$$A = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{pmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{pmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} = U$$

pivots

→ Echelon form (staircase form)

Rank of A

↳ no. of pivots $= 2$

Choice 1:

$$x = \begin{pmatrix} x_1 \\ 1 \\ x_3 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0$$

$x_2 = 1$

$x_4 = 0$,

$$2x_3 + 4x_4 = 0$$

$$x_1 + 2 + 2x_4 = 0$$

$$x_1 + 2x_4 + 2 = 0$$

$x = c\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ → vector in nullspace

↳ describes a line

choice: $\begin{pmatrix} x_1 \\ 0 \\ x_3 \\ 1 \end{pmatrix}$

$x_1 - 4 + 2 = 0$

$x_1 = 2$

$\Rightarrow x_3 = -2$

$x_4 = 1$

$x = d\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$

$$x = c\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + d\begin{pmatrix} 2 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

no. of free variables: $n - r$

n = no. of variables

r = rank.

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

R = reduced row echelon Matrix

↳ zeroes above & below the pivots

$R_1 \to R_1 - R_2$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$R_2 \to \frac{1}{2} R_2$

$$\begin{pmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

free columns

rref(A) → in matlab

rref form: $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$

pivot columns

$N = \begin{pmatrix} -F \\ I \end{pmatrix}$ ← null space matrix

$Rx = 0 \qquad (I \; F) \begin{pmatrix} x_{pivot} \\ x_{free} \end{pmatrix} = 0$

$x_{pivot} = -F x_{pivot}$

$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \\ 1 & 3 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 4 & 4 \end{pmatrix}$

1 pivot

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

free: 3 − 2 = 1 variables

$r = 2$

$x = c \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$x = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$R = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$c \begin{pmatrix} -F \\ I \end{pmatrix} = x$$