MA 227-Multivariable Calculus   Hw-4   Due: Thursday March 12th, by 9:00 am

Name (Printed):

Pledge and Sign:

A high quality scan of the solutions in pdf format is to be uploaded to Canvas before the deadline. You need to pledge and sign on the cover page of your solutions. You may use this page as the cover page.

Legibility, organization of the solution, and clearly stated reasoning where appropriate are all important. Points will be deducted for sloppy work or insufficient explanations.

1. **[10 pts.]** Evaluate $\oint_C (x\sin(y^2) - y^2)dx + (x^2 y\cos(y^2) + 3x)dy$, where C is the counterclockwise boundary of the trapezoid with vertices $(0,-2)$, $(1,-1)$, $(1,1)$, and $(0,2)$. If you are using a theorem to evaluate this line integral, you need to quote it. You also need to verify that the hypotheses of theorem are satisfied.

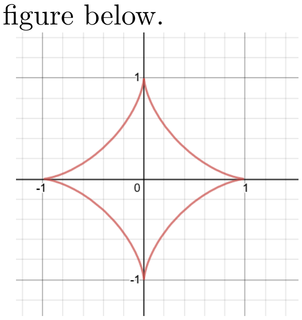
2. **[10 pts.]** Recall that by Green's Theorem, if D is the region enclosed by a piecewise smooth simple closed curve C, then

$$\text{Area of } D = \oint_C xdy = \oint_C -ydx = \oint_C \frac{-y}{2}dx + \frac{x}{2}dy$$

since in all these cases $Q_x - P_y = 1$. Choose (think about your choice) any of the line integrals above to find the area of the region bounded by the curve C given by:

$$C : \vec{\mathbf{r}}(t) = \langle \cos^3 t, \sin^3 t\rangle, \quad 0 \le t \le 2\pi$$

A sketch of the curve is given in the figure below.



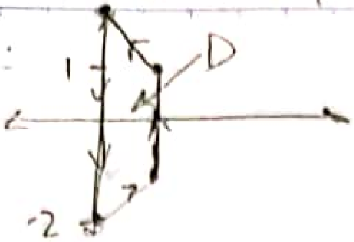
3. **[10 pts.]** For a vector field $\vec{\mathbf{F}}(x,y) = \langle P(x,y), Q(x,y)\rangle$ we know that $Q_x(x,y) - P_y(x,y) = 3$ for all (x,y) on an open set including the region D, which is bounded by the circles $C_1 : x^2+y^2 = 1$ and $C_2 : x^2+y^2 = 4$. Assume both C_1 and C_2 are oriented counterclockwise, and we know that $\oint_{C_1} Pdx + Qdy = 3$. Find the value of $\oint_{C_2} Pdx + Qdy$. [Hint: Use the generalized version of Green's Theorem.]

No.

Date 3 · 12 · 20

MA 227 HW 4 Max Shi I pledge my honor that I have abided by the Stevens Honor System

1. Trapezoid :



$P = x\sin(y^2) - y^2$

$Q = x^2y\cos(y^2) + 3x$

$P_y = 2yx\cos(y^2) - 2y$

$Q_x = 2xy\cos y^2 + 3$

P_y and Q_x are defined and continuous for all x, y, and are therefore defined inside the boundary D.

C is positively oriented, closed, simple, and piecewise smooth. Thus Green's Theorem applies.

$y_{top} = 2-x$

$y_{bottom} = x-2 \Rightarrow \oint_C P\,dx + Q\,dy = \int_0^1 \int_{x-2}^{2-x} Q_x - P_y \, dy\,dx$

$Q_x - P_y = 2xy\cos y^2 + 3 - (2xy\cos y^2 - 2y)$
$= 3 + 2y$

$\int_0^1 \int_{x-2}^{2-x} 3 + 2y \; dy\,dx$

$\int_0^1 [3y + y^2]_{x-2}^{2-x} dx$

$\int_0^1 3(2-x) + (2-x)^2 - 3(x-2) - (x-2)^2 \, dx$

$\int_0^1 6 - 3x + 4 - 4x + x^2 - 3x + 6 - x^2 + 4x - 4 \, dx$

$\int_0^1 12 - 6x \, dx$

$[12x - 3x^2]_0^1 = 12 - 3 = 9.$

2. $dx = 3\cos^2 t(\sin t)$. $dy = 3\sin^2 t\cos t.$

$x\,dy = \cos^3 t(3\sin^2 t\cos t)$

$-y\,dx = \sin^3 t(-\sin(t))(3\cos^2 t))$

$-\frac{y}{2}dx + \frac{x}{2}dy = -\frac{\sin^3 t}{2} \cdot 3\cos^2 t(-\sin t) + \frac{\cos^3 t}{2}(3\sin^2 t\cos t)$

$= +\frac{3}{2}\sin^4 t\cos^2 t + \frac{3}{2}\cos^4 t\sin^2 t$

$= \sin^2 t\cos^2 t\left(\frac{3}{2}\sin^2 t + \frac{3}{2}\cos^2 t\right)$

$= \sin^2 t \cdot \cos^2 t\left(\frac{3}{2}\right)$

$= \frac{3}{2}(\sin^2 t - \sin^4 t)$ ✓

$\int_0^{2\pi} \frac{3}{2}(\sin^2 t - \sin^4 t)\,dt$

$= \frac{3}{2}\int_0^{2\pi} \sin^2 t\,dt - \frac{3}{2}\int_0^{2\pi} \sin^4 t\,dt$

$= \frac{3}{2}\left[\frac{t}{2} - \frac{\cos t\sin t}{2}\right]_0^{2\pi} - \frac{3}{2}\left[-\frac{\cos t\sin^3 t}{4} - \frac{3\cos t\sin t}{8} + \frac{3t}{8}\right]_0^{2\pi}$

$= \frac{3}{2}\left[\frac{\cos t\sin^3 t}{4} - \frac{\cos t\sin t}{8} + \frac{t}{8}\right]_0^{2\pi}$

$= \frac{3}{2}\left[\frac{-\sin 4t + 4t}{32}\right]_0^{2\pi} = \frac{3}{2}\left(\frac{-\sin 8\pi + 8\pi}{32} + \frac{\sin 0 + 0}{32}\right) = \frac{24\pi}{64} = \frac{3}{8}\pi$

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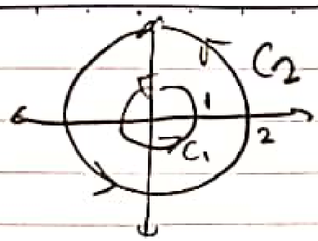
3.



C_1 is CC, but not positively oriented

thus

$$\iint_D Q_x - P_y\, dA = -\oint_{C_1} P\,dx + Q\,dy + \oint_{C_2} P\,dx + Q\,dy.$$

Solving the left side, =>

$$\iint_D Q_x - P_y\, dA = \iint_D 3\, dA$$

convert to polar: . $1 \le r \le 2,\ 0 \le \theta \le 2\pi$

$$\int_0^{2\pi}\int_1^{2} 3\, r\,dr\,d\theta$$

$$\int_0^{2\pi} \left[\frac{3}{2} r^2\right]_1^2 d\theta$$

$$\int_0^{2\pi} (6 - 3/2)\, d\theta = 2\pi \cdot (6 - 3/2) = 9\pi$$

$$9\pi = -3 + \oint_{C_2} P\,dx + Q\,dy$$

$$9\pi + 3 = \oint_{C_2} P\,dx + Q\,dy$$

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