Max Shi HW 11

I pledge my honor that I have abided by the Stevens Honor System

1. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

sine series $\Rightarrow a_n = 0$, $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$L = 2$

$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

$= \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx$ using $\int u\sin(bu)\,du = \frac{1}{b^2}(-bu\cos(bu) + \sin(bu))$

$= \left[\frac{4}{n^2\pi^2}\left(-\frac{n\pi}{2}x\cos\left(\frac{n\pi x}{2}\right) + \sin\left(\frac{n\pi}{2}x\right)\right)\right]_0^1$

$= \frac{4}{n^2\pi^2}\left(-\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) + 0 - 0\right)$

$= \frac{4}{n^2\pi^2}\left(\sin\left(\frac{n\pi}{2}\right) - \frac{n\pi}{2}\cos\frac{n\pi}{2}\right)$

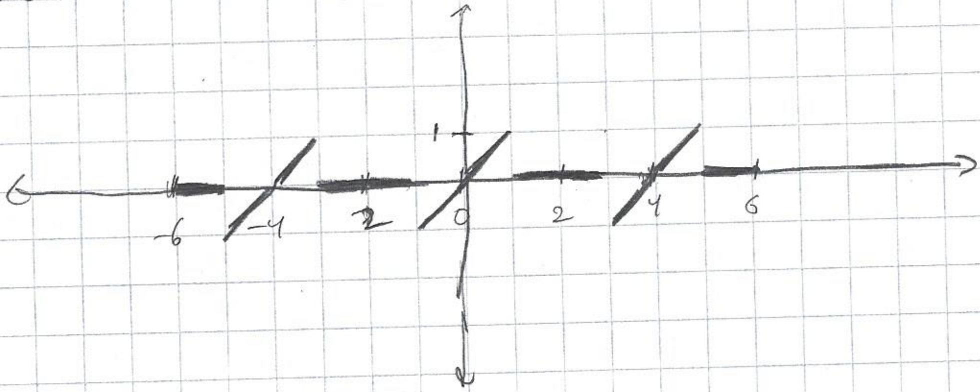
$n=1 \Rightarrow \frac{4}{\pi^2}\left(\sin\frac{\pi}{2} - \frac{\pi}{2}\cos\frac{\pi}{2}\right) = \frac{4}{\pi^2}$

$n=2 \Rightarrow \frac{4}{4\pi^2}\left(\sin\pi - \frac{2\pi}{2}\cos\pi\right) = \frac{1}{\pi^2}(-\pi)(-1) = \frac{1}{\pi}$

$n=3 \Rightarrow \frac{4}{9\pi^2}\left(\sin\frac{3\pi}{2} - \frac{3\pi}{2}\cos\frac{3\pi}{2}\right) = \frac{-4}{9\pi^2}$

$n=4 \Rightarrow \frac{4}{16\pi^2}\left(\sin 2\pi - 2\pi(\cos 2\pi)\right) = \frac{1}{4\pi^2}(-2\pi(1)) = \frac{-1}{2\pi}$

$f(x) \sim \frac{4}{\pi^2}\sin\left(\frac{\pi x}{2}\right) + \frac{1}{\pi}\sin(\pi x) - \frac{4}{9\pi^2}\sin\left(\frac{3\pi}{2}x\right) - \frac{1}{2\pi}\sin(2\pi x)$



let $f_p(x)$ be the periodic extension

$f_p(x) = -1/2$ at $x = \{-5, -1, 3\}$

$f_p(x) = 1/2$ at $x = \{-3, 1, 5\}$

2. $u_t = 2u_{xx}$ $\quad u_{xx} = X''T \quad u_t = T'X$

(a) $XT' = 2X''T \Rightarrow \frac{T'}{2T} = \frac{X''}{X} = -\lambda \Rightarrow \frac{T'}{2T} = -\lambda,\ \frac{X''}{X} = -\lambda$

$T' = -2\lambda T$
$T'' + 2\lambda T = 0$

$X'' = -\lambda X$
$X'' + \lambda X = 0.$

(b) $\frac{dT}{dt} = -2\lambda T$

$\int \frac{dT}{T} = \int -2\lambda\, dt$
$\ln T = -2\lambda t$
$T = ce^{-2\lambda t}$

$m^2 + \lambda = 0$
$m = \sqrt{-\lambda}$

$-\lambda = 0 \Rightarrow m = 0,\ 0 = X = c_1 + c_2 x$
$X'(0) = 0,\ X(2\pi) = 0$
$X' = c_2$
$0 = c_2 \Rightarrow c_2 = 0$
$0 = c_1 + c_2(2\pi)$
$0 = c_1 \Rightarrow$ trivial sol.

$-\lambda = \alpha^2 > 0$
$m = \pm\alpha$
$X = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$
$X(2\pi) = 0$
$0 = c_1 e^{\alpha 2\pi} + c_2 e^{-\alpha 2\pi}$

$X' = \alpha c_1 e^{\alpha x} - \alpha c_2 e^{-\alpha x}$
$X'(0) = 0$
$0 = \alpha c_1 - \alpha c_2$
$\alpha c_1 = \alpha c_2 \Rightarrow c_1 = c_2$

$0 = c_1 e^{\alpha 2\pi} + c_1 e^{-\alpha 2\pi}$
$\Rightarrow c_1 e^{\alpha 2\pi} = -c_1 e^{-\alpha 2\pi}$
c_1 must $= 0$, so trivial solution

$-\lambda = -\alpha^2 < 0$
$m = \pm\alpha i \Rightarrow X = c_1 \sin\alpha x + c_2 \cos\alpha x.$
$X' = \alpha c_1 \cos\alpha x - \alpha c_2 \sin\alpha x$
$X'(0) = 0.$
$0 = \alpha c_1 \cos 0 - \alpha c_2 \sin 0.$
$c_1 = 0.$
$X = c_2 \cos\alpha x,\ X(2\pi) = 0.$
$0 = c_2 \cos(2\pi\alpha),\ c_2 \neq 0.$
$n\pi - \frac{\pi}{2} = 2\pi\alpha$
$\frac{n}{2} - \frac{1}{4} = \alpha,\quad \lambda = \left(\frac{n}{2} - \frac{1}{4}\right)^2$

$T_n = ce^{-2\left(\frac{n}{2}-\frac{1}{4}\right)^2 t}$

$X_n = c_n \cos\left(\left(\frac{n}{2}-\frac{1}{4}\right)x\right)$

$u(x,t) = \sum_{n=1}^{\infty} c e^{-2\left(\frac{n}{2}-\frac{1}{4}\right)^2 t} \cos\left(\left(\frac{n}{2}-\frac{1}{4}\right)x\right)$

(c) Cos Series:
$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$

$L = 2\pi \Rightarrow a_n = \frac{1}{\pi}\int_0^{\pi} \cos\left(\frac{nx}{2}\right) dx. = \frac{2}{\pi n}\left[\sin\left(\frac{nx}{2}\right)\right]_0^{\pi} = \frac{2}{\pi n}\left(\sin\frac{\pi n}{2} - \sin 0\right)$
$= \frac{2}{\pi n}\sin\left(\frac{\pi n}{2}\right)$

$a_0 = \frac{1}{\pi}\int_0^{\pi} 1\, dx = 1 \Rightarrow \frac{a_0}{2} = \frac{1}{2}$

$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n} \sin\left(\frac{\pi n}{2}\right) e^{-2\left(\frac{n}{2}-\frac{1}{4}\right)^2 t} \cos\left(\left(\frac{n}{2}-\frac{1}{4}\right)x\right)$

$u(x,t) \sim \frac{1}{2} + \frac{2}{\pi} e^{-\frac{t}{8}} \cos\left(\frac{x}{4}\right) - \frac{2}{3\pi} e^{-\frac{25t}{8}} \cos\left(\frac{5}{4}x\right) + \frac{2}{5\pi} e^{-\frac{81t}{8}} \cos\left(\frac{9}{4}x\right)$

$n=1 \qquad n=3 \qquad n=5$

3. $T''X = 16X''T$

$\frac{T''}{16T} = \frac{X''}{X} = -\lambda$

$T'' = -\lambda 16T$ $\quad X'' = -\lambda X$

$T'' + \lambda 16T = 0$ $\quad X'' + \lambda X = 0$ $\quad m^2 = \sqrt{-\lambda}$

$X(0) = 0,\ X(2) = 0.$

$-\lambda = 0.$ $\quad -\lambda = \alpha^2 > 0.$

$X = c_1 + c_2 t$ $\quad X = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$0 = c_1 + c_2(0)$ $\quad 0 = c_2(2)$ $\quad 0 = c_1 + c_2$ $\quad 0 = c_1 e^{\alpha 2} - c_1 e^{-\alpha 2}$

$c_1 = 0.$ $\quad c_2 = 0.$ $\quad c_1 = -c_2$ $\quad c_1 e^{2\alpha} = c_1 e^{-2\alpha}$

trivial. $\quad c_1 = 0,\ c_2 = 0$, trivial solution

$-\lambda = -\alpha^2 \leq 0.$

$x = c_1 \sin \alpha x + c_2 \cos \alpha x$ $\quad X = c_1 \sin \alpha x$

$0 = c_1 \sin(0) + c_2 \cos(0)$ $\quad 0 = c_1 \sin 2\alpha$

$c_2 = 0.$ $\quad n\pi = 2\alpha$

$\frac{n\pi}{2} = \alpha,\ \lambda = \left(\frac{n\pi}{2}\right)^2$ $\quad X_n = c_n \sin\left(\frac{n\pi}{2}x\right)$

$T'' + \lambda 16T = 0$

$m^2 = -16\lambda$

$m = \sqrt{-16\lambda}$

$m = \sqrt{-16\left(\frac{n\pi}{2}\right)^2}$

$m = 4\frac{n\pi}{2} i = 2\pi n$

$T = c_1 \sin(2\pi n t) + c_2 \cos(2\pi n t)$

$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \sin(2\pi n t) + B_n \cos(2\pi n t)\right] \sin\left(\frac{n\pi}{2}x\right)$

$u(x,0) = \sum_{n=1}^{\infty} B_n \cos(2\pi n(0)) \sin \frac{n\pi}{2}x$

$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{2}x\right)$

$\frac{\sin\left(\frac{\pi x}{2}\right)}{4} - \frac{\sin\left(\frac{3\pi}{2}x\right)}{16} = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{2}x$

$n=1$ $\quad n=3$

$B_1 = \frac{1}{4},\ B_3 = -\frac{1}{16},\ B_n = 0,\ n \neq 1, 3.$

$u_t(x,t) = \sum_{n=1}^{\infty} 2\pi n\left(a_n \cos(2\pi n t) - B_n \sin(2\pi n t)\right) \sin\left(\frac{n\pi}{2}x\right)$

$\frac{\sin\frac{\pi x}{2}}{4} - \frac{\sin\left(\frac{5\pi x}{2}\right)}{20} = \sum_{n=1}^{\infty} 2\pi n\, a_n \sin\left(\frac{n\pi}{2}x\right)$

$n=1$ $\quad n=5$

$$\boxed{u(x,t) = \left(\frac{1}{8\pi}\sin 2\pi t + \frac{1}{4}\cos 2\pi t\right)\sin\left(\frac{\pi x}{2}\right) - \frac{1}{16}\cos(6\pi t)\sin\left(\frac{3\pi x}{2}\right) - \frac{1}{200\pi}\sin(10\pi t)\sin\left(\frac{5\pi x}{2}\right)}$$

$2\pi a_1 = \frac{1}{4}$ $\quad 10\pi a_5 = -\frac{1}{20}$

$a_1 = \frac{1}{8\pi}$ $\quad a_5 = -\frac{1}{200\pi}$ $\quad a_n = 0,\ n \neq 1, 5$