DMpD ⊳ **Exercice** - *Dérivées partielles*

Soit la fonction $V\colon \mathbb{R}^2 \to \mathbb{R}$ définie par

$$V(r,\theta) = \frac{\cos\theta}{r^2}$$

1. Donner l'ensemble de définition de V.

   $\mathcal{D}_V = \mathbb{R}^* \times \mathbb{R}$

2. Calculer les dérivées partielles $\dfrac{\partial V}{\partial r}$ et $\dfrac{\partial V}{\partial \theta}$.

   pour $(r,\theta) \in \mathcal{D}_V$, on a
   $$\frac{\partial V}{\partial r} = -\frac{2}{r^3}\cos\theta, \quad \frac{\partial V}{\partial \theta} = -\frac{1}{r^2}\sin\theta$$

3. Calculer les dérivées partielles secondes $\dfrac{\partial^2 V}{\partial r^2}$, $\dfrac{\partial V}{\partial \theta}$ et $\dfrac{\partial^2 V}{\partial r\, \partial \theta}$.

   pour $(r,\theta) \in \mathcal{D}_V$, on a

   $$\begin{aligned}
   \frac{\partial^2 V}{\partial r^2} &= \frac{\partial}{\partial r}\left[-\frac{2}{r^3}\cos\theta\right] = \frac{6}{r^4}\cos\theta \\
   \frac{\partial^2 V}{\partial \theta^2} &= \frac{\partial}{\partial \theta}\left[-\frac{1}{r^2}\sin\theta\right] = -\frac{1}{r^2}\cos\theta \\
   \frac{\partial^2 V}{\partial r\, \partial \theta} &= \frac{\partial}{\partial r}\left[-\frac{1}{r^2}\sin\theta\right] = \frac{2}{r^3}\sin\theta
   \end{aligned}$$

4. En déduire que
   $$\sin\theta\, \frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] = 0.$$

   On a

   $$\begin{aligned}
   &\sin\theta\, \frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] \\
   &= \sin\theta\left[2r\frac{\partial V}{\partial r} + r^2\frac{\partial^2 V}{\partial r^2}\right] + \left[\cos\theta\frac{\partial V}{\partial \theta} + \sin\theta\frac{\partial^2 V}{\partial \theta^2}\right] \\
   &= \sin\theta\left[2r(-\frac{2}{r^3}\cos\theta) + r^2\frac{6}{r^4}\cos\theta\right] + \left[\cos\theta(-\frac{1}{r^2}\sin\theta) + \sin\theta(-\frac{1}{r^2}\cos\theta)\right] \\
   &= \frac{\cos\theta\sin\theta}{r^2}(-4+6-1-1) \\
   &= 0
   \end{aligned}$$