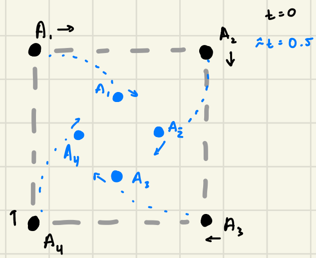
Lecture I

Puzzle:

Suppose we have a perfect square

$t = 0$

$|v| = 1$



Q: How long will it take for Agents to converge?

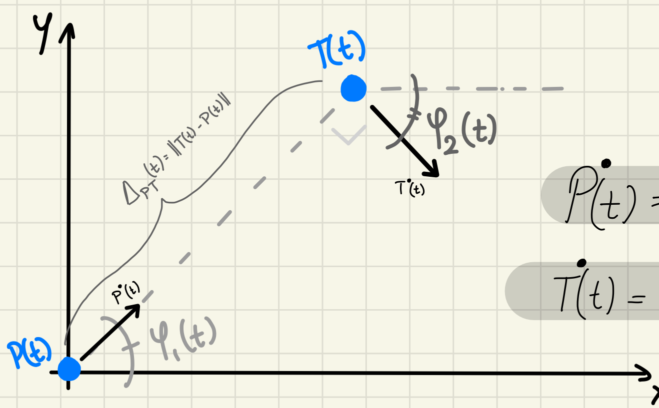
A: Equation of Continuous Persuit

$P(t)$: location of Persuer

$T(t)$: location of Target

at time 't'

Velocity of Persuer: $\frac{d}{dt}\left(P(t)\right) = \frac{T(t) - P(t)}{\|T(t) - P(t)\|}$



$$\dot{P}(t) = \left[\cos(\varphi_1(t)),\ \sin(\varphi_1(t))\right]$$

$$\dot{T}(t) = \left[\cos(\varphi_2(t)),\ \sin(\varphi_2(t))\right]$$

Velocity of Pursuer: $\underbrace{\frac{d}{dt}\left(P(t)\right)}_{\dot{P}(t)} = \frac{T(t)-P(t)}{\underbrace{\|T(t)-P(t)\|}_{\Delta_{PT}(t)}}$ — length of side of square???

$$\dot{P}(t) = \frac{1}{\Delta_{PT}(t)} \cdot \left[T(t) - P(t)\right]$$

$$\ddot{P}(t) = \frac{1}{\Delta_{PT}(t)}\left[\dot{T}(t) - \dot{P}(t)\right] - \frac{\dot{\Delta}(t)}{\Delta_{PT}^{\bullet}(t)}\left[T(t) - P(t)\right]$$

$$= \frac{1}{\Delta_{PT}(t)}\left[\dot{T}(t) - \dot{P}(t)\right] - \frac{\dot{\Delta}_{PT}(t)}{\Delta_{PT}(t)}\left[\dot{P}(t)\right] \quad *1$$

*Check this derivation later

Also, we can differentiate $\dot{P}(t)$'s vector form:

equivalent

$$\dot{P}(t) = \left[\cos(\varphi_1(t)), \sin(\varphi_1(t))\right]$$

$$\ddot{P}(t) = \left[-\sin(\varphi_1(t)) \cdot \dot{\varphi}_1(t), \cos(\varphi_1(t)) \cdot \dot{\varphi}_1(t)\right]$$

$$= \left[-\dot{\varphi}_1(t) \cdot \sin(\varphi_1(t)), \dot{\varphi}_1(t) \cdot \cos(\varphi_1(t))\right] \quad *2$$

$$= \frac{1}{\Delta_{PT}(t)}\left[\dot{T}(t) - \dot{P}(t)\right] - \frac{\dot{\Delta}_{PT}(t)}{\Delta_{PT}(t)}\left[\dot{P}(t)\right] \quad *1$$

$$= \frac{1}{\Delta_{PT}(t)}\left[\cos(\varphi_2(t)) - \cos(\varphi_1(t)), \sin(\varphi_2(t)) - \sin(\varphi_1(t))\right] - \frac{1}{\Delta_{PT}(t)} \dot{\Delta}_{PT}\left[\cos\varphi_1(t), \sin\varphi_1(t)\right] \quad *3 \quad = \ddot{P}(t)$$

Now we will derive 2 equations by setting components of *2 equal to components of *3

1. $$-\dot{\varphi}_1(t)\sin(\varphi_1(t)) = \frac{1}{\Delta_{PT}(t)} \cdot \left[\cos(\varphi_2(t)) - \cos(\varphi_1(t))\right] - \frac{\dot{\Delta}_{PT}(t)}{\Delta_{PT}(t)} \cdot \cos(\varphi_1(t))$$

2. $$\dot{\varphi}(t)\cos(\varphi_1(t)) = \frac{1}{\Delta_{PT}(t)} \cdot \left[\sin(\varphi_2(t)) - \sin(\varphi_1(t))\right] - \frac{\dot{\Delta}_{PT}(t)}{\Delta_{PT}(t)} \cdot \sin(\varphi_1(t))$$

Unknowns we are interested in finding: $\dot{\varphi}(t)$, $\dot{\Delta}_{PT}(t)$

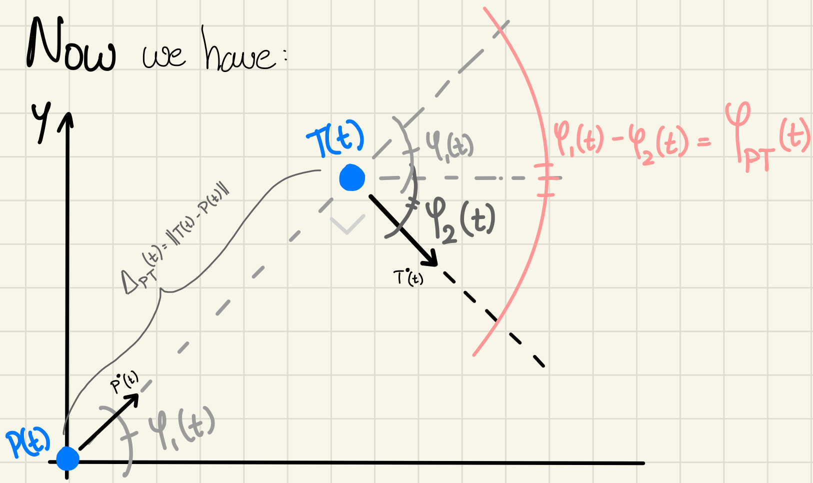
Now, We can use matrices to solve for $\dot{\varphi}(t)$, $\dot{\Delta}_{PT}(t)$ using equations 1. and 2.

$$\begin{bmatrix} -\sin\varphi_1(t) & \frac{1}{\Delta}\cos\varphi_1(t) \\ \cos\varphi_1(t) & \frac{1}{\Delta}\sin\varphi_1(t) \end{bmatrix} \begin{bmatrix} \dot{\varphi}_1(t) \\ \dot{\Delta}_{PT}(t) \end{bmatrix} = \frac{1}{\Delta_{PT}(t)} \begin{bmatrix} \cos\varphi_2(t) - \cos\varphi_1(t) \\ \sin\varphi_2(t) - \sin\varphi_1(t) \end{bmatrix}$$

$$\dot{\varphi}_1(t) = -\frac{1}{\Delta_{PT}(t)} \cdot \sin\left(\varphi_1(t) - \varphi_2(t)\right)$$ ← Rate of change of angle between Persuer (P(t)) and Target T(t)

$$\dot{\Delta}_{PT}(t) = \cos(\varphi_1(t) - \varphi_2(t)) - 1$$ ← Rate of change of distance between Persuer (P(t)) and Target T(t). Since the derivative is (nearly) always negative (<1), the distance is decreasing.

Now we have:



$$\dot{\varphi}_1(t) = -\frac{1}{\Delta_{PT}(t)} \cdot \sin\left(\varphi_{PT}(t)\right)$$

$$\dot{\Delta}_{PT}(t) = \cos\left(\varphi_{PT}(t)\right) - 1$$

In our problem, what is $\varphi_{PT} = 90°$
and what is $\Delta_{PT}(t)$

So, $\dot{\Delta}_{PT}(t) = 0 - 1 = -1$
$\dot{\varphi}_1(t) = -\frac{1}{\Delta_{PT}(t)} \cdot 1 = -\frac{1}{\Delta_{PT}(t)}$

Finally, What is the solution:

$$\frac{d}{dt}\left(\Delta_{PT}(t)\right) = -1 \quad ; \quad \Delta_{PT}(t=0) = 1$$

$$\Delta_{PT}(t) = \Delta_{PT}(0) - t$$

$$\boxed{\Delta_{PT}(t) = 1 - t}$$

Next class: Think about the proof for the theorem: "n agents in cyclic persuit gather in finite time."

Also: Think about what other problems we can solve

Papers: Persuit problems paper in Prof's webpage
"Why the n-trails look so straight & nice"