Volatility Harvesting in Theory and Practice

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Rebalancing is an important tool for managing a portfolio. It keeps the portfolio close to the investor's desired weights and ensures that concentrations are not allowed to build up. Rebalancing can also be a source of return—the act of maintaining constant weights generates a buy-low, sell-high trading pattern that is designed to harvest extra return from the volatility of the underlying assets. Of course, there is also the risk that this type of strategy will underperform over a particular period as a result of differences in the growth rates of the assets. For example, a portfolio rebalanced to constant sector weights during the 1990s would have underperformed a buy-and-hold portfolio. By controlling the weight of technology stocks over the course of the decade, a rebalancing portfolio would have benefited less from the extraordinary performance of those stocks in 1998 and 1999. Yet over longer periods, rebalancing portfolios appear to have a higher growth rate.

In this article, we explore the theory behind the concept of *volatility harvesting*, with a focus on quantifying both the potential long-term performance advantage and the risk of underperforming in the short term. We derive a formula that decomposes the excess returns of a portfolio strategy versus the market into three terms: a *volatility return*, a *dispersion return*, and a *drift return*. This formula is an extension of the energy–entropy decomposition of Pal and Wong [2013]. We adopt some of the naming conventions of Hallerbach [2014] to help bridge the language barrier between the mathematical and mainstream finance literatures. This approach represents a new way of thinking about the benchmark-relative risks involved with rebalancing.

Our volatility-harvesting formula is an improvement over earlier attempts to render this idea in a mathematically precise manner. First, the formula can be generalized to any portfolio strategy, whereas previous approaches were limited to only constant-weight and functionally generated portfolios.[1] Second, the formula does not require stochastic modeling assumptions: There is no need to assume that stocks follow a random walk with constant covariances. Using a discrete-time framework avoids the technicalities of stochastic calculus and allows us to calculate the components of return precisely by observing stock prices. As a result, the main ideas are easier to understand. Finally, the formula has been adjusted to deal with the fact that market capitalizations are not entirely driven by returns—initial public offerings (IPOs), new share issuance, float adjustments, bankruptcies, market index reconstitutions, etc., all have an impact on the level of concentration in the market. This makes the formula more useful when dealing with historical datasets.

We illustrate the return decomposition using historical data on developed- and emerging-market country indexes from 1997 to 2015. The cumulative return components are plotted as a time series, which provides a visual depiction of the potential benefits and risks of a rebalancing portfolio. We also use the theory of volatility harvesting to help clarify some of the misconceptions that have arisen in the recent debate on rebalancing. Specifically, we show that the mean reversion of asset prices is not a required condition for a rebalancing portfolio to outperform the market.

Volatility-harvesting formulas provide investors and advisors with a practical tool for understanding a different sort of outperformance. Traditional portfolio managers attempt to select securities that are expected to outperform on the basis of the manager's forecasting skill, an informational advantage, or a factor risk premium. In contrast, rebalancing is a dynamic allocation strategy that relies on the long-term stability of the capital distribution and the presence of volatility. A broad class of volatility-harvesting strategies—not only constant-weight or functionally generated portfolios—can be expected to outperform whenever the market-capitalization distribution is stable in the long run. We do not claim that rebalancing is a riskless opportunity that will always outperform the market, only that it is a dependable strategy with risks and rewards that can be analyzed.

A TALE OF TWO LITERATURES

It has long been observed that the compounded growth rate of a portfolio is greater than the weighted-average growth rate of its underlying assets. The articles published on this topic can be roughly sorted into two camps: those that use stochastic calculus and those that do not. The calculus group includes Fernholz and Shay [1982], Luenberger [1998], Fernholz [2002], Dempster, Evstigneev, and Schenk-Hoppe [2007], Karatzas and Fernholz [2008], and Platen and Rendek [2012]. These authors generally agree that rebalancing leads to extra portfolio growth under appropriate conditions and label this growth the *excess growth rate*.

A separate line of discussion began when Booth and Fama [1992] introduced the same concept under the name *diversification return*. A series of papers exploring the topic followed, including work by Bernstein and Wilkinson [1997], Erb and Harvey [2006], Willenbrock [2011], Qian [2012], and Rulik [2013]. Bouchey et al. [2012] attempted to bridge the gap between these two segments of the literature by deriving the formula for diversification return using stochastic calculus. That paper described rebalancing as one type of volatility harvesting and used a number of examples to show how a portfolio that is systematically rebalanced to constant weights will have a higher expected growth rate than a buy-and-hold portfolio.

More recently, critics of diversification return—including Chambers and Zdanowicz [2014], Qian [2014], Cuthbertson et al. [2015], and Blitz [2015]—have called into question whether rebalancing is a source of added value. Their primary arguments are the following:

1. Rebalancing does not add value because the portfolio's expected wealth does not increase.
2. Rebalancing only increases returns when stock prices are mean reverting or negatively autocorrelated.
3. There is a benefit, but it is due to diversification or risk factor premiums, not rebalancing.

Counter-arguments in defense of rebalancing are provided by Banner [2015], who showed that median wealth outcomes are more descriptive of the central tendency of the distribution than the expected wealth. He also argued that mean reversion of asset prices is not required for rebalancing to add value.

In the midst of this debate, two key papers have emerged that derive formulas for decomposing extra return into terms that are more easily analyzed. Hallerbach [2014] separated rebalancing return into two terms: a *volatility return* and a *dispersion discount*. Pal and Wong [2013] created a similar decomposition, splitting the portfolio excess return into *free energy*, *relative entropy*, and *control* terms. Once the naming conventions are resolved, it is easy to see that Hallerbach actually discusses a special case of the energy–entropy decomposition. If the initial portfolio weights are set equal to the initial market weights and the strategy is rebalanced to constant weights, the two formulas are identical: The general case treated by Pal and Wong allows for any portfolio weights and any dynamic allocation strategy. An analysis of the components of excess return significantly clarifies several issues that, to date, have created much confusion in the literature. In the next section, we introduce the energy–entropy framework. We translate this framework into volatility return terminology because this terminology is more descriptive than others and seems to have gained acceptance in the mainstream finance literature.[2]

THE VOLATILITY HARVESTING FORMULAS

To measure the relative performance of a portfolio with return $r(t)$ versus the market $R(t)$ we define the *excess log return* as:

$$\begin{aligned} \ln V(t) &= \ln \frac{\text{growth of \$1 of the portfolio}}{\text{growth of \$1 of the market}} \\ &= \ln\big(1+r(t)\big) - \ln\big(1+R(t)\big) \end{aligned} \tag{1}$$

We use the natural logarithm to turn the simple returns into continuously compounded growth rates. This is convenient because it makes the terms in the return decomposition additive. The extra return can be split into three parts that are useful for analyzing the sources of return (see the appendix for a derivation of this relationship):

$$\begin{aligned} \text{Excess log return} = \text{Volatility return} \\ - \text{Dispersion return} + \text{Drift return} \end{aligned} \tag{2}$$

For a portfolio with weights $w_i(t)$ and asset returns $r_i(t)$, we define the volatility return as the difference between the portfolio growth rate and the weighted-average asset growth rate:

$$\text{Volatility return} = \ln\big(1+r(t)\big) - \sum_{i=1}^{n} w_i(t) \cdot \ln\big(1+r_i(t)\big) \tag{3}$$

This return is always positive in markets in which the assets have returns that are different from one another. In markets with only one asset, or over periods during which all assets have identical returns, the value is zero. The volatility return can also be approximated as one half of the difference between the weighted-average variance of the security returns and the portfolio variance. This expression highlights the fact that the higher the volatility and lower the correlation of the underlying assets, the greater the volatility return will be. Essentially, a higher cross-sectional volatility leads to a higher volatility return.[3]

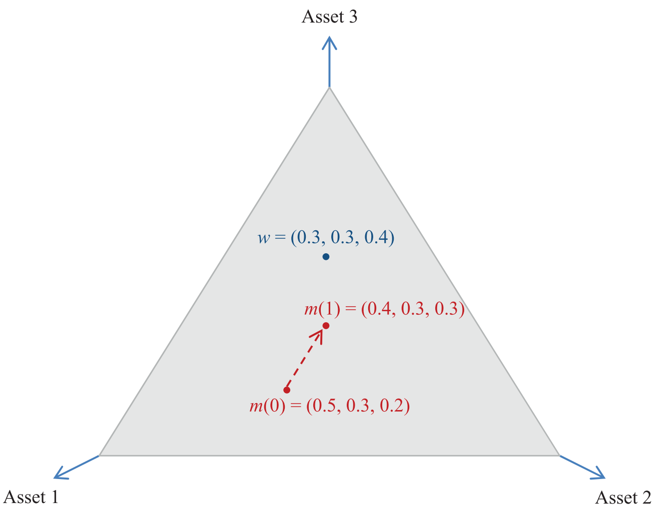
To understand the risk associated with rebalancing, it is useful to measure the difference between the portfolio and the market in terms of weights. For example, a constant-weighted portfolio that is close to the market in its weight distribution will track the market fairly well. In contrast, an equal-weighted portfolio that is much more diverse than the market faces much higher relative-performance risk. To measure the "distance" between a portfolio and the market, we borrow the concept of relative entropy from information theory.[4] For portfolio weights w and market weights m, relative entropy is defined as the weighted-average logarithm of the ratio of portfolio weights to market weights, the value of which is always non-negative:

$$\text{Relative entropy} = H\left(w|m\right) = \sum_{i=1}^{n} w_i \cdot \ln \frac{w_i}{m_i} \tag{4}$$

To get a feel for the geometry involved, consider a three-asset case. The set of all possible portfolio weights (long only, unlevered) can be visualized as a triangular sheet. Mathematically, this object is called the *simplex*. The corners of the simplex represent concentrated portfolios with 100% in a single asset. The edges represent the possible two-asset portfolios. The center of the sheet represents an equal-weighted portfolio with one third weight in each asset. Suppose, as in Exhibit 1, we hold the constant-weighted portfolio (0.3, 0.3, 0.4) of three stocks. If the market moves toward the portfolio, say from (0.5, 0.3, 0.2) to (0.4, 0.3, 0.3), the constant-weighted portfolio will have a positive excess return because it has less weight in stock 1 and more weight in stock 3 than the market. Conversely, if the market were to move away from the portfolio, the excess returns would be negative. Relative entropy calculates the appropriate distance needed to measure this effect in units of excess return.

For a constant-weighted portfolio, the changes in distance between the portfolio and the market are due entirely to movements of market weights (caused by the dispersion of asset returns); this part of the excess return is defined as the *dispersion return*. The dispersion return is related to the long-term active bets of the portfolio. If the portfolio is positioned well, it will have more weight in securities that have higher growth rates. If not, the active bets will act as a drag on performance.

EXHIBIT 1
Relative Entropy and the Geometry of Rebalancing



$$\text{Dispersion return} = H\big(w(T)\big|m(T)\big) - H\big(w(0)\big|m(0)\big) \tag{5}$$

For a more general case with portfolio weights that are not constant, we need to consider the change in distance caused by moving to a new set of weights. Sometimes this change is due to the portfolio manager, but often changes in portfolio weights are caused by allowing the weights to drift. We label this the *drift return*:

$$\begin{aligned}\text{Drift return} &= \text{Post-trade distance} - \text{Pre-trade distance}\\ &= H\big(w(t+1)\big|m(t+1)\big) - H\big(w(t)\big|m(t+1)\big)\end{aligned} \tag{6}$$

If left untended, a portfolio can drift toward concentration, which undermines the risk reduction typically associated with diversification. The drift return is the only part of the formula over which portfolio managers have significant control; they can choose to drift, rebalance, follow recent performance trends, or implement some other dynamic allocation strategy. To further illustrate this concept, a constant-weight strategy has a drift return of zero, whereas a contrarian value strategy might trade against the market even more aggressively than a rebalancing portfolio, and a momentum strategy trades with the market.

As an example calculation, assume the three assets from Exhibit 1 had simple returns of −20, 0, and 50%. The portfolio return is 14%, and its growth rate is 13.1%. The weighted average growth rate of the underlying assets is 9.52%, so the volatility return is 3.58%:

$$\begin{aligned}&\ln(1+0.14) = 13.10\%\\ &0.3\cdot\ln(1-0.20) + 0.3\cdot\ln(1+0)\\ &\quad +0.4\cdot\ln(1+0.50) = 9.52\%\\ &\text{Volatility return} = 13.10\% - 9.52\% = 3.58\%\end{aligned} \tag{7}$$

The relative entropy distance changes from 0.124 to 0.0288. The dispersion return is the difference, −9.52%, between these two values:

$$\begin{aligned}\text{Initial relative entropy} &= 0.3\cdot\ln\frac{0.3}{0.5} + 0.3\cdot\ln\frac{0.3}{0.3}\\ &\quad + 0.4\cdot\ln\frac{0.4}{0.2} = 0.124\end{aligned} \tag{8}$$

$$\begin{aligned}\text{Final relative entropy} &= 0.3\cdot\ln\frac{0.3}{0.4} + 0.3\cdot\ln\frac{0.3}{0.3}\\ &\quad + 0.4\cdot\ln\frac{0.4}{0.3} = 0.0288\\ \text{Dispersion return} &= 2.88\% - 12.40\% = -9.52\%\end{aligned} \tag{9}$$

The drift return is zero because the portfolio was rebalanced to constant weights over this period. The return decomposition is:

$$\begin{aligned}\text{Excess log return} &= \text{Volatility return} - \text{Dispersion return}\\ &\quad + \text{Drift return}\\ 13.10\% &= 3.58\% - (-9.52\%) + 0\%\end{aligned} \tag{10}$$

Absolute returns are not as important as the relative returns. For example, we could scale up to −12, 10, and 65% and find the same change in weightings, relative returns, and values for the volatility return and dispersion return.

Extending the example, we assume that in the next period, returns are such that the market weights revert to their original values. In Exhibit 2, we plot the return decomposition over two time steps for the market portfolio, the constant-weighted portfolio, and a drifting portfolio that starts at (0.3, 0.3, 0.4) and does not rebalance.

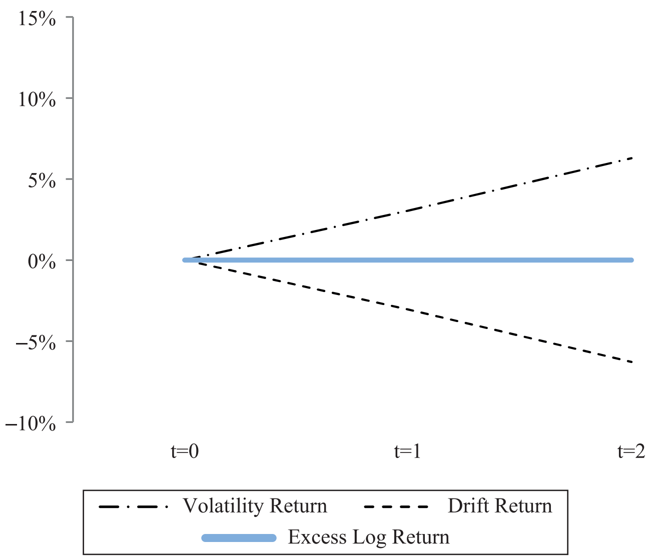
For the market-weighted portfolio, the excess log return and relative entropy terms are zero by definition.

The volatility return is canceled out by the drift return. The constant-weight portfolio has a drift return that is always zero. The dispersion return is beneficial in the first period and detrimental in the second period. The portfolio outperforms the market by exactly the cumulative volatility return of 7%; the volatility in the market is

EXHIBIT 2

Volatility Harvesting Return Decomposition for a Three-Asset Portfolio with Weights 30/30/40, in a Market That Drifts from 50/30/20 to 40/30/30 and Back to 50/30/20

Panel A: Market Weights



Panel B: Constant Weights

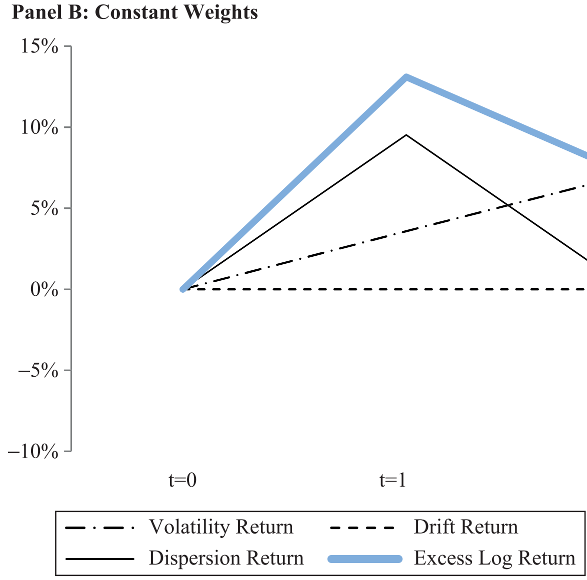
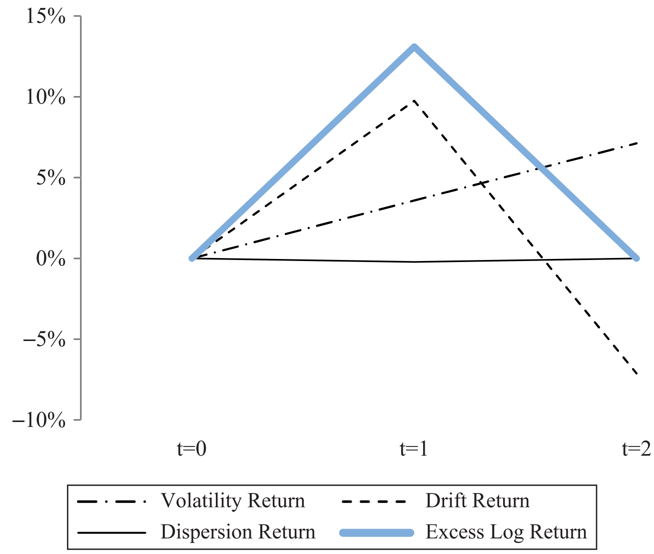


EXHIBIT 2 (*Continued*)

Panel C: Drifting Weights



harvested by the act of rebalancing and results in excess return over the drifting market portfolio.

What about the buy-and-hold portfolio that never rebalances? There is still relative risk, but in this case the drift return dominates, whereas the dispersion return is close to zero. This result occurs because the buy-and-hold portfolio's distance to the market is nearly constant—both portfolios drift in concert. At first, drifting benefits excess return, but ultimately it gives back all its former gains. There is no rebalancing, and thus no excess return is accumulated over the full cycle. The portfolio became concentrated in Asset 3 in the first period and was hurt more in the second period as a result.

This example is an idealized one in which asset prices do revert. However, price reversals are not a requirement for a rebalancing strategy to outperform. For a constant-weight portfolio, excess return will exactly equal volatility return when the market starts and ends at the same relative-entropy distance from the portfolio. In Exhibit 1, this can be visualized as a set of contour lines around the portfolio: If, instead of the asset prices reverting, the market followed a random walk until it reached the same contour line as (0.5, 0.3, 0.2), then the excess return would be equal to the 7% volatility return. Based on these relationships, it is clear, then, that the mean reversion of asset prices is not a prerequisite for a rebalancing portfolio to outperform.

Several mathematical results follow directly from the volatility harvesting formulas:

- Cumulative volatility return will always increase as long as assets have returns that are different from one another.
- The dispersion return will remain finite, except in the unlikely event that the market devolves into a single security and there are no further opportunities for diversification or rebalancing.[5]
- The drift return will always be zero for a portfolio that rebalances to constant weights.
- Taken together, the previous three statements imply that the volatility return will eventually overtake the dispersion return and produce a positive excess return.
- Any dynamic strategy can become a volatility-harvesting strategy so long as the portfolio manager trades in a way that does not allow the drift return to be more negative than the volatility return realized in each time period.

Of course, investors are interested in what happens over a particular investment period, not what happens in the limit as the investment horizon approaches infinity. A more practical observation is that a constant-weight strategy will outperform by an amount exactly equal to the volatility return when the distance between the market and the portfolio is the same as it was at the starting point, highlighting the importance of understanding the current level of market concentration.

How negative can the dispersion return become? The answer depends on how concentrated the market becomes. For example, a value strategy must sell stocks that have become expensive and buy stocks that have become cheap to maintain a value tilt. This strategy is contrarian; thus, the drift return will tend to be positive. Value stocks can suffer during periods of market concentration, as in the 1990s; as a trading strategy, however, a value strategy is likely to benefit from volatility return in the long term. On the other hand, a momentum strategy is trend-following and benefits from a concentrated market. The most extreme form of the momentum strategy (i.e., investing 100% in the top performing asset) does not accumulate any volatility return because a single asset cannot have cross-sectional volatility. A more diversified form of the momentum strategy will have a positive volatility return, but this will be overwhelmed by the magnitude of the drift return and dispersion return.

Not all factor strategies take advantage of volatility harvesting. In the following section, we explore the use of these formulas with historical data.

EMPIRICAL ILLUSTRATIONS USING COUNTRY INDEX PORTFOLIOS

To illustrate volatility harvesting using real data, we analyze an emerging-market equity portfolio using monthly S&P Global BMI country index data from March 1997 to May 2015. The stocks within each country index are capitalization weighted. We selected the 20 emerging-market countries with a complete data history over the sample period. We excluded those countries (like Israel) that graduated to the developed market index, but retained countries (like Argentina) that were demoted to the frontier market index.[6] Using market capitalization data, we constructed historical market weights for these countries by drifting the weights back in time, based on the total returns of the country indexes. This approach excludes changes in capitalization that were due to IPOs, changes in float adjustment, and stocks that left the index, allowing us to focus our analysis on the changes in market capitalization that are a result of returns.

To compare and contrast different approaches to rebalancing, we look at four portfolio construction approaches:

1. Capitalization weight (CAP): Countries are cap-weighted and allowed to drift.
2. Equal weight, rebalanced (EWR): Countries are rebalanced to a 5% weight each month.
3. Equal weight, drifted (EWD): Countries start at a 5% weight but are allowed to drift over the sample period.
4. Equal weight, +/−2% trigger (TRIG): Countries start at a 5% weight but are allowed to drift so long as the weights are between 3% and 7%. If a country breaches one of these triggers, the weight for that country is rebalanced to the target. For example, if Brazil drifts to a 7% weight, it will be sold down to 5%. The 2% cash generated by the trade will be used to purchase the country that is most underweight to target. In some cases, more than one country will need to be purchased (or sold) to yield portfolio weights that sum to 100%.

Exhibit 3 summarizes the return, risk, and benchmark-relative characteristics of the four portfolios. All three of the equal-weighting strategies outperform the CAP portfolio over this period with slightly less risk, although the EWD portfolio trails the other two strategies. All three have similar tracking errors versus CAP. Exhibit 3 also shows the cumulative volatility return, dispersion return, and drift return over the period. These three numbers add up to the total excess log return. By definition, the CAP portfolio has zero excess return; in other words, the cost of drifting cancels the volatility return in the capitalization-weighted portfolio. Also note that the CAP portfolio has slightly less cumulative volatility return because it is more concentrated than the other strategies.

For a volatility-harvesting portfolio to outperform the market, the dispersion and drift terms need to remain bounded so that the volatility term can accumulate over time. Exhibit 4 shows the time series for the EWR strategy: Dispersion return remains within a +/−20% range, whereas the volatility return grows to 60%, approximating the total excess return. The drift return is zero because the portfolio always rebalances back to the policy weight each period. Most of the returns from this portfolio are due to volatility harvesting

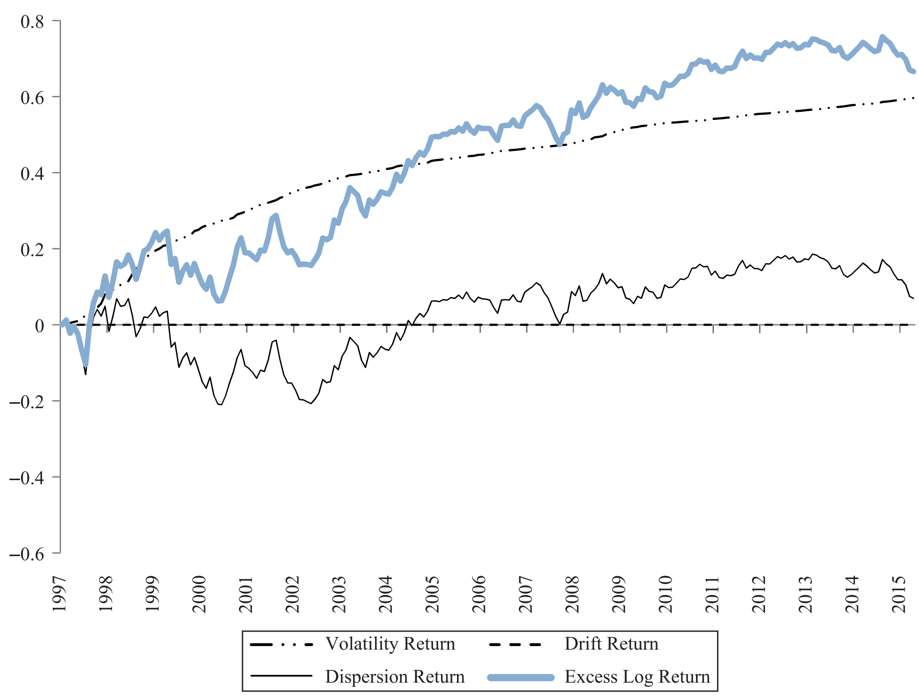
EXHIBIT 3

Return, Risk, and Benchmark-Relative Characteristics for Emerging-Market Strategies (March 1997 to May 2015)

	Capitalization Weight (CAP)	Equal Weight, Rebalanced (EWR)	Equal Weight, Drifted (EWD)	Equal Weight, +/–2% Trigger (TRIG)
Return	6.70%	10.67%	7.83%	10.74%
Volatility	25.45%	23.04%	22.49%	22.92%
Return/Volatility	0.26	0.46	0.35	0.47
Excess Return	—	3.97%	1.13%	4.04%
Tracking Error	—	8.09%	8.63%	8.14%
Information Ratio	—	0.49	0.13	0.50
Volatility Return	44%	60%	55%	58%
Dispersion Return	0%	7%	–10%	20%
Drift Return	–44%	0%	–25%	–10%
Excess Log Return	0%	67%	20%	68%

EXHIBIT 4

Cumulative Excess Returns for the Equal Weight, Rebalancing (EWR) Emerging-Market Strategy (March 1997 to May 2015)



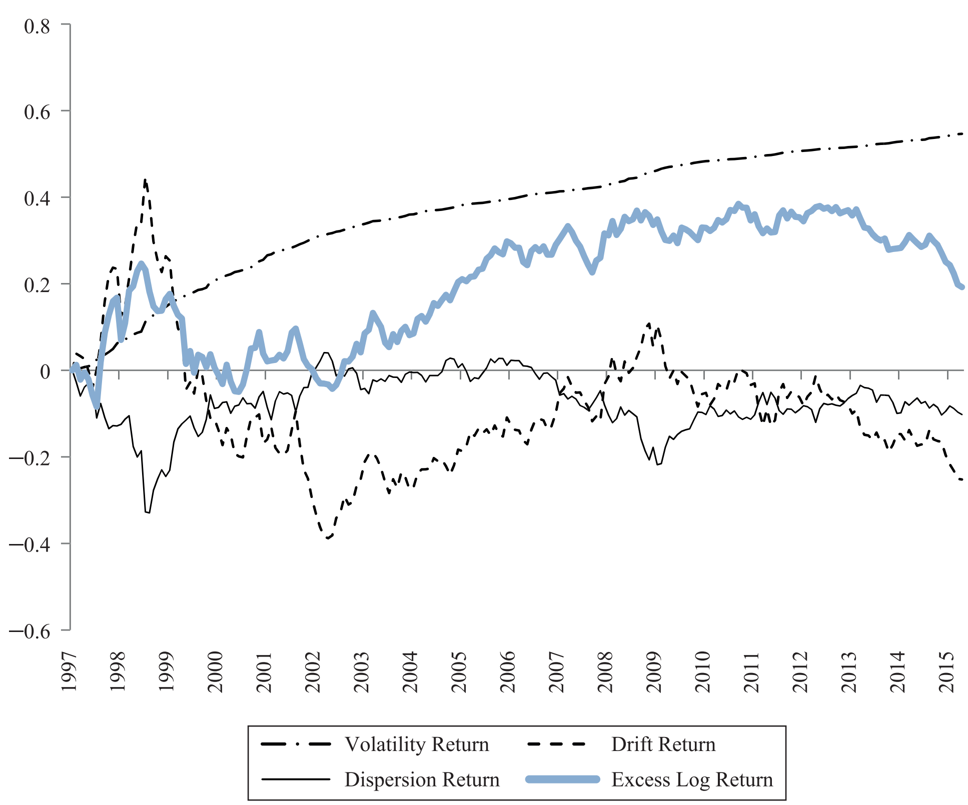
rather than to active bets on differential country growth rates. The first half of the period seems to have a greater accumulation of volatility return than the latter half.

Turning our attention to the EWD portfolio in Exhibit 5, we see that the dispersion term is no longer bounded in a narrow range and that the drift term is no longer zero. The portfolio started with 5% in each country, but by 1998 had drifted to 16% in Morocco because of that country's outperformance. Initially, the drifting portfolio benefited from the concentration; by 2000, however, Morocco had fallen to a 7% weight, and Turkey had become the most concentrated holding at a weight of 15%. This pattern continued through the period, with Peru, Mexico, and other countries taking turns dominating the weight in the drifting portfolio. This dynamic is captured by the drift term, adding to excess return in the first two years and subtracting from it in the next two. The absence of rebalancing is the primary reason this equal-weight portfolio underperformed the others.

These results are shown without consideration of the transaction costs. To reduce turnover and costs, we consider a middle case between full periodic rebalancing and drifting: In the +/−2% trigger case, we allow countries to drift until they stray outside the 3% to 7% range. The trigger strategy will rebalance more frequently during periods of high cross-sectional volatility but will be relatively inactive during quiet periods. Exhibit 6 shows the results of this strategy. The drift term is non-zero, which represents the effects of drifting; however, both drift and dispersion are bounded. The two time series are negatively correlated with one another and tend to cancel each other out, which will be true for other trigger-based rebalancing strategies in which the target weights are more diverse than the market portfolio. A market that is diversifying helps an equal-weight strategy, but some of that benefit is wasted by allowing the portfolio to drift within a range. In general, however, trigger-based rebalancing is much more efficient than calendar-based rebalancing. This method significantly reduces turnover

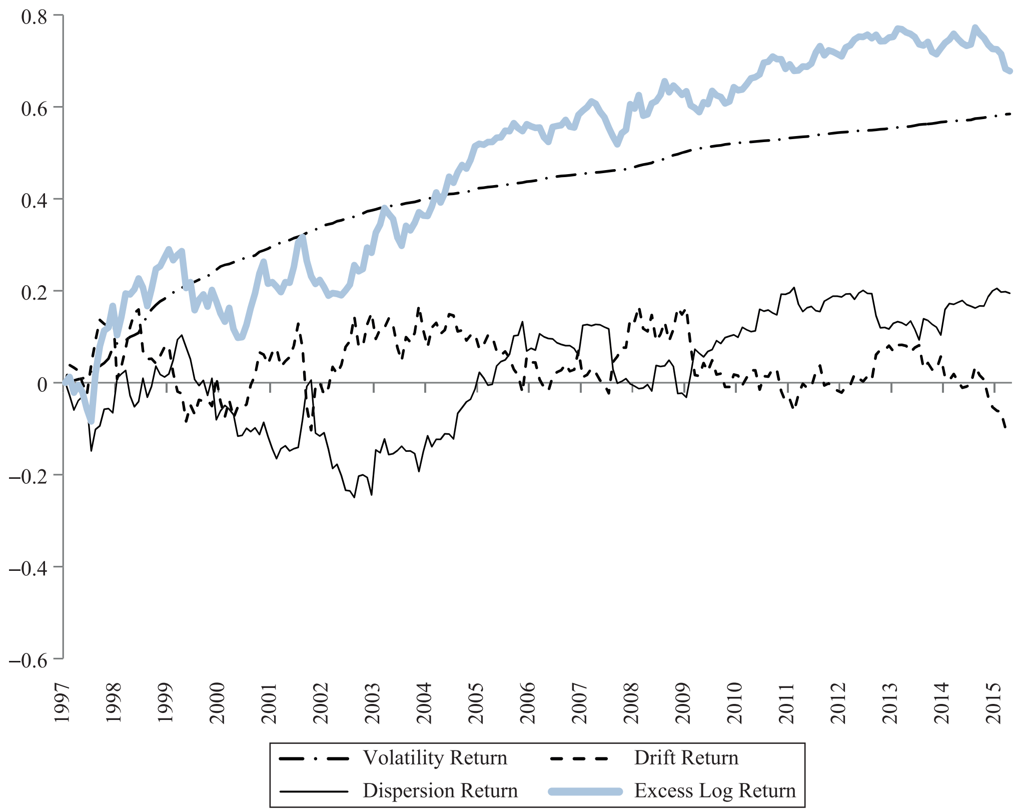
EXHIBIT 5

Cumulative Excess Returns for the Equal Weight, Drifting (EWD) Emerging-Market Strategy (March 1997 to May 2015)



E X H I B I T 6

Cumulative Excess Returns for the Equal Weight With +/–2% Trigger (TRIG) Emerging-Market Strategy (March 1997 to May 2015)



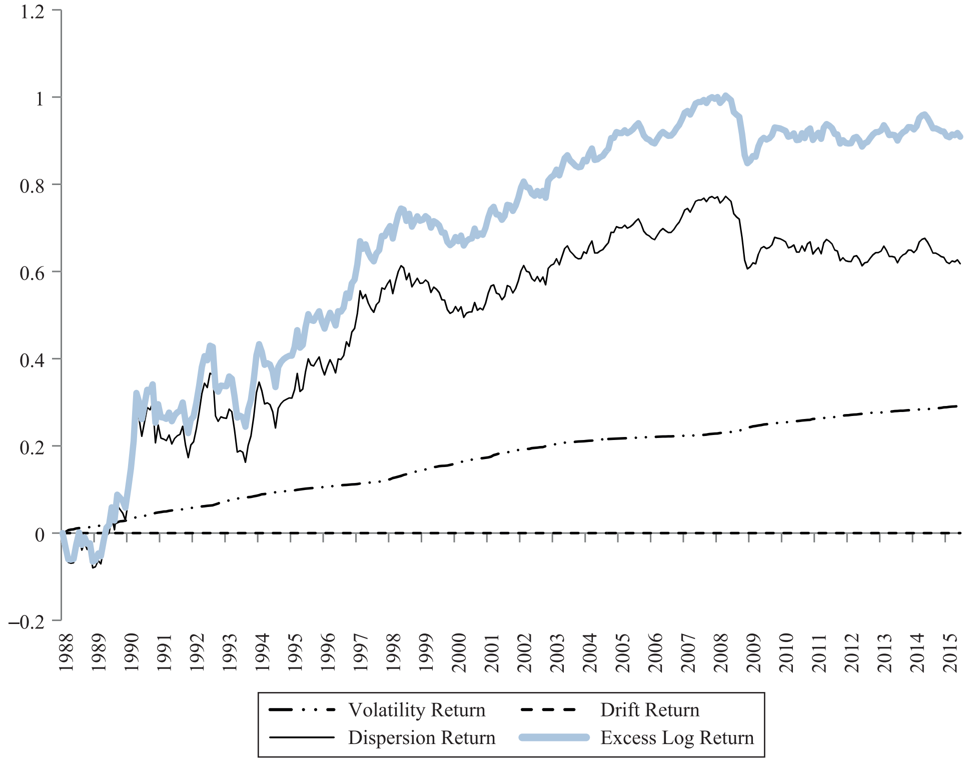
and implementation costs while still capturing the benefits of diversification and rebalancing.

Of course, rebalancing does not win every time because there are situations in which the dispersion return can dominate the excess return profile of a portfolio for long periods of time. An excellent example is found in the developed markets. Exhibit 7 displays the excess return for an equal-weighting strategy that rebalances monthly for developed-market countries (excluding the United States) from January 1988 to May 2015. More than two thirds of the excess return are a result of the dispersion return. Over this period, Japan had a capitalization that started at more than 60% of the index weight; the top of the Japanese equity and real estate bubble was reached in 1989, and in the subsequent 25 years, Japanese equities had lackluster performance relative to other countries, causing its weight in the index to drop to 20%. This increase in market diversity was a boon to an equal-weighted strategy that was underweight in Japanese equities. The equal-weight drifting portfolio also outperformed in this scenario by about the same amount. If Japan is excluded from the developed dataset, the change in relative entropy no longer dominates the excess returns, and the equal-weighted, rebalancing portfolio outperforms the equal-weight, drifting portfolio.

Throughout the 10 years before the bubble, during Japan's boom period, an equal-weight strategy would have underperformed because of the increase in market concentration. Rebalancing in the subperiods appears to be unimportant, but over the whole market cycle, the dispersion return cancels out and a rebalancing portfolio is left with the volatility return as an excess return. The bubble in Japan was widely recognized by institutional investors, and very few held the market weight to Japan in developed-market (excluding U.S.) portfolios, even if they were passive in their outlook. Diversification and rebalancing can be painful during extended asset bubbles. However, from a risk management perspective,

EXHIBIT 7

Cumulative Excess Returns for Equal Weight, Rebalancing Developed Markets Excluding the United States Strategy (January 1988 to May 2015)



the risk of allowing concentrations to build up is usually found to be unpalatable.

CONCLUSION

We showed that rebalancing is a source of excess return that relies on the cross-sectional volatility of the securities in the market, as well as the long-term stability of the market-capitalization distribution. The risk of underperforming can be quantified and depends both on how active weights are managed through time and the level of concentration that accumulates during a market cycle. Importantly, the potential for increasing the long-term growth rate of the portfolio does not require stocks to behave in a certain way; that is, prices do not have to reverse their directions and returns do not need to be negatively correlated with previous returns. Rebalancing is not about exploiting a trading anomaly, a risk factor premium, or a behavioral finance bias; it only requires the presence of multiple assets that have returns that are different from one another. These differences drive both the long-term return potential of a rebalancing strategy and the short-term risk of underperforming.

APPENDIX

DERIVATION OF THE VOLATILITY HARVESTING FORMULA

Let $X_1(t), \ldots X_n(t)$ denote the capitalizations of n stocks at discrete time points, $t = 0, 1, \ldots, T$. If the value of the portfolio is $\tilde{V}(t)$, then the *relative value* is given by

$$V(t) = \frac{\tilde{V}(t)}{S(t)} = \frac{\text{growth of \$1 of the portfolio}}{\text{growth of \$1 of the market}} \tag{A-1}$$

where $S(t) = X_i(t) + \cdots + X_n(t)$ is the total value of the cap-weighted market. The market weight of asset i is $m_i(t) = X_i(t)/S(t)$, and the portfolio weight is $w_i(t)$. Weights are long only and add to 100%.

We define *relative entropy* as the distance between the portfolio and the market.

$$H\left(w|m\right) = \sum_{i=1}^{n} w_i \cdot \ln \frac{w_i}{m_i} \tag{A-2}$$

We also define the *volatility return* as

$$\gamma(t) = \ln \frac{\tilde{V}(t+1)}{\tilde{V}(t)} - \sum_{i=1}^{n} w_i(t) \cdot \ln \frac{X_i(t+1)}{X_i(t)}$$
$$= \text{Portfolio growth rate}$$
$$- \text{ Weighted average asset growth rate} \tag{A-3}$$

The volatility return can be written using relative value and the change in market weights.

$$\gamma(\mathrm{t}) = \ln \frac{V(t+1)}{V(t)} - \sum_{i=1}^{n} w_i(t) \cdot \ln \frac{m_i(t+1)}{m_i(t)} \tag{A-4}$$

To see that this is true, note that the first term can be written:

$$\ln \frac{V(t+1)}{V(t)} = \ln \frac{\tilde{V}(t+1)}{\tilde{V}(t)} \cdot \frac{S(t)}{S(t+1)}$$
$$= \ln \frac{\tilde{V}(t+1)}{\tilde{V}(t)} - \ln \frac{S(t+1)}{S(t)} \tag{A-5}$$

Similarly, the second term can be written:

$$\sum_{i=1}^{n} w_i(t) \cdot \ln \frac{m_i(t+1)}{m_i(t)} = \sum_{i=1}^{n} w_i(t) \cdot \ln \frac{X_i(t+1)}{X_i(t)}$$
$$- \left(\sum_{i=1}^{n} w_i\right) \cdot \ln \frac{S(t+1)}{S(t)} \tag{A-6}$$

Because the portfolio weights sum to 1, subtracting the two terms results in equation A-3. This shows that the volatility return stays the same whether we use dollar values or market-relative values.

We can solve equation A-4 for the log relative return to get

$$\ln \frac{V(t+1)}{V(t)} = \gamma(t) + \sum_{i=1}^{n} w_i(t) \cdot \ln \frac{m_i(t+1)}{m_i(t)}$$
$$= \gamma(t) + \sum_{i=1}^{n} w_i(t) \cdot \ln \frac{m_i(t+1)}{m_i(t)} \cdot \frac{w_i(t)}{w_i(t)}$$
$$= \gamma(t) + \sum_{i=1}^{n} w_i(t) \left[\ln \frac{w_i(t)}{m_i(t)} - \ln \frac{w_i(t)}{m_i(t+1)} \right]$$
$$= \gamma(t) + H\left(w(t)|m(t)\right) - H\left(w(t)|m(t+1)\right)$$
$$\tag{A-7}$$

Next, we can calculate the cumulative log excess return from time 0 to time T. First we set the cumulative volatility return equal to

$$\Gamma(T) = \sum_{t=0}^{T-1} \gamma(t) \tag{A-8}$$

Then we can write

$$\ln \frac{V(T)}{V(0)} = \Gamma(T) + \sum_{t=0}^{T-1} H\left(w(t)|m(t)\right) - H\left(w(t)|m(t+1)\right)$$
$$= \Gamma(T) + \sum_{t=0}^{T-1} \left[H\left(w(t)|m(t)\right) - H\left(w(t+1)|m(t+1)\right) \right]$$
$$+ \sum_{t=0}^{T-1} \left[H\left(w(t+1)|m(t+1)\right) - H\left(w(t)|m(t+1)\right) \right]$$
$$= \Gamma(T) + H\left(w(0)|m(0)\right) - H\left(w(T)|m(T)\right)$$
$$+ \sum_{t=0}^{T-1} \left[H\left(w(t+1)|m(t+1)\right) - H\left(w(t)|m(t+1)\right) \right]$$
$$= \text{Volatility return} - \text{Dispersion return} + \text{Drift return}$$
$$\tag{A-9}$$

When market capitalizations are not entirely driven by the returns—as happens in practice with new share issuance, IPOs, float adjustments, bankruptcies, market benchmark reconstitutions, etc.—then we introduce the market weights $\widetilde{m}(t+1)$, that are implied by the returns and adjust the computation accordingly. These equations, and the associated plots, are coded in the publically available R computing package RelValAnalysis on CRAN.

ENDNOTES

[1]*Functionally generated portfolios* are a generalization of constant-weighted portfolios. An example of this portfolio type is a diversity-weighted portfolio that weights stocks according to a power function of the capitalization weights. See Fernholz [2002].

[2]To be clear, excess growth rate, diversification return, free energy, and volatility return are all names for the same concept: the difference between the growth rate of a portfolio and the weighted-average growth rate of the underlying securities.

[3]See Bouchey, Fjelstad, and Vadlamudi [2011] for a review of cross-sectional volatility and its effects on active strategies.

[4]For a review of information theory and its relationship to portfolio selection, see Cover and Thomas [2006].

[5]Mathematically speaking, relative entropy will become infinite whenever the market weight hits a boundary of the simplex. In practice, it is possible to stop rebalancing into an asset on its way to zero.

[6]The countries used are Argentina, Brazil, Chile, China, Colombia, Egypt, India, Indonesia, Malaysia, Mexico, Morocco, Peru, the Philippines, Poland, Russia, South Africa, South Korea, Taiwan, Thailand, and Turkey.

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