Trilateration Using Motion Models

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***Abstract*—In this paper, we present a framework for doing localization from distance measurements, given an estimate of the local motion. We show how we can register the local motion of a receiver, to a global coordinate system, using trilateration of given distance measurements from the receivers to senders in known positions. We describe how many different motion models can be formulated within the same type of registration framework, by only changing the transformation group. The registration is based on a test and hypothesis framework, such as RANSAC, and we present novel and fast minimal solvers that can be used to bootstrap such methods. The system is tested on both synthetic and real data with promising results.**

***Index Terms*—Trilateration, time-of-arrival, motion model, odometry, IMU, minimal solvers, RANSAC.**

I. Introduction

Where am I? This is a question that is fundamental to most living organisms in the world. Both the methods and sensor inputs used for answering this question vary wildly. In this paper, we will specifically look into how one can combine a local motion model with distance measurements to a global coordinate system in order to do positioning.

Many animals—from desert ants [1] to mammals [2]—use dead reckoning or path integration based on locomotion estimates to navigate, and such techniques have also been used by humans throughout history for navigating both at sea and land. Typically one gets quite accurate results from dead reckoning based on, e.g., estimated headings and speed, but these results invariably suffer from drift.

Using (direct or indirect) distance measurements has also been used for a long time, e.g., using sound waves from as early as the eighteenth century [3]. Given a small set of distance measurements (in the plane at least two and in space at least three) to known locations, one can estimate the position using trilateration. These distances are often acquired using approaches based on Time Of Arrival (TOA), Received Signal Strength (RSS), or Time Difference Of Arrival (TDOA). In such cases a signal, e.g, radio, sound, or light, is typically emitted from the senders and received at the sought position, or vice versa. For TOA and TDOA the speed of the signal in the medium is most often assumed to be known, which then directly gives distance measurements. For RSS, a signal propagation model can be used to translate power measurements to distances. One problem with trilateration techniques is that one often has gross outliers in the data, arising from errors when correlating signals, from the geometry of the problem, or multipath effects.

Our contribution in this paper is a way of robustly handling outliers, and making use of distance measurements, at the same time as we use any available motion model or odometry data to locally constrain the motion, and by this way aggregating distance measurements. Specifically, we will formulate our trilateration problem as registering a local receiver coordinate system to a global sender coordinate system, so that the distance measurements are realized. We propose to do this robustly in a RANSAC framework [4]–[6], so that we can handle outliers in the distance measurements. We will focus on the most tractable case, when the receivers are in a plane, i.e., the two-dimensional case. In order to do this, we need to develop a number of minimal trilateration registration solvers, that are used as components in bootstrapping the solutions. In Section II we describe the geometry of our setup. Then in Section III we describe the proposed solvers, and how they were developed. We also test our approach using the proposed solvers, both on synthetic and real data, in Section IV.

The Matlab-Mex implementations are publicly available[1].

A. Related Work

The problem considered presently can be described as the simultaneous trilateration and registration of points. Regarding trilateration, also known as single source localization, there has been a large body of previous work. The minimal problem, where the number of distance measurements equals the spatial dimension, has a closed-form solution [7]–[9]. However, when the problem is over-determined, some form of maximum likelihood (ML) estimator must be constructed. Finding the ML estimate given Gaussian noise in the measurements is a nonlinear, non-smooth and non-convex problem, and unlike the minimal case, it lacks a closed-form solution. Multiple iterative methods, with various convergence guaranties, have nevertheless been proposed [10]–[13]. There has also been several contributions where the ML problem is relaxed, and the error in the squared distance measurements are minimized [14]–[17]. Various forms of linear solvers have also been proposed, see [18] and references therein. Common for the works listed above is that they do not treat the scenario when gross outliers are present. These outliers can result from non-line-of-sight (NLOS) measurements, in which case they are

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[1]The code for all presented solvers is publicly available at *https://github.com/hamburgerlady/motion-model-trilateration*

necessarily longer than the line-of-sight (LOS) distance. These measurements can either be identified and removed [19], [20] or incorporated into the model [21].

The positioning resulting from trilateration can be improved by incorporating a motion model. The most common approach here is using filtering, to fuse the data from different sources, e.g., using a Kalman filter [22]–[24] or hidden Markov models [25]. This can give significantly better result, especially in the case of noisy RSS distance measurements. However, by the real-time nature of these filters, outlier detection might not be trivial, and such approaches also rely on a reasonable initialization. The most similar approach, to our proposed framework, is perhaps within robotics and autonomous vehicles, using angle measurements combined with odometry [26]. Although, the underlying governing measurement equations are completely different.

When it comes to registration of corresponding point sets, it has been long known that the globally optimal solution is found by singular value decomposition (SVD) [27]–[29]. Robust methods that allows for missing data and outliers have since then been proposed, see [30]–[32] and references therein. However, the problem considered here is slightly different, in that the corresponding points are required to be a specific distance from each other rather than being coincident. This fundamentally changes the problem and requires a more involved approach. Closer similarities can be found within the study of forward kinematics of parallel manipulators. In particular, forward kinematics of the well known Stewart platform [33], [34] and the planar 3-R$\underline{\text{P}}$R manipulator [35] is identical to the minimal rigid registration problem considered here, in 3D and 2D, respectively. However, to the best of our knowledge, this type of problem has never before been applied in the context of robust trilateration with motion priors. Furthermore, when the scale of the registration transform is unknown, the problem becomes different from the forward kinematics one.

II. Problem Formulation

We will now describe our approach in more detail and give motivation and use cases where it is suitable. We assume that we have measured the distances d_{ij} from a number of receivers $\mathbf{r}_i \in \mathbb{R}^N$ and senders $\mathbf{s}_j \in \mathbb{R}^N$,

$$d_{ij}^2 = |T(\mathbf{r}_i) - \mathbf{s}_j|^2, \tag{1}$$

where the receivers are given in a local coordinate system and T is a transformation from the receiver coordinate system to the global sender coordinate system. We will now consider the problem when $\mathbf{r}_i$ and $\mathbf{s}_j$ are known but the transformation T is unknown. Note that the problem is agnostic in what we label as receivers and senders. We will henceforth, for notation purposes, assume that the receivers are in the local coordinate system. Furthermore we can w.l.o.g. assume that we enumerate our senders and receivers so that $i = j$, (with the possible need of duplicating receiver or sender positions), so that

$$d_i^2 = |T(\mathbf{r}_i) - \mathbf{s}_i|^2. \tag{2}$$

Depending on the degrees of freedom of T, we would need differently many distance measurements d_i in order to minimally estimate T. We will now give a number of examples when the above formulation is an appropriate model for localizing a number of unknown receivers.

We will in this paper assume that the receivers are in the plane, i.e., $N = 2$. One can do the same analysis in 3D, but the specific problems become much harder. In many cases, we have quite accurate information on the local motion of a moving receiver. This could for instance be measurements from odometry systems or Inertial Measurement Unit (IMU) measurements, but it could also be a constrained motion model. Consider the following hierarchy of knowledge about the motion of a receiver, where we assume that motion is:

1) in a straight line with constant speed.
2) in a straight line and with known relative speed.
3) in a straight line and with known absolute speed.
4) with known direction and with constant speed.
5) with known direction and with known relative speed.
6) with known direction and with known absolute speed.

Relative speeds could be realized using, e.g., a step counter with unknown step length, absolute speeds with a calibrated step counter or odometry, and directions using an IMU. All these cases can be parameterized using a local coordinate systems for the receiver positions, and can be registered to the sender coordinate using either a similarity transform (Cases 1, 2, 4 and 5) or a Euclidean transformation (Cases 3 and 6). Since a similarity transformation in the plane has four degrees of freedom, we would need at least four distance measurements to solve (2). For the rigid Euclidean transformation, we would need at least three measurements.

III. Solvers

In this section, we will describe our proposed minimal solvers for the 2D cases. Since these solvers typically are used in a RANSAC framework [4]–[6], the most important aspect of them is that they are fast. We will use an approach based on the action matrix method [36] and use an automatic solver generator [37], [38]. The process is based on formulating problem instances—the system of multivariate polynomial equations that we want to solve—with random coefficient from a finite field, where the analysis can be done in exact arithmetic using a computer algebra system such as Macaulay2 [39]. From these calculations the solver can be constructed. The resulting solver contains two major steps, a linear system—the so-called elimination template—to construct the action matrix, and then the solution is extracted using an eigenvalue solver of the action matrix. Please see [37] for details on the process. Even though the solver generation is automatic, the properties of the resulting solver (in terms of speed, accuracy and robustness) will heavily rely on the parametrization used.

A. Minimal Euclidean Three-Point Solver

We assume that we have three receivers, $\mathbf{r}_i \in \mathbb{R}^2$ in a local coordinate system, that we want to register to the corresponding senders, $\mathbf{s}_j \in \mathbb{R}^2$, in the global coordinate system, so that,

$$d_i^2 = |R\mathbf{r}_i + \mathbf{t} - \mathbf{s}_i|^2, \quad i = 1, 2, 3, \tag{3}$$

where R is the unknown two-dimensional rotation and $\mathbf{t} \in \mathbb{R}^2$ is the unknown translation. Since $R^T R = I$ we can simplify (3) to

$$d_i^2 = \mathbf{r}_i^T \mathbf{r}_i + 2\mathbf{r}_i^T R^T (\mathbf{t} - \mathbf{s}_i) + \mathbf{t}^T \mathbf{t} - 2\mathbf{s}_i^T \mathbf{t} + \mathbf{s}_i^T \mathbf{s}_i, \tag{4}$$

for $i = 1, 2, 3$. We can see that these equations are linear in R, but there is a cross-term between R and $\mathbf{t}$ and also a quadratic term in $\mathbf{t}$. We will parameterize the rotation using two parameters (a, b), so that

$$R = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad a^2 + b^2 = 1. \tag{5}$$

With the non-linear constraint on (a, b) and (4) we have four equations in four unknowns. If we use this system and construct a solver using the automatic solver from [37], we get in general six (possibly complex) solutions, and an elimination template of size 45×51. However, we can get a faster solver by reducing the problem to a single univariate sextic polynomial. This was done in [35] using Cayley-Menger determinants, but our approach is as follows. If we take differences between two equations in (4), the quadratic terms $\mathbf{t}^T\mathbf{t}$ cancel out. This means that from two such differences, we get linear expressions in $\mathbf{t}$ from which we can solve directly for the translation (as a function of the rotation R). Inserting this expression for $\mathbf{t}$ in (4) gives an equation in only (a, b), with terms up to total degree five in (a, b). We simplify this expression further, by substituting all powers of b^2 and higher using $b^2 = 1 - a^2$. This gives us a new equation in a and b,

$$c_1 a^3 + c_2 a^2 b + c_3 a^2 + c_4 ab + c_5 a + c_6 b + c_7 = 0, \tag{6}$$

where $c_k, k = 1, \ldots, 7$, are coefficients only depending on the measured input data. Note that it is now linear in b. Multiplying this equation with b and again substituting all powers of b^2 using $b^2 = 1 - a^2$ gives another equation that is linear in b. We now have two equations on the form

$$\underbrace{\begin{bmatrix} f_{11}(a) & f_{12}(a) \\ f_{21}(a) & f_{22}(a) \end{bmatrix}}_{M} \begin{bmatrix} b \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7}$$

where $f_{kl}(a)$ are polynomials in a with coefficients that only depend on the input data. Since (7) should have a non-trivial solution, we get our final equation in a by

$$\det M = e_6 a^6 + e_5 a^5 + e_4 a^4 + e_3 a^3 + e_2 a^2 + e_1 a + e_0 = 0. \tag{8}$$

This equation can be solved very fast and efficiently using the companion matrix or by Sturm sequences [40], [41].

B. Minimal Similarity Four-Point Solver

For the similarity transformation case we get the same type of equations as for the Euclidean case. The difference is that we use four equations on the form,

$$d_i^2 = |S\mathbf{r}_i + \mathbf{t} - \mathbf{s}_i|^2, \quad i = 1, \ldots, 4, \tag{9}$$

where S is a scaled rotation matrix. We can parametrize the problem in more or less the same way as for the Euclidean case, using four parameters, but now without the scale constraint on (a, b), so that

$$S = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}. \tag{10}$$

We could solve for the translation also in this case, using the same approach as for the Euclidean case. However, this will for the similarity case introduce spurious solutions (arising from the denominator of the solution in $\mathbf{t}$). One could use saturation to automatically eliminate the spurious solutions using the technique describe in [42], but this will in this case lead to a much slower solver. Instead, we keep the initial formulation (9) and use this as input to the automatic solver generator. The system of equations has in general up to six solutions, and the generated solver has an elimination template of size 53×59.

IV. EXPERIMENTS

In this section, we will evaluate our solvers and test them as system components for robust trilateration. We will do a number of systematic tests on synthetic data, and show a proof-of-concept evaluation on a dataset with real Round-Trip Time (RTT) measurements. We have implemented our two minimal solvers in MATLAB. A number of the time-consuming steps were implemented as compiled Mex-C++ subroutines. On a standard laptop (Macbook pro 2.5 GHz Dual-Core Intel Core i7 running MATLAB 2020b), the average execution times are $25\,\mu$s for the Euclidean solver and $57\,\mu$s for the similarity solver.

A. Solver Tests on Synthetic Data

In order to test the numerical stability of our solvers, without noise, we generated synthetic minimal problem instances (three sender-receiver-distance measurements for the Euclidean solver and four sender-receiver-distance measurements for the similarity solver). We then ran our minimal solvers and evaluated the equation residuals. We repeated this 10 000 times in order to generate error statistics. In Fig. 1 the error distributions for the two minimal solvers are shown. One can see that we get small errors in general. The errors for the similarity solver are slightly larger than for the Euclidean solver. This is to be expected since the elimination template is larger. In order to test our solvers in the presence of noise we ran our solvers in a RANSAC framework. We randomly placed ten receivers and ten senders in an area of size 50×50. We then calculated the distances between the receivers and senders. In order to simulate a local coordinate system, we randomly chose a transformation (Euclidean respectively similarity) and applied this to the receiver positions. Finally,

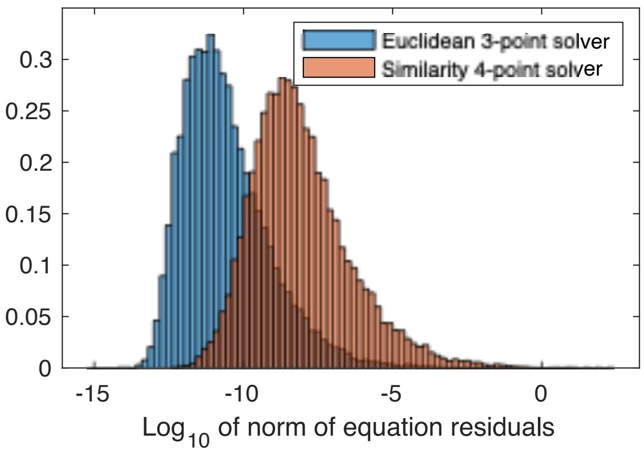

Fig. 1. Distribution of the norm of equations residuals for 10 000 instances of running our minimal solvers. Note that the errors are on a logarithmic scale.

we added Gaussian noise to all positions, with a standard deviation of σ.

The unknown transformation between the local receiver coordinate system and the global sender coordinate system was then estimated using RANSAC and our minimal respective solvers. The estimated transformation was compared to the ground truth transformation in terms of rotation angle difference and relative distance between translations. For the similarity case we also compared the scale. In the left of Fig. 2, the resulting errors are shown as functions of σ for the added noise (on a logarithmic scale). One can see that the errors degrade gracefully. This first test was done without outliers in the data. For a second test we repeated the experiment, but we also corrupted a certain percentage of the data grossly, in order to test the sensitivity to outliers. The right of Fig. 2 shows the errors in this case, with $20\,\%$ outliers. Here we get more or less the same error characteristic as without outliers. Since we only have ten measurements in this case, and we need three or four for an estimate, more than around $30\,\%$ outliers will lead to a large degradation of the results.

B. System Tests on Synthetic Data

In order to test our proposed approach in a more realistic scenario, we simulated an indoor positioning scenario, with a person walking with a step counter, heading estimation and distance measurements to a set of known sender positions. There are of course a large number of parameters that one can set in such a scenario, but we have tried to make a simple test, with typical error characteristics, based on, e.g., sound measurements. We generated 20 random positioned senders in a $50\,\mathrm{m} \times 50\,\mathrm{m}$ area. We then simulated a path with 50 receiver positions. At each position we measure the distance to three randomly chosen senders. We also assume that we have measured the number of steps and heading between each receiver position. We add Gaussian noise to all measurements, and also corrupt a certain ratio of the distance measurements in order to simulate, e.g., NLOS measurements. The used error statistics for the experiment are summarized in

TABLE I
THE ERROR CHARACTERISTICS USED IN THE SYNTHETIC SYSTEM EXPERIMENT

	Standard deviation	Outlier ratio
Step Counter	$0.5\,\mathrm{steps}$	-
Heading measurement	$0.005\,\mathrm{rad}$	-
Distance measurement	$0.1\,\mathrm{m}$	$10\,\%$

TABLE II
COMPARISON TO GROUND TRUTH POSITIONS ON SYNTHETIC SYSTEM

Method	Optimal trilateration [14]	Dead reckoning*	Proposed (Euclidean)	Proposed (similarity)
RMS (m)	16.2	2.00	**0.179**	0.498

*Assumes ground truth initialization of first position.

Table I. In order to test our method, we divided the set of 50 receiver positions into ten segments of five positions in each segment. In each segment we parameterized a local coordinate system using the measured number of steps and the measured heading. For the Euclidean case, we also assume that the step length is known ($0.75\,\mathrm{m}$), but for the similarity case this parameter is estimated. For each segment we have 15 distance measurements, since we have three distance measurements for each receiver position. For each segment we estimate the transformation to the global coordinate system using RANSAC with our minimal solvers. We repeated the experiment 100 times. For a simple comparison, we also report the results using optimal trilateration [14] for the individual receiver positions (based on three measurements, without the use of odometry data) and dead reckoning based on the odometry data without the use of distance measurements. The results are shown in Table II. One example reconstruction is shown in Fig. 3. One can see that the receiver positions are generally reconstructed well. Also shown are the reconstructions using dead reckoning and optimal trilateration. These reconstructions suffer from drift and outliers respectively.

C. System Test on Semi-Synthetic Data

We have done some preliminary tests also based on semi-synthetic data. Using a phone, we have recorded real RTT measurements in an office environment. A person walked around the office while collecting measurements to Wi-Fi hotspots. At a number of positions the user marked the location manually on a map of the office. This corresponds to the selective ground truth for the experiment. In addition, we also know the location of all beacons. Here, we do not have access to any reliable motion estimates, so we have used synthetic motion data (heading and step counter) based on the ground truth positions and assuming a linear motion between the ground truth positions. We added Gaussian noise to the motion data with standard deviations of $0.5\,\mathrm{steps}$ and $0.01\,\mathrm{rad}$ for the step counter and heading, respectively. Using the real distance measurements (with standard deviation on the order of meters) we ran our system and compared the result with the sparse

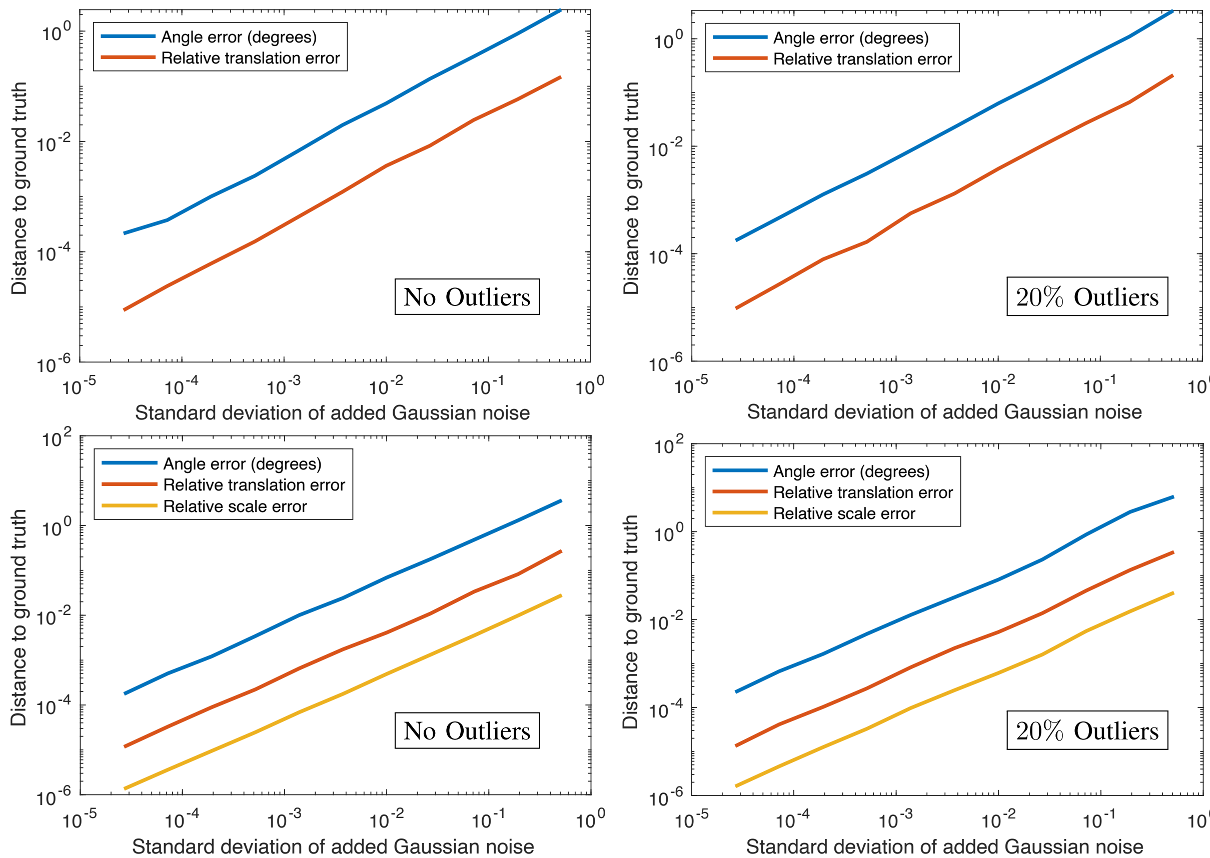

Fig. 2. Results on synthetic data for varying degrees of noise and outliers (on a log-log scale). Top left shows comparison to ground truth after RANSAC using the proposed three-point Euclidean solver, as a function of the added noise. Here there are no outliers in the data. Top right shows the same plot for 20 % outliers in the input data. Bottom row shows the result after RANSAC using the proposed four-point similarity solver. Bottom left is without outliers and bottom right with 20 % outliers. One can see that the errors degrade gracefully for all cases.

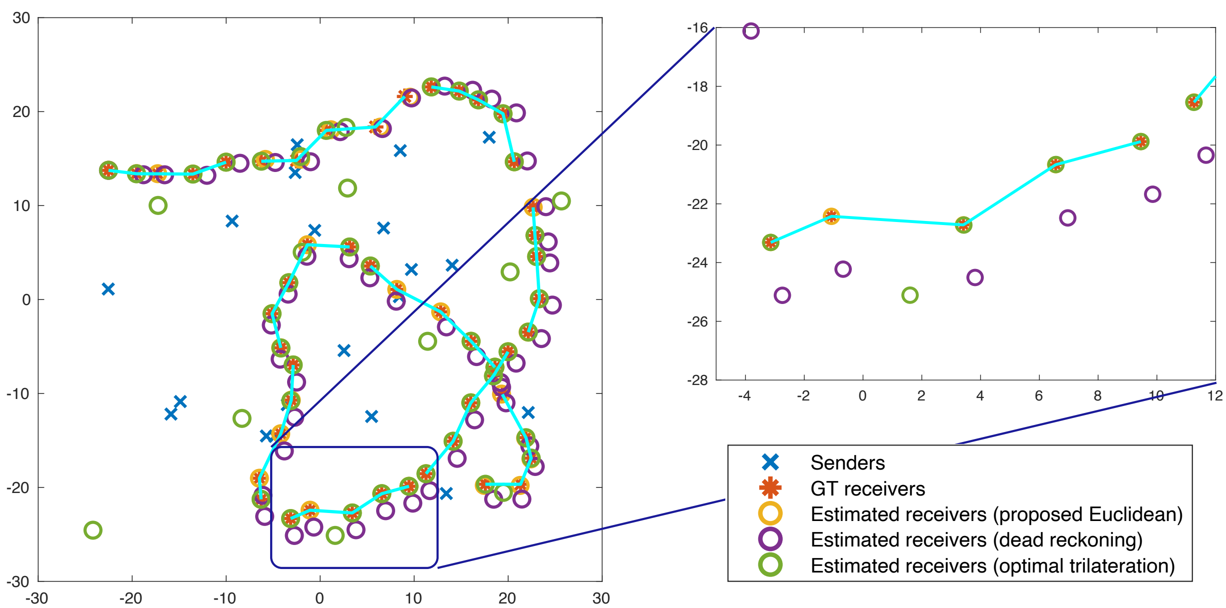

Fig. 3. Reconstruction and comparison with ground truth for the synthetic system experiment. Left shows the complete reconstruction. The proposed solution is very close to the ground truth positions in this case. On the right, a magnification of one segment is shown. The solution based on only odometry data (in purple) suffers from drift. The solution based on optimal trilateration (in green) is very accurate for many receiver positions, but breaks down for cases where there are outliers in the distance measurements.

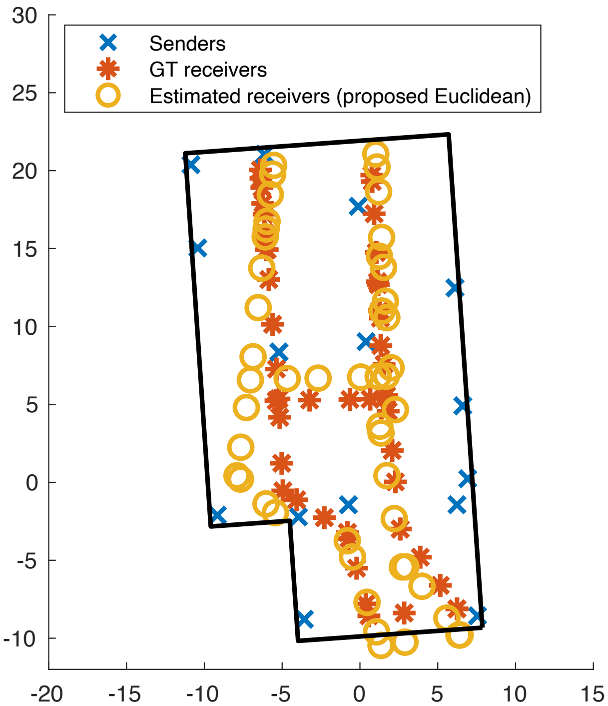

Fig. 4. Reconstruction and comparison with ground truth for the semi-synthetic system experiment. Real Wi-Fi RTT distance measurements were used together with synthetic motion data in the proposed approach.

ground truth. In this case we got an average RMS of $2.6\,\mathrm{m}$ compared to $3.9\,\mathrm{m}$ using optimal trilateration on the individual ground truth receivers. One of the reconstructions is shown in Fig. 4. This experiment should be seen as a proof-of-concept, since the results are based on synthetically generated motion measurements, and also depend on a number of parameter settings in the system. However, it shows promising results of being able to aggregate distance measurements over several receiver positions and eliminate outliers, to get more robust and accurate results.

V. Conclusion

In this paper, we have presented a framework for performing robust trilateration using motion measurements or a constrained motion model. We have shown that a number of different motion priors can be formulated as registration of the local motion coordinate system to a global coordinate system, so that the estimated distance measurements are realized. Depending on the motion model, we use either a similarity transform or a rigid Euclidean transform. We have presented fast minimal solvers for the two-dimensional versions of these problem, that can be used to bootstrap efficient and robust RANSAC-type hypothesize and test methods. We have tested the solvers on synthetic data, and also shown preliminary tests, based on real RTT measurements, with promising results. Future work includes developing the systems, so that they are robust to different scenarios. We would also like to tackle the more challenging three-dimensional case.

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