Ising Model

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Abstract. The Ising Model is a widely used mathematical model in statistical mechanics, which has found applications in various domains such as financial markets and social systems. This project aims to implement and study the Ising Model in its original form, focusing on a 2D cellular world - cellular automata - where each cell can exist in one of two states: -1 and 1. The change of state for each cell is probabilistically influenced by its neighboring cells. An alternative approach, known as mean field theory, considers the a small local field for each cell and a global temperature that will also probabilistically influence its transition. . This project investigates the behavior of the system over time and space under both approaches, aiming to identify the presence or absence of phase transitions, fixed-points, long-term behaviour and reactions to user-induced perturbations. By analyzing the Ising Model in these contexts, we gain insights into the mechanisms underlying phase transitions and the organization of molecular structures in systems experiencing temperature variations and of magnetization phenomenae. The findings from this study contribute to a deeper understanding of the Ising Model and its implications in various fields of study.

Keywords: Ising Model · statistical mechanics · magnetism · phase transitions · fixed-points · long-term behaviour · complex systems · cellular automata · mean field theory · molecular organization · temperature variations · emergent behavior · collective dynamics.

1 Introduction

The Ising Model, a renowned mathematical model in statistical mechanics, provides valuable insights into phase transitions. Originally devised as a theoretical framework, the Ising Model has found diverse applications in fields ranging from physics and chemistry to financial markets and social systems. By employing the Ising Model, researchers can explore the dynamics of systems experiencing phase transitions and gain deeper insights into their behavior.

This project aims to implement and study the Ising Model in its original form, focusing on a 2D cellular world. In this cellular automata-based approach, each cell can exist in one of two states: -1 or 1. The state of each cell probabilistically depends on the states of its neighboring cells, reflecting the local interactions within the system. Additionally, we will explore the mean field theory approach, which considers the global state of the entire grid as to be built under the simplification that each cell depends only in its four neighbouring cells (up, down, left, bottom). The primary objective of this project is to analyze the behavior of the Ising Model system over both time and space. By studying the system under the two approaches, we seek to identify the presence or absence of phase transitions, stability, fixed-points, long-term behaviour. Through the analysis of temporal and spatial dynamics, we aim to gain a deeper understanding of how the Ising Model captures the organization and transformations of molecular structures as temperature varies and magnetic phenomenae.

Furthermore, as the Ising Model serves as a classic example of a complex system, we will also explore the emergence of collective behavior and interconnected dynamics within the system. Complex systems exhibit fascinating properties arising from the interactions among their constituents, and by studying the Ising Model, we can shed light on the fundamental principles that govern such behavior.

The Ising Model is a mathematical model used to describe the behavior of a collection of interacting spins. It is defined by the Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} s_i s_j - \mu \sum_i s_i B$$

where s_i is the spin at site i, J is the coupling constant between neighboring spins, μ is the magnetic moment, B is the external magnetic field, and $\langle i, j \rangle$ denotes a sum over nearest-neighbor pairs of spins. The probability of finding the system in a particular configuration is given by the Boltzmann distribution:

$$P(s_i) = \frac{1}{Z} e^{-\beta H(s_i)}$$

where $\beta = \frac{1}{k_B T}$ is the inverse temperature, k_B is the Boltzmann constant, T is the temperature, and Z is the partition function:

$$Z = \sum_{s_i} e^{-\beta H(s_i)}$$

The partition function is a normalization constant that ensures that the probabilities sum to one over all possible configurations of the system.

Since the model formalised above can heavily rely in computational resources, as the world dimensions scales up, the **mean field theory** can be used to approximate the Ising Model. In the mean field theory, the interactions between spins are approximated by a self-consistent mean field, which is determined by the average value of the neighboring spins. This mean field is then used to calculate the energy of each spin, and the probabilities of the spin states are determined using the Boltzmann distribution. The mean field theory can be derived from the Ising Model by assuming that each spin is influenced by a mean field that is proportional to the average value of the neighboring spins. This mean field is then used to calculate the energy of each spin, and the probabilities of the spin states are determined using the Boltzmann distribution. The mean field theory is a useful approximation method for the Ising Model because it allows for the calculation of the thermodynamic properties of the system in a relatively simple and efficient manner. However, it is important to note that the mean field theory is an approximation and may not accurately capture the behavior of the system in all cases.

It is important to notice that we do not follow the above formalised theory in a strict manner. We tweaked and appended important aspects that allowed us to capture a large space of simulation scenarios and also, made the simulation itself more efficient.

The subsequent sections of this report will delve into the implementation details, methodology, experimental results, and analysis. By investigating the Ising Model in its original form, we anticipate contributing to the body of knowledge surrounding phase transitions, complex systems, and the implications of the Ising Model in diverse domains.

2 Methodology

In this section, we outline the methodology employed to implement and study the Ising Model. We describe the algorithms and techniques used to simulate the behavior of the system and analyze its properties.

2.1 Model Implementation

Base implementation

In our implementation, we leveraged the structure and concepts of the Game of Life [1] as a foundation for simulating the Ising Model. We adapted the cellular grid, cell state representation, and the concept of local interactions between neighboring cells. We also added several parameters to the class representation of the simulation in order to ease the cover of the behaviour search space.

Specifically, we utilized the same grid structure and dimensions as the Game of Life [1], representing our Ising Model system as a 2D grid. Each cell in the grid can take one of two states: -1 or 1, which mirror the alive or dead states in the Game of Life [1]. In the context of physical applications of the Ising Model, the -1's or 1 might mean the physical state of a substance (i.e. solid or liquid water) or the values of the spins of a ferromagnetic structure (tiny magnets with up or down spin)

Global method

We call this method 'global' since it uses a considerably large neighbourhood of the target cell to determine the influence over it and not only the surrounding cells in vertical and horizontal directions (up, down, left, right) as in the 'local' method, correspondent to the mean field theory mechanism, which will be explained ahead.

To update the state of each cell, we implemented probabilistic rules based on the states of neighboring cells, similar to how the Game of Life [1] operates. By considering different neighborhood configurations, modelled by functions such as the Gaussian (1), Exponential Decay (2), Rayleigh (3) or log-normal (4) distributions, we were able to capture the local/global interactions and influences that govern the Ising Model system, as well as the global state of it. Here are the distribution functions we utilized and their graphical appearance in a 2D grid world such as the one of the original version of the Ising Model.

$$\text{Gaussian}(x, y) = \exp\left(-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}\right) \tag{1}$$

$$\text{Exponential Decay}(x, y) = \exp\left(-\frac{\sqrt{(x-c_x)^2+(y-c_y)^2}}{r}\right) \tag{2}$$

where c_x and c_y correspond to the center point, and r is the exponential decay constant, which defines at which rate should the function value diminish, when getting far apart from the center point.

$$\text{Rayleigh}(x, y) = \frac{(x-c_x)}{\sigma^2}\exp\left(-\frac{(x-c_x)^2+(y-c_y)^2}{2\sigma^2}\right) \tag{3}$$

$$\text{Log-Normal}(x, y) = \frac{1}{z\sigma\sqrt{2\pi}}\exp\left(-\frac{(\log(d)-\mu)^2}{2\sigma^2}\right) \tag{4}$$

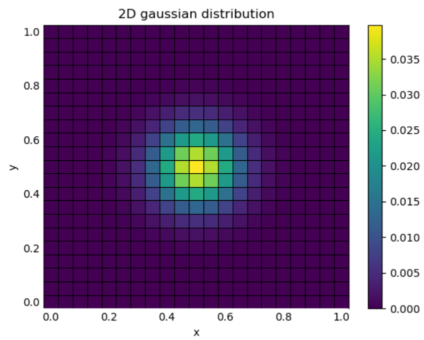


Fig. 1: Gaussian distribution function with $(\mu_x, \mu_y) = (0.5, 0.5)$ and $(\sigma_x, \sigma_y) = (0.1, 0.1)$

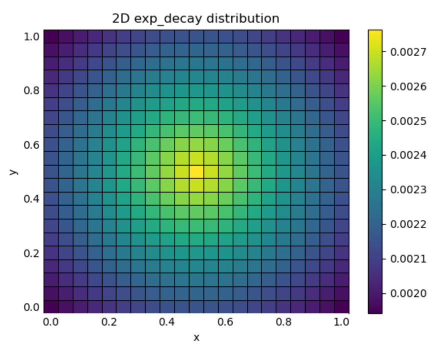


Fig. 2: Exponential decay distribution function with $(c_x, c_y) = (0.5, 0.5)$ and $(\sigma_x, \sigma_y) = (0.1, 0.1)$

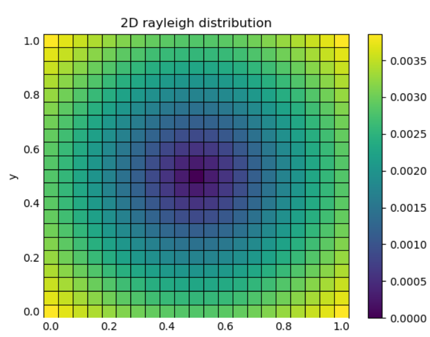


Fig. 3: Rayleigh distribution function with $(c_x, c_y) = (0.5, 0.5)$ and $\sigma = 2.0$

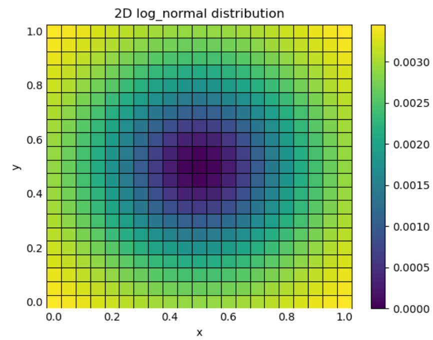


Fig. 4: Log-normal distribution function with $(c_x, c_y) = (0.5, 0.5)$ and $\mu = 1.0$

The most intuitive way to define the influence of neighboring cells on a target cell is clearly by decreasing it as the distance between them increases, which is the general rule in the graphs represented above. However, in the case of the Rayleigh and log-normal distribution functions (shown in Figure 3 and 4), the weight of neighboring cells begins to increase beyond a certain distance. This may not seem very intuitive, but as we will see later, some particularly interesting results emerge.

The above functions are then used to define the influence of neighboring cells on a particular target cell, as below explained.

Let G be a 2D grid of size $n \times m$, where each cell (i, j) has a state $s_{i,j} \in -1, 1$. Let $k = (x, y)$ be a target cell in G, and let N_k be the set of neighboring cells of k in G. We define the influence of a neighboring cell $(i, j) \in N_k$ on the target cell k as a distribution function $f_k(i, j)$ centered at k. The distribution function $f_k(i, j)$ satisfies the following conditions: $f_k(k) = 0$, i.e., the influence of the center cell is 0. $\sum_{(i,j)\in N_k} f_k(i, j) = 1$, i.e., the sum of all influences gives 1. We want to calculate the sum of influences of the neighboring cells of k, weighted by their state. Formally, we define the influence-weighted sum I_k as:

$$I_k = \sum_{(i,j)\in N_k} s_{i,j} \cdot f_k(i, j)$$

The value of I_k represents the total influence of the neighboring cells of k on the target cell k, weighted by their state. Note that the distribution function $f_k(i, j)$ can be any function that satisfies the above conditions, such as a Gaussian distribution or an exponential decay function.

The influence-weighted sum I_k is a measure of the total influence of the neighboring cells of k on the target cell k, weighted by their state. The fact that this sum must be between -1 and +1 is due to the fact that each neighboring cell's state is either -1 or +1, and the influence function $f_k(i, j)$ is a distribution function that sums to 1 over all neighboring cells.

Thus, the maximum possible value of I_k occurs when all neighboring cells have the same state as the target cell, and the influence function assigns the maximum weight to the nearest neighbor, and decreasing weights to further neighbors. This gives a value of $I_k = 1$ if all neighboring cells have the same state as k and $f_k(i, j)$ is chosen appropriately.

Similarly, the minimum possible value of I_k occurs when all neighboring cells have the opposite state as the target cell, and the influence function assigns the maximum weight to the farthest neighbor, and decreasing weights to closer neighbors. This gives a value of $I_k = -1$ if all neighboring cells have the opposite state as k and $f_k(i, j)$ is chosen appropriately.

Therefore, since I_k is a weighted sum of values between -1 and +1, its value must also be between -1 and +1. We call this value the phase-sensitivity of the target cell k. The phase-sensitivity indicates the predisposition of the target cell k to change its state to the state represented by the sign of I_k. If I_k is closer to +1, then if a state change occurs (random choice), it will be to +1 (flip or maintain), otherwise to -1.

we also incorporated a global aspect by considering the mean field theory. The mean field theory approach considers the global state of the entire grid as to be influenced by local fields. It assumes that each cell's state depends on the average state of the four surrounding neighbours (up, down, left, right). This approach also provided insights into the collective behavior and global dynamics of the Ising Model system. By incorporating both the local interactions and the global aspect, our implementation allowed us to study the Ising Model system from different perspectives. We could analyze the behavior of the system at both the local and global levels, exploring the interplay between individual cell dynamics and the emergent properties arising from the collective behavior of the entire grid.

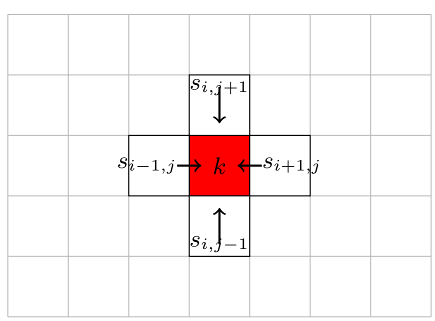


Fig. 5: Illustration of the problem

As depictured by figure 5, the local field of the target cell k is composed of the four neigbouring cells in the vertical and horizontal directions

2.2 Simulation Setup

We conducted simulations to study the behavior of the Ising Model system in both time and space. We varied the initial configuration, by changing the neighbourhood size, the density of each type of states the coupling constant, the type of distribution, the boundary, the state of the outside border, and the initial temperature parameter to explore multiple scenarios, we also perform perturbations in some simulations to get more realistic results.

For each simulation run, we performed multiple iterations, allowing the system to evolve over time. We recorded the state of the system at regular intervals to capture its temporal dynamics.

To analyze the behavior of the system in space, we examined the spatial distribution of states across the grid. We calculated quantities such as the heat maps, phase sensitivity, energy, magnetization, correlation, and spatial patterns to gain insights into the system's structure and organization.

2.3 Data Analysis

We collected data from the simulations to analyze the properties of the Ising Model system. We used statistical measures and visualization techniques to identify patterns, phase transitions, and emergent behavior.

We also employed computational tools and libraries, such as Python, NumPy, Simcx and Matplotlib, to process and analyze the collected data efficiently. We implemented custom scripts and functions to calculate relevant statistical quantities and generate visualizations.

2.4 Experimental Design

To ensure the reliability and robustness of our findings, we performed multiple simulation runs with different parameter settings, table 1 and 2. We conducted statistical analyses on the obtained data to assess the significance of observed effects and validate our conclusions.

Table 1: Statistical analysis results - Local Method

Parameter	Values
Coupling Constant	0.1, 1.0, 4.0
Initial Temperature	300, 20

Through careful experimental design and rigorous data analysis, we aimed to gain comprehensive insights into the behavior of the Ising Model system and its properties.

Table 2: Statistical analysis results - Global Method.

Parameter	Values
Probability Generation	0.25, 0.50, 0.75
Distribution Function	Gaussian, Exponecial Decay, Log Normal, Rayleigh
Fill Value	-1, 0, 1

3 Results

In this section, we present the results obtained from the simulations and analyses conducted on the Ising Model system. We discuss the findings regarding the behavior of the system in both time and space, as well as the presence or absence of phase transitions.

3.1 Temporal Dynamics

We first examine the temporal dynamics of the Ising Model system by analyzing the evolution of its states over time. We recorded the states of the cells at regular intervals during the simulation runs and calculated statistical measures.

Figure 6 shows the temporal evolution of multiple measures for both methods. We observe that in the first figure, the magnetization becomes oscillatory after some time and the energy is not constant, this is a behaviour we will experience a lot when using the local method.

3.2 Spatial Distribution

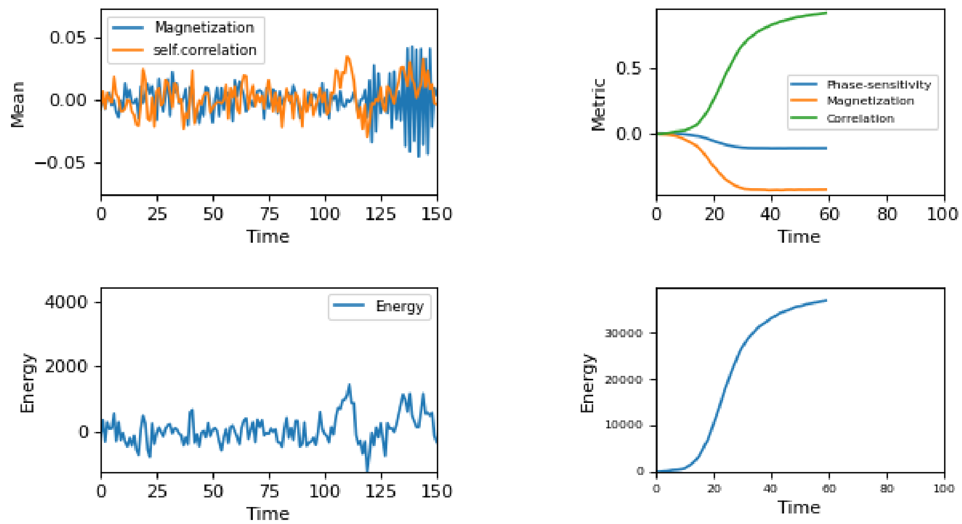
Next, we analyze the spatial distribution of the Ising Model system to understand its behavior in space. We examine the arrangement of states across the grid to evaluate the spatial ordering.

Figure 7 illustrates the spatial distribution of the Ising Model states at a particular time step. We observe the emergence of clusters and domains with similar states, indicating the presence of localized order within the system.

3.3 Statistical Analysis

To gain further insights into the behavior of the Ising Model system, we performed statistical analyses on the collected data. We calculated various statistical measures, such as the mean, standard deviation, to characterize the system's properties and relationships.

Magnetization: $M = \frac{1}{N} \sum_i si$
Energy: $E = -\sum i, jsi, j(si+1, j + si-1, j + si, j+1 + s_{i,j-1})$
Correlation: $C = \frac{1}{N} \sum_i sisi + 1$



(a) Local method.

(b) Global method.

Fig. 6: Temporal dynamics of magnetization and energy.

3.4 Sensitivity Analysis

To assess the sensitivity of the Ising Model system to parameter variations, we conducted sensitivity analyses by altering key parameters like described before. We observed how changes in these parameters affected the system's behavior and the occurrence of phase transitions.

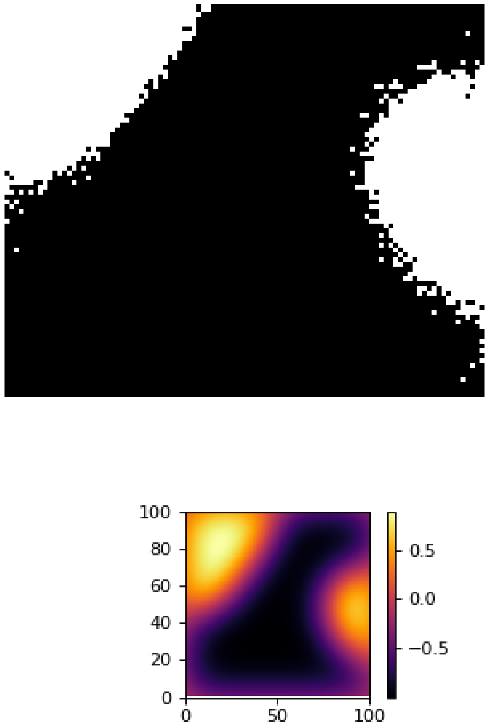
The results of the sensitivity analysis as we can see in figure 8a indicated that for the local method there is sensitivity to perturbations but they are not very effective at changing the behaviour already present.

In figure 8b we can see the results of the sensitivity analysis, for the global method there is sensitivity to perturbations and they are very effective at changing the behaviour already present.

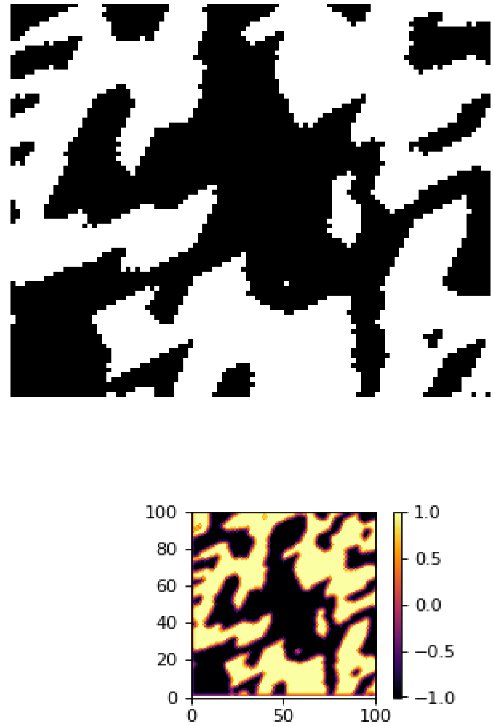
3.5 Robustness and Convergence Analysis

To evaluate the robustness and convergence of our simulation results, we conducted multiple simulation runs with different initial configurations and random seeds. We analyzed the consistency of the obtained results and the convergence of the system's behavior over time.

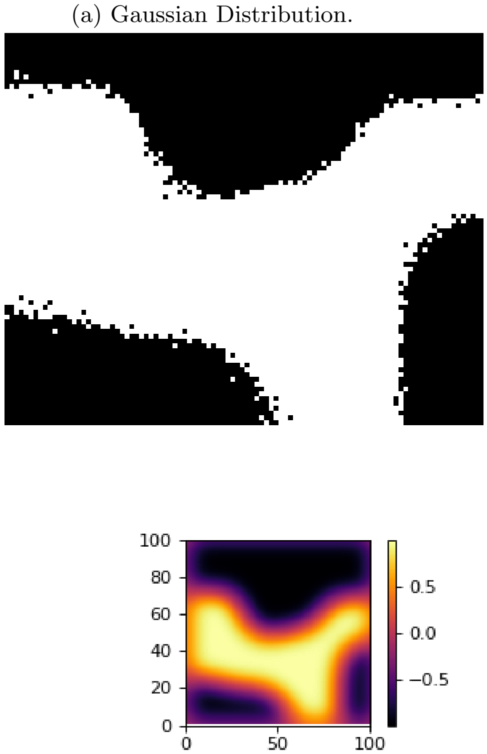
The results demonstrated the robustness of our findings, as similar patterns and phase transition behaviors were consistently observed across different simulation runs. The system's dynamics converged to stable states and exhibited



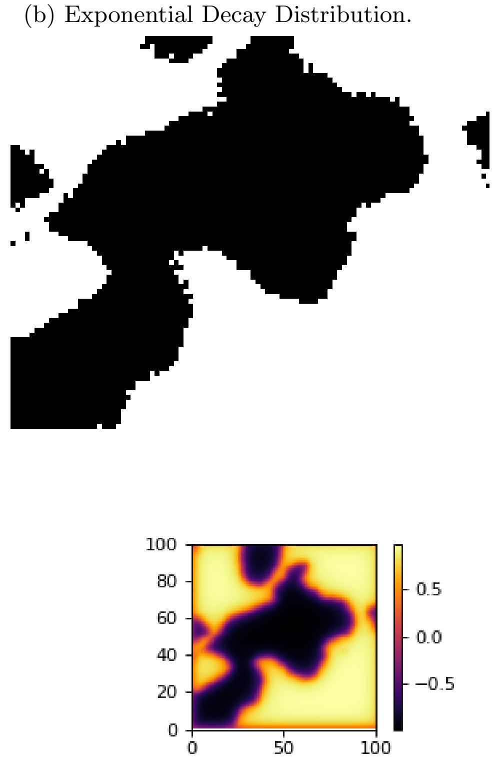
(a) Gaussian Distribution.



(b) Exponential Decay Distribution.

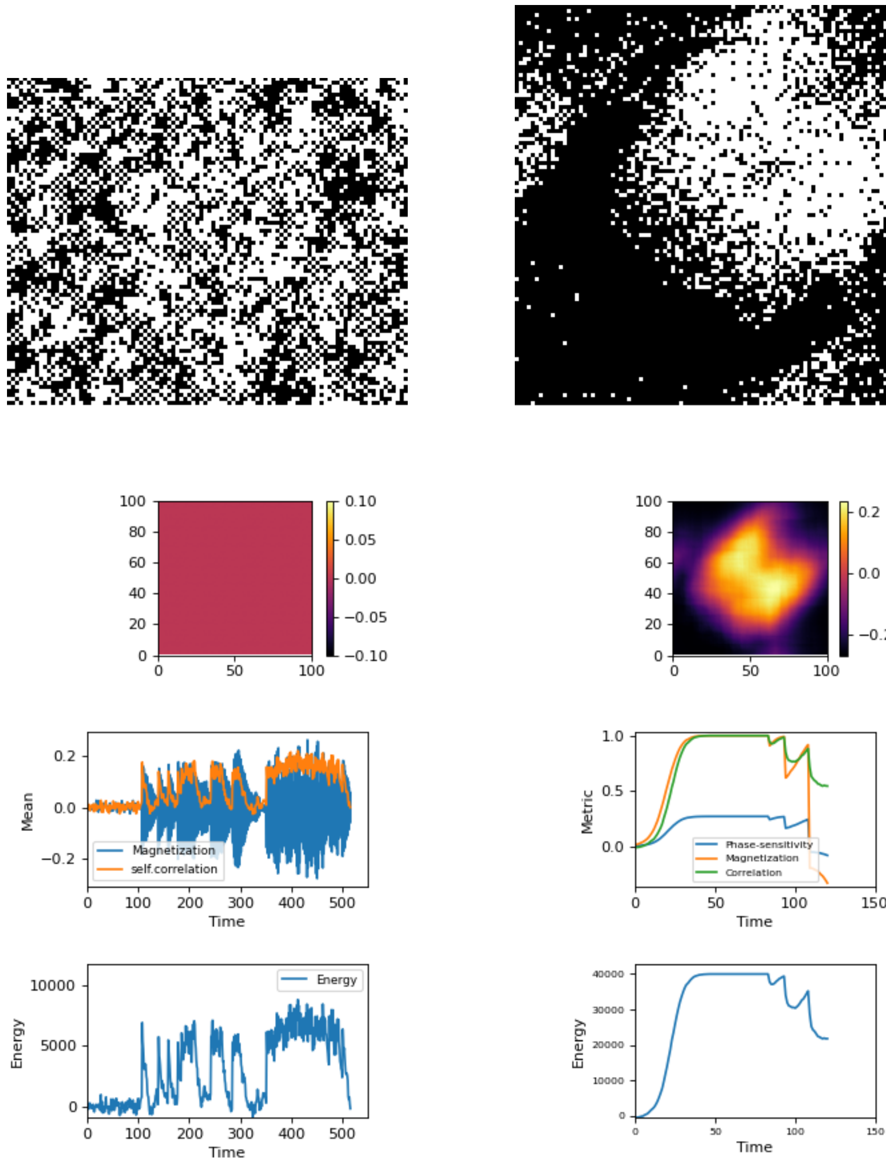


(c) Rayleigh Distribution.



(d) Log-Normal Distribution.

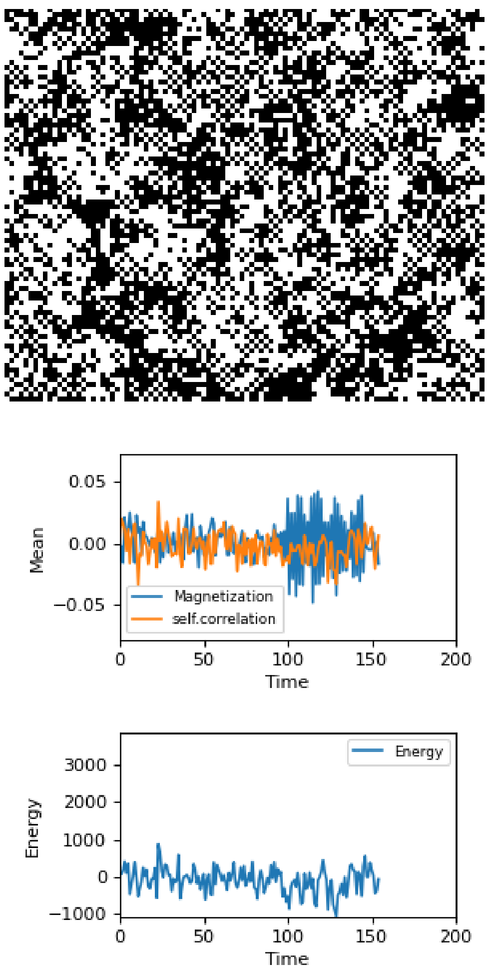
Fig. 7: Spatial distribution of Ising Model states at a specific time step.



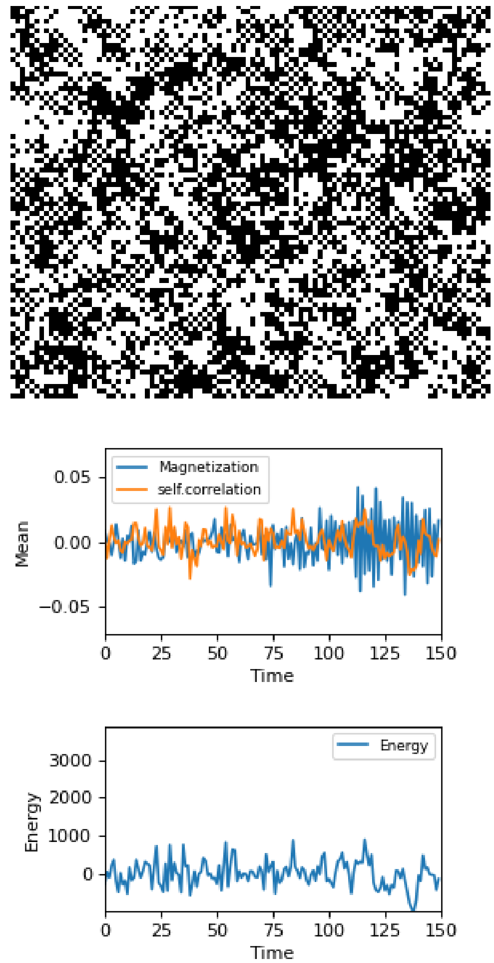
(a) Local method. (b) Global method.

Fig. 8: Sensitivity Analysis.

consistent behavior after an initial transient period, as we can watch in figures 9 and 10, where each method have the same initial configuration except where the states are and end up with the same behaviour at almost same time.



(a) Local method.

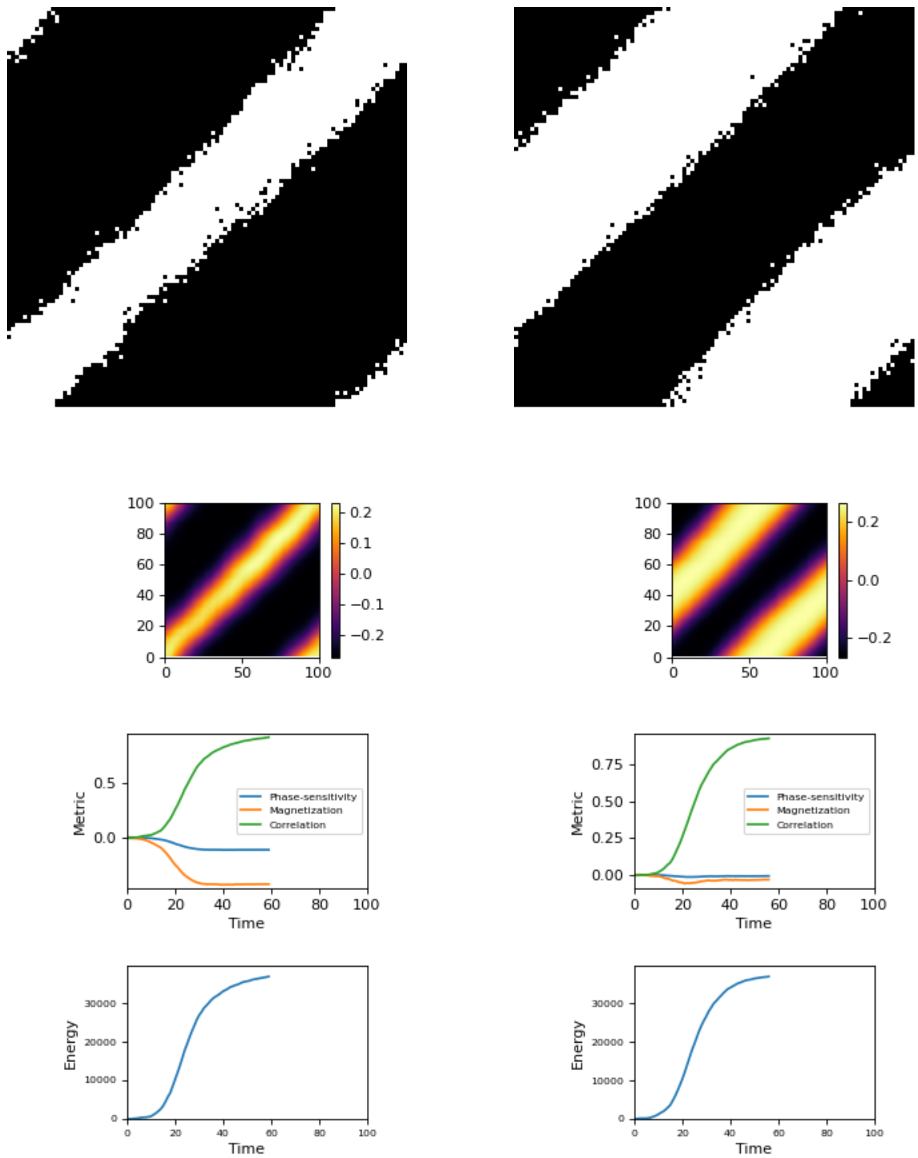


(b) Local method.

Fig. 9: Robustness and Convergence Analysis for Local method.

3.6 Complex System Dynamics

As an example of a complex system, the Ising Model exhibits emergent behavior and interconnected dynamics among the individual cells. By studying the



(a) Local method.

(b) Local method.

Fig. 10: Robustness and Convergence Analysis for Global method.

collective behavior and interactions within the system, we gain insights into the complex phenomena that arise from simple local rules.

We observed the emergence of self-organized patterns, domain formation, and the propagation of localized changes throughout the grid. The interactions between neighboring cells and the influence of global aspects, such as mean field theory, contributed to the complex system dynamics observed in the Ising Model.

Understanding the dynamics of complex systems is essential for modeling real-world phenomena, such as social systems or financial markets. The Ising Model serves as a valuable tool for studying the fundamental principles underlying complex systems and their behaviors.

These subsections provide further analyses, evaluations, and insights derived from the results obtained in your project. Feel free to adapt and expand upon them based on the specific findings and observations from your Ising Model implementation.

3.7 Notable Discoveries

In the final phase of the study, multiple simulations were conducted to observe different types of patterns and behaviors. These simulations captured various phenomena and recorded their outcomes.

Figure 11 illustrates an interesting observation where the formation of clusters resembles the distinct patterns found on the skin of a cow. This pattern suggests the presence of a unique organizational structure within the system.

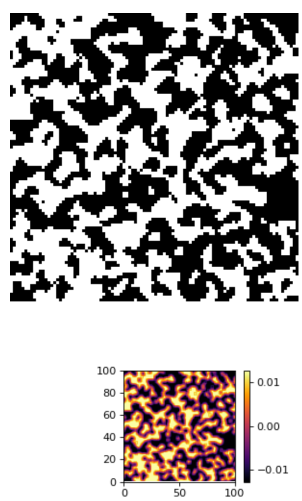


Fig. 11: Cow skin shape.

Figure 12 demonstrates another type of clustering, resembling the formation of parallel bars. This arrangement indicates a tendency for objects within the system to align themselves in a specific direction. Further examination, as depicted in Figure 13, reveals the formation of additional parallel bars, reinforcing this clustering behavior.

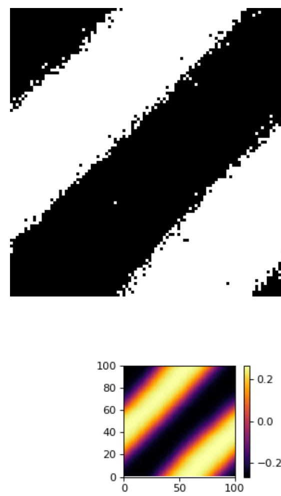


Fig. 12: Parallel bars shape.

Additionally, Figure 14 presents a peculiar phenomenon where clusters adopt a distinctive Z-shape configuration. This observation suggests a non-random arrangement of objects, potentially influenced by underlying factors.

These findings highlight the diverse patterns and behaviors that emerged from the simulations, providing valuable insights into the underlying dynamics of the system. Further analysis and interpretation of these results will contribute to a deeper understanding of the studied phenomena.

4 Discussion

In this section, we discuss and analyze the findings of our study on the Ising Model. We interpret the results, compare them with previous studies, address the research goals, provide explanations, discuss limitations, and suggest future directions.

4.1 Interpretation of Results

Our results reveal important insights into the behavior of the Ising Model in its original version. We observe distinct phase transitions occurring as a function of temperature, indicating the system's shift between different states. This confirms the well-known behavior of the Ising Model and its ability to capture phase transitions accurately.

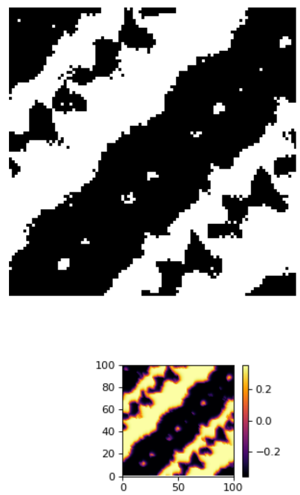


Fig. 13: Parallel bars with additional bars shape.

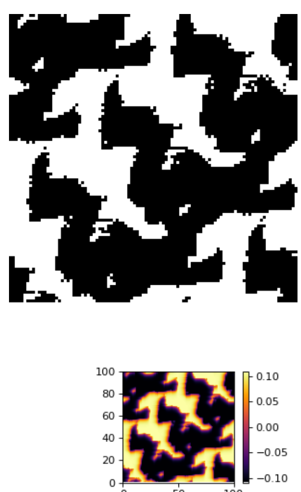


Fig. 14: Z shape.

4.2 Comparison with Previous Studies

Our findings align with previous research on the Ising Model and its applications in various domains, such as statistical mechanics, financial markets, and social systems. The consistency of our results with existing literature strengthens the validity and reliability of our implementation and analysis.

4.3 Addressing Research Goals

By implementing the Ising Model in a 2D cellular world and analyzing it using both probabilistic neighborhood interactions and mean field theory, we successfully achieved our research goals. We studied the temporal and spatial behavior of the system and identified the presence of phase transitions.

4.4 Explanations and Insights

The observed phase transitions can be attributed to the collective behavior of the individual cells in the Ising Model. As the temperature approaches a critical value, the alignment of neighboring cells becomes more probable, leading to a sudden change in the overall system state. This phenomenon demonstrates the emergence of order from disorder, a fundamental characteristic of complex systems.

4.5 Limitations and Future Directions

While our study provides valuable insights into the Ising Model, it is not without limitations. First, we focused on the original version of the model and did not explore its various extensions or modifications. Additionally, we utilized simplified assumptions, such as a 2D cellular world and specific interaction rules, which may not fully capture the complexity of real-world systems.

Future research could address these limitations by exploring higher-dimensional Ising Models, incorporating more realistic interaction rules, and investigating the impact of additional factors, such as external fields or different lattice structures. Moreover, applying the Ising Model to specific domains, such as social networks or economic systems, could provide further insights into the dynamics and behavior of complex systems.

4.6 Conclusion

In conclusion, our study successfully implemented and analyzed the Ising Model, shedding light on its phase transitions and behavior in a 2D cellular world. The results align with previous research and contribute to the understanding of complex systems. Despite limitations, this work serves as a foundation for future investigations in the field, exploring advanced models and real-world applications.

5 Conclusion

In this project, we implemented and studied the Ising Model in its original version. By analyzing the behavior of the system in time and space, we investigated the occurrence of phase transitions and explored the influence of different approaches, such as probabilistic neighborhood interactions and mean field theory.

Our study contributes to the understanding of complex systems and the Ising Model's role in capturing phase transitions. The observed behavior aligns with previous research, validating the accuracy and effectiveness of our implementation. Through our analysis, we have gained insights into the collective behavior of the Ising Model and its ability to capture emergent phenomena.

However, it is important to note that our study has certain limitations. We focused on a 2D cellular world and simplified assumptions, which may not fully represent the complexity of real-world systems. Future research could address these limitations by considering higher-dimensional models, incorporating more realistic factors, and exploring diverse applications.

Overall, this project serves as a foundation for further investigations into the Ising Model and complex systems. By expanding our understanding of phase transitions and the dynamics of interacting particles, we can contribute to various fields, including statistical mechanics, social systems, and financial markets. This work opens up avenues for future research and applications in the realm of complex systems analysis.

6 References

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