Point-set Topology for TQEC

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Topological Quantum Error Correction

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Agenda

- What's topology?
- Metric spaces
- Open sets
- Topological spaces
- Continuity
- Homeomorphisms
- Manifolds
- Open cover and finite subcover
- Compactness
- The circle S^1
- The torus $\mathbb{T}^2$

We assume existing familiarity with definitions used in linear algebra (set theory, bijection, etc.)

Donuts, is there anything they can't do?
—Homer Simpson

What's topology?

The popular analogy

Topology can be thought of as the study of structures whose properties are preserved under stretching like clay.

More formally

We require the precise notions of topological spaces and continuous functions.

Metric spaces

Definition

A metric space is a set X equipped with a "distance" function $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying three properties for all $x, y, z \in X$:

- ("Obvious") $d(x, y) = 0$ if and only if $x = y$
- (Positivity) If $x \neq y$ then $d(x, y) > 0$
- (Reflexivity) $d(x, y) = d(y, x)$
- (Triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$

Example

The real numbers $\mathbb{R}$ form a metric space with the "standard" distance function $d(x, y) = |x - y|$.

Open sets

Definition

A subset U of a metric space (M, d) is open if for all $x \in U$, there exists a positive real number ϵ such that any point $y \in M$ satisfying $d(x, y) < \epsilon$ belongs to U.

Remark

An open set is a generalization of to idea of an open interval $(x, y) \subseteq \mathbb{R}$.

Topological spaces

Definition

A topology on a metric space X is a collection $\mathcal{T}$ of open subsets of X satisfying the following axioms:

- The empty set $\emptyset$ and X itself belong to $\mathcal{T}$.
- Any arbitrary (finite or infinite) union of members of $\mathcal{T}$ belong to $\mathcal{T}$
- The intersection of any finite number of members of $\mathcal{T}$ belong to $\mathcal{T}$.

Definition

A topological space is the pair $(X, \mathcal{T})$, where $\mathcal{T}$ (also denoted $\mathcal{T}_X$) is a topology on the metric space X.

Continuity

Definition

A function $f : X \to Y$ between two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ is continuous if the inverse image of every open set in Y is an open set in X, meaning that for all $V \in \mathcal{T}_Y$, $f^{-1}(V) \in \mathcal{T}_X$.

Homeomorphisms

Definition

A function $f : X \to Y$ between two topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ is a homeomorphism if:

- f is bijective;
- f is continuous;
- its inverse, f^{-1}, is continuous.

Manifolds

Definition
The basis $\mathcal{B}$ of a topological space $\mathcal{T}$ is a collection of open sets such that every open set in $\mathcal{T}$ can be written as a union of elements of $\mathcal{B}$.

Definition
A set is countable if its elements can be put into a one-to-one correspondance with a subset of the natural numbers $\mathbb{N}$.

Definition
A topological space M is an n-dimensional manifold (or n-manifold) if:

- (Locally Euclidean) Every point $p \in M$ has a neighborhood that is homemorphic to an open set in $\mathbb{R}^n$.
- (Hausdorff) Any two distinct points in M can be separated by disjoint sets.
- (Countable Basis) Its basis is a countable set.

Open cover and finite subcover

Definition

An open cover of a topological space $\mathcal{T}$ is a (possibly infinite) collection of open sets $\{U_i\}$ in $\mathcal{T}$ such that the union of all these sets contains the entire space $\mathcal{T}$:

$$\mathcal{T} \subseteq \bigcup_i U_i$$

Definition

A finite subcover of a topological space $\mathcal{T}$ is a finite collection of open sets $\{U_1, \ldots, U_n\}$ such that

$$\mathcal{T} \subseteq \bigcup_{i=1}^{n} U_i$$

Compactness

Definition

A topological space $\mathcal{T}$ is compact if every open cover of $\mathcal{T}$ has a finite subcover.

The circle S^1

Definition

The circle S^1 is the one-dimensional unit sphere, defined using Euclidean distance between two points:

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

Remark

$\mathcal{T}_{S^1}$ is a topology defined by the open sets

$$\mathcal{T}_{S^1} = \{V \cap S^1 : V \text{ is an open set in } \mathbb{R}^2\}$$

The torus $\mathbb{T}^2$

Definition

The torus is defined as $S^1 \times S^1$.