Expectation of function of Random Variable

Let X be a random variable with PMF P_X & Let $g(X)$ be the function of X. Then,

$$\boxed{E[g(x)] = \sum_x g(x) p_X(x)}$$

Eg: Consider the previous example:

$$p_X(x) = \begin{cases} 1/9 & x \in \{-4,-3,\dots,3,4\} \\ 0 & \text{otherwise} \end{cases}$$

$$Var(X) = E[(X-E[X])^2] = E[g(x)]$$

$$= \sum_x g(x) p_x(x)$$

$$= \sum_x (x-0)^2 p_X(x) = \sum x^2 p_X(x)$$

$$= \frac{2}{9}[1+4+9+16] = \frac{2}{9_3} \times 30^{10}$$

$$= 6.66$$

Properties of Mean & Variance:

Let X be a random variable & Let $Y = aX+b = g(X)$

$$E[Y] = E[g(x)] = \sum (ax+b) p_x(x)$$

$$= a\sum x p_X(x) + b\sum p_x(x)$$

$$= a E[X] + b$$

$$\Rightarrow \boxed{E[Y] = aE[X] + b}$$

Prove that $Var(Y) = a^2 Var(X)$

$$Var(Y) = E[(Y-E(Y))^2] = E[(Y-aE[x]+b)^2]$$

$$= E[(aX - aE(X))^2] = E[a^2(X-E(X))^2]$$

$$\Rightarrow Z = a^2(X-E(X))^2 \qquad (X-E(X))^2 = V$$

So, $E[Z] = a^2 E[V]$

$$\Rightarrow E[a^2(X-E(x))^2] = a^2 E[(X-E(x))^2]$$

$$\Rightarrow \boxed{Var(Y) = a^2 Var(X)}$$

Alternative formula for Variance

$$\boxed{Var(X) = E[X^2] - (E[X])^2}$$

$$Var(X) = \sum_x (x - E[X])^2 p_X(x)$$

$$= \sum_x (x^2 - 2xE[x] + (E[x])^2) p_X(x)$$

$$= \sum_x x^2 p_X x - 2E[x]\underbrace{\sum_x x p_x(x)}_{E[x]}$$

$$+ (E[X])^2 \sum p_x(x)$$

$$= \sum_x x^2 p_X x - 2(E[X])^2 + (E[X])^2$$

$$\Rightarrow E[X^2] - (E[X])^2$$

Thus, $Var(X) = E[X^2] - (E[X])^2$