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Complex Systems

2022/2023

Exercise Sheet 2

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Continuous Dynamical Systems

2.1 Exercises

Exercise 2.1 **E**

Consider the following differential equation:

$$\frac{df}{dt} = 3$$

.

Let the initial condition be $f(0) = 2$.

(a) Use Euler's method with $\Delta t = 2.0$ to determine an approximate solution to the differential equation.

(b) Use Euler's method with $\Delta t = 1.0$ to determine an approximate solution to the differential equation.

(c) Does the solution depend on Δt? Why or why not?

(d) Determine the formula for the solution $f(t)$ to this differential equation.

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Exercise 2.2 **E**

Consider the following differential equation:

$$\frac{df}{dt} = 2f(t)$$

.

Let the initial condition be $f(0) = 2$.

(a) Use Euler's method with $\Delta t = 2.0$ to determine an approximate solution to the differential equation.

(b) Use Euler's method with $\Delta t = 1.0$ to determine an approximate solution to the differential equation.

(c) What is the long-term behaviour of the solutions to this differential equation? Explain.

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Exercise 2.3 **M**

Consider the following differential equation:

$$\frac{df}{dt} = 2f(t)(1 - f(t))$$

.

This can be thought of as the differential equation version of the logistic equation. The population in this instance varies continuously instead of in discreet steps. For each of the following initial conditions $f(0)$ and time steps Δt, use Euler's method to determine an approximate solution to the differential equation.

(a) $f(0) = 0.5$ and $\Delta t = 1.0$

(b) $f(0) = 0.5$ and $\Delta t = 0.5$

(c) $f(0) = 2.0$ and $\Delta t = 1.0$

(d) $f(0) = 2.0$ and $\Delta t = 0.5$

How would you describe the global behaviour of this equation? Are there any attractors?

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Exercise 2.4 **M**

Using the simcx framework, implement a simulator for continuous-time dynamical systems. Use Euler's method to approximate the solutions for the differential equations. Test the simulator with the equation from exercise 2.3.

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Exercise 2.5 **M**

Simulate the following continuous-time Lotka-Volterra (predator-prey) model using the simulator implemented previously, with $x(0) = y(0) =$

$0.1, a = b = c = d = 1$ and $\Delta t = 0.01$. Visualize the simulation results over time and also in a phase space.

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = -cy + dxy$$

Then try to reduce the value of Δt to even smaller values and see how the simulation results are affected by such changes.

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Exercise 2.6 **M**

There are many other more sophisticated methods for the numerical integration of differential equations, such as the Runge-Kutta methods. Implement a new simulator using fourth order Runge-Kutta. Investigate that method to see how it works in comparison to Euler's.

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Exercise 2.7 **M**

Develop a continuous-time mathematical model of two species competing for the same resource, and simulate its behaviour.

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Exercise 2.8 **E**

Consider a phase plane for the Lotka–Volterra system such as that shown in fig 2.1. Imagine instead of rabbits, we have aphids, which are considered an agricultural pest. And instead of foxes, we have wasps, which eat aphids.

(a) Suppose that in an effort to get rid of aphids, a farmer applies a pesticide that kills both aphids and wasps. How might this be represented on the phase plane? That is, in what direction on the phase plane does one move if the aphids and wasps both decrease?

(b) Suppose that the effect of the pesticide is to kill almost all the aphids and almost all the wasps. Argue that, based on the LV model, the result will be a larger aphid population than there was before the pesticide was applied. (This is sometimes used as an argument against broad-spectrum pesticides.)

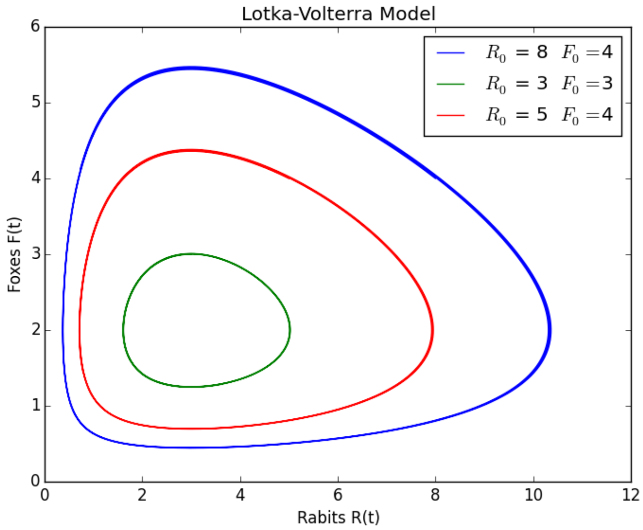

Figure 2.1: Lotka-Volterra

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Exercise 2.9 E

Consider the Lorenz equations with $\sigma = 10.0$, $\rho = 28.0$, and $\beta = 2.667$.

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

These parameter values yield the famous Lorenz attractor. At the center of each lobe there is an unstable fixed point. Verify that the coordinates of these fixed points are $x = 6\sqrt{2}$, $y = 6\sqrt{2}$, $z = 27$, and $x = -6\sqrt{2}$, $y = -6\sqrt{2}$, $z = 27$.

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Exercise 2.10 M

Consider the Lorenz equations from exercise 2.9. Test different values for σ, ρ, and β to find different behaviours.

Further Reading

- ✔ David P. Feldman, *Chaos and Fractals – An Elementary Introduction.*
- ✔ Hiroki Sayama, *Introduction to the Modeling and Analysis of Complex Systems.*