魔方陣

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第 1 章

魔方陣を量子アニーリングで解く

魔方陣は、$N \times N$ の各セルに $1 \sim N^2$ の数値を一つずつ入れ、縦横斜めのどの列についても、その列の数値の総和が等しくなる様に、数値を配置するもの。

例えば 3×3 の魔方陣であれば、縦横斜めのどの列にある数値の総和も等しく 15 になる。

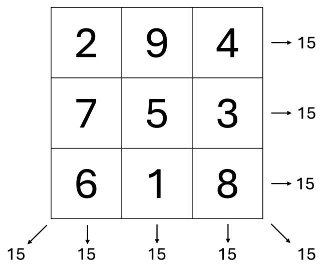

図 1.1　参考資料 [2] より

3×3 は、対称な形を 1 つと数えることにすると 1 通り、4×4 では 880 通り、5×5 では 275305224 通り、6×6 では 17753889197660635632 通り存在することがわかっている。[2]

1.1　問題の構成

決定変数 q を各セル毎に $N \times N$ 個用意する。$q_{i,j,n}$ は、i 行 j 列目のセル内の $N \times N$ 個の決定変数。ここで、$i, j\ \in \{1, 2, \cdots, N\},\ n\ \in \{1, 2, \cdots, N \times N\}$。表は $N = 3$ の場合。

	1 列目 $(j=1)$				2 列目 $(j=2)$				3 列目 $(j=3)$			
1 行目	セル $(i=1, j=1)$				セル $(i=1, j=2)$				セル $(i=1, j=3)$			
	1	2	$\cdots$	9	1	2	$\cdots$	9	1	2	$\cdots$	9
$(i=1)$	q_{111}	q_{112}	$\cdots$	q_{119}	q_{121}	q_{122}	$\cdots$	q_{129}	q_{131}	q_{132}	$\cdots$	q_{139}
2 行目	セル $(i=2, j=1)$				セル $(i=2, j=2)$				セル $(i=2, j=3)$			
	1	2	$\cdots$	9	1	2	$\cdots$	9	1	2	$\cdots$	9
$(i=2)$	q_{211}	q_{212}	$\cdots$	q_{219}	q_{221}	q_{222}	$\cdots$	q_{229}	q_{231}	q_{232}	$\cdots$	q_{239}
3 行目	セル $(i=3, j=1)$				セル $(i=3, j=2)$				セル $(i=3, j=3)$			
	1	2	$\cdots$	9	1	2	$\cdots$	9	1	2	$\cdots$	9
$(i=3)$	q_{311}	q_{312}	$\cdots$	q_{319}	q_{321}	q_{322}	$\cdots$	q_{329}	q_{331}	q_{332}	$\cdots$	q_{339}

この決定変数は 0 か 1 かの 2 値変数で、当該数値 $1 \sim N \times N$ をそのセルに置く（1）か置かない（0）かを表すものとする。すると制約条件は、次の様に考えることができる。($M = N \times N$ とする)

1. 各セルの中では 1 ∼ 9 の中のどれか 1 つだけが 1 になる（セルに 2 つ以上の数値は入らない）
   $\rightarrow \sum_i q_i = 1\ (for\ cell = 1 \sim 9)$

$$f_1 = \sum_i^N \sum_j^N \left(\sum_n^M q_{i,j,n} - 1 \right)^2$$

2. あるセルの数値と同じ数値は、他のセルには入らない
   $\rightarrow (q_i + q_{i+9} + q_{i+2\times9} + \cdots + q_{i+8\times9}) = \sum_{cell=1}^{9} q_{i+(cell-1)\times9} = 1\ (for\ i = 1 \sim 9)$

$$f_2 = \sum_n^M \left(\sum_i^N \sum_j^N q_{i,j,n} - 1 \right)^2$$

3. いずれの行方向の数値の和も同じ値 S(= 15) になる
   $\rightarrow \sum_{cell=1}^{3}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$
   $\sum_{cell=4}^{6}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$
   $\sum_{cell=7}^{9}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$

$$f_3 = \sum_i^N \left(\sum_j^N \sum_n^M n \cdot q_{i,j,n} - S \right)^2$$

4. いずれの列方向の数値の和も同じ値 S(= 15) になる
   $\rightarrow \sum_{cell=1,4,7}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$
   $\sum_{cell=2,5,8}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$
   $\sum_{cell=3,6,9}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$

$$f_4 = \sum_j^N \left(\sum_i^N \sum_n^M n \cdot q_{i,j,n} - S \right)^2$$

5. 右下がりの対角要素の和と右上がりの対角要素の和も同じ値 S(= 15) になる
   $\rightarrow \sum_{cell=1,5,9}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$
   $\sum_{cell=3,5,7}(\sum_{i=1}^{9} i \times q_{i+(cell-1)\times9}) = 15$

$$f_5 = \left(\sum_d^N \sum_n^M n \cdot q_{d,d,n} - S \right)^2 + \left(\sum_d^N \sum_n^M n \cdot q_{d,N-d+1,n} - S \right)^2$$

1.1.1　式の展開

$$
\begin{aligned}
f_1 &= \sum_i^N \sum_j^N \left(\sum_n^M q_{i,j,n} - 1 \right)^2 \\
&= \sum_i \sum_j \left(\sum_{n_1} \sum_{n_2} q_{i,j,n_1} q_{i,j,n_2} - 2 \sum_n q_{i,j,n} \right) \\
f_2 &= \sum_n^M \left(\sum_i^N \sum_j^N q_{i,j,n} - 1 \right)^2 \\
&= \sum_n \left(\sum_{i_1} \sum_{j_1} q_{i_1,j_1,n} \sum_{i_2} \sum_{j_2} q_{i_2,j_2,n} - 2 \sum_i \sum_j q_{i,j,n} \right) \\
f_3 &= \sum_i^N \left(\sum_j^N \sum_n^M n \cdot q_{i,j,n} - S \right)^2 \\
&= \sum_i \left(\sum_{j_1} \sum_{n_1} n_1 \cdot q_{i,j_1,n_1} \sum_{j_2} \sum_{n_2} n_2 \cdot q_{i,j_2,n_2} - 2 \cdot S \sum_j \sum_n n \cdot q_{i,j,n} \right) \\
f_4 &= \sum_j^N \left(\sum_i^N \sum_n^M n \cdot q_{i,j,n} - S \right)^2 \\
&= \sum_j \left(\sum_{i_1} \sum_{n_1} n_1 \cdot q_{i_1,j,n_1} \sum_{i_2} \sum_{n_2} n_2 \cdot q_{i_2,j,n_2} - 2 \cdot S \sum_i \sum_n n \cdot q_{i,j,n} \right) \\
f_5 &= \left(\sum_d^N \sum_n^M n \cdot q_{d,d,n} - S \right)^2 + \left(\sum_d^N \sum_n^M n \cdot q_{d,N-d+1,n} - S \right)^2 \\
&= \left(\sum_{d_1} \sum_{n_1} n_1 \cdot q_{d_1,d_1,n_1} \sum_{d_2} \sum_{n_2} n_2 \cdot q_{d_2,d_2,n_2} - 2 \cdot S \sum_d \sum_n n \cdot q_{d,d,n} \right) \\
&+ \left(\sum_{d_1} \sum_{n_1} n_1 \cdot q_{d_1,N-d_1+1,n_1} \sum_{d_2} \sum_{n_2} n_2 \cdot q_{d_2,N-d_2+1,n_2} - 2 \cdot S \sum_d \sum_n n \cdot q_{d,N-d+1,n} \right)
\end{aligned}
$$

1.2　実装

1.2.1　class

```
from openjij import SASampler, SQASampler
from collections import defaultdict, Counter
import numpy as np
```

```
class MagicCircle:
  def __init__(self, N=3):
      self.N = N      # self.N = 3
      self.M = N * N      # self.M = 9
      self.S = N * (N**2 + 1) // 2   # self.S = 15
      self.idx = {}
```

```
        k = 0
        for i in range(self.N):
            for j in range(self.N):
                for n in range(self.M):
                    self.idx[(i,j,n)] = k
                    k += 1
        samplers = [SASampler(), SQASampler()]
        self.sampler = samplers[0]

    def get_param(self):
        return self.N, self.M, self.S, self.idx
```

1.2.2　制約：f1

```
    def sub1(self, i, j, L, Q):
      N, M, _, idx = self.get_param()
      for n1 in range(M):
          Q[(idx[(i, j, n1)], idx[(i, j, n1)])] -= 2.0 * L
          for n2 in range(M):
              Q[(idx[(i, j, n1)], idx[(i, j, n2)])] += 1.0 * L

    def f1(self, L, Q):
      N, _, _, _ = self.get_param()
      for i in range(N):
          for j in range(N):
              self.sub1(i, j, L, Q)
      return Q
```

1.2.3　制約：f2

```
    def sub2(self, n, L, Q):
      N, _, _, idx = self.get_param()
      for i1 in range(N):
          for j1 in range(N):
              Q[(idx[(i1, j1, n)], idx[(i1, j1, n)])] -= 2.0 * L
              for i2 in range(N):
                  for j2 in range(N):
                      Q[(idx[(i1, j1, n)], idx[(i2, j2, n)])] += 1.0 * L

    def f2(self, L, Q):
      _, M, _, _ = self.get_param()
      for n in range(M):
          self.sub2(n, L, Q)
      return Q
```

1.2.4　制約：f3

```
def sub3(self, i, L, Q):
  N, M, S, idx = self.get_param()
  for j1 in range(N):
      for n1 in range(M):
          Q[(idx[(i, j1 ,n1)], idx[(i, j1, n1)])] -= 2.0 * (n1+1) * S * L
          for j2 in range(N):
              for n2 in range(M):
                  Q[(idx[(i, j1, n1)], idx[(i, j2, n2)])] += (n1+1) * (n2+1) * L

def f3(self, L, Q):
  N, _, _, _ = self.get_param()
  for i in range(N):
      self.sub3(i, L, Q)
  return Q
```

1.2.5　制約：f4

```
def sub4(self, j, L, Q):
  N, M, S, idx = self.get_param()
  for i1 in range(N):
      for n1 in range(M):
          Q[(idx[(i1, j, n1)], idx[(i1, j, n1)])] -= 2.0 * (n1+1) * S * L
          for i2 in range(N):
              for n2 in range(M):
                  Q[(idx[(i1, j, n1)], idx[(i2, j, n2)])] += (n1+1) * (n2+1) * L

def f4(self, L, Q):
  N, _, _, _ = self.get_param()
  Q = defaultdict(lambda: 0)
  for j in range(N):
      self.sub4(j, L, Q)
  return Q
```

1.2.6　制約：f5

```
def f5(self, L, Q):
  N, M, S, idx = self.get_param()
  Q = defaultdict(lambda: 0)
  for d1 in range(N):
      for n1 in range(M):
          Q[(idx[(d1, d1, n1)], idx[(d1, d1, n1)])] -= 2.0 * (n1+1) * S * L
          Q[(idx[(d1, N-d1-1, n1)], idx[(d1, N-d1-1, n1)])] -= 2.0 * (n1 + 1) * S
   * L
          for d2 in range(N):
              for n2 in range(M):
                  Q[(idx[(d1, d1, n1)], idx[(d2, d2, n2)])] += (n1+1) * (n2+1) *
  L
                  Q[(idx[(d1, N-d1-1, n1)], idx[(d2, N-d2-1, n2)])] += (n1 + 1) *
   (n2 + 1) * L
```

```
    return Q
```

1.2.7　評価関数：f

$$f = \lambda_1 \cdot f_1 + \lambda_2 \cdot f_2 + \lambda_3 \cdot (f_3 + f_4 + f_5)$$

```
def f(self, lagrange1=1.0, lagrange2=1.0, lagrange3=1.0):
  Q = defaultdict(lambda: 0)
  _ = self.f1(lagrange1, Q)
  _ = self.f2(lagrange2, Q)
  _ = self.f3(lagrange3, Q)
  _ = self.f4(lagrange3, Q)
  _ = self.f5(lagrange3, Q)
  return Q

def solv(self, Q, num_reads=1):
  sampleset = self.sampler.sample_qubo(Q, num_reads=num_reads)
  return sampleset

def result(self, sampleset):
  N, M, S, idx = self.get_param()
  result = [i for i in sampleset.first[0].values()]
  ans = [[None] * N for _ in range(N)]
  for i in range(N):
      for j in range(N):
          for n in range(N**2):
              if result[idx[(i,j,n)]] == 1:
                  ans[i][j] = n+1
  return ans
```

出力された結果を、ふるいにかける仕掛け

```
def evaluate(self, sampleset, prn=True):
  # Extract sample solutions, energies, and sort them by frequency
  samples = sampleset.record['sample']
  energies = sampleset.record['energy']
  # Combine solutions and corresponding energies
  sample_data = [(tuple(sample), energy) for sample, energy in zip(samples,
  energies)]
  # Sort the results by appearance frequency and then energy
  sample_frequency = Counter(sample for sample, _ in sample_data)
  # Print sorted results by frequency and include energy
  if prn:
      print("\nSorted samples by frequency and energy:")
      for solution, freq in sample_frequency.most_common():
          energy = next(energy for sample, energy in sample_data if sample ==
  solution)
          print(f"Sample: {solution}, Frequency: {freq}, Energy: {energy:+.2f}")
  return sample_data, sample_frequency
 
  def check1(self, a):
    N, M, _, _ = self.get_param()
    b = np.array(a).reshape(M, M)
    for i in range(M):
        s = 0
        for j in range(M):
            s += b[i][j]
        if s!=1:
            return False
    for j in range(M):
        s = 0
        for i in range(M):
            s += b[i][j]
        if s!=1:
            return False
    return True

  def check2(self, a):
    N, M, S, _ = self.get_param()
    b = np.array(a).reshape(N, N)
    for i in range(N):
        s = 0
        for j in range(N):
            s += b[i][j]
        if s!= S:
            return False
    #
    for j in range(N):
        s = 0
        for i in range(N):
            s += b[i][j]
        if s!= S:
            return False
    #
    s = 0
    for i in range(N):
        for j in range(N):
            if i==j:
                s += b[i][j]
    if s!=S:
        return False
    #
    s = 0
    for i in range(N):
        k = N-i-1
        s += b[i][k]
    if s!=S:
        return False
    #
    return True

  def decode(self, a):
    N, M, _, _ = self.get_param()
    b = np.array(a).reshape(M, M)
    mat = []
    for i in range(M):
        for j in range(M):
            if b[i][j]==1:
                mat.append(j+1)
    return mat
```

1.2.8　main

```
if __name__ == '__main__':
  N = 3
  mc = MagicCircle(N)
  lagrange1 = 10.0
  lagrange2 = 10.0
  lagrange3 =  1.0
  Q = mc.f(lagrange1, lagrange2, lagrange3)
  num_reads = 10000
  sampleset = mc.solv(Q, num_reads)
  #ans = mc.result(sampleset)
  #print(*ans, sep='\n')
  #
  for sample in sampleset.record['sample']:
      if mc.check1(sample):
          a = mc.decode(sample)
          if mc.check2(a):
              print(np.array(a).reshape(N, N))
              print()
```

1.3　実行結果

```
[[6 7 2]
 [1 5 9]
 [8 3 4]]

[[4 3 8]
 [9 5 1]
 [2 7 6]]
```

1.4　全体プログラム

```
from openjij import SASampler, SQASampler
from collections import defaultdict, Counter
import numpy as np
```

```
class MagicCircle:
    def __init__(self, N=3):
        self.N = N      # self.N = 3
        self.M = N * N      # self.M = 9
        self.S = N * (N**2 + 1) // 2    # self.S = 15
        self.idx = {}
        k = 0
        for i in range(self.N):
            for j in range(self.N):
                for n in range(self.M):
                    self.idx[(i,j,n)] = k
                    k += 1
        samplers = [SASampler(), SQASampler()]
        self.sampler = samplers[0]

    def get_param(self):
        return self.N, self.M, self.S, self.idx

    def sub1(self, i, j, L, Q):
        N, M, _, idx = self.get_param()
        for n1 in range(M):
            Q[(idx[(i, j, n1)], idx[(i, j, n1)])] -= 2.0 * L
            for n2 in range(M):
                Q[(idx[(i, j, n1)], idx[(i, j, n2)])] += 1.0 * L

    def f1(self, L, Q):
        N, _, _, _ = self.get_param()
        for i in range(N):
            for j in range(N):
                self.sub1(i, j, L, Q)
        return Q

    def sub2(self, n, L, Q):
        N, _, _, idx = self.get_param()
        for i1 in range(N):
            for j1 in range(N):
                Q[(idx[(i1, j1, n)], idx[(i1, j1, n)])] -= 2.0 * L
                for i2 in range(N):
                    for j2 in range(N):
                        Q[(idx[(i1, j1, n)], idx[(i2, j2, n)])] += 1.0 * L

    def f2(self, L, Q):
        _, M, _, _ = self.get_param()
        for n in range(M):
            self.sub2(n, L, Q)
        return Q

    def sub3(self, i, L, Q):
        N, M, S, idx = self.get_param()
        for j1 in range(N):
            for n1 in range(M):
                Q[(idx[(i, j1 ,n1)], idx[(i, j1, n1)])] -= 2.0 * (n1+1) * S * L
                for j2 in range(N):
                    for n2 in range(M):
                        Q[(idx[(i, j1, n1)], idx[(i, j2, n2)])] += (n1+1) * (n2+1)
    * L

    def f3(self, L, Q):
        N, _, _, _ = self.get_param()
        for i in range(N):
            self.sub3(i, L, Q)
        return Q

    def sub4(self, j, L, Q):
        N, M, S, idx = self.get_param()
        for i1 in range(N):
            for n1 in range(M):
                Q[(idx[(i1, j, n1)], idx[(i1, j, n1)])] -= 2.0 * (n1+1) * S * L
                for i2 in range(N):
                    for n2 in range(M):
                        Q[(idx[(i1, j, n1)], idx[(i2, j, n2)])] += (n1+1) * (n2+1)
    * L

    def f4(self, L, Q):
        N, _, _, _ = self.get_param()
        Q = defaultdict(lambda: 0)
        for j in range(N):
            self.sub4(j, L, Q)
        return Q

    def f5(self, L, Q):
        N, M, S, idx = self.get_param()
        Q = defaultdict(lambda: 0)
        for d1 in range(N):
            for n1 in range(M):
                Q[(idx[(d1, d1, n1)], idx[(d1, d1, n1)])] -= 2.0 * (n1+1) * S * L
                Q[(idx[(d1, N-d1-1, n1)], idx[(d1, N-d1-1, n1)])] -= 2.0 * (n1 + 1)
     * S * L
                for d2 in range(N):
                    for n2 in range(M):
                        Q[(idx[(d1, d1, n1)], idx[(d2, d2, n2)])] += (n1+1) * (n2
    +1) * L
                        Q[(idx[(d1, N-d1-1, n1)], idx[(d2, N-d2-1, n2)])] += (n1 +
    1) * (n2 + 1) * L
        return Q

    def f(self, lagrange1=1.0, lagrange2=1.0, lagrange3=1.0):
        Q = defaultdict(lambda: 0)
        _ = self.f1(lagrange1, Q)
        _ = self.f2(lagrange2, Q)
        _ = self.f3(lagrange3, Q)
        _ = self.f4(lagrange3, Q)
        _ = self.f5(lagrange3, Q)
        return Q

    def solv(self, Q, num_reads=1):
        sampleset = self.sampler.sample_qubo(Q, num_reads=num_reads)
        return sampleset

    def result(self, sampleset):
        N, M, S, idx = self.get_param()
        result = [i for i in sampleset.first[0].values()]
        ans = [[None] * N for _ in range(N)]
        for i in range(N):
            for j in range(N):
                for n in range(N**2):
                    if result[idx[(i,j,n)]] == 1:
                        ans[i][j] = n+1
        return ans

    def evaluate(self, sampleset, prn=True):
        # Extract sample solutions, energies, and sort them by frequency
        samples = sampleset.record['sample']
        energies = sampleset.record['energy']
        # Combine solutions and corresponding energies
        sample_data = [(tuple(sample), energy) for sample, energy in zip(samples,
    energies)]
        # Sort the results by appearance frequency and then energy
        sample_frequency = Counter(sample for sample, _ in sample_data)
        # Print sorted results by frequency and include energy
        if prn:
            print("\nSorted samples by frequency and energy:")
            for solution, freq in sample_frequency.most_common():
                energy = next(energy for sample, energy in sample_data if sample ==
     solution)
                print(f"Sample: {solution}, Frequency: {freq}, Energy: {energy:+.2f
    }")
        return sample_data, sample_frequency

    def check1(self, a):
        N, M, _, _ = self.get_param()
        b = np.array(a).reshape(M, M)
        for i in range(M):
            s = 0
            for j in range(M):
                s += b[i][j]
            if s!=1:
                return False
        for j in range(M):
            s = 0
            for i in range(M):
                s += b[i][j]
            if s!=1:
                return False
        return True

    def check2(self, a):
        N, M, S, _ = self.get_param()
        b = np.array(a).reshape(N, N)
        for i in range(N):
            s = 0
            for j in range(N):
                s += b[i][j]
            if s!= S:
                return False
        #
        for j in range(N):
            s = 0
            for i in range(N):
                s += b[i][j]
            if s!= S:
                return False
        #
        s = 0
        for i in range(N):
            for j in range(N):
                if i==j:
                    s += b[i][j]
        if s!=S:
            return False
        #
        s = 0
        for i in range(N):
            k = N-i-1
            s += b[i][k]
        if s!=S:
            return False
        #
        return True

    def decode(self, a):
        N, M, _, _ = self.get_param()
        b = np.array(a).reshape(M, M)
        mat = []
        for i in range(M):
            for j in range(M):
                if b[i][j]==1:
                    mat.append(j+1)
        return mat

if __name__ == '__main__':
    N = 3
    mc = MagicCircle(N)
    lagrange1 = 10.0
    lagrange2 = 10.0
    lagrange3 =  1.0
    Q = mc.f(lagrange1, lagrange2, lagrange3)
    num_reads = 10000
    sampleset = mc.solv(Q, num_reads)
    #ans = mc.result(sampleset)
    #print(*ans, sep='\n')
    #
    for sample in sampleset.record['sample']:
        if mc.check1(sample):
            a = mc.decode(sample)
            if mc.check2(a):
                print(np.array(a).reshape(N, N))
                print()
```

プログラム 1.1　魔方陣

参考文献

[1] 西森秀稔、大関真之, 量子アニーリングの基礎, 共立出版

[2] https://zenn.dev/luna_moonlight/articles/38de858bdc855f