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Clustering-Based Model Reduction for Heat Diffusion Systems

Internship Report

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***Abstract*— In this paper we use finite difference technique to model the 2-D heat diffusion system as a network system. The obtained model is then reduced to a lower order system using clustering technique by computing the characteristic (aggregation) matrix. Furthermore we compare the results with an "average state observer"[8] for network systems.**

I. Introduction

Network systems or multi-agent systems are made up of numerous sub-systems which exhibit various behaviors and dynamics of interconnected physical systems. These network systems can be found in numerous applications such as power grids[1], chemical reaction networks[2] and cooperative control[3], etc. Increase in number of sub-systems in the network systems leads to the increase in complexity of the system which ultimately leads to difficulty in formulation of a complex controller for the system. Thus we need model reduction techniques to reduce the complexity and dimension of such complex systems.

Classical model reduction approaches include methods such as balanced truncation[10], [11] and Hankel-norm approximation[12]. The main disadvantage of these methods is that, they do not preserve the network structure. Hence we reside to clustering techniques. In clustering technique we simply group the similar behaving systems or nodes together, which is made possible due to the use of graph theory [6] and this in turn achieves the preservation of network structure.

In this paper we apply the clustering technique to a heat diffusion system, as we are often required to measure the temperature accurately for systems having temperature distribution playing a vital role for its operation. As the size of a heat diffusion system increases we need more sensors and more complex controllers to control the heat distribution. Hence it becomes an important task to approximate the higher order model to obtain the best possible lower order model to perceive the approximated heat distribution.

The paper is organised as follows. Section II contains the modelling of the heat diffusion system. The Clustering technique is elaborated in Section III. The resulting reduced order model is shown in Section IV along with its error analysis in section V. Specification of parameters for the heat diffusion system under consideration is provided in Section VI followed by simulation results in section VII. A brief explanation about the average state observer and its comparison with our results are shown in Section VIII and Section IX respectively. Section X provides the conclusion.

II. Modelling

At first it is important that we model our system and analyse its structure using the PDE's. Here we consider a system consisting of metal plate and a heating element which is surrounded by air as shown in Figure 1

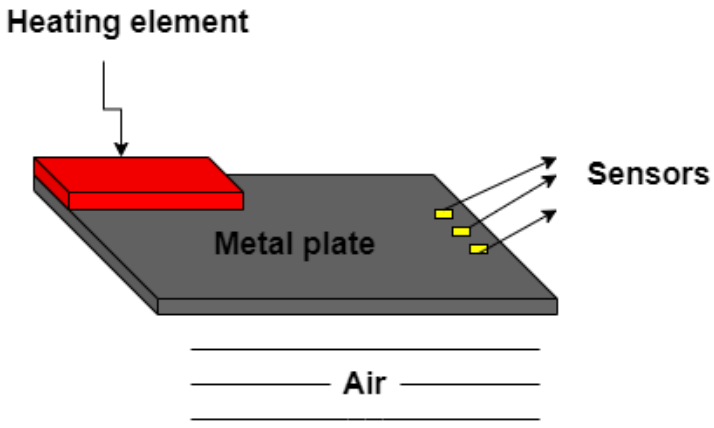


Fig. 1: Thermal diffusion system

A. Heat equation

Since we consider a system consisting of a plate of size $L_x \times L_y$ which is two-dimensional in nature, we apply the 2D heat equation with a heat source which is given by [4] :

$$\frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} \right) + \frac{\dot{q}}{\beta} \tag{1}$$

Where λ is thermal diffusivity, β is thermal conductivity and $\dot{q}$ is heat transfer rate.
Here, T(X,Y,t) denotes temperature at location X $\in [0, L_x]$, Y$\in [0, L_Y]$ and t$\geq$ 0.

B. Choice of Boundary condition

For the above Partial differentiation Equation (PDE) in (1) we need appropriate boundary conditions. Typically for such partial differentiation problems we have 3 types of Boundary conditions:

- Dirichlet Boundary condition
- Neumann Boundary Condition
- Robin Boundary condition

For the given system, Neumann Boundary condition is selected due to following reasoning:

- In heat transfer problems, the Neumann condition corresponds to a given rate of change of temperature. In other words, this condition assumes that the heat flux at the surface of the material is known.

- Since the given system consists of metal plate with heating element surrounded by air. In this case the temperature exchange between the air and metal is very slow and hence we can consider this as a perfectly insulated case and the heat flux at boundary is assumed to be zero.

The equation of boundary condition corresponding (1) is given by :

$$\frac{\partial T}{\partial n} = 0 \tag{2}$$

where

- $\frac{\partial T}{\partial n}$ is the heat flux
- n is the unit vector normal to plate's surface.

C. Heat source equation

The equation for heat source corresponding to (1) is given by[4]

$$\dot{q} = \beta \times H(X,Y) \times \frac{(U(t) - T(t))}{d}, \tag{3}$$

where d is the thickness of the material, H an indicator function on where the heat source is located, U(t) is the heater temperature and T(t) is temperature of the node at time t. Substituting the above equation in (1) we get:

$$\frac{\partial T}{\partial t} = \lambda \left(\frac{\partial^2 T}{\partial^2 X} + \frac{\partial^2 T}{\partial^2 Y} \right) + H(X,Y) \times \frac{(U(t) - T(t))}{d} \tag{4}$$

D. Discretization

The clustering technique makes use of laplacian matrix structure [6] and hence it is required to convert the PDE in (4) to a state space model. In order to convert the PDE in (1) to state space model we use finite difference method[5] for discretization procedure.

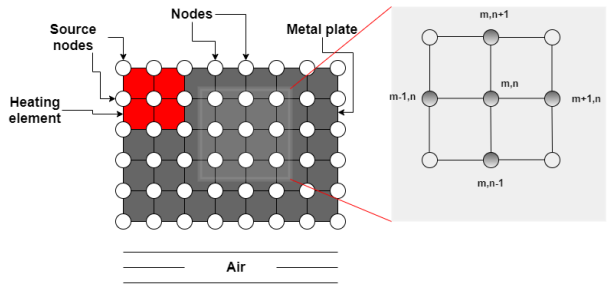


Fig. 2: Grid

We convert the rectangular plate into a grid structure creating a network of sub-systems, where each node is treated as a sub-system who's dynamics is given by (4). Finite difference method is used to derive the relationship between each nodes with its neighbouring nodes in the network system. The temperature gradients in (4) may be written as follows:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{m,n} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{m+1/2,n} - \left.\frac{\partial T}{\partial x}\right|_{m-1/2,n}}{\Delta x} = \frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} \tag{5}$$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{m,n} \approx \frac{\left.\frac{\partial T}{\partial y}\right|_{m,n+1/2} - \left.\frac{\partial T}{\partial y}\right|_{m,n-1/2}}{\Delta y} = \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} \tag{6}$$

Substituting equation (5) and (6) in equation (4) gives us:

$$\frac{\partial T}{\partial t} = \lambda \left(\frac{T_{m+1,n} + T_{m-1,n} - 2T_{m,n}}{(\Delta x)^2} + \frac{T_{m,n+1} + T_{m,n-1} - 2T_{m,n}}{(\Delta y)^2} \right) + \frac{(U-T)}{d} \tag{7}$$

E. State Space Formulation

Applying (7) to every node as shown in Figure 2 gives us the state matrix L as shown

$$\begin{bmatrix} \dot{T}_1 \\ \dot{T}_2 \\ \dot{T}_3 \\ \dot{T}_4 \\ \dot{T}_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} \frac{-\lambda_x}{\Delta X^2} - \frac{\lambda_y}{\Delta Y^2} & \frac{\lambda_y}{\Delta Y^2} & 0 & \cdots & \frac{\lambda_x}{\Delta X^2} & 0 & 0 & \cdots & \\ \frac{\lambda_y}{\Delta Y^2} & \frac{-\lambda_x}{\Delta X^2} - \frac{2\lambda_y}{\Delta Y^2} & \frac{\lambda_y}{\Delta Y^2} & \cdots & \frac{\lambda_x}{\Delta X^2} & 0 & 0 & \cdots & \\ 0 & \frac{\lambda_y}{\Delta Y^2} & \frac{-\lambda_x}{\Delta X^2} - \frac{2\lambda_y}{\Delta Y^2} & \frac{\lambda_y}{\Delta Y^2} & 0 & \cdots & \frac{\lambda_x}{\Delta X^2} & 0 \cdots & \\ 0 & 0 & \frac{\lambda_y}{\Delta Y^2} & \frac{-\lambda_x}{\Delta X^2} - \frac{2\lambda_y}{\Delta Y^2} & \frac{\lambda_y}{\Delta Y^2} & 0 & \cdots & \frac{\lambda_x}{\Delta X^2} & 0 & \cdots \\ \frac{\lambda_x}{\Delta X^2} & 0 & \cdots & \frac{2\lambda_y}{\Delta Y^2} & \frac{-2\lambda_x}{\Delta X^2} - \frac{2\lambda_y}{\Delta Y^2} & \frac{\lambda_y}{\Delta Y^2} & 0 & 0 & \cdots & \frac{\lambda_x}{\Delta X^2} \\ \ddots & 0 & \cdots & \ddots & \ddots & \ddots & 0 & 0 & \cdots & \ddots \end{bmatrix}$$

Here the input to the system is taken as heat source. Thus taking (3) into account we get:

$$\frac{\dot{q}}{\beta} = H(X,Y) \times \frac{(U-T)}{d} \tag{8}$$

Input matrix B has term $\frac{1}{d}$, depending on source location specified by function H(X,Y).

$$F = \frac{1}{d} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} \times (U - T)$$

The above modeled state matrix satisfies the following laplacian properties

- State matrix is symmetric.
- Both the row and column sums are zero.
- State has only one zero eigenvalue, and all the other eigenvalues are positive

Further analysis of our system shows us that state matrix has a laplacian structure and can be treated as undirected graph [6]. Due to presence of a zero eigenvalue the system under consideration is said to be semi-stable. Thus our network system can be represented as a LTI system :

$$\Sigma : \dot{x} = Ax + Bu \tag{9}$$

where $A = -M^{-1}L$, $B = M^{-1}F$ and M is inertia matrix, for our system we consider it to be an identity matrix.

III. Clustering

After obtaining the network system model we can proceed to clustering technique. In order to group two similar nodes in same cluster it is important to know if both the nodes have similar behaviour, this is quantified by characterizing the behaviour of each node using there transfer function from inputs to individual states. The dissimilarity between two nodes is quantified by the H2−norm of there transfer function difference. The difference in transfer function between every pair of nodes in the network system is given by the dissimilarity matrix which is given by[6]

$$D_{ij}^2 = \left\|C(sI-A)^{-1}B\right\|_{H_2}^2 = \text{Trace}\left(\int_0^\infty \underbrace{Ce^{A\tau}B}_{g(\tau)} B^\top e^{A^\top \tau} C^\top d\tau\right) \tag{10}$$

with $g(\tau) = Ce^{A\tau}B$ the impulse response of the system and $C = (e_i - e_j)^\top$ where e_i and e_j are is the i^{th} and j^{th} column of Identity Matrix, I$\in n \times n$

$$D_{ij} = \sqrt{Trace(CPC)} \tag{11}$$

Where P is the controlablity gramain given as

$$P = \int_0^\infty \underbrace{Ce^{A\tau}B}_{g(\tau)} B^\top e^{A^\top \tau} C^\top d\tau$$

When we have a stable system ie. A is Hurwitz, we can use the lyapunov equation to compute the infity gramian given by

$$0 = AP + PA^\top + BB^\top$$

As mentioned in Section II the network system under consideration is semistable and hence the solution for P from the above lyapunov equation is not unique. Thus we introduce the concept of pseudo gramian.

Definition:

Consider a semistable system (A, B, C). The pseudo controllability Gramian [6] is defined as:

$$P = \int_0^\infty \left(e^{A\tau} - J\right) BB^\top \left(e^{A^\top \tau} - J^\top\right) d\tau \tag{12}$$

where

$$J = \lim_{t\to\infty} e^{At} = \frac{11^\top M}{1^\top M 1} \tag{13}$$

The psuedo controllability gramian is unique symmetric solution of the following equations:

$$\begin{cases} 0 = AP + PA^\top + (I-J)BB^\top \left(I - J^\top\right) \\ 0 = JPJ^\top \end{cases} \tag{14}$$

Based on dissimilarity matrix in (11) we can cluster the nodes using various clustering algorithms such as K-means clustering, iterative clustering and hierarchical clustering etc [9]. We choose hierarchical clustering as it costs the least computation time and gives least error approximation compared to other algorithms[6].

As a result of hierarchical clustering we can aggregate all the nodes into various clusters and this leads to the Characteristic matrix Π.

Definition(Characteristic matrix) $\Pi \in R^{n\times r}$:

Let $\{C_1, C_2, \ldots, C_r\}$ be a clustering of G with n nodes. The characteristic matrix $\Pi \in R^{n\times r}$ is defined by:

$$\Pi_{ij} := \begin{cases} 1 & \text{if vertex } i \in \mathscr{C}_j \\ 0 & \text{otherwise} \end{cases} \tag{15}$$

Example 1:

For a given system of the form:

$$\Sigma : \dot{x} = Ax + Bu$$

with n=5 nodes, consider a cluster set $C_1 = \{1\}$, $C_2 = \{2,5\}$, $C_3 = \{3,4\}$, Π is given by:

$$\Pi = \begin{matrix} \\ cluster1 \\ cluster2 \\ cluster3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}^T$$

IV. Reduced order system

The Π matrix discussed in the previous section is used for approximating the n^{th} order system in (9) to a r^{th} order reduced order model.

Reduced Model($\hat{x} \approx$ x)

$$\hat{\Sigma} : \hat{M}\dot{z} = -\hat{L}z + \Pi^\top F u \tag{16}$$

where

$\hat{x} = \Pi z$, $\hat{M} = \Pi^\top M \Pi$ and $\hat{L} = \Pi^\top L \Pi$

$\hat{M}$ is the diagonal positive definite

$\hat{L}$ is the Laplacian matrix of a reduced undirected graph

$\hat{\Sigma}$ can be interpreted as a reduced network system

V. Error analysis

In order to analyse how well our reduced order model captures the behaviour of the original system, we calculate the approximation error between Σ from (9) and $\hat{\Sigma}$ from (16), which is given by[7]

$$\|\Sigma - \hat{\Sigma}\|_{H_2}^2 = trace\left[H\left(P_o + \Pi P_r \Pi^T - 2\Pi P_x\right)H^T\right] \tag{17}$$

where,

P_o is pseudo controllability Gramians of the full-order

P_r is pseudo controllability Gramians of the reduced order and

$$P_x = \tilde{P}_x - \Pi^\dagger J \Pi \tilde{P}_x J^T \in \mathbb{R}^{r\times n}$$

with $\tilde{P}_x$ an arbitrary symmetric solution ofthe following Sylvester equation:

$$\hat{L}^T \tilde{P}_x + \tilde{P}_x L^T - \Pi^\dagger (I-J) F F^T \left(I - J^T\right) = 0$$

VI. Parameters of thermal diffusion network

TABLE I: Parameter values

Parameter	Notation	Value
Thermal diffusivity (mm^2/s)	λ	23
Thermal conductivity (W/ mm^2K)	β	28.2×10^3
Thickness of the plate (mm)	d	2
Width of Plate (mm)	Lx	180
Length of Plate (mm)	Ly	20
Distance between two nodes in x direction(mm)	Δ_X	2
Distance between two nodes in y direction(mm)	Δ_Y	2

Using Δ_X and Δ_Y for metal plate we get total number of nodes in network system as n = 1001. In the next section the simulations have been conducted for the above provided parameters.

VII. Simulation results

The number of clusters is taken as r=24 and the clustering technique is applied to the given network system using above mentioned parameters. The cluster formation for the rectangular plate is visualized in Figure 3, where each colour section represents a cluster. The heating element is considered to be supplying heat at 75°C with initial temperature of nodes at 0°C. The temperature distribution over time for different order models is shown in Figure 4.

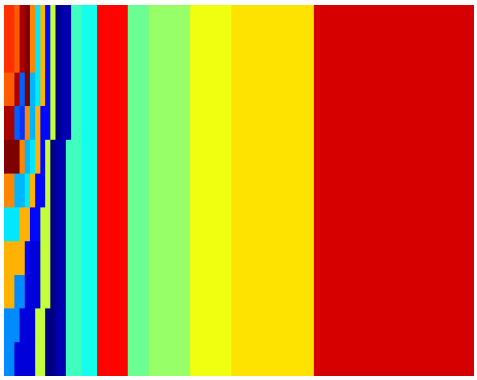


Fig. 3: Clusters formation

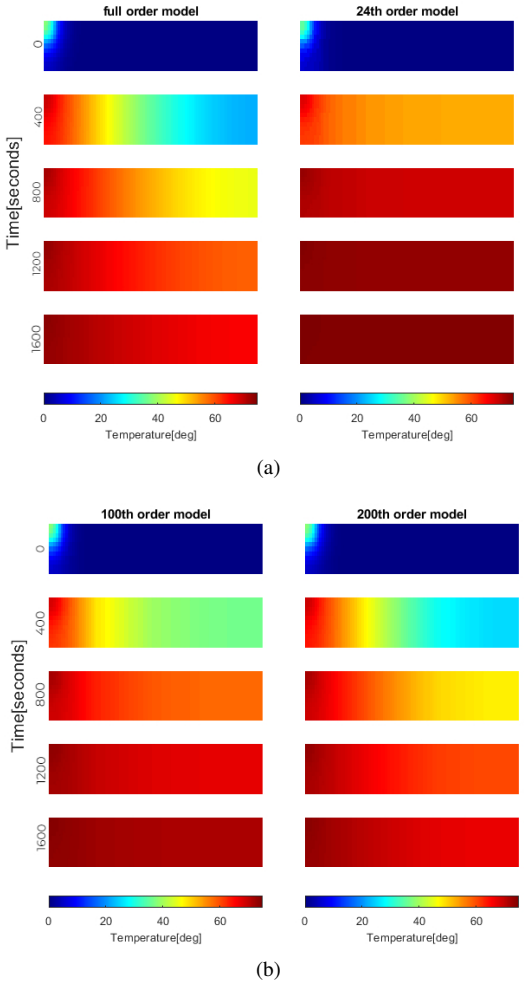


Fig. 4: Temperature distribution over time for different order models

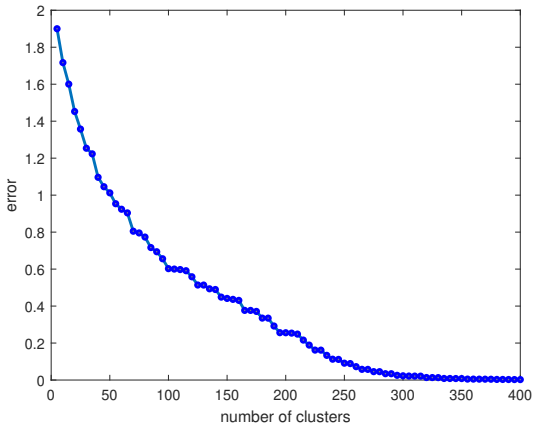


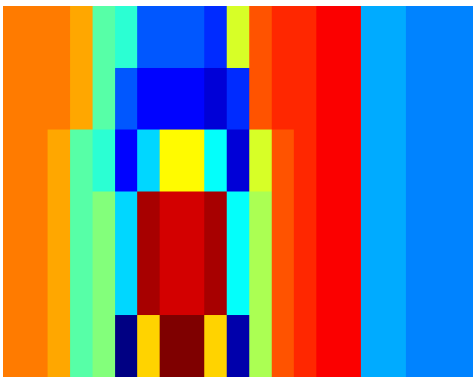
Fig. 5: Number of clusters vs approximation error

As can be seen from Figure 4(a) that, there is huge difference in heat distribution of full order model and 24^{th} order model from 400^{th} till 1600^{th} second. Hence we can conclude that the 24^{th} order does not capture the behaviour of the full order model well. Thus we will look into the error approximation presented in Figure 5 and increase the number

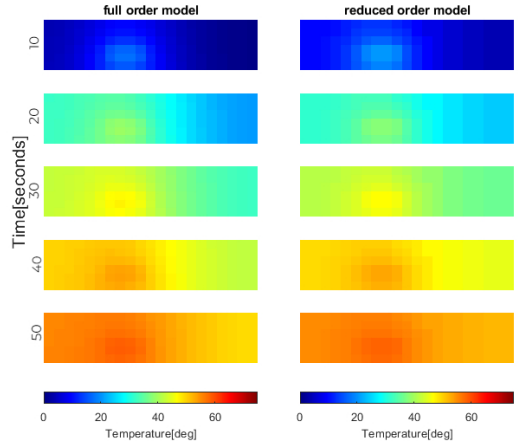
to clusters to compare the results.

Figure 5 shows that, there is in significant improvement in the error after choosing number of clusters, r= 200. Hence we choose the number of clusters, r = 200 and compare it with the full order model. Moreover in order to provide further insight on the performance of reduced order model we make the comparison of behaviour with 100^{th} order model as well.

Upon studying Figure 4 we see that, increasing the order to 200 has caused a significant improvement in the performance of reduced order model whereas the 100^{th} order model is still not able to capture the full order dynamics quite well. We



(a) Source location: Inside plate



(b)

Fig. 6: Cluster formation and temperature distribution over time for source present inside plate

can further analyse the clustering technique by studying the effect of heater location on the cluster formation shown in Figure 6, Figure 7 and Figure 8. For the purpose of studying the effect of heater we choose the following parameters given in the Table II. Using Δ_X and Δ_Y from Table II for metal plate we get total number of nodes in network system as n = 126.

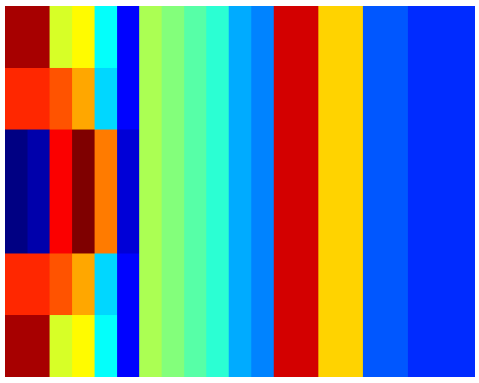
As can be seen from Figure 6(a) , Figure 7(a) and Figure 8(a) the cluster are more finely distributed near the heat source and more roughly distributed as we get far away from

TABLE II: Parameter values

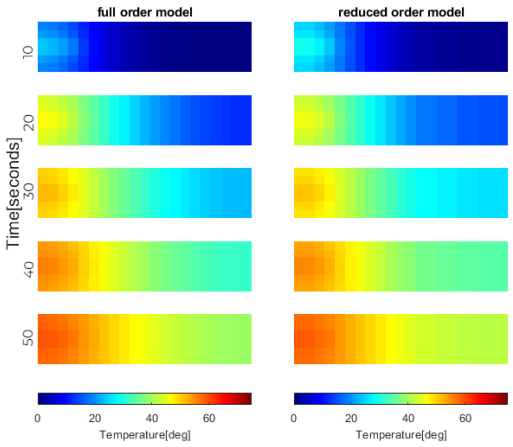
Parameter	Notation	Value
Thermal diffusivity (mm^2/s)	λ	23
Thermal conductivity ($W/\ mm^2K$)	β	28.2×10^3
Thickness of the plate (mm)	d	2
Width of Plate (mm)	Lx	40
Length of Plate (mm)	Ly	10
Distance between two nodes in x direction(mm)	Δ_X	2
Distance between two nodes in y direction(mm)	Δ_Y	2
Number of clusters	r	25

the heat source. This implies that the dynamics of the nodes nearby the heat source vary more whereas nodes away from heat source behave more uniformly.

After studying the effect of heat source location, we compare the results of clustering method used in reference [6] and the clustering result of "Average State Observers for Large-Scale Network Systems" [8].



(a) Source location: Left edge

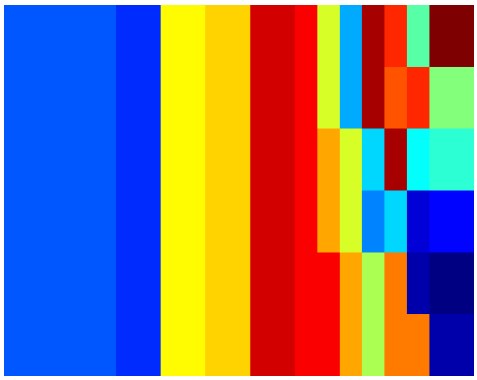


(b)

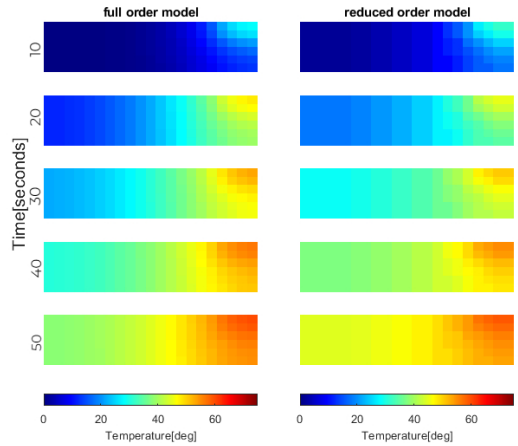
Fig. 7: Cluster formation and temperature distribution over time for source present at left edge

VIII. Average State Observer

Considering the clustering technique used "Average State Observers for Large-Scale Network Systems", we have the



(a) Source location: Top right



(b)

Fig. 8: Cluster formation and temperature distribution over time for source present at top right

following equations under consideration. For a given network system :

$$\Sigma : \dot{x} = Ax + Bu \tag{18}$$

Where A is stable and Hurwitz. The characteristic matrix for Average state matrix A is defined as follows:
Given system in (18), the characteristic matrix P is given by[8]

$$P = \text{diag}\left(\frac{1}{n_1}\mathbf{1}_{n_1}^{\top}, \ldots, \frac{1}{n_L}\mathbf{1}_{n_L}^{\top}\right)\Pi_a \in \mathbb{R}^{L\times n} \tag{19}$$

where $n_l = |I_l|$ and $\Pi_a \in \mathbb{R}^{n\times n}$ is the permutation matrix of average state observer given by

$$\Pi_a = \left[e_{I_1}^n, \ldots, e_{I_L}^n\right]^{\top} \in \mathbb{R}^{n\times n}$$

Considering **example 1** from section III:
For a system in (18) with n=5 consider a cluster set $I_1 = \{1\}$, $I_2 = \{2,5\}$, $I_3 = \{3,4\}$, P using (19) is given by

$$P = \begin{array}{c} \\ cluster1 \\ cluster2 \\ cluster3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0 \end{bmatrix}$$

IX. Average state observer Vs Gramian based clustering technique

Running both the clustering techniques for system with same parameters provided in section 6 we get the following results:

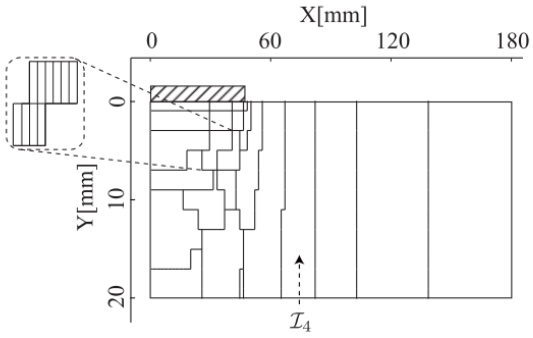


Fig. 9: Set of clusters in the case L =24 using [8]

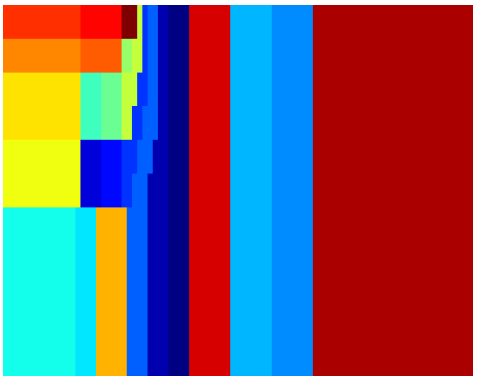


Fig. 10: Set of clusters in the case L =24 using [6]

On comparing both the Figure 9 and Figure 10 we see that the cluster formation matches our previous observation , where cluster are more finely distributed near the heat source and more roughly distributed as we get far away from the heat source. We can further see that the clusters formed in this paper have same shape whereas the clusters in the reference paper are non-uniform in nature. This is due to the difference in characteristic matrix used in both the methods.

X. Conclusion

In this paper we used finite differencing technique to obtain the state space model from the PDE. Later we applied the theory of graph clustering [6] to obtain the reduced

order model with structure preservation and analysed the performance of various reduced order model using error approximation. Further analysis showed that clusters are distributed finely about the heating element and roughly distributed as we move away from the heater in both clustering methods from [6] and [8]. For future work we can compare the performances of various clustering techniques, obtain the relationship between the thermal diffusivity and cluster formation as well as compute the most optimal node to represent each cluster.

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