Mao Shi MA 227 HW 5

I pledge my honor that I have abided by the Stevens Honor System. [signature]

1. (a) $\oint_C \vec{F}\cdot d\vec{r}$, $C = x^2+y^2=1$, $F = \langle y^2\sin z, x^3+y, z^2\rangle$

$r(t) = \langle \cos t, \sin t, 0\rangle$, $0 \le t \le 2\pi$

$r'(t) = \langle -\sin t, \cos t, 0\rangle$

$F(r(t)) = \langle \sin^2(t)\sin 0, \cos^3(t)+\sin(t), 0^2\rangle$

$= \langle 0, \cos^3(t)+\sin(t), 0\rangle$

$\vec{F}\cdot d\vec{r} = \cos^4(t)+\sin(t)\cos(t)$

$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \cos^4(t)+\sin(t)\cos(t)\,dt$

$= \int_0^{2\pi}\cos^4(t)\,dt + \int_0^{2\pi}\sin(t)\cos(t)\,dt$

$= \frac{1}{32}\left[\sin(4t)+8\sin(2t)+12t\right]_0^{2\pi} + \frac{1}{2}\left[\sin^2(t)\right]_0^{2\pi}$

$= \frac{1}{32}\left[(\sin 8\pi + 8\sin 4\pi + 24\pi) - \sin 0 - 8\sin 0 + 0\right] + \frac{1}{2}\left[\sin^2(2\pi)-\sin^2(0)\right]$

$= \frac{1}{32}\left[0+0+24\pi-0-0-0\right] + \frac{1}{2}(0-0)$

$\boxed{= \frac{3\pi}{4}}$

(b) $S_1 = x^2+y^2 \le 1$, $z=0$.

$S_1(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$, $0\le r\le 1$, $0\le\theta\le 2\pi$

$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\sin z & x^3+y & z^2 \end{vmatrix} = \left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(x^3+y)\right]i - \left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}y^2\sin z\right]j + \left[\frac{\partial}{\partial x}(x^3+y) - \frac{\partial}{\partial y}(y^2\sin z)\right]k$

$= (0-0)i - (0 - y^2\cos z)j + (3x^2 - 2y\sin z)k$

$\text{curl } F(S_1(r,\theta)) = (r^2\sin^2\theta)\cos(0)j + 3(r^2\cos^2\theta) - 2(r\sin\theta)\sin 0\, k$

$= r^2\sin^2\theta\, j + 3r^2\cos^2\theta\, k$

$S_{1r} = \langle\cos\theta, \sin\theta, 0\rangle$
$S_{1\theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle$
$\Rightarrow |S_{1r}\times S_{1\theta}| = \begin{vmatrix} i & j & k \end{vmatrix} = (0-0)i - (0-0)j + (r\cos^2\theta + r\sin^2\theta)k$

$= rk \Rightarrow d\vec{S} = rk\,dr\,d\theta$

(upward, as $r\ge 0$)

$\text{curl }\vec{F}\cdot d\vec{S} = 3r^3\cos^2\theta\,dr\,d\theta$

$\iint_{S_1} \text{curl }F\cdot dS = \int_0^{2\pi}\int_0^1 3r^3\cos^2\theta\,dr\,d\theta$

$= \int_0^{2\pi}\left[\frac{3}{4}r^4\cos^2\theta\right]_0^1 d\theta$

$= \int_0^{2\pi}\frac{3}{4}\cos^2\theta\,d\theta = \frac{3}{4}\int_0^{2\pi}\cos^2\theta\,d\theta$

$= \frac{3}{4}\left[\frac{\cos\theta\sin\theta+\theta}{2}\right]_0^{2\pi}$

$= \frac{3}{4}\left[\frac{\cos 2\pi\sin 2\pi + 2\pi}{2} - \frac{\cos 0\sin 0 + 0}{2}\right]$

$= \frac{3}{4}(\pi) = \boxed{\frac{3\pi}{4}}$

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(a) $\iint_S \vec{F}\cdot d\vec{s} = \iint_{S_1} \vec{F}\cdot d\vec{s} + \iint_{S_2} \vec{F}\cdot d\vec{s}$, $F = \langle y^2, -x^2, z^2 \rangle$

$S_1: z = x^2+y^2$ $\quad S_1 = \langle x, y, x^2+y^2 \rangle$

$S_1(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$S_{1r} = \langle \cos\theta, \sin\theta, 2r \rangle \Rightarrow S_{1r} \times S_{1\theta} = -2r^2\cos\theta\, i - 2r^2\sin\theta\, j + (r\cos^2\theta + r\sin^2\theta)k$
$S_{1\theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$
$= -2r^2\cos\theta\, i - 2r^2\sin\theta\, j + r k$

as it is downward oriented, at $\langle 0,0,0 \rangle$, k should be negative. Thus,
$dS = \langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle\, dr\, d\theta$
$F = \langle r^2\sin^2\theta, -r^2\cos^2\theta, r^4 \rangle$
$F\cdot ds = 2r^4\cos\theta\sin^2\theta - 2r^4\sin\theta\cos^2\theta - r^5\, dr\, d\theta$

$\iint_{S_1} \vec{F}\cdot d\vec{s} = \int_0^{2\pi}\int_0^2 2r^4\cos\theta\sin^2\theta - 2r^4\sin\theta\cos^2\theta - r^5\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^2 2r^4(\cos\theta\sin^2\theta - \sin\theta\cos^2\theta)\, dr\, d\theta - \int_0^{2\pi}\int_0^2 r^5\, dr\, d\theta$

$= \int_0^2 2r^4\, dr \cdot \int_0^{2\pi} \cos\theta\sin^2\theta - \sin\theta\cos^2\theta\, d\theta \quad - 2\pi \cdot \left[\frac{r^6}{6}\right]_0^2$

$= \left[\frac{2r^5}{5}\right]_0^2 \cdot \left(\int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta - \int_0^{2\pi}\cos^2\theta\sin\theta\, d\theta\right) - \frac{64}{3}\pi$

$= \frac{64}{5} \cdot 0 - \frac{64}{3}\pi$

$\boxed{\iint_{S_1} \vec{F}\cdot d\vec{s} = -\frac{64}{3}\pi}$

$S_2: x^2+y^2 \le 4$ $\quad S_2 = \langle r\cos\theta, r\sin\theta, 4 \rangle$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$

$S_{2r} = \langle \cos\theta, \sin\theta, 0 \rangle \Rightarrow S_{2r} \times S_{2\theta} = 0i + 0j + rk$
$S_{2\theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$dS = \langle 0, 0, r \rangle\, dr\, d\theta$ $\quad$ upwards at $(0,0,1)$ $k > 0$. ✓

$\iint_{S_2} \vec{F}\cdot d\vec{s} = \iint_{S_2} 4^2 \cdot r\, d\theta \Rightarrow \int_0^2\int_0^{2\pi} 16r\, d\theta\, dr = 2\pi\,[8r^2]_0^2 = \boxed{64\pi}$

(b) $\text{div } F = \frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial y} x^2 + \frac{\partial}{\partial z} z^2 = 2z$.

$x^2+y^2 = z$
$r^2 = 2 \quad r = \sqrt{2}$

E (cylindrical) = $0 \le \theta \le 2\pi$
$0 \le z \le 4$
$0 \le r \le \sqrt{2}$
$\Rightarrow \iiint_E \text{div} F\, dV = \int_0^{2\pi}\int_0^4\int_0^{\sqrt{2}} 2zr\, dr\, dz\, d\theta$
$= \int_0^{2\pi}\int_0^4 z[r^2]_0^{\sqrt{2}}\, dz\, d\theta$
$= \int_0^{2\pi}\int_0^4 z^2\, dz\, d\theta$
$= \int_0^{2\pi}\left[\frac{z^3}{3}\right]_0^4 d\theta$
$= 2\pi \cdot \frac{64}{3} = \boxed{\frac{128\pi}{3}}$

$\iint_S F\cdot dr = \iint_{S_1} F\cdot dr + \iint_{S_2} F\cdot dr$
$= 64\pi - \frac{64}{3}\pi$
$= \frac{128}{3}\pi$

(c) They are the same value,
in accordance with the Divergence Theorem.