Max Shi MA227 HW 6. I pledge my honor that I have abided by the Stevens Honor System. [signature]

1 (a) By the right hand rule, we can say $\vec{w} \times \vec{r}$ is parallel with the xy plane and perpendicular to the line to the z-axis. Thus, $\vec{w}\times\vec{r}$ and $\vec{v}$ have the same direction, as $\vec{v}$, as a rotation vector, is tangent to the line to the z-axis and parallel to the xy-plane.

Next, as $\omega = \frac{|\vec{v}|}{d}$, by definition, $|\vec{v}| = \omega d$.
$|\vec{w}\times\vec{r}| = |w||r|\sin\theta$, and using θ, $|d| = |r|\sin\theta$. Thus,
$|\vec{w}\times\vec{r}| = \omega d = |\vec{v}|$.
As magnitude and direction are the same, these two vectors are equal.

(b) $\vec{w} = \langle 0, 0, \omega \rangle$
$\vec{r} = \langle x, y, z \rangle$

$$|\vec{w}\times\vec{r}| = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = (0-\omega y)i - (0-\omega x)j + (0-0)k$$
$$= \boxed{-\omega y i + \omega x j}$$

(c) $$\text{curl}\,\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}\omega x\right)i - \left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}(-\omega y)\right)j$$
$$+ \left(\frac{\partial}{\partial x}\omega x - \frac{\partial}{\partial y}(-\omega y)\right)k$$
$$= (0-0)i - (0-0)j + (\omega + \omega)k$$
$$= 2\omega k = \boxed{2\vec{w}.}$$

2 (a) $\iint_{S_2} \vec{F}\cdot d\vec{S} = \iiint_{S_2} \text{div}\,F\, dV.$

$\text{div}\,F = \frac{\partial}{\partial x}(ze^y) + \frac{\partial}{\partial y}(z^3\ln(x^2)) + \frac{\partial}{\partial z}(z)$
$= z$

Using cylindrical coordinates:
$1 \le z \le 2-r^2$
$0 \le \theta \le 2\pi$
$0 \le r \le 1$

$$\int_0^{2\pi}\int_0^1\int_1^{2-r^2} z\, r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left[\frac{1}{2}z^2\right]_1^{2-r^2} r\,dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 ((2-r^2)^2 - 1)\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^1 (4-4r^2+r^4-1)\,r\,dr\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (3r - 4r^3 + r^5)\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\left[\frac{3r^2}{2} - r^4 + \frac{r^6}{6}\right]_0^1$$
$$= \frac{1}{2}\int_0^{2\pi}\left(\frac{3}{2} - 1 + \frac{1}{6}\right)d\theta = \boxed{\pi\left(\frac{2}{3}\right)}$$

(b) As S_1 is oriented positively but the positive orientation of S_2 has S_1 oriented negatively:

$$\boxed{\iint_S \vec{F}\cdot d\vec{s} - \iint_{S_1} \vec{F}\cdot d\vec{s} = \iint_{S_2} \vec{F}\cdot d\vec{s}}$$

(c) $\iint_{S_1} \vec{F}\cdot d\vec{s}$:

$$S(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 1\rangle \qquad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi$$

$$S_r = \langle \cos\theta,\ \sin\theta,\ 0\rangle$$

$$S_\theta = \langle -r\sin\theta,\ r\cos\theta,\ 0\rangle$$

$$S_r \times S_\theta = (0-0)i - (0-0)j + (r\cos^2\theta + r\sin^2\theta)k = rk.$$

$$F(S)\cdot(S_r \times S_\theta) = 0 + 0 + r$$

$$\int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \frac{2\pi}{2} = \boxed{\pi}$$

$$\iint_S \vec{F}\cdot d\vec{s} - \iint_{S_1} \vec{F}\cdot d\vec{s} = \iint_{S_2} \vec{F}\cdot d\vec{s}$$

$$\boxed{\iint_S \vec{F}\cdot d\vec{s} = \pi + \frac{2}{3}\pi = \frac{5}{3}\pi}$$