Note 1 (Systems of Linear Equations)

- Linear Function:
$$f(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \dots, \alpha x_n + \beta y_n) \implies \alpha f(x_1, x_2, \dots, x_n) + \beta f(y_1, y_2, \dots, y_n)$$
- If $f : \mathbb{R}^n \to \mathbb{R}$ is linear, then
$$f(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$
- Affine function:
$$g(x_1, x_2, \dots, x_n) = f(x_1, x_2, \dots, x_n) + c_0$$
for a linear function $f : \mathbb{R}^n \to \mathbb{R}$ and constant $c_0 \in \mathbb{R}$.

Note 2 (Vectors/Matrices)

- Vectors $\vec{x} \in \mathbb{R}^n$
  - each x_i = component/element
  - size = # of elements (n)
  - vectors are equal if same size and elements are equal
- Standard unit vector: all elements 0 except one 1
- Vector multiplication:
  - $\vec{y}^T \vec{x}$ = dot product; $\sum x_i y_i$
  - $\vec{x}\vec{y}^T$ = matrix: $\begin{bmatrix} x_1 y_1 & x_1 y_2 & \cdots \\ x_2 y_1 & \ddots & \vdots \\ \vdots & \cdots & x_n y_n \end{bmatrix}$
- Matrix multiplication:
  - $\mathbf{AB}$: for each row of $\mathbf{A}$, multiply and sum for each col of $\mathbf{B}$
  - Associative, not commutative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$; $\mathbf{AB} \neq \mathbf{BA}$
- Identity matrix: $\mathbf{I}$; 1's along diagonal, 0's everywhere else

Note 3 (Linear Independence/Span)

- Set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly dependent (LD) if $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n = \vec{0}$, where some $\alpha_i \neq 0$
- Or, LD if v_i can be written as $\sum \alpha_j v_j$, where some $\alpha_j \neq 0$
- Linearly independent (LI) if $\sum \alpha_i \vec{v}_i = \vec{0}$ only if all $\alpha_i = 0$
- Span: set of all linear combinations of the vectors

Note 5 (Water Pumps)

- Transition matrix (flow from cols to rows):
$$\begin{array}{c|ccc} & A & B & C \\ \hline A & A \to A & B \to A & C \to A \\ B & A \to B & B \to B & C \to B \\ C & A \to C & B \to C & C \to C \end{array}$$
- Columns sum to 1 $\implies$ conservative system (everything goes somewhere)

Note 6 (Matrix Inversion)

- $\mathbf{A}$ is invertible if there exists a $\mathbf{B}$ s.t. $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$
- Finding inverse: Gaussian elimination, like solving $\mathbf{AX} = \mathbf{I}$; i.e. reduce $\left[\mathbf{A} \mid \mathbf{I}\right] \to \left[\mathbf{I} \mid \mathbf{A}^{-1}\right]$
- If invertible, then: (baby version of IMT)
  - rows and cols LI
  - $\mathbf{A}\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$
  - has a trivial nullspace
  - $\det \mathbf{A} \neq 0$

Note 7 (Vector Spaces)

- V is a vector space if:
  - $\vec{0} \in V$
  - $\vec{x}, \vec{y} \in V$ implies $\alpha\vec{x} + \beta\vec{y} \in V$ for all $\alpha, \beta \in \mathbb{R}$; i.e.
    * closed under vector addition (if $\vec{x}, \vec{y} \in V$ then $\vec{x} + \vec{y} \in V$)
    * closed under scalar multiplication (if $\vec{x} \in V$ then $\alpha\vec{x} \in V$ for all $\alpha \in \mathbb{R}$)
  - (among other things, but these are the most important)
- Basis of a vector space:
  - LI vectors, can express any $\vec{v} \in V$ as a linear combination of basis vectors, and is a minimum set of vectors that does so (implied by being LI)
- Dimension of a vector space = # of basis vectors
- All equivalent bases of a vector space must have the same dimension

Note 8 (Matrix subspaces)

- U is a subspace if it satisfies the 3 points in Note 7 (a subspace is a subset of a vector space)
- Column space: $\text{range}(\mathbf{A}) = \text{span}(\mathbf{A}) = \text{C}(\mathbf{A})$ = span of cols of $\mathbf{A}$
- $\text{rank}(\mathbf{A}) = \dim(\text{span}(\mathbf{A}))$
- Nullspace: $\text{Null}(\mathbf{A}) = \text{N}(\mathbf{A})$ = set of all $\vec{x}$ s.t. $\mathbf{A}\vec{x} = \vec{0}$
- $\text{nullity}(\mathbf{A}) = \dim(\text{N}(\mathbf{A}))$
- rank–nullity theorem: $\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A})$ = # of columns of $\mathbf{A}$

Note 9 (Eigenvalues/Eigenvectors)

- If $\mathbf{A}\vec{x} = \lambda\vec{x}$, then $\vec{x}$ is an eigenvector, λ is an eigenvalue of $\mathbf{A}$
- Calculating eigenvalues: solve $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ for λ
- Calculating eigenvectors: solve $(\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$ for $\vec{v}$
- Repeated eigenvalues: multiple eigenvectors, same eigenvalue; forms an eigenspace
- If $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$ are two distinct eigenpairs, then $\vec{v}_1, \vec{v}_2$ are LI.
- Characteristic polynomial: $\det(\mathbf{A} - \lambda I)$
- Steady states (water pumps): $\vec{x}$ s.t. $\mathbf{A}\vec{x} = \vec{x}$ (i.e. eigenspace for $\lambda = 1$)
- $\lim\limits_{n\to\infty} \mathbf{A}^n \vec{x} = \lim\limits_{n\to\infty} \lambda^n \vec{x}$ if $(\lambda, \vec{x})$ is an eigenpair of $\mathbf{A}$

Note 10 (Change of Basis/Diagonalization)

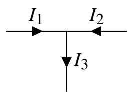
- If $\mathbf{T}\vec{u} = \vec{v}$, and $\vec{u}_A$ and $\vec{v}_B$ are vectors in the $\mathbf{A}$ and $\mathbf{B}$ bases respectively (i.e. columns of $\mathbf{A}$ and $\mathbf{B}$ are basis vectors in the new coordinate system), then arrows represent consecutive left-multiplication:
$$\begin{tikzcd} \vec{u} \arrow[r, "\mathbf{T}"] \arrow[d, bend right, "\mathbf{A}^{-1}"'] & \vec{v} \arrow[d, bend left, "\mathbf{B}^{-1}"] \\ \vec{u}_A \arrow[r, "\mathbf{B}^{-1}\mathbf{TA}"'] \arrow[u, bend right, "\mathbf{A}"'] & \vec{v}_B \arrow[u, bend left, "\mathbf{B}"] \end{tikzcd}$$
- Diagonalization: $\mathbf{A} = \mathbf{PDP}^{-1}$; $\mathbf{P}$ = matrix of eigenvectors, $\mathbf{D}$ = diagonal matrix of eigenvalues such that $\mathbf{A}\vec{v}_i = \lambda_i \vec{v}_i$
$$\mathbf{P} = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix} \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$
- Only diagonalizable if eigenvalues are linearly independent (i.e. if all eigenvalues are distinct)

Other

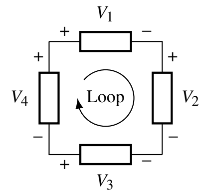
- $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$
- rotation matrix by θ counterclockwise: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$
- $\det\left(\mathbf{A}^{-1}\right) = \det(\mathbf{A})^{-1}$
- $\det(\mathbf{A})$ = product along diagonal if $\mathbf{A}$ is triangular
  - eigenvalues of a triangular matrix are the values along its diagonal

Note 11 (Circuits)

- Ohm's law: $V = IR$
- KCL: $I_{in} = I_{out}$
- KVL: $\sum_{\text{loop}} V_k = 0$; $- \to + =$ add, $+ \to - =$ subtract

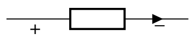


$I_1 + I_2 = I_3$



$-V_1 - V_2 + V_3 + V_4 = 0$

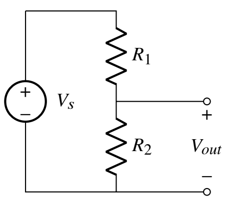
- NVA:
  - Label everything
  - Passive sign convention: current goes into +, out of −



  - Write KCL at each unknown node
  - Substitute Ohm's law for each current
  - Solve for desired values

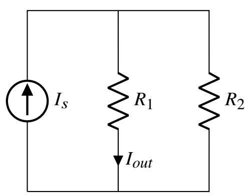
Note 12 (Resistive Touchscreen)

- Voltage divider:



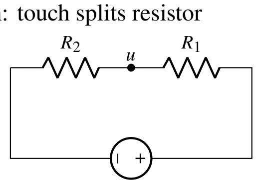
$V_{out} = V_s \frac{R_2}{R_1+R_2}$

- Current divider:



$I_{out} = I_s \frac{R_2}{R_1+R_2}$

- $R = \rho \frac{L}{A} = \rho \frac{\text{length}}{\text{area}}$, where ρ = resistivity
- Resistive touchscreen: touch splits resistor



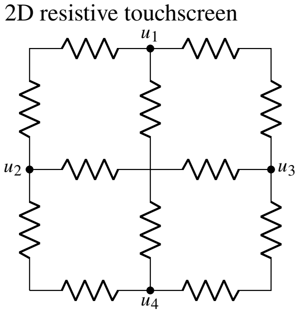
$$R_2 = \rho \frac{L_{touch}}{A} \qquad R_1 = \rho \frac{L - L_{touch}}{A} \qquad u = V_s \frac{L_{touch}}{L}$$

Note 13 (Power)

- Power: $P = VI = \frac{V^2}{R} = I^2 R$
- Voltmeter: connected *in parallel* to element to measure voltage drop
- Ammeter: embedded in the circuit *in series* to measure current

Note 14 (2D Resistive Touchscreen)

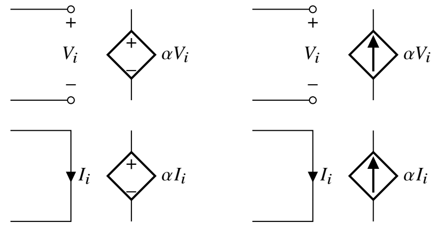
- 2D resistive touchscreen



- Powering $u_1 \to u_4$: measure y position
$$V_{out} = V_s \frac{L_{touch,vertical}}{L}$$
- Powering $u_2 \to u_3$: measure x position
$$V_{out} = V_s \frac{L_{touch,horizontal}}{L}$$

Note 15 (Superposition, Equivalences)

- Dependent sources:



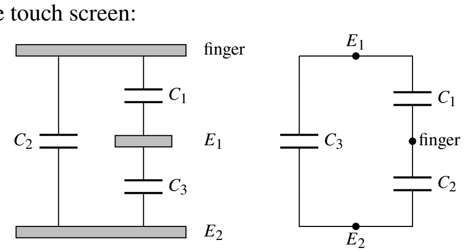
- Superposition:
  - for each independent source:
    * replace voltage source with wire, current source with open circuit
    * leave everything alone, find value (keep the same signs!)
  - sum up everything
- Resistor equivalences:
  - Parallel: $R_{eq} = \frac{R_1 R_2}{R_1+R_2}$
  - Series: $R_{eq} = R_1 + R_2$
- Voltage drop is equal through parallel branches (adjacent to same nodes)
- Current is equal through elements in series (by KCL)

Note 16 (Capacitors)

- Capacitors:
  - charge (on positive plate) $= Q = CV$; C = capacitance
  - $I = C \frac{\mathrm{d}V}{\mathrm{d}t}$
  - if constant current, then $I = C \frac{\Delta V}{\Delta t}$ and $It = C(V(t) - V(0))$
- Capacitor equivalences:
  - Parallel: $C_{eq} = C_1 + C_2$
  - Series: $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$
- $C = \varepsilon \frac{A}{d} = \varepsilon \frac{\text{area}}{\text{distance}}$, where ε = permitivity
- Energy: $E = \frac{1}{2} C V^2$

Note 17 (Capacitive Touchscreen)

- Capacitive touch screen:



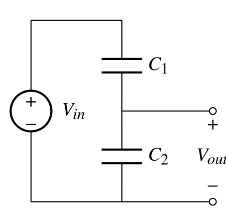
- Touch adds parallel capacitors (C_1, C_2) $\implies$ increased capacitance

Note 17B (Charge Sharing)

- Charge sharing steps:
  - Draw/label phases, keep polarity/signs for elements consistent through phases
  - For all floating nodes in phase 2, use charge conservation; find total charge on adjacent plates (keep + and − plates in mind!)
  - Equate with the total charge on the *same* plates in phase 1

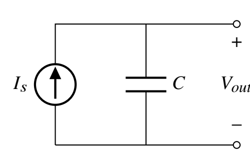
Other

- Capacitive divider:



$V_{out} = V_{in} \frac{C_1}{C_1+C_2}$

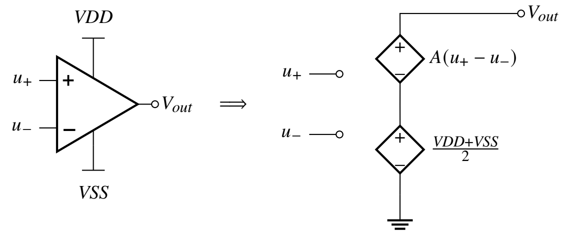
- Charging a capacitor:



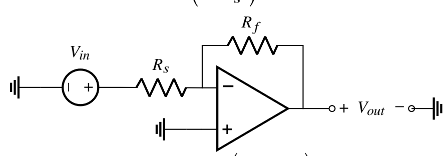
$V_{out} = \frac{I_s}{C} t$

Note 18/19 (Op Amps)

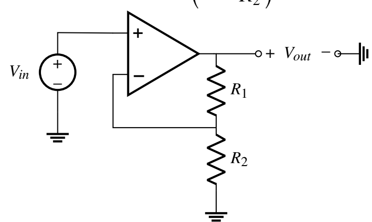
- Op amp:



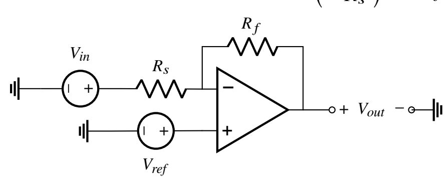
- Ideal op amp:
  - $A \to \infty$
  - No current through u_+, u_-
  - $u_+ - u_- = 0$
- Inverting Amplifier: $V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right)$



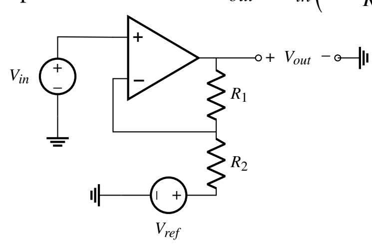
- Noninverting Amplifier: $V_{out} = V_{in}\left(1 + \frac{R_1}{R_2}\right)$



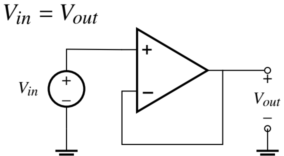
- Inverting Amplifier w/ reference: $V_{out} = V_{in}\left(-\frac{R_f}{R_s}\right) + V_{ref}\left(1 + \frac{R_f}{R_s}\right)$



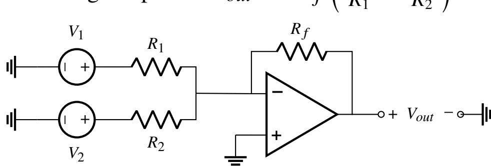
- Noninverting Amplifier w/ reference: $V_{out} = V_{in}\left(1 + \frac{R_1}{R_2}\right) - V_{ref}\left(\frac{R_1}{R_2}\right)$



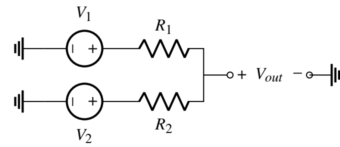
- Unity Gain Buffer: $V_{in} = V_{out}$



- Inverting Summing Amplifier: $V_{out} = -R_f\left(\frac{V_{in1}}{R_1} + \frac{V_{in2}}{R_2}\right)$



- Voltage Summer: $V_{out} = V_1\left(\frac{R_2}{R_1+R_2}\right) + V_2\left(\frac{R_1}{R_1+R_2}\right)$



Note 21 (Inner Products)

- (Euclidean) Inner product: $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y} = \vec{x}^T \vec{y} = \sum x_i y_i$
- $\vec{x}, \vec{y}$ are orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$
- $\langle a\vec{x}, \vec{y} \rangle = a \langle \vec{x}, \vec{y} \rangle$ and $\langle \vec{x} + \vec{y}, \vec{z} \rangle = \langle \vec{x}, \vec{z} \rangle + \langle \vec{y}, \vec{z} \rangle$
- Norm: $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ = length/magnitude of vector
- Alternate definition: $\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \|\vec{y}\| \cos\theta$
- Cauchy-Schwarz inequality: $|\langle \vec{x}, \vec{y} \rangle| \le \|\vec{x}\| \|\vec{y}\|$

Note 22 (Correlation)

- Cross-correlation: $\text{corr}_{\vec{x}}(\vec{y})[k] = \sum_{i=-\infty}^{\infty} x[i]y[i-k]$
- $\vec{x}[i], \vec{y}[i] = 0$ outside of defined range
- $\text{corr}_{\vec{x}}(\vec{y})[k] = \text{corr}_{\vec{y}}(\vec{x})[-k]$; they're mirrored
- Autocorrelation: $\text{corr}_{\vec{x}}(\vec{x})$
- Circular correlation:

$$\text{circcorr}(\vec{x}, \vec{y}) = \begin{bmatrix} \text{——} & \text{rows are all} & \text{——} \\ \text{——} & \text{circular shifts} & \text{——} \\ \text{——} & \text{of } \vec{y} & \text{——} \end{bmatrix} \vec{x}$$

Note 23 (Projection/Least Squares)

- Projection of $\vec{b}$ onto $\vec{a}$: $\text{proj}_{\vec{a}}(\vec{b}) = \frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle} \vec{a}$
- Scalar projection of $\vec{b}$ onto $\vec{a}$: $\frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle}$
- Projection onto subspace: if columns of $\mathbf{A}$ are orthogonal, then $\text{proj}_{\mathbf{A}}(\vec{b}) = \sum \text{proj}_{\vec{a}_i}(\vec{b})$ where $\vec{a}_i$ are columns of $\mathbf{A}$; if not, use least squares
- Least squares: to minimize the error $e = \left\|\mathbf{A}\vec{x} - \vec{b}\right\|$, we have $\hat{\vec{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\vec{b}$
- Setting up least squares:
  - $\mathbf{A}$ = matrix of known values/coefficients
  - $\vec{x}$ = vector of variables
  - $\vec{b}$ = vector of constants
- $\mathbf{A}^T\mathbf{A}$ is invertible if $\mathbf{A}$ has LI columns (i.e. can only apply least squares if $\mathbf{A}$ has LI columns)

Other

- Trilateration:
  - n variables $\Longrightarrow$ n equations if linear, $n+1$ equations if nonlinear (subtract from one equation to linearize)
  - in space: n dimensions $\Longrightarrow$ $n+1$ equations for circles/spheres; one is sacrificed to linearize
  - if delays are unknown, need $n+2$ equations; sacrifice one for reference, sacrifice another to linearize
- Units (good to double check calculations)
  - Current: A = C/s = charge/time
  - Voltage: V = J/C = energy/charge
  - Resistance: Ω = V/A
  - Power: W = J/s = energy/time
  - Capacitance: F = C/V = charge/volt