Random Variable

Experiment $\longrightarrow$ outcome $\longrightarrow$ Numerical / non-numerical

Idea: Given an experiment & the possible outcomes, a random variable associates a particular no with each outcome.

Random Variable is a real valued function of the experimental outcome, i.e. $X : \Omega \to \mathbb{R}$

X – random variable
Ω – Sample space
$\mathbb{R}$ – Set of real numbers

Ex: Experiment $\to$ rolling two 6-sided die
outcomes $\to$ $(1,1)$ .... $(6,6)$

the following random variables can be defined
(i) Sum of the two rolls
(ii) Number of 6 in the two rolls and so on,

Suppose X = Sum of the two rolls
$\Rightarrow X \in \{2, 3, \dots 12\}$

Suppose Y = no. of 6 in the two rolls
$Y \in \{0, 1, 2\}$

$X : \Omega \longrightarrow R$

*Definition: A random variable is called discrete if its range is either finite or countably infinite
$\mathbb{N} = \{1, 2 \dots\} \to$ countably infinite ✓
$\mathbb{R} = \{0, \dots\} \to$ uncountably infinite ⊗

*A random variable that can take on uncountably infinite number of values is Not discrete.

Ex: choosing a point 'a' from the interval $[-1, 1]$. Random Variable is giving a^2 to the outcome 'a'.

Concepts Related to Discrete random variables:

Probability Mass function (PMF)

If x is any possible value of random variable X, the probability mass of x is the probability of event $\{X = x\}$ i.e.

$$\boxed{\text{probability mass of } x = P_X(x) = P\{X = x\}}$$

Ex: Tossing a fair coin twice
random variable X: no. of heads obtained
$\Omega = \{HH, HT, TH, TT\}$
$X = 0, 1, 2$

PMF of random variable X:

$$P_X(x) = \begin{cases} 1/4 & x = 0 \\ 1/2 & x = 1 \\ 1/4 & x = 2 \end{cases}$$

Prove the following identity:

(a) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!}$$

$$\frac{n!}{k!(n-k-1)!}\left[\frac{1}{n-k} + \frac{1}{k+1}\right] = \frac{n!}{k!(n-k-1)!}\left[\frac{k+1+n-k}{(n-k)(k+1)}\right]$$

$$= \frac{n!(n+1)}{k!(n-k)!(k+1)} = \frac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}$$

(b) $\sum_{k=0}^{n} \binom{n}{k}(-1)^k = 0$

Consider the binomial expansion of $(1+x)^n$

$$(1+x)^n = {}^nC_0 + x\,{}^nC_1 + x^2\,{}^nC_2 + \dots x^n\,{}^nC_n$$

for $x = -1$

$$(1-1)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k = 0 = \underline{RHS}$$

(c) $\sum_{k=1}^{n}\binom{n}{k} k = n\,2^{n-1}$

Consider the Binomial expansion:

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k} p^k q^{n-k}$$

Differentiating w.r.t. p:

$$\frac{d}{dp}\left[(p+q)^n\right] = \sum_{k=1}^{n} k\binom{n}{k} p^{k-1} q^{n-k}$$

$$\Rightarrow n(p+q)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} p^{k-1} q^{n-k}$$

$p = 1, q = 1$

$$n\,2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k} \qquad \underline{H.P.}$$

(d) $\sum_{k=0}^{n}\binom{n}{k} k(-1)^k = 0$

Consider the expansion:

$$(p+q)^n = \sum_{k=0}^{n}\binom{n}{k} p^k q^{n-k}$$

diff^n w.r.t. p:

$$n(p+q)^{n-1} = \sum_{k=1}^{n}\binom{n}{k} k\,p^{k-1} q^{n-k}$$

$$\Rightarrow p\,n(p+q)^{n-1} = \sum_{k=1}^{n}\binom{n}{k} k\,p^{k} q^{n-k}$$

$p = -1, q = 1$

$$\Rightarrow \sum_{k=1}^{n}\binom{n}{k} k(-1)^k = 0 \qquad \underline{H.P.}$$

Q) Cards are drawn from a standard 52-card deck until the third club is drawn. After each card is drawn, it is put back in the deck & the cards are reshuffled so that each card drawn is independent of all others

① Find the probability that the 3rd club is drawn on the 8th selection

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

probability of 2 clubs in 7 selections:

$$P_7(2) = \binom{7}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^5$$

probability of 2 clubs in 7 selections and 3rd club in the 8th selection:

$$p(X) = \binom{7}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)$$

$$= \binom{7}{2}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^5 \approx \underline{0.0779}$$

(b) Find the probability that atleast 8 cards are drawn before the 3rd club appears

P(atleast 8 cards are drawn before 3rd club)
$= P$(we draw either 0, 1 or 2 clubs in 8 trials)

$$= \binom{8}{0}\left(\tfrac{1}{4}\right)^0\left(1-\tfrac{1}{4}\right)^8 + \binom{8}{1}\left(\tfrac{1}{4}\right)^1\left(\tfrac{3}{4}\right)^7 + \binom{8}{2}\left(\tfrac{1}{4}\right)^2\left(\tfrac{3}{4}\right)^6 = \underline{0.7119}$$

Bernoulli Random Variable :-

X is said to be Bernoulli random variable

$$X = \begin{cases} 1 & \text{if success} \\ 0 & \text{if fail} \end{cases}$$

Suppose probability of success is p

Its PMF is:

$$P_X(k) = \begin{cases} p & \text{if } k = 1 \\ 1-p & \text{if } k = 0 \end{cases}$$

Let X be the no. of successes in n-trials. X is called Binomial random variable with parameters n & p & the PMF of X is:

$$p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The normalization property gives:

$$\sum_{k=0}^{n}\binom{n}{k} p^k (1-p)^{n-k} = 1$$

Plot of PMF of Binomial random Variable