Max Sui

I pledge my honor that I have abided by the Stevens Honor Code
_Max Sui

Problem Set 4

1. The function $g \circ f$ would result in the transformation $A \to C$ through B. Because f is bijective, every unique A put into f will have a unique value B. Because g is also bijective, every unique B plugged into g will have a unique value for C. Thus, combining these two ideas, each unique value of A has a unique output C, this meets the definition of a bijective function, where each element in the domain has a ~~one~~ ~~unique output~~ not shared with any other element in the domain.

2a. Let x and y be integers such that $x \leq y$.
Let $g \circ f$ be rewritten as $g(f)$.
Because $x \leq y$, by definition, $f(x) \leq f(y)$
substituting $f(x)$ for x_f and $f(y)$ for y_f,
we can express the same inequality as $x_f \leq y_f$
Plugging these into g, ~~it~~ follows that $g(x_f) \leq g(y_f)$.
Substituting back we get $g(f(x)) \leq g(f(y))$,
which is the same as $g \circ f(x) \leq g \circ f(y)$.
This implies that $x \leq y \Rightarrow g \circ f(x) \leq g \circ f(y)$,
therefore $g \circ f$ is monotonically non decreasing.

b. $f(x) = 4x$ (non-decreasing)
$g(x) = |1-x|$ (decreasing) $0 < 1$ but $|1-0| > |1-1|$

$g \circ f(x) = |1-4x|$ is non-decreasing because:
$g \circ f(0) \leq g \circ f(1)$
$1 \leq 3$
and for all $x \in N$ $x \geq 1$
$g \circ f(x) \leq g \circ f(x+1)$
$|1-4x| \leq |1-4(x+1)|$
$4x-1 \leq 4(x+1)-1$
$4x-1 \leq 4x+3$
$0 \leq 4$

Because $1-4x$ for all $x \geq 1$ = $4x-1$, substitute.

Therefore because $x \geq y \Rightarrow g \circ f(x) > g \circ f(y)$, $g \circ f$ is monotonically non-decreasing.

3. Let $A \subseteq \mathbb{N}$ (infinite subset), and let f be a function that maps $A \to \mathbb{N}$.

Let there be an x_0, such that $\forall x \in A, (x_0 \le x)$ by WOP.

let $f(x_0) = 0$.

Let $A - \{x_0\}$ be A without x_0.

Let there be an x_1 such that $\forall x \in (A - \{x_0\}), (x_1 \le x)$ by WOP

Let $f(x_1) = 1$.

This process is a bijection because this process is infinitely repeatable, therefore because there is a bijective function between A and N, A is countably infinite and all infinite subsets of N are countably infinite.

4a. If h is surjective, that means that all elements C have at least one value of A associated with them.

However, because h is surjective, $\forall c \exists a \; h(a) = c$.

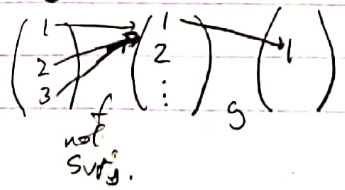
Substituting for h, we get $\forall c \exists a \; g(f(a)) = c$

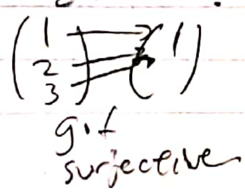
Let b be some element such that $f(a) = b$, so $\forall c \exists b \; g(b) = c$.

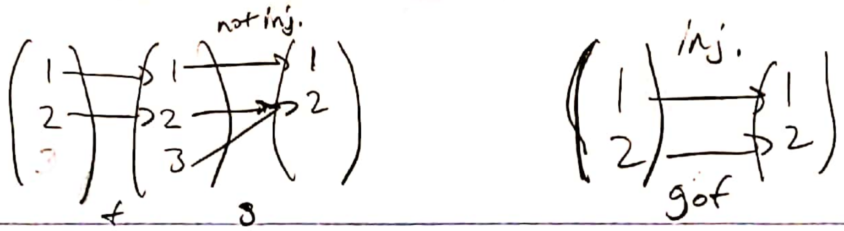
This reads as for all c, there is at least one element b in the domain associated with it.

Therefore, as this is the definition of surjective, h = surjective ⇒ g = surjective.

b. Let f be a function such that there exists an element x such that $(\forall y \; f(y) = x) \wedge (x \in B) \wedge (|B| > 1)$. Thus, f is not surjective because $\exists b \; b \in B \wedge f(y) \neq b$. Then, taking the same x from before, let there exist an element z such that $C = \{z\}$ and $g(x) = z$. Thus, $h = g(f(x))$, and $\forall a \in A \; g(a) = z$. because $C = \{z\}$, h is surjective, but f isn't, therefore h = surjective does not imply f = surjective.







4c. Let g be a non injective function such that $\exists x \exists y \exists z$ where $x, y \in B$ and $z \in C$ such that $g(x) = z \land g(y) = z$. If f is an injective function such that $\forall i \forall j \forall k$ where $i, j \in A$ and $k \in B$ such that $\neg(f(i) = k \land f(j) = k) \land f(i) \neq y \land f(j) \neq y$, then the double mapping to z does not make h non-injective, and h is still injective while g is non-injective. Thus h being injective does not imply g is injective.

d. If h is injective, then $\forall x, y \in A$, $x \neq y \Rightarrow h(x) \neq h(y)$.
If f is not injective then $\exists i, j \in A$, $f(i) = f(j) \land i \neq j$
Expanding step 1, $\forall x, y \in A$, $x \neq y \Rightarrow g(f(x)) \neq g(f(y))$
However, if $x = i$ and $y = j$, this becomes $i \neq j \Rightarrow g(f(i)) \neq g(f(j))$.
Because $f(i) = f(j)$, substitution gives $i \neq j \Rightarrow g(f(i)) \neq g(f(i))$.
$g(f(i)) \neq g(f(i))$ is false, and because the predicate is true and the antecedent is false, h is not injective. This is a contradiction therefore f must be injective.