Max Shi MA227 Exam 3 Section A.

I pledge my honor that I have abided by the Stevens Honor System - Max Shi

1, (a) S_1: $S(\theta, z) = \langle \cos\theta, \sin\theta, z\rangle$ $0 \le \theta \le 2\pi$, $0 \le z \le 2$

$S_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$
$S_z = \langle 0, 0, 1\rangle$
$\Rightarrow S_\theta \times S_z = \langle \cos\theta, \sin\theta, 0\rangle$
at $\theta = 0$, $= \langle 1, 0, 0\rangle$ outward ✓.

$F(S) = \langle 2\cos\theta, 2\sin\theta, z-2\rangle$

$F(S) \cdot (S_\theta \times S_z) = 2\cos^2\theta + 2\sin^2\theta = 2.$

$\int_0^2\int_0^{2\pi} 2\, d\theta\, dz = 2 \cdot 2\pi \cdot 2 = 8\pi = \iint_{S_1} \vec{F}\cdot d\vec{S}$

S_2: $S(\theta, r) = \langle r\cos\theta, r\sin\theta, 2\rangle$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$S_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$
$S_r = \langle \cos\theta, \sin\theta, 0\rangle$
$\Rightarrow S_\theta \times S_r = \langle 0, 0, -r\sin^2\theta - r\cos^2\theta\rangle$
positive oriented so
$S_\theta \times S_r = \langle 0, 0, r\rangle$

$F(S) = \langle 2r\cos\theta, 2r\sin\theta, 0\rangle$

$F(S) \cdot (S_\theta \times S_r) = 0 \Rightarrow \iint_{S_2} \vec{F}\cdot d\vec{S} = 0$

S_3: $S(\theta, r) = \langle r\cos\theta, r\sin\theta, 0\rangle$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$

$S_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$
$S_r = \langle \cos\theta, \sin\theta, 0\rangle$
$\Rightarrow S_\theta \times S_r = \langle 0, 0, -r\rangle$
outward orientation = down ✓.

$F(S) = \langle 2r\cos\theta, 2r\sin\theta, -2\rangle$

$F(S) \cdot (S_\theta \times S_r) = 2r.$

$\int_0^{2\pi}\int_0^1 2r\, dr\, d\theta = \int_0^{2\pi} [r^2]_0^1\, d\theta = \int_0^{2\pi} d\theta = 2\pi = \iint_{S_3} \vec{F}\cdot d\vec{S}$

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S} = 8\pi + 0 + 2\pi = 10\pi}$$

(b) $\operatorname{div} F = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(z-2)$

$= 2 + 2 + 1$

$= 5$

$E: \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 0 \le z \le 2 \end{array} \Rightarrow \int_0^{2\pi}\int_0^1\int_0^2 5\, r\, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^1 5r[z]_0^2\, dr\, d\theta = \int_0^{2\pi}\int_0^1 10r\, dr\, d\theta$

$= \int_0^{2\pi}[5r^2]_0^1\, d\theta = 5\int_0^{2\pi} d\theta = \boxed{10\pi}$

(c) $\boxed{10\pi = 10\pi}$

The answers should be the same by the Divergence Theorem.

2. (a) $\operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & z+x & \sin(z^3)+x^2 \end{vmatrix} = \left(\frac{\partial}{\partial y}(\sin(z^3)+x^2) - \frac{\partial}{\partial z}(z+x)\right)i - \left(\frac{\partial}{\partial x}(\sin(z^3)+x^2) - \frac{\partial}{\partial z}(-y)\right)j + \left(\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(-y)\right)k$

$= (0-1)i - (2x-0)j + (1-(-1))k$

$\operatorname{curl} F = -1i - 2xj + 2k = \boxed{\langle -1, -2x, 2 \rangle}$

(b) $\oint_C F \cdot dr = \iint_S \operatorname{curl} F \cdot dS$

$S(x,y) = \langle x, y, 2-x \rangle,\ 0 \le x \le 2,\ 0 \le y \le 2$

$S_x = \langle 1, 0, -1 \rangle$
$S_y = \langle 0, 1, 0 \rangle \Rightarrow S_x \times S_y = \langle 1, 0, 1 \rangle$

positively oriented!

$\operatorname{curl} F(S) \cdot (S_x \times S_y) = -1 + 0 + 2 = 1$

$\int_0^2\int_0^2 1\, dx\, dy = \int_0^2\int_0^2 dx\, dy = 2 \cdot 2 = \boxed{4}$

3.(a) $A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}$

$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ $\quad \det A = 16 - 15 = 1$

$A^{-1} = \frac{1}{1}\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \boxed{\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}}$

check: $\begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 16-15 & -6+6 \\ 40-40 & -15+16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8-6 \\ -5+4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\boxed{x = 2,\ y = -1.}$

4. $\begin{aligned} x + y + 3z &= 1 \\ 2x + y + 4z &= -3 \\ 3x + 0y + 2z &= -14 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 2 & 1 & 4 & -3 \\ 3 & 0 & 2 & -14 \end{array}\right] \begin{array}{l} r_2 - 2r_1 \\ \Longrightarrow \\ r_3 - 3r_1 \end{array}$

$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & -1 & -2 & -5 \\ 0 & -3 & -7 & -17 \end{bmatrix} \begin{array}{l} -1\cdot r_2 \\ \Rightarrow \\ -1\cdot r_3 \end{array} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 3 & 7 & 17 \end{bmatrix} \xrightarrow{r_3 - 3r_2} \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

$\Rightarrow \begin{aligned} x + y + 3z &= 1 \Rightarrow x + y + 3(2) = 1 \Rightarrow x + 1 + 6 = 1 \Rightarrow x = -6. \\ y + 2z &= 5 \Rightarrow y + 2(2) = 5 \Rightarrow y = 1 \\ z &= 2 \end{aligned}$

$\boxed{x = -6,\ y = 1,\ z = 2}$