I pledge my honor that I have abided by the Stevens Honor System
[signature]

MA346 Final Exam

1a. $\cos h + \frac{1}{2}h^2 = 1 + O(h^4)$

$\cos h + \frac{1}{2}h^2 - 1 = O(h^4)$ $\alpha_n = \cos h + \frac{1}{2}h^2$ $\alpha = 1$, $\beta_h = h^4$

True if $|\alpha_h - \alpha| \leq k|\beta_h|$ for large h.

$|\cos h + \frac{1}{2}h^2 - 1| \leq 1 + \frac{1}{2}h^2 - 1 = \frac{1}{2}h^2 \leq h^4$

$-1 \leq \cos h \leq 1$ for all values $h > 1$.

Thus, $\cos h + \frac{1}{2}h^2 = 1 + O(h^4)$ is true, by Definition 1.18 in the textbook. I agree.

(b) 3^{-2^n} $\lim_{n\to\infty} \frac{|3^{-2^{n+1}}|}{|3^{-2^n}|} = \lim_{n\to\infty} \frac{3^{2^n}}{3^{2^{n+1}}} = \lim_{n\to\infty} \frac{3^{2^n}}{(3^{2^n})^2} = 0$

$p = 0$

$\alpha = 1$

Converges linearly to 0, I agree.

(c) $E_n = (2n + \frac{1}{e^{0.1n}} + 3)E_0$

This is linear, as $\frac{1}{e^{0.1n}} \to 0$ for large n. Thus, this error is stable.

(d) This is the midpoint formula, with error $= -\frac{h^2}{6}f^{(3)}(\xi_1)$.
This means the error is dependent on h. Therefore, I agree. Reducing h will increase precision, but one must be careful of round off errors with small values of h.

2 (a) $g(2) = 2 - c\frac{2^3 - 8}{2^2} = 2 - c\left(\frac{8-8}{4}\right) = 2$ yes.

(b) $g'(x) \leq 1$ for all x. $g(x) = x - c\frac{x^3}{x^2} + c\frac{8}{x^2} = x - cx + \frac{8c}{x^2}$

$g'(x) = 1 - c - \frac{16c}{x^3}$

If interval is $[1, 3]$, $g'(x)$ has max at $x = 1$.

$|1 - c - \frac{16c}{1}| \leq 1$

$-1 \leq 1 - c - 16c \leq 1$

$-2 \leq -17c \leq 0$

$0 \leq c \leq \frac{2}{17}$ for guaranteed linear convergence on $[1, 3]$.

c. for quadratic convergence $g'(p)=0$. $p=2$

$$g'(x) = 1 - c - \frac{16c}{x^3}$$
$$g'(2) = 1 - c - \frac{16c}{8}$$
$$0 = 1 - c - 2c$$
$$3c = 1$$
$$c = \frac{1}{3}$$

3. $$a_i = \int_{x_0}^{x_n} L_i(x)\,dx = \int_{x_0}^{x_n} \prod_{\substack{j=0 \\ j\neq i}}^{4} \frac{(x-x_j)}{(x_i-x_j)}\,dx$$

$$a_3 = \int_{x_0}^{x_n} L_3(x)\,dx = \int_{x_0}^{x_n} \underbrace{\prod_{\substack{j=0 \\ j\neq 3}}^{4} \frac{(x-x_j)}{(x_3-x_j)}}\,dx$$

$x_3 = -2+3\cdot h = 1$

$h = 1$

$$\frac{x-(-2)}{1-(-2)} \cdot \frac{x-(-1)}{1-(-1)} \cdot \frac{x-0}{1-0} \cdot \frac{x-2}{1-2}$$
$$= \frac{1}{3}(x+2) \cdot \frac{1}{2}(x+1) \cdot x \cdot -(x-2)$$
$$-\left(\frac{x^4+x^3-4x^2-4x}{6}\right)$$

$$\int_{-2}^{2} -\left(\frac{x^4+x^3-4x^2-4x}{6}\right)dx =$$

$$-\frac{1}{6}\int_{-2}^{2} x^4+x^3-4x^2-4x\,dx$$
$$= -\frac{1}{6}\left(\frac{x^5}{5}+\frac{x^4}{4}-\frac{4x^3}{3}-\frac{x^2}{2}\right)\Big|_{-2}^{2}$$
$$= \frac{64}{45}$$

	x_i	y_i				
4(a)	-2	4				
			$\frac{7-4}{-1-(-2)} = 3$			
	-1	7		$\frac{-8-3}{0-(-2)} = -\frac{11}{2}$		
			$\frac{-1-7}{0-(-1)} = -8$		$\frac{5-(-\frac{11}{2})}{1-(-2)} = \frac{7}{2}$	
	0	-1		$\frac{2-(-8)}{1-(-1)} = 5$		$-\frac{17}{12}$
			$\frac{1-(-1)}{1-0} = 2$		$\frac{-\frac{3}{2}-5}{2-(-1)} = -\frac{13}{6}$	
	1	1		$\frac{-1-(2)}{2-0} = -\frac{3}{2}$		
			$\frac{0-1}{2-1} = -1$			
	2	0				

$$\frac{-\frac{13}{6}-\frac{7}{2}}{4} = -\frac{17}{12}$$

(b) $P_4(x) = 4 + 3(x+2) - \frac{11}{2}(x+1)(x+2) + \frac{7}{2}(x+1)(x+2)(x)$
$- \frac{17}{12}(x+2)(x+1)(x)(x-1)$

(c) $f(0.5) = 4 + 3(2.5) - \frac{11}{2}(1.5)(2.5) + \frac{7}{2}(1.5)(2.5)(0.5)$
$- \frac{17}{12}(2.5)(1.5)(0.5)(-0.5)$
$= -1.23438$

(d) error $= |f(x) - P_n(x)| \le \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i)$

$|f(0.5) - P_4(0.5)| \le \frac{f^5(\xi(x))}{5!} (0.5+2)(0.5+1)(0.5)(0.5-1)(0.5-2)$

error $\le \frac{8}{5!}(2.5)(1.5)(0.5)(-0.5)(-1.5)$
error ≤ 0.09375

5. $S_0'(1) = S_1'(1)$

$S_0'(x) = 6x^2 \Rightarrow S_0'(1) = 6$

$S_1'(x) = 3x^2 + 2\beta x - 3 \Rightarrow S_1'(1) = 3 + 2\beta - 3 = 2\beta$

$2\beta = 6 \Rightarrow \beta = 3.$

$S_1'(2) = S_2'(2)$

$S_1'(x) = 3x^2 + 6x - 3 \Rightarrow S_1'(2) = 12 + 12 - 3 = 21$

$S_2'(x) = 18x - 3\delta \Rightarrow S_2'(2) = 36 - 3\delta$

$36 - 3\delta = 21 \Rightarrow 15 = 3\delta \Rightarrow \delta = 5$

$S_0(1) = S_1(1) \Rightarrow 2(1)^3 + \alpha = 1^3 + 3(1) - 3(1) + \gamma$

$2 + \alpha = 1 + 3 - 3 + \gamma$

$1 + \alpha = \gamma$

$S_1(2) = S_2(2) \Rightarrow 2^3 + 3(2)^2 - 3(2) + \gamma = 9(2)^2 - 15(2) + 9$

$8 + 12 - 6 + \gamma = 36 - 30 + 9$

$\gamma = 1 \quad \beta = 3$

$\alpha = 0. \quad \delta = 5$

b) $S_0''(0) = 12(0) = 0$

$S_2''(0) = 18 = 18$

Because $S_2''(0) \neq 0$, This is not a natural cubic spline

6. (a) $\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}$ $s_1 = 2, \quad s_2 = 3, \quad s_3 = 4$

$\frac{|a_{11}|}{s_1} = \frac{1}{2} \quad \frac{|a_{21}|}{s_2} = \frac{1}{3} \quad \frac{|a_{31}|}{s_3} = \frac{1}{2}$ (no pivot)

$\therefore m_{21} = \frac{a_{21}}{a_{11}} = 1 \quad m_{31} = \frac{a_{31}}{a_{11}} = 2$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 4 \\ 0 & -5 & 6 \end{bmatrix}$ $\frac{|a_{22}|}{s_2} = 0 \quad \frac{|a_{32}|}{s_3} = \frac{5}{4}$ $(R_3 \leftrightarrow R_2)$

$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 4 \end{bmatrix}$ $m_{32} = 0$

(complete)

$$U = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$ (Result of elimination)

$R_3 = R_2' = R_2 - R_1 \quad R_2 = R_3' = R_3 - 2R_1$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \Rightarrow P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P^t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A = P^t L U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

7.(a) Closed Newton Cotes $n=4$:

$$\int_1^5 f(x)\,dx = \frac{2(1)}{45}\left(7f(1) + 32\cancel{f}f(2) + 12f(3) + 32(f(4)) + 7f(5)\right)$$

$h = \frac{b-a}{n} = \frac{5-1}{4} = 1$

$$= \frac{2}{45}\left(16.8994 + 85.5488 + 34.7688 + 99.1232 + 22.9628\right)$$

$$= 11.52457778$$

- Simpsons $n=2$:

$$\int_1^5 f(x)\,dx = \frac{h}{3}\left[f(1) + 4f(3) + f(5)\right]$$

$h = \frac{5-1}{2} = 2$

$$= \frac{2}{3}\left[2.4142 + 4(2.8974) + 3.2804\right]$$

$$= 11.5228$$

b. $R_{1,1} = \frac{b-a}{2}\left[f(1) + f(5)\right] = \frac{4}{2}\left[2.4142 + 3.2804\right] = 11.3892$

$R_{2,1} = \frac{4}{4}\left[f(1) + 2(f(3)) + f(5)\right] = 1\left[2.4142 + 2(2.8974) + 3.2804\right]$
$= 11.4894$

$R_{3,1} = \frac{4}{8}\left[f(1) + 2(f(2) + f(3) + f(4)) + f(5)\right]$
$= \frac{1}{2}\left[2.4142 + 2(2.6734 + 2.8974 + 3.0976) + 3.2804\right]$
$= 11.5157$

Table

11.3892		
11.4894	11.5228	
11.5157	11.524467	11.524578

$R_{2,2} = R_{2,1} + \frac{1}{3}(R_{2,1} - R_{1,1}) = 11.4894 + \frac{1}{3}(11.4894 - 11.3892)$
$= 11.5228$

$R_{3,2} = R_{3,1} + \frac{1}{3}(R_{3,1} + R_{2,1}) = 11.5157 + \frac{1}{3}(11.5157 - 11.4894)$
$= 11.52446667$

$R_{3,3} = R_{3,2} + \frac{1}{15}(R_{3,2} - R_{2,2}) = 11.524467 - \frac{1}{15}(11.524767 - 11.5228)$
$= 11.52457778$

$\int_1^5 f(x)\,dx = 11.52457778$

8a. $\int_0^2 xe^x dx$

$a=0 \quad b=2$

$t = \frac{2x-a-b}{b-a} = \frac{2x-0-2}{2-0} = \frac{2x-2}{2} = x-1$

$x = t+1$

$\int_0^2 xe^x = \int_{-1}^1 (t+1)e^{t+1}\left(\frac{2-0}{2}\right)dt = \int_{-1}^1 (t+1)(e^{t+1})dt$

$f(t) = (t+1)e^{t+1}$

$(n=3)$

$\int_{-1}^1 f(t)dt = 0.\bar{5} f(0.7745966692) + 0.\bar{8} f(0) + 0.\bar{5} f(-0.7745966692)$

$= 5.814664989 + 2.416250514 + 0.156884214$

$= 8.387799717$

(b) $\int_0^2 xe^x dx = \frac{2}{2}[0 + 2e^2] = 14.7781122$

$h = \frac{b-a}{1} = 2$ $\quad \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f(x_0) + f(x_1))$

relative error

(a) $\frac{|1+e^2 - 8.387799717|}{1+e^2} = 1.49764 \times 10^{-4}$

(b) $\frac{|1+e^2 - 14.7781122|}{1+e^2} = 0.761594$

9(a) $f(t, y(t)) = y^2 + ye^{t} = y''$

$f'(t, y(t)) = 2y\,y' + y'e^{t} + ye^{t}$
$= 2y(y^2+ye^{t}) + (y^2+ye^{t})(e^{t}) + ye^{t}$
$= 2y^3 + 2y^2e^{t} + y^2e^{t} + ye^{2t} + ye^{t}$
$= 2y^3 + 3y^2e^{t} + ye^{2t} + ye^{t}$

$T_2 = y^2+ye^{t} + \frac{h}{2}(2y^3+3y^2e^{t}+ye^{2t}+ye^{t})$

$w_0 = \alpha = 1$
$w_1 = w_0 + hT^{(2)}$
$w_1 = 1 + 0.1(1^2+1e^0+\frac{0.1}{2}(2(1)^3+3(1)^2e^0+1e^0+1e^0))$
$= 1+0.1(1+1+0.05(2+3+1+1))$
$= 1.235$
$y(0.1) \approx 1.235$

(b) y' is defined on convex set, as on $-2 \le t \le 2$, $-\infty \le y' \le \infty$.

Then, $\frac{df}{dy}(t, y) = 2(t-\cos\xi(y)) \cdot \sin(\xi(y))$
$-1 \le \cos\xi \le 1$
$-1 \le \sin\xi \le 1 \quad -2 \le t \le 2$

$|2(t-\cos\xi(y))) \cdot \sin(\xi(y))| \le 2(2-(-1)) \cdot 1$
≤ 6

By theorem 5.4, because $f(t, y)$ is defined on a convex set and satisfies the Lipschitz condition with $L = 6$, the IVP has a unique solution $y(t)$ for $-2 \le t \le 2$.

10a. $M = \phi(h) + a_1 h + a_3 h^3 + a_5 h^5$ (E1)

$M = \phi\left(\frac{h}{2}\right) + a_1\left(\frac{h}{2}\right) + a_3 \frac{h^3}{8} + a_5 \frac{h^5}{32} \ldots$ (E2)

$2\cdot(E2) - (E1) \Rightarrow M = \phi\left(\frac{h}{2}\right) + \left[\phi\left(\frac{h}{2}\right) - \phi(h)\right] + \left(\frac{a_3}{4} - a_3\right)h^3 + \left(\frac{a_5}{16} - a_5\right)h^5$

"Let $\phi_1(h) = \phi\left(\frac{h}{2}\right) + \left[\phi\left(\frac{h}{2}\right) - \phi(h)\right]$

value b_3 " ..... value b_5

$M = \phi_1(h) + b_3 h^3 + b_5 h^5$

By Richardson's extrapolation, $\phi_1(h)$ now approximates M to an order of h^3 instead of h. This can be repeated to further increase the precision, as long as the second equation after $h/2$ replacement is multiplied enough

b. Replace h with $h/2$. $\lim_{h\to 0} f(h) = \lim_{h \to 0} f\left(\frac{h}{2}\right)$, so L stays the same.

$L - f(h) = a_6 h^6 + a_9 h^9$ (E1)

$L - f\left(\frac{h}{2}\right) = a_6 \frac{h^6}{64} + a_9 \frac{h^9}{512}$ (E2)

to get rid of the leading error term upon subtraction. e.g. next iteration is 8E4 - E3.

64 E2 - E1

$\Rightarrow 64L - 64 f\left(\frac{h}{2}\right) - L + f(h) =$

$a_6 h^6 - a_6 h^6 + \left(\frac{a_9}{8} - a_9\right) h^9$

$\Rightarrow 63L = 64\, f\left(\frac{h}{2}\right) - f(h) + b_9 h^9$

$L = \left(\frac{64 f\left(\frac{h}{2}\right) - f(h)}{63}\right) + b_9 h^9$

This term will now be accurate to h^9 instead of h^6.