Marco Shi   MA227 Final Exam   Lecture A.

I pledge my honor that I have abided by the Stevens Honor System - [signature]

MI.1.

(a) $\int_0^\infty \int_0^\infty p(x,y)\, dx\, dy = 1$

$\int_0^4 \int_0^2 C(2xy + x)\, dy\, dx = 1$

$C \int_0^4 \int_0^2 (2xy + x)\, dy\, dx = 1$

$C \int_0^4 \left[xy^2 + xy \right]_0^2 dx = 1$

$C \int_0^4 4x + 2x\, dx = 1$

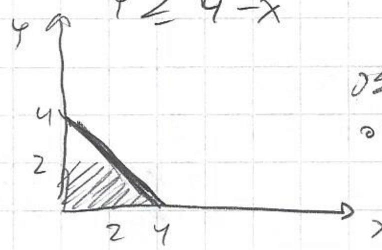
$C \int_0^4 6x\, dx = 1$

$C \left[3x^2 \right]_0^4 = 1$

$\boxed{C = \frac{1}{48}}$

(b) $x + y \leq 4$

$y \leq 4 - x$



$0 \leq x \leq 4$

$0 \leq y \leq 2$

$x = 4 - y$

$P(x + y \leq 4) =$

$\int_0^2 \int_0^{4-y} \frac{1}{48}(2xy + x)\, dx\, dy$

$\frac{1}{48} \int_0^2 \int_0^{4-y} (2xy + x)\, dx\, dy$

$\frac{1}{48} \int_0^2 \left[x^2 y + \frac{x^2}{2} \right]_0^{4-y} dy$

$\frac{1}{48} \int_0^2 (4-y)^2 y + \frac{(4-y)^2}{2}\, dy$

$\frac{1}{48} \int_0^2 16y - 8y^2 + y^3 + 8 - 4y + \frac{y^2}{2}\, dy$

$\frac{1}{48} \int_0^2 12y - \frac{15}{2} y^2 + y^3 + 8\, dy$

$\frac{1}{48} \left[6y^2 - \frac{5}{2} y^3 + \frac{1}{4} y^4 + 8y \right]_0^2$

$\frac{1}{48} (24 - 20 + 4 + 16)$

$= \frac{24}{48} = \boxed{\frac{1}{2}}$

MI.3. Volume $\iiint dV$

sphere - $x^2 + y^2 + z^2 - 8z + 16 = 16$

$x^2 + y^2 + (z-4)^2 = 16.$

$z^2 + (z-4)^2 = 16.$ satisfied at $z = 4$.

Cone — $0 \leq z \leq 4.$ $z = \sqrt{x^2 + y^2} = \sqrt{r^2} = r.$

$0 \leq r \leq z$

$0 \leq \theta \leq 2\pi$

$\int_0^{2\pi} \int_0^4 \int_0^z r\, dr\, dz\, d\theta$

$2\pi \int_0^4 \int_0^z r\, dr\, dz$

$2\pi \int_0^4 \left[\frac{r^2}{2} \right]_0^z dz$

$2\pi \int_0^4 \frac{z^2}{2}\, dz$

$2\pi \left[\frac{z^3}{6} \right]_0^4 = \frac{64\pi}{3}$

sphere - $\rho^2 = 8\rho\cos\phi \Rightarrow \rho = 8\cos\phi$

$0 \leq \phi \leq \frac{\pi}{4}$

$0 \leq \theta \leq 2\pi$

$0 \leq \rho \leq 8\cos\phi$

$\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{8\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

$2\pi \int_0^{\frac{\pi}{4}} \int_0^{8\cos\phi} \rho^2 \sin\phi\, d\rho\, d\phi$

$2\pi \int_0^{\frac{\pi}{4}} \left[\frac{\rho^3}{3} \sin\phi \right]_0^{8\cos\phi} d\phi$

$2\pi \int_0^{\frac{\pi}{4}} \frac{8^3}{3} \cos^3\phi \sin\phi\, d\phi$ $u = \cos\phi$ $\phi = \frac{\pi}{4} \Rightarrow u = \frac{\sqrt{2}}{2}$

$du = -\sin\phi$ $\phi = 0 \Rightarrow u = 1$

$2\pi \cdot \frac{8^3}{3} \int_1^{\frac{\sqrt{2}}{2}} -u^3\, du$

$2\pi \cdot \frac{8^3}{3} \int_{\frac{\sqrt{2}}{2}}^1 u^3\, du$

$2\pi \cdot \frac{8^3}{3} \cdot \left(\frac{u^4}{4} \right)_{\frac{\sqrt{2}}{2}}^1 = \left(\frac{1}{4} - \frac{1}{16} \right) 2\pi \cdot \frac{8^3}{3} = \frac{3}{16} \cdot 8\pi \cdot \frac{8^2}{3} \cdot 2 = 64\pi$

$\boxed{V = \frac{64\pi}{3} + 64\pi}$

MI. 4.

(a) $z = x$, $y = 4 - x^2$
when $y = 0$, $x^2 = 4$, $x = 2$.
$z = 2$.

$0 \le z \le 2$
$0 \le x \le z$
$0 \le y \le 4 - x^2$

$$\int_0^2 \int_0^z \int_0^{4-x^2} dy\, dx\, dz$$

(b) $z = x$, $y = 4 - x^2$
$y = 0 \Rightarrow x = 2$

$0 \le x \le 2$
$0 \le y \le 4 - x^2$
$0 \le z \le x$

$$\int_0^2 \int_0^{4-x^2} \int_0^x dz\, dy\, dx$$

VC.1. (a) conservative if curl $F = 0$.

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ z & y^3 & x \end{vmatrix} = \left(\frac{d}{dy} x - \frac{d}{dz} y^3\right) i - \left(\frac{d}{dx} x - \frac{d}{dz} z\right) j + \left(\frac{d}{dx} y^3 - \frac{d}{dy} z\right) k$$

$(0 - 0) i - (1 - 1) j + (0 - 0) k$
$= 0$. $\boxed{\vec{F} \text{ is conservative}}$

(b) finding f -

$\int z\, dx = xz + h(y) + g(z) \Rightarrow$ $y^2 - \frac{d}{dy}(xz) = h'(y)$
$y^2 - 0 = h'(y)$
$y^2 = h'(y)$
$h(y) = \frac{y^3}{3}$

$z - \frac{d}{dz}\left(xz - \frac{y^3}{3}\right) = g'(z)$
$z - z = g'(z)$
$0 = g'(z)$

$f = xz + \frac{y^3}{3} + C$

$$\oint_C F \cdot dr = f(Q) - f(P) = \left(0 \cdot 3 + \frac{2^3}{3}\right) - \left(1 \cdot 2 + \frac{0^3}{3}\right) = \frac{8}{3} - 2 = \boxed{\frac{2}{3}}$$

VC.3. (a)

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ xy & yz & y+z \end{vmatrix} = \left(\frac{d}{dy}(y+z) - \frac{d}{dz}(yz)\right) i - \left(\frac{d}{dx}(y+z) - \frac{d}{dz}(xy)\right) j + \left(\frac{d}{dx}(yz) - \frac{d}{dy}(xy)\right) k$$

$= (1 - y) i - (0 - 0) j + (0 - x) k$

$$\boxed{= \langle 1 - y,\ 0,\ -x \rangle}$$

(b). bounded by $x^2+y^2=9$, so use that as surface.

$S = \langle r\cos\theta, r\sin\theta, 0 \rangle$

$S_r = \langle \cos\theta, \sin\theta, 0 \rangle \Rightarrow S_r \times S_\theta = 0i - 0j + (r\cos^2\theta + r\sin^2\theta)k$

$S_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle \qquad = r k$ upward!

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$\text{curl } F(S) = \langle 1 - r\sin\theta, 0, -r\cos\theta \rangle$

$F(S) \cdot S_r \times S_\theta = r^2\cos\theta$

$\int_0^{2\pi}\int_0^3 r^2\cos\theta \, dr \, d\theta = \int_0^{2\pi}\left(\frac{r^3}{3}\cos\theta\right)_0^3 d\theta = \int_0^{2\pi} 9\cos\theta \, d\theta = (9\sin\theta)_0^{2\pi}$

$= 0 - 0$

$\boxed{\iint_S \text{curl} F \cdot dS = 0}$

VC. 4 (a) $S_2(r,\theta) = \langle r\cos\theta, r\sin\theta, 0 \rangle$

from prev problem, $S_r \times S_\theta = r k$

$0 \le r \le 1, \quad 0 \le \theta \le 2\pi$

$F(S_2) = \langle r\cos\theta, 2r\sin\theta, -1 \rangle$

$F(S_2) \cdot S_r \times S_\theta = -r$

$\int_0^{2\pi}\int_0^1 -r \, dr \, d\theta = 2\pi \cdot \left(-\frac{r^2}{2}\right)_0^1 = \boxed{-\pi}$

(b) $\text{div} F = \nabla \cdot F = \frac{d}{dx} xe^2 + \frac{d}{dy} 2y + \frac{d}{dz} -e^z = e^2 + 2 - e^2 = \boxed{2}$

(c) $\iint_S \vec{F} \cdot d\vec{s} = \iint_{S_1} \vec{F} \cdot d\vec{s} - \underbrace{\iint_{S_2} \vec{F} \cdot d\vec{s}}_{\text{flip orientation}} = \iiint_S \text{div} F \, dV$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \frac{\pi}{2}$

$0 \le \rho \le 1$

$\iint_{S_1} \vec{F} \cdot d\vec{s} + \pi = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 2\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$

$= 4\pi \int_0^{\frac{\pi}{2}}\int_0^1 \rho^2\sin\phi \, d\rho \, d\phi$

$= 4\pi \int_0^{\frac{\pi}{2}} \left(\frac{\rho^3}{3}\sin\phi\right)_0^1 d\phi$

$= \frac{4}{3}\pi \int_0^{\frac{\pi}{2}} \sin\phi \, d\phi$

$\frac{4}{3}\pi \left(-\cos\phi\right)_0^{\frac{\pi}{2}} = \frac{4}{3}\pi$

$\iint_{S_1} F \cdot ds = \frac{4}{3}\pi - \pi = \boxed{\frac{1}{3}\pi}$

LS. 2 $\begin{vmatrix} 5-\lambda & 10 \\ -2 & 1-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)+20 = 5-6\lambda+\lambda^2+20 = 25-6\lambda+\lambda^2 = 0$

$9-6\lambda+\lambda^2 = -16$

$(\lambda-3)^2 = -16$

$\lambda = 3 \pm 4i \qquad \lambda_1 = 3+4i$

$A-\lambda_1 I = \begin{matrix} 2-4i & 10 \\ -2 & -2+4i \end{matrix}$

$(2-4i)k_1 + 10k_2 = 0$

$-2k_1 + (2+4i)k_2 = 0 \qquad 2k_1 = (-2+4i)k_2 \qquad k_1 = -1+2i$

$k_1 = -1+2i\, k_2 \qquad k_2 = 1$

$k = \begin{bmatrix} -1+2i \\ 1 \end{bmatrix} \Rightarrow Re(k) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \ Im(k) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$\alpha = 3, \quad \beta = 4,$

$$\boxed{X = e^{3t}\left[C_1\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\cos(4t) - \begin{bmatrix} 2 \\ 0 \end{bmatrix}\sin(4t)\right) + C_2\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\cos(4t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\sin(4t)\right)\right]}$$

LS. 3 $\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix} = (2-\lambda)(5-\lambda)(5-\lambda)+0+0-0-0-0.$

$\lambda = 2, 5, 5$

$A-\lambda_1 I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{bmatrix} \Rightarrow \begin{matrix} k_2 = 0 \\ 3k_2 = 0 \\ k_1 + 3k_3 = 0 \end{matrix} \quad k_2 = 0, \ k_1 = -3k_3 \quad k_3 = 1, k_1 = -3. \Rightarrow \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$A-\lambda_2 I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Rightarrow C_3$ is free so $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector.

$\begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} P = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow P_1 = 1, \ P_2 = 3, \ P_3 = 0. \ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ is another eigenvector

$$\boxed{X = C_1\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}e^{2t} + C_2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}e^{5t} + C_3\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}te^{5t} + \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}e^{5t}\right)}$$

LS. 4 (a) $\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)-2 = 6-5\lambda+\lambda^2-2 = 4-5\lambda+\lambda^2 = (\lambda-1)(\lambda-4)$

$A - I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} k_1 = 0 \Rightarrow k_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$A - 4I = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} k_2 = 0 \Rightarrow k_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore X = C_1 e^{t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + C_2 e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(b) $F = \begin{pmatrix} 9e^{t} \\ 27e^{t} \end{pmatrix}$

$$\phi = \begin{bmatrix} -2e^{t} & e^{4t} \\ e^{t} & e^{4t} \end{bmatrix} \qquad \det(\phi) = -2e^{5t} - e^{5t} = -3e^{5t}$$

$$\phi^{-1} = \frac{-1}{3e^{5t}} \begin{bmatrix} e^{4t} & -e^{4t} \\ -e^{t} & -2e^{t} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -e^{-t} & e^{-t} \\ e^{-4t} & 2e^{-4t} \end{bmatrix}$$

$$\phi^{-1}F = \frac{1}{3}\begin{bmatrix} -e^{-t} & e^{-t} \\ e^{-4t} & 2e^{-4t} \end{bmatrix}\begin{bmatrix} 9e^{t} \\ 27e^{t} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -9+27 \\ 9e^{-3t} + 54e^{-3t} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 18 \\ 63e^{-3t} \end{bmatrix} = \begin{bmatrix} 6 \\ 21e^{-3t} \end{bmatrix}$$

$$\int \phi^{-1}F\,dt = \begin{bmatrix} 6t \\ -7e^{-3t} \end{bmatrix} = u(t)$$

$$\phi\, u(t) = \begin{bmatrix} -2e^{t} & e^{4t} \\ e^{t} & e^{4t} \end{bmatrix}\begin{bmatrix} 6t \\ -7e^{-3t} \end{bmatrix} = \begin{bmatrix} -12te^{t} - 7e^{t} \\ 6te^{t} - 7e^{t} \end{bmatrix}$$

$$\boxed{X_p = \begin{bmatrix} -12 \\ 6 \end{bmatrix} te^{t} - \begin{bmatrix} 7 \\ 7 \end{bmatrix} e^{t}}$$