# Linear Algebra for Quantum Computing and ZX Calculus

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Topological Quantum Error Correction

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# Agenda

- Vector spaces
- Basis vectors
- Linear maps
- Dirac notation
- Eigenstates
- Unitary operators
- Tensor products
- The commutator
- Matrix trace

We assume existing familiarity with sets, functions, matrices, vectors, and complex numbers.

# Vector spaces

## Definition

$V$ is a vector space if it is a set of vectors coupled with the addition of vectors and scalar multiplication.
Vector addition: $\mathbf{u}, \mathbf{v} \in V$ means that $\mathbf{u} + \mathbf{v} \in V$.
Scalar multiplication: $\alpha\mathbf{v} \in V$ for any $\mathbf{v} \in V, \alpha \in \mathbb{C}$.

## Remark

The vector space we often use in quantum computing is $\mathbb{C}^n$, where $n$ is a power of 2.

## Example

$$\begin{pmatrix} 42 \\ 1.618 \\ e^{i\pi/3} \\ 1 + i \end{pmatrix} \in \mathbb{C}^4$$

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# Linear independence and basis vectors

### Definition

Given a set of vectors $S = \{\mathbf{v}_1, \dots \mathbf{v}_m\}$ and associated nonzero scalars $\alpha_1 \dots, \alpha_m$, we say that $S$ is linearly independent if

$$\sum_{k=1}^{m} \alpha_k \mathbf{v}_k = \mathbf{0},$$

where $\mathbf{0}$ is the zero vector.

### Definition

If a set of vectors $T$ can be combined in a linear fashion to produce every element of $S$, we say that $T$ spans $S$.

### Definition

Suppose $V$ is a vector space. Then a minimum cardinality subset $\mathcal{B} \subseteq V$ that is linearly independent and spans $V$ is a basis of $V$.

# Linear maps

## Definition

A linear map is a function $T : U \to V$, where $U$ and $V$ are vector spaces, such that

$$T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) \text{ for all } \mathbf{u}_1, \mathbf{u}_2 \in U$$

and

$$T(\alpha \mathbf{u}) = \alpha T(\mathbf{u}) \text{ for all } \mathbf{u} \in U, \alpha \in \mathbb{C}.$$

## Example

The Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is a linear map $H : \mathbb{C}^2 \to \mathbb{C}^2$.

# Dirac notation

## Definition

A ket (or state vector) $|\psi\rangle$ is a vector in a vector space, where $\sum_{k=1}^{n} |\psi_k|^2 = 1$ and $\psi_k$ is the $k$th element of $|\psi\rangle$ .

## Definition

A bra-ket $\langle\phi|\psi\rangle$ is the inner product of $|\phi\rangle$ and $|\psi\rangle$,

$$\sum_{k=1}^{n} \phi_k^* \psi_k.$$

## Example

For any quantum state $|\psi\rangle$, $\langle\psi|\psi\rangle = 1$, since $\psi_k \psi_k^* = |\psi_k|^2$.

# Dirac notation (cont'd)

### Definition

A ket-bra $|\phi\rangle\langle\psi|$ is a linear map

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_n \end{pmatrix} \left(\psi_1^*, \ldots, \psi_n^*\right) = \begin{pmatrix} \phi_1\psi_1^* & \ldots & \phi_1\psi_n^* \\ \vdots & \ddots & \vdots \\ \phi_n\psi_1^* & \ldots & \phi_n\psi_n^* \end{pmatrix}$$

# Eigenstates

### Definition

A state vector $|\psi\rangle$ is the $\lambda$-eigenstate of a linear map $U$ if $U|\psi\rangle = \lambda|\psi\rangle$ for some $\lambda \in \mathbb{C}$.

### Example

If $U|\psi\rangle = |\psi\rangle$, we say that $|\psi\rangle$ is the $+1$ eigenstate of $U$.

# Unitary matrices

## Definition

The transpose of a matrix $U$ is denoted $U^T$ and is obtained by systematically exchanging (i.e., swapping) the values of the rows with the values in the columns in $U$.

## Definition

The adjoint of a matrix $U$ is the conjugate transpose

$$U^\dagger = \begin{pmatrix} u_{1,1}^* & \cdots & u_{1,n}^* \\ \vdots & \ddots & \vdots \\ u_{n,1}^* & \cdots & u_{n,n}^* \end{pmatrix}^T = \begin{pmatrix} u_{1,1}^* & \cdots & u_{n,1}^* \\ \vdots & \ddots & \vdots \\ u_{1,n}^* & \cdots & u_{n,n}^* \end{pmatrix}$$

## Definition

A matrix $U$ is unitary if it is self-adjoint, that is, $UU^\dagger = I$.

# Tensor products

## Definition

For two-dimensional quantum states, we can define the tensor product between them as

$$|\phi\rangle \otimes |\psi\rangle = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \otimes \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ \phi_2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \phi_1\psi_1 \\ \phi_1\psi_2 \\ \phi_2\psi_1 \\ \phi_2\psi_2 \end{pmatrix}$$

## Remark

You can take tensor products of matrices as well; the new object's dimension is the product of the dimensions of the objects multiplied, meaning that tensor products of higher-dimensional objects quickly become unmanageable to compute by hand.

# The commutator

### Definition

Two matrices $A$ and $B$ are said to commute if $AB = BA$.

### Definition

The commutator of two matrices is defined as $[A, B] = AB - BA$.

### Remark

For commuting (or, "Abelian") matrices, $[A, B] = \mathcal{O}$ (the zero matrix).

# Matrix trace

### Definition

The trace of a matrix $A$ is the sum of its diagonal elements; that is

$$\operatorname{tr}(A) = \sum_{k=1}^{n} A_{k,k}$$

### Remark

Trace is "commutativity-preserving," meaning $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for any matrices $A, B$.