0$, most rapid divergence occurred for x near to 0.2, and results improved as x moved from 0.2 towards 0.5.

It is clear that this method of calculating upstream hydrographs is unsatisfactory.

4.2 Reverse Routing Moving Backward in Time

Equation (10) diverges because the term C_1/Co exceeds 1.0. If equation (8) is re-arranged to

$$I_i = \frac{1}{C_1} \cdot Q_{i+1} - \frac{C_2}{C_1} \cdot Q_i - \frac{Co}{C_1} \cdot I_{i+1} \qquad (11)$$

then the term $Co/C_1 = (-Kx + \frac{\Delta t}{2}) / (Kx + \frac{\Delta t}{2})$ is always less than 1.0 and any error entering into the calculated I_{i+1} value is carried into the calculation and diminishes towards zero. Equation (11) is therefore numerically stable.

Equation (11) requires that the calculation be carried out backwards in time, starting at the end of the hydrograph tail and moving backward to the start of rise of the hydrograph. This appears to be a drastic step to take, but since we are using a recorded downstream hydrograph to estimate an upstream hydrograph which has already occurred, no forecasting is involved. Another possible problem is that we may not know the starting discharge (at the tail of the upstream hydrograph). However, since Co/C_1 is always less than 1.0, any uncertainty in this discharge rapidly diminishes to zero. Figure 5 shows upstream hydrographs calculated from various assumed starting discharges, computed from the exact calculated downstream hydrograph for the ARR87 example. Convergence is rapid and good upstream hydrograph reproduction is obtained.

4.3 Solution Using Instantaneous Discharges

An alternative solution has been developed using calculations moving forward in time. The approach is similar to that used in the solution of the concentrated reservoir case, since the continuity equation (1) is re-arranged into the form of equation (4). The first derivative dS/dt can be approximated by equations (5a) and (5c) where the storage S at any specified discharge is given by equation (7). For the case of distributed river reach storage however, equation (4) becomes implicit because the storage S is expressed in terms of the upstream discharge I, the value of which is itself being sought by equation (4).

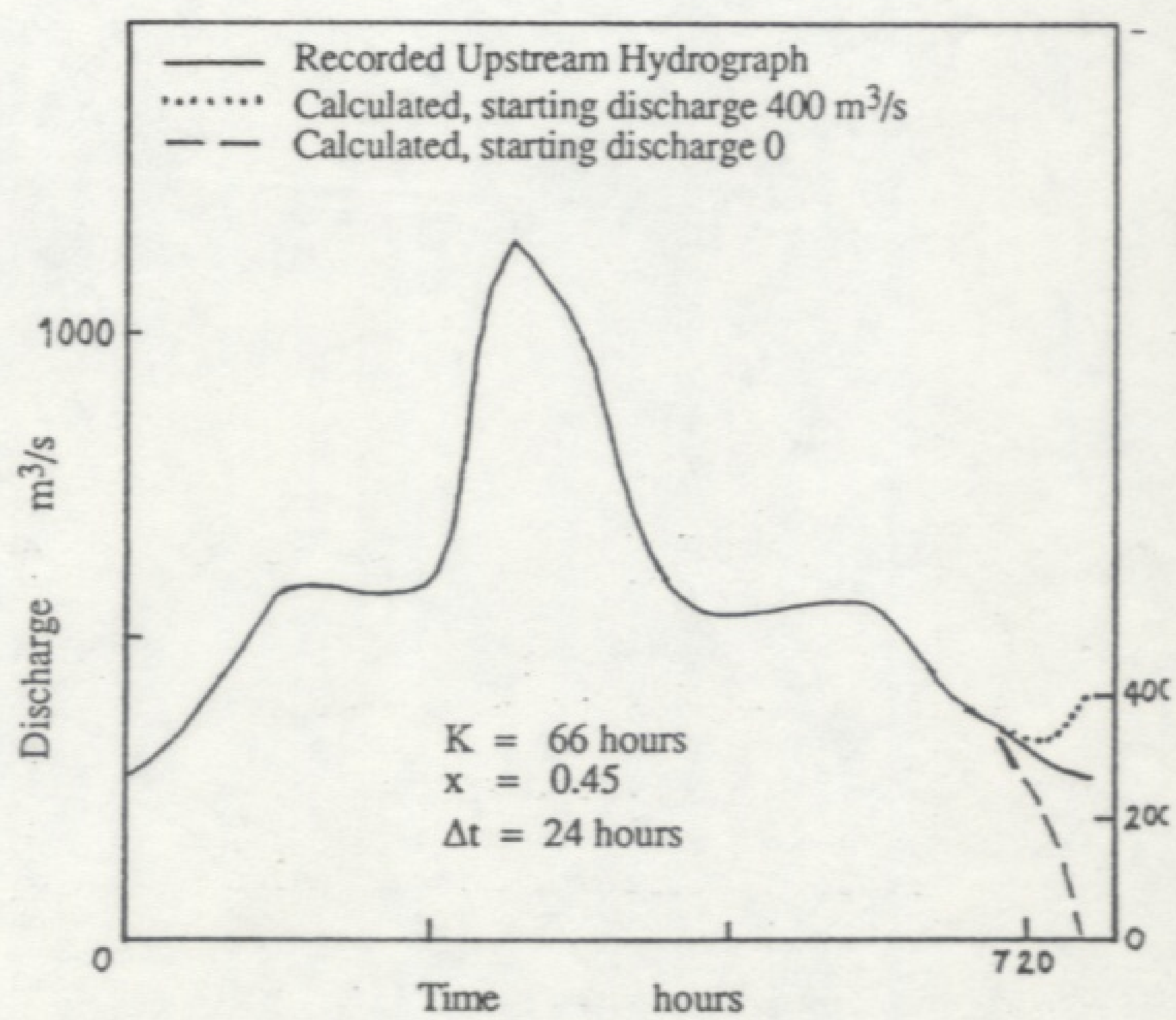

Figure 5 River Reach Routing Moving Backward in Time

An iterative solution is required, and the method of solution used was to adopt the downstream hydrograph ordinates Q as the first estimate of the upstream hydrograph ordinates I, use equation (7) to calculate values of storage S, use equations (5a) and (5c) to determine the derivative dS/dt, then use equation (4) to make an improved estimate of I. These steps can then be repeated until successive calculated upstream hydrographs converge.

First attempts using this method of solution were unsatisfactory, with oscillations occurring in the estimated upstream hydrograph. These oscillations became greater with each iteration. The reason for this was that the first derivature dS/dt estimated from equation (5) contained slight oscillations. When these were removed using the smoothing algorithm equation (6), much more satisfactory results were obtained. Figure 6 shows results. The successively calculated upstream hydrographs were found to converge, as long as the time step was greater than a value of $\Delta t = Kx/2$, since this kept the multiplying factors for I less than 1.0. The number of iterations depended on the time step, approximately 50 being needed for Δt near to this limit, and fewer iterations for larger Δt. Results could be dramatically improved, with fewer iterations required, if the upstream hydrograph ordinates at iteration N were combined with those from iteration N-1 as a weighted average, before commencing iteration N+1.

4.4 Tests Using Recorded Downstream Hydrographs

The methods of sections 4.2 and 4.3 were applied to recorded upstream and downstream hydrographs, to examine the case which occurs in practice where the known downstream hydrograph is used to estimate an unknown upstream hydrograph. The upstream and downstream hydrographs from ARR87 Table 7.1, page 134 were used, and Figure 7a shows results for K = 66 hours, x = 0.45 and Δt = 24 hours. As for conventional Muskingum routing from upstream to downstream, the estimated and recorded hydrographs cannot be expected to agree exactly because the movement of flood waves in river reaches does not exactly conform with the behaviour assumed in the linear Muskingum equation. Note however that the two methods give calculated hydrographs that agree reasonably well with the recorded upstream hydrograph. Figure 7b shows conventional Muskingum forward routing to estimate the downstream hydrograph from a recorded upstream hydrograph. Comparison of figures 7a and 7b shows that the reverse routing methods developed in this study have the same order of accuracy as conventional Muskingum forward routing.

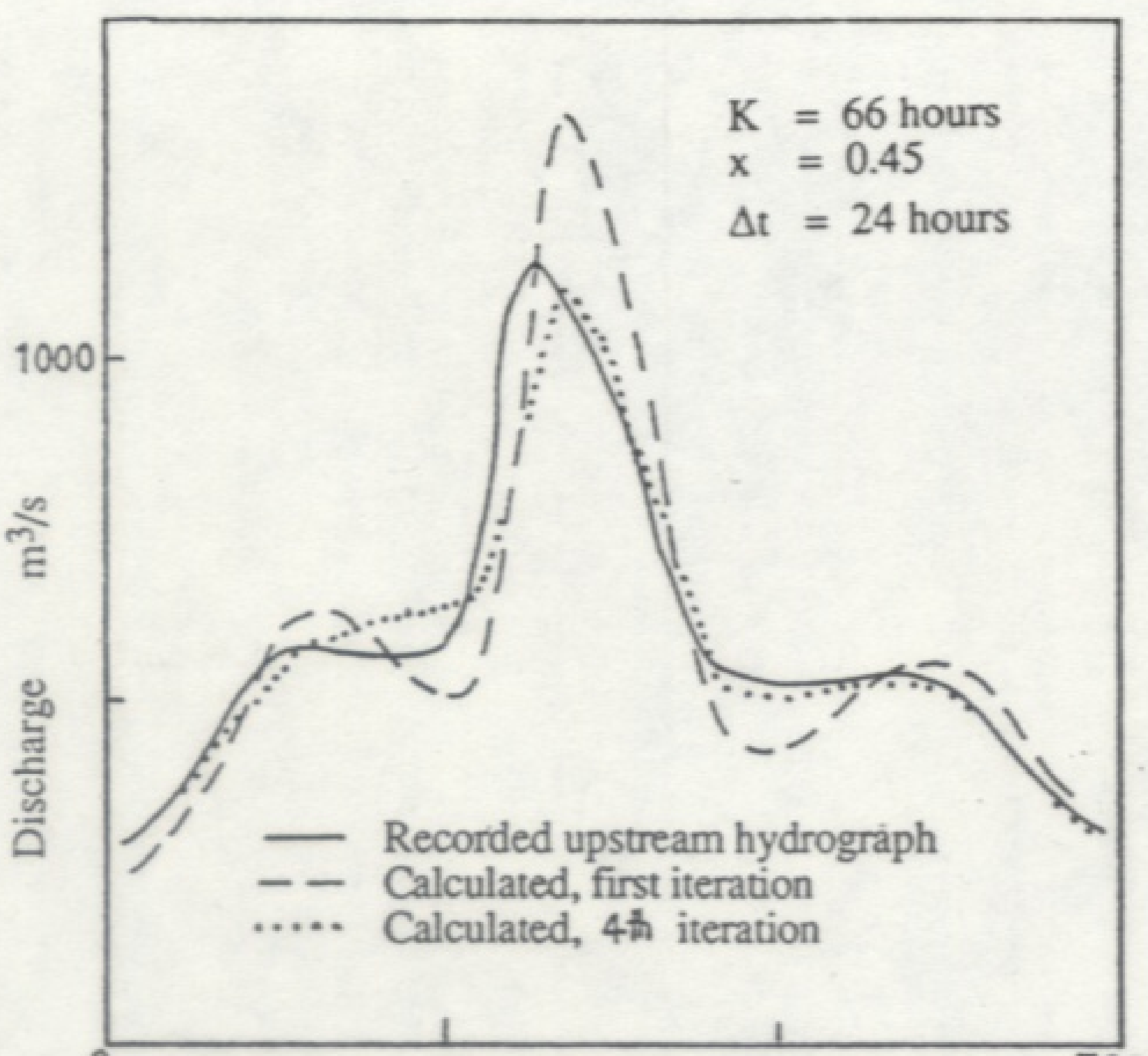

Figure 6 River Reach Routing using Instantaneous Discharges

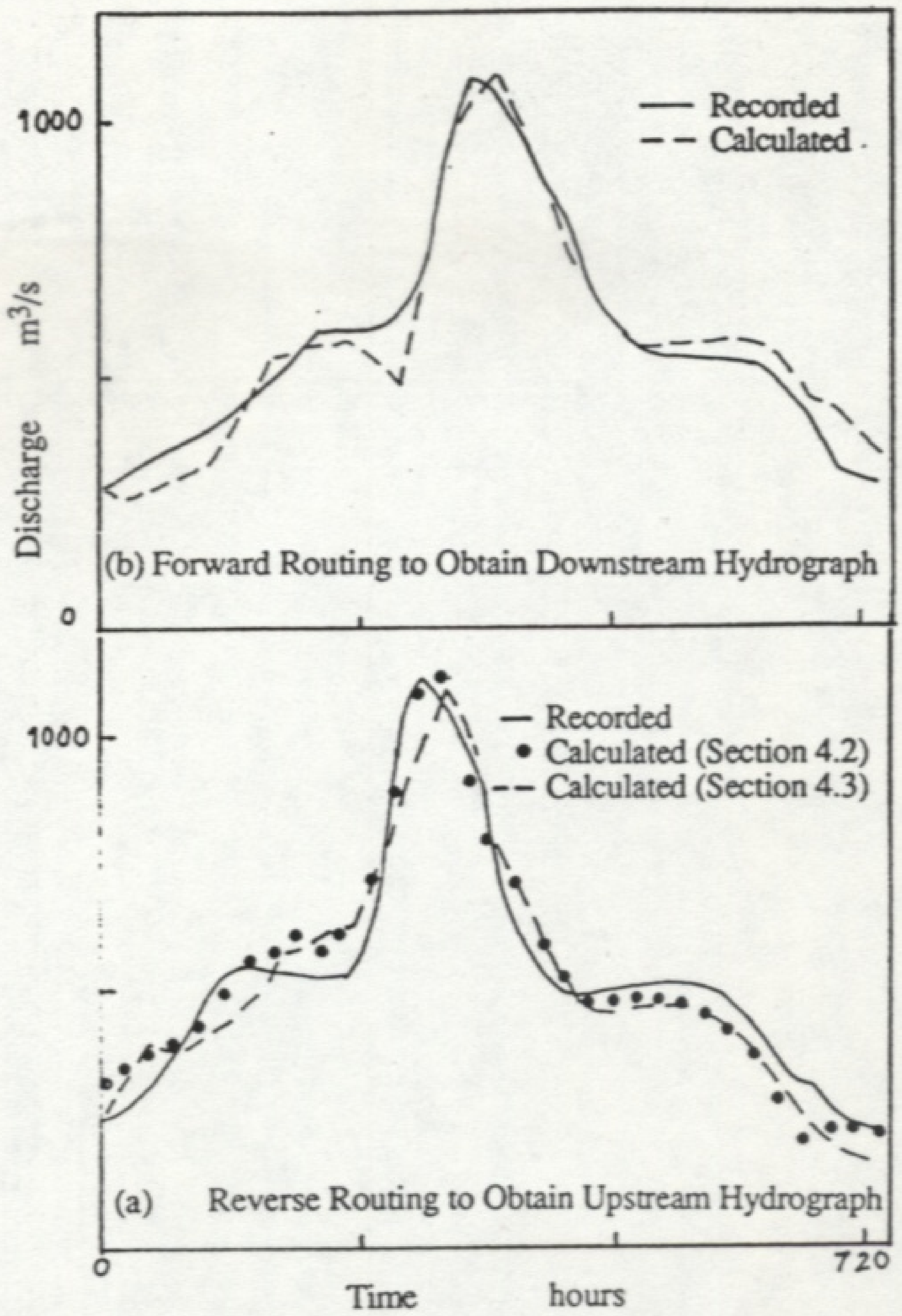

Figure 7 Comparison of Calculated and Recorded Hydrographs

5. CONCLUSIONS

The reverse application of conventional routing procedures to obtain upstream hydrographs I from downstream hydrographs Q is numerically unstable for both concentrated reservoir and distributed river reach storages, and in practice, satisfactory upstream hydrographs cannot be obtained. However, satisfactory solutions can be obtained using re-formulation of the solution method, together with smoothing and averaging algorithms.

For distributed river reach routing, re-arrangement of the usual finite difference form of the Muskingum equation to solve for I_{i+1} given I_i and (i.e. moving forward in time) does not yield a solution, and very rapid divergence of the estimated upstream hydrograph occurs. If the Muskingum equation is re-arranged to solve for I_i given I_{i+1} (i.e. moving backward in time), the solution converges and very accurate estimates of the upstream hydrograph are obtained.

Re-arrangement of the continuity equation to solve for I given Q and dS/dt, yields a solution for both concentrated storage reservoirs and for distributed river reach storage. In both cases some oscillations occur and smoothing algorithms must be used. In the case of concentrated reservoir routing, the equation is explicit and a direct solution can be obtained. Oscillations occur and increase in severity as the time step is made smaller. These oscillations can be largely removed by smoothing the calculated I ordinates. With the correct choice of time step, very good estimates of I are obtained.

In the case of distributed river reach routing, the equation is implicit and an iterative method of solution must be used. Oscillations occur and smoothing must be utilised. The technique is greatly improved if weighted averaging of the successive iterations is used.

Results presented here for both reservoir routing and river reach routing with "perfect" data used downstream hydrographs calculated from the original upstream hydrograph, retaining all significant figures. The river reach routing methods were also applied to recorded upstream and downstream hydrographs, with good results. Recorded hydrographs were not used for the concentrated reservoir, but since this method yields a very accurate direct solution, it can be expected to work equally well with recorded data.

REFERENCES

1. Bree, T. "The stability of parameter estimation in the general linear model". J. Hydrology, Vol. 37, 1987 pp. 47-66

2. Institution of Engineers, Australia. "Australian Rainfall and Runoff. A Guide to Flood Estimation". Vol. 1., 1987

3. Pilgrim, D.H. and Watson, K.K. "A comparative study of response correction methods for linear ratemeters". Nuclear Instruments and Methods, Vol. 46, 1967, pp 77-85.

4. Salvadori, M.G. and Baron. "Numerical Methods in Engineering". Prentice Hall.