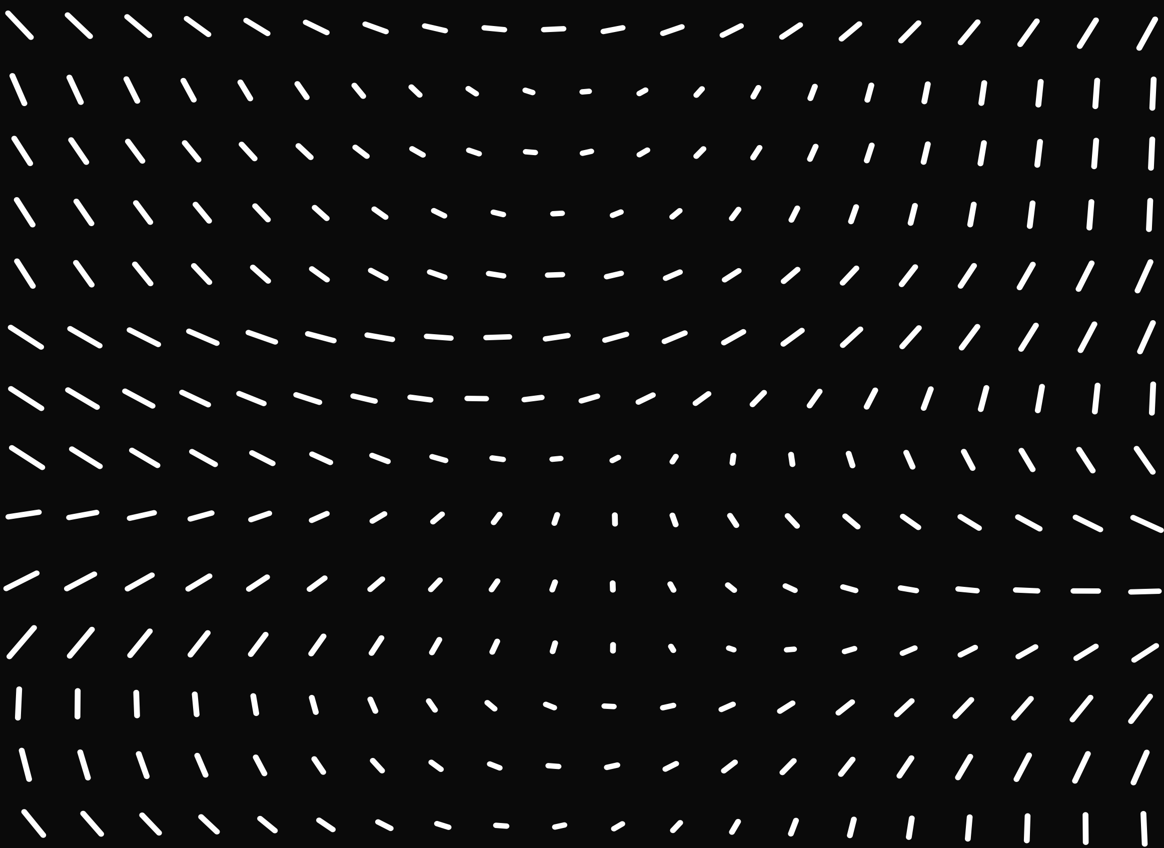
电动力学听课随记

Electronic Dynamics Lecture Notes



-1. Foreword

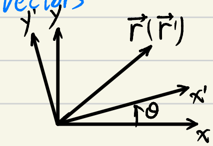
2023 Spring 读陶鑫老师的ED，助教是某热心的Au moon him大佬，收获颇丰，在pk社区上也写了长评好评。可以说是学的最认真的一集。

fall学期粗粗整了一下notes，其中不足，还请各路大佬多多批评。

0. Contents

Chapter 1 Vector & Tensor Analysis

Vectors



$$\vec{r}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = T\vec{r}$$

In Minkowski space for special relativity has the displacement

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \rightarrow (ct, \vec{r})$$

so do all the 4-vectors like $(\frac{\omega}{c}, \vec{k})$, $(\frac{\varepsilon}{c}, \vec{p})$, $(\rho c, \vec{j})$

Einstein Summation Rule

Rule #1 : repeated indices → dummy indices

Rule #2 : non-repeated indices → free indices

Rule #3 : each index can appear at most twice in any term !

Kronecker delta $\delta_i^j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$ Levi-Civita symbol 置換

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } i,j,k \text{ form an even permutation of } 1,2,3. \\ -1 & \text{if } i,j,k \text{ form an odd permutation of } 1,2,3. \\ 0 & \text{otherwise} \end{cases}$$

Useful : $\boxed{\varepsilon_{ijk}\varepsilon_{imn} = \delta_m^j \delta_n^k - \delta_m^k \delta_n^j}$

nubla operator : $\nabla = e_i \frac{\partial}{\partial x_i}$ ⟶ vector & operator Both !

then $\nabla \times (\vec{A} \times \vec{B}) = \nabla_A \times (\vec{A} \times \vec{B}) + \nabla_B \times (\vec{A} \times \vec{B})$

$$= (\vec{B} \cdot \nabla)\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\nabla \cdot \vec{B})\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$$

$$\nabla(\vec{A} \cdot \vec{B}) = \nabla_A(\vec{A} \cdot \vec{B}) + \nabla_B(\vec{A} \cdot \vec{B})$$

$$= \vec{B} \times (\nabla \times \vec{A}) + (\vec{B} \cdot \nabla)\vec{A} + \vec{A} \times (\nabla \times \vec{B}) + (\vec{A} \cdot \nabla)\vec{B}$$

not simply replace !

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

also we have to use indices to calculate these formulas !

$$\nabla \times (\vec{A} \times \vec{B}) = \vec{e_i}\, \varepsilon_{ijk} \frac{\partial}{\partial x_j}(\varepsilon_{kmn} A_m B_n) = \varepsilon_{kij}\varepsilon_{kmn} \vec{e_i} \frac{\partial}{\partial x_j}(A_m B_n)$$

$$= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\left(\frac{\partial A_m}{\partial x_j} B_n + \frac{\partial B_n}{\partial x_j} A_m\right)\vec{e_i}$$

$$= \left(\frac{\partial A_i}{\partial x_j} B_j + \frac{\partial B_j}{\partial x_j} A_i - \frac{\partial A_j}{\partial x_j} B_i - \frac{\partial B_i}{\partial x_j} A_j\right)\vec{e_i}$$

$$= (\vec{B} \cdot \nabla)\vec{A} + (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} - (\vec{A} \cdot \nabla)\vec{B}$$

this process it quite difficult !

Tensors (2^{nd} order)

$\vec{g} = \dfrac{\Delta m \vec{v}}{\Delta V} = \rho_M \vec{v}$

mass density

define : flux (amount of X passing through unit area per unit time)

Energy flux : $\varepsilon \vec{v}$ (ε : energy density)

Momentum flux : $(g_x \vec{v}, g_y \vec{v}, g_z \vec{v}) \Longleftrightarrow \overleftrightarrow{T} = \vec{g}\vec{v} = \rho_M \vec{v}\vec{v}$ (量纲 $J \cdot m^{-3}$)

应力张量

here T_{xx} T_{yy} T_{zz} 压强力

T_{yx} ⋯ ⋯ 剪切力

unit tensor $\overleftrightarrow{I} = \delta_{ij} \vec{e_i}\vec{e_j} = \nabla \vec{r}$

passing through area $\vec{S}$? $\overleftrightarrow{T} \cdot \vec{S}$

transform for tensor : $T'_{ij} = M_{ik} M_{jl} T_{kl}$

prove : $\nabla \cdot (\vec{f}\vec{g} \times \vec{r}) = \nabla_f \cdot (\vec{f}\vec{g} \times \vec{r}) + \nabla_g \cdot (\vec{f}\vec{g} \times \vec{r}) + \nabla_r \cdot (\vec{f}\vec{g} \times \vec{r})$

$(\nabla_f \cdot \vec{f})\, \vec{g} \times \vec{r} = (\nabla \cdot \vec{f})\, \vec{g} \times \vec{r}$ $\quad (\vec{f} \cdot \nabla \vec{g}) \times \vec{r}$ $\quad$ Scalar $\boxed{(\nabla_r \cdot \vec{f})}\, \vec{g} \times \vec{r} = \vec{g} \times [(\nabla_r \cdot \vec{f})\vec{r}]$

$= \vec{g} \times (\vec{f} \cdot \nabla \vec{r}) = \vec{g} \times \vec{f}$

Dirac delta function

definition $\delta(x) = \begin{cases} 0 , & x \neq 0 \\ \infty , & x = 0 \end{cases}$ & $\displaystyle\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$, $f(x)\delta(x-a) = f(a)\delta(x-a)$

related to †Fourier Transformation $\displaystyle\int e^{i(\omega - \omega')t} dt = 2\pi\, \delta(\omega - \omega')$

Thus $\displaystyle\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$ 用于微观(尺度足够小!A)

e.g. 1D particle density distribution $\rho(x,t)$ where $x_p(t)$. then $\rho(x,t) = m\,\delta(x - x_p)$, $x_p = x_p(t)$

† $\delta[g(x)] = \displaystyle\sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}$, where $g(x_i) = 0$ † $\displaystyle\int_{-\infty}^{\infty} f(x) D_1(x)\, dx = \int_{-\infty}^{\infty} f(x) D_2(x)\, dx$ then $D_1(x) = D_2(x)$

let $y = g(x)$, then $\delta(g(x)) = \delta(y)$ $\therefore \displaystyle\int_{y_0 - \Delta}^{y_0 + \Delta} \delta(y)\, dy = \int_{x_0 - \Delta}^{x_0 + \Delta} \delta(g(x)) \cdot g'(x)\, dx \Rightarrow \delta(g(x)) = \frac{\delta(x)}{|g'(x)|}$ ← ? Jacob

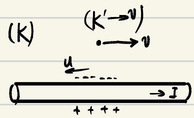
3-D dirac function $\delta(\vec{r}) = \delta(x)\delta(y)\delta(z)$, $\vec{r} = (x, y, z)$

六维位形空间 $(\vec{x}, \vec{v})$ 中的 δ 函数 $\quad f = \displaystyle\sum_i \left[\delta(\vec{r} - \vec{x}_i(t))\, \delta(\vec{r} - \vec{v}_i(t)) \right]$

Chapter 2 Foundations of Theory of Relativity

Electrodynamics of Moving Body:

$\begin{cases} \text{Principle of Relativity} \\ \text{Constancy of Speed of Light} \end{cases}$



f_K? $f_{K'}$?

interval : $ds = \left(c^2dt^2 - dx^2 - dy^2 - dz^2\right)^{1/2}$ OR $ds^{\alpha} = (c\,dt, d\vec{r})$

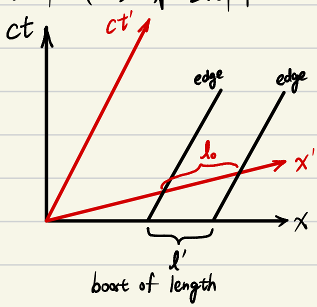
s is invariant between internal references !

proper time $d\tau = \frac{ds}{c} = \frac{dt}{\gamma} \Rightarrow d\tau < dt$

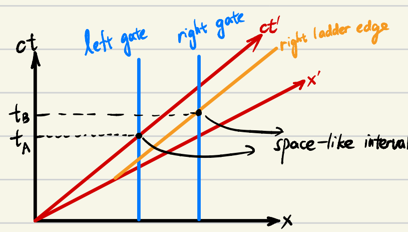
Lorentz Transformation

$$\begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} A'^0 \\ A'^1 \\ A'^2 \\ A'^3 \end{pmatrix} \Leftrightarrow A^{\mu} = \Lambda^{\mu}_{\ \nu} A'^{\nu}$$

世界线观点来理解 Paradox



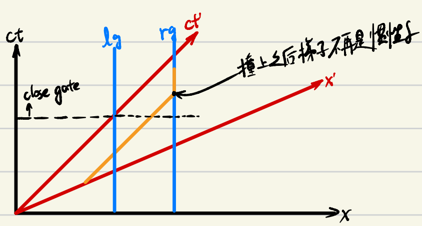
$\therefore l_0 = \gamma l'$



ladder paradox v1

Most paradox is based on space-like intervals !

In v2：前端撞上钢门后，"停止"的信号传播到后端的速度上限为光速，故后端不会瞬时同时停止，因此仍会继续前进，能够被关进门.



ladder paradox v2

volume in spacetime

$$dV_0 = dV\gamma \ , \quad d\tau = dt/\gamma \Rightarrow dV_0 d\tau \equiv dV dt = d\Omega \ (\text{unit volumn in 4-D spacetime})$$

Conversion between covarient and contravariant components IN MINKOWSKI SPACE

the metric coefficent $g^{\alpha\beta} = g_{\alpha\beta} = \begin{cases} 0 & \alpha\neq\beta \\ 1 & \alpha=\beta=0 \\ -1 & \text{otherwise} \end{cases}$, $g^{\alpha}_{\cdot\beta} = \begin{cases} 0 & \alpha\neq\beta \\ 1 & \alpha=\beta=0,1,2,3 \end{cases} \equiv \delta^{\alpha}_{\beta}$ (unit tensor)

数值上一致

definition: $g^{ij} = \vec{e}^{i}\cdot\vec{e}^{j}$, $g^{i}_{\ j} = \vec{e}^{i}\cdot\vec{e}_{j} = \delta^{i}_{j}$

thus $g_{\alpha}^{\ \beta} \triangleq \delta^{\beta}_{\alpha}$ and $A_\alpha = g_{\alpha\beta}A^\beta$ e.g. $\begin{cases} A_0 = g_{0\beta}A^\beta = g_{00}A^0 = A^0 \\ A_i = g_{ij}A^j = g_{ii}A^i \quad (\text{no summation}) \end{cases}$

Minkowski space : 4-scalar

invariant between conversions !

Minkowski space : 4-tensor

$$F^{\mu\nu} \sim A^\mu B^\nu \quad \& \quad F^{\mu\nu\alpha} \sim A^\mu B^\nu C^\alpha \ (\text{dyad form})$$

$$F^{00} \sim A^0 B^0 = \gamma(A'^0 + \beta A'^1)\gamma(B'^0 + \beta B'^1) = \gamma^2(A'^0B'^0 + \beta A'^0B'^1 + \beta A'^1B'^0 + \beta^2 A'^1B'^1)$$

$$= \gamma^2\left[F'^{00} + \underbrace{\beta(F'^{01} + F'^{10})}_{\text{如果反对称则该项消失}} + \beta^2 F'^{11}\right]$$

thus

$$A^\alpha = \Lambda^\alpha_{\ \beta} A'^\beta \ , \quad A^\alpha B^\beta = \Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} A'^\mu B'^\nu$$

$$\Leftrightarrow F^{\alpha\beta} = \Lambda^\alpha_{\ \mu}\Lambda^\beta_{\ \nu} F'^{\mu\nu}$$

and

$$F^{\alpha\beta} = g^{\alpha\mu} F_{\mu\nu} g^{\beta\nu}$$

$\Rightarrow$ useful rules $\begin{cases} F^0 \longleftrightarrow F_0 \\ F^i \longleftrightarrow (-1)F_i \ , \quad i=1,2,3 \end{cases}$ i.e. $\begin{cases} F_{01} = -F_0^{\ 1} \\ F_{11} = F^{11} \end{cases}$

Construct 4-scalar

$$F^{\alpha\beta}F_{\beta\alpha} \qquad F^{\alpha}_{\cdot\alpha}\ (\text{trace}) \qquad A^\alpha F_{\alpha\beta} B^\beta = \overleftrightarrow{BA} : \overleftrightarrow{F}$$

Basic Differential Calculus

4-scalar gradient $d\phi = \frac{\partial \phi}{\partial x^\mu} dx^\mu$, $\underbrace{\frac{\partial \phi}{\partial x^\mu}}_{\text{Co-variant}} = \left(\frac{\partial \phi}{\partial x^0}, \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2}, \frac{\partial \phi}{\partial x^3} \right) = \partial_\mu \phi = \left(\frac{1}{c} \frac{\partial \phi}{\partial t}, \nabla \phi \right)$

Co-variant

Contra-variant

$$\frac{\partial \phi}{\partial x_\mu} = \partial^\mu \phi = \left(\frac{1}{c} \frac{\partial \phi}{\partial t}, -\nabla \phi \right)$$

and $\partial_\mu \partial^\mu = \boxed{\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2} = \square$

4-vector divergence $\partial_\alpha A^\alpha \left(= \partial^\alpha A_\alpha \right) = \frac{\partial A^\alpha}{\partial x^\alpha} \left(= \frac{\partial A_\alpha}{\partial x_\alpha} \right) = \frac{1}{c} \frac{\partial A^0}{\partial t} + \nabla \cdot \vec{A}$

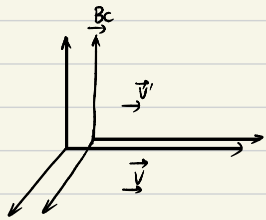
e.g.: $\partial_\mu j^\mu = \frac{1}{c} \frac{\partial (\rho c)}{\partial t} + \nabla \cdot \vec{j} \equiv 0$

Important 4-vectors:

(1) 4-velocity

$u^\alpha = dx^\alpha / d\tau$ 四标量方可使用, dt则不行

$= (\gamma c, \gamma \vec{v})$



transformation:

$$\begin{cases} \gamma c = Y(\gamma' c + B \gamma' v_x') & ① \\ \gamma v_x = Y(\gamma' v_x' + B \gamma' c) & ② \\ \gamma v_y = \gamma' v_y' & ③ \\ \gamma v_z = \gamma' v_z' & ④ \end{cases}$$

$② / ① \Rightarrow \frac{v_x}{c} = \frac{v_x' + Bc}{c + B v_x'} = \frac{v_x'/c + B}{1 + v_x' B/c}$, $③ / ① \Rightarrow \frac{v_y}{c} = \frac{1}{Y} \cdot \frac{v_y'}{c + B v_x'} = \frac{1}{Y} \cdot \frac{v_y'/c}{1 + B v_x'/c}$

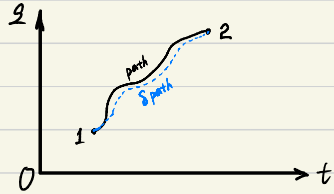
& $u^\alpha u_\alpha \equiv c^2$ (inv)

$p^\alpha = m u^\alpha = (\gamma m c, \gamma m \vec{v}) = \left(\frac{\varepsilon}{c}, \vec{p} \right)$

$p^\alpha p_\alpha = \frac{\varepsilon^2}{c^2} - \vec{p}^2 = m^2 c^2$ (inv)

$\varepsilon^2 = (mc^2)^2 + p^2 c^2$

Chapter 3. Relastic Dynamics



The Euler-Lagrange eq.

$$S=\int_{t_1}^{t_2} dS=\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$$

扰动 $q=q+\delta q$, $\dot{q}=\dot{q}+\delta\dot{q}$, 则有

$$\delta S=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right)dt \underset{\text{target}}{=\!=} \int_{t_1}^{t_2}(\cdots\cdots)\,\delta q\,dt=0$$

$$\boxed{\begin{array}{l}\text{同理对于某场 }\phi,\ \phi=\phi(\vec{x},t),\ \text{其 Lagrangian density } \mathcal{L}=\mathcal{L}(\phi,\phi_{t,x})\\ \text{求解其运动的第一步便为 } \delta S=\int(\cdots\cdots)\,\delta\phi=0\end{array}}$$

注意到 $\delta\dot{q}=\frac{d}{dt}\delta q$ ，故 $\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}}\delta\dot{q}\,dt=\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}}\,d\delta q=\left[\frac{\partial L}{\partial \dot{q}}\delta q\right]\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\delta q\,d\left(\frac{\partial L}{\partial \dot{q}}\right)$

即 $\delta S=\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q\,dt=0 \Rightarrow \frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=0$

$$L'=L+\frac{d}{dt}f(q,t)$$

Total Lagrangian for the uncoupled system

$$L=L_1+L_2 \Rightarrow \begin{cases}\frac{\partial L_1}{\partial q_1}-\frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_1}=0\\ \frac{\partial L_2}{\partial q_2}-\frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}_2}=0\end{cases}$$

Total Lagrangian for the coupled system

$$L=L_1+L_2+L_c\ ,\ L_c=\alpha q_1 q_2 \quad \begin{cases}\frac{\partial L_1}{\partial q_1}-\frac{d}{dt}\frac{\partial L_1}{\partial \dot{q}_1}=-\alpha q_2\\ \frac{\partial L_2}{\partial q_2}-\frac{d}{dt}\frac{\partial L_2}{\partial \dot{q}_2}=-\alpha q_1\end{cases}$$

Lagrangian for particle in a potential field

$L=\frac{1}{2}mv^2+\alpha\phi$ $\quad\alpha$: coupling coefficient

The covariant eq. of motion

operater : $\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2=\partial_\alpha\partial^\alpha$ 达朗贝尔算符，方具有相对论不变性.

Key : $x^\alpha \rightarrow \underline{x^\alpha+\delta x^\alpha}$

Chapter 4 Charges in Given EM Field

covariant eq. of motion

$$S=\int_a^b\left(-mc^2 d\tau - q A_\alpha dx^\alpha\right) \qquad \delta S_p = -\int mc\,\delta(ds) = \int \underset{\sim\sim\sim}{\frac{dp_\alpha}{d\tau}}\cdot \delta x^\alpha \cdot d\tau$$

$\delta ds = \delta\sqrt{dx^\alpha dx_\alpha}$

$$\delta S_{pf} = -q\int_a^b\left(\delta A_\alpha \cdot dx^\alpha + A_\alpha \cdot d\,\delta x^\alpha\right)$$

$$\overset{\text{integrating by parts}}{=\!=\!=} -q\int_a^b \delta A_\alpha dx^\alpha + q\int_a^b \delta x^\alpha dA_\alpha$$

according to: $\delta A_\alpha = \frac{\partial A_\alpha}{\partial x_\beta}\delta x_\beta$, $dA_\alpha = \frac{\partial A_\alpha}{\partial x_\beta} dx_\beta$, we have:

$$\delta S_{pf} = q\int_a^b\left(\frac{\partial A_\alpha}{\partial x^\beta}\cdot dx^\beta \delta x^\alpha - \frac{\partial A_\alpha}{\partial x^\beta}\cdot dx^\alpha \delta x^\beta\right)$$

$$= q\int_a^b\left(\frac{\partial A_\alpha}{\partial x^\beta} dx^\beta \delta x^\alpha - \frac{\partial A_\beta}{\partial x^\alpha}\cdot dx^\beta\cdot \delta x^\alpha\right)$$

$$= q\int_a^b\left(\frac{\partial A_\alpha}{\partial x^\beta}\cdot u^\beta d\tau\, \delta x^\alpha - \frac{\partial A_\beta}{\partial x^\alpha}\cdot u^\beta d\tau\, \delta x^\alpha\right)$$

$$= -\int_a^b q\left(\frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\right) u^\beta d\tau\cdot \delta x^\alpha$$

we define 4-tensor $F_{\alpha\beta}$ by:

$$\boxed{F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha}$$

$F_{ij} = -F_{ji}$ 反对称张量

then

$$\delta S = \int_a^b\left[\frac{dp_\alpha}{dt} - q F_{\alpha\beta} u^\beta\right] d\tau \cdot \delta x^\alpha$$

单粒子在外场(电磁场)中的行为用此描述

$$\Longrightarrow \boxed{\frac{dp_\alpha}{dt} = q F_{\alpha\beta} u^\beta}$$

(co- form of eq. of motion)

$$F_{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}.$$

Moreover: $\begin{cases} F_{0i} = E_i/c \\ F_{i0} = -E_i/c \end{cases}$ $\quad F_{ij} = -\epsilon_{ijk}B^k \qquad i=1,2,3\ (i\neq 0!)$

Further More $\left\{\begin{aligned} &E_i = c F_{0i} \\ &B^k = -\frac{1}{2}\varepsilon^{ijk} F_{ij} \end{aligned}\right.$

Prove: $$-\frac{1}{2}\varepsilon^{ijk} F_{ij} = -\frac{1}{2}\varepsilon^{ijk}\left(-\varepsilon_{ijl} B^l\right) = \frac{1}{2}\left[\overset{3}{\boxed{\delta^j_j}}\delta^k_l - \delta^k_j \delta^j_l\right] B^l = \frac{1}{2}\left[3\delta^k_l - \delta^k_l\right] B^l = \delta^k_l B^l = B^k$$

For $\boxed{\text{eq. motion}}$, it has

$$\frac{dp^0}{d\tau} = q F^{0\beta} u_\beta \Rightarrow \frac{d(\varepsilon/c)}{dt/\gamma} = q\left(+\frac{E^i}{c}\right)\gamma v_i \Rightarrow \frac{d\varepsilon}{dt} = q\vec{E}\cdot\vec{v}$$

$$\frac{dp^i}{dt/\gamma} = q F^{i\beta} u_\beta = q F^{i0} u_0 + q F^{ij} u_j = q \frac{E^i}{c}\cdot\gamma c + q\cdot\varepsilon^{ijk} B_k \cdot \gamma v_j$$

$$\Longleftrightarrow \frac{d\vec{p}}{dt} = q\vec{E} + q\vec{v}\times\vec{B}$$

$C^T C = I$ 酉矩阵 (转置共轭×自己=unit)

$$\begin{tikzcd} x_0 \arrow[r] \arrow[d] & C x_0 \arrow[d] \\ A x_0 \arrow[r] & A C x_0 \end{tikzcd}$$

Transformation of the EM field $\left\{\begin{aligned} &A^\alpha = (\phi/c, \vec{A}) \\ &F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \end{aligned}\right.$

Lorentz's Matrix $\Lambda^\alpha{}_\beta = \begin{pmatrix} \gamma & \gamma\beta & & \\ \gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$$A^\alpha = \Lambda^\alpha{}_\beta A'^\beta$$

$$F^{12} = \gamma F'^{12} + \gamma\beta F'^{02} \Rightarrow B_z = \gamma B'_z + \gamma\beta E'_y/c$$

$$\Rightarrow \left\{\begin{aligned} &E_x = E'_x, \quad E_y = \gamma E'_y + \gamma\beta B'_z, \quad E_z = \gamma E'_z - \gamma\beta B'_y \\ &B_x = B'_x, \quad B_y = \gamma B'_y - \gamma\beta E'_z, \quad B_z = \gamma B'_z + \gamma\beta E'_y \end{aligned}\right.$$

for the non-relativistic situation $V/c \ll 1$, we have

$$\boxed{\vec{E} = \vec{E}' + \vec{B}' \times \vec{V} \ , \quad \vec{B} = \vec{B}' - \frac{1}{c^2}\vec{E}' \times \vec{V}}$$

Important invariants related to EM field

*1) $F^{\alpha\beta} F_{\alpha\beta} = \text{inv.} = c^2B^2 - E^2$

2) 4^{th} order unit tensor $\boxed{e^{\alpha\beta\mu\nu}}$

i) $e^{0123} = 1$

ii) $e^{\alpha\beta\mu\nu} = 0$ if any two indices are the same

iii) $e^{\alpha\beta\mu\nu}$ changes sign if interchanging any pair of indices

thus $F_{\alpha\beta}\, e^{\alpha\beta\mu\nu} F_{\mu\nu} = \text{inv.} \Rightarrow \vec{E} \cdot \vec{B} = \text{inv.}$

e.g. ① If $E > cB$ in K, then $E' > cB'$ in ALL K'

② If $\vec{E} \perp \vec{B}$ in K, then $\vec{E}' \perp \vec{B}'$ in ALL K'.

e.g: Non-Relativity, Faraday: $\frac{c\vec{k}}{\omega} \times \vec{E} = \vec{B}c$, profe: $(v_{\parallel} - \omega/k)^2 + v_{\perp}^2 = \text{const.}$, when: $n = ck/\omega \gg 1$

似动能守恒关系

Proof: $cB - E = (n-1)E > 0$ and $\vec{E} \cdot \vec{B} = 0$

thus, we can find a reference frame K', where $E' = 0$.

$$\Rightarrow \vec{E} = \vec{B} \times \vec{V}$$

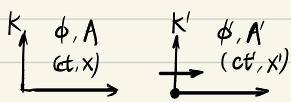
therefore

$$\vec{V} = \frac{\omega}{k} \hat{k}$$

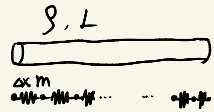
then in K' frame, we have $v_{\parallel}'^2 + v_{\perp}'^2 = \left(v_{\parallel} - \frac{\omega}{k}\right)^2 + v_{\perp}^2 = \text{Const.}$

Example: Fields of a moving particle

refer to TX's lecture P52.



Chapter 5 The EM Field Equations



$\Delta x = \frac{L}{N}$

$m = \rho \frac{L}{N}$

Principle of Least Action in $\boxed{\text{Scalar Field}}$

$L = L(q_i, \dot{q}_i, t)$

对应n个 Eular-Largrange eq.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad (j=1,2,\cdots,N)$$

令 $N \to +\infty$, 上述N个偏微分方程变为连续方程

$$\delta S_p = 0 \Rightarrow f(q, \dot{q}, t) = 0$$

$$\delta S_f = 0 \Rightarrow f(\phi, \partial_\mu \phi) = 0$$

$$L_f = \int \mathcal{L}_f dV, \quad \boxed{\mathcal{L}_f = \mathcal{L}_f(\phi, \partial_\mu \phi)}$$

thus $S_f = \int L_f \, dt = \boxed{\int \mathcal{L}_f \, dV \, dt}$, here $c\,dV\,dt = d\Omega$ (4-Dimension Unit Volumn)

For thoses scalar field, we can have its evolve eq.

※对φ进行扰动!

1) $\phi = \phi(t)$, $S_f = \int \mathcal{L} \, dt$, $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

*2) $\phi = \phi(x, y, z)$, $S_f = \int \mathcal{L} \, dx dy dz dt$, $\partial_i \left(\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

*3) $\phi = \phi(x^\mu)$, $S_f = \int \mathcal{L} \, dx dy dz dt$, $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$

*Example: $\mathcal{L} = \frac{1}{2} \partial_\alpha \phi \, \partial^\alpha \phi$

Lagrang eq. $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) - \underbrace{\boxed{\frac{\partial \mathcal{L}}{\partial \phi}}}_{=0} = 0$ (求导时固定 $\partial_\alpha \phi$, 故认为零)

$\partial_\nu \phi = \delta_\nu{}^\alpha \partial_\alpha \phi \qquad \partial^\nu \phi = g^{\nu\alpha} \partial_\alpha \phi$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)} = \frac{1}{2} \frac{\partial}{\partial(\partial_\alpha \phi)} (\partial_\nu \phi \, \partial^\nu \phi) = \frac{1}{2} \left[\frac{\partial(\partial_\nu \phi)}{\partial(\partial_\alpha \phi)} \partial^\nu \phi + \frac{\partial(\partial^\nu \phi)}{\partial(\partial_\alpha \phi)} \partial_\nu \phi\right]$$

$$= \frac{1}{2} \left[\delta_\nu{}^\alpha \partial^\nu \phi + g^{\nu\alpha} \partial_\nu \phi\right]$$

$$= \frac{1}{2} \left[\partial^\alpha \phi + \partial^\alpha \phi\right] = \partial^\alpha \phi$$

thus $\Rightarrow \partial_\alpha (\partial^\alpha \phi) = \partial_\alpha \partial^\alpha \phi = \underline{\left(\frac{1}{c^2} \frac{\partial}{\partial t^2} - \nabla^2\right)} \phi = 0$

$\Box$: d'Alembert Operator

*Example field + particle

$\mathcal{L}_f = \frac{1}{2} \partial_\alpha \phi \, \partial^\alpha \phi$, $S_f = \int \mathcal{L}_f \, dV dt$, $S_{pf} = \int -q \phi(t, x_p) \, dt$; $S = S_f + S_{pf}$

along the trace

To do this, we have to construct $\mathcal{L}_{pf}$ for S_{pf} !!!

Only

i.e: $\int \mathcal{L}_{pf}\, dx\,dy\,dz = -q\phi(t,x_p) \Rightarrow \mathcal{L}_{pf} = -q\phi(t,x_p)\,\delta(\vec{x}-\vec{x}_p)$

For Largrang eq.: $\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$ Only

$$\Rightarrow \Box\phi = -q\,\delta(\vec{x}-\vec{x}_p)$$

$\boxed{\text{Vector Field}}$

field: $A^\nu = A^\nu(x^\mu)$, its Largrangion density: $\mathcal{L} = \mathcal{L}(A^\nu, \partial_\mu A^\nu)$

$$\xRightarrow{\text{E-L eq.}} \boxed{\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0}$$

from now we difine

$$S = \int \mathcal{L}\, d\Omega, \quad d\Omega = c\,dt\cdot dx\,dy\,dz$$

About (Simply) EM Field

Noting this:

1. $\mathcal{L}$ a 4-scalar; 2. Gauge invariance; 3. accord with Experiments !!!

$\Rightarrow$ linear in E, B $\Rightarrow$ linear in $\partial_\mu A^\nu$

......

$$\Rightarrow \mathcal{L} = -\frac{1}{4\mu_0 c} F_{\alpha\beta}F^{\alpha\beta}$$

$$= -\frac{1}{4\mu_0 c}\cdot(\partial_\alpha A_\beta - \partial_\beta A_\alpha)(\partial^\alpha A^\beta - \partial^\beta A^\alpha)$$

$$= -\frac{1}{4\mu_0 c}\left(\boxed{\partial_\alpha A_\beta \partial^\alpha A^\beta}\boxed{-\partial_\alpha A_\beta \partial^\beta A^\alpha}\boxed{-\partial_\beta A_\alpha \partial^\alpha A^\beta} + \boxed{\partial_\beta A_\alpha \partial^\beta A^\alpha}\right)$$

$$= -\frac{1}{2\mu_0 c}\left(\partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha\right)$$

put the simplified $\mathcal{L}$ into E-L eq., we have

ALSO: $\partial_\alpha A_\beta = \delta_\alpha^\mu g_{\beta\nu}\partial_\mu A^\nu$, $\partial^\alpha A^\beta = \delta^\alpha_\mu g^{\beta\nu}\partial_\mu A^\nu$

THUS: $$\frac{\partial(\partial_\alpha A_\beta \partial^\alpha A^\beta)}{\partial(\partial_\mu A^\nu)} = \underbrace{\delta_\alpha^\mu g_{\beta\nu}\,\partial^\alpha A^\beta}_{\partial^\mu A_\nu} + \underbrace{\delta^\alpha_\mu g^{\beta\nu}\,\partial_\alpha A_\beta}_{\partial_\mu A^\nu} = \partial^\mu A_\nu + \partial_\mu A^\nu = 2\,\partial_\mu A^\nu$$

$$\frac{\partial(\partial_\alpha A_\beta \partial^\beta A^\alpha)}{\partial(\partial_\mu A^\nu)} = \underbrace{\delta^\mu_\alpha g_{\beta\nu}\partial^\beta A^\alpha}_{\partial_\nu A^\mu} + \qquad = \partial_\nu A^\mu + \partial^\nu A_\mu = 2\partial_\nu A^\mu$$

AND: $\frac{\partial \mathcal{L}}{\partial A^\nu} = \frac{\partial(\cdots\cdots)}{\partial A^\nu} = 0$

SO: $\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\nu)} - \frac{\partial \mathcal{L}}{\partial A^\nu} = 0 \Rightarrow \frac{\partial}{\partial x^\mu}\left(\frac{1}{\mu_0 c}(\partial_\nu A^\mu - \partial_\mu A^\nu)\right) = \frac{\partial}{\partial x^\mu}\left(\frac{1}{\mu_0 c}\cdot F_\nu{}^\mu\right) = 0$ (*)

Considering (*) $\quad 0 = \frac{\partial}{\partial x^\mu}\left(\partial_\nu A^\mu - \partial^\mu A_\nu\right) = \partial_\nu \boxed{\partial_\mu A^\mu} - \partial_\mu \partial^\mu A_\nu \overset{\text{Lorenz G}}{=\!=} -\Box A_\nu$

Chose Lorenz Gauge AND IT becomes "0": $\frac{1}{c^2}\frac{\partial \phi}{\partial t} + \nabla\cdot\vec{A} = 0$

$$\Rightarrow \Box A^\nu = 0 \xRightarrow{\text{components}} \begin{aligned} &\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0, \quad \nu = 0, &&\Rightarrow \nabla\cdot\vec{E} = 0 \\ &\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\boldsymbol{A} = 0, \quad \nu = 1,2,3. &&\Rightarrow \nabla\times\vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$\nabla\cdot\vec{E} = -\nabla^2\varphi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A}) = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi + \frac{\partial}{\partial t}(\nabla\cdot\vec{A})$

Lorenz G

(NEXT PAGE)

而光光从 $\vec{E}$、$\vec{B}$ 的定义上即可获得

$$\begin{cases} \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \end{cases}$$

Consider the effects of particles

$$S = S_f + S_{pf} = \int \mathcal{L}_f \, d\Omega + \int \sum_i -q_i A_\alpha(t,\vec{x}_i)\, dx^\alpha$$

$$= \int \mathcal{L}_f \, d\Omega + \int \sum_i -q_i A_\alpha(t,\vec{x})\,\delta(\vec{x}-\vec{x}_i)\, dV\, dx^\alpha$$

AND: $\sum_i q_i \delta(\vec{x}-\vec{x}_i) = \rho(t,x) \quad$ SO: $\underline{\underline{\mathcal{L}_{pf} = -\frac{1}{c}\rho A_\alpha \frac{dx^\alpha}{dt}}}$

Here, we can define the current density 4-vector $\quad j^\alpha = \rho\frac{dx^\alpha}{dt} = (\rho c, \vec{j})$

Then

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_{pf} = -\frac{1}{4\mu_0 c}\cdot F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{c} j^\alpha A_\alpha$$

Put it into E-L eq. $\quad \frac{\partial F_{\mu\nu}}{\partial x_\mu} - \mu_0 j_\nu = 0$

Using Lorenz Gauge: $\quad \Box A_\nu - \mu_0 j_\nu = 0 \xrightarrow{\text{Components}} \begin{cases} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi - \frac{\rho}{\varepsilon_0} = 0 & ,\nu = 0 \\ \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} - \mu_0\vec{j} = 0 & ,\text{otherwise} \end{cases}$ (*)

According to (*):

$$\nabla\cdot\vec{E} = \nabla\cdot\frac{\partial\vec{A}}{\partial t} - \nabla^2\varphi \overset{(*)}{=\!=} -\nabla\cdot\frac{\partial\vec{A}}{\partial t} + \frac{\rho}{\epsilon_0} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi \overset{LG}{=\!=} \frac{\rho}{\epsilon_0}$$

(Lorenz Gauge $(\frac{\partial}{\partial t})$: $0 = \frac{\partial}{\partial t}\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\varphi$)

$$\Box\vec{A} = \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} - \nabla^2\vec{A} \overset{LG}{=\!=} \boxed{\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} + \frac{1}{c^2}\nabla\frac{\partial\varphi}{\partial t}} + \boxed{\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}} = -\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial\vec{A}}{\partial t}\right) + \nabla\times(\nabla\times\vec{A})$$

$$= -\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \nabla\times\vec{B}$$

(Lorenz Gauge (∇): $0 = \frac{1}{c^2}\nabla\frac{\partial\varphi}{\partial t} + \nabla(\nabla\cdot\vec{A})$)

$$\Leftrightarrow \nabla\times\vec{B} = \mu_0\vec{j} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

Charge Conservation Law, Equation of Continuity

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0 \quad \text{(charge)}$$

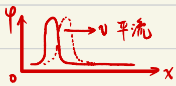
$$\frac{\partial\rho_A}{\partial t} + \nabla\cdot\vec{F}_A = 0 \qquad \text{conservative form}$$

e.g. $\begin{cases} \frac{\partial n}{\partial t} + \nabla\cdot n\vec{v} = 0 & \text{(num)} \\ \frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{v} = 0 & \text{(mass)} \end{cases}$

what if A has a source (源) or a sink (汇)

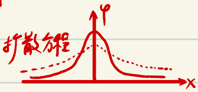
$$\frac{\partial\rho_A}{\partial t} + \nabla\cdot\vec{F}_A = S_A$$

1) $\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} = 0$，通解 $\varphi = \varphi(x-vt)$



2) $\left(\frac{\partial^2}{\partial t^2} - v^2\frac{\partial^2}{\partial x^2}\right)\varphi = 0$ 波动方程

边值&初值 3) $\frac{\partial\varphi}{\partial t} = D\frac{\partial^2\varphi}{\partial x^2}$ 扩散方程



扩散至稳态 $\Rightarrow \frac{\partial\varphi}{\partial t} + \frac{\partial}{\partial x}\left[-D\frac{\partial\varphi}{\partial x}\right] = 0$ 连续性方程

边值 4) $\nabla^2\varphi = \begin{cases} 0 \\ -\rho/\epsilon_0 \end{cases}$ Laplacian eq.

The Continuity Equation of Field Energy and Momentum

noting that 1. ε is only related to field
2. E、B is symetric
3. satisfy continuity equation

$$\boxed{\frac{\partial \varepsilon}{\partial t} + \nabla\cdot\vec{P} = S_\varepsilon}$$

ALSO $\frac{d\varepsilon_p}{dt} = q\vec{E}\cdot\vec{v}$ FOR dV : $\frac{d\varepsilon_p}{dt} = \rho dV\,\vec{E}\cdot\vec{v} = \vec{E}\cdot\vec{j}\,dV$

FOR Unit volume : $S_\varepsilon = -\frac{d\varepsilon_p}{dt} = -\vec{j}\cdot\vec{E}$

Ampère Thr : $\vec{j} = \frac{1}{\mu_0}\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{j}\cdot\vec{E} = \frac{1}{\mu_0}\nabla\times\vec{B}\cdot\vec{E} - \varepsilon_0\frac{\partial\vec{E}}{\partial t}\cdot\vec{E}$

Applying Faraday Thr.

$$= \frac{1}{\mu_0}\left(\nabla\times\vec{B}\cdot\vec{E} - \nabla\times\vec{E}\cdot\vec{B}\right) + \frac{1}{\mu_0}\underbrace{\nabla\times\vec{E}}_{-\frac{\partial\vec{B}}{\partial t}}\cdot\vec{B} - \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}$$

$$= -\frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B}) - \left(\frac{1}{2\mu_0}\cdot\frac{\partial B^2}{\partial t} + \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}\right)$$

Thus

$$\frac{\partial\varepsilon}{\partial t} + \nabla\cdot\vec{P} = \frac{1}{\mu_0}\nabla\cdot(\vec{E}\times\vec{B}) + \left(\frac{1}{2\mu_0}\cdot\frac{\partial B^2}{\partial t} + \frac{\varepsilon_0}{2}\frac{\partial E^2}{\partial t}\right)$$

$$\Rightarrow \begin{cases} \varepsilon = \frac{1}{2\mu_0}\cdot B^2 + \frac{\varepsilon_0}{2}E^2 \\ \vec{P} = \frac{1}{\mu_0}\vec{E}\times\vec{B} \ (= \vec{E}\times\vec{H} \text{ Poynting flux}) \end{cases}$$

FOR momentum $\frac{\partial\vec{g}}{\partial t} + \nabla\cdot\vec{\vec{G}} = S_g$

AND FOR S_g, we have $S_g = -\rho(\vec{E} + \vec{v}\times\vec{B}) = -\rho\vec{E} + \vec{j}\times\vec{B}$

ALSO $\rho = \varepsilon_0\nabla\cdot\vec{E}$ $\vec{j} = \frac{1}{\mu_0}\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$

i.e.

$$\frac{\partial\vec{g}}{\partial t} + \nabla\cdot\vec{\vec{G}} = -\varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} + \underline{\varepsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}}$$

$$= -\varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} + \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \varepsilon_0\vec{E}\times(\nabla\times\vec{E})$$

$$= \varepsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \varepsilon_0(\nabla\cdot\vec{E})\vec{E} - \frac{1}{\mu_0}(\nabla\cdot\vec{B})\vec{B} - \frac{1}{\mu_0}(\nabla\times\vec{B})\times\vec{B} + \varepsilon_0\vec{E}\times(\nabla\times\vec{E})$$

mathematical trick : $\vec{E}\times(\nabla\times\vec{E}) = (\nabla\cdot\vec{E})\vec{E} - \nabla\cdot\left(\vec{E}\vec{E} - \frac{1}{2}E^2\vec{\vec{I}}\right)$

SO $\frac{\partial\vec{g}}{\partial t} + \nabla\cdot\vec{\vec{G}} = \frac{\partial}{\partial t}(\varepsilon_0\vec{E}\times\vec{B}) - \nabla\cdot\left[\varepsilon_0\vec{E}\vec{E} + \frac{1}{\mu_0}\vec{B}\vec{B} - \frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)\vec{\vec{I}}\right]$

$$\Rightarrow \begin{cases} \vec{g} = \varepsilon_0\vec{E}\times\vec{B} \\ \vec{\vec{G}} = -\vec{\vec{T}}, \quad \vec{\vec{T}} = \varepsilon_0\vec{E}\vec{E} + \frac{1}{\mu_0}\vec{B}\vec{B} - \frac{1}{2}\left(\varepsilon_0E^2 + \frac{1}{\mu_0}B^2\right)\vec{\vec{I}} \end{cases}$$

T is Maxwell stress tensor.

$$\Leftrightarrow E\vec{v} = m\vec{v}\,c^2$$

Noting that $\vec{P} = c^2\vec{g} \Rightarrow$ energy flux $= c^2 \times$ momentum density

Force on Particles in unit volumn : $$\vec{f} = -\frac{\partial \vec{g}}{\partial t} - \nabla \cdot \overleftrightarrow{G}$$

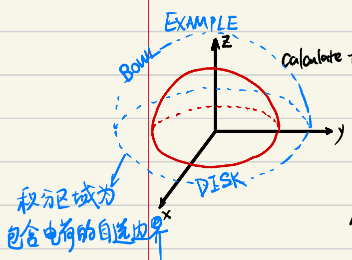
For static EM Field, $\frac{\partial \vec{g}}{\partial t} = 0$, $$\vec{f} = \nabla \cdot \overleftrightarrow{T}$$

For all particles within a volumn V: $$\vec{F} = \int \nabla \cdot \overleftrightarrow{T}\, dV = \oint d\vec{s} \cdot \overleftrightarrow{T}$$

$$d\vec{F} = \vec{e_i}\, T_{ij}\, dS_j$$

$$\Rightarrow dF_x = T_{xj}\, dS_j = T_{xx}\, dS_x + \underbrace{T_{xy}\, dS_y + T_{xz}\, dS_z}_{\text{Sheer}}$$

when $\vec{E} = 0$, $\vec{f} = \frac{1}{\mu_0}(\vec{B} \cdot \nabla)\vec{B} - \underline{\nabla\left(\frac{1}{2\mu_0}B^2\right)}$ magetic pressure (plasma physics)



Calculate the force on the northern hemisphere ?

$$F = \int \nabla \cdot \overleftrightarrow{T}\, dV = \int d\vec{S} \cdot \overleftrightarrow{T} = \int \overleftrightarrow{T} \cdot d\vec{s}$$

$$\because \left(\overleftrightarrow{T} \cdot d\vec{s}\right)_z = T_{zj}\, dS_j = T_{zx}\, dS_x + T_{zy}\, dS_y + T_{zz}\, dS_z$$

AND $\overleftrightarrow{T} = \varepsilon_0 \vec{E}\vec{E} - \frac{1}{2}\varepsilon_0 E^2 \overleftrightarrow{I}$ $(\vec{B} = 0)$

$$d\vec{s} = R^2 \sin\theta\, d\theta\, d\varphi\, \vec{e_r},\quad \vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{e_r},\quad \vec{e_r} = \cos\theta\, \vec{e_z} + \sin\theta\cos\varphi\, \vec{e_x} + \sin\theta\sin\varphi\, \vec{e_y}$$

Bowl + Disk

$$T_{zx} = \varepsilon_0 E_z E_x = \varepsilon_0 \left(\frac{Q}{4\pi\varepsilon_0 R^2}\right)^2 \sin\theta\cos\theta\sin\varphi$$

$$T_{zz} = \varepsilon_0 E_z^2 - \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \left(E_z^2 - E_x^2 - E_y^2\right)$$

......

Chapter 6 General Electrostatics

Energy of Electrostatic Field

$$U=\frac{1}{2}\sum_i q_i\varphi_i=\underbrace{U_{self}}_{\text{donot consider}}+U_{inter}\ ,\quad U_{self}=\frac{1}{2}q_a\varphi_a(\vec{x}_a)=+\infty \text{ for a point charge}$$

Assume the electron is a sphere of uniform charge with radius R_0 $\quad U=\frac{1}{2}\int\rho\phi\, dV\sim\frac{e^2}{4\pi\varepsilon_0 R_0}=mc^2$

$\Rightarrow R_0=\frac{e^2}{4\pi\varepsilon_0 mc^2}\sim 10^{-15}\,m$ classical electron radius

General Solution of Electrostatic Problems

$$\nabla^2\phi=-q_i\delta(\vec{x}-\vec{x}_i')/\epsilon_0$$

[Green Function]

$\hat{L}\phi=$ source ($\hat{L}$: linear operator, $\phi=\hat{L}^{-1}$ source)

$$\nabla^2 G(\vec{x},\vec{x}')=\delta(x-\vec{x}')\qquad \int\delta(\vec{x}-\vec{x}')s(\vec{x}')\,dV'=s(\vec{x})$$

$$\int\nabla^2 G(\vec{x},\vec{x}')s(\vec{x}')\,dV'=s(\vec{x})$$

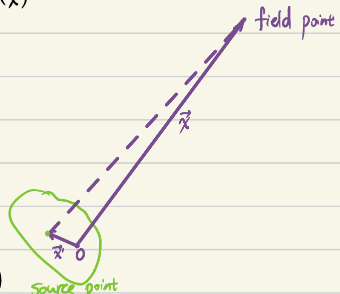
Electrostatics $\Rightarrow \phi(\vec{x})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{x}')\,dV'}{|\vec{x}-\vec{x}'|}$

↓Expand

Magnetostatics $\nabla^2\vec{A}=-\mu_0\vec{J}\Rightarrow\vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{x}')\,dV'}{|\vec{x}-\vec{x}'|}\xrightarrow[\nabla\,(!!\text{ not }\nabla')]{\text{rotation}}$ (Biot-Savart Th.)

Approximate Solution (using Taylor Expand 电多极展开)

Taylor expansion: $f(\vec{x}-\vec{x}')=f(\vec{x})+\nabla f(\vec{x})\cdot(-\vec{x}')+\frac{1}{2}x_i'x_j'\frac{\partial^2}{\partial x_i\partial x_j}f(\vec{x})$



For here: $\varphi(\vec{x}-\vec{x}')=\frac{1}{4\pi\varepsilon_0}\sum_a\frac{q_a}{|\vec{x}-\vec{x}'|}$, $\boxed{\epsilon=\frac{|\vec{x}'|}{|\vec{x}|}\ll 1}$

$$=\underset{O(1)}{\varphi_0}+\underset{O(\epsilon)}{\varphi_1}+\underset{O(\epsilon^2)}{\varphi_2}+\cdots\cdots$$

Applying them, we arrive the conclusion that:

$$\varphi_0=\frac{1}{4\pi\varepsilon_0}\cdot\frac{\sum_a q_a}{r}=\frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{r}\qquad r\equiv|\vec{x}|$$

$Q=\sum_a q_a$ (zero order) 电零阶矩

$$\varphi_1=\frac{1}{4\pi\varepsilon_0}\cdot\sum_a -q_a\vec{x}'\cdot\nabla\frac{1}{r}=\frac{\vec{p}\cdot\vec{x}}{4\pi\varepsilon_0 r^3}$$

$\vec{p}=\sum_a q_a\vec{x}'$ (first order moment of electron) 电一阶矩

$$\varphi_2=\frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{2}\sum_a q_a x_i'x_j'\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r}$$

Consider φ_2, define quadrupole moment $\overleftrightarrow{D}$ as

$$\overleftrightarrow{D}=\sum_a q_a(3\vec{x}'\vec{x}'-r'^2\overleftrightarrow{I})\ ,\quad D_{ij}=\sum_a q_a(3x_i'x_j'-r'^2\delta_{ij})$$

$$\therefore\ \varphi_2=\frac{1}{24\pi\varepsilon_0}D_{ij}\frac{\partial^2}{\partial x_i\partial x_j}\frac{1}{r}$$

Consider φ_1, the Electric field can be obtained by adding an operator on φ, i.e:

$$\vec{E}_1 = -\nabla\left(\frac{1}{4\pi\varepsilon_0}\cdot\frac{\vec{p}\cdot\vec{x}}{r^3}\right) = -\frac{1}{4\pi\varepsilon_0}\cdot\nabla(\vec{p}\cdot\vec{x})\cdot\frac{1}{r^3} - \frac{1}{4\pi\varepsilon_0}\cdot\nabla\frac{1}{r^3}\cdot\vec{p}\cdot\vec{x}$$

$$= -\frac{1}{4\pi\varepsilon_0}\cdot\frac{1}{r^3}\,\vec{p}\cdot\overbrace{(\nabla\vec{x})}^{=\vec{I}} - \frac{1}{4\pi\varepsilon_0}\,\vec{p}\cdot\vec{x}\,(-3)\frac{1}{r^4}\nabla r$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\vec{p}\cdot\vec{x})\vec{x}}{r^5} - \frac{\vec{p}}{r^3}\right) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{3(\vec{p}\cdot\vec{n})\vec{n}-\vec{p}}{r^3}$$

Potential energy of a system charges in an external field

spatial scale of $\begin{cases}\phi : \dfrac{\phi}{\nabla\phi} \sim \dfrac{\phi}{\phi/L} = L \\ q : x' \sim l\end{cases}$ define $\epsilon = \dfrac{l}{L} \ll 1$

potential is $\quad U = \sum_a q_a \phi(\vec{x}_a') = \sum_a q\,\phi(\vec{x}')$

here expand ϕ: $\begin{cases}\phi(\vec{x}') = \phi(0) + \vec{x}'\cdot\nabla\phi + \frac{1}{2}x_i'x_j'\dfrac{\partial^2\phi}{\partial x_i\partial x_j} \\ U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots\cdots\end{cases}$

thus

$$U^{(0)} = \sum_a q\,\phi(0) = Q\phi(0)$$

$$U^{(1)} = \sum_a q\,\vec{x}'\cdot(\nabla\phi)_0 = \vec{p}\cdot(\nabla\phi)_0 = -\vec{p}\cdot\vec{E}$$

$$U^{(2)} = \sum_a \frac{1}{2}q\,x_i'x_j'\frac{\partial^2\phi}{\partial x_i\partial x_j} = \frac{1}{6}D_{ij}\frac{\partial^2\phi}{\partial x_i\partial x_j} = \frac{1}{6}\vec{\vec{D}} : \nabla\nabla\phi = -\frac{1}{6}\vec{\vec{D}} : \nabla\vec{E}$$

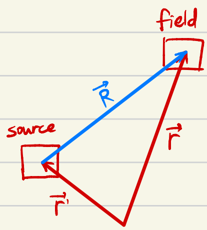
the Force on the charges, can be written as

$$\vec{F} = -\nabla U \simeq -\nabla U^{(0)} - \nabla U^{(1)}$$

$$= Q\vec{E}(0) + \underline{\nabla(\vec{p}\cdot\vec{E})} \equiv (\vec{p}\cdot\nabla)\vec{E}$$

the torque felt by an electric dipole

$$\vec{\tau} = \vec{p}\times\vec{E}$$



Polarized

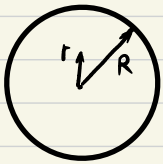
$$\varphi = \frac{1}{4\pi\varepsilon_0}\int\frac{\vec{P}\cdot\vec{R}}{R^3}\,dV', \quad \vec{R} = \vec{r} - \vec{r}'$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\vec{R}}{R^3}, \quad \nabla' = \frac{\partial}{\partial\vec{r}'} \qquad \therefore \varphi = \frac{1}{4\pi\varepsilon_0}\int\vec{P}\cdot\nabla'\left(\frac{1}{R}\right)dV' = \frac{1}{4\pi\varepsilon_0}\int\frac{\vec{P}}{R}\cdot d\vec{s}' + \frac{1}{4\pi\varepsilon_0}\int\frac{-\nabla'\cdot\vec{P}}{R}\,dV'$$

$$= \frac{1}{4\pi\varepsilon_0}\int\frac{\sigma_b}{R}\,ds' + \frac{1}{4\pi\varepsilon_0}\int\frac{\rho_b}{R}\,dV'$$

i.e: $\quad \sigma_b = \vec{P}\cdot\vec{n}, \quad \rho_b = -\nabla\cdot\vec{P}$

Electric Field inside a Dielectric



微观上足够大，宏观上足够小.（见hw#4题解）

$$\vec{E}(\vec{r}) = \vec{E}_{in} + \vec{E}_{out}$$

$$\langle\varphi\rangle_{out} = \frac{1}{4\pi\varepsilon_0}\int_{out} \frac{\vec{p}\cdot\vec{R}}{R^3}dV' \quad , \langle\varphi\rangle_{in} = \frac{1}{4\pi\varepsilon_0}\int_{in} \frac{\vec{p}\cdot\vec{R}}{R^3}dV'$$

$$\Rightarrow \langle\varphi\rangle = \frac{1}{4\pi\varepsilon_0}\int_{ALL} \frac{\vec{p}\cdot\vec{R}}{R^3}dV' \text{ 仍成立}$$

Boundary Problems

1. Laplace eq. $\nabla^2\varphi = 0$

$$\begin{cases} \vec{j} = -D\nabla n \\ \nabla\cdot\vec{j} = -\frac{\partial n}{\partial t} \end{cases}$$

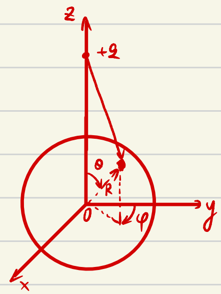
$$\Rightarrow \frac{\partial n}{\partial t} = D\nabla^2 n$$

stable solution $\nabla^2 n = 0$

1-D Laplace eq. $\frac{\partial^2\varphi}{\partial x^2} = 0$ Solution: $\varphi = Ax + B$

2-D Laplace eq. $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$ Solution: $\varphi = \frac{1}{2\pi R}\oint\varphi\, dl$

3-D Laplace eq. $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$, Solution: $\underline{\varphi = \frac{1}{4\pi R^2}\oiint\varphi\, ds}$



Proof: $\varphi_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{z}$

$$\frac{1}{4\pi R^2}\oiint\varphi\, ds = \frac{1}{4\pi R^2}\int_0^{2\pi}\int_0^{\pi}\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{\sqrt{z^2+R^2-2zR\cos\theta}}R^2\sin\theta\, d\theta\, d\varphi$$

$$= \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{2zR}\sqrt{z^2+R^2-2zR\cos\theta}\Big|_0^{\pi} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{z} \quad \text{QED.}$$

What about adding boundary conditions on ∂V ?

first uniqueness theorem: φ is uniquely determined in φ on ∂V is given.

Proof: assume that φ_1 & φ_2. thus $\begin{cases} \nabla^2\varphi_1 = 0,\ \nabla^2\varphi_2 = 0. \\ \varphi_1 = \varphi_2 \text{ on } \partial V \end{cases}$ define $\varphi_3 = \varphi_2 - \varphi_1$

Considering $\begin{cases} \nabla^2\varphi_3 = 0 \\ \varphi_3 = 0 \text{ on } \partial V \end{cases}$ φ_3 has the max, minium only on boundary $\Rightarrow \varphi_3 = 0$ (inside V) $\Rightarrow \varphi_1 = \varphi_2$

*first theorem can also be applied to $\nabla^2\varphi = \rho/\varepsilon_0$

Second uniqueness theorem:

Proof: $\nabla\cdot\vec{E}_1 = \frac{\rho}{\varepsilon_0}$, $\nabla\cdot\vec{E}_2 = \frac{\rho}{\varepsilon_0}$ $\oint_{i^{th}\text{ conductor}}\vec{E}_1\cdot d\vec{S} = \frac{Q_i}{\varepsilon_0}$ $\oint_{i^{th}\text{ conductor}}\vec{E}_2\cdot d\vec{S} = \frac{Q_i}{\varepsilon_0}$ $\oint_{\text{boundary}}\vec{E}_1\cdot d\vec{S} = \frac{Q_{tot}}{\varepsilon_0}$ $\oint_{\text{boundary}}\vec{E}\cdot d\vec{S} = \frac{Q_{tot}}{\varepsilon_0}$

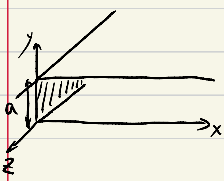
define $\vec{E}_3 = \vec{E}_2 - \vec{E}_1$. $\begin{cases} \nabla\cdot\vec{E}_3 = 0 \\ \oint\vec{E}_3\cdot d\vec{S} = 0 \end{cases}$ $\varphi_3 = \text{const}$ & $\nabla\cdot(\varphi_3\vec{E}_3) = -E_3^2$, Considering $\int\nabla\cdot(\varphi_3\vec{E}_3)dV = -\int E_3^2 dV = 0 \Rightarrow \vec{E}_3 = 0$

Separation of Variables

Completeness $f(y)=\sum_n C_n f_n(y)$

Orthogonality $\int_0^a f_n(y) f_{n'}(y)\,dy=0$ if $n\neq n'$

Example:



$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\varphi=0\ ,\quad \varphi=X(x)Y(y)$$

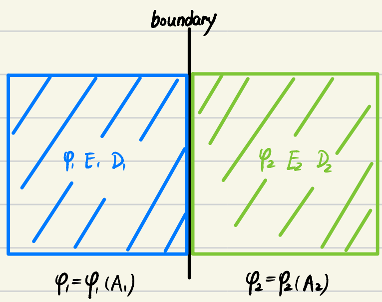
$$\Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=0$$

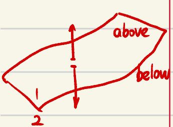
$$\begin{cases} \frac{1}{X}\frac{d^2X}{dx^2}=C_1=k^2, & X=Ae^{kx}+Be^{-kx}=Be^{-kx} \\ \frac{1}{Y}\frac{d^2Y}{dy^2}=C_2=-k^2, & Y=A\sin ky+B\cos ky=A\sin ky \end{cases}$$

$$\therefore \varphi=Ce^{-kx}\sin ky\ ,\quad k=\frac{n\pi}{a}\ ,\ n=1,2,3,\cdots$$

Connection boundary conditions for electric field and potential

we can solve $\begin{cases}\varphi_1=\varphi_1(A_1)\\ \varphi_2=\varphi_2(A_2)\end{cases}$ where A_1, A_2 should be determined by the boundary conditions on 'Black Line'.





$$\begin{cases} D_{1n}-D_{2n}=\sigma_f \\ E_{1\tau}=E_{2\tau} \\ \varphi_1=\varphi_2 \end{cases}$$

Example: 线性介质球在匀强场中的解

$$\nabla^2\varphi=0\ ,\quad \frac{\partial\varphi}{\partial\phi}=0 \Rightarrow \varphi(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)=\begin{cases}\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)\underline{\underline{-E_0 r\cos\theta}}\ , & \text{outside}\\ \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)\ , & \text{inside}\end{cases}$$

对应 1st-order 勒让德

B/C: $\varphi_{in}=\varphi_{out}$ at $r=R$, $\epsilon\frac{\partial\varphi_{in}}{\partial\vec{r}}=\epsilon_0\frac{\partial\varphi_{out}}{\partial\vec{r}}$

$$\Rightarrow A_l=B_l=0\ (l\neq1),\quad A_1=\frac{-3}{\epsilon_r+2}E_0\ ,\quad B_1=$$

$$\therefore \varphi_{in}=\frac{-3}{\epsilon_r+2}E_0 r\cos\theta\ ,\quad \vec{E}_{in}=\frac{3}{\epsilon_r+2}\vec{E}_0$$

Chapter 7 General Magnetostatics

$\nabla\times\vec{B}=0$
$\Rightarrow \vec{B}=-\nabla\varphi$ (磁标势)

Using Coulomb Gauge, we have

$$\nabla\times(\nabla\times\vec{A})=\nabla(\nabla\cdot\vec{A})-\nabla^2\vec{A}=-\nabla^2\vec{A}=\mu_0\vec{j}$$

similar to $\nabla^2\phi=-\frac{\rho}{\varepsilon_0}$

$$\Rightarrow \nabla^2\vec{A}=-\mu_0\vec{j} \Rightarrow \vec{A}=\frac{\mu_0}{4\pi}\int\frac{\vec{j}(\vec{x}')dV'}{|\vec{x}-\vec{x}'|}$$

$$\vec{B}=\nabla\times\vec{A}\xlongequal{\vec{R}=\vec{x}-\vec{x}'}\frac{\mu_0}{4\pi}\int\nabla\times\left(\frac{\vec{j}(\vec{x}')}{R}\right)dV'=\frac{\mu_0}{4\pi}\int\frac{\vec{j}\times\vec{R}}{R^3}dV'$$

let $\vec{j}dV'=Id\vec{l}$

magnetic multiple moments

Considering current $(\vec{x}')$ get magnetic field at $\vec{x}$, where $|\vec{x}'|/|\vec{x}|=\epsilon\ll 1$, & $r=|\vec{x}|$

Taylor Expand:

$$f(\vec{x}-\vec{x}')=f(\vec{x})+\nabla f(\vec{x})\cdot(-\vec{x}')+\cdots$$

$$\Leftrightarrow \frac{1}{|\vec{x}-\vec{x}'|}\simeq\frac{1}{r}+\nabla\frac{1}{r}(-\vec{x}')+\cdots$$

Hence we have

$$\vec{A}=\vec{A}^{(0)}+\vec{A}^{(1)}+\cdots,$$

$$\vec{A}^{(0)}=\frac{\mu_0}{4\pi r}\int\vec{j}dV',\quad \vec{A}^{(1)}=-\frac{\mu_0}{4\pi}\int\vec{j}\left(\vec{x}'\cdot\nabla\frac{1}{r}\right)dV'=-\frac{\mu_0\vec{x}}{4\pi r^3}\cdot\int\vec{x}'\vec{j}dV'.$$

And: $\nabla'\cdot(\vec{j}fg)=\nabla'\cdot\vec{j}fg+\vec{j}\cdot\nabla'fg=\vec{j}\cdot\nabla'fg$, Gauss Law: $\int\nabla'\cdot(fg\vec{j})dV'=\int fg\,\underbrace{\boxed{\vec{j}\,d\vec{S}}}_{\text{boundary}}=0$

f-g eg.: $\Leftrightarrow\int\vec{j}\cdot\nabla'fg\,dV'=\int(f\vec{j}\cdot\nabla'g+g\vec{j}\cdot\nabla f)dV'=0$

Considering $\vec{A}^{(0)}$, let $f=1$, $g=x_i'$, hence $\nabla fg=\nabla x_i'=\vec{e}_i$

$$A_i^{(0)}=\int\vec{j}\cdot\nabla fg\,dV'=\int\vec{j}\cdot\vec{e}_i\,dV'=0\quad i=1,2,3\Rightarrow\vec{A}^{(0)}=0$$

Considering $\vec{A}^{(1)}$, let $f=x_i'$, $g=x_l'$, thus

与粒子的 Angular momentum 关系 $\vec{\mu}=\frac{q}{2m}\vec{L}$

The force felt by a small current system

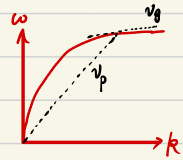
$$\vec{F}=\int\vec{j}\times\vec{B}(\vec{x})dV'=\vec{F}^{(0)}+\vec{F}^{(1)}+\cdots\cdots$$

$$\vec{F}^{(0)}=\left(\int\vec{j}dV'\right)\times\vec{B}(0)=0\times\vec{B}(0)=0$$

$$\vec{F}^{(1)} = \int \vec{j} \times (\vec{x}' \cdot \nabla) \vec{B}(0)\, dV'$$

$$\Rightarrow F_i^{(1)} =$$

Chapter 8 Time-varying EM field



前置知识

1. 色散 $n(\omega)=\dfrac{ck}{\omega} \Rightarrow \dfrac{\omega}{k}=f(\omega)$ 色散关系

2. phase velocity & group velocity $v_{phase}=\dfrac{\omega}{k}$, $v_{group}=\dfrac{\partial\omega}{\partial k}$

3. $\phi=\phi_0 e^{i(kx-\omega t)}$, (虚波矢代表了振幅的衰减)

replace-relationship $\boxed{\dfrac{\partial}{\partial t}=-i\omega,\ \dfrac{\partial}{\partial x}=ik \text{ OR } \nabla=i\vec{k}}$

e.g.: $\hbar\omega=\frac{1}{2m}(\hbar k)^2+V \Rightarrow i\hbar\frac{\partial}{\partial t}=\frac{\hbar^2}{2m}\left(-\frac{\partial^2}{\partial x^2}\right)+V \Rightarrow \underline{i\hbar\frac{\partial\psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\psi=V\psi}$

Time-dependent EM field in vacuum

$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{B}=0 \quad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{E}=0$

Properties for:

Plane wave ~ only one spacial coordinate

Complex Notation for monochromatic $f=\mathrm{Re}[\hat{A}e^{i(kx-\omega t)}]=\mathrm{Re}[\hat{f}]$

here, $\hat{A}$ is complex Amplitude $\hat{A}=A_0e^{i\phi_0}$

more general: $\hat{f}=\hat{A}e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, hence $\nabla=i\vec{k}$

$\Rightarrow$ only for Linear Operator

$$\left.\begin{aligned} f(t,x)&=\sum_i \hat{A}(\vec{k}_i)e^{i(\vec{k}\cdot\vec{x}-\omega t)}\\ &=\int \hat{B}(\vec{k})e^{i(\vec{k}\cdot\vec{x}-\omega t)}dk\end{aligned}\right\} \text{Two type of Fourier Transformation}$$

Thus $\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times\vec{E}=\omega\vec{B}$

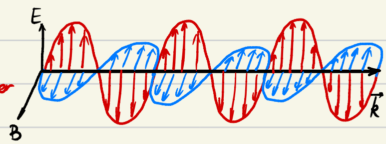
$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\vec{E}=\left(-\frac{1}{c^2}\omega^2+k^2\right)\vec{E}=0$

E, B have the same phase

E, B are perpendicular to each other

$B_0=E_0\frac{k}{\omega}=E_0/c$

$c=\frac{\omega}{k}$



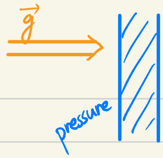
Field Energy Density: $\begin{cases} w=\frac{1}{2}\varepsilon_0E^2+\frac{1}{2\mu_0}B^2\\ w_B=w_E\\ w=\varepsilon_0E^2,\ \langle w\rangle=\frac{1}{2}\varepsilon_0E_0^2\end{cases}$, Poynting flux: $\vec{S}=w\cdot c\,\hat{k}$ (group velocity), momentum density: $\vec{g}=\vec{S}/c^2=\frac{w}{c}\hat{k}$

then how to use complex notation to express these density?

$\langle w\rangle=\frac{\epsilon_0}{2}\hat{E}_0\hat{E}_0^*$ $\langle\vec{S}\rangle=\frac{1}{2}c\epsilon_0E_0^2\hat{k}=c\langle w\rangle\hat{k}$, $\langle\vec{g}\rangle=\frac{\langle w\rangle}{c}\hat{k}=\frac{\epsilon_0E_0^2}{2c}\hat{k}$

即:波的强度 I

辐射光压 $p = \begin{cases} \langle g\rangle c = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c} \\ 2\langle g\rangle c = \varepsilon_0 E_0^2 = 2\frac{I}{c} \end{cases}$



EM field in homogeneous linear media

$\vec{E}, \vec{B}$ satisfies: $\frac{1}{v^2}\frac{\partial^2}{\partial t^2}u - \nabla^2 u = 0$ where $v = \frac{1}{\sqrt{\varepsilon\mu}}$ is the phase velocity

Solve for plane wave: $\tilde{u} = u_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)}$

$\tilde{n} = c\tilde{k}/\omega$ (real & imagie part) $\tilde{n} = n_r + i n_i$

A model for refraction index and its frequency dependence

$F = -\frac{\partial V}{\partial x} \approx -x\frac{\partial^2 V}{\partial x^2} \triangleq -m\omega_0^2 x \Rightarrow m\ddot{x} = -m\omega_0^2 x$ (model equation without EM field)

Considering $\frac{Bc}{E} \sim \frac{v}{c} \sim 0$ (for Non-relativity), and δx is small,

$$F_{EM} \approx -eE_0 e^{i(\vec{k}\cdot\vec{x} - \omega t)} \approx -eE_0 e^{-i\omega t}$$

thus $m\ddot{x} = -m\omega_0^2 x - eE_0 e^{-i\omega t} - m\gamma\dot{x} \iff \ddot{x} + \gamma\dot{x} + \omega_0^2 x = -\frac{e}{m}E_0 e^{-i\omega t}$

$$\tilde{x} = x_0 e^{-i\omega t}, \quad \tilde{x} = -\frac{e}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}\tilde{E}$$

the dipole moment is $\tilde{p} = -e\tilde{x}$

thus $\tilde{P} = n\tilde{p}$, $\epsilon_r = 1 + \chi_e = 1 + \frac{ne^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$

$$n = \sqrt{\epsilon_r} \approx 1 + \frac{1}{2}\cdot\frac{ne^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega\gamma_j}$$

we define $\tilde{k} = \tilde{n}\frac{\omega}{c} = \boxed{\beta + \frac{\alpha}{2}i}$, then the intense will decline with $\boxed{e^{-\alpha x}}$

Fourier Series

$I(x)=\int_0^\infty e^{-t^2}\cos 2xt\,dt$

$I'(x)=\int_0^\infty e^{-t^2}(-2t\sin 2xt)\,dt$

$\overset{\text{by parts}}{=}\left[e^{-t^2}\sin 2xt\right]_0^\infty-2x\int_0^\infty e^{-t^2}\cos 2xt\,dt=-2xI(x)$

hence $\frac{dI}{dx}=-2xI \Rightarrow \frac{dI}{I}=-2\frac{dx}{x} \Rightarrow I=I_0e^{-x^2}$

Considering $I(x=0)=\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}$, thus $I=\frac{\sqrt{\pi}}{2}e^{-x^2}$

1. 周期函数的 Fourier Series

对于周期为2L的$f(x)$

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}f(t)dt,\quad a_n=\frac{1}{L}\int_{-L}^{L}f(t)\cos\frac{n\pi}{L}t\,dt,\quad b_n=\frac{1}{L}\int_{-L}^{L}f(t)\sin\frac{n\pi}{L}t\,dt$$

e.g. 锯齿函数 $\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{1}{2}(\pi-x)$, $0<x<2\pi$

2. 半幅 Fourier Seires

$$\boxed{\int_0^L\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}dx=\frac{L}{2}\delta_{mn}}$$

对于定义在 $0<x<L$ 上的 $\phi(x)$, 有几种展开方式

正弦式 $\phi(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{L}$, $C_n=\frac{2}{L}\int_0^L\phi(x)\sin\frac{n\pi x}{L}dx$

余弦式 $\phi(x)=D_0+\sum_{n=1}^{\infty}D_n\cos\frac{n\pi x}{L}$, $D_0=\frac{1}{L}\int_0^L\phi(x)dx$, $D_n=\frac{2}{L}\int_0^L\phi(x)\cos\frac{n\pi x}{L}dx$

更一般地 在$[a,b]$上, $l=\frac{b-a}{2}$, $a_0=\frac{1}{b-a}\int_a^b\phi(x)dx$, $a_n=\frac{1}{b-a}\int_a^b\phi(x)\cos\frac{n\pi x}{l}dx$

$b_n=\frac{1}{b-a}\int_a^b\phi(x)\sin\frac{n\pi x}{l}dx$

3. Fourier Intergrate

定义在$(-\infty,+\infty)$上的函数$f(x)$绝对可积, 即 $\int_{-\infty}^{\infty}f(t)dt=A$ (有限值), 展开有

$$f(x)=\int_0^\infty[A(\omega)\cos\omega x+B(\omega)\sin\omega x]d\omega$$

$$\underset{\text{偶}}{A(\omega)}=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\cos\omega t\,dt,\quad \underset{\text{奇}}{B(\omega)}=\frac{1}{\pi}\int_{-\infty}^{\infty}f(t)\sin\omega t\,dt$$

Dirichlet Intergrate: $\int_0^\infty\frac{\sin\omega}{\omega}d\omega=\frac{\pi}{2}$ & $\int_0^\infty\frac{\cos(\omega\pi/2)}{1-\omega^2}d\omega=\frac{\pi}{2}$

Fourier Transform

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\underbrace{\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt}_{F(\omega)}\,e^{i\omega x}d\omega$$

Parseval eq.
if $F(f(x))=F(\lambda)$, then $\int_{-\infty}^{\infty}f^2(x)dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(\lambda)|^2d\lambda$

得到 Fourier transform: $F(\omega)=\int_{-\infty}^{\infty}f(x)e^{-i\omega x}dx$, $-\infty<\omega<\infty$ $\Leftrightarrow$ $F(\omega)=\mathcal{F}\{f(x)\}$

Fourier inverse transform: $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{i\omega x}d\omega$, $-\infty<x<\infty$ $\Leftrightarrow$ $f(x)=\mathcal{F}^{-1}\{F(\omega)\}$

$f(x)\longleftrightarrow F(\omega)$

Fourier Transform is a Linear Transform !!!

Properties of Fourier Transform:

1) Differential th.1: if $f(x)\longleftrightarrow F(\omega)$

then $\frac{df(x)}{dx}\longleftrightarrow i\omega F(\omega)$, $f^{(n)}(x)\longleftrightarrow(i\omega)^nF(\omega)$

2) Differential th.2: $xf(x)\longleftrightarrow i\frac{d}{d\omega}F(\omega)$

3) Intergral th.: $\int_{x_0}^{x}f(x)dx\longleftrightarrow\frac{F(\omega)}{i\omega}$

$\vec{D}=\varepsilon\vec{E}$

$\Rightarrow \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\vec{D}\cdot\frac{\partial\vec{E}}{\partial t}$?

4) Displacement th.: $f(x+\xi) \longleftrightarrow e^{i\omega\xi} F(\omega)$

5) Convolution

① defination: $f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(\xi) f_2(x-\xi) d\xi$

② convolution th.: $f_1(x) * f_2(x) \longleftrightarrow F_1(\omega) F_2(\omega)$

Method of separation of variables

$\mathbb{L}$ is a Linear Operator, then we have

$$\underbrace{\mathbb{L} u(x,t) = 0}_{\text{齐次!}} \Rightarrow \begin{cases} \mathbb{L}_x X(x) = 0 \\ \mathbb{L}_t T(t) = 0 \end{cases}, \text{ thus } u(x,t) = X(x) T(t)$$

*是数理方程里的东西，不作为重点

EM Waves in conductors

For atoms in conductors $\begin{cases} \text{bound ele (束缚电荷)} \sim \epsilon \\ \text{free ele (自由电荷)} \sim \sigma \end{cases}$

$$\begin{cases} \nabla\cdot\vec{E} = \frac{\rho_f}{\epsilon} \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} + \mu\sigma\vec{E} \end{cases}$$

$$\frac{\partial\rho_f}{\partial t} = -\nabla\cdot\vec{j}_f = -\nabla\cdot\sigma\vec{E} = -\frac{\sigma}{\epsilon}\rho_f \Rightarrow \rho_f = \rho_{f0}\, e^{-t/\left(\frac{\epsilon}{\sigma}\right)} \sim \tau$$

for good conductors: $\tau = \frac{\epsilon}{\sigma} \ll \frac{1}{\omega}$

after a few τ's, $\rho_f \to 0$.

then we have

$$\begin{cases} \nabla\cdot\vec{E} = 0 \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ \nabla\cdot\vec{B} = 0 \\ \nabla\times\vec{B} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} + \mu\sigma\vec{E} \end{cases}$$

$$\because \nabla\times(\nabla\times\vec{E}) = -\nabla^2\vec{E} = -\mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} - \mu\sigma\frac{\partial\vec{E}}{\partial t}$$

$$\because \nabla\times(\nabla\times\vec{B}) = -\nabla^2\vec{B} = -\mu\epsilon\frac{\partial^2\vec{B}}{\partial t^2} - \mu\sigma\frac{\partial\vec{B}}{\partial t}$$

$$\Rightarrow \nabla^2\vec{E} = \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} + \mu\sigma\frac{\partial\vec{E}}{\partial t} \sim m\ddot{x} + m\omega_0^2 x = -m\gamma\dot{x}$$

sim to oscillator with damping term

using <u>Plane Wave Solution</u> $\vec{E} = \vec{E}_0 e^{i(kz-\omega t)}$, then we have

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega$$

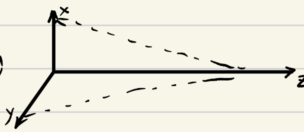
Put $k = k_r + ik_i$ into it, then

$$\begin{cases} k_r = \beta = \sqrt{\frac{\mu\epsilon}{2}}\,\omega\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2}, \\ k_i = \frac{\alpha}{2} = \sqrt{\frac{\mu\epsilon}{2}}\,\omega\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2}. \end{cases}$$

◆ $\boxed{\text{for good conductor}}$, $\frac{\sigma}{\epsilon\omega} \gg 1$, $k_r \simeq k_i \simeq \sqrt{\frac{\mu\sigma\omega}{2}}$, $\tilde{k} = \sqrt{\mu\sigma\omega}\, e^{i\frac{\pi}{4}} = Ke^{i\phi}$

then $\vec{E} = E_0 e^{-k_i z} e^{i(k_r z-\omega t)}\hat{x}$

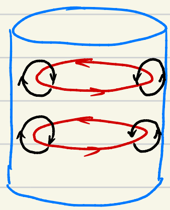
using Faraday th., we got $\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = \sqrt{\frac{\mu\sigma}{\omega}}\, E_0 e^{-k_i z} e^{i(k_r z-\omega t+\frac{\pi}{4})}\hat{y}$



Also, the attenuation of the field, reduce by a factor of e^{-1} is the skin depth (趋肤深度) δ,

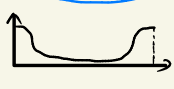
$$\delta = \frac{1}{k_i} = \sqrt{\frac{2}{\mu\sigma\omega}}$$

for a higher frequency ω, the EM Wave has a smaller skin depth

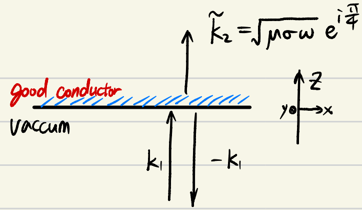


Considering the EM energy,

$$\frac{W_B}{W_E} = \frac{B^2/\mu}{\epsilon E^2} = \frac{\frac{\mu\sigma}{\omega}/\mu}{\epsilon} = \frac{\sigma}{\omega\epsilon} \gg 1 \quad \left(\text{however } \frac{W_B}{W_E} = 1 \text{ in vaccum}\right)$$



Refection at a conducting surface



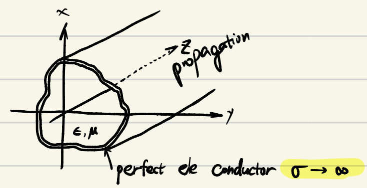
$$\begin{cases} E_{I0}+E_{R0}=E_{T0} \\ \dfrac{k_1}{\mu_1\omega}(E_{I0}-E_{R0})=\dfrac{\tilde{k}_2}{\mu_2\omega}E_{T0} \Leftrightarrow E_{I0}-E_{R0}=\dfrac{\mu_1\tilde{k}_2}{\mu_2 k_1}E_{T0}=\tilde{\beta}E_{T0}\end{cases}$$

thus $\begin{cases} E_{R0}=\dfrac{1-\tilde{\beta}}{1+\tilde{\beta}}E_{I0} \\ E_{T0}=\dfrac{2}{1+\tilde{\beta}}E_{I0}\end{cases}$ | $\sigma\to\infty$, $\tilde{k}_2$ & $\tilde{\beta}\to\infty$

$E_{R0}=-E_{I0}$ (half-wave loss)

we define $R=\left|\dfrac{E_{R0}}{E_{I0}}\right|^2=\left|\dfrac{\mu_2k_1-\mu_1\tilde{k}_2}{\mu_2k_1+\mu_1\tilde{k}_2}\right|^2 \simeq 1-2\sqrt{\dfrac{2\omega\epsilon_0}{\sigma}}$

Propagation of EM Waves in Wave Guide (波导)



$$\begin{cases}\vec{E}(x,y,z,t)=\vec{E}(x,y)\,e^{i(kz-\omega t)} \\ \vec{B}(x,y,z,t)=\vec{B}(x,y)\,e^{i(kz-\omega t)}\end{cases}$$

boundary condition: $\vec{E}^{\parallel}=0$, $\vec{B}^{\perp}=0$.

wave eq.: $\nabla^2=\partial_x^2+\partial_y^2-k^2$, $\dfrac{\partial^2}{\partial t^2}=-\omega^2$

$$\Rightarrow \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k^2\right)\right]\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix}=0$$

also with Faraday th.: $\nabla\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$

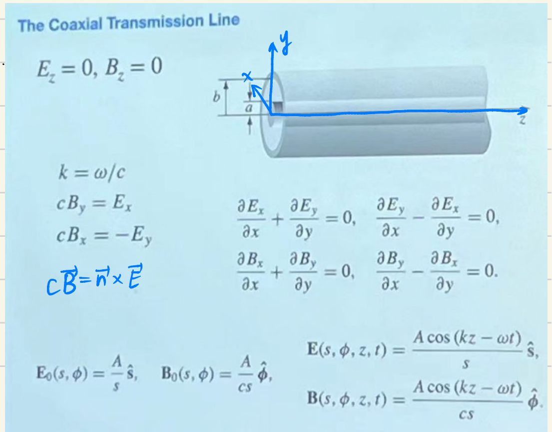
Ampère th.: $\nabla\times\vec{B}=\dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}$ $\Longrightarrow$

$$\boxed{\begin{aligned} E_x&=\frac{i}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right), &① \\ E_y&=\frac{i}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right), &② \\ B_x&=\frac{i}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\mu\epsilon\omega\frac{\partial E_z}{\partial y}\right), &③ \\ B_y&=\frac{i}{\mu\epsilon\omega^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\mu\epsilon\omega\frac{\partial E_z}{\partial x}\right). &④\end{aligned}}$$

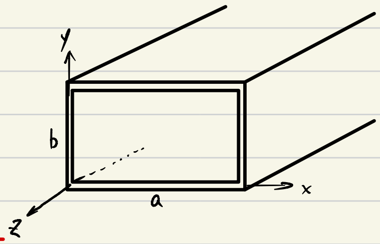
here we gave three types of Wave Mode $\begin{cases} 1)\ E_z=0\ (TE) \\ 2)\ B_z=0\ (TM) \sim \text{called transverse wave mode} \\ 3)\ \text{Both}=0\ (TEM)\end{cases}$

for 3) TEM we get $\begin{cases}\dfrac{\partial E_x}{\partial x}+\dfrac{\partial E_y}{\partial y}=0 \sim \nabla\cdot\vec{E} \quad \vec{E}=E_x\vec{e}_x+E_y\vec{e}_y \\ \dfrac{\partial E_y}{\partial x}-\dfrac{\partial E_x}{\partial y}=0 \sim \nabla\times\vec{E}\end{cases} \Rightarrow$ divergence & curl free with B.C.: $\vec{E}^{\parallel}=0 \Longrightarrow \vec{E}\equiv 0$

however, for a coaxial transmission line it will be possible



for 1) TE mode in rectangular wave guide



Using separation of variables $B_z = X(x)Y(y)$

$(\partial_x^2+\partial_y^2+\mu\epsilon\omega^2-k^2)B_z=0 \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}+\mu\epsilon\omega^2-k^2=0$

Considering B.C. with ③④ : $\frac{\partial B_z}{\partial x}=0$ at $x=0,a$; $\frac{\partial B_z}{\partial y}=0$ at $y=0,b$

$$X(x) = B\cos(k_x x), \text{ with } k_x = m\pi/a,$$
$$Y(y) = D\cos(k_y y), \text{ with } k_y = n\pi/b.$$

$\Rightarrow B_z = B_0 \cos(m\pi x/a)\cos(n\pi y/b)\, e^{ikz}$, $k=\sqrt{\mu\epsilon\omega^2-\pi^2\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]}$

phase velocity $v_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}\sqrt{1-(\omega_{mn}/\omega)^2}}$.

group velocity $v_g = \frac{d\omega}{dk} = \frac{1}{dk/d\omega} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{1-(\omega_{mn}/\omega)^2}$.

for 1) TE mode in cavity (腔)

$z=0,d$, $B_z=0 \Rightarrow B_z = B_0\cos(m\pi x/a)\cos(n\pi y/b)\sin(p\pi z/d)$

$$\omega_{mnp} = c\pi\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2}$$

$\lambda_{110} = \sqrt{2}a$ — 紫金葫芦

Chapter 9 Fields of Moving Particle

Solution of the EM field eq.:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\phi=\frac{\rho(t,\vec{x})}{\epsilon_0}=\frac{dq(t)\,\delta(\vec{x}-\vec{x}')}{\epsilon_0}$$ ~ Green Fuction Methods

let $\vec{R}=\vec{x}-\vec{x}'$

$$=\frac{dq(t)\,\delta(\vec{R})}{\epsilon_0}=\begin{cases}0 & \text{, away from charge} \quad ①\\ \frac{dq(t)\,\delta(\vec{R})}{\epsilon_0} & \text{, near the charge} \quad ②\end{cases}$$

general solution for ① (in sphere coordinate):

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\phi}{\partial R}\right)=\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2}{\partial R^2}(R\phi)=0$$

$$\Rightarrow \phi=\frac{f(t-R/c)}{R}+\frac{g(t+R/c)}{R}=\frac{\chi(t-R/c)}{R}$$

propagate in $+R$ direction propagate in $-R$ direction, give up

solution for ② when $R\to 0$,

$$\nabla^2\phi=-\frac{dq(t)\,\delta(\vec{R})}{\epsilon_0}, \quad \frac{\partial^2\phi}{\partial t^2} \text{ is alway finite}$$

$$\Rightarrow \phi=\frac{dq(t)}{4\pi\epsilon_0 R}$$

Boundary Condition:

$$R\to 0,\quad \phi_A\to\phi_N \Rightarrow \chi(t)=\frac{dq(t)}{4\pi\epsilon_0}$$

hence we get

$$\phi=\frac{dq(\boxed{t-R/c})}{4\pi\epsilon_0 R}$$ R/c eariler then we get the signal.

$t-R/c$ is the retarded time

moreover

$$\boxed{\begin{aligned}\phi(\vec{x},t)&=\frac{1}{4\pi\epsilon_0}\int\frac{1}{R}\rho(\vec{x}',t-R/c)\,dV'\\ \vec{A}(\vec{x},t)&=\frac{\mu_0}{4\pi}\int\frac{1}{R}\vec{j}(\vec{x}',t-R/c)\,dV'\end{aligned}}$$ retarded potential (solution of general EM field eq.)

Fields of a moving charge: The Lienard-Wiechert potential

$$\rho(\vec{x},t)=q\,\delta[\vec{x}-\vec{x}_p(t)]$$

$$\rho(\vec{x}',t)=q\,\delta[\vec{x}'-\vec{x}_p(t-R/c)]$$

$$\phi(\vec{x},t)=\frac{q}{4\pi\epsilon_0}\int\frac{\delta[\vec{x}'-\vec{x}_p(t_r)]}{|\vec{x}-\vec{x}'|}\,dV'$$