Max Shi MA227 HW 8 Brady LECH

I pledge my honor that I have abided by the Stevens Honor System
                                    — The AW

1. $4y'' + \pi^2 y = g(x)$, $0 \leq x < 4$, $y(0) = 0$, $y'(4) = 0$.

(a) $m^2 + \pi^2 = 0$

$m = \pm \pi i$

$y_c = C_1 \sin(\pi x) + C_2 \cos(\pi x)$

$y = C_1 \sin(\pi x) + C_2 \cos(\pi x)$

$y(0) = 0$

$0 = C_1 \sin(0) + C_2 \cos(0)$

$0 = C_2 \Rightarrow C_2 = 0$

$g(x) = 0 \Rightarrow$

$y_p = 0$

$y(4) = 0$

$0 = C_1 \sin(4\pi)$

$0 = 0 \checkmark$

$C_1 = \mathbb{R}$

$$\boxed{y = C_1 \sin(x\pi)}$$

(b) $4y'' + \pi^2 y = -x^2 + 4x$

$y_c = C_1 \sin(\pi x) + C_2 \cos(\pi x)$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

$4(2A) + \pi^2(Ax^2 + Bx + C) = -x^2 + 4x$

$8A + \pi^2 A x^2 + \pi^2 B + \pi^2 C = -x^2 + 4x$

$\pi^2 A = -1$, $\pi^2 B = 4$, $\pi^2 C + 8A = 0$

$A = -\frac{1}{\pi^2}$, $B = \frac{4}{\pi^2}$, $\pi^2 C + 8 \frac{1}{\pi^2} = 0$

$C = \frac{8}{\pi^4}$

$y = y_c + y_p$

$y = C_1 \sin(\pi x) + C_2 \cos(\pi x) - \frac{1}{\pi^2} x^2 + \frac{4}{\pi^2} x + \frac{8}{\pi^4}$

$y(0) = 0$

$0 = C_1 \sin 0 + C_2 \cos 0 - 0 + 0 + \frac{8}{\pi^4}$

$0 = C_2 + \frac{8}{\pi^4} \Rightarrow C_2 = -\frac{8}{\pi^4}$

$y(4) = 0 \Rightarrow$ $0 = C_1 \sin 4\pi + \frac{8}{\pi^4} \cos 4\pi - \frac{1}{\pi^2}(4)^2 + \frac{4}{\pi^2}(4) + \frac{8}{\pi^4}$

$0 = -\frac{8}{\pi^4} - \frac{16}{\pi^2} + \frac{16}{\pi^2} + \frac{8}{\pi^4} = 0 \Rightarrow C_1 = \mathbb{R}$

$$\boxed{y = C\sin(\pi x) - \frac{8}{\pi^4} \cos(\pi x) - \frac{1}{\pi^2} x^2 + \frac{4}{\pi^2} x + \frac{8}{\pi^4}}$$

2. $y'' + 4y' + \lambda y = 0$, $0 < x < \pi$, $y'(0) = 0$, $y'(\pi) = 0$.

$m^2 + 4m + \lambda = 0$

$m^2 + 4m = -\lambda$

$m^2 + 4m + 4 = -\lambda + 4$

$(m+2)^2 = -\lambda + 4$

$m = -2 \pm \sqrt{-\lambda + 4}$

(i) $-\lambda + 4 < 0 \Rightarrow 4 < \lambda$
(ii) $-\lambda + 4 = 0 \Rightarrow 4 = \lambda$
(iii) $-\lambda + 4 > 0 \Rightarrow 4 > \lambda$

$y'(0) = 0 \Rightarrow 0 = \alpha C_1 - 2C_2 \Rightarrow 2C_2 = \alpha C_1$

$y' = e^{-2x} C_1 [-2\sin\alpha x + \alpha\cos(\alpha x)] - \alpha\cos(\alpha x) - \frac{1}{2}\alpha^2 \sin(\alpha x)$

$y' = \sin(\alpha x) e^{-2x} C_1 [-2 - \frac{1}{2}\alpha^2]$

$y'(\pi) = 0 \Rightarrow 0 = \sin(\pi\alpha) e^{-2\pi} C_1 [-2 - \frac{1}{2}\alpha^2]$

back. ——————→

(b) $-\lambda + 4 < 0 \Rightarrow -\lambda + 4 = -\alpha^2$

$m = -2 \pm \alpha i$

$y = e^{-2x}(C_1 \sin(\alpha x) + C_2 \cos(\alpha x))$

$y = e^{-2x} C_1 \sin(\alpha x) + e^{-2x} C_2 \cos(\alpha x)$

$y' = -2e^{-2x} C_1 \sin\alpha x + \alpha e^{-2x} C_1 \cos(\alpha x) - 2e^{-2x} C_2 \cos\alpha x - \alpha e^{-2x} C_2 \sin(\alpha x)$

$0 = \sin(\alpha\pi)\, e^{2\pi}\underbrace{C_1}_{\neq 0}\underbrace{[-2 - \tfrac{1}{2}\alpha^2]}_{\neq 0}$

underbraces: this $= 0$. $=0$ $\Rightarrow$ trivial solution.

$0 = \sin\alpha\pi$

$\alpha\pi = 0, \pi, 2\pi \ldots$

$\alpha = 0, 1, 2 \ldots$

$\alpha = n, \quad n = 0, 1, 2, \ldots$

$-\lambda + 4 = -\alpha^2$

$-\lambda + 4 = -n^2$

$\boxed{\lambda_n = 4 + n^2, \; n = 0, 1, 2, \ldots \quad y_n = e^{-2x}\left[C\sin(nx) + \tfrac{C_n}{2}\cos(nx) \right], n = 0, 1, 2 \ldots}$

3. $y'' + (\lambda - 1)y = 0, \; 0 < x < \pi, \; y(0) = 0, \; y'(\pi) = 0$

$\overset{\longleftarrow}{} \lambda \le 1 \overset{\longrightarrow}{}$

$m^2 + (\lambda - 1) = 0$

$m = \sqrt{1 - \lambda} \qquad \lambda = 1$

$\qquad\qquad m = 0, 0$

$y = C_1 + C_2 x$

$y' = C_2$

$y(0) = 0 \qquad\qquad y'(\pi) = 0$

$0 = C_1 + C_2(0) \qquad 0 = C_2$

$0 = C_1$

| Trivial solution |

$\downarrow$

Thus, no eigenvalues/eigenfunctions for $\lambda \le 1$ for this problem.

$\overset{\longleftarrow}{}$

Right column ($\lambda < 1$):

$\lambda < 1$

$1 - \lambda = \alpha^2$

$m = \pm\alpha$

$y = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$

$y' = \alpha C_1 e^{\alpha x} - \alpha C_2 e^{-\alpha x}$

$y(0) = 0 \qquad\qquad y'(\pi) = 0$

$0 = C_1 e^0 + C_2 e^0 \qquad y' = \alpha C_1 e^{\alpha x} + \alpha C_1 e^{-\alpha x}$

$0 = C_1 + C_2 \qquad 0 = \alpha C_1 e^{\alpha\pi} + \alpha C_1 e^{-\alpha\pi}$

$C_1 = -C_2 \qquad 0 = \alpha C_1 (e^{\alpha\pi} + e^{-\alpha\pi})$

$e^{\alpha\pi} > 0, \; e^{-\alpha\pi} > 0$

so $(e^{\alpha\pi} + e^{-\alpha\pi}) \neq 0$,

$1 - \lambda = \alpha^2, \; \lambda < 1 \Rightarrow \alpha \neq 0,$

$C_1 \neq 0$ as that would be Trivial solution.

4. $4x^2 y'' + 4xy' + \lambda y = 0, \; 1 < x < 4, \; y(1) = 0, \; y'(4) = 0$

$4m^2 + (4 - 4)m + \lambda = 0$

$4m^2 + \lambda = 0$

$m^2 = -\tfrac{1}{4}\lambda$

$m = \sqrt{-\tfrac{1}{4}\lambda}$

Complex roots: $-\tfrac{1}{4}\lambda < 0$

$-\tfrac{1}{4}\lambda = -\alpha^2$

$m = \pm\alpha i$

$y = C_1 \cos(\alpha \ln x) + C_2 \sin(\alpha \ln x)$

$y' = -\tfrac{\alpha C_1}{x}\sin(\alpha \ln x) + \tfrac{\alpha C_2}{x}\cos(\alpha \ln x)$

$y(1) = 0$

$0 = C_1 \cos(\alpha \ln 1) + C_2 \sin(\alpha \ln 1)$

$0 = C_1 \cos(0) + C_2 \sin 0$

$0 = C_1$

$y'(4) = 0, \; C_1 = 0.$

$0 = \tfrac{\alpha C_2}{4}\cos(\alpha \ln 4)$

$\alpha \neq 0, \; C_2 \neq 0$

$-\tfrac{1}{4}\lambda < 0 \quad \rightsquigarrow$ non-trivial solution.

$\cos \alpha \ln 4 = 0.$

$\alpha \ln 4 = \tfrac{\pi}{2}, \tfrac{3\pi}{2}, \tfrac{5\pi}{2} \ldots = \pi n + \tfrac{\pi}{2}, \; n = 0, 1, 2 \ldots$

$\alpha = \tfrac{\pi}{2\ln 4}, \tfrac{3\pi}{2\ln 4}, \tfrac{5\pi}{2\ln 4}$

$\alpha = \tfrac{\pi}{\ln 4}n + \tfrac{\pi}{2\ln 4} = \tfrac{2\pi n + \pi}{2\ln 4}$

$-\tfrac{1}{4}\lambda = -\left(\tfrac{2\pi n + \pi}{2\ln 4}\right)^2$

$\boxed{\lambda_n = \dfrac{(2\pi n + \pi)^2}{\ln 4}}$

$\boxed{y_n = C_2 \sin\left(\left(\tfrac{2\pi n + \pi}{2\ln 4}\right)\ln x\right)}$

$n = 0, 1, 2, \ldots$