Max Shi

I pledge my honor that I have abided by the Stevens Honor System
— Max Shi

Problem Set #6

1\. (a) Base case: $h = 0$

Height 0 has a single vertex $\Rightarrow$ 1 leaf

$2^0 = 1$ $\qquad 1 \leq 1$ ✓

Inductive hyp: for $k \geq 0$, there exists a binary tree s.t. the amount of leaves is less than 2^k.

Inductive step: A binary tree of height $k+1$ has two children, where one is a binary tree of height k and the other has height at most k. The latter child has an upper bound on its amount of leaves of of its height k, where it has $\leq 2^k$ leaves. ~~the former tree~~ has at most 2^k leaves. Thus, combining the leaves of the two children, $2^k + 2^k = 2^{k+1}$, which is an upper bound for the number of leaves for a tree of height of $k+1$. Thus, this proves the claim by principle of induction.

(b) Base: Tree of $h=0$, $N=1$, 1 leaf, 1 vertex.

$2N - 1 = 1 \Rightarrow 2(1) - 1 = 1 \Rightarrow 1 = 1$

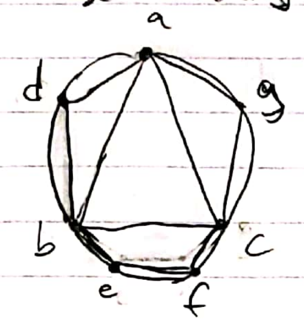
Ind hyp: for $k \geq 1$, a binary tree with k leaves has $2k-1$ vertices.

Ind step:

Let there be a binary tree with $k+1$ leaves. At the bottom of the tree, there will always be a vertex and its sibling, which are both leaves. Removing these two nodes will make their shared parent a leaf, making a binary tree of height k. By IH, this tree has $2k-1$ vertices. Adding the two nodes that were removed restores the tree back to $k+1$ leaves and adds two vertices, making the total $2k-1+2$, which is $2k+1$, which is equal to $2(k+1) - 1 = 2k+1$, therefore the claim is proven through PI.

2a(a) K_4 is impossible because it would require the opposite vertices on the circle to connect with each other, which is impossible when the other two opposite vertices must also have an edge connecting them. →

(b)



(a,b) (a,c) (a,d) (a,g)
(b,c) (b,d) (b,e) (c,f) (c,g)
(e,f).

1(b). Base case: $n=2$. The graph with two vertices has at most one edge and must be outerplanar. $2(2)-3=1$, which is consistent with the fact that two vertices can have at most 1 edge.

Ind hyp: For $k \geq 2$, all outerplanar graphs have at most $2k-3$ edges.

Ind step:

Let there be an outerplanar graph with $k+1$ vertices.
This graph has a vertex with degree at most 2.
If this vertex is removed, then the graph is still outerplanar, but with k vertices.
By IH, this graph has at most $2k-3$ edges.
Adding the removed vertex back adds at most 2 edges back to the graph.
Thus, the number of edges on the graph is less than $2k-3+2$, or $\leq 2(k+1)-3$.
Thus, the claim follows from PI.

c Basis: $n=1$ a graph with 1 vertex is 3-colorable.
IH: for $k \geq 1$, there exists a graph that is 3 colorable
IS: let there be an outerplanar graph with $k+1$ vertices. This graph has at least one vertex with degree ≤ 2. Removing this vertex gives us an outerplanar graph with k vertices. By IH, this graph is 3-colorable. Adding back the vertex removed earlier, because this vertex has two edges, it has at most two adjacent vertices. These adjacent vertices can be colored at most with two different colors, resulting in the added vertex being colored with the third color. Therefore, a maximum of three colors is needed to color the graph of $k+1$ vertices, therefore every outerplanar graph with $k+1$ vertices is 3-colorable, and the claim follows from PI.

d If there is an outerplanar drawing with vertices from v_1 to v_n drawn on the circle, and there is no edge connecting two non-adjacent vertices, then each vertex can at most be connected to its two adjacent neighbors, and therefore all vertices in this case have max degree 2.
If an edge connecting two non-adjacent vertices exists, choose the one that connects the two closest vertices on the circle. If there is one vertex between the two non-adjacent vertices connected by an edge, then the vertex in the middle can only have edges to its two adjacent vertices, because any other edges would cross the shortest non-adjacent edge. If there are more than two vertices between this edge E, then there cannot exist an edge connecting two non-adjacent vertices from these enclosed vertices, or else E would not be the shortest edge. Therefore, all enclosed edges can at most be connected to their neighbors and also have a maximum degree of 2.

let this be E