# Math 210++: Generating Functions

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# First, back to recurrences

Normally we define functions like this: $f(x) = x^2$

But sometimes, we define a function like this:

$$f(n) = f(n-1) + f(n-2),$$

$$f(0) = 0, \qquad f(1) = 1.$$

Analogous to a recursive computer procedure

Think of the function as a mapping between the integers and a *sequence* of integers (this will show up later!)

# Main Technique: Unrolling (Math 210)

Intuit the closed form by expanding the sequence and looking for a pattern

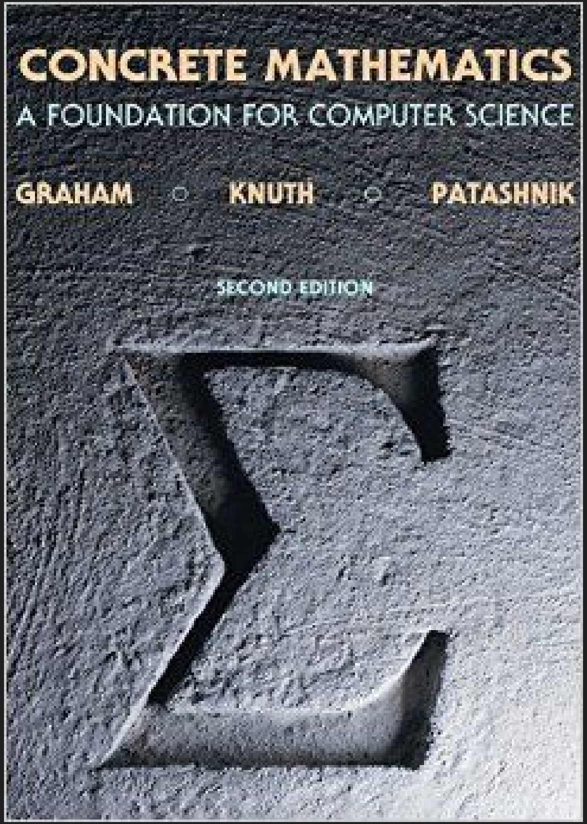
Can’t solve *any* recurrence this way

# Real Life Example (CS 475, Operating Systems)

- Determining burst time of a CPU-intensive job based on its previous burst times (moving averages)
- Fortunately we could use the unrolling technique to see the answer. For other recurrences, like Fibonacci, we're not as lucky

# Cool Technique: Generating Functions

- Turn the problem into a sum of formal power series, then read off the answer
- "A closeline on which we hang up a sequence of numbers for display" --Herbert Wilf
- Invented by Abraham de Moivre in 1730 to solve general linear recurrences
- "The most important idea in this whole book" --Concrete Mathematics (a primer to all the math needed to solve any kind of Computer Science problem)



# Ordinary generating function

(There are other kinds too, which add more to the coefficient of $x^n$)

$a_n$ is the sequence; we're not interested in $x$ since we're not plugging in (that means we avoid convergence issues!)

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n.$$

# Simplest generating function: the geometric series

What’s the generating function for the sequence of 1’s, {1,1,1,...}?

How can we use this to derive other series? (e.g, {1,2,3,4,…}?)

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$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

# Enumerative Combinatorics (counting) problem

How many solutions of the equation

$$x_1 + x_2 + x_3 + \cdots + x_m = n \qquad \text{(nonnegative integer } x_i\text{)}$$

are there? We can represent the set of choices by

$$(x^0 + x^1 + x^2 + \cdots)^m.$$

The $x_i$'s in the problem are the powers of $x$ being added in the product.

The coefficient of $x^n$ is the number of solutions.

The answer ends up using a binomial coefficient, due to Newton's Generalized Binomial Theorem

# Takeaway: can be used to solve *any* linear recurrence!

- Closed form of Fibonacci recurrence (can alternatively be found via a characteristic polynomial)
- Generally requires complicated algebra, calculus, knowledge of power series (and, with that, occasionally real and/or complex analysis), and binomial identities
- Bridges the gap between *continuous* and *discrete* mathematical tools needed for computer science (algorithm analysis, etc.)

(Hence the title “Concrete Mathematics”)

# Further generating functions

- *Generatingfunctionology* (Wilf, 1994) (online)
- Concrete Mathematics (Graham/Knuth/Patashnik, 1994)
- The Art of Computer Programming (Knuth, 1997)
- The Art of Proving Binomial Identities (Spivey, 2017)
- They show up in other places (probability theory, for example)