MA 346 HW3

I pledge on my honor that I have abided by the Stevens Honor System

6.2. 2a. $\begin{bmatrix} 5 & 1 & -6 & 7 \\ 2 & 1 & -1 & 8 \\ 6 & 12 & 1 & 9 \end{bmatrix}$ $i=1, p=1.$ $i=p$ ✓ (no swap)

$\begin{bmatrix} 5 & 1 & -6 & 7 \\ 0 & \frac{3}{5} & \frac{7}{5} & \frac{26}{5} \\ 0 & \frac{54}{5} & \frac{41}{5} & \frac{3}{5} \end{bmatrix}$ $i=2, p=2$ $i=p$ (no swap necessary)

$\begin{bmatrix} 5 & 1 & -6 & 7 \\ 0 & 3/5 & \frac{7}{5} & \frac{26}{5} \\ 0 & 0 & -\frac{85}{5} & -\frac{465}{5} \end{bmatrix}$ no (row) swaps necessary.

2d. $\begin{bmatrix} 1 & -1 & 1 & 5 \\ 7 & 5 & -1 & 8 \\ 2 & 1 & 1 & 7 \end{bmatrix}$ $i=1, p=1$ $i=p$ ✓

$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -27 \\ 0 & 3 & -1 & -3 \end{bmatrix}$ $i=2, p=2$ $i=p$ ✓

$\begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 12 & -8 & -27 \\ 0 & 0 & 1 & \frac{15}{4} \end{bmatrix}$ ✓ no row swaps necessary

2b. $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & 6 & 8 & 5 \\ 6 & 7 & 10 & 5 \end{bmatrix}$

a. 2b, $s_1 = 3, s_2 = 8, s_3 = 10$

$\frac{|a_{1,1}|}{s_1} = \frac{2}{3}$ $\frac{|a_{2,1}|}{s_2} = \frac{4}{8}$ $\frac{|a_{3,1}|}{s_3} = \frac{6}{10}$

a_3 is greatest, so no row interchange.

$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{bmatrix}$

$\frac{|a_{22}|}{s_2} = \frac{4}{8}$ $\frac{|a_{32}|}{10} = \frac{3}{10}$

$\frac{4}{8} > \frac{3}{10}$, so no row interchange

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 0 & -0.5 & -0.25 \end{bmatrix}$$

No row interchanges

b. $\alpha = 9$ $s_1 = 3, s_2 = 8, s_3 = 10.$

Does not change first elimination.

$\frac{|a_{11}|}{3} = \frac{2}{3}$ $\frac{|a_{21}|}{8} = \frac{4}{8}$ $\frac{|a_{31}|}{10} = \frac{6}{10}$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & 6 & 1 & 2 \end{bmatrix}$$

$\frac{|a_{22}|}{s_2} = \frac{4}{8}$ $\frac{|a_{32}|}{10} = \frac{6}{10}.$ $E_2 \leftrightarrow E_3$

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 6 & 1 & 2 \\ 0 & 4 & 2 & 3 \end{bmatrix}$$

Elimination to produce $\rightarrow$ $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 6 & 1 & 2 \\ 0 & 0 & \frac{4}{3} & \frac{5}{3} \end{bmatrix}$

1 row interchange needed.

c. $\alpha = -3$, $s_1 = 3$, $s_2 = 8$, $s_3 = 10$

$\frac{|a_{11}|}{3} = \frac{2}{3},$ $\frac{|a_{21}|}{8} = \frac{4}{8},$ $\frac{|a_{31}|}{10} = \frac{6}{10}.$ $\Rightarrow$ $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 4 & 2 & 3 \\ 0 & -6 & 1 & 2 \end{bmatrix}$

$\frac{|a_{21}|}{s_2} = \frac{4}{8}$ $\frac{|a_{31}|}{s_3} = \frac{6}{10} \Rightarrow E_2 \leftrightarrow E_3 \Rightarrow$ $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -6 & 1 & 2 \\ 0 & 4 & 2 & 3 \end{bmatrix}$

Elimination $\Rightarrow$ $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -6 & 1 & 2 \\ 0 & 0 & \frac{8}{3} & \frac{13}{3} \end{bmatrix}$

1 row interchange needed.

6.3. 5a $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$

$\det M = 4\cdot(0\cdot(-3) - 7(-1)) +$
$(-2)\cdot(3\cdot -3 - 7\cdot(-2)) +$
$6\cdot(3\cdot -1 - 0\cdot -2)$
$= 4\cdot 7 + -2\cdot 5 + 6\cdot -3$
$= 28 - 10 - 18 = 0$ (Matrix is singular)

5b. $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \Rightarrow$ $\det M = 1\cdot(1 - (-1)(1)) +$
$-2\cdot(2\cdot 1 - -1\cdot 3) = 1\cdot 2 + -2\cdot 5 = -8$
(Nonsingular)

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & -5 & 1 & -3 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -3 & -1 & -2 & 1 & 0 \\ 0 & 0 & \frac{8}{3} & \frac{1}{3} & -\frac{5}{3} & 1 \end{array}\right]$$

$b_{31} = \frac{1}{3}\cdot\frac{3}{8} = \frac{1}{8}$
$-3b_{21} - b_{31} = -2$
$-3b_{21} - \frac{1}{8} = -2$
$-3b_{21} = -\frac{15}{8}$
$b_{21} = \frac{5}{8}$
$b_{11} + 2b_{21} = 1$
$b_{11} + 2(\frac{5}{8}) = 1$
$b_{11} = -\frac{1}{4}$

$b_{32} = -\frac{5}{3}\cdot\frac{3}{8} = -\frac{5}{8}$
$-3b_{22} - b_{32} = 1$
$-3b_{22} - -\frac{5}{8} = 1$
$-3b_{22} = \frac{3}{8}$
$b_{22} = -\frac{1}{8}$
$b_{12} + 2b_{22} = 0$
$b_{12} - \frac{2}{8} = 0$
$b_{12} = \frac{1}{4}$

$b_{33} = 1\cdot\frac{3}{8} = \frac{3}{8}$
$-3b_{23} - b_{33} = 0$
$b_{23} = -\frac{1}{3}\cdot\frac{3}{8} = -\frac{1}{8}$
$b_{13} + 2b_{23} = 0$
$b_{13} = -2b_{23} = \frac{2}{8} = \frac{1}{4}$

$A^{-1} = B = \begin{bmatrix} -1/4 & 1/4 & 1/4 \\ 5/8 & -1/8 & -1/8 \\ 1/8 & -5/8 & 3/8 \end{bmatrix}$

8c. B takes more operations because inverse is gaussian elimination + backward sub. Adding on matrix multiplication means more operations than a.

8a. $Ab = \begin{bmatrix} 2 & -3 & 1 & 2 & 6 & 0 & -1 \\ 0 & 2.5 & -1.5 & -2 & 1 & 1 & 0.5 \\ 0 & 0 & -2.8 & 0.6 & 8.2 & -2.8 & -0.4 \end{bmatrix}$

b. $A^{-1} = \begin{bmatrix} 0.1429 & 0.5714 & -0.1429 \\ -0.2857 & 0.3571 & 0.2143 \\ -0.1429 & -0.0714 & -0.3571 \end{bmatrix}$

$x = (A^{-1}B)^T = \begin{bmatrix} -0.2857 & -0.9286 & -0.2143 \\ 2.4286 & -1.3571 & -2.9286 \\ 1 & 1 & 1 \\ -0.1429 & 0.2857 & 0.1429 \end{bmatrix}$

6.4 2b. $\begin{bmatrix} 2 & 2 & 1 \\ 3 & 4 & -1 \\ 3 & 0 & 5 \end{bmatrix}$ $\det M = 3\cdot(2(-1)-4(1)) - 0 + 5(2\cdot 4 - 2\cdot 3)$

$= 3(-6)+5(2) = -8$

6 $\begin{bmatrix} 1 & 2 & -1 \\ 1 & \alpha & 1 \\ 2 & \alpha & -1 \end{bmatrix}$ $\det M = 1\cdot(\alpha\cdot -1 - \alpha\cdot 1) - 2(1\cdot -1 - 1\cdot 2) + -1\cdot(1\cdot\alpha - 2\cdot\alpha)$

$= -2\alpha - 2\cdot(-3) + -1(-\alpha)$

$= -\alpha + 6$

singular when $\det = 0 \Rightarrow \alpha = 6$.

6.5 2a. $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$

$Ly = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix}$

$y_1 = 1$, $-2y_1 + y_2 = 0 \Rightarrow y_2 = 2$, $3y_1 + y_3 = -5 \Rightarrow y_3 = -8$

$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$

$5x_3 = -8 \Rightarrow x_3 = -8/5$ $\quad 4x_2 + 2x_3 = 2 \Rightarrow x_2 = 13/10$

$2x_1 + x_2 - x_3 = 1 \Rightarrow x_1 = \frac{1 + x_3 - x_2}{2} = -\frac{19}{20}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -19/20 \\ 13/10 \\ -8/5 \end{bmatrix}$

and a vector x of size n

8b: Let there be an input vector b of size n, and a vector y of size n,

Step 8: $y_1 = \frac{b_1}{l_{11}}$

Step 9: for $i = 2, 3 \ldots n$

set $y_i = \frac{1}{l_{ii}}\left[b_i - \sum_{j=1}^{i-1} l_{ij} y_j\right]$

Step 10: $x_n = \frac{y_n}{u_{nn}}$

for $i = n-1, n-2 \ldots 1$

set $x_i = \frac{1}{u_{ii}}\left[y_i - \sum_{j=i+1}^{n} u_{ij} x_j\right]$

$$A = \begin{bmatrix} 1/3 & 1/2 & -1/4 \\ 1/5 & 2/3 & 3/8 \\ 2/5 & -2/3 & 5/8 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$$

Step 1: $l_{11} = 1/3, \quad u_{11} = 1$

Step 2: $u_{12} = \frac{1}{2}/\frac{1}{3} = 3/2 \quad u_{13} = -\frac{1}{4}/\frac{1}{3} = -3/4$

$l_{21} = \frac{1}{5}/1 = 1/5 \quad l_{31} = \frac{2}{5}/1 = 2/5$

Step 3: $i = 2$

Step 4: $l_{22}u_{22} = a_{22} - \sum_{k=1}^{1} l_{2k}u_{k2} = 2/3 - l_{21}u_{12} = 2/3 - 1/5 \cdot 3/2$

$= \frac{11}{30}$

$l_{22} = \frac{11}{30}, \quad u_{22} = 1$

Step 5: ~~for~~ $j = 3$

$u_{23} = \frac{1}{l_{22}}\left[a_{23} - l_{21}u_{13}\right] = \frac{1}{\frac{11}{30}}\left[3/8 - 1/5 \cdot \left(-\frac{3}{4}\right)\right] = \frac{63}{44}$

$l_{32} = \frac{1}{u_{22}}\left[a_{32} - l_{31}u_{12}\right] = \frac{1}{1}\left[-2/3 - 2/5 \cdot 3/2\right] = -\frac{19}{15}$

Step 6:

$l_{33}u_{33} = a_{33} - \sum_{k=1}^{2} l_{3k}u_{k3} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

$= \frac{5}{8} - 2/5 \cdot -3/4 - -\frac{19}{15} \cdot \frac{63}{44} = \frac{241}{88}$

$l_{33} = \frac{169}{152}, \quad u_{33} = 1$

$$l = \begin{bmatrix} 1/3 & 0 & 0 \\ 1/5 & 11/30 & 0 \\ 2/5 & -\frac{19}{15} & \frac{241}{88} \end{bmatrix} \quad u = \begin{bmatrix} 1 & 3/2 & -3/4 \\ 0 & 1 & \frac{63}{44} \\ 0 & 0 & 1 \end{bmatrix} \qquad y = \begin{bmatrix} 3 \\ \frac{42}{11} \\ \frac{56}{241} \end{bmatrix}$$

Step 8: $y_1 = \frac{b_1}{l_{11}} = 3$

Step 9: $y_2 = \frac{1}{l_{22}}\left[b_2 - \sum_{j=1}^{i-1} l_{2j}y_j\right] = \frac{30}{11}\left[2 - 1/5 \cdot 3\right] = \frac{42}{11}$

$y_3 = \frac{1}{l_{33}}\left[b_3 - l_{31}y_1 - l_{32}y_2\right] = \frac{88}{241}\left[-3 - 2/5 \cdot 3 - -\frac{19}{15} \cdot \frac{42}{11}\right]$

$= 0.232 = \frac{56}{241}$

Step 10: $x_3 = \frac{y_3}{u_{33}}$ $x_3 = \frac{\frac{56}{241}}{1} = \frac{56}{241}$

$x_2 = \frac{42}{4} - \frac{63}{44}\left(\frac{56}{241}\right)$

$x_2 = \frac{840}{241}$

$x = \begin{bmatrix} -\frac{495}{241} \\ \frac{840}{241} \\ \frac{56}{241} \end{bmatrix}$

$x_1 = 3 - \frac{3}{2} \cdot \frac{840}{241} - \left(-\frac{3}{4}\right)\left(\frac{56}{241}\right)$

$x_1 = -\frac{495}{241}$

10b. $\begin{bmatrix} 1 & -2 & 3 & 0 \\ 1 & -2 & 3 & 1 \\ 1 & -2 & 2 & -2 \\ 2 & 1 & 3 & -1 \end{bmatrix} \xrightarrow{r_4 \leftrightarrow r_1} \begin{array}{c} \begin{bmatrix} 2 & 1 & 3 & -1 \\ 1 & -2 & 3 & 1 \\ 1 & -2 & 2 & -2 \\ 1 & -2 & 3 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ -0.5r_1 \\ -0.5r_1 \\ -0.5r_1 \end{array}$

$\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & -2.5 & 1.5 & 1.5 \\ 0 & -2.5 & 0.5 & -1.5 \\ 0 & -2.5 & 1.5 & 0.5 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & -2.5 & 0.5 & -1.5 \\ 0 & -2.5 & 1.5 & 1.5 \\ 0 & -2.5 & 1.5 & 0.5 \end{bmatrix} \begin{array}{l} \\ \\ -r_2 \\ -r_2 \end{array}$

$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2.5 & 0.5 & -1.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ -r_3 \end{array} \rightarrow \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2.5 & 0.5 & -1.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

$r_4 \leftrightarrow r_1,\ r_2 \leftrightarrow r_3 \Rightarrow P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \Rightarrow Pe = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$r_1 = r_1$, $r_2 = r_2 - 0.5r_1$, $r_3 = r_3 - 0.5r_1 - r_2$

$r_4 = r_4 - 0.5r_1 - r_2 - r_3$

$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0.5 & 1 & 1 & 1 \end{bmatrix}$

$$A = P^t L U = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0.5 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & -2.5 & 0.5 & -1.5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$