lift, exposure, and impulse based entrainment theory

Kevin Pierce

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Consider a row of uniform bed particles, each having diameter d. For now, neglect drag, so that at an instant, Newton's third law governing the motion of particle i is

$$ma_i = L - W. \tag{1}$$

Assume the lift force is driven by a turbulent fluctuation which exists for a length of time T, then ceases to exist; that is,

$$L(T) = L_0\{1 - \Theta(t - T)\}, \tag{2}$$

and assume further that the exposures of particles are uniformly distributed away from mean bed elevation. If the exposure $e = 0$, then the top of the particle is flush with the mean bed elevation. If the exposure $e = d$, then the particle is maximally exposed away from the mean bed elevation. For our purposes, considering two particles is enough. Call them e_1 and e_2, with e_1 upstream of e_2. Then, following Paintal (1971), the joint distribution of particle exposures is

$$g(e_1, e_2) = \begin{cases} 1/d^2 & \text{if } 0 \leq e_1 \leq d \text{ and } 0 \leq e_2 \leq d \\ 0 & \text{else.} \end{cases} \tag{3}$$

Integrating the force balance on particle 1 considering only the case of $L_0 > W$, the elevation of its highest point as a function of time for $t \leq T$ is

$$h_1(t) = \frac{L_0 - W}{2m} t^2 + e_1, \tag{4}$$

and the particle elevation only decreases for $t > T$, so the maximum height attained by the particle is

$$h_1^{max} = \frac{L_0 - W}{2m} T^2 + e_1. \tag{5}$$

We define the condition for entrainment by lifting of particle 1 as the situation when its lowest point exceeds the highest point of its downstream contact for $t \leq T$. That is, entrainment occurs if

$$h_1^{max} - d \geq e_2, \tag{6}$$

or equivalently

$$\psi(e_1, e_2; L_0, T) \geq 0, \tag{7}$$

with

$$\psi(e_1, e_2; L_0, T) = \frac{L_0 - W}{2m/T^2} + e_1 - d - e_2. \tag{8}$$

The probability of entrainment from lift force of magnitude L_0 and duration T is then the density of configurations $\{e_1, e_2\}$ respecting the entrainment force balance $\psi(e_1, e_2; L_0, T) \geq 0$:

$$p_E(L_0, T) = \int\int_{\psi(e_1,e_2;L_0,T)\geq 0} g(e_1, e_2) de_1 de_2. \tag{9}$$

Incorporating the exposure distribution equation 3, the range of exposures, $0 \leq e_1 \leq d$, and the entrainment condition, equation 6, equation 9 becomes

$$p_E(L_0, T) = \frac{1}{d^2} \int_0^d de_2 \int_{e_{1,min}}^d de_1 = \frac{1}{d^2} \int_0^d de_2 (d - e_{1,min}), \tag{10}$$

where

$$e_{1,min} = \begin{cases} 0 & \text{if } e_2 + d - \frac{L_0 - W}{2m/T^2} < 0, \text{ and} \\ e_2 + d - \frac{L_0 - W}{2m/T^2} & \text{else.} \end{cases} \tag{11}$$

Denoting $e_2^* = \frac{L_0 - W}{2m/T^2} - d$ as a pseudo-exposure, defined on the range $-d \leq e_2^* < \infty$, this relationship eq. 11 can be written more compactly as

$$e_{1,min} = (e_2 - e_2^*)\Theta(e_2 - e_2^*), \tag{12}$$

once again invoking the Heaviside step function Θ . From this point, integrating equation 10 over e_2 requires the investigation of three cases.

1. $e_2^* > d$: In this case, $e_{1,min} = 0$ for all e_2, so the probability of entrainment is

$$p_E(L_0, T) = 1 \quad \text{(case 1)}. \tag{13}$$

2. $-d/2 \leq e_2^* < 0$: Now $e_{1,min} = e_2 - e_2^*$, from which it follows that

$$p_E(L_0, T) = \frac{1}{2} + \frac{e_2^*}{d} = \frac{L_0 - W}{2md/T^2} - \frac{1}{2} \quad \text{(case 2)}. \tag{14}$$

3. $0 \leq e_2^* \leq d$: This time, the integral over e_2 splits around e_2^*:

$$p_E(L_0, T) = \frac{1}{d^2}\Big(\int_{e_2^*}^d de_2 (d - e_2 + e_2^*) + \int_0^{e_2^*} de_2 d \Big),$$

which evaluates to

$$p_E(L_0,T) = \frac{1}{d^2}\Big(\frac{d^2}{2} - \frac{e_2^{*2}}{2} + e_2^* d\Big) \tag{15}$$
$$= \frac{1}{2}\Big[1 - \Big(\frac{L_0 - W}{2md/T^2} - 1\Big)^2 + \Big(\frac{L_0 - W}{2md/T^2} - 1\Big)\Big] \quad \text{(case 3)}. \tag{16}$$

In summary, the probability of entrainment of a particle in a random bed configuration, subjected to lift force L_0 for duration T, is

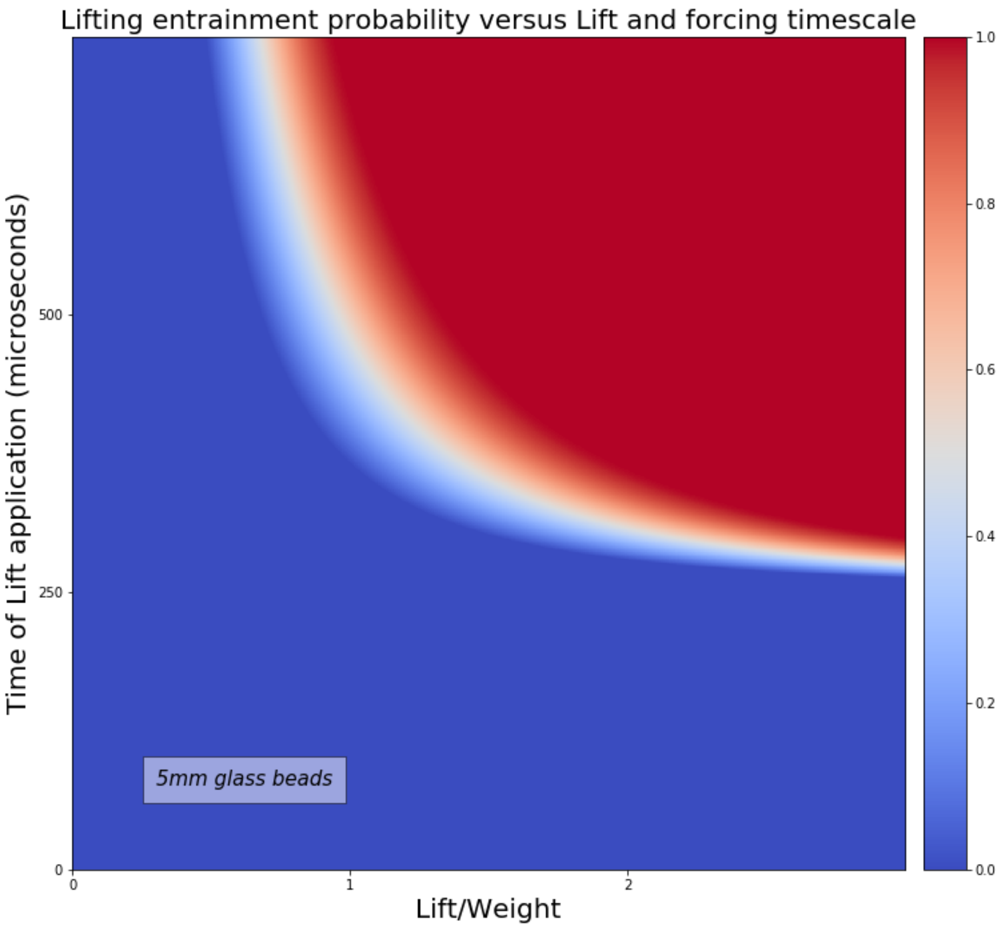
$$p_E(L_0,T) = \begin{cases} 1 & \text{if } e_2^* > d, \\ \frac{1}{2} + \frac{e_2^*}{d} & \text{if } -d/2 \le e_2^* < 0, \\ \frac{1}{d^2}\Big(\frac{d^2}{2} - \frac{e_2^{*2}}{2} + e_2^* d\Big) & \text{if } 0 \le e_2^* \le d, \\ 0 & \text{else.} \end{cases} \tag{17}$$

The dependence on L_0 and T is implicit in $e_2^* = \frac{L_0 - W}{2m/T^2} - d$. This entrainment probability is plotted over the normalized lift (L_0/W) and T for 5mm glass beads in the figure on the last page. On the figure, the time-scale of force application is between 1 microsecond and 1 millisecond. Interestingly, entrainment can happen with small probability for $L_0 < W$, provided the lift force is sustained for more than 0.5 milliseconds. If the time of force application were taken very large, there would be a sharp lift threshold for entrainment.

Paintal's PhD Thesis indicates he observed sub-threshold transport, and this must have inspired his research direction to some degree, because he undermined the entrainment threshold notion with an examination of granular configurational influences on entrainment. Diplas et al. (2008) also attack the entrainment threshold idea, but from an examination of the variability in forcing time-scale.

Generalizing one of the last equations of Paintal (1971)'s section on the entrainment probability, if the joint distribution of lift magnitude and duration were known, $f(L_0,T)$, then the mean entrainment probability would be

$$\bar{p}_E = \frac{\int dL_0 dT p_E(L_0,T) f(L_0,T)}{\int dL_0 dT f(L_0,T)}. \tag{18}$$


Lifting entrainment probability versus Lift and forcing timescale
Time of Lift application (microseconds)
Lift/Weight
5mm glass beads
0
250
500
0
1
2
0.0
0.2
0.4
0.6
0.8
1.0