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I pledge my honor that I have abided by the Stevens Honor System
— Max Shi

CS135 Problem Set 3

1. This proof is invalid because if there exists an element c in set A such that $\forall d\ (c,d) \notin R$, then by the proof, (c,c) does not necessarily exist in R, and R does not meet the reflexive relation definition that states for all elements a in A, (a,a) must be in the relation. Here, because c is in the set but not in the relation, R is not reflexive by this proof.

2. $R: \forall x\,(x \in A \rightarrow (x,x) \in R)$
$S: \forall x\,(x \in A \rightarrow (x,x) \in S)$ } Definition of reflexive relations.

a. $R \cup S \equiv \forall x(x \in A \rightarrow (x,x) \in R) \vee \forall x(x \in A \rightarrow (x,x) \in S)$

$\forall x\ (\neg(x \in A) \vee (x,x) \in R) \vee \neg(x \in A) \vee (x,x) \in S)$

$\forall x\ (\neg(x \in A) \vee \neg(x \in A)) \vee ((x,x) \in R \vee (x,x) \in S)$

$\forall x\ \neg(x \in A \wedge x \in A) \vee [(x,x) \in R \vee (x,x) \in S]$

$\forall x\ (x \in A \wedge x \in A) \rightarrow [(x,x) \in R \vee (x,x) \in S]$

$\forall x\ \ x \in A \rightarrow ((x,x) \in R \vee (x,x) \in S)$

It follows that for all x, where $x \in A$, because $(x,x) \in R$ or $(x,x) \in S$, then (x,x) is in $R \cup S$, thus $R \cup S$ is reflexive.

b. $R \cap S \equiv \forall x(x \in A \rightarrow (x,x) \in R) \wedge \forall x(x \in A \rightarrow (x,x) \in S)$

$\equiv \forall x(\neg(x \in A) \vee (x,x) \in R) \wedge (\neg(x \in A) \vee (x,x) \in S)$

$\equiv \forall x\ \neg(x \in A) \vee ((x,x) \in R \wedge (x,x) \in S)$

$\equiv \forall x\ (x \in A) \rightarrow ((x,x) \in R \wedge (x,x) \in S)$

It follows that for all x where $x \in A$, $(x,x) \in R$ and $(x,x) \in S$, the intersection of this set would include all $(x,x) \in R$, therefore the relation is reflexive.

c. Subtraction is also defined as $R \cap \bar{S}$. Because $\bar{S}$ is defined as all elements in the universal set not in S, and $\forall x\,(x,x) \in S$, then $\forall x\ (x,x) \notin \bar{S}$. Therefore, $\bar{S}$ is anti-reflexive, and the intersection $R \cap \bar{S}$ would remove all reflexive relations in R because $\bar{S}$ is anti-reflexive. Therefore, $R \cap \bar{S}$ is anti-reflexive, and $R - S$ is anti-reflexive.

$\forall a \forall c$

d. $(a,c) \in S \circ R \longleftrightarrow \exists b((a,b) \in R) \wedge ((b,c) \in S)$

$\forall x\ (x,x) \in S \circ R \longleftrightarrow \exists b\ ((x,b) \in R \wedge (b,x) \in S))$

let $b = x$.

$\forall x\ (x,x) \in S \circ R \longleftrightarrow ((x,x) \in R \wedge (x,x) \in S))$

$\forall x\ (x,x) \in S \circ R \longleftrightarrow T \wedge T$

$\forall x\ (x,x) \in S \circ R \longleftrightarrow T$

Because the right-side is true, it follows that $\forall x\ (x,x) \in S \circ R$, therefore $S \circ R$ is reflexive.

e. By the definition of inverses: $\forall a \forall b\ ((a,b) \in R \Leftrightarrow (b,a) \in R^{-1})$

By definition of R: $\forall x\ ((x \in A) \rightarrow ((x,x) \in R))$

Let $a = b = x$: $\forall x\ ((x,x) \in R \Leftrightarrow (x,x) \in R^{-1})$

Because both R and R^{-1} are over set A, and $(x,x) \in R$ for all x, line 3 suggests that $(x,x) \in R^{-1}$ for all x, therefore R^{-1} is reflexive.

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3A. i. $[0]_R$ is the set of all numbers divisible by 7.
$[1]_R$ is the set of all numbers with remainder 1 when divided by 7.

ii. $14 \in [0]_R$

iii. $75 \in [5]_R$

iv. 7, for all possible remainders when divided by 7.

v. Yes, the same number cannot have two different remainders when divided by 7.

B. i. Let a and b be elements in set A. Define the equivalence class as $\forall x(x \in [a] \longleftrightarrow (a,x) \in R)$.
Given that $(a,b) \in R$, if we let x be all elements s.t. $(a,x) \in R$, and because R is an equivalence relation, then symmetry implies that $(b,a) \in R$. By transitive definition, because $(b,a) \in R \land (a,x) \in R$, then $(b,x) \in R$. Going back to the definition, replacing a for b, $\forall x(x \in [b] \longleftrightarrow (b,x) \in R)$. Thus, for all x, because $(a,x) \in R \Leftrightarrow (b,x) \in R$, for all x, $x \in [a] \land x \in [b]$. Therefore, [a] and [b] are equivalent sets, and $[a]=[b]$

ii. Because $[a]=[b]$, we can substitute for either on the right side to make $[a] \cap [a] \neq \emptyset$. This is true if there is at least one element in [a]. Because the problem states "Let a and b be any two elements in set A," we know that $[a] \neq \emptyset$. Therefore, $[a] \cap [a] \neq \emptyset$, proving $[a]=[b] \Rightarrow [a] \cap [b] \neq \emptyset$

iii. $[a] \cap [b] \neq \emptyset$ implies that [a] and [b] share a common element. Let this element be element x, where $x \in [a]$ and $x \in [b]$. Thus, by definition of equivalence class in 3.B.i., this implies that $(a,x) \in R$ and $(b,x) \in R$. By the symmetric element of equivalence relations, $(b,x) \in R \Longleftrightarrow (x,b) \in R$. By transitivity of equivalence relations, $(a,x) \in R \land (x,b) \in R \longrightarrow (a,b) \in R$. Thus, $[a] \cap [b] \neq \emptyset \longrightarrow (a,b) \in R$.