Note on the chasing of zero diagonal elements of bidiagonal matrices

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We briefly describe a bulge-chasing procedure (see, e.g., [1]) to isolate the zero singular values of a square bidiagonal matrix using orthogonal similarity transformations. The procedure requires Givens rotations. We recall that given a vector $\mathbb{R}^2 \ni [a\ b]^T \neq 0$, the Givens rotation $G_{a,b}$ is defined by

$$G_{a,b}\begin{bmatrix} a \\ b \end{bmatrix} := \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} \sqrt{a^2+b^2} \\ 0 \end{bmatrix}. \tag{1}$$

Given a matrix $n \times n$ matrix $\left[\begin{smallmatrix} 0 & -\widetilde{B}^T \\ \widetilde{B} & 0 \end{smallmatrix}\right]$ where $\widetilde{B}$ is $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$, bidiagonal and has $\lceil \frac{r}{2} \rceil$ zero singular values, the goal is to obtain $\left[\begin{smallmatrix} 0 & 0 & -B^T \\ 0 & 0_r & 0 \\ B & 0 & 0 \end{smallmatrix}\right]$ where B is bidiagonal and full rank. We assume $\widetilde{B}$ to be square because a bulge-chasing procedure described in [2] allows to eliminate the zero singular value associated with n odd. If $\widetilde{B}$ is not full rank, then there is at least one zero diagonal element since $\widetilde{B}$ is upper triangular. This zero can be isolated by the bulge chasing procedure described below where we consider a small matrix example. The method readily extends to higher dimensions. We assume that Givens rotations are extended to match matrix dimensions that are not 2×2. The pairs of scalars eliminated by the successive Givens rotations are colored in blue.

$$\widetilde{B} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & \beta_2 & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_4 \\ 0 & 0 & 0 & 0 & \alpha_4 \end{bmatrix} \xrightarrow{\bullet G^T_{\alpha_2,\beta_2}} \begin{bmatrix} \alpha_1 & \widetilde{\beta}_1 & \gamma_1 & 0 & 0 \\ 0 & \widetilde{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_4 \\ 0 & 0 & 0 & 0 & \alpha_4 \end{bmatrix} \xrightarrow{\bullet G^T_{\alpha_1,\gamma_1}}$$

$$\begin{bmatrix} \widetilde{\alpha}_1 & \widetilde{\beta}_1 & 0 & 0 & 0 \\ 0 & \widetilde{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_3 & 0 \\ 0 & 0 & 0 & \alpha_3 & \beta_4 \\ 0 & 0 & 0 & 0 & \alpha_4 \end{bmatrix} \xrightarrow{G_{\alpha_3,\beta_3}\bullet} \begin{bmatrix} \widetilde{\alpha}_1 & \widetilde{\beta}_1 & 0 & 0 & 0 \\ 0 & \widetilde{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma_3 \\ 0 & 0 & 0 & \widetilde{\alpha}_3 & \widetilde{\beta}_4 \\ 0 & 0 & 0 & 0 & \alpha_4 \end{bmatrix} \xrightarrow{G_{\alpha_4,\gamma_3}\bullet} \begin{bmatrix} \widetilde{\alpha}_1 & \widetilde{\beta}_1 & 0 & 0 & 0 \\ 0 & \widetilde{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widetilde{\alpha}_3 & \widetilde{\beta}_4 \\ 0 & 0 & 0 & 0 & \widetilde{\alpha}_4 \end{bmatrix}.$$

After row and column permutation of the last matrix, we can extract the submatrix

$$B = \begin{bmatrix} \widetilde{\alpha}_1 & \widetilde{\beta}_1 & 0 & 0 \\ 0 & \widetilde{\alpha}_2 & 0 & 0 \\ 0 & 0 & \widetilde{\alpha}_3 & \widetilde{\beta}_4 \\ 0 & 0 & 0 & \widetilde{\alpha}_4 \end{bmatrix} \text{ with } \widetilde{B} = G^T_{\alpha_3,\beta_3} G^T_{\alpha_4,\gamma_3} P_1^T \begin{bmatrix} 0 & 0_1 \\ B & 0 \end{bmatrix} P_2 G_{\alpha_1,\gamma_1} G_{\alpha_2,\beta_2}.$$

If we define $G = \begin{bmatrix} G^T_{\alpha_2,\beta_2} G^T_{\alpha_1,\gamma_1} P_2^T & 0 \\ 0 & G^T_{\alpha_3,\beta_3} G^T_{\alpha_4,\gamma_3} P_1^T \end{bmatrix}$, it follows that

$$\begin{bmatrix} 0 & -\widetilde{B}^T \\ \widetilde{B} & 0 \end{bmatrix} = G \begin{bmatrix} 0 & 0 & -B^T \\ 0 & 0_2 & 0 \\ B & 0 & 0 \end{bmatrix} G^T. \tag{2}$$

In higher dimensions, if B is still not full rank, the method can recursively be applied on the top left and the bottom right blocks of B to isolate more zero singular values.

References

[1] Raf Vandebril. Chasing bulges or rotations? A metamorphosis of the QR-algorithm. *SIAM Journal on Matrix Analysis and Applications*, 32(1):217–247, 2011.

[2] R. C. Ward and L. J. Gray. Eigensystem computation for skew-symmetric and a class of symmetric matrices. *ACM Trans. Math. Softw.*, 4(3):278–285, sep 1978.