Quantum Field Theory

期中前内容

U(1)对称性 ⟺ 守恒流

$\boxed{\bar\psi = \psi^\dagger \gamma^0}$

- 自由Dirac场 $\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi$

  做U(1)整体变换，$\psi'(x) = e^{ig\theta}\psi(x)$ 则 $\left.\begin{cases} [\psi^\dagger(x)]' = \psi^\dagger(x)\, e^{-ig\theta} \\ [\bar\psi(x)]' = [\psi'(x)]^\dagger \gamma^0 = \bar\psi(x)\, e^{-ig\theta} \end{cases}\right\}$ $\delta\psi = \bar\delta\psi = ig\theta\psi$

  在此变化下的 $\mathcal{L}$ 不变：$\mathcal{L}' = \bar\psi e^{-ig\theta}(i\partial\!\!\!/ - m)e^{ig\theta}\psi = \bar\psi(i\partial\!\!\!/ - m)\psi = \mathcal{L}$

  相应的Noether Current为 $j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi = i\bar\psi\gamma^\mu(ig\theta\psi) = -g\theta\bar\psi\gamma^\mu\psi$

  验证 $\partial_\mu j^\mu = \partial_\mu(\bar\psi\gamma^\mu\psi) \sim \left.\begin{cases} (i\gamma^\mu\partial_\mu - m)\psi = 0 \\ \bar\psi(i\gamma^\mu\overleftarrow{\partial_\mu} + m) = 0 \end{cases}\right\} \Rightarrow \partial_\mu j^\mu = 0$

- QED的U(1)对称性 $\mathcal{L}_{QED} = \bar\psi(i D\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$，$D_\mu = \partial_\mu + ieA_\mu$

  做U(1)局域规范变换 $\psi' = e^{i\alpha(x)}\psi$，$\bar\psi' = \bar\psi e^{-i\alpha(x)}$，$A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha(x)$ $\Rightarrow \delta\psi = \bar\delta\psi = i\alpha(x)\psi(x)$

  Noether Current $j^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\bar\delta\psi = i\alpha\bar\psi\gamma^\mu\psi$

实标场 $\mathcal{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 \sim$ KG

共轭动量密度 $\pi = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi$

Hamiltonian密度 $\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}(\partial_0^2 + m^2 + \nabla^2)\phi$

总动量 $\vec{P} = -\int d^3x\, \pi\cdot\nabla\phi$

复标场 $\mathcal{L} = (\partial^\mu\phi)(\partial_\mu\phi) - m^2\phi^\dagger\phi \sim$ KG(2个) $\sim$ U(1)整体对称性

共轭动量密度 $\pi = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^\dagger$，$\pi^\dagger = \dfrac{\partial\mathcal{L}}{\partial(\partial_0\phi^\dagger)} = \partial_0\phi$

Hamiltonian密度 $\mathcal{H} = \pi\partial_0\phi + \pi^\dagger\partial_0\phi^\dagger - \mathcal{L} = (\partial^0\phi^\dagger)\partial_0\phi + (\partial^0\phi)\partial_0\phi^\dagger - (\partial^\mu\phi^\dagger)\partial_\mu\phi + m^2\phi^\dagger\phi$

总动量 $\vec{P} = -\int d^3x\,(\pi\cdot\nabla\phi + \pi^\dagger\cdot\nabla\phi^\dagger)$

Lorentz Group

$[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$

$[A,BC]=[A,B]C+B[A,C]$

$\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$, $S^{\mu\nu}=\frac{i}{4}[\gamma^\rho,\gamma^\sigma]$, $(J^{\rho\sigma})^\mu{}_\nu = g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu$

then $[\gamma^\mu, S^{\rho\sigma}] = \frac{i}{4}[\gamma^\mu,[\gamma^\rho,\gamma^\sigma]] = \frac{i}{4}\left([[\gamma^\mu,\gamma^\rho],\gamma^\sigma]+[\gamma^\rho,[\gamma^\mu,\gamma^\sigma]]\right)$

$$= (g^{\mu\rho}\gamma^\sigma - g^{\mu\sigma}\gamma^\rho)$$

$$= (J^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

which is equivalently $\Leftrightarrow (1+\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1-\frac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}) = (1-\frac{i}{2}\omega_{\rho\sigma}J^{\rho\sigma})^\mu{}_\nu\gamma^\nu$

$\Leftrightarrow S^{-1}(\Lambda)\gamma^\mu S(\Lambda)$

$j^\mu = \bar\psi\gamma^\mu\psi$ 是 Lorentz Vector

Recall: $S(\Lambda)=\exp(-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu})$

Proof:

transformation $\left\{\begin{array}{l} x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \\ \psi(x)\to\psi'(x')=S(\Lambda)\psi(x) \\ \bar\psi(x)\to\bar\psi'(x')=\bar\psi(x)S^{-1}(\Lambda)\end{array}\right\} \Rightarrow j'^\mu(x')=\bar\psi'(x')\gamma^\mu\psi'(x') = \bar\psi(x)\underbrace{S^{-1}(\Lambda)\gamma^\mu S(\Lambda)}_{=\Lambda^\mu{}_\nu\gamma^\nu}\psi(x)$

$$= \Lambda^\mu{}_\nu j^\nu(x)$$

为什么需要 QFT ?

- NR-QM

  描述单粒子系统

  波函数满足 Schödinger Eq.

  无法处理粒子产生与湮灭

- SR-QM

  描述单粒子系统，满足相对论不变性

  波函数满足 K-G Eq. or Dirac Eq.

  无法处理粒子产生与湮灭

  存在不自恰处，并非完整理论

- QFT

  处理多粒子系统与相互作用

  基本变量为场 $\phi(x)$ $\psi(x)$，

  统一处理相对论性和粒子数变化

Ward Identity 是量子 Noether theorem 的体现

$\downarrow$

规范对称性引发的物理约束

三个相互作用场理论

(1) ϕ^4 theory $\quad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

(2) Yukawa theory $\quad \mathcal{L} = \mathcal{L}_{Dirac} + \mathcal{L}_{KG} + \mathcal{L}_{int} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - g\bar{\psi}\psi\phi$

(3) QED $\quad \mathcal{L} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}(F_{\mu\nu})^2 - e\bar{\psi}\gamma^\mu\psi A_\mu$

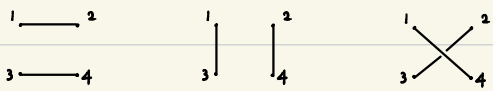
可重整性

耦合常数 $[\lambda] \geq 0$ 方可

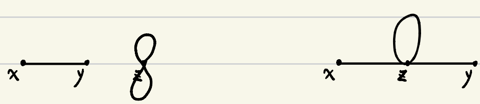
action 无量纲: $S = \int \mathcal{L}\, d^4x \Rightarrow \mathcal{L} \sim (\text{mass})^4$ 即 $[\mathcal{L}] = 4$

为确保这一点, $[\mathcal{L}_{int}] = 4$, $\Rightarrow \begin{cases} \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Rightarrow [A_\mu] = 1 \\ \bar{\psi}(i\partial\!\!\!/ - m)\psi \Rightarrow [\psi] = \frac{3}{2} \\ \frac{1}{2}m^2\phi^2 \Rightarrow [\phi] = 1 \end{cases}$ $\quad \lambda\phi^4 \Rightarrow [\lambda] = 0$

Feynman Diagram



$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle = D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_2-x_4)D_F(x_1-x_3)$$



$$\langle 0|\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)|0\rangle = 3\langle 0|\phi(x)\phi(y)|0\rangle\left(\langle 0|\phi(z)\phi(z)|0\rangle\right)^2 + 12\langle 0|\phi(x)\phi(z)|0\rangle\langle 0|\phi(y)\phi(z)|0\rangle\langle 0|\phi(z)\phi(z)|0\rangle$$

thus $\langle 0|T\{\phi(x)\phi(y)(-i)\int dt\int d^3z\,\frac{\lambda}{4!}\phi^4|0\rangle = 3\cdot\left(-\frac{i\lambda}{4!}\right)D_F(x-y)\int d^4z\, D_F(z-z)D_F(z-z)$

$\quad + 12\cdot\left(-\frac{i\lambda}{4!}\right)\int d^4z\, D_F(x-z)D_F(y-z)D_F(z-z)$

Wick 定理

一组场算符的时序乘积可以分解为它们的正规乘积及所有可能缩并的正规乘积之和，也就是说，

T N

$$\mathsf{T}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n)] = \mathsf{N}[\Phi_{a_1}(x_1)\Phi_{a_2}(x_2)\cdots\Phi_{a_n}(x_n) + (\Phi_{a_1}\Phi_{a_2}\cdots\Phi_{a_n}\text{的所有可能缩并})]. \quad (6.166)$$

即 $T(\Phi_a\Phi_b\Phi_c\Phi_d) = N(\Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d$
$+ \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d$
$+ \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d)$

对応的真空期待值

$$\langle 0|T(\Phi_a\Phi_b\Phi_c\Phi_d)|0\rangle = \langle 0|N(\Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d + \Phi_a\Phi_b\Phi_c\Phi_d)|0\rangle$$
$$= \langle 0|T(\Phi_a\Phi_b)|0\rangle\langle 0|T(\Phi_c\Phi_d)|0\rangle + \varepsilon_{bc}\langle 0|T(\Phi_a\Phi_c)|0\rangle\langle 0|T(\Phi_b\Phi_d)|0\rangle$$
$$+ \varepsilon_{dc}\varepsilon_{bd}\langle 0|T(\Phi_a\Phi_d)|0\rangle\langle 0|T(\Phi_b\Phi_c)|0\rangle$$

fermion 交换反对称

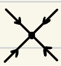
Feynman Propagator

实/复标量场 $D_F(x-y) = \langle 0|T(\phi(x)\phi(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$

无质量矢量场 $\Delta_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{-ig^{\mu\nu}}{p^2+i\varepsilon} e^{-ip\cdot(x-y)}$

Dirac 旋量场 $S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$

Momentum-space Feynman Rules

1. For each propagator, $\longrightarrow = \frac{i}{p^2-m^2+i\varepsilon}$
2. For each vertex,  $= -i\lambda$
3. For each external point, $x \bullet\!\xleftarrow[p]{}\!\!- = e^{-ip\cdot x}$
4. Impose momentum conservation at each vertex
5. Integrate over each undermined momentum $\int \frac{d^4p}{(2\pi)^4}$
6. Divide by the symmetry factor

S 矩阵与散射截面 σ、衰变率 Γ

末态有全同粒子时，需 $\times \frac{1}{n!}$

$$\sigma = \frac{\text{单位时间跃迁几率}}{\text{flux}}$$

Decay Rate $\Gamma = -\frac{dN/dt}{N}$, $N = N_0 e^{-\Gamma t}$, $\tau\,(\text{lifetime}) = \frac{1}{\Gamma}$

$H = H_0 + H_{int}$ (or V)

$$\begin{cases} \boxed{\begin{matrix}|i\rangle_{in} \\ |f\rangle_{out} \\ H\end{matrix}} \end{cases} \qquad \begin{cases} \text{initial state @ } t=-\infty \\ \text{final state @ } t=+\infty \end{cases} \boxed{\begin{matrix}|i\rangle \\ |f\rangle \\ H_0\end{matrix}} \Rightarrow \begin{cases} |i\rangle_{in} = U(t_0,-\infty)|i\rangle \\ |f\rangle_{out} = U(+\infty,t_0)|f\rangle \end{cases}$$

$i\to f$ 的概率振幅 $S_{if} = {}_{out}\langle f|i\rangle_{in} \iff {}_{out}\langle p_1 p_2 \cdots | p_A p_B\rangle_{in}$ $\quad A+B \longrightarrow 1+2+\cdots\cdots$

$$= \langle f|U(\infty,t_0)U(t_0,-\infty)|i\rangle$$

$$= \langle f|U(\infty,-\infty)|i\rangle = \langle f|S|i\rangle$$

$$\Rightarrow S = U(\infty,-\infty) = T e^{-i\int d^4x\, \mathcal{H}_I(x)} = T e^{-i\int_{-\infty}^{+\infty} dt\, H_I(t)}$$

$$= I + iT$$

相互作用

相互作用导致的 $i\to f$ 跃迁几率 $\quad |\langle f|iT|i\rangle|^2 = [(2\pi)^4 \delta^{(4)}(p_i - \sum_f p_f)]^2 \, |i\mathcal{M}(i\to f)|^2 \cdot VT$ (四维时空体积)

进一步考虑 $\begin{cases}\text{末态粒子所有可能态求和} \\ \text{初末态的归一化}\end{cases}$ 单位时间到确定末态的跃迁几率: $\displaystyle\int \prod_f \frac{V d^3p_f}{(2\pi)^3} \frac{[(2\pi)^4 \delta^{(4)}(p_i - \sum_f p_f)]^2 |i\mathcal{M}(i\to f)|^2 V}{\langle i|i\rangle \langle f|f\rangle}$

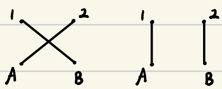
$$d\sigma = \frac{1}{2E_A 2E_B |v_A - v_B|} \underbrace{\boxed{\prod_f \left(\frac{d^3 p_f}{(2\pi)^3} \frac{1}{2E_f}\right) (2\pi)^4 \delta^{(4)}(p_A + p_B - \sum_f p_f)}}_{d\Pi_n \ \text{相空间}} |\mathcal{M}(A+B\to f)|^2$$

for $A+B\to 1+2$ $\qquad \displaystyle\int d\Pi_n = \int d\Omega \frac{1}{16\pi^2} \frac{|\vec{p}_1|}{E_{cm}}$, $\displaystyle\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_A E_B |v_A - v_B|} \frac{|\vec{p}_1|}{16\pi^2 E_{cm}} |\mathcal{M}(A+B\to 1+2)|^2$

$$= \frac{|\mathcal{M}|^2}{64\pi^2 E_{cm}^2} \quad \text{(4质量相同)}$$

微分衰变率 $\displaystyle d\Gamma = \frac{1}{2m_A} \left(\prod_f \frac{d^3p_f}{(2\pi)^3} \frac{1}{2E_f}\right) (2\pi)^4 \delta^{(4)}(p_A - \sum_f p_f) \, |\mathcal{M}(m_A \to \{p_f\})|^2$

$A+B\to 1+2$ in ϕ^4 theory, Expanding S:



first term $\langle p_1 p_2 | p_A p_B\rangle = \sqrt{2E_1 2E_2 2E_A 2E_B} \langle 0 | a_1 a_2 a_A^\dagger a_B^\dagger |0\rangle$

$$= 2E_A 2E_B (2\pi)^6 [\delta^{(3)}(p_1 - p_B)\delta^{(3)}(p_2 - p_A) + \delta^{(3)}(p_1 - p_A)\delta^{(3)}(p_2 - p_B)]$$

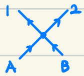
Second term $\displaystyle\langle p_1 p_2 | T\left(-\frac{i\lambda}{4!}\int d^4x\, \phi_I^4(x)\right)|p_A p_B\rangle \overset{Wick}{=\!=} \langle p_1 p_2 | N\left\{-\frac{i\lambda}{4!}\int d^4x\, \phi_I^4(x) + \text{contractions}\right\}|p_A p_B\rangle$

现在并非真空态$|0\rangle$, terms that are not fully contracted do not vanish

$$\begin{cases} \phi_I^+(x)|p\rangle = \displaystyle\int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} a_k e^{-ik\cdot x} \cdot \sqrt{2E_p}\, a_p^\dagger |0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} e^{-ik\cdot x} \sqrt{2E_p} \cdot (2\pi)^3 \delta(k-p)|0\rangle = e^{-ip\cdot x}|0\rangle \\ \Rightarrow \overline{\phi_I |p}\rangle = e^{-ip\cdot x}|0\rangle \ ; \ \langle \overline{p|\phi_I}(x) = e^{+ip\cdot x} \end{cases}$$

N-product term $\phi\phi\phi\phi$ $\quad \contraction{}{\phi}{}{\phi}\phi\phi\phi\phi \xrightarrow{} \text{trivial}$ $\quad \phi\phi\phi\phi \rightarrow \text{trivial}$

Only fully connected diagrams in which ALL external lines are CONNECTED contribute to the T-matrix



$$= (4!)\cdot\left(-i\frac{\lambda}{4!}\right)\int d^4x\, e^{-i(p_A+p_B-p_1-p_2)\cdot x}$$

$$= -i\lambda(2\pi)^4\delta^{(4)}(p_A+p_B-p_1-p_2) \sim \mathcal{M}=-\lambda \Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^2}{64\pi^2E_{cm}^2}, \quad \sigma_{tot} = 4\pi\cdot\frac{\lambda^2}{64\pi^2E_{cm}^2}\cdot\frac{1}{2!}$$

Summary: S-matrix element $\langle f|iT|i\rangle = \underbrace{i\mathcal{M}}_{\text{sum of all connected, amputated diagrams}}\cdot(2\pi)^4\delta^{(4)}(p_A+p_B-\sum_f p_f)$

Also the Feynman Rules

Feynman Rules for Fermions

for Yukawa theory, $H = H_{Dirac} + H_{KG} + \int d^3x\, g\bar\psi\psi\phi$

expanding S-matrix

$$\langle f|Te^{-i\int d^4x\, g\bar\psi\psi\phi}|i\rangle = \langle f|1 + (-ig)T\int d^4x\,\bar\psi\psi\phi + \frac{1}{2!}(-ig)^2T\int d^4x d^4y\,\bar\psi\psi\phi\,\bar\psi\psi\phi + \cdots|i\rangle$$

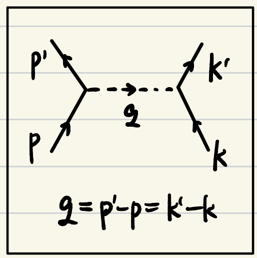
$$\psi(x)|p,s\rangle = \int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\sum_{s'}a_{p'}^{s'}u^{s'}(p')e^{-ip'\cdot x}\sqrt{2E_p}\,a_p^{s\dagger}|0\rangle = e^{-ip\cdot x}u^s(p)|0\rangle$$

$$\langle p,s|\bar\psi(x) = \langle 0|\bar u^s(p)e^{+ip\cdot x}$$

对于过程 fermion p + fermion k $\longrightarrow$ fermion p' + fermion k'

$$\langle p',k'|\frac{1}{2!}(-ig)\int d^4x\,\bar\psi\psi\phi\,(-ig)\int d^4y\,\bar\psi\psi\phi|p,k\rangle$$

$$\Rightarrow (-ig)^2\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m_\phi^2}(2\pi)^4\delta(p'-p-q)\times(2\pi)^4\delta(k'-k+q)\,\bar u(p')u(p)\,\bar u(k')u(k)$$



形如 $i\mathcal{M}(2\pi)^4\delta(\Sigma p)$

对比可得 $i\mathcal{M} = (-ig)^2\bar u(p')u(p)\dfrac{i}{q^2-m_\phi^2}\bar u(k')u(k)$

Rules:

(1) 传播子 $\contraction{}{\phi}{}{\phi}\phi\phi$ $\overset{q}{\dashrightarrow} = \dfrac{i}{q^2-m_\phi^2+i\epsilon}$ $\qquad$ (2) 顶点, $= -ig$

$\contraction{}{\psi}{}{\bar\psi}\psi\bar\psi$ $\overset{p}{\longrightarrow} = \dfrac{i(\not p-m)}{p^2-m^2+i\epsilon}$ $\qquad$ (3) 外线项 $\phi|q\rangle = 1$ $\quad \psi|p,s\rangle = u^s(p)$ (fermion) $\quad \bar\psi|k,s\rangle = \bar v^s(k)$ (anti fermion)

$\langle q|\phi = 1$ $\quad \langle p,s|\bar\psi = \bar u^s(p)$ $\quad \langle k,s|\psi = v^s(k)$

Recall that $\boxed{\begin{aligned}\bar{u}^{s'}(p')u^{s}(p) &= 2m\xi^{s'\dagger}\xi^{s} = 2m\delta^{ss'},\\ \bar{v}^{r'}(k')v^{r}(k) &= -2m\xi^{r'\dagger}\xi^{r} = -2m\delta^{rr'}.\end{aligned}}$

以及NR limit 代入 $\Rightarrow i\mathcal{M} = (-ig)^2\, 2m\delta^{ss'}\frac{i}{(p'-p)^2-m_\phi^2}\, 2m\delta^{rr'}$

$$= \frac{ig^2}{|\vec{p}'-\vec{p}|^2+m_\phi^2}\, 2m\delta^{ss'}\, 2m\delta^{rr'}$$, 即 $\langle p'k'|iT|pk\rangle = i\mathcal{M}(2\pi)^4\delta^{(4)}(p+k-p'-k')$

Compare it with Born approximation

$$\Rightarrow \tilde{V}(\vec{q}) = \frac{-g^2}{|\vec{q}|^2+m_\phi^2} = \int d^3x\, V(x)\, e^{-iq\cdot x}$$

$$\Rightarrow V(x) = -\frac{g}{4\pi}\frac{1}{r}e^{-m_\phi r}$$

Feynman Rules for QED

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int}$$

$$= \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \underline{- e\bar{\psi}\gamma^\mu\psi A_\mu}$$ $H_{int} = \int d^3x\, e\bar{\psi}\gamma^\mu\psi A_\mu$

S matrix: $S = T\{\exp[-i\int d^4x\, e\bar{\psi}\gamma^\mu\psi A_\mu]\}$

Rules:

Yukawa theory | QED

(1) 传播子 $\phi\phi$ $\quad q \quad = \frac{i}{q^2-m_\phi^2+i\epsilon}$ $\qquad$ $\mu \;\; q \;\; \nu = \frac{-ig_{\mu\nu}}{q^2+i\epsilon}$

(2) 顶点 $\quad = -ig$ $\qquad$ $\mu \quad = iQe\gamma^\mu$ ($Q=-1$ for electron)

(3) 外线项 $\phi|q\rangle = \quad q \quad = 1$ $\qquad$ $A_\mu|p\rangle = \mu \; q \; = \epsilon_\mu(p)$

$\langle q|\phi = \quad q \quad = 1$ $\qquad$ $\langle p|A_\mu = q \; \mu = \epsilon^*_\mu(p)$

E.g. Coulomb potential

$$i\mathcal{M} = \quad (p,\, p',\, \mu,\, \nu,\, k,\, k') \quad = (-ie)^2\,\bar{u}(p')\gamma^\mu u(p)\frac{-ig_{\mu\nu}}{(p-p')^2}\bar{u}(k')\gamma^\nu u(k) \qquad \Rightarrow V(r) = \frac{e^2}{4\pi}\frac{1}{r} = \frac{\alpha}{r}$$

NR limit $\begin{cases} \mu=0 & \bar{u}(p')\gamma^0 u(p) = u^\dagger(p')u(p) \approx 2m\xi'^\dagger\xi \\ \mu=i & \text{忽略}\end{cases}$

γ 矩阵求迹 Trick

γ^μ 性质：$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}$ $\begin{cases} \gamma^0 \text{ 厄米} \\ \gamma^i \text{ 反厄米} \end{cases}$

γ^5 性质：$(\gamma^5)^2 = \mathbb{1}$ $\quad \{\gamma^5,\gamma^\mu\}=0 \Rightarrow \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu \Rightarrow \boxed{\mathrm{tr}[\cdots\gamma\cdots\gamma] = \begin{cases} 0, & \text{if } \gamma \text{ 奇数个} \\ & \text{otherwise} \end{cases}}$

$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

这步则是 trAB=trBA

反对易

$$\mathrm{tr}\,\gamma^\mu = \mathrm{tr}(\gamma^\mu\gamma^5\gamma^5) = -\mathrm{tr}(\gamma^5\gamma^\mu\gamma^5) = -\mathrm{tr}(\gamma^5\gamma^5\gamma^\mu) = -\mathrm{tr}\,\gamma^\mu \Rightarrow \mathrm{tr}\,\gamma^\mu = 0$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu) = \mathrm{tr}(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2g^{\mu\nu}\times 4 - \mathrm{tr}(\gamma^\mu\gamma^\nu) \Rightarrow \mathrm{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

$$\mathrm{tr}\,\gamma^5 = \mathrm{tr}(\gamma^5\gamma^0\gamma^0) = -\mathrm{tr}(\gamma^0\gamma^5\gamma^0) = -\mathrm{tr}(\gamma^0\gamma^0\gamma^5) = -\mathrm{tr}(\gamma^5) \Rightarrow \mathrm{tr}\,\gamma^5 = 0$$

$$\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) = \pm\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5\gamma^\alpha\gamma^\alpha) = \mp\mathrm{tr}(\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^5\gamma^\alpha) = \mp\mathrm{tr}(\gamma^\alpha\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^5) = -\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5)$$
$$\Rightarrow \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$$

以及一些缩并公式，

$$\gamma^\mu\gamma_\mu = \tfrac{1}{2}g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} = g_{\mu\nu}g^{\mu\nu} = 4$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = 2g^{\mu\nu}\gamma_\mu - \gamma^\nu\gamma^\mu\gamma_\mu = (2-4)\gamma^\nu = -2\gamma^\nu$$

考虑四动量后，

$$\gamma^\mu \not{p} = p_\nu\gamma^\mu\gamma^\nu = p_\nu(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2p^\mu - \not{p}\gamma^\mu$$

$$\not{p}\not{k} = p_\mu k_\nu(2g^{\mu\nu} - \gamma^\nu\gamma^\mu) = 2p\cdot k - \not{k}\not{p}$$

$$\not{p}\not{p} = \tfrac{1}{2}2p^2 = p^2$$

$$\gamma^\mu\not{p}\gamma_\mu = p_\nu\gamma^\mu\gamma^\nu\gamma_\mu = -2\not{p}$$

$$\mathrm{tr}(\not{p}\gamma^\mu\not{k}\gamma^\nu) = p_\rho k_\sigma\,\mathrm{tr}(\gamma^\rho\gamma^\mu\gamma^\sigma\gamma^\nu) = p_\rho k_\sigma\cdot 4(g^{\rho\mu}g^{\sigma\nu} - g^{\rho\sigma}g^{\mu\nu} + g^{\rho\nu}g^{\mu\sigma})$$
$$= 4(p^\mu k^\nu - p\cdot k\, g^{\mu\nu} + p^\nu k^\mu)$$

$$\mathrm{tr}(\not{p}\gamma^\mu\not{k}\gamma^\nu)\,\mathrm{tr}(\not{q}\gamma_\mu\not{l}\gamma_\nu) = 4(p^\mu k^\nu - p\cdot k\, g^{\mu\nu} + p^\nu k^\mu)\cdot 4(q_\mu l_\nu + q_\nu l_\mu - g_{\mu\nu}\, q\cdot l)$$
$$= 32[(p\cdot q)(k\cdot l) + (p\cdot l)(k\cdot q)]$$

Casimir's Trick

$$\sum_{\text{spins}} \mathrm{tr}[u(p,\lambda)\bar{u}(p,\lambda)v(p,\lambda)\bar{v}(p,\lambda)]$$
$$= \mathrm{tr}[(\not{p}+m)(\not{q}-m)]$$

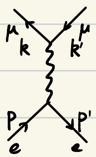
Chap 8 习题

(a) $\mathrm{tr}(\not{p}\gamma^\mu) = p_\nu\,\mathrm{tr}(\gamma^\nu\gamma^\mu) = p_\nu 4g^{\nu\mu} = 4p^\mu$

(b) $\mathrm{tr}(\not{p}\not{k}\not{q}\gamma^\mu) = p_\alpha k_\beta q_\nu\,\mathrm{tr}(\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu)$

$$= p_\alpha k_\beta q_\nu\cdot 4(g^{\alpha\beta}g^{\nu\mu} - g^{\alpha\nu}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\nu})$$
$$= 4[(p\cdot k)q^\mu - (p\cdot q)k^\mu + (k\cdot q)p^\mu]$$

1. $e^+e^- \longrightarrow \mu^+\mu^-$



$$= i\mathcal{M} = \bar{v}^{s'}(p')(-ie\gamma^{\mu})u^{s}(p)\frac{-ig_{\mu\nu}}{q^2}\bar{u}^{r}(k)(-ie\gamma^{\nu})v^{r'}(k')$$

considering $(\bar{v}\gamma^{\mu}u)^* = u^{\dagger}(\gamma^{\mu})^{\dagger}(\gamma^0)^{\dagger}v = u^{\dagger}(\gamma^{\mu})^{\dagger}\gamma^0 v = u^{\dagger}\gamma^0\gamma^{\mu}v = \bar{u}\gamma^{\mu}v$

then $|\mathcal{M}|^2 = \frac{e^4}{q^4}\left(\bar{v}(p')\gamma^{\mu}u(p)\bar{u}(p)\gamma^{\nu}v(p')\right)\left(\bar{u}(k)\gamma_{\nu}v(k')\bar{v}(k')\gamma_{\mu}u(k)\right)$

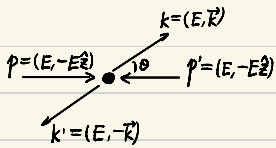
作自旋平均需考虑 $\frac{1}{4}\sum_{ss'}\sum_{rr'}|\mathcal{M}(ss'\to rr')|^2$

$$\boxed{\begin{aligned}\sum_s u^s(p)\bar{u}^s(p) &= \not{p}+m\\ \sum_s v^s(p)\bar{v}^s(p) &= \not{p}-m\end{aligned}}$$

$$\Rightarrow \sum_{ss'}\bar{v}_a^{s'}(p')\gamma^{\mu}_{ab}u^s_b(p)\bar{u}^s_c(p)\gamma^{\nu}_{cd}v^{s'}_d(p') = (\not{p}'-m)_{ad}\gamma^{\mu}_{ab}(\not{p}+m)_{bc}\gamma^{\nu}_{cd}$$
$$= \mathrm{tr}\left[(\not{p}'-m)\gamma^{\mu}(\not{p}+m)\gamma^{\nu}\right]$$

Finally we got $\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4q^4}\mathrm{tr}\left[(\not{p}'-m_e)\gamma^{\mu}(\not{p}+m_e)\gamma^{\nu}\right]\mathrm{tr}\left[(\not{k}+m_{\mu})\gamma_{\mu}(\not{k}'-m_{\mu})\gamma_{\nu}\right]$

$$= \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_{\mu}^2\, p\cdot p'\right]$$



$e^+e^-\to\mu^+\mu^-$ 反应中
$$\begin{cases} q^2 = (p+p')^2 = 4E \\ p\cdot p' = 2E^2 \\ p\cdot k = p'\cdot k' = E^2 - E|\vec{k}|\cos\theta \\ p\cdot k' = p'\cdot k = E^2 + E|\vec{k}|\cos\theta \end{cases}$$

therefore $\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = e^4\left[\left(1+\frac{m_{\mu}^2}{E^2}\right) + \left(1-\frac{m_{\mu}^2}{E^2}\right)\cos^2\theta\right].$

recall that $d\sigma = \frac{1}{2E_A 2E_B|v_A - v_B|}\boxed{\prod_f\left(\frac{d^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)(2\pi)^4\delta^{(4)}(p_A+p_B-\sum_f p_f)}|\mathcal{M}(A+B\to f)|^2$

with $E_A = E_B = E_f = E = \frac{1}{2}E_{cm}$, $|v_A - v_B| \simeq 2$ $(m_e\simeq 0)$

we get $d\sigma = \frac{1}{2E\cdot 2E\cdot 2}\frac{d^3k}{(2\pi)^3 2E}\frac{d^3k'}{(2\pi)^3 2E}\cdot(2\pi)^4\delta^{(4)}(p+p'-k-k')\cdot\left(\frac{1}{4}\sum_{spins}|\mathcal{M}|^2\right)$

$$= \frac{d^3k}{(2E)^4 2(2\pi)^2}\delta(E_{cm}-2E)\left(\frac{1}{4}\sum_{spins}|\mathcal{M}|^2\right)$$

$$= \frac{|\vec{k}|^2 d|\vec{k}|\, d\Omega}{(2E)^4 2(2\pi)^2}\delta(E_{cm}-2E)\left(\frac{1}{4}\sum_{spins}|\mathcal{M}|^2\right)$$

$$|\vec{k}| = \sqrt{E^2 - m_\mu}$$
$$|\vec{k}|\cdot d|\vec{k}| = E\,dE$$

$$= \frac{|\vec{k}|\,E\,dE\,d\Omega}{(2E)^4\,2\,(2\pi)^2}\,\delta(E_{cm}-2E)\left(\frac{1}{4}\sum_{spins}|\mathcal{M}|^2\right)$$

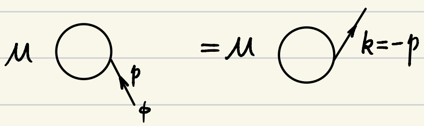
$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{|\vec{k}|\,E}{(2E)^4\,2\,(2\pi)^2}\cdot\frac{1}{2}\cdot\left(\frac{1}{4}\sum_{spins}|\mathcal{M}|^2\right)$$

$$= \frac{\alpha^2}{4E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left[\left(1+\frac{m_\mu^2}{E^2}\right)+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$\sigma_{total} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)$$

Crossing Symmetry theorem $\mathcal{M}(\phi(p)+\cdots\to\cdots) = \mathcal{M}(\cdots\to\cdots+\bar{\phi}(k))$

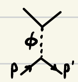
$k=-p$



$$\sum u(p)\bar{u}(p) = \not{p}+m \longrightarrow -(\not{k}-m) = -\sum v(k)\bar{v}(k)$$

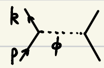
Mandelstam variables

$s = (p+p')^2 = (k+k')^2$



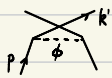
$\mathcal{M} \propto \frac{1}{s-m_\phi^2}$ s-channel

$t = (k-p)^2 = (k'-p')^2$



$\mathcal{M} \propto \frac{1}{t-m_\phi^2}$ t-channel

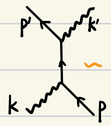
$u = (k'-p)^2 = (k-p')^2$



$\mathcal{M} \propto \frac{1}{u-m_\phi^2}$ u-channel

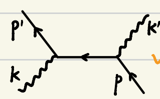
2. Compton Scattering $e^-\gamma\to e^-\gamma$

$e^-\gamma \to e^-\gamma =$



s-channel

$+$



t-channel

$$i\mathcal{M} = \bar{u}(p')(-ie\gamma^\mu)\,\epsilon_\mu^*(k')\,\frac{i(\not{k}+\not{p}+m)}{(k+p)^2-m^2}\,\epsilon_\nu(k)(-ie\gamma^\nu)\,u(p) + \bar{u}(p')(-ie\gamma^\mu)\,\epsilon_\mu(k)\,\frac{i(\not{p}-\not{k}'+m)}{(p-k')^2-m^2}\,u(p)(-ie\gamma^\nu)\,\epsilon_\nu^*(k')$$

$$= -ie^2\epsilon_\mu^*(k')\epsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu\not{k}\gamma^\nu+2\gamma^\mu p^\nu}{2p\cdot k}+\frac{-\gamma^\nu\not{k}'\gamma^\mu+2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p)$$

with Spin sums: $\begin{cases}\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m \\ \sum_{polariz} \epsilon_\mu^*\epsilon_\nu = -g_{\mu\nu}\end{cases}$

根据QED相互作用拉氏量

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial}-m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\,\bar{\psi}\gamma^\mu\psi A_\mu \xRightarrow{U(1)\text{ symmetry}} J^\mu = e\bar{\psi}\gamma^\mu\psi$$

$\partial_\mu J^\mu = 0$

thus $\mathcal{L}_{int} = -A_\mu J^\mu_{EM}$

再考虑 E-L Eq. 后可得 $0=\partial_\mu \dfrac{\partial \mathcal{L}_{QED}}{\partial(\partial_\mu A_\nu)}-\dfrac{\partial \mathcal{L}_{QED}}{\partial A_\nu}=\partial_\mu(-F^{\mu\nu})+J^\nu_{EM} \Rightarrow \partial_\mu F^{\mu\nu}=J^\nu_{EM}$

铺垫完毕，接下来看 动量空间的 Ward Identity:

$$i\mathcal{M}=i\mathcal{M}(k)\equiv i\mathcal{M}^\mu(k)\,\epsilon^*_\mu(k)\ ,\ \text{then}\quad k_\mu \mathcal{M}^\mu(k)=0$$

$$\parallel\ i\mathcal{M}^{\mu\nu}\epsilon^*_\mu(k')\,\epsilon_\nu(k)\ ,\quad \text{then}\quad k'_\mu\mathcal{M}^{\mu\nu}=k_\nu\mathcal{M}^{\mu\nu}=0$$

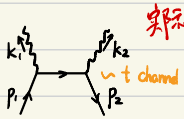
electron
photon

$$\frac{1}{2}\cdot\frac{1}{2}\sum_{spins}|\mathcal{M}|^2=\cdots(\text{complex!})\cdots=2e^4\left[\frac{p\cdot k'}{p\cdot k}+\frac{p\cdot k}{p\cdot k'}+2m^2\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)+m^4\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)^2\right]$$

Klein-Nishina Formula $\quad \dfrac{d\sigma}{d\cos\theta}=\dfrac{\pi\alpha^2}{m^2}\left(\dfrac{\omega'}{\omega}\right)^2\left[\dfrac{\omega'}{\omega}+\dfrac{\omega}{\omega'}-\sin^2\theta\right]$

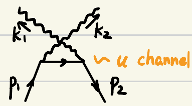
3. Pair Annihilation $e^+e^-\to\gamma\gamma$

$e^+e^-\to\gamma\gamma\quad =$



实际上是"−"

+



Using crossing symmetry, do the replacement $\quad p\to p_1\ ,\ k\to -k_1\ ,\ p'\to -p_2\ ,\ k'\to k_2$

We can obtain $\quad \dfrac{1}{4}\sum_{spins}|\mathcal{M}|^2=2e^4\left[-\dfrac{p_1\cdot k_2}{p_1\cdot k_1}-\dfrac{p_1\cdot k_1}{p_1\cdot k_2}-2m^2\left(\dfrac{1}{p_1\cdot p_2}+\dfrac{1}{p_1\cdot k_1}\right)+m^4\left(\dfrac{1}{p_1\cdot p_2}+\dfrac{1}{p_1\cdot k_1}\right)^2\right]$

Or calculate it directly $\quad i\mathcal{M}=u(p_1)(-ie\gamma^\mu)\epsilon^*_\mu(k_1)\dfrac{i(\not{p}_1-\not{k}_1+m)}{(p_1-k_1)^2-m^2}\epsilon^*_\nu(k_2)(-ie\gamma^\nu)\bar{u}(p_2)$