

Contents

Table of Contents	1
1 Introduction	2
2 How do you obtain a 4–manifold with a specific boundary?	3
3 Constructive proof that a smooth, closed, orientable 3–manifold is the boundary of some 4–manifold	5
3.1 Projections from 3–manifolds to \mathbb{R}^2	6
3.2 Decomposing \mathbb{R}^2	6
3.3 Stratifying M	7
3.4 Handle attachment	7
3.5 The constructed 4–manifold	7
4 Algorithm for constructing a triangulated 4–manifold with prescribed 3–manifold boundary	8
4.1 Define projection	9
4.2 Induce subdivision	10
4.3 Form base 4–manifold	11
4.4 Attach 2–handles	11
4.5 Attach 3–handles	11
4.6 Cone away final boundary components	12
5 Conclusion	13
Bibliography	14

Chapter 1

Introduction

The main goal of this document is to provide an algorithm whose input is a closed, orientable 3-manifold triangulation M and whose output is a 4-manifold triangulation whose boundary is M . This algorithm mirrors the constructive proof that a smooth, closed, orientable 3-manifold bounds some 4-manifold presented in Chapter 3.

Chapter 2

How do you obtain a 4-manifold with a specific boundary?

Given a smooth, closed 3-manifold M , there are infinitely many 4-manifolds with boundary M . We do not ensure that the constructed 4-manifold has any properties other than a specified boundary, so our construction ensures easy verification that the constructed 4-manifold's boundary is exactly M . This is done by setting $W = M \times [0, 1]$, so W has boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

We then attach handles to the boundary of W away from M_0 until only M_0 remains.

The concept of a stratified handle attachment needs some explanation. First, we define attachment of topological spaces, and use that language to define handle attachment.

Definition 2.0.1 (Attachment). Let X and Y be topological spaces, $A \subset X$ a subspace, and $f : A \rightarrow Y$ a continuous map. We define a relation \sim by putting $f(x) \sim x$ for every x in A . Denote the quotient space $X \sqcup Y / \sim$ by $X \cup_f Y$. We call the map f the *attaching map*. We say that X is *attached* or *glued* to Y over A . A space obtained through attachment is called an *adjunction space* or *attachment space*.

Alternatively, we let A be a topological space and let $i_X : A \rightarrow X$, $i_Y : A \rightarrow Y$ be inclusions. Here, the adjunction is formed by taking $i_X(a) \sim i_Y(a)$ for every $a \in A$ and we denote the adjunction space by $X \cup_A Y$.

Definition 2.0.2 (Handle). Take $n = \lambda + \mu$ and M a smooth n -manifold with

nonempty boundary ∂M . Let D^λ be the closed λ -disk and put $H^\lambda = D^\lambda \times D^\mu$. Let $\varphi : \partial D^\lambda \times D^\mu \rightarrow \partial M$ be an embedding and an attaching map between M and H^λ . The attached space H^λ is an n -dimensional λ -handle, and $M \cup_\varphi H^\lambda$ is the result of an n -dimensional λ -handle attachment.

Handle attachment is defined for smooth manifolds, but the resulting attachment space is not a smooth manifold. Rather, the result is a stratified manifold. We use the definition from [3].

Definition 2.0.3 (Stratification). X is a *filtered space* on a finite partially ordered indexing set S if

1. there is a closed subset X_s for each $s \in S$,
2. $s \leq s'$ implies that $X_s \subset X_{s'}$, and
3. the inclusions $X_s \hookrightarrow X_{s'}$ satisfy the homotopy lifting property.

The X_s are the *closed strata* of X , and the differences

$$X_s \setminus \bigcup_{r < s} X_r$$

are *pure strata*. The pure strata are denoted X^s . The singular declension of the word strata is *stratum*.

A *filtered map* between spaces filtered over the same indexing set is a continuous function $f : X \rightarrow Y$ such that $f(X_s) \subset Y_s$, and such a map is *stratified* if $f(X^s) \subset Y^s$. This leads to definitions of stratified homotopy, therefore stratified homotopy equivalence.

Immediate examples of stratified manifolds are manifolds with boundary and manifolds with corners. The result of a smooth handle attachment is a manifold with corners at $\varphi(\partial D^\lambda \times \partial D^\mu)$, hence a stratified manifold.

Chapter 3

Constructive proof that a smooth, closed, orientable 3-manifold is the boundary of some 4-manifold

We prove that every smooth, closed, orientable 3-manifold is the boundary of some 4-manifold. We do so by explicitly constructing such a 4-manifold from a given 3-manifold. This construction is mirrored in Chapter 4 where we prove the same for a given closed, orientable 3-manifold triangulation and provide an algorithm.

Let M be a smooth, closed, orientable 3-manifold and take $W = M \times [0, 1]$. Then W is a 4-manifold with boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

A 4-manifold with only one boundary component, M_0 , is obtained from W by iteratively attaching 4-dimensional 2-, then 3-, then 4-handles to W over the M_1 boundary component until that component has been “filled in.”

Instructions for handle attachment come from defining a projection $f : M_1 \rightarrow \mathbb{R}^2$ that induces a stratification of $M_1 \subset \partial W$, indexed by dimension of the pure strata as a submanifold of M_1 , such that the pure stratum M_1^3 is a disjoint collection of open submanifolds of M_1 . We call the closure of a connected component of M_1^3 a *block*, and we impose conditions on f to ensure that every block can be classified as one of the following:

face-block: An attachment neighbourhood for a stratified 2-handle. Each face-block is diffeomorphic to $S^1 \times G_n$, the product of the circle with an n -gon for some n .

edge-block: A partial attachment neighbourhood for a stratified 3-handle. Each edge-block is diffeomorphic to one of $D^2 \times [0, 1]$, $A \times [0, 1]$, or $P \times [0, 1]$, where A is the annulus $S^1 \times [0, 1]$ and P is a pair-of-pants surface. Attachment of stratified 2-handles over our face-blocks “fill in” the annular boundary components of edge-blocks, forming full attachment neighbourhoods for stratified 3-handles.

vertex-block: A partial attachment neighbourhood for a stratified 4-handle. Each vertex-block is homeomorphic to a genus 2 or 3 (3,1)-handlebody. When stratified 2- and 3-handles are attached to W , the remaining boundary is M_0 union a collection of stratified 3-spheres which are then coned away.

The remainder of this chapter is spent ensuring that such a stratification can be achieved for any smooth, orientable, closed 3-manifold, detailing how the stratification is induced, proving that the attachment of stratified 2- and 3-handles has the previously stated effects, and discussing the resulting 4-manifold.

3.1 Projections from 3-manifolds to \mathbb{R}^2

We begin by stating the conditions required of the projection $f : M \rightarrow \mathbb{R}^2$ for the desired decomposition of \mathbb{R}^2 , hence stratification of M , to exist. For proof that an f satisfying these conditions exists and that such functions are generic, see [1].

3.2 Decomposing \mathbb{R}^2

We use a projection f satisfying the conditions in Section 3.1 to induce a particular decomposition on \mathbb{R}^2 . We fit closed neighbourhoods around the singular values of f and classify these sleeves by the maximum codimension (with respect to \mathbb{R}^2) of singular values they contain. Because S_f , the singular values of f , consists of codimension 1 and codimension 2 singular values (i.e. arcs and arc-crossings respectively), we decompose \mathbb{R}^2 into face-regions that contain no singular values, edge-regions that contain only codimension 1 singular values, and vertex-regions, each of which contain exactly 1 codimension 2 singular value.

3.3 Stratifying M

Decomposing \mathbb{R}^2 via the singular values of f also induces a stratification of M by considering the preimage through f of the decomposing regions. The interiors of a face-regions have preimage through f a disjoint collection of face-blocks, the interiors of edge-regions have preimage of edge-blocks, and of vertex-regions, vertex-blocks.

3.4 Handle attachment

We now investigate the consequences of stratified handle attachment over the face-blocks of the stratified M_1 boundary component of $W = M \times [0, 1]$. This investigation is performed by comparing ∂W to $\partial W'$, where

$$W' = W \cup \{H_\alpha^2\}_{\alpha \in A} / \sim,$$

the 4-manifold obtained by attaching stratified 2-handles to W .

Attaching these stratified 2-handles to W alters the boundary of W via surgery on M_1 that is equivalent to replacing the interiors each face-block with a solid torus whose meridians are longitudes of the replaced face-block. The annular boundary components of each edge-block are “filled-in” by cylinders found between meridians of these newly introduced solid tori, in each case forming a stratified $S^2 \times [0, 1]$. These stratified $S^2 \times [0, 1]$ are taken as attachment neighbourhoods for stratified 3-handles, forming W'' .

Finally, we compare the boundaries of W' and W'' to show that $\partial W''$ is the disjoint union of M_0 with a collection of stratified 3-spheres. The 3-sphere boundary components are then coned away.

3.5 The constructed 4-manifold

Chapter 4

Algorithm for constructing a triangulated 4-manifold with prescribed 3-manifold boundary

The steps to construct a triangulated 4-manifold with prescribed 3-manifold boundary broadly follow the steps to construct a 4-manifold with prescribed smooth, orientable 3-manifold boundary. Let N be a 3-manifold triangulation. Then the steps of construction are:

- Step 1: Define a projection $f : N \rightarrow \mathbb{R}^2$.
- Step 2: Induce a subdivision of N from f . The result is a 3-manifold triangulation M that is equivalent to N in the sense of triangulations.
- Step 3: Let $W = M \times [0, 1]$ be a 4-manifold with boundary components $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$.
- Step 4: Attach 4-dimensional 2-handles to W over its M_1 boundary as prescribed by the subdivision of M from f . Call the result W' and call the boundary of W' different from M_0 by M'_1 .
- Step 5: Attach 4-dimensional 3-handles to W' over M'_1 as prescribed by the subdivision induced by f and the surgery induced by 2-handle attachment. Call the result W'' .
- Step 6: The boundary of W'' consists of M_0 and a collection of copies of S^3 that we now cone off. The result is a 4-manifold whose boundary is exactly M_0 .

Each of these steps is made algorithmic, and these algorithms are chained in series to form a single algorithm. The result has input a closed, orientable 3-manifold triangulation M and output a 4-manifold triangulation W whose sole boundary component is a triangulated 3-manifold that is equivalent to M in the sense of triangulations. In this case, we find that ∂W is a subdivision of M , and this subdivision is the subdivision induced by the projection f in Step 1.

Throughout this chapter N refers to the initial input 3-manifold triangulation, f to the projection defined in Section 4.1, M is the subdivision of N induced by f , W is the 4-manifold $M \times [0, 1]$, W' is the result of attaching 2-handles to W , and W'' is the result of attaching 3-handles to W' .

4.1 Define projection

The projection's utility is in defining a subdivision of the initial closed, orientable 3-manifold triangulation N such that attaching regions for triangulated 2- and 3-handles can be found. This is done before forming the base 4-manifold so that the subdivided triangulation is used in the algorithm that provides W .

Our subdivision is obtained by imposing four conditions on the $f : N \rightarrow \mathbb{R}^2$:

1. f maps vertices to the circle. i.e. for each vertex $v \in N^0$, $f(v)$ lies on the unit circle in \mathbb{R}^2 .
2. The images of vertices are distinct. i.e. for every pair of vertices $u, v \in N^0$, $f(u) \neq f(v)$.
3. f is linear on each simplex of N and piecewise-linear on N . i.e. if $x \in \sigma$ is a point in the simplex σ with vertices v_i , then $x = \sum_i a_i v_i$ with $\sum_i a_i = 1$ and $f(x) = \sum_i a_i f(v_i)$.
4. Edge intersections are distinct. i.e. for every triple of edges $e_1, e_2, e_3 \in N^1$ that share no vertices, $f(e_1) \cap f(e_2) \neq f(e_2) \cap f(e_3)$.

Conditions 1 and 2 ensure that every simplex of N is mapped to the plane in standard position (i.e. the images of the vertices in the plane form a convex set). This, along with conditions 3 and 4, allows us to use concepts and language from normal surface theory to describe the subdivision of N in the next section. We call these four conditions the *subdivision conditions* on f .

All conditions are satisfied by fixing an odd integer k greater than or equal to the number of vertices in N , injecting the vertices of N to the k^{th} complex roots of unity in the plane, then extending linearly over the skeletons of N . The first three conditions are clearly satisfied by this procedure, and the last is satisfied by applying the results in [2].

The algorithm presented in this section takes as input the triangulated 3-manifold N and produces a projection $f : N \rightarrow \mathbb{R}^2$ satisfying the subdivision conditions.

4.2 Induce subdivision

The goal of subdividing N is to create and identify analogues to the face- edge- and vertex-blocks of Chapter 3 where we may iteratively attach 2-, 3-, then 4- handles. We use a similar technique to that found in Chapter 3, first decomposing \mathbb{R}^2 with the projection, then examining preimages to define our subdivision.

Decomposition of \mathbb{R}^2 is done through $f(N_1)$. A point in $f(N_1)$ is the image of either a vertex, exactly one edge, or exactly two edges (i.e. is an edge crossing), so we refer to these as the *vertices*, *edges*, and *crossings* of the decomposition. Because $f(N) \setminus f(N_1)$ is a disjoint collection of simply connected regions, we call the connected components of $f(N) \setminus f(N_1)$ the *faces* of the decomposition.

We construct our subdivision of N using the decomposition component preimages. The preimage of a face component defines substructures analogous to face-blocks, of edge components to edge blocks, and vertices and crossings to vertex-blocks. Inside of an individual tetrahedron of N , edge and crossing preimages are well-defined, supplying a natural subdivision of N into a cell complex. Then, the subdivision of a cell complex into a triangulation is well-defined.

The algorithm presented in this section takes as input a closed, orientable 3-manifold triangulation N and a projection $f : N \rightarrow \mathbb{R}^2$ and produces as output a closed, orientable 3-manifold triangulation M that is a subdivision of N . Furthermore,

1. the 3-cells of M are partitioned into subsets that serve the same purpose as the face-blocks of Chapter 3: attaching regions for 2-handles,
2. the 2-cells of M are partitioned into subsets that are either interior to face-blocks, or subsets that serve the same purpose as the edge-blocks of Chapter 3, i.e. buffers between face-blocks.

4.3 Form base 4-manifold

The algorithm presented in this chapter takes as input a closed, orientable 3-manifold triangulation M and produces as output a 4-manifold triangulation W whose boundary is the disjoint union of two copies of M . We do this by explicitly triangulating the cell complex $W \times [0, 1]$.

4.4 Attach 2-handles

At this point in our procedure we have a 4-manifold W with triangulated boundary components M_0 and M_1 . We aim to attach handles to W over the boundary of W away from M_0 until the boundary of W is exactly M_0 . The first step is to attach 2-handles to W over the closed solid torus triangulations that partition M_1 .

The algorithm of this section takes as input a closed solid torus triangulation T with a collection of triangulated parallel longitudes Γ in its boundary (each γ_i in Γ is an explicit 0-framing for the 2-handle attachment) and produces as output a D^4 triangulation whose S^3 boundary triangulation has genus 1 Heegard splitting over ∂T , and γ_i bounds a triangulated disk in $\partial D^4 \setminus T$ for each i . Such a D^4 triangulation is taken as a 4-dimensional 2-handle and attached to W over T . We then attach such a 2-handle over each closed solid torus in our partition of M_0 .

4.5 Attach 3-handles

We now have a triangulated 4-manifold with boundary components M_0 and the result of surgery on M_1 induced by 2-handle attachment. This surgery had the following effect on M_1 :

1. Each solid torus in the partition of M_1 induced by the projection from Section 4.1 is associated with a triangulated D^4 in Section 4.4.
2. For each (torus, 2-handle) pair (T, H^2) , the boundary of H^2 contains T and $T^* = H^2 \setminus \text{int}(T)$ is another solid torus whose boundary triangulation is identical to that of T .
3. Thus the effect of surgery on M_1 is of replacing each T with T^* over their shared boundary.

4.6 Cone away final boundary components

Chapter 5

Conclusion

Bibliography

- [1] Harold Levine. Elimination of cusps. *Topology*, 3(2):263–296, 1965.
- [2] Bjorn Poonen and Michael Rubinstein. The number of intersection points made by the diagonals of a regular polygon. *SIAM Journal on Discrete Mathematics*, (1) 11:135–156, 1998.
- [3] Shmuel Weinberger. *The topological classification of stratified spaces*. Chicago, 1994.