Max Shi   MA 221 Exam 4   I pledge my honor that I have abided by the Stevens Honor System
Lecture H   Brady.   Max Shi

1. $u_x - u_y + u = 0$

$u = XY$

$u_x = X'Y \quad u_y = XY'$

$X'Y - XY' + XY = 0$

$X'Y + XY = XY'$

$Y(X' + X) = XY'$

$\frac{X'+X}{X} = \frac{Y'}{Y} = -\lambda$

$\frac{X'+X}{X} = -\lambda \qquad \frac{Y'}{Y} = \lambda$

$X' + X = -\lambda X \qquad Y' - \lambda Y = 0$

$X' + X + \lambda X = 0 \qquad m - \lambda = 0$

$X' + (1+\lambda)X = 0. \qquad m = \lambda$

$m + 1 + \lambda = 0 \qquad Y = c_1 e^{\lambda y}$

$m = -\lambda - 1 \qquad Y' = \lambda c_1 e^{\lambda y}$

$X = c_2 e^{(-\lambda-1)x} \qquad \frac{\lambda c_1 e^{\lambda y}}{c_1 e^{\lambda y}} = \lambda$ ✓

$X' = (-\lambda-1)c_2 e^{(-\lambda-1)x}$

$\frac{X'+X}{X} = \frac{X'}{X} + 1 \Rightarrow \frac{(-\lambda-1)c_2 e^{(-\lambda-1)x}}{c_2 e^{-\lambda-1x}} + 1 = -\lambda$

$= -\lambda - 1 + 1 = -\lambda$

$-\lambda = -\lambda$

$$\boxed{u = XY = ce^{(-\lambda-1)x} \cdot e^{\lambda y} = ce^{(-\lambda-1)x + \lambda y}}$$

$u_x = (-\lambda-1)c\, e^{(-\lambda-1)x + \lambda y}$

$u_y = \lambda c e^{(-\lambda-1)x + \lambda y} \qquad -\lambda + \lambda - 1 + 1 = 0$ ✓

2. $y'' + y = 0, \ 0 < x < \pi, \ y(0) = 0, \ y(\pi) = 0.$

$m^2 + 1 = 0$

$m = \pm i \Rightarrow y = C_1\cos(x) + C_2\sin(x)$

$y(0) = 0 \Rightarrow 0 = C_1\cos(0) + C_2\sin(0)$

$0 = C_1 + 0$

$C_1 = 0.$

$y(\pi) = 0 \Rightarrow 0 = C_1\cos(\pi) + C_2\sin(\pi)$

$0 = C_2\sin(\pi)$

$0 = 0$ ✓ $\qquad C_1 = 0, \ C_2 = \text{free}.$

$$\boxed{y = C_2\sin(x), \ C_2 \in \mathbb{R}}$$

3. $y'' + (\lambda+1)y = 0, \quad 0 < x < 1, \quad y(0) = 0, \quad y'(1) = 0.$

$m^2 + \lambda + 1 = 0$

$m = \pm\sqrt{-\lambda-1}$

$\boxed{\text{iii. } -\lambda - 1 > 0 \Rightarrow -1 > \lambda}$

$\boxed{\text{ii. } -\lambda - 1 = 0 \Rightarrow \lambda = -1}$

$\boxed{\text{i. } -\lambda - 1 = -\alpha^2 < 0, \ -\lambda - 1 < 0 \Rightarrow -1 < \lambda}$

(b) $m = \pm\sqrt{-\alpha^2} = \pm\alpha i \Rightarrow y = c_1\cos(\alpha x) + c_2\sin(\alpha x)$

$y(0) = 0 \Rightarrow 0 = c_1\cos(0) + c_2\sin(0)$

$\Rightarrow 0 = c_1$

$y' = \alpha c_2 \cos(\alpha x)$

$y'(1) = 0 \Rightarrow 0 = \alpha c_2 \cos(\alpha) \qquad \alpha \neq 0, \ c_2 \neq 0.$

$\cos(\alpha) = 0$

$\cancel{\alpha = n\pi \pm \frac{\pi}{2}, \ n = 0, 1, 2, \dots}$

$\cancel{\alpha = n\pi \pm \frac{\pi}{2}, \ n = 1, 2, 3}$

$\cancel{y = c_n \sin((n\pi \pm \frac{\pi}{2})x)}$

$\alpha = n\pi - \frac{\pi}{2}, \ n = 1, 2, 3 \dots$

$-\alpha^2 = -\lambda - 1$

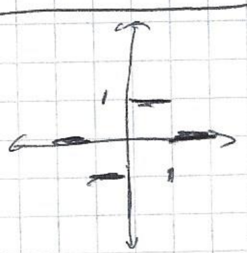
$\alpha^2 = \lambda + 1$

$\alpha^2 - 1 = \lambda$

$(n\pi - \frac{\pi}{2})^2 - 1 = \lambda$

$$\boxed{\begin{aligned} &\lambda_n = (n\pi - \tfrac{\pi}{2})^2 - 1 \\ &y_n = c_n \sin((n\pi - \tfrac{\pi}{2})x) \\ &n = 1, 2, 3 \dots \end{aligned}}$$

4.



$L = 2$

$\boxed{\text{(a) } f(x) \text{ has odd symmetry.}}$

(b) $a_0 = 0, \quad a_n = 0,$ by odd symmetry.

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\int_{-1}^{0}(-1)\sin\left(\frac{n\pi x}{2}\right)dx + \int_{0}^{1}\sin\left(\frac{n\pi x}{2}\right)dx\right]$$

$$= \frac{1}{2}\left[-\left[\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{-1}^{0} + \left[\frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\right]_{0}^{1}\right]$$

$$= \frac{1}{2}\left[-\left(\frac{-2}{n\pi} + \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right) + \left(\frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{2}{n\pi}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{n\pi} - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{-2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi}\right]$$

$$\boxed{b_n = \frac{2}{n\pi} - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) = \frac{2}{n\pi}\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)}$$

4(b). $n=1$ $\quad$ $\frac{2}{1\pi}\left(1-\cos\frac{\pi}{2}\right)=\frac{2}{\pi}$

$n=2$ $\quad$ $\frac{2}{2\pi}\left(1-\cos(\pi)\right)=\frac{2}{2\pi}(1-(-1))=\frac{2}{\pi}$

$n=3$ $\quad$ $\frac{2}{3\pi}\left(1-\cos\frac{3\pi}{2}\right)=\frac{2}{3\pi}$

$n=4$ $\quad$ $\frac{2}{4\pi}\left(1-\cos 2\pi\right)=0.$

$n=5 \Rightarrow \frac{2}{5\pi}\left(1-\cos\frac{5\pi}{2}\right)=\frac{2}{5\pi}$

$$\boxed{f(x)\sim\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)+\frac{2}{\pi}\sin(\pi x)+\frac{2}{3\pi}\sin\left(\frac{3\pi x}{2}\right)+\frac{2}{5\pi}\sin\left(\frac{5\pi x}{2}\right)}$$

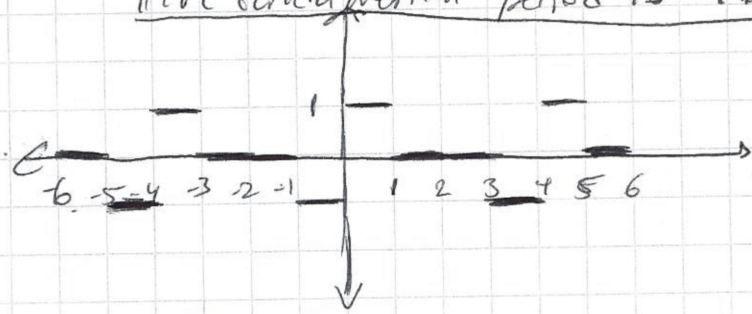
(C) smallest term is $\frac{\pi x}{2}$

let $x = x+c$, $\frac{\pi}{2}(x+c)=\frac{\pi x}{2}+2\pi$

$x+c = x+4$

$c=4$

$\boxed{\text{The fundamental period is } 4.}$

(d). 

$f_p(x)=0$ at $x=\{-4,0,4\}$

$f_p(x)=-1/2$ at $x=\{-5,-1,3\}$

$f_p(x)=1/2$ at $x=\{-3,1,5\}$