Geometric Random Variable:

First success –

It is the number X of trials to get the first success. Its PMF is given by $X = 1, 2, 3 \ldots$

Suppose probability of success is p

$$p(X=1) = p$$
$$p(X=2) = (1-p)\,p$$
$$p(X=k) = \boxed{P_X(k) = (1-p)^{k-1} \cdot p\,, \quad k = 1, 2, 3 \ldots}$$

Prove that:

$$\sum_{k=1}^{\infty} P_X(k) = 1$$

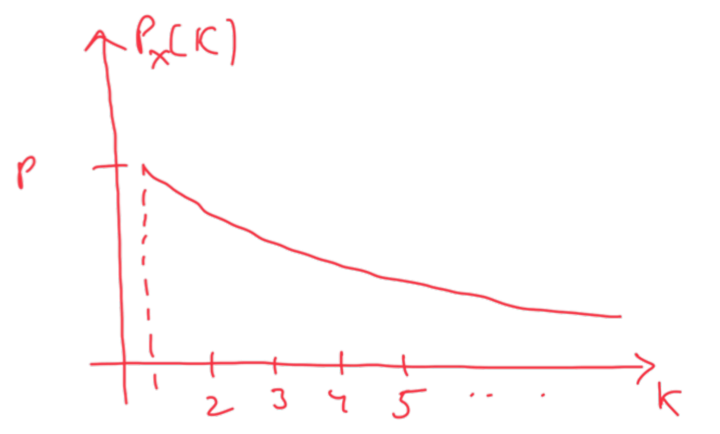
Proof $= \displaystyle\sum_{k=1}^{\infty} P_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1} \cdot p$

$a = p$ So, Sum of infinite G.P. $= \dfrac{a}{1-r}$

$r = 1-p$

So, $\displaystyle\sum_{k=1}^{\infty} P_X(k) = \frac{p}{1-(1-p)} = \frac{p}{1-1+p} = \frac{\cancel{p}}{\cancel{p}} = 1$

Thus $\displaystyle\sum_{k=1}^{\infty} P_X(k) = 1$ Hence proved



The Poisson Random Variable:

A poisson Random variable has the following PMF:

$$P_X(k) = \frac{e^{-\lambda} \lambda^k}{k!} \qquad k = 0, 1, 2 \ldots \quad \& \ \lambda > 0$$

[This is used when we want to know that in a fixed time interval how many times an event happens.]

Prove that $\displaystyle\sum_{k=0}^{\infty} \frac{e^{-\lambda} \lambda^k}{k!} = 1$

$$= e^{-\lambda}\left(\frac{1}{1} + \frac{\lambda}{1} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\right)$$

Taylor expansion of e^{λ}

$$= e^{-\lambda} \cdot e^{\lambda} = 1$$

Take a binomial random variable with very small 'p' & very large n.
The poisson PMF with parameter λ is a good approximation for a binomial PMF with parameter n & p, i.e.

$$\frac{e^{-\lambda} \lambda^k}{k!} \approx \binom{n}{k} p^k (1-p)^{n-k}$$
$$k = 0, 1, 2, \ldots n$$

provided $\boxed{\lambda = np}$

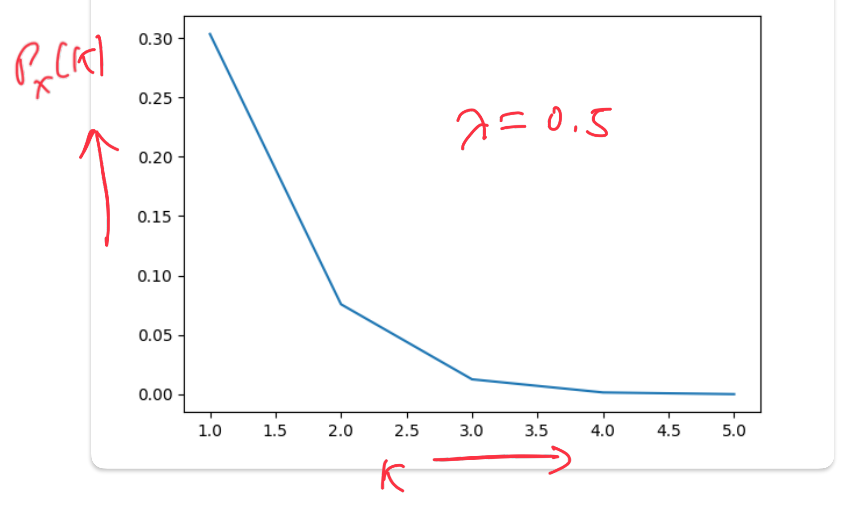
Ex: $n = 100$, $p = 0.01$, then probability of 5 successes in 100 trials is

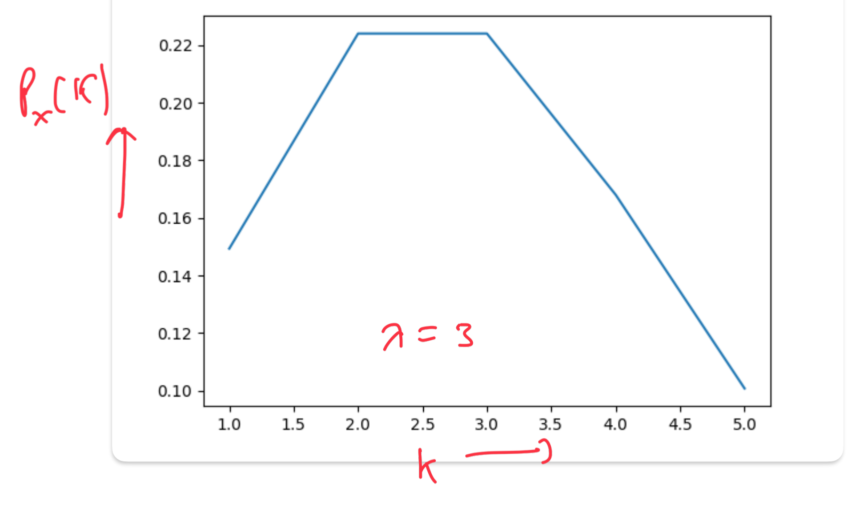
binomial $\rightarrow \binom{100}{5} (0.01)^5 (1-0.01)^{100-5} = 0.0029$

Using Poisson PMF with $\lambda = np = 100 \times 0.01 = 1$

$$\frac{e^{-1} (1)^5}{5!} = 0.0030$$

Plot of Poisson PMF:





Function of random variables:–

Let X be a random variable. If $Y = g(X)$ is a function of X, then Y is also a random variable, since it provides a numerical value for each possible outcome.

If X is discrete the Y is also a discrete random variable. The PMF of Y is calculated from PMF of X.

The PMF of Y is

$$\boxed{P_Y(y) = \sum_{\{x \mid g(x) = y\}} P_X(x)}$$

Ex: $P_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in } [-4, 4] \\ 0 & \text{otherwise} \end{cases}$

Let $Y = |X| \Rightarrow Y$ is a discrete random variable

Possible values of $Y = 0, 1, 2, 3, 4$.

$$P(Y=0) = P(X=0) = 1/9$$

$$P(Y=1) = P(X=1) + P(X=-1) = \frac{2}{9}$$
$$P(Y=2) = P(X=2) + P(X=-2) = \frac{2}{9}$$
$$P(Y=3) = P(X=3) + P(X=-3) = 2/9$$
$$P(Y=4) = P(X=4) + P(X=-4) = 2/9$$

The PMF of Y is:

$$P_Y(y) = \begin{cases} 1/4 & y = 0 \\ 2/9 & y = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$