Small Scales and Homogenisation (SmaSH)

Abstracts

Claude Bardos (University of Paris 7-Denis Diderot)

A Baby Version of the Landau Damping with Scalings. Application to the Quasilinear Approximation.

Considering the solution of the Vlasov equation on a finite time interval and introducing some convenient and may be physical scalings one obtains an elementary version of the Landau Damping and a road map to the proof of the quasilinear approximation.

Marco Bravin (Université de Bordeaux)

On the Asymptotic Limit of a Shrinking Source

In this talk I will present a recent result on the study of the asymptotic limit of a shrinking source in a perfect two dimensional fluid. The system consists of an Euler type system in an exterior domain with non-homogeneous boundary condition. The boundary conditions lead to the creation of a point source and a vortex point in the limit. Similar type of systems have been already study by Chemetov and Starovoitov in [1], where a different approximation approach was considered.

References:

[1] Chemetov, N. V., Starovoitov, V. N. (2002). On a Motion of a Perfect Fluid in a Domain with Sources and Sinks. Journal of Mathematical Fluid Mechanics, 4(2), 128-144.

Mikhail Cherdantsev (Cardiff University)

Stochastic homogenisation of high-contrast media.

Using a suitable stochastic version of the compactness argument of [V. V. Zhikov, 2000. On an extension of the method of two-scale convergence and its applications. Sb. Math., 191(7-8), 973-1014], we develop a probabilistic framework for the analysis of heterogeneous media with high contrast. We show that an appropriately defined multi-scale limit of the field in the original medium satisfies a system of equations corresponding to the coupled "macroscopic" and "microscopic" components of the field, giving rise to an analogue of the "Zhikov function", which represents the effective dispersion of the medium. We demonstrate that in the bounded domain setting, under some lenient conditions within the new framework, the spectra of the original problems converge to the spectrum of their homogenisation limit. We also discuss the behaviour of the spectrum in the whole space setting.

Kirill Cherednichenko (University of Bath)

Periodic PDEs with Micro-Resonators: Unified Approach to Homogenisation and Links to Time-Dispersive Media.

I shall discuss a novel approach to the homogenisation of critical-contrast periodic PDEs, which yields an explicit construction of their norm-resolvent asymptotics. A practically relevant outcome of this result is that it interprets composite media with micro-resonators as a class of time-dispersive media. This is joint work with Yulia Ershova and Alexander Kiselev.

Hong Duong (University of Birmingham)

Thermodynamic Limit of the Transition Rate of a Crystalline Defect

The presence of defects in crystalline materials significantly affects their mechanical and chemical properties, hence determining defect geometry, energies, and mobility is a fundamental problem of materials modelling. The inherent discrete nature of defects requires that any "ab initio" theory should start from an atomistic description. While there is a substantial literature on the scaling limit (free energy per particle), a rigorous analysis of the thermodynamic limit of crystalline defects in a finite temperature setting is still missing.

In this talk, we consider an isolated point defect embedded in a homogeneous crystalline solid. We show that, in the harmonic approximation, a periodic supercell approximation of the formation free energy as well as of the transition rate between two stable configurations converge as the cell size tends to infinity. We characterise the limits and establish sharp convergence rates. Both cases can be reduced to a careful renormalisation analysis of the vibrational entropy difference, which is achieved by identifying an underlying spatial decomposition and its locality estimates.

This talk is based on collaborative works, see references, with Mathew Dobson (UMass Amherst), Julian Braun and Christoph Ortner (University of Warwick).

References:

[1] Matthew Dobson, Manh Hong Duong and Christoph Ortner. On assessing the accuracy of defect free energy computations. ESAIM: Mathematical Modelling and Numerical Analysis, Volume 52, Number 4, 1315 - 1352, 2018.
[2] Julian Braun, Manh Hong Duong, Christoph Ortner. Thermodynamic Limit of the Transition Rate of a Crystalline Defect. Submitted, https://arxiv.org/abs/1810.11643.

Pavel Exner (Doppler Institute for Mathematical Physics and Applied Mathematics, Prague)

Leaky Quantum Structures

The subject of the talk are Schrödinger operators with an attractive singular 'potential', supported by a manifold or a geometric complex Γ of codimension one, formally written as $-\Delta - \alpha\delta(x - \Gamma)$. We show how they can be approximated by regular potentials or by arrays of point interactions, where the last named procedure can be regarded as a homogenization of a sort. We discuss the ways in which spectral properties of such systems are influenced by the geometry of the interaction support with the main attention paid to situations when the coupling constant is large or the geometric perturbation is weak, and asymptotic expansions can be derived. We also discuss effects arising from the presence of a magnetic field, in particular, the influence of an Aharonov-Bohm flux on the so-called Welsh eigenvalues.

Antoine Gloria (Sorbonne Université , Paris)

Stochastic Homogenization: Regularity, Oscillations, and Fluctuations.

In this talk I'll give an overview of recent results in stochastic homogenization. Stochastic homogenization is the analysis of PDEs, say linear elliptic equations in divergence form, with random oscillating coefficients. When the typical correlation-length of the coefficient field gets small, the solution of the PDE with random coefficients can be replaced at first order by the solution of a similar PDE with constant and deterministic coefficients – this is the qualitative theory of stochastic homogenization. The description of the solution of the original problem by that of the homogenized problem lacks accuracy because the former oscillates (due to the oscillations of the coefficients) and fluctuates (due to the fluctuations of the coefficients) whereas the latter does not. The aim of this talk is precisely to characterize oscillations and fluctuations in this context, for a family of Gaussian coefficient fields (in the full range of decay of the covariance function)..

Mitsuo Higaki (Université de Bordeaux)

Navier Wall Law for Nonstationary Viscous Incompressible Flows

In fluid mechanics, it is a basic subject to understand the mathematical structure of flows near a rough-surfaced wall. The Navier wall law is a method to model a rough boundary as a flat one and to impose an appropriate condition on it reflecting the roughness of the original boundary. In this talk, we consider the two-dimensional Navier-Stokes equations in a half-plane with a small perturbation, and report the results on the Navier wall law to the initial-boundary value problem.

Cristophe Lacave (Université Grenoble Alpes)

Incompressible Fluids through a Porous Medium.

In a perforated domain, the asymptotic behavior of the fluid motion depends on the rate (inter-hole distance)/(size of the holes). We will present the standard framework and explain how to find the critical rate where "strange terms" appear for the Laplace and Navier-Stokes equations. Next, we will study Euler equations where the critical rate is totally different than for parabolic equations. These works are in collaboration with V. Bonnaillie-Noel, M. Hillairet, N. Masmoudi, C. Wang and D. Wu..

Yulia Meshkova (Chebyshev Laboratory, St Petersburg State University)

On Homogenization of Periodic Hyperbolic Systems

The talk is devoted to homogenization of periodic differential operators. Let $B_\varepsilon = B_\varepsilon^* > 0$ be a second order matrix strongly elliptic differential operator acting in $L_2(\mathbb{R}^d;\mathbb{C}^n)$. The coefficients of the operator B_ε depend on $\mathbf{x}/\varepsilon$, $0 < \varepsilon \leqslant 1$. Consider the hyperbolic system

$$\partial_t^2 \mathbf{u}_\varepsilon(\mathbf{x},t) = -B_\varepsilon \mathbf{u}_\varepsilon(\mathbf{x},t) + \mathbf{F}(\mathbf{x},t), \quad \mathbf{u}_\varepsilon(\mathbf{x},0)=0, \quad (\partial_t \mathbf{u}_\varepsilon)(\mathbf{x},0) = \boldsymbol{\psi}(\mathbf{x}),$$

where $\boldsymbol{\psi} \in L_2(\mathbb{R}^d;\mathbb{C}^n)$ and $\mathbf{F} \in L_1((0,T);L_2(\mathbb{R}^d;\mathbb{C}^n))$ for some $0 < T \leqslant \infty$. Then

$$\mathbf{u}_\varepsilon(\cdot,t) = B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})\boldsymbol{\psi} + \int_0^t B_\varepsilon^{-1/2}\sin((t-\widetilde{t})B_\varepsilon^{1/2})\mathbf{F}(\cdot,\widetilde{t})\,d\widetilde{t}.$$

We are interested in the behaviour of the solution $\mathbf{u}_\varepsilon(\cdot,t)$ in the small period limit $\varepsilon \to 0$. It turns out that, for sufficiently smooth $\boldsymbol{\psi}$ and $\mathbf{F}$, the error estimates in approximations for the solution $\mathbf{u}_\varepsilon$ depend on suitable norms of $\boldsymbol{\psi}$ and $\mathbf{F}$ explicitly. In other words, we can approximate the operator $B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2})$ in a uniform operator topology:

$$\begin{aligned}
&\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2})\|_{H^1(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \\
&\quad \leqslant C\varepsilon|t|, \qquad (1)\\
&\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon K_1(\varepsilon;t)\|_{H^2(\mathbb{R}^d;\mathbb{C}^n)\to H^1(\mathbb{R}^d;\mathbb{C}^n)} \\
&\quad \leqslant C\varepsilon(1+|t|), \qquad (2)\\
&\|B_\varepsilon^{-1/2}\sin(tB_\varepsilon^{1/2}) - (B^0)^{-1/2}\sin(t(B^0)^{1/2}) - \varepsilon K_2(\varepsilon;t)\|_{H^3(\mathbb{R}^d;\mathbb{C}^n)\to L_2(\mathbb{R}^d;\mathbb{C}^n)} \\
&\quad \leqslant C\varepsilon^2(1+t^2). \qquad (3)
\end{aligned}$$

Here B^0 is the so-called effective operator with constant coefficients, $K_1(\varepsilon;t)$ and $K_2(\varepsilon;t)$ are the correctors. The correctors contain rapidly oscillating factors and so depend on ε.

We derive estimates (1) and (2) from the corresponding approximations for the resolvent B_ε^{-1}, obtained by T. A. Suslina (2010) with the help of the spectral theory approach to homogenization problems. Our method is a modification of the classical proof of the Trotter-Kato theorem. The analogue of estimate (3) is also known (Suslina, 2014). But the author have no idea how to modify the Trotter-Kato theorem for this case. So, to prove estimate (3), we directly apply the spectral theory approach in a version developed by M. Sh. Birman and T. A. Suslina. The technique is based on the unitary scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory.

Svetlana Pastukhova (Russian Technological University)

Homogenization of Monotone Operators under Nonstandard Growth Conditions

In a bounded Lipschitz domain in $\mathbb{R}^d$, we consider the Dirichlet boundary value problem for an elliptic strictly monotone operator A_ε such that $A_\varepsilon u = -\mathrm{div}\,(a_\varepsilon(x, \nabla u))$, for which growth estimates of power type with a variable exponent are satisfied. This variable exponent $p_\varepsilon(x)$ and also the symbol $a_\varepsilon(x, \xi)$ oscillate with a small period ε with respect to the space variable x. We prove a homogenization result for this problem, thus, obtaining, in the limit as $\varepsilon \to 0$, a strictly monotone operator A that is much simpler than the original one. Namely, $Au = -\mathrm{div}\,(a_0(\nabla u))$, $a_0 = a_0(\xi)$ does not depend on the space variable but satisfies new type growth estimates which are formulated in terms of a convex function $f(\xi)$ instead of the power function $|\xi|^{p_\varepsilon(x)}$ used in growth conditions for a_ε. The functions $f(\xi)$ and $a_0(\xi)$ are found via auxiliary problems stated on the unit cell of periodicity. The Dirichlet problem for the operator A_ε is posed in the variable order Sobolev space $W_0^{1,p_\varepsilon(\cdot)}(\Omega)$, while the limit problem with the operator A is posed in the Sobolev-Orlicz space $W_0^f(\Omega)$ (with generalized Orlicz integrability defined by the function $f(\xi)$).

For these problems, the appropriate Compensated Compactness Lemma is proved or two-scale convergence technique is adapted.

These are the results [1], [2] joint with professor V.V. Zhikov.

References:

[1] Zhikov V. V., Pastukhova S. E., Homogenization of monotone operators under conditions of coercitivity and growth of variable order// (Russian) Mat. Zametki 90 (1-2) (2011) 53–69; English transl.: Math. Notes 90 (1-2) (2011) 48–63.

[2] Zhikov V. V., Pastukhova S. E., Homogenization and two-scale convergence in Sobolev space with oscillating exponent// (Russian) Algebra i Analiz, 30:2 (2018), 114144; English transl.: St. Petersburg Math. Journal, 30:2 (2019), 231251.

Olaf Post (Trier University)

Operator Estimates for the Crushed Ice Problem

We show norm resolvent convergence for the Laplacian on a family of domains with many small holes taken out. The critical parameter here is the capacity of the small holes. If the density of holes is below the critical parameter, then the limit operator is the free Laplacian, in the critical case, an extra potential term appears. The proof relies on an abstract convergence scheme for operators acting in varying Hilbert spaces.

Cristophe Prange (Université de Bordeaux)

Regularity Above Bumpy Boundaries

In this talk, we will study regularity for elliptic equations above highly oscillating boundaries. We will show an improved regularity effect. Our focus will be on the study of the boundary layer correctors needed to prove such results. This is joint work with Carlos Kenig and Mitsuo Higaki.

Valery Smyshlyaev (University College London)

Homogenization for General Classes of High-Contrast Problems: From Two-Scale Convergence to Operator-Type Error Bounds.

There has been considerable recent interest in multiscale analysis of high-contrast problems where macroscopic properties can display a number of interesting effects, often due to the so-called "micro-resonances". We will review some background, as well as some more recent developments and applications. One is two-scale analysis of general "partially-degenerating" periodic PDE systems [1], where strong two-scale (pseudo-) resolvent convergence appears to hold under a rather generic decomposition assumption, implying in particular (two-scale) convergence of semigroups with applications to a wide class of micro-resonant dynamic problems. A substantial additional effort is required for establishing not only the convergence but also its *rate* i.e. error bounds. In [2], we establish such error bounds for eigenvalues and eigenmodes due to a localized defect in a high-contrast periodic medium by, in particular, controlling the effect of a high-contrast boundary layer.

Finally we briefly review a most recent generic approach for establishing *operator-type error bounds*, i.e. error estimates in strongest possible sense, for high-contrast problems in unbounded domains [3]. The latter is achieved for a rather general class of high-contrast problems via an abstract generalisation of the Floquet transform and careful analysis of related quadratic forms and of projection operators on relevant "macro" and "micro" subspaces and their interactions. This allows to establish the desired operator-type error bounds in a general setting, and as one particular implication to obtain error bounds on the spectral convergence including on the spectral band gaps. The approach can be illustrated by various examples ranging from PDE systems to differential-difference equations and periodic quantum graphs and their generalisations. Various parts of the work are joint with Ilia Kamotski and Shane Cooper.

References:

[1] I.V. Kamotski, V.P. Smyshlyaev, Two-scale homogenization for a general class of high contrast PDE systems with periodic coefficients, *Applicable Analysis* 98:64–90, 2019
[2] I. V. Kamotski, V. P. Smyshlyaev, Localised modes due to defects in high contrast periodic media via two-scale homogenization, *J. Math. Sci (NY)* 232 (3), 349-377
[3] S. Cooper, I.V. Kamotski, V.P. Smyshlyaev. To be published, 2019.

Tatiana Suslina (St Petersburg State University)

PART 1: Spectral Approach to Homogenization of Periodic Differential Operators

The talk is devoted to the operator-theoretic (spectral) approach to homogenization suggested by M. Birman and T. Suslina. We study selfadjoint matrix second order elliptic operators A_ε in $\mathbb{R}^d$. The coefficients are periodic with respect to some lattice and depend on $\mathbf{x}/\varepsilon$. For small ε the coefficients oscillate rapidly. We show that the resolvent $(A_\varepsilon + I)^{-1}$ converges to the resolvent of the effective operator A^0 in the $(L_2 \to L_2)$-operator norm, as $\varepsilon \to 0$, error estimate being of order $O(\varepsilon)$. We also find more accurate approximation for the resolvent $(A_\varepsilon + I)^{-1}$ in the $(L_2 \to L_2)$-norm, as well as approximation in the $(L_2 \to H^1)$-norm. The method is based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory. Next, we discuss the operator error estimates for the similar elliptic operators $A_{D,\varepsilon}$ and $A_{N,\varepsilon}$ in a bounded domain with the Dirichlet or Neumann boundary conditions.

PART 2: Homogenization of the Stationary Maxwell System with Periodic Coefficients

We study homogenization of a stationary Maxwell system in $\mathbb{R}^3$ and in a bounded domain $\mathcal{O} \subset \mathbb{R}^3$ with sufficiently smooth boundary. The coefficients (electric permittivity and magnetic permeability) are periodic with respect to some lattice and depend on $\mathbf{x}/\varepsilon$. So, for small ε they oscillate rapidly. We are interested in the behavior of the solutions for small ε. The classical result is the weak L_2-convergence of the solutions to the solution of the effective problem, as $\varepsilon \to 0$. We find approximations for the solutions in the L_2-norm with error estimates of operator type.