Point-set Topology for TQEC

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Topological Quantum Error Correction

November 7, 2025

Agenda

- What's topology?
- Metric spaces
- Topological spaces
- Open sets
- Continuity
- Homeomorphisms
- Manifolds
- Open cover and finite subcover
- Compactness
- The circle S^1
- The torus $\mathbb{T}^2$

We assume existing familiarity with definitions used in linear algebra (set theory, bijection, etc.)

Donuts, is there anything they can't do?
—Homer Simpson

What's topology?

The popular analogy

Topology can be thought of as the study of structures whose properties are preserved under stretching like clay.

More formally

We require notions of open sets, metric spaces, topological spaces, and continuity to *rigorously* define topology.

Metric spaces

Definition

A **metric space** is a set X equipped with a distance function $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying three properties for all $x, y, z \in X$:

- (Positivity) If $x \neq y$ then $d(x, y) > 0$
- (Symmetry) $d(x, y) = d(y, x)$
- (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$

Example

The real numbers $\mathbb{R}$ form a metric space with the **standard distance function** $d(x, y) = |x - y|$. Likewise, real-valued vectors $\mathbb{R}^n$ for $n \geq 2$ also form metric spaces with distance functions defined by the Euclidean distance function.

Topological spaces

Definition

A **topology** on a set X is a collection $\mathcal{T}$ of subsets (called **open sets**) of X satisfying the following axioms:

- The empty set $\emptyset$ and X itself belong to $\mathcal{T}$.
- Any arbitrary (finite or infinite) union of members of $\mathcal{T}$ belong to $\mathcal{T}$.
- The intersection of any finite number of members of $\mathcal{T}$ belong to $\mathcal{T}$.

Definition

A **topological space** is the pair $(X, \mathcal{T})$, where $\mathcal{T}$ (also denoted $\mathcal{T}_X$) is a topology on a set X.

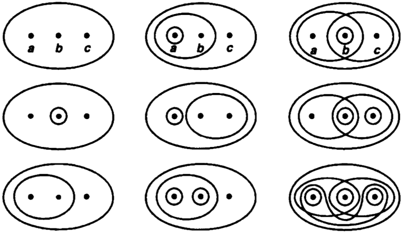
Theorem

Every metric space admits a topological space.

Topological spaces (cont'd)

Example

The set containing three points $X = \{a, b, c\}$ admits many possible topologies; some are illustrated below. The top-left example represents the topology $\{\emptyset, X\}$. The center one contains the subsets $\emptyset, X, \{a\}, \{b, c\}$. The strict subsets needn't be disjoint, as shown in the top-center: $\emptyset, X, \{a\}, \{a, b\}$ also form a topological space over X.



Open subsets of metric spaces

Definition

A subset U of a metric space (X, d) is **open** if for all $x \in U$, there exists a positive real number ε such that any point $y \in X$ satisfying $d(x, y) < \varepsilon$ belongs to U.

Example

An open set is a generalization of the idea of an open interval $(x, y) \subseteq \mathbb{R}$. Can you show that $U = (-1, 1)$ is an open subset of $\mathbb{R}$?

Remark

Open subsets must be defined in terms of a super-structure like topological spaces or metric spaces.