Max Sw   MA227 HW 8   Brady LECH

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- [signature]

1. $4y''+\pi^2 y = g(x),\ 0<x<4,\quad y(0)=0,\ y(4)=0.$

(a) $m^2+\pi^2=0$ $\qquad g(x)=0 \Rightarrow$

$m=\pm\pi i$ $\qquad y_p=0$

$y_c = C_1\sin(\pi x)+C_2\cos(\pi x)$

$y = C_1\sin(\pi x)+C_2\cos(\pi x)$

$y(0)=0$ $\qquad y(4)=0$

$0=C_1\sin(0)+C_2\cos(0)$ $\qquad 0=C_1\sin(4\pi)$

$0=C_2 \Rightarrow C_2=0$ $\qquad 0=0\ \checkmark$

$C_1=\mathbb{R}$

$\boxed{y=C_1\sin(4\pi)}$

(b) $4y''+\pi^2 y=-x^2+4x$

$y_c=C_1\sin(\pi x)+C_2\cos(\pi x)$

$y_p=Ax^2+Bx+C$

$y_p'=2Ax+B$

$y_p''=2A$

$4(2A)+\pi^2(Ax^2+Bx+C)=-x^2+4x$

$8A+\pi^2Ax^2+\pi^2B+\pi^2C=-x^2+4x$

$\pi^2A=-1,\ \pi^2B=4,\ \pi^2C+8A=0$

$A=-\frac{1}{\pi^2},\ B=\frac{4}{\pi^2},\ \pi^2C+8\cdot-\frac{1}{\pi^2}=0$

$C=\frac{8}{\pi^4}$

$y=y_c+y_p$

$y=C_1\sin(\pi x)+C_2\cos(\pi x)-\frac{1}{\pi^2}x^2+\frac{4}{\pi^2}x+\frac{8}{\pi^4}$

$y(0)=0$

$0=C_1\sin 0+C_2\cos 0-0+0+\frac{8}{\pi^4}$

$0=C_2+\frac{8}{\pi^4}\Rightarrow C_2=-\frac{8}{\pi^4}$

$y(4)=0\Rightarrow\quad 0=C_1\sin 4\pi+-\frac{8}{\pi^4}\cos 4\pi-\frac{1}{\pi^2}(4)^2+\frac{4}{\pi^2}(4)+\frac{8}{\pi^4}$

$0=-\frac{8}{\pi^4}-\frac{16}{\pi^2}+\frac{16}{\pi^2}+\frac{8}{\pi^4}=0\Rightarrow C_1=\mathbb{R}$

$\boxed{y=C\sin(\pi x)-\frac{8}{\pi^4}\cos(\pi x)-\frac{1}{\pi^2}x^2+\frac{4}{\pi^2}x+\frac{8}{\pi^4}}$

2. $y''+4y'+\lambda y=0,\ 0<x<\pi,\ y'(0)=0,\ y'(\pi)=0$

$m^2+4m+\lambda=0$

$m^2+4m=-\lambda$

$m^2+4m+4=-\lambda+4$

$(m+2)^2=-\lambda+4$

$m=-2\pm\sqrt{-\lambda+4}$

$$\boxed{\begin{aligned}&(i)\ -\lambda+4<0\Rightarrow 4<\lambda\\&(ii)\ -\lambda+4=0\Rightarrow 4=\lambda\\&(iii)\ -\lambda+4>0\Rightarrow 4>\lambda\end{aligned}}$$

(i) $-\lambda+4<0\Rightarrow -\lambda+4=-\alpha^2$

$m=-2\pm\alpha i$

$y=e^{-2x}(C_1\sin(\alpha x)+C_2\cos(\alpha x))$

$y=e^{-2x}C_1\sin(\alpha x)+e^{-2x}C_2\cos(\alpha x)$

$y'=-2e^{-2x}C_1\sin\alpha x+\alpha e^{-2x}C_1\cos(\alpha x)-2e^{-2x}C_2\cos\alpha x-\alpha e^{-2x}C_2\sin(\alpha x)$

$y'(0)=0\Rightarrow 0=\alpha C_1-2C_2\Rightarrow 2C_2=\alpha C_1$

$y'=e^{-2x}C_1[-2\sin\alpha x+\cancel{\alpha\cos(\alpha x)-\alpha\cos(\alpha x)}-\frac{1}{2}\alpha^2\sin(\alpha x)]$

$y'=\sin(\alpha x)e^{-2x}C_1[-2-\frac{1}{2}\alpha^2]$

$y'(\pi)=0\Rightarrow 0=\sin(\pi\alpha)e^{-2\pi}C_1[-2-\frac{1}{2}\alpha^2]$

back →

$0 = \underbrace{\sin(\alpha\pi)}_{\text{this} = 0} \underbrace{e^{2\pi}}_{\neq 0} \underbrace{c_1}_{=0 \Rightarrow \text{trivial solution}} \underbrace{[-2 - \tfrac{1}{2}\alpha^2]}_{\neq 0}$

$0 = \sin \alpha\pi$

$\alpha\pi = 0, \pi, 2\pi \ldots$

$\alpha = 0, 1, 2 \ldots$

$\alpha = n, \; n = 0, 1, 2 \ldots$

$-\lambda + 4 = -\alpha^2$

$-\lambda + 4 = -n^2$

$\boxed{\lambda_n = 4 + n^2, \; n = 0, 1, 2 \ldots \quad y_n = e^{-2x}\left[c \sin(nx) + \frac{cn}{2}\cos(nx)\right], n = 0, 1, 2 \ldots}$

3. $y'' + (\lambda - 1)y = 0, \; 0 < x < \pi, \; y(0) = 0, \; y'(\pi) = 0$

$\lambda \leq 1$

$m^2 + (\lambda - 1) = 0$

$m = \sqrt{1 - \lambda}$

$\lambda = 1$

$m = 0, 0$

$y = c_1 + c_2 x$

$y' = c_2$

$y(0) = 0$

$0 = c_1 + c_2(0)$

$0 = c_1$

$y'(\pi) = 0$

$0 = c_2$

Trivial Solution

$\lambda < 1$

$1 - \lambda = \alpha^2$

$m = \pm\alpha$

$y = c_1 e^{\alpha x} + c_2 e^{-\alpha x}$

$y' = \alpha c_1 e^{\alpha x} - \alpha c_2 e^{-\alpha x}$

$y(0) = 0$

$0 = c_1 e^0 + c_2 e^0$

$0 = c_1 + c_2$

$c_1 = -c_2$

$y'(\pi) = 0$

$y' = \alpha c_1 e^{\alpha x} + \alpha c_1 e^{-\alpha x}$

$0 = \alpha c_1 e^{\alpha\pi} + \alpha c_1 e^{-\alpha\pi}$

$0 = \alpha c_1 (e^{\alpha\pi} + e^{-\alpha\pi})$

$e^{\alpha\pi} > 0, \; e^{-\alpha\pi} > 0$

so $(e^{\alpha\pi} + e^{-\alpha\pi}) \neq 0,$

$1 - \lambda = \alpha^2, \; \lambda < 1 \Rightarrow \alpha \neq 0,$

$c_1 \neq 0$ as that would be trivial solution.

Thus, no eigenvalues / eigenfunctions for $\lambda \leq 1$ for this problem.

4. $4x^2 y'' + 4xy' + \lambda y = 0, \; 1 < x < 4, \; y(1) = 0, \; y'(4) = 0$

$4m^2 + (4 - 4)m + \lambda = 0$

$4m^2 + \lambda = 0$

$m^2 = -\frac{1}{4}\lambda$

$m = \sqrt{-\frac{1}{4}\lambda}$

Complex roots: $-\frac{1}{4}\lambda < 0$

$-\frac{1}{4}\lambda = -\alpha^2$

$m = \pm\alpha i$

$y = c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)$

$y' = -\frac{\alpha c_1}{x}\sin(\alpha \ln x) + \frac{\alpha c_2}{x}\cos(\alpha \ln x)$

$y(1) = 0$

$0 = c_1 \cos(\alpha \ln 1) + c_2 \sin(\alpha \ln 1)$

$0 = c_1 \cos(0) + c_2 \sin 0$

$0 = c_1$

$y'(4) = 0, \; c_1 = 0.$

$0 = \frac{\alpha c_2}{4}\cos(\alpha \ln 4)$

$\alpha \neq 0, \; c_2 \neq 0 \rightarrow$ non-trivial solution.

$-\frac{1}{4}\lambda < 0$

$\cos \alpha \ln 4 = 0.$

$\alpha \ln 4 = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots = \pi n + \frac{\pi}{2}, \; n = 0, 1, 2 \ldots$

$\alpha = \frac{\pi}{2\ln 4}, \frac{3\pi}{2\ln 4}, \frac{5\pi}{2\ln 4}$

$\alpha = \frac{\pi}{\ln 4} n + \frac{\pi}{2\ln 4} = \frac{2\pi n + \pi}{2\ln 4}$

$-\frac{1}{4}\lambda = -\left(\frac{2\pi n + \pi}{2\ln 4}\right)^2$

$\lambda_n = 4\frac{(2\pi n + \pi)^2}{\ln 4}$

$y_n = c_2 \sin\left(\left(\frac{2\pi n + \pi}{2\ln 4}\right)\ln x\right)$

$n = 0, 1, 2, \ldots$