Mean & Variance of Bernoulli random variable

Let X be a Bernoulli random variable

$$p_X(x) = \begin{cases} p & x=1 \\ 1-p & x=0 \end{cases}$$

$$\boxed{\text{Mean}: \; E[X] = 1\cdot p + 0\cdot(1-p) = p}$$

Variance: $E[X^2] = 1^2\cdot p + 0^2(1-p) = p$

$$\boxed{Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)}$$

Mean & Variance of Poisson random variable

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2 \ldots$$

$$E[X] = \sum x \cdot p = e^{-\lambda}\left(0\cdot 1 + \frac{\lambda}{1} + 2\frac{\lambda^2}{2!} + 3\frac{\lambda^3}{3!} + \ldots\right)$$

$$= e^{-\lambda}\lambda\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\right]$$

$$= e^{-\lambda}\cdot\lambda\cdot e^{\lambda} = \lambda \Rightarrow \boxed{E[X] = \lambda}$$

Variance: $E[X^2] = \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{x!}$

$$= \sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{(x-1)!} = \lambda e^{-\lambda}\left(1 + 2\frac{\lambda}{1!} + 3\frac{\lambda^2}{2!} + 4\frac{\lambda^3}{3!} \ldots\right)$$

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots$$

$$\lambda e^{\lambda} = \lambda + \lambda^2 + \frac{\lambda^3}{2!} + \frac{\lambda^4}{3!} + \ldots$$

$$e^{\lambda} + \lambda e^{\lambda} = 1 + 2\lambda + \frac{3\lambda^2}{2!} + \ldots$$

So, $\sum_{x=1}^{\infty} x \frac{e^{-\lambda}\lambda^x}{(x-1)!} = \lambda \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}}(1+\lambda) = \lambda(1+\lambda)$

$$Var(X) = E[X^2] - (E[X])^2$$
$$= \lambda + \lambda^2 - \lambda^2 = \lambda$$

So, $\boxed{Var(X) = \lambda}$

Joint PMF of Multiple random variables

Consider two discrete random variables X & Y associated with same experiment

Then Joint PMF of X & Y is

$$\boxed{p_{X,Y}(x,y) = p(X=x, Y=y)}$$

Eg: $X = 1, 2, 3$

$Y = 1, 2, 3$

$$\underbrace{p_{X,Y}(x,y)}_{\text{Joint PMF}} = \begin{cases} 1/9 & (1,1) \\ 1/9 & (2,1) \\ 1/9 & (3,1) \\ 1/9 & (1,2) \\ 1/9 & (2,2) \\ 1/9 & (3,2) \\ 1/9\;1/9\;1/9 & (1,3)\;(2,3)\;(3,3) \end{cases}$$

Can we get PMF of X & Y from this?

Eg: $\left.\begin{array}{l} X = 1,2,3,4 \\ Y = 1,2,3,4 \end{array}\right\}$

Valid PMF

Joint PMF of X & Y

Y \ X	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

$\to p(Y=4) = 3/20$

$p(X=3) = 8/20$

$$p_X(x) = \sum_y p_{X,Y}(x,y) \quad -(1)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y) \quad -(2)$$

The PMF $P_X(x)$ & $P_Y(y)$ are reffered to as marginal PMFs of X & Y respectively.

Function of multiple random Variable

Let, $Z = g(X,Y)$

$$\boxed{p_Z(z) = \sum_{\{(x,y) \mid g(x,y) = z\}} p_{X,Y}(x,y)}$$

Eg: Consider the previous example of joint PMF given as:

$X = 1,2,3,4$

$Y = 1,2,3,4$

Y \ X	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

Find the PMF of $Z = X + 2Y$

Z can be b/w 3 and 12

$Z=3$ when $(1,1) \to 1/20$

$Z=4$ $(2,1) \to 1/20$

$Z=5$ $(1,2), (3,1)$ $\frac{1}{20} + 1/20 = \frac{2}{20}$

$Z=6$ $(2,2), (4,1)$ $2/20 + 0 = \frac{2}{20}$

$Z=7$ $(3,2), (1,3)$ $\frac{3}{20} + \frac{1}{20} = \frac{4}{20}$

$Z=8$ $(2,3), (4,2)$ $\frac{2}{20} + \frac{1}{20} = \frac{3}{20}$

$Z=9$ $(3,3)\;(1,4)$ $3/20 + 0 = 3/20$

$Z=10$ $(4,3), (2,4)$ $\frac{1}{20} + 1/20 = 2/20$

$Z=11$ $(3,4)$ $1/20$

$Z=12$ $(4,4)$ $1/20$

Find $E[Z]$

$E[Z] = E[X] + 2E[Y]$

$$E[X] = \sum x p_X(x) = 1\cdot\frac{3}{20} + 2\cdot\frac{6}{20} + 3\cdot\frac{8}{20} + 4\cdot\frac{3}{20}$$

$$= \frac{15 + 24 + 12}{20} = \frac{15 + 36}{20} = \frac{51}{20}$$

$$E[Y] = \sum y \cdot p_Y(y) = 1\cdot\frac{3}{20} + 2\cdot\frac{7}{20} + 3\cdot\frac{7}{20} + 4\cdot\frac{3}{20}$$

$$= \frac{3}{20} + \frac{14}{20} + \frac{21}{20} + \frac{12}{20} =$$

$$\frac{17}{20} + \frac{33}{20} = \frac{50}{20}$$

$$E[Z] = \frac{51}{20} + \frac{100}{20} = \frac{151}{20}$$

In general, $X_1 \ldots X_n$ are random variables

$$E[a_1X_1 + a_2X_2 \ldots + a_nX_n] = a_1E[X_1] + a_2E[X_2] \ldots + a_nE[X_n]$$

where, $a_1, a_2 \ldots a_n$ are constants.

Mean of Binomial :-

Eg: Your CSDA101 class has 300 students, each student has probability 1/3 of getting an 'A', independent of any other student. Let X be the random variable defined as:

X = no. of students that get an 'A',

Find $E[X]$

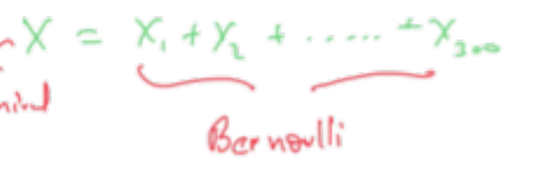
Let $X_1, X_2 \ldots X_{300}$ represents the students 1, 2, ... 300 respectively. Define

$$X_i = \begin{cases} 1 & \text{getting 'A' grade} \\ 0 & \text{otherwise} \end{cases}$$

Each X_i is a Bernoulli Random Variable

We know then $E[X_i] = p = 1/3$

Illustration:



we have:

$$\underset{\text{Binomial}}{X} = \underbrace{X_1 + X_2 + \ldots + X_{300}}_{\text{Bernoulli}}$$

Since X is the no. of success in 300 independent trials, it is a binomial random variable.

$$E[X] = E[X_1] + E[X_2] + \ldots + E[X_{300}]$$
$$= \frac{1}{3} + \frac{1}{3} + \ldots + \frac{1}{3} = 100$$

In general, if we have n bernoulli random variables having probability of success p, then

$$\boxed{E[X] = \sum_{i=1}^{n} E[X_i] = np}$$

The mean of binomial random variable is np