

1. Calculer la division euclidienne de $X^5 + 1$ par $X^4 - 2X^3 + X^2$.



$$X^5 + 1 = (X + 2) \cdot (X^4 - 2X^3 + X^2) + 3X^3 - 2X^2 + 1$$

2. Factoriser dans $\mathbb{R}[X]$ le polynôme $X^4 - 2X^3 + X^2$.



$$X^4 - 2X^3 + X^2 = X^2 \cdot (X^2 - 2X + 1) = X^2 \cdot (X - 1)^2$$

3. On note $F(X) = \frac{X^5 + 1}{X^4 - 2X^3 + X^2}$. Donner la forme de la décomposition en éléments simples de $F(X)$ dans $\mathbb{R}(X)$.



$$F(X) = \frac{X^5 + 1}{X^4 - 2X^3 + X^2} = \frac{X^5 + 1}{X^2 \cdot (X - 1)^2} = X + 2 + \frac{A}{X} + \frac{B}{X^2} + \frac{C}{(X - 1)} + \frac{D}{(X - 1)^2}$$

4. Calculer $\lim_{X \rightarrow 0} (X^2 \cdot F(X))$



$$\begin{aligned} \lim_{X \rightarrow 0} (X^2 \cdot F(X)) &= \lim_{X \rightarrow 0} \frac{X^2 \cdot (X^5 + 1)}{X^2 \cdot (X - 1)^2} = 1 \\ &= \lim_{X \rightarrow 0} X^2 \cdot \left[X + 2 + \frac{A}{X} + \frac{B}{X^2} + \frac{C}{(X - 1)} + \frac{D}{(X - 1)^2} \right] = B \end{aligned}$$

5. Calculer $\lim_{X \rightarrow 1} ((X - 1)^2 \cdot F(X))$.



$$\begin{aligned} \lim_{X \rightarrow 1} ((X - 1)^2 \cdot F(X)) &= \lim_{X \rightarrow 1} \frac{(XA)^2 \cdot (X^5 + 1)}{X^2 \cdot (X - 1)^2} = 2 \\ &= \lim_{X \rightarrow 1} (X - 1)^2 \cdot \left[X + 2 + \frac{A}{X} + \frac{B}{X^2} + \frac{C}{(X - 1)} + \frac{D}{(X - 1)^2} \right] = D \end{aligned}$$

6. Calculer $F(-1)$ et $F(2)$ de deux façons différentes.



$$\begin{aligned} F(-1) &= \frac{(-1)^5 + 1}{(-1)^2 \cdot (-1 - 1)^2} = 0 \\ &= -1 + 2 + \frac{A}{-1} + \frac{1}{(-1)^2} + \frac{C}{(-1 - 1)} + \frac{2}{(-1 - 1)^2} = 2 - A - \frac{C}{2} + \frac{1}{2} \\ &\Rightarrow 2A + C = 5 \\ F(2) &= \frac{(2)^5 + 1}{(2)^2 \cdot (2 - 1)^2} = \frac{33}{4} \\ &= 2 + 2 + \frac{A}{2} + \frac{1}{(2)^2} + \frac{C}{(2 - 1)} + \frac{2}{(2 - 1)^2} = 4 + \frac{A}{2} + \frac{1}{4} + C + 2 \\ &\Rightarrow A + 2C = 4 \\ \left\{ \begin{array}{l} 2A + C = 5 \\ A + 2C = 4 \end{array} \right. &\Leftrightarrow \left\{ \begin{array}{l} A = 2 \\ C = 1 \end{array} \right. \end{aligned}$$

7. En déduire la décomposition en éléments simples de $F(X)$ dans $\mathbb{R}(X)$.



$$F(X) = \frac{X^5 + 1}{X^4 - 2X^3 + X^2} = \frac{X^5 + 1}{X^2 \cdot (X - 1)^2} = X + 2 + \frac{2}{X} + \frac{1}{X^2} + \frac{1}{(X - 1)} + \frac{2}{(X - 1)^2}$$