$$x_1 + 2x_2 + 2x_3 + 2x_4 = b_1$$
$$2x_1 + 4x_2 + 6x_3 + 8x_4 = b_2$$
$$3x_1 + 6x_2 + 8x_3 + 10x_4 = b_3$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right) \leftarrow \text{Augmented matrix} = [A \; b]$$

$$\downarrow \; R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 3R_1$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right) \longrightarrow R_3 \to R_3 - R_2$$

Pivots

pivot columns

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right) \Rightarrow \boxed{b_3 - b_2 - b_1 = 0}$$

Let us say $b = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}$

OK !

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & 2 & 1 \\ 0 & 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Solvability condition on b:

* Ax = b is solvable when b is in C(A)

* If a combination of rows of A gives zero rows the same combination of the entries of b must give 0.

To find complete sol^n to $Ax = b$:

1. $X_{particular}$: → Set all free variables to 0.
   → Solve $Ax = b$ for pivot variables

$x_2, x_4 = 0$

$$x_1 + 2x_3 = 1 \quad \Rightarrow \quad x_1 = -2$$

$$2x_3 = 3 \qquad\qquad x_3 = 3/2$$

$$x_p = \begin{pmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{pmatrix}$$

② $X_{nullspace}$:

$Ax_p = b$

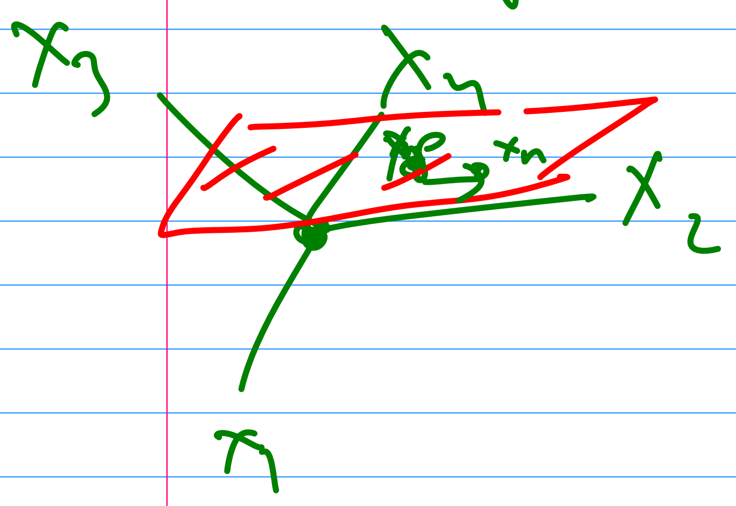
$Ax_n = 0$

* $\boxed{X_{complete} = X_p + X_n}$ $\Rightarrow A(X_p + X_n) = b$

Here, $X_{complete} = \begin{pmatrix} -2 \\ 0 \\ 3/2 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

obtained in last lecture.

Plot all solutions of X in $\mathbb{R}^4$



* m by n matrix A of rank r:
(know $r \le m$, $r \le n$)

* Full column rank means $r = n$:
  ↳ no free variables
  $N(A) = \{\text{zero vector}\}$: $\boxed{x = x_p}$
  ↳ Unique soln if it exists; → (0 or 1 solution)

Eg: $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{pmatrix}$

$$\boxed{R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}$$ rref of A

Full row rank means $r = m$

Can solve $Ax = b$ for every b.

Exists

Left with $n - r$ free variables:

$\hookrightarrow n - m$

Eg. $\begin{pmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{pmatrix} \to$ rank $= 2$

$R = \begin{pmatrix} 1 & 0 & - & - \\ 0 & 1 & - & - \end{pmatrix}$ F

$r = m = n \to$ full rank

$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \to$ invertible

$R = I$

rref

$N(A) = 0$

zero vector

$Ax = b$

$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \to$ no condition on b_1 & b_2

$\to$ X exists for every b

$r = m = n$

$R = I$

1 solution

$r = n < m$

$R = \begin{bmatrix} I \\ 0 \end{bmatrix}$

(0 or 1 solution)

$r = m < n$

$R = \begin{bmatrix} I & F \end{bmatrix}$

(∞ solutions)

$r < m, \ r < n$

$R = \begin{pmatrix} I & F \\ 0 & 0 \end{pmatrix}$ (0 or ∞ solutions)