Ex

Soit $n \in \mathbb{N} \setminus \{0,1\}$. Montrer que $P_n(X) = (X+1)^{2n} - X^{2n} - 2X - 1$ est divisible par $Q(X) = X(X+1)(2X+1)$.

✓

On a :

$$P_n(0) = (1)^{2n} - 0 - 0 - 1 = 0$$
$$P_n(-1) = (-1+1)^{2n} - (-1)^{2n} - 2\cdot(-1) - 1 = 0 - 1 + 2 - 1 = 0$$
$$P_n\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}+1\right)^{2n} - \left(-\frac{1}{2}\right)^{2n} - 2\cdot\left(-\frac{1}{2}\right) - 1 = \left(\frac{1}{2}\right)^{2n} - \left(\frac{1}{2}\right)^{2n} + 1 - 1 = 0$$

0 racine signifie que X divise $P_n(X)$: $P_n(X) = X.Q_1(X)$; -1 racine signifie que $(X+1)$ divise $P_n(X)$: $P_n(X) = (X+1)\cdot \underbrace{Q_2(X)}_{X\cdot Q_3(X)} = X\cdot(X+1)\cdot Q_3(X)$; $-\frac{1}{2}$ racine signifie que $(2X+1)$ divise $P_n(X)$:

$$P_n(X) = (2X+1)\cdot Q_4(X) = \underbrace{X\cdot(X+1)\cdot(2X+1)}_{Q(X)}\cdot Q_5(X)$$

donc $Q(X) = X(X+1)(2X+1)$ divise $P_n(X) = (X+1)^{2n} - X^{2n} - 2X - 1$.