矢量场的量子化

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负号与约定

本作业讨论矢量场 (有质量与无质量情形) 的量子化问题.

默认采用 Minkowski Gauge

$$g_{\mu\nu} = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}, \quad \text{with } \mu, \nu = 0, 1, 2, 3. \tag{1}$$

未特殊注明默认使用 Einstein 求和约定.

1 有质量矢量场的量子化

仿照电磁场，对于一个自由的有质量的实矢量场，其 Lagrangian 写作

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{2}$$

其中的 $F^{\mu\nu}$ 为反对称张量，满足

$$F^{\mu\nu} = -F^{\nu\mu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\Rightarrow \partial_\nu \partial_\mu F^{\mu\nu} = 0$$

Lagrangian 带入 Euler-Lagrange 方程，得 A^μ 运动方程: Proca 方程

$$0 = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} - m^2 A^\nu$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + m^2 A^\nu = 0 \tag{3}$$

对于 Proca 方程 3两边同时作用 ∂_ν 得到

$$0 = \partial_\nu(\partial_\mu F^{\mu\nu} + m^2 A^\nu) = \partial_\nu \partial_\mu F^{\mu\nu} + m^2 \partial_\nu A^\nu = m^2 \partial_\nu A^\nu$$

$$m \neq 0 \Longrightarrow \partial_\mu A^\mu = 0 \quad \text{(Lorenz 条件)}$$

Lorenz 条件回代 Proca 方程，即可得 KG 方程：

$$\begin{aligned}\partial_\mu F^{\mu\nu}+m^2A^\nu &= \partial_\mu(\partial^\mu A^\nu-\partial^\nu A^\mu)+m^2A^\nu \\ &= \partial^2A^\nu-\partial^\nu\partial_\mu A^\mu+m^2A^\nu \\ &= \partial^2A^\nu+m^2A^\nu=0\end{aligned} \tag{4}$$

根据定义，A^μ 对应的共轭动量密度为

$$\pi_\mu=\frac{\partial\mathcal{L}}{\partial(\partial^0A^\mu)}=-\partial_0A_\mu+\partial_\mu A_0=-F_{0\mu}, \tag{5}$$

$$\pi_0=-F_{00}=0,\quad \pi_i=-\partial_0A_i+\partial_iA_0=-F_{0i}. \tag{6}$$

可以发现 π_0 是平庸的 0 值，它不能作为与 A^0 对应的正则共轭场，因而不能为 A^0 构造正则对易关系。然而由于 Lorenz 条件的约束，自由度减一，A^μ 只有 3 个独立分量，我们可以将 A^0 视作依赖于 3 个空间分量 A^i 的量。接下来走流程，正则量子化程序要求**独立**的正则变量满足等时对易关系，故指标 0 不纳入考虑，有

$$[A^i(\mathbf{x},t),\pi_j(\mathbf{y},t)]=\mathrm{i}\delta^i{}_j\delta^{(3)}(\mathbf{x}-\mathbf{y}),\quad [A^i(\mathbf{x},t),A^j(\mathbf{y},t)]=[\pi_i(\mathbf{x},t),\pi_j(\mathbf{y},t)]=0. \tag{7}$$

我们回过头来再看 A^i 的正则动量，

$$\pi^i=-\partial^0A^i+\partial^iA^0=-F^{0i}=F^{i0} \tag{8}$$

$$\Longrightarrow \boldsymbol{\pi}=-\dot{\mathbf{A}}-\nabla A_0,\quad \text{OR}\quad \dot{\mathbf{A}}=-\boldsymbol{\pi}-\nabla A_0 \tag{9}$$

对 Proca 方程取 $\nu=0$, 得 $\partial_\mu F^{\mu0}+m^2A^0=0$, 因此

$$A^0=-\frac{1}{m^2}\,\partial_\mu F^{\mu0}=-\frac{1}{m^2}\,\partial_iF^{i0}=-\frac{1}{m^2}\,\partial_i\pi^i=-\frac{1}{m^2}\,\nabla\cdot\boldsymbol{\pi}.$$

通过上式即可将 A^0 表达为 $\boldsymbol{\pi}$ 的函数。

1.1 极化矢量与平面波展开

矢量场 $A^\mu(x)$ 满足 KG 方程，那应有两个平面波解，即正能解 $\exp(-\mathrm{i}p\cdot x)$ 和负能解 $\exp(\mathrm{i}p\cdot x)$。由于 $A^\mu(x)$ 带有一个 Lorentz 指标，平面波展开式的系数也必须具有一个这样的指标。

对于确定的动量 $\boldsymbol{p}$, 矢量场的正能解模式具有如下形式：

$$\varphi^\mu(x,\mathbf{p},\sigma)=e^\mu(\mathbf{p},\sigma)\exp(-\mathrm{i}p\cdot x),\quad p^0=E_{\mathbf{p}}=\sqrt{|\mathbf{p}|^2+m^2}\,. \tag{10}$$

这里的系数 $e^\mu(\mathbf{p},\sigma)$ 是 Lorentz 矢量，为极化矢量，它依赖于动量 $\mathbf{p}$, 而且具有另外一个指标 σ 以描述矢量粒子的极化态。我们希望一组极化矢量能够构成 Lorentz 矢量空间的一组基底，从而用它们可以展开任意的 Lorentz 矢量。因此，一组极化矢量应当是线性独立且正交完备的。Lorentz 矢量空间是一个 4 维线性空间，可将它的一组基底简单地取为

$$\tilde{e}^\mu(0)=(1,0,0,0),\quad \tilde{e}^\mu(1)=(0,1,0,0),\quad \tilde{e}^\mu(2)=(0,0,1,0),\quad \tilde{e}^\mu(3)=(0,0,0,1).$$

基底的基本性质：

$$\begin{aligned}&\text{正交归一性：}\quad e_\mu(\mathbf{p},\sigma)e^\mu(\mathbf{p},\sigma')=g_{\sigma\sigma'}.\\&\text{完备性：}\quad \sum_{\sigma=0}^{3}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=g_{\mu\nu}.\end{aligned}\tag{11}$$

值得注意的是，正交归一关系和完备性关系都是 Lorentz 协变的，也即只要在某个惯性系中取定符合这两个关系的一组极化矢量，Boost 至另一惯性系后仍能满足这两个关系。任意 Lorentz 矢量 V_μ 可以展开成

$$V_\mu=g_{\mu\nu}V^\nu=\sum_{\sigma=0}^{3}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)V^\nu=\sum_{\sigma=0}^{3}v_\sigma(\mathbf{p})e_\mu(\mathbf{p},\sigma)\tag{12}$$

接下来，我们根据在壳动量 p^μ 的关系选择一组极化矢量。首先, 选取 2 个只有空间分量的**类空**的横向与纵向极化矢量

$$e^\mu(\mathbf{p},1)=(0,\mathbf{e}(\mathbf{p},1)),\quad e^\mu(\mathbf{p},2)=(0,\mathbf{e}(\mathbf{p},2)),\quad e^\mu(\mathbf{p},3)=\left(\frac{|\mathbf{p}|}{m},\frac{p^0\mathbf{p}}{m|\mathbf{p}|}\right).\tag{13}$$

$$\mathbf{e}(\mathbf{p},1)=\frac{1}{|\mathbf{p}||\mathbf{p}_\mathrm{T}|}(p^1p^3,p^2p^3,-|\mathbf{p}_\mathrm{T}|^2),\quad \mathbf{e}(\mathbf{p},2)=\frac{1}{|\mathbf{p}_\mathrm{T}|}(-p^2,p^1,0),\quad |\mathbf{p}_\mathrm{T}|\equiv\sqrt{(p^1)^2+(p^2)^2}.\tag{14}$$

以及**类时**极化矢量取为正比于 p^μ 的矢量

$$e^\mu(\mathbf{p},0)=\frac{p^\mu}{m}=\frac{1}{m}(p^0,\mathbf{p}).\tag{15}$$

如此定义便有

$$\begin{aligned}&\text{四维横向条件：}p_\mu e^\mu(\mathbf{p},1)=p_\mu e^\mu(\mathbf{p},2)=p_\mu e^\mu(\mathbf{p},3)=0,\quad(\text{类时矢量并不满足})\\&\text{正交归一性：}e_\mu(\mathbf{p},i)e^\mu(\mathbf{p},j)=g_{ij},\quad i,j=0,1,2,3,\\&\text{完备性：}\sum_{\sigma=0}^{3}g_{\sigma\sigma}e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=g_{\mu\nu}\end{aligned}\tag{16}$$

由于有质量矢量场 A^μ 必须满足 Lorenz 条件约束，代入正能解模式便有

$$\begin{aligned}0=\partial_\mu\varphi^\mu(x,\mathbf{p},\sigma)&=-\mathrm{i}p_\mu e^\mu(\mathbf{p},\sigma)\exp(-\mathrm{i}p\cdot x)\\&\Longrightarrow p_\mu e^\mu(\mathbf{p},\sigma)=0.\end{aligned}\tag{17}$$

也即，描述有质量矢量场的极化矢量必须满足四维横向条件。因此，类时极化矢量 $e^\mu(\mathbf{p},0)$ 不能用于描述有质量矢量场 A^μ。这说明 A^μ 只有 **3 个物理的极化状态**，由类空的极化矢量 $e^\mu(\mathbf{p},1)$,$e^\mu(\mathbf{p},2)$ 和 $e^\mu(\mathbf{p},3)$ 描述。据完备性关系, 这三个物理极化矢量满足极化求和关系

$$\sum_{\sigma=1}^{3} e_\mu(\mathbf{p},\sigma)e_\nu(\mathbf{p},\sigma)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}.$$

根据有质量矢量场的 3 个极化态，线性组合定义另一套物理的**极化矢量** $\varepsilon^\mu(\mathbf{p},\lambda)$, 其中 $\lambda=+1,0,-1$:

$$\varepsilon^\mu(\mathbf{p},\pm)\ \equiv\ \frac{1}{\sqrt{2}}\left[e^\mu(\mathbf{p},1)\pm \mathrm{i}e^\mu(\mathbf{p},2)\right],\quad \varepsilon^\mu(\mathbf{p},0)\equiv e^\mu(\mathbf{p},3).$$

这样定义的 $\varepsilon^\mu(p,\pm)$ 是复的，而 $\varepsilon^\mu(p,0)$ 是实的。它们满足

$$\begin{aligned}&\text{四维横向条件：}p_\mu\varepsilon^\mu(\mathbf{p},\lambda)=0,\quad \lambda=\pm,0.\\&\text{正交归一性：}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon^\mu(\mathbf{p},\lambda')=-\delta_{\lambda\lambda'},\\&\text{极化求和关系：}\sum_{\lambda=\pm,0}\varepsilon^*_\mu(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda)=-g_{\mu\nu}+\frac{p_\mu p_\nu}{m^2}.\end{aligned}\tag{18}$$

定义螺旋度矩阵算符为粒子的自旋角动量在动量方向上的投影

$$\hat{\mathbf{p}}\cdot\mathcal{J}=\frac{1}{|\mathbf{p}|}\mathbf{p}\cdot\mathcal{J}=\frac{1}{|\mathbf{p}|}\begin{pmatrix}0&&&\\&0&-\mathrm{i}p^3&\mathrm{i}p^2\\&\mathrm{i}p^3&0&-\mathrm{i}p^1\\&-\mathrm{i}p^2&\mathrm{i}p^1&0\end{pmatrix}.\tag{19}$$

代入极化矢量 $\varepsilon_\nu(\mathbf{p},\lambda)$ 的分量式有

$$(\hat{\mathbf{p}}\cdot\mathcal{J})^\mu{}_\nu\varepsilon^\nu(\mathbf{p},\lambda)=\lambda\varepsilon^\mu(\mathbf{p},\lambda),\quad \lambda=\pm,0.\tag{20}$$

上式说明极化矢量 $\varepsilon^\mu(\mathbf{p},\lambda)$ 是螺旋度矩阵的本征矢量，本征值 λ。也即 ,$\varepsilon^\mu(\mathbf{p},\lambda)$ 描述动量为 $\mathbf{p}$、螺旋度为 λ 的矢量粒子的极化态。螺旋度 $\lambda=\pm1$ 对应于两种横向极化 (transverse polarization) , 包括右旋极化 $(\lambda=+)$ 和左旋极化 $(\lambda=-)$; $\lambda=0$ 对应于纵向极化 (longitudinal polarization).

弄清楚有质量实矢量场的极化特性后，其 $A^\mu(\mathbf{x},t)$ 的**平面波展开式**应当包含正能解和负能解的所有动量模式的所有极化态，并满足自共轭条件，符合要求的形式为

$$A^\mu(\mathbf{x},t)=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=\pm,0}\left[\varepsilon^\mu(\mathbf{p},\lambda)a_{\mathbf{p},\lambda}\mathrm{e}^{-\mathrm{i}p\cdot x}+\varepsilon^{\mu*}(\mathbf{p},\lambda)a^\dagger_{\mathbf{p},\lambda}\mathrm{e}^{\mathrm{i}p\cdot x}\right],$$

$$
\begin{aligned}
\pi_i = -\partial_0 A_i + \partial_i A_0 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \sum_{\lambda=\pm,0} \Big\{ & [\mathrm{i}p_0 \varepsilon_i(\mathbf{p},\lambda) - \mathrm{i}p_i \varepsilon_0(\mathbf{p},\lambda)] a_{\mathbf{p},\lambda} \mathrm{e}^{-\mathrm{i}p\cdot x} \\
& + [-\mathrm{i}p_0 \varepsilon_i^*(\mathbf{p},\lambda) + \mathrm{i}p_i \varepsilon_0^*(\mathbf{p},\lambda)] a^\dagger_{\mathbf{p},\lambda} \mathrm{e}^{\mathrm{i}p\cdot x} \Big\}.
\end{aligned} \tag{21}
$$

1.2 产生湮灭算符的性质与对易关系

场 $A^\mu(\mathbf{x},t)$, 共轭动量密度 $\pi_\mu \quad (\mu=0$不纳入考虑) 的平面波展开式子代入等时对易关系, 经计算得出产生湮灭算符的对易关系:

$$
[a_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = (2\pi)^3 \delta_{\lambda\lambda'} \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [a_{\mathbf{p},\lambda}, a_{\mathbf{q},\lambda'}] = [a^\dagger_{\mathbf{p},\lambda}, a^\dagger_{\mathbf{q},\lambda'}] = 0. \tag{22}
$$

这与实标量场类似，有质量实矢量场对应的矢量子是**纯中性**的 Boson，无非是加上了极化自由度，表明该粒子为 Vector Boson, Spin=1. 进一步看产生湮灭算符的性质，推导矢量场的 Lorentz Boost, 应有

$$
A'^\mu(x') = U^{-1}(\Lambda) A^\mu(x') U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(x), \tag{23}
$$

$$
\Longrightarrow U^{-1}(\Lambda) A^\mu(x) U(\Lambda) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x). \tag{24}
$$

其中

$$
\begin{aligned}
U^{-1}(\Lambda) A^\mu(x) U(\Lambda) &= \left(\mathbb{I} + \frac{\mathrm{i}}{2}\omega_{\gamma\delta} J^{\gamma\delta}\right) A^\mu(x) \left(\mathbb{I} - \frac{\mathrm{i}}{2}\omega_{\alpha\beta} J^{\alpha\beta}\right) \\
&= A^\mu(x) - \frac{\mathrm{i}}{2}\omega_{\alpha\beta} A^\mu(x) J^{\alpha\beta} + \frac{\mathrm{i}}{2}\omega_{\gamma\delta} J^{\gamma\delta} A^\mu(x) = A^\mu(x) - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}[A^\mu(x), J^{\rho\sigma}] \\
\Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x) &= \left[\delta^\mu{}_\nu - \frac{\mathrm{i}}{2}\omega_{\rho\sigma} (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\right] \left[A^\nu(x) - \frac{\mathrm{i}}{2}\omega_{\alpha\beta} \hat{L}^{\alpha\beta} A^\nu(x)\right] \\
&= A^\mu(x) - \frac{\mathrm{i}}{2}\omega_{\rho\sigma}[\hat{L}^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x)].
\end{aligned} \tag{25}
$$

对比得到

$$
[A^\mu(x), J^{\rho\sigma}] = \hat{L}^{\rho\sigma} A^\mu(x) + (\mathcal{J}^{\rho\sigma})^\mu{}_\nu A^\nu(x). \tag{26}
$$

将我们设定的**平面波展开式**代入左右两边，得到

$$
(\hat{\mathbf{p}} \cdot \mathbf{J}) \left|\mathbf{p},\lambda\right\rangle = \lambda \left|\mathbf{p},\lambda\right\rangle \tag{27}
$$

$$
\text{with} \quad \left|\mathbf{p},\lambda\right\rangle \equiv \sqrt{2E_{\mathbf{p}}} a^\dagger_{\mathbf{p},\lambda} \left|0\right\rangle, \quad \lambda = \pm, 0. \tag{28}
$$

1.3 哈密顿量与总动量

有质量矢量场的哈密顿量密度为

$$\begin{aligned}\mathcal{H} &= \pi_\mu \partial_0 A^\mu - \mathcal{L} = 0 - \boldsymbol{\pi} \cdot \dot{\mathbf{A}} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu \\ &= \frac{1}{2} \boldsymbol{\pi}^2 + \nabla \cdot (A_0 \boldsymbol{\pi}) + \frac{1}{2m^2} (\nabla \cdot \boldsymbol{\pi})^2 + \frac{1}{2} (\nabla \times \mathbf{A})^2 + \frac{1}{2} m^2 \mathbf{A}^2.\end{aligned} \tag{29}$$

对空间积分即得哈密顿量为

$$\begin{aligned}H &= \int \mathrm{d}^3 x \mathcal{H} = \frac{1}{2} \int \mathrm{d}^3 x \left[\boldsymbol{\pi}^2 + \frac{1}{m^2} (\nabla \cdot \boldsymbol{\pi})^2 + (\nabla \times \mathbf{A})^2 + m^2 \mathbf{A}^2 \right] \\ &= \sum_{\lambda = \pm, 0} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{3}{2} E_{\mathbf{p}}.\end{aligned} \tag{30}$$

场的动量为

$$\begin{aligned}\mathbf{P} &= -\int \mathrm{d}^3 x \pi_a \nabla \Phi_a = -\int \mathrm{d}^3 x \ (\pi_i \nabla A^i + 0) \\ &= \sum_{\lambda = \pm, 0} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathbf{p} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda} + \frac{3}{2} \delta^{(3)}(\mathbf{0}) \int \mathrm{d}^3 p \mathbf{p} \\ &= \sum_{\lambda = \pm, 0} \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathbf{p} a^\dagger_{\mathbf{p},\lambda} a_{\mathbf{p},\lambda}.\end{aligned} \tag{31}$$

2 无质量矢量场的量子化

2.1 极化求和关系

由于无质量情况下 $m = 0, p^2 = 0$, 因此最明显的一个问题就是 $e^\mu(\mathbf{p}, 3)$ 无法像之前那样定义成正比于 $\mathbf{p}$; 同时 $e^\mu(\mathbf{p}, 0)$ 也不可能是四维横向的. 因此我们需要重新定义 $e^\mu(\mathbf{p}, 0)$, $e^\mu(\mathbf{p}, 3)$

$$\begin{aligned}e^\mu(\mathbf{p}, 0) &= n^\mu = (1, 0, 0, 0), \\ e^\mu(\mathbf{p}, 3) &= \frac{p^\mu - (p \cdot n) n^\mu}{p \cdot n} 0 = \frac{p^\mu - p^0 n^\mu}{p^0} = \left(0, \frac{\mathbf{p}}{|\mathbf{p}|}\right) = \frac{1}{|\mathbf{p}|}(0, p^1, p^2, p^3).\end{aligned}$$

可以验证新引入的这组极化矢量基满足完备性.

仍然按照有质量的情景定义横向圆极化矢量 $\varepsilon^\mu(\mathbf{p}, \pm)$,

则部分求和关系表明

$$\sum_{\lambda = \pm} \varepsilon^*_\mu(\mathbf{p}, \lambda) \varepsilon_\nu(\mathbf{p}, \lambda) = \sum_{\sigma = 1}^{2} e_\mu(\mathbf{p}, \sigma) e_\nu(\mathbf{p}, \sigma),$$

因而极化求和关系为

$$\sum_{\text{spins}} \varepsilon_\mu^*(p)\varepsilon_\nu(p) = \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) = -g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}.$$

光子场的极化关系可以导出与 **Ward 恒等式** $p_\mu M^\mu(p) = 0$. 作业题中有体现，这里不再赘述。应用 Ward 恒等式，我们发现

$$\begin{aligned}
\sum_{\text{spins}} |\mathcal{M}(p)|^2 &= \sum_{\lambda=\pm} \varepsilon_\mu^*(\mathbf{p},\lambda)\varepsilon_\nu(\mathbf{p},\lambda) M^\mu(p) M^{\nu *}(p) \\
&= \left[-g_{\mu\nu} - \frac{p_\mu p_\nu}{(p\cdot n)^2} + \frac{p_\mu n_\nu + p_\nu n_\mu}{p\cdot n}\right] M^\mu(p) M^{\nu *}(p) \\
&= -g_{\mu\nu} M^\mu(p) M^{\nu *}(p).
\end{aligned}$$

也就是说，在 QED 计算中可以使用替换关系

$$\sum_{\text{spins}} \varepsilon_\mu^*(p)\varepsilon_\nu(p) \to -g_{\mu\nu}.$$

2.2 规范对称性

基于先前对与有质量矢量场的讨论, 当 $m=0$ 时, 无质量的矢量场应有

- Lagrangian 变为 $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$;
- $\partial_\mu A^\mu = 0$ (Lorenz 条件) 不再是必然成立的，场多出了一个自由度;
- A^μ 运动方程 (Proca 方程) 变为
$$\partial_\mu F^{\mu\nu} = 0. \tag{32}$$

这便是我们熟悉的无源电磁场方程 (Maxwell Eq.), 若对 $A^\mu(x)$ 作规范变换

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \chi(x)$$

其中，作为变换参数的 $\chi(x)$ 是一个任意的 Lorentz 标量函数，依赖于时空坐标，因而这样的变换是 local 变换。在此规范变换下，$F_{\mu\nu}$、$\mathbf{E}$、$\mathbf{B}$ 均不变，称为**规范对称性 (gauge symmetry)**.

在对有质量矢量场 $A^\mu(x)$ 的正则量子化中，我们最后是通过 Proca Eq. 导出

$$A^0 = -\frac{1}{m^2}\nabla\cdot\boldsymbol{\pi}, \tag{33}$$

来解决 $A^0(x)$ 不拥有非零的共轭动量密度的诘难，因而 $A^0(x)$ 顺理成章没有作为独立的正则运动变量纳入考量。但当 $m=0$ 时，上式显然不成立。对于无质量矢量场，可以让 $A^0(x)$ 也作为独立的正则变量，这需要给它安排非零的共轭动量密度。为此，在拉氏量中增加一个不会影响最终物理结果的项，

$$\mathcal{L}_1 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \boxed{\frac{1}{2\xi}(\partial_\mu A^\mu)^2}$$

其中 ξ 是一个可以自由选取的实参数。A^μ 对应的共轭动量密度为

$$\pi_\mu = \frac{\partial\mathcal{L}_1}{\partial(\partial^0 A^\mu)} = -\partial_0 A_\mu + \partial_\mu A_0 - \frac{1}{\xi}(\partial_\nu A^\nu)\frac{\partial(\partial_\rho A^\rho)}{\partial(\partial_0 A^\mu)} = -F_{0\mu} - \frac{1}{\xi}\, g_{0\mu}\partial_\nu A^\nu,$$

即

$$\pi_i = -F_{0i} = -\partial_0 A_i + \partial_i A_0, \quad \pi_0 = -\frac{1}{\xi}\partial_\mu A^\mu.$$

可以看出，在 $A^\mu(x)$ 满足 Lorenz Gauge $\partial_\mu A^\mu = 0$ 的情况下，新构造的 $\mathcal{L}_1$ 等价于原先的 $\mathcal{L}$。新增的 **gauge-fixing term** $-(2\xi)^{-1}(\partial_\mu A^\mu)^2$ 不仅为 A^0 提供共轭动量密度 π_0, 另一作用便是破坏规范对称性，固定规范.

老套路，正则量子化要求算符 A^μ 和 π_μ 满足等时对易关系

$$A^\mu(\mathbf{x},t),\pi_\nu(\mathbf{y},t)] = \mathrm{i}\delta^\mu{}_\nu\delta^{(3)}(\mathbf{x}-\mathbf{y}), \quad [A^\mu(\mathbf{x},t),A^\nu(\mathbf{y},t)] = [\pi_\mu(\mathbf{x},t),\pi_\nu(\mathbf{y},t)] = 0. \tag{34}$$

从 $\mathcal{L}_1$ 导出关于 A^μ 的 Euler-Lagrange 方程

$$0 = \partial_\mu\frac{\partial\mathcal{L}_1}{\partial(\partial_\mu A_\nu)} - \frac{\partial\mathcal{L}_1}{\partial A_\nu} = -\partial^2 A^\nu + \partial^\nu\partial_\mu A^\mu - \frac{1}{\xi}g^{\mu\nu}\partial_\mu\partial_\rho A^\rho = -\partial^2 A^\nu + \left(1-\frac{1}{\xi}\right)\partial^\nu\partial_\rho A^\rho,$$

即 A^μ 的经典运动方程是

$$\partial^2 A^\mu - \left(1-\frac{1}{\xi}\right)\partial^\mu\partial_\nu A^\nu = 0.$$

取 $\boxed{\xi=1}$, 称为 **Feynman Gauge** , 运动方程化为 **d'Alembert** Eq.

$$\partial^2 A^\mu(x) = 0.$$

其本质上也就是 KG Eq. 的无质量情形. d'Alembert Eq. 的平面波正负能解分别带因子 $\exp(-\mathrm{i}p\cdot x)$ 和 $\exp(\mathrm{i}p\cdot x)$. 使用之前得到的实极化矢量组 $e^\mu(\mathbf{p},\sigma)$, 对无质量实矢量场 $A^\mu(\mathbf{x},t)$ 作平面波展开，得

$$A^\mu(\mathbf{x},t) = \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\sigma=0}^{3}e^\mu(\mathbf{p},\sigma)\left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} + b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right),$$

相应的共轭动量密度展开式为

$$\pi_\mu(\mathbf{x},t) = -\partial_0 A_\mu = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \frac{\mathrm{i}p_0}{\sqrt{2E_\mathbf{p}}} \sum_{\sigma=0}^{3} e_\mu(\mathbf{p},\sigma) \left(b_{\mathbf{p},\sigma} \mathrm{e}^{-\mathrm{i}p\cdot x} - b^\dagger_{\mathbf{p},\sigma} \mathrm{e}^{\mathrm{i}p\cdot x} \right).$$

容易验证自共轭条件 $[A^\mu(\mathbf{x},t)]^\dagger = A^\mu(\mathbf{x},t)$, $[\pi_\mu(\mathbf{x},t)]^\dagger = \pi_\mu(\mathbf{x},t)$.

2.3 产生湮灭算符的对易关系

场 $A^\mu(\mathbf{x},t)$, 共轭动量密度 π_μ 的平面波展开式子代入等时对易关系, 经计算得出产生湮灭算符的对易关系:

$$[b_{\mathbf{p},\sigma}, b^\dagger_{\mathbf{q},\sigma'}] = -(2\pi)^3 g_{\sigma\sigma'} \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad [b_{\mathbf{p},\sigma}, b_{\mathbf{q},\sigma'}] = [b^\dagger_{\mathbf{p},\sigma}, b^\dagger_{\mathbf{q},\sigma'}] = 0. \tag{35}$$

2.4 物理的极化态

将真空态 $|0\rangle$ 定义为被任意 $b_{\mathbf{p},\sigma}$ 湮灭的态，满足

$$b_{\mathbf{p},\sigma}|0\rangle = 0, \quad \langle 0|0\rangle = 1, \quad H|0\rangle = E_{\mathrm{vac}}|0\rangle.$$

动量为 $\mathbf{p}$、极化指标为 σ 的单粒子态定义为

$$|\mathbf{p},\sigma\rangle \equiv \sqrt{2E_\mathbf{p}}\, b^\dagger_{\mathbf{p},\sigma}|0\rangle,$$

详细计算 Hamilton

$$\begin{aligned}\mathcal{H} &= \pi_\mu \partial^0 A^\mu - \mathcal{L}_2 = -(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_\mu A_\nu)\partial^\mu A^\nu \\ &= -\frac{1}{2}(\partial_0 A_\mu)\partial^0 A^\mu + \frac{1}{2}(\partial_i A_\mu)\partial^i A^\mu = -\frac{1}{2}\left[\pi_\mu \pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)\right].\end{aligned} \tag{36}$$

$$\begin{aligned} H &= \int \mathrm{d}^3 x \mathcal{H} = -\frac{1}{2}\int \mathrm{d}^3 x [\pi_\mu \pi^\mu + (\nabla A_\mu)\cdot(\nabla A^\mu)] \\ &= -\frac{1}{2}\sum_{\sigma\sigma'} \int \frac{\mathrm{d}^3 x \mathrm{d}^3 p \mathrm{d}^3 q}{(2\pi)^6 \sqrt{4E_\mathbf{p}E_\mathbf{q}}} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \times \Big[(\mathrm{i}p_0)(\mathrm{i}q_0) \left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} - b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right) \left(b_{\mathbf{q},\sigma'}\mathrm{e}^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q},\sigma'}\mathrm{e}^{\mathrm{i}q\cdot x}\right) \\ &\qquad + (\mathrm{i}\mathbf{p})\cdot(\mathrm{i}\mathbf{q}) \left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} - b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right) \left(b_{\mathbf{q},\sigma'}\mathrm{e}^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q},\sigma'}\mathrm{e}^{\mathrm{i}q\cdot x}\right) \Big] \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_\mathbf{p} \left(-b^\dagger_{\mathbf{p},0} b_{\mathbf{p},0} + \sum_{\sigma=1}^{3} b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma} \right) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} 2E_\mathbf{p}. \end{aligned} \tag{37}$$

利用 H 表达式

$$[H, b^\dagger_{\mathbf{p},\sigma}] = E_{\mathbf{p}} \sum_{\sigma'=0}^{3} g_{\sigma'\sigma'} g_{\sigma'\sigma} b^\dagger_{\mathbf{p},\sigma'} = E_{\mathbf{p}} b^\dagger_{\mathbf{p},\sigma} \tag{38}$$

$$\begin{aligned} \Longrightarrow H\,|\mathbf{p},\sigma\rangle &= \sqrt{2E_{\mathbf{p}}} H b^\dagger_{\mathbf{p},\sigma}|0\rangle = \sqrt{2E_{\mathbf{p}}}(b^\dagger_{\mathbf{p},\sigma} H + E_{\mathbf{p}} b^\dagger_{\mathbf{p},\sigma})|0\rangle \\ &= \sqrt{2E_{\mathbf{p}}}\,(E_{\text{vac}} + E_{\mathbf{p}})\, b^\dagger_{\mathbf{p},\sigma}\,|0\rangle = (E_{\text{vac}} + E_{\mathbf{p}})\,|\mathbf{p},\sigma\rangle\,. \end{aligned} \tag{39}$$

利用产生湮灭算符的对易关系计算单粒子态的内积，可得

$$\begin{aligned} \langle \mathbf{q},\sigma' \mid \mathbf{p},\sigma\rangle &= \sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|\, b_{\mathbf{q},\sigma'} b^\dagger_{\mathbf{p},\sigma}\,|0\rangle \\ &= \sqrt{4E_{\mathbf{q}}E_{\mathbf{p}}}\,\langle 0|\left[b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{q},\sigma'} - (2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right]|0\rangle \\ &= -2E_{\mathbf{p}}(2\pi)^3 g_{\sigma\sigma'}\delta^{(3)}(\mathbf{p}-\mathbf{q}). \end{aligned}$$

也即

$$\langle \mathbf{p},0 \mid \mathbf{p},0\rangle = -2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0}), \quad \langle \mathbf{p},i \mid \mathbf{p},i\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{0}), \quad i=1,2,3. \tag{40}$$

也即，单粒子态 $|\mathbf{p},0\rangle$ 的内积是负的，这不符合 Hilbert 空间中态矢的要求，且相应的能量期望值 $\langle \mathbf{p},0 \mid H \mid \mathbf{p},0\rangle$ 也是负值，出现了很大的问题。为解决负能问题，引入 **Gupta-Bleuler** Condition. 首先, 将平面波展开式分解成正负能两部分

$$\begin{aligned} A^\mu(\mathbf{x},t) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma)\left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} + b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right) \\ &= \underbrace{\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}}_{A^{\mu(-)}(x)} + \underbrace{\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x}}_{A^{\mu(+)}(x)} \end{aligned}$$

Gupta-Bleuler Condition 表述为

$$\partial_\mu A^{\mu(+)}(x)|\Psi\rangle = 0, \quad \langle\Psi|\partial_\mu A^{\mu(-)}(x) = \langle\Psi|[\partial_\mu A^{\mu(+)}(x)]^\dagger = 0 \tag{41}$$

其意味着

$$\partial_\mu A^{\mu(+)}(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-\mathrm{i}\mathrm{e}^{-\mathrm{i}p\cdot x}}{\sqrt{2E_{\mathbf{p}}}}\left[p_\mu\sum_{\sigma=0}^{3} e^\mu(\mathbf{p},\sigma) b_{\mathbf{p},\sigma}\right] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\frac{-\mathrm{i}\mathrm{e}^{-\mathrm{i}p\cdot x}}{\sqrt{2E_{\mathbf{p}}}} E_{\mathbf{p}}(b_{\mathbf{p},0} - b_{\mathbf{p},3}), \tag{42}$$

$$\text{Gupta-Bleuler Condition} \Longrightarrow \langle\Psi|\, b^\dagger_{\mathbf{p},0} b_{\mathbf{p},0}\,|\Psi\rangle = \langle\Psi|\, b^\dagger_{\mathbf{p},3} b_{\mathbf{p},3}\,|\Psi\rangle\,. \tag{43}$$

这样一来, 物理态 $|\Psi\rangle$ 的能量期待值为

$$\begin{aligned} \langle\Psi|\,H\,|\Psi\rangle &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}}\langle\Psi|\left(-b^\dagger_{\mathbf{p},0} b_{\mathbf{p},0} + \sum_{\sigma=1}^{3} b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma}\right)|\Psi\rangle + E_{\text{vac}}\langle\Psi|\Psi\rangle \\ &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}}\sum_{\sigma=1}^{2}\langle\Psi|\, b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma}\,|\Psi\rangle + E_{\text{vac}}\,\langle\Psi|\,\Psi\rangle. \end{aligned} \tag{44}$$

2.5 哈密顿量与总动量

Hamiltonian 的计算上一小节已经呈现:

$$
\begin{aligned}
H &= \int \mathrm{d}^3 x \mathcal{H} = -\frac{1}{2}\int \mathrm{d}^3 x [\pi_\mu \partial^0 A^\mu - \mathcal{L}_2] \\
&= -\frac{1}{2}\sum_{\sigma\sigma'}\int \frac{\mathrm{d}^3 x \mathrm{d}^3 p \mathrm{d}^3 q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \times \Big[(\mathrm{i}p_0)(\mathrm{i}q_0)\left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} - b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right)\left(b_{\mathbf{q},\sigma'}\mathrm{e}^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q},\sigma'}\mathrm{e}^{\mathrm{i}q\cdot x}\right) \\
&\qquad + (\mathrm{i}\mathbf{p})\cdot(\mathrm{i}\mathbf{q})\left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} - b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right)\left(b_{\mathbf{q},\sigma'}\mathrm{e}^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q},\sigma'}\mathrm{e}^{\mathrm{i}q\cdot x}\right)\Big] \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_{\mathbf{p}} \left(-b^\dagger_{\mathbf{p},0} b_{\mathbf{p},0} + \sum_{\sigma=1}^{3} b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma}\right) + (2\pi)^3 \delta^{(3)}(\mathbf{0}) \int \frac{\mathrm{d}^3 p}{(2\pi)^3} 2E_{\mathbf{p}}.
\end{aligned} \tag{45}
$$

总动量的计算依葫芦画瓢:

$$
\begin{aligned}
\mathbf{P} &= -\int \mathrm{d}^3 x \pi_\mu \nabla A^\mu \\
&= -\sum_{\sigma\sigma'}\int \frac{\mathrm{d}^3 x \mathrm{d}^3 p \mathrm{d}^3 q}{(2\pi)^6\sqrt{4E_{\mathbf{p}}E_{\mathbf{q}}}} e_\mu(\mathbf{p},\sigma) e^\mu(\mathbf{q},\sigma') \\
&\times (\mathrm{i}p_0)\left(b_{\mathbf{p},\sigma}\mathrm{e}^{-\mathrm{i}p\cdot x} - b^\dagger_{\mathbf{p},\sigma}\mathrm{e}^{\mathrm{i}p\cdot x}\right)(\mathrm{i}\mathbf{q})\left(b_{\mathbf{q},\sigma'}\mathrm{e}^{-\mathrm{i}q\cdot x} - b^\dagger_{\mathbf{q},\sigma'}\mathrm{e}^{\mathrm{i}q\cdot x}\right) \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathbf{p}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma} b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma}) + \delta^{(3)}(\mathbf{0})\int \mathrm{d}^3 p \frac{\mathbf{p}}{2}\sum_{\sigma=0}^{3}(-g_{\sigma\sigma})^2 \\
&= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\mathbf{p}\left(-b^\dagger_{\mathbf{p},0} b_{\mathbf{p},0} + \sum_{\sigma=1}^{3} b^\dagger_{\mathbf{p},\sigma} b_{\mathbf{p},\sigma}\right).
\end{aligned} \tag{46}
$$

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