Derivatives: An Overview

Based on the book "Options, Futures & Other Derivatives" by John C. Hull

Stefano Mauloni

July 2024

Contents

1 Markets

1.1 Types of derivatives markets

Financial securities can be exchanged on Exchange-Traded or Over-The-Counter markets (OTC).

In the former, the Exchange provides standardized contracts to traders. Traders do not have to worry about credit risk since the exchange clearing house acts as an intermediary body and takes care of that for both parts of the contract. Usually, this is done by requiring a margin (deposit funds) for both traders.

On the other hand, OTC markets are the ones in which the majority of the derivatives trade takes part. The main participants in OTC derivatives markets are banks, large financial institutions, and corporations.
Once an OTC trade has been agreed upon, the two parties can either present it to a central counterparty (CCP) or clear the trade bilaterally. A CCP is like an exchange clearing house.
Often, banks act as market makers for the most common instruments. Thus, they quote bid and ask prices for them.
Nowadays OCT markets are heavily regulated to improve transparency and reduce systemic risk, but before 2007 regulations were little to none.

1.2 Contracts on derivatives markets

The following are some important types of contracts that one can encounter when dealing with derivatives.

1.2.1 Forwards

A forward contract is an agreement between two parties to buy/sell an asset at a certain time in the future for a certain price, and they are usually traded OTC. When the contract acts at an immediate time, we often refer to those contracts as spot contracts.
Parties can assume a long (resp. short) position if they want to buy (resp. sell) the underlying asset. The other party necessarily assumes the other position.
The payoff of a forward for one unit of the asset is:

$$sign(P)(S_T - K)$$

where S_T is the spot price of the asset at maturity T, K is the delivery price of the forward, P is positive if we have a long position, negative otherwise.
Note that the payoff is equal to the total gain or loss from the contract since the cost of entering a forward contract is zero.

1.2.2 Futures

Like a forward contract, a futures contract is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. Unlike forward contracts, futures contracts are normally traded on an exchange. To make trading possible, the exchange specifies standardized features of the contract and there is an exchange clearing house between the parties.

Two large exchanges on which futures contracts are traded are the Chicago Board of Trade (CBOT) and the Chicago Mercantile Exchange (CME), which have now merged to form the CME Group.

1.2.3 Options

Options can be traded in both types of markets. We shall distinguish between the kinds of options:
A first distinction can be made on the right that the option gives to the holder:
Call option = gives the holder the right to **buy** the underlying asset by a certain date for a certain price
Put option = gives the holder the right to **sell** the underlying asset by a certain date for a certain price

Another distinction is the time one can exercise the option. if at time t_0 we buy an option with maturity T. Then the option is: *American* if it can be exercised at any time $\mathbf{t} \in [\mathbf{t_0}, \mathbf{T}]$, *European* if it can be exercised only at time $\mathbf{t} = \mathbf{T}$

Note that one does not necessarily have to exercise the option. It acts like a sort of "insurance" on the price of an underlying asset. Thus, there is a cost to acquire an option. On top of that, Exchanges or institutions OTC quote bid and ask prices with a much larger spread than that for the underlying stock and it depends on the volume of trading.

Given the distinctions we made so far, we can identify four types of positions:
It is possible to *buy* or *sell a call* and *buy* or *sell a put*. In jargon, selling the option is often referred to as *writing the option*.

As a side note, it can be useful to know that In the United States, an option contract is a contract to buy or sell 100 shares.

1.3 Subjects acting on derivatives markets

In a market, we can identify three categories of traders: hedgers, speculators, and arbitrageurs.

1.3.1 Hedgers

Hedgers use derivatives to reduce the risk that they face from potential future movements in a market variable.
They can use both forwards or options. The fundamental difference between them when hedging is that Forward contracts are designed to neutralize risk by fixing the price that the hedger will pay or receive for the underlying asset. Option contracts, by contrast, provide insurance. They offer a way for investors to protect themselves against adverse price movements in the future while still allowing them to benefit from favorable price movements.
An example of hedging using such contract is by fixing the currency conversion rate at present day to neutralize currency conversion risk.

1.3.2 Speculators

Speculators use derivatives to bet on the future direction of a market variable.
They can decide to use forwards over buying directly the underlying asset at the spot price and selling it afterward. If one makes the calculations, it turns out that using forwards results in a worse outcome in both cases (right & wrong prediction of the market direction). The reason is simple: if one buys the asset at a spot price, a great amount of liquidity is needed. On the other hand, the future requires only a fraction of the liquidity (that is, the margin) and one can leverage the position. Also, one should account for the "risk-free" interest rate earned or paid for the liquidity that is not immediately used in the case of the forward. (*See section* **??** *for a definition of "risk-free"*).
There is a difference between options and futures: when a speculator uses futures, the potential loss as well as the potential gain is very large. When options are purchased, no matter how bad things get, the speculator's loss is limited to the amount paid for the options.

1.3.3 Arbitrageurs

Arbitrageurs take offsetting positions in two or more instruments — even in different markets — to lock in a riskless profit.
In general, (mathematical) arbitrage opportunities cannot last for long. For this reason, in the book most of the arguments concerning derivatives are made under the assumption that no arbitrage opportunity exist.

2 Futures market

2.1 Specification of the contract

The exchange specifies the **asset** and the **contract size**.
When the asset is a commodity, there may be variations in the quality of what is available in the marketplace. When the asset is specified, the exchange stipulate the grade or grades of the commodity that are acceptable and, in some cases, the difference in prices of the various grades.
The exchange specifies also:
Contract size: Specifies the amount of the asset that has to be delivered under one contract.
Delivery arrangements: the place where delivery will be made.
Delivery months: the precise period during the month when delivery can be made.
Price Limits and Position Limits: for most contracts, daily price movement limits are specified, to prevent large price movements from occurring because of speculative excesses. Position limits are the maximum number of contracts that a speculator may hold.
Limit up/down: price movement up/down whose magnitude is equal to the daily price limit.

2.2 Convergence of futures price to spot price

As the delivery period for a futures contract is approached, the futures price converges to the spot price of the underlying asset. When the delivery period is reached, the futures price equals — or is very close to — the spot price. If this was not the case, than there would exist a clear arbitrage opportunity.

2.3 must continue... WIP

3 Hedging with futures

WIP

4 Interest Rates

4.1 Types of Rate

One important factor influencing interest rates is credit risk. This is the risk that there will be a default by the borrower of funds. The extra amount added to a risk-free interest rate to allow for credit risk is known as a credit spread.

Treasury Rates Rates an investor earns on Treasury bills and Treasury bonds (more in general, bonds from a developed country). It is usually assumed that there is no chance that the government of a developed country will default on an obligation denominated in its own currency. A developed country's Treasury rates are therefore regarded as risk-free.

Overnight Rates At the end of a day, some financial institutions typically have surplus funds in their accounts with the central bank while others have requirements for funds, given the requirements of reserves that the central bank makes. This leads to borrowing and lending overnight. There are many rates, depending on the country of interest: effective federal funds rate (US), SONIA (UK), ESTER (EU), etc...

Repo Rates Unlike the overnight federal funds rate, repo rates are secured borrowing rates.
In a repo (or repurchase agreement), a financial institution that owns securities agrees to sell the securities for a certain price and buy them back at a later time for a slightly higher price. The interest rate is referred to as the *repo rate*. If structured carefully, a repo involves very little credit risk.

4.2 Reference Rates

In a financial transaction, the parties frequently enter into contracts where the future interest rate paid or received is set equal to the value of an agreed reference interest rate.

LIBOR = London Interbank Offered Rate.
It is set by asking a panel of global banks to provide quotes estimating the unsecured rates of interest at which they could borrow from other banks just prior to 11 a.m. (U.K. time). Thus, they are estimates of unsecured borrowing rates for creditworthy banks.
A problem with LIBOR is that there is not enough borrowing between banks for a bank's estimates to be determined by market transactions. As a result. it can be subject to manipulation. Bank regulators are uncomfortable with this and have developed plans to phase out the use of LIBOR.

New Reference The plan is to base reference rates on the overnight rates (like ESTER). [Note that in the US the rate used will be SOFR, based on repo rates, thus they are secured rates].
Longer rates such as three-month rates, six-month rates, or one-year rates can be determined from overnight rates by compounding them daily:

$$\left(\prod_{i=1}^{n}(1 + r_i\hat{d}_i)\right) - 1 * \frac{360}{D}$$

With d_i = number of days to which r_i is applied, $\hat{d}_i = \frac{d_i}{D}$, and $D = \sum_i d_i$.
e.g. rate on friday is assumed to be applied also on saturday and sunday.

The new reference rates are regarded as risk-free because they are derived from one-day loans to creditworthy financial institutions. LIBOR, by contrast, incorporates a credit spread. Since credit spreads increase in stessed market conditions, the new risk-free reference rates may also be augmented by credit spread measures in the future.
Also, LIBOR rates are forward looking. They are determined at the beginning of the period to which they will apply. The new reference rates are backward looking: The rate applicable to a particular period is not known until the end of the period when all the relevant overnight rates have been observed.

4.3 Risk-free Rate

the usual approach to valuing derivatives involves setting up a riskless portfolio and arguing that the return on the portfolio should be the risk-free rate.
For that purpose, Treasury rates are are artificially low. Therefore, The risk-free reference rates created from from overnight rates are the ones used in valuing derivatives.

4.4 Measure interest rates - Compounding

The compounding frequency defines the units in which an interest rate is measured. A rate expressed with one compounding frequency can be converted into an equivalent rate with a different compounding frequency.

Given an amount A invested for n years at interest rate R:

Discrete compounding The interest rate is compounded m times a year. Then at the end of the period we get:

$$A\left(1+\frac{R}{m}\right)^{mn}$$

When $m = 1$ the rate is the *equivalent annual interest rate.*

If we want to derive the equivalent rate given a compounding m_2, we can just solve

$$A\left(1+\frac{R_1}{m_1}\right)^{m_1 n} = A\left(1+\frac{R_2}{m_2}\right)^{m_2 n} \implies R_2 = m_2\left[\left(1+\frac{R_1}{m_1}\right)^{\frac{m_1}{m_2}} - 1\right]$$

Continuous compounding At the limit $lim_{m\to 0}$, we get that A, at the end of the period, becomes

$$Ae^{Rn}$$

For practical purposes, one can think of it as being equivalent to daily compounding. To get the equivalent rate with discrete compounding m, we solve:

$$Ae^{R_c n} = A\left(1+\frac{R_m}{m}\right)^{mn} \implies R_m = m(e^{\frac{R_c}{m}} - 1)$$

On the other hand, continuous compounding is widely used for derivative pricing. From now on, every interest rate is to be intended as continuously compounded, if not specified.

If we want to discount the rate, we simply compute Ae^{-Rn}

4.5 Zero-Rates

The **n-year zero-coupon interest rate** is the rate of interest earned on an investment that starts today and lasts for n years. All the interest and principal is realized at the end of n years. *There are no intermediate payments.* Sometimes it is also called the n-year spot rate, the n-year zero rate, or just the n-year zero.
In the market, most rates are not pure zero rates.

4.6 Bond Pricing

A bond's principal — or face/par value — is the amount that is paid at the end of its life. Thus, the theoretical price of a bond should be the present value plus the (discounted) cash flow that will come from the bond ownership over time.
To discount cashflows, one can use the same rate as the bond's one, but a more accurate approach is to use a different zero-rate for each cash flow.

Bond Price The bond price can be calculated as:

$$B = \sum_{i=1}^{N} c_i e^{-z_{fi} * fi}$$

c_i is the value of the coupon that the bond pays at period i, f is the frequency of the coupon (annual, semiannual, etc...), and z_{fi} is the zero-rate with maturity fi.

For every coupon, we pick a different zero-rate and we compound it for its respective maturity.
[Note that c_N is equal to the face value of the bond plus the last coupon, if present].

Bond Yield A bond's yield is the single discount rate that, if applied to all cashflows, gives a bond price equal to its market price. If we let M be the market price of the bond, we write:

$$\sum_{i=1}^{N} c_i e^{-\mathbf{y}*fi} = M$$

Which can be solved using numerical methods (i.e. Newton-Raphson)

Par Yield The par yield for a certain bond maturity is the coupon rate that causes the bond price to equal its par value.

$$\sum_{i=1}^{N-1} \mathbf{c}e^{-z_{fi}*fi} + (Par\ Value + \mathbf{c})e^{-z_{fN}*fN} = Par\ Value$$

4.7 Determination of Zero Rates

A chart showing the zero rate as a function of maturity is known as the zero curve. A common assumption is that the zero curve is linear between the points determined using the bootstrap method.

Boothstrap method In practice, we do not usually have bonds with maturities equal to exactly 1.5 years, 2 years, 2.5 years, and so on. One approach is to interpolate between the bond price data before it is used to calculate the zero curve. Assume a piecewise linear curve with corners at these times. Use an iterative "trial and error" procedure to determine the rate at time t1 that matches the price of the first instrument, then use a similar procedure to determine the rate at time t2 that matches the price of the second instrument, and so on. For any trial rate, the rates used for coupons are determined by linear interpolation. A more sophisticated approach is to use polynomial or exponential functions, rather than linear functions, for the zero curve between times ti and ti+ 1 for all i. The functions are chosen so that they price the bonds correctly and so that the gradient of the zero curve does not change at any of the ti. This is referred to as using a spline function for the zero curve.

4.8 Forward Rates

Forward interest rates are the rates of interest implied by current zero rates for periods of time in the future.
e.g. when considering a 2-years timespan, it is the rate for year 2 that, when combined with the zero-rate for year 1, gives the zero-rate at 2 years from today (compounded for 2 years).

$$A(1 + z_1)(1 + \mathbf{x}) = A(1 + z_2)^2$$

When interest rates are continuously compounded and rates in successive time periods are combined, the overall equivalent rate is simply the average rate during the whole period.
Assume R_1, R_2 are zero rates for maturities T_1 and T_2, respectively. R_F is the forward interest rate for the period of time between T_1 and T_2. Then:

$$R_F = \frac{R_2T_2 - R_1T_1}{T_2 - T_1} \implies R_F = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1} \implies R_F = R + \frac{\partial R}{\partial T} \quad as \quad T_2 \to T_1$$

Where R is the zero-rate for a maturity of T and R_F is the *instantaneous forward rate* for a maturity of T.
This is the forward rate applicable to a very short future time period that begins at time T. Defining $P(0, T) = e^{-RT}$, the equation becomes

$$R_F = -\frac{\partial}{\partial T} \ln P(0, T)$$

This shows that, if the zero curve is upward sloping between T_1 and T_2 so that $R_2 > R_1$, then $R_F > R_2$ (i.e., the forward rate for a period of time ending at T_2 is greater than the T_2 zero rate) and viceversa.

If a large investor thinks that rates in the future will be different from today's forward rates, there are many trading strategies that the investor can explore. One of these involves entering into a contract known as a forward rate agreement (FRA).

4.9 Forward Rate Agreements

...

4.10 Duration

The duration of a bond measures how long the holder of the bond has to wait before receiving the present value of the cash payments. For instance, a zero-coupon bond that lasts n years has a duration of n years.
Recalling from the formula for the bond price B and bond yield y, with cashflows c_i:

$$B = \sum_{i=1}^{N} c_i e^{-y*t_i}$$

Then the *duration* is defined as:

$$D = \sum_{i=1}^{N} t_i \left[\frac{c_i e^{-y*t_i}}{B} \right]$$

In square brackets is the ratio of the present value of the cashflow at time t_i to the bond price.

The duration is therefore a weighted average of the times when payments are made, with the weight applied to time t_i being equal to the proportion of the bond's total present value provided by the cash flow at time t_i.

When we consider a small change in yield Δy, then

$$\Delta B \approx -\Delta y \sum_{i=1}^{N} c_i t_i e^{-y t_i}$$

From this we can retrieve the key duration relationship:

$$\Delta B \approx -\Delta y B D \implies \frac{\Delta B}{B} \approx -\Delta y D$$

Which is an approximate relationship between percentage changes in a bond price and changes in its yield. It is easy to use and is the reason why duration has become such a popular measure.

For the sake of definitions, $DV01$ is the price change from a 1-basis-point increase in all rates, while **Gamma** is the change in $DV01$ from a 1-basis-point increase in all rates.

Modified Duration can be used only when the interest rate is compounded m times per year ($m \in \mathbb{N}$). In this case,

$$\Delta B = -\frac{BD\Delta y}{1 + y/m}$$

We define D^* as the *bond's modified duration.*

$$D^* = \frac{D}{1 + y/m} \quad \implies \quad \Delta B = -BD^* \Delta y$$

Modified duration calculation gives good accuracy for small yield changes.

Another term that is sometimes used is *dollar duration*: $D_\$ = D^* B$, so that $\Delta B = -D_\$ \Delta y$

Duration of a bond portfolio The duration D of a bond portfolio is defined as a weighted average of the duration of the single bonds in the portfolio, with weights being proportional to the bond prices. Then, the previous equations apply (still for small changes in Δy).
It is essential to note that, in this case, we are implicitly assuming that the yields of all bonds in the portfolio change by almost the same amount. When bonds have different maturities, this happens only when there is a parallel shift in the zero-coupon yield curve. Hence, the equations apply for small, parallel shifts of Δy in the zero curve.
This implies that, if the net duration of the assets in a portlofio is zero, then we eliminate exposure to parallel shifts in the yield curve, but we are still exposed to either nonparallel or large deviations.

4.11 Convexity

Duration applies only to small changes in yields. Two portfolios can have the same duration (gradients of the curves are equal at the origin), but react differently for large yield changes: one has more curvature than the other. *Convexity* measures this curvature.
A measure of convexity is:

$$C = \frac{1}{B}\frac{d^2B}{dy^2} = \frac{1}{B}\sum_{i=1}^{N} c_i t_i^2 e^{-yt_i}$$

Using Taylor series expansion,

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}\Delta y^2 \implies \frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

Given a duration D, the convexity of a bond portfolio tends to be greatest when the portfolio provides payments evenly over a long period of time, while is smaller when payments are concentrated around a point in time.

By choosing a portfolio of assets and liabilities with a net duration of zero and a net convexity of zero, a financial institution can make itself immune to relatively large parallel shifts in the zero curve. However, it is still exposed to nonparallel shifts.

4.12 Theories of The Term Structure of Interest Rates

It is natural to ask what determines the shape of the zero curve. Why is it sometimes downward sloping, sometimes upward sloping, and sometimes partly upward sloping and partly downward sloping?
expectations theory,
market segmentation theory,
liquidity preference theory,
...WIP