

3-manifolds algorithmically bound 4-manifolds

by

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B.Sc., University of Victoria, 2013

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Chapter 1

Introduction

When we begin studying manifolds, we find the basic problems of surface theory rife with algorithms and elegant solutions. Moving up by one dimension, a core of algorithms cover 3-manifold theory. These algorithms, though sophisticated, are historically unexpected in form and possibly hard to implement, leading some to prefer more conceptually pleasant pseudo-algorithms such as SnapPy [4]. Many problems in high dimensions (5 and up) become impossible due to the complexities of the word problem in fundamental groups and the difficulties in resolving smoothing of CW-complexes. By comparison, the landscape of 4-manifold theory is not so easily generalized. High dimensional complications such as the word problem crop up, but dimension 4 is unique when considering the corresponding relation between CW complexes and smoothing theory. Moreover, algorithmic recognition of the n -sphere is resolved in every dimension but 4, where it is an open problem. Costantino & Thurston's results in shadow theory resonate traditional 3-manifold theory techniques, suggesting that some basic cobordism questions may not be computationally difficult in dimension 4.

Our assemblage of a constructive proof that 3-manifolds bound 4-manifolds was inspired by the discussion in Section 2.2 of Costantino & Thurston's "3-manifolds efficiently bound 4-manifolds" [3]. An appropriately well-behaved smooth map from a 3-manifold M to \mathbb{R}^2 induces a stratification of M into a union of handlebodies. This stratification serves as a set of surgery instructions for turning M into S^3 . It can also be interpreted as a set of 4-dimensional handle attachment sites, and attaching these handles to one boundary component of $M \times [0, 1]$ produces a 4-manifold whose boundary is precisely M .

Our algorithm is the adaptation of this smooth proof to the setting of triangulated

manifolds. We define a map from the input 3-manifold triangulation to \mathbb{R}^2 , subdivide the manifold into handlebodies, then attach handles to one boundary component the manifolds' 4-thickening until we obtain the desired outcome.

1.1 Expected Background

This document is aimed at a reader with some understanding of the tools of low-dimensional manifold theory. We do not present the basics of manifold theory, calculus on manifolds, or triangulations of manifolds.

For the fundamentals of manifolds and calculus on manifolds see Lee's "Introduction to Smooth Manifolds" [6]. For more on 3-manifolds and an introduction to triangulations, see Thurston's "Three-dimensional geometry and topology" [11].

1.2 Agenda

What follows is a brief summary of the contents of each chapter in this document. Each chapter has a purpose, and that purpose is also stated.

Chapter 1 lays out what the central thesis problem is and why it is interesting. It also sets expectations for the rest of the document in terms of what to expect and what not to expect.

Chapter 2 presents some specialized tools of low-dimensional topology that are utilized throughout the document. It serves as a reference or refresher depending on the reader's familiarity with the subject.

Chapter 3 presents the constructive proof in the smooth case. It fills a void in the current literature and anchors the abstraction of Chapter 4 to something accessible and visual.

Chapter 4 provides the algorithm. It is a precursor to an implementation of the algorithm in a low-dimensional topology software package.

Chapter 5 restates our results and delves into implications on future work. It, along with Chapter 1 and the Abstract, provide an overview of the entirety of the document.

Chapter 2

Tools of low-dimensional topology

Given a smooth, closed 3-manifold M , there are infinitely many 4-manifolds with boundary M . For example, consider $M = S^3$. Removing a 4-ball from any 4-manifold W produces a 4-manifold with S^3 boundary. Our construction begins with the 4-manifold $W = M \times [0, 1]$ which has boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \sqcup M_1.$$

We attach stratified handles to the boundary of W away from M_0 until only M_0 remains. Handle attachment sites are obtained through a stratifying procedure in Sections 3.1 and 3.2, but only 2-handle attachment sites are immediate. The sites for 3-handles appear after considering the effect of 2-handle attachment, and the sites for 4-handles appear after considering the effect of 2- and 3-handle attachment. The remainder of this chapter sets the foundation for our analysis of the effect of handle attachments.

2.1 Handles

The concept of a stratified handle attachment needs some explanation. First, we define attachment of topological spaces, and use that language to define handle attachment. We use Gompf & Stipsicz's "4-Manifolds and Kirby Calculus" [5] as our main reference for handle attachment.

Definition 2.1.1 (Attachment). Let X and Y be topological spaces, $A \subset X$ a subspace, and $f : A \rightarrow Y$ a continuous map. We define a relation \sim by putting

$f(x) \sim x$ for every x in A . Denote the quotient space $X \sqcup Y / \sim$ by $X \cup_f Y$. We call the map f the *attaching map*. We say that X is *attached* or *glued* to Y over A . A space obtained through attachment is called an *adjunction space* or *attachment space*.

Alternatively, we let A be a topological space and let $i_X : A \rightarrow X$, $i_Y : A \rightarrow Y$ be inclusions. Here, the adjunction is formed by taking $i_X(a) \sim i_Y(a)$ for every $a \in A$ and we denote the adjunction space by $X \cup_A Y$.

Definition 2.1.2 (Handle). Take $n = \lambda + \mu$ and M a smooth n -manifold with nonempty boundary ∂M . Let D^λ be the closed λ -disk and put $H^\lambda = D^\lambda \times D^\mu$. Let $\varphi : \partial D^\lambda \times D^\mu \rightarrow \partial M$ be an embedding and an attaching map between M and H^λ . The attached space H^λ is an n -dimensional λ -handle, abbreviated (n, λ) -handle, and $M \cup_\varphi H^\lambda$ is the result of an n -dimensional λ -handle attachment, abbreviated (n, λ) -handle attachment.

Let $n = \lambda + \mu$, let $H^\lambda = D^\lambda \times D^\mu$ be a (n, λ) -handle, and let $\varphi : \partial D^\lambda \times D^\mu \rightarrow \partial M$ be an attaching map from H^λ to the n -manifold M . The sphere $\partial D^\lambda \times \{0\}$ in H^λ is the *attaching sphere* of H^λ , and the sphere $\{0\} \times \partial D^\mu$ is the *belt sphere* of H^λ .

Handles H_1 and H_2 attached to M are extraneous when $(M \cup H_1) \cup H_2 \approx M$. When this happens, we say that the handles *cancel*. In the proof of Theorem 3.4.1 we attach 2-handles with the intention of canceling 1-handles.

Proposition 2.1.3 (Proposition 4.2.9 in [5]). A $(k-1)$ -handle H^{k-1} and a k -handle H^k ($1 \leq k \leq n$) can be canceled, provided the attaching sphere of H^k intersects the belt sphere of H^{k-1} transversely in a single point.

Handle attachment is defined for smooth manifolds, but the resulting attachment space is not a smooth manifold. Rather, the result is a manifold with corners. Some other formulations of handle attachment (e.g. [5]) implicitly smooth the corners resulting from handle attachment, so in their formulation of handle cancellation the equivalence relation is strong (i.e. \approx is diffeomorphism). Instead of smoothing corners, we keep everything in the language of stratified spaces (introduced in Section 2.2). We therefore take \approx to be homeomorphism for the purposes of handle cancellation.

2.2 Stratification

We use Weinberger's "The topological classification of stratified spaces" [13] as our main reference for stratified spaces.

Definition 2.2.1 (Stratification). X is a *filtered space* on a finite partially ordered indexing set S if

1. there is a closed subset X_s for each $s \in S$,
2. $s \leq s'$ implies that $X_s \subset X_{s'}$, and
3. the inclusions $X_s \hookrightarrow X_{s'}$ satisfy the homotopy lifting property.

The X_s are the *closed strata* of X , and the differences

$$X^s = X_s \setminus \bigcup_{r < s} X_r$$

are *pure strata*.

A *filtered map* between spaces filtered over the same indexing set is a continuous function $f : X \rightarrow Y$ such that $f(X_s) \subset Y_s$, and such a map is *stratified* if $f(X^s) \subset Y^s$. This leads to definitions of stratified homotopy, therefore stratified homotopy equivalence.

Immediate examples of stratified manifolds are manifolds with boundary and manifolds with corners. Many handles (e.g. $[0, 1] \times [0, 1]$) are manifolds with corners, and the result of a smooth handle attachment is a manifold with corners at $\varphi(\partial D^\lambda \times \partial D^\mu)$. Hence both are stratified manifolds.

A *stratified handle attachment* is a handle attachment where the handle, the manifold to which we attach the handle, and the attaching map are each stratified. The main distinctions between stratified handle attachment and handle attachment are:

1. the handle is necessarily stratified, though the stratification is not necessarily induced by the corners that occur in the standard formation of a handle as the Cartesian product of a pair of disks,
2. the manifold to which we attach the handle is necessarily stratified, and
3. the attaching map ensures that there is a coherent identification between the strata of the handle and the strata of the manifold (i.e. the stratification of the resulting attachment space is well-defined).

2.3 Handlebodies

Handlebodies are objects central to the arguments of Chapters 3 and 4. We use Gompf & Stipsicz [5] and Schleimer [10] as our main references for handlebodies. A handlebody is formed by attaching some number of (n, λ) -handles to an $(n, 0)$ -handle. The name is evoked by the type of handlebody that we examine in this section: the $(3, 1)$ -handlebody.

Definition 2.3.1. A connected n -manifold M that has a handle decomposition consisting of exactly one 0-handle and g λ -handles is called an (n, λ) -*handlebody* of *genus* g .

Let V be an $(n, 1)$ -handlebody of genus g . A simple closed curve in ∂V is called *essential* if it is not homotopic to a point. A simple closed curve J in ∂V that is essential in ∂V and that bounds a 2-disc in V is called a *meridian*. The properly embedded disc in V that has boundary J is called a *meridinal disc*.

The special case of the oriented genus 1 $(m, 1)$ -handlebody is called a *solid torus*. More generally, any space that is homeomorphic to $S^1 \times D^{n-1}$ is called a *solid n -torus*. In our most common case of $n = 3$, we just say that $S^1 \times D^2$ is a *solid torus*. A simple closed curve J in the boundary of a solid torus that intersects a meridian at a single point is called a *longitude*. A longitude is essential in a solid torus and there are infinitely many isotopy classes of longitudes, but there is exactly one isotopy class of meridians.

We apply handlebodies to 3-manifold classification through Heegaard splittings.

Definition 2.3.2. Let U and V be 3-dimensional handlebodies of genus g and let $f : \partial U \rightarrow \partial V$ be an orientation preserving diffeomorphism. The adjunction $M = U \cup_f V$ is a *Heegaard splitting* of M , the shared boundary $H = \partial U = \partial V$ is the *Heegaard surface* of the splitting, and the shared genus of U and V is the *genus* of the splitting as well. A Heegaard splitting is also denoted by the pair (M, H) .

We use the notion of equivalence between splittings from [10]: A pair of splittings (M, H) and (M, H') are *equivalent* if there is a homeomorphism $h : M \rightarrow M$ such that h is isotopic to id_M and $h|_H$ is an orientation preserving homeomorphism $H \rightarrow H'$.

Example 2.3.3. The 3-sphere S^3 has two standard Heegaard splittings. The first is the genus 0 splitting, which is realized by considering S^3 as the set of unit vectors in \mathbb{R}^4 . Take the Heegaard surface to be the intersection of S^3 with the xyz -hyperplane

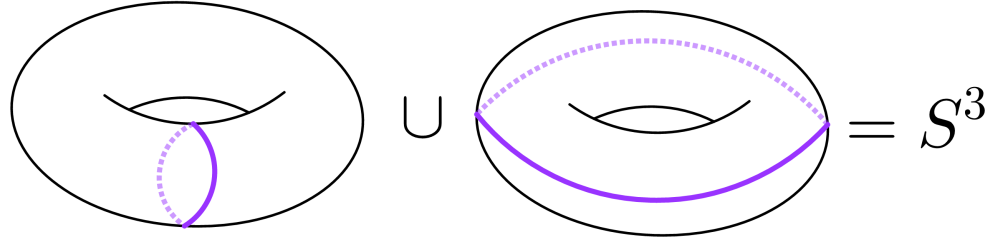


Figure 2.1: **Genus 1 Heegaard splitting of S^3 .** The purple curve is essential in each solid torus pictured. In the left torus it is a meridian, on the right a longitude.

in \mathbb{R}^4 . This is a copy of S^2 that separates S^3 into two connected components. This splitting is written (S^3, S^2)

The second is the genus 1 splitting, which is visualized using the realization of S^3 as the one-point compactification of \mathbb{R}^3 . Take solid tori U and V and identify ∂U with ∂V by the homeomorphism that swaps a meridian with a longitude. The adjunction is S^3 , and this splitting is written (S^3, T^2) . Figure 2.1 displays the tori of the splitting. This description of S^3 is also obtained by examining the boundary of a 4-dimensional 2-handle $\partial(D^2 \times D^2)$.

Definition 2.3.4. Let $(M, H) = U \cup_f V$ be a Heegaard splitting of M . The connected sum $(M, H) \# (S^3, T^2)$ is called an *elementary stabilization* of M , and itself is a splitting $(M, H \# T^2)$. A Heegaard splitting (M, H) is called a *stabilization* of another splitting (M, H') if it obtained from (M, H') via a finite number of elementary stabilizations.

Consider the meridians of the solid tori in the standard genus 1 splitting of S^3 . They each bound a disc in their respective handlebody, and they intersect in exactly one point. We would expect to be able to find such curves in any 3-manifold obtained as a stabilization.

Definition 2.3.5. Let $(M, H) = U \cup_f V$ be a Heegaard splitting of genus g and let α, β a pair of simple, closed, essential curves in H . Let α be a meridian of U and β be a meridian of V with associated meridinal discs D_α, D_β . If α and β intersect exactly once, then the pair (α, β) is a *meridinal pair* or *destabilizing pair* of the splitting.

To see why (α, β) would be called a destabilizing pair, remove a tubular neighbourhood of D_α from U and add it to V as a 2-handle along a tubular neighbourhood of α in H . In fact, the altered spaces U' and V' are handlebodies of genus $g - 1$

with $H' = \partial U' = \partial V'$, and (M, H) is a stabilization $(M, H') \# (S^3, T^2)$. We say that (M, H') is a *destabilization* of (M, H) over (α, β) . Note that when we consider β to be the belt sphere of a 1-handle and α to be the attaching sphere of a 2-handle, a destabilization is also a handle cancellation.

Chapter 3

Construction for smooth manifolds

We prove that every smooth, closed, orientable 3-manifold is the boundary of some 4-manifold. We do so by explicitly constructing such a 4-manifold from a given 3-manifold. This construction is mirrored in Chapter 4 where we prove the same for a given closed, orientable 3-manifold triangulation and provide an algorithm.

Let M be a smooth, closed, orientable 3-manifold and take $W = M \times [0, 1]$. Then W is a 4-manifold with boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

We construct a manifold with only one boundary component from W by attaching 4-dimensional 2-, 3-, and 4-handles to W over the part of its boundary away from M_0 . We start by attaching 2-handles to W over M_1 to produce a 4-manifold W' with boundary $M_0 \sqcup M'_1$, where $M'_1 = \partial W' \setminus M_0$ and is described via surgery on M_1 . We then attach 3-handles to W' over M'_1 to produce a 4-manifold W'' with boundary $M_0 \sqcup M''_1$. As before, $M''_1 = \partial W'' \setminus M_0$ and M''_1 is described via surgery on M'_1 . At this point in the construction, M''_1 is the disjoint union of a finite number of copies of S^3 . We attach 4-handles to W'' over M''_1 to produce a 4-manifold whose boundary is exactly M_0 .

Instructions for handle attachment come from defining a projection $f : M_1 \rightarrow \mathbb{R}^2$ that induces a stratification of $M_1 \subset \partial W$.

We call a closed 3-dimensional stratum of M_1 a *block*, and we impose conditions on f to ensure that every block can be classified up to *stratified homeomorphism*. We say that X, Y are *stratified homeomorphic* if X, Y are stratified spaces and there exists a homeomorphism $f : X \rightarrow Y$ such that f and f^{-1} are each stratified maps.

The blocks of M_1 are described below:

Face block: An attachment neighbourhood for a stratified 2–handle. Each face block is stratified-homeomorphic to $S^1 \times G_n$, the product of the circle with an n -gon for some n .

Edge block: A partial attachment neighbourhood for a stratified 3–handle. Each edge block is stratified-homeomorphic to one of $D^2 \times [0, 1]$, $A \times [0, 1]$, or $P \times [0, 1]$, where A is the annulus $S^1 \times [0, 1]$ and P is a pair-of-pants surface. Attachment of stratified (4,2)–handles over our face blocks “fill in” the annular boundary components of edge blocks, forming full attachment neighbourhoods for stratified 3–handles.

Vertex block: A partial attachment neighbourhood for a stratified 4–handle. Each vertex block is homeomorphic to a (3,1)-handlebody of genus at most 3. When stratified (4,2)– and (4,3)–handles are attached to W , the genus of these handlebodies are reduced until the remaining boundary of W consists of M_0 union a collection of stratified 3–spheres. The 3–spheres are then coned away.

The remainder of this chapter is spent ensuring that such a stratification can be achieved for any smooth, orientable, closed 3–manifold, detailing how the stratification is induced, proving that the attachment of stratified (4,2)– and (4,3)–handles has the previously stated effects, and discussing the resulting 4–manifold.

3.1 Projections from 3–manifolds to \mathbb{R}^2

Our stratification of M is induced by a decomposition of the plane, itself induced by the singular values of a smooth map $M \rightarrow \mathbb{R}^2$. To prove that a stratification suitable for our construction exists for any smooth orientable 3–manifold, we show first that an inducing decomposition of \mathbb{R}^2 exists. To prove that such a decomposition of \mathbb{R}^2 exists, we present the properties of $f : M \rightarrow \mathbb{R}^2$ required to induce the decomposition, and argue why a map possessing such properties exists for any smooth, orientable 3–manifold.

Let $f : M \rightarrow \mathbb{R}^2$ be a smooth map, let df be the differential of f , and let $S_r(f)$ be the set of points in M such that df has rank r . Then we require that the following be true of f :

1. $S_0(f)$ is empty.
2. $S_1(f)$ consists of smooth non-intersecting curves. We call these the *fold curves* of f .
3. The set of points where $f|_{S_1(f)}$ has zero differential is empty.
4. If $p \in S_1(f)$ then there exist coordinates (u, z_1, z_2) centred at p and (x, y) centred at $f(p)$ such that f takes the form of either
 - (a) $(x, y) = (u, \pm(z_1^2 + z_2^2))$, or
 - (b) $(x, y) = (u, \pm(z_1^2 - z_2^2))$
 in a neighbourhood of p . If f takes the form of 4a then we further classify p as a *definite fold*, and if f takes the form of 4b then p is an *indefinite fold*.
5. Let $\gamma_i, \gamma_j, \gamma_k \in S_1(f)$ be fold curves. Then each of $f(\gamma_i)$, $f(\gamma_j)$, $f(\gamma_k)$ are submanifolds of \mathbb{R}^2 such that
 - (a) $f(\gamma_i)$ and $f(\gamma_j)$ intersect transversely,
 - (b) $f(\gamma_i) \cap f(\gamma_j) \cap f(\gamma_k)$ is empty (i.e. there are no triple-intersections), and
 - (c) self-intersections of $f(\gamma_i)$ are transverse
6. The set of singular values of f in the plane is connected
7. The image of M through f is bounded in the plane.

We call these the *stratification conditions* on f , and we call a map satisfying the stratification conditions a *stratifying map*. The existence of smooth maps satisfying conditions 1–4 is discussed in [7], and 5–7 are straightforward. A smooth map satisfying conditions 1–4 can be smoothly perturbed inside of a tubular neighbourhood of $S_1(f)$ to satisfy conditions 5 & 6. Finally, condition 7 is always satisfied because the image of a compact set through a continuous map is compact.

Let $f : M \rightarrow \mathbb{R}^2$ be a stratifying map and let $X_f = f(S(f))$, the set of singular values of f . We first consider $f(M) \setminus X_f$. Because X_f is a connected collection of arcs in the plane that intersect only transversely, $f(M) \setminus X_f$ is a collection of connected regions in the plane, each of which consists entirely of regular values. The sixth stratification condition guarantees that each of these regions is simply connected — the boundary components of these regions are formed by the singular values of f

in the plane, and different boundary components are necessarily disjoint. The sixth stratification condition does not restrict the class of maps we consider, as a map satisfying the first five stratification conditions can be smoothly homotoped to satisfy the sixth.

3.2 Stratifying \mathbb{R}^2

Let $f : M \rightarrow \mathbb{R}^2$ be a stratifying map with singular values X_f . We fit closed neighbourhoods (*sleeves*) around the singular values of f and classify these sleeves by the maximum codimension (with respect to \mathbb{R}^2) of singular values they contain. Because X_f consists of codimension 1 and codimension 2 singular values (arcs and arc-crossings respectively) in the plane and the endpoints of each arc of codimension 1 singular values is a pair of (not necessarily distinct) codimension 2 singular values, we stratify \mathbb{R}^2 into face regions that contain no singular values, edge regions that contain only codimension 1 singular values, and vertex regions, each of which contain exactly 1 codimension 2 singular value.

The naming convention is due to the natural structure X_f has of an embedded planar graph whose vertices are codimension 2 singular values and whose edges are arcs of codimension 1 singular values. We refer to the codimension 2 singularities as the vertices of X_f , the arcs of codimension 1 singularities as the edges of X_f , and the regions of regular values comprising $f(M) \setminus X_f$ as the faces of X_f . Figures 3.1-3.4 are used to illustrate the stratification resulting from sleeve-fitting.

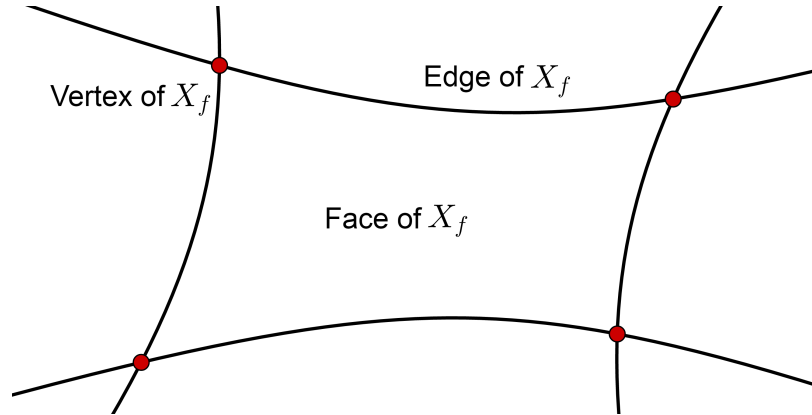


Figure 3.1: **Singular values of f .** Vertices are arc crossings, edges are arcs, and connected components of $f(M) \setminus X_f$ are faces.

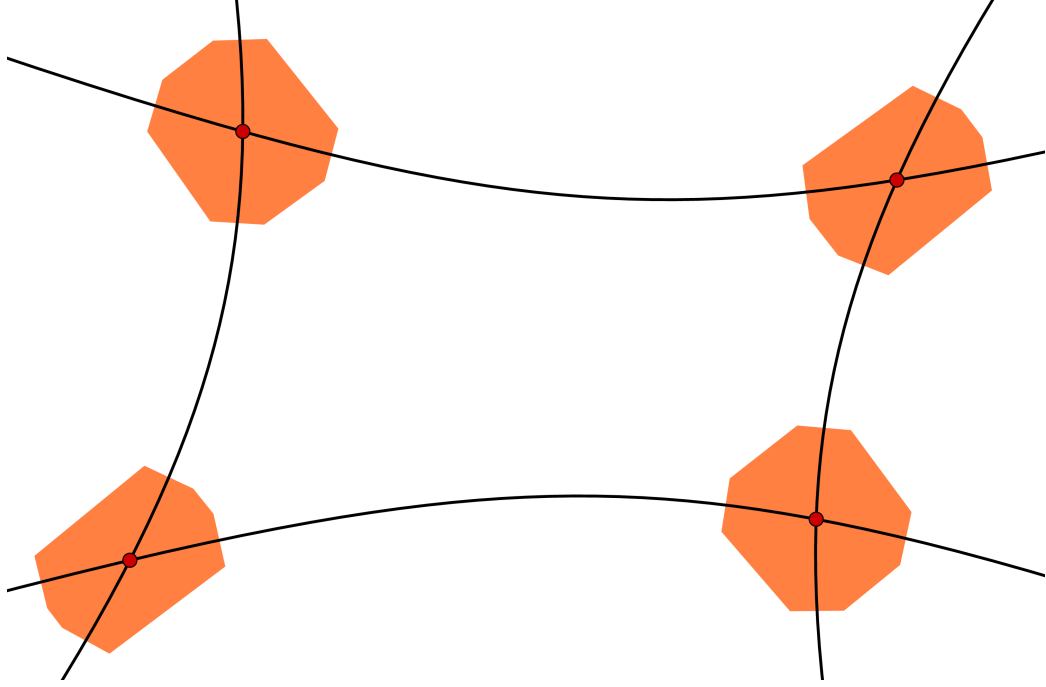


Figure 3.2: **Forming vertex regions.** Octagonal sleeves are fit around the vertices of X_f to form vertex regions. Note that the 1-strata vertex sleeves alternate between containing exactly one singular value, its intersection with an edge of X_f , or consisting entirely of regular values. New vertex regions are shaded orange.

We begin by fitting sleeves around codimension 2 singular values as in Figure 3.2. These sleeves are each boundary-stratified 2-discs with exactly eight 1-dimensional boundary strata (i.e. octagons). Following the notation patterns already present in this document, we call the k -dimensional strata of a stratified manifold the k -strata of that manifold. If x is a codimension 2 singular value then x is the result of an arc crossing, and a small neighbourhood around an arc crossing is divided into four regions of regular values. The octagon around x is fit so that its 1-strata alternate between being fully contained in a region of regular values and orthogonally intersecting one of the arcs of singular values that creates x , as depicted by the example fitting in Figure 3.2.

The closed octagons form the vertex regions of the stratification of \mathbb{R}^2 . The octagons are chosen to be small enough that no two vertex regions overlap and such that the 1-strata that intersect arcs of codimension 1 singular values are all the same length.

Let γ be an edge of X_f with endpoints a pair of vertices. γ orthogonally intersects one 1-strata from each of the octagonal vertex regions fit around its endpoints, and

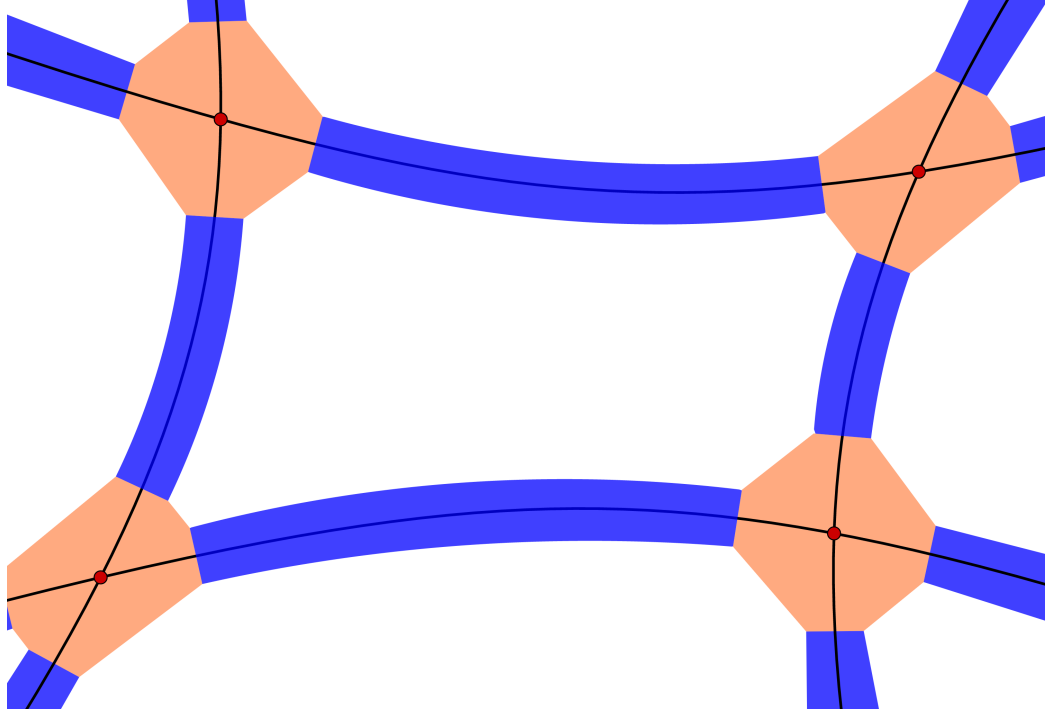


Figure 3.3: **Forming edge regions.** Vertex region corners are connected to fit sleeves around arcs of codimension 1 singular values to form edge regions. New edge regions are shaded blue.

we use these 1-strata to form the edge region associated to γ by connecting the 0-strata boundaries of the 1-strata to one another using a pair of arcs parallel to γ , as illustrated in the example edge region fitting of Figure 3.3.

We form the boundary of the edge region associated with γ as the union of:

1. the arcs parallel to γ connecting vertex region 0-strata, and
2. the vertex region 1-strata that intersect γ

This union is a simple closed curve J . The component of $\mathbb{R}^2 \setminus J$ that contains no vertices of X_f is a boundary stratified 2-disc that shares two of its four boundary 1-strata with the vertex regions on either end of γ . This boundary stratified 2-disc is then the edge region of the stratification of \mathbb{R}^2 associated with γ .

Removing from $f(M)$ all vertex and edge regions, we are left with a collection of connected regions in the plane, each of which is a deformation retract of a face of X_f . We take the closures of these to be the face regions of the stratification of \mathbb{R}^2 . The boundary of each face region is an alternating collection of boundary 1-strata

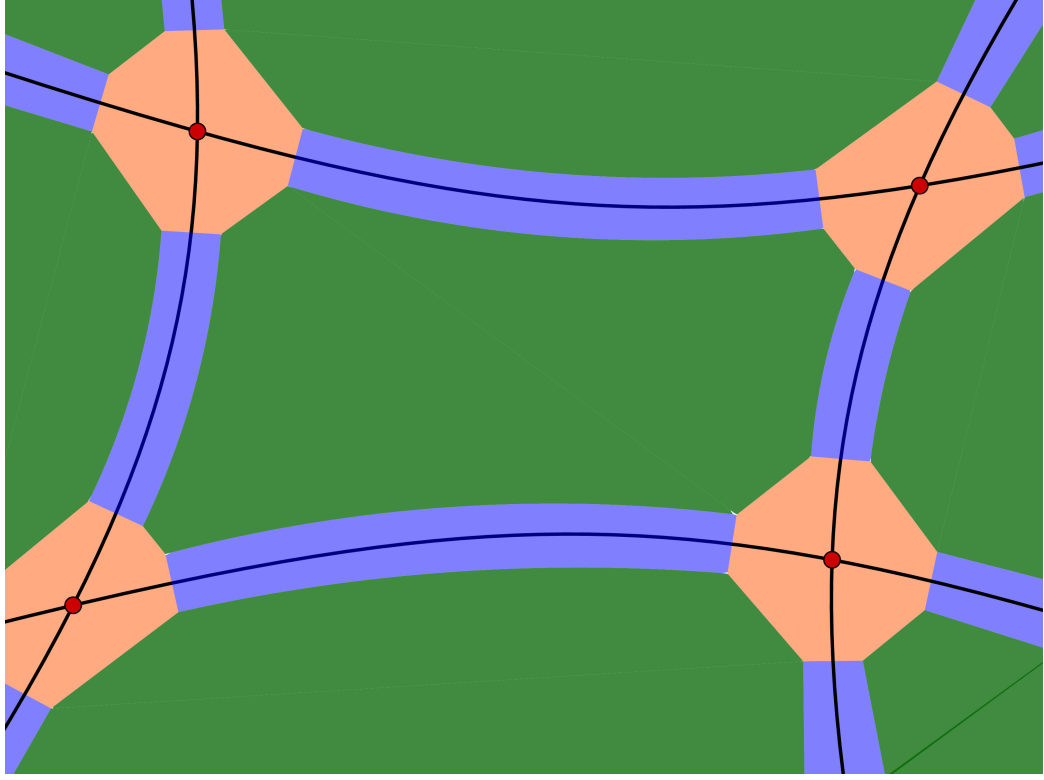


Figure 3.4: **Forming face regions.** All remaining regions contain no singular values, and we take these to be the face regions. New face regions are shaded green.

from edge regions and boundary 1-strata from vertex regions. See Figure 3.4 for an example fitting.

With all of the regions defined, we can describe precisely how f stratifies \mathbb{R}^2 . The stratification of \mathbb{R}^2 is a stratification into subsets $R_{(i,j)}$ where i, j are integers. Subset indexing is defined so that a subset $R_{(i,j)}$ is an i -dimensional submanifold of \mathbb{R}^2 , thus $R_{(i,j)} \not\subseteq R_{(k,l)}$ if $i \not\leq k$. The first collection of subsets used to filter \mathbb{R}^2 are the 0-strata of the octagonal vertex sleeves. We assign to these subsets the indices $(0, i)$ for $i = 1 \dots N_0$, where N_0 is the number of 0-strata. Our 0-strata are disjoint, so for any i, j with $i \neq j$, $R_{(0,i)}$ is not contained in $R_{(0,j)}$, hence $(0, i) \not\leq (0, j)$.

The boundary 1-strata connect the $(0, i)$ -level strata. These 1-strata are indexed by $(1, j)$ for $j = 1 \dots N_1$, where N_1 is the number of arcs. The boundary points of 1-strata are 0-strata and are subsets of the filtration indexed by the $(0, i)$ indices, so $(0, i) \leq (1, j)$ if and only if $R_{(0,i)}$ is one of the boundary points of $R_{(1,j)}$. Our 1-strata intersect only at their boundary points, so $(1, j) \not\leq (1, k)$ for any j, k .

The regions themselves are indexed by $(2, k)$ for $k = 1 \dots N_2$, where N_2 is the

number of regions. These indices work similarly to the 1-strata indices. The boundary of a region consists of 0- and 1-strata, so $(n, i) \leq (2, k)$ if and only if $R_{(n,i)}$ is contained in the boundary of $R_{(2,k)}$.

With \mathbb{R}^2 stratified, we move onto a stratification of M . This stratification is induced by the preimages of the strata of \mathbb{R}^2 .

3.3 Stratifying M

Stratifying \mathbb{R}^2 via the singular values of f also induces a stratification of M by considering the fibers of f above the stratifying regions. The interiors of face regions have preimage through f a disjoint collection of face blocks, the interiors of edge regions have preimage of edge blocks, and of vertex regions, vertex blocks.

To understand the structure of face, edge, and vertex blocks we investigate the preimages of regular and singular values of f in the plane.

Definition 3.3.1. Because M is closed, f is proper. Thus, for any point q in $f(M)$, a fiber of f above q (i.e. a connected component of $f^{-1}(q)$) is either a closed 1-manifold (i.e. S^1) or contains a critical point of f .

We define a *singular fiber* to be a fiber that contains a critical point of f , and a *regular fiber* to be a fiber consisting entirely of regular points.

The subsets used to stratify M are the fibers of f that lie above the individual strata of our \mathbb{R}^2 stratification. Because the $(0, \cdot)$ -strata are regular values, their fibers are regular, hence a finite collection of disjointly embedded circles in M . We take these circles as the first collection of subsets that filter M , and assign to them the indices $(1, i)$ for $i = 1 \dots N_1$, where N_1 is the number of circles. These circles are disjoint, so $(1, i) \not\leq (1, j)$ for any i, j .

The $(1, \cdot)$ -strata of the \mathbb{R}^2 stratification connect the $(0, \cdot)$ -strata of the \mathbb{R}^2 stratification and either consist entirely of regular values or contain exactly one singular value. As pictured in Figure 3.4, a $(1, \cdot)$ -strata contains exactly one singular value precisely when it is the shared boundary of a vertex region and an edge region. When a $(1, \cdot)$ -strata contains exactly one singular value, exactly one fiber above that value is a singular fiber, with the rest regular fibers. A $(1, \cdot)$ -strata is diffeomorphic to the unit interval and f is a smooth submersion between smooth manifolds, so a fiber above a $(1, \cdot)$ -strata is a surface whose boundary circles are the fibers above the

strata's endpoints. When the fiber is regular, the surface is diffeomorphic to an annulus $S^1 \times [0, 1]$. When the fiber is singular, the surface classification depends on the type of singularity. Theorem 3.3.2 and Figure 3.5 show that the fiber containing the singularity either has the structure of a figure-of-eight (when the singularity is part of an indefinite fold) or is a single point (when the singularity is part of a definite fold), hence the singular fiber above the arc is diffeomorphic to either a 2-disk or a pair-of-pants.

Theorem 3.3.2 refers to a *stable map*, and we'll denote the set of smooth stable maps $X \rightarrow Y$ by $Stab(X, Y)$. When X is a smooth, closed, orientable 3-manifold and Y is the plane, $Stab(X, Y)$ consists of all maps $X \rightarrow Y$ satisfying the first five stratification conditions. The last stratification condition is trivially satisfied because X is closed, hence the set of stratifying maps $X \rightarrow Y$ is the subset of $Stab(X, Y)$ consisting of maps f such that $f(S(f))$ is connected.

Theorem 3.3.2 (Adapted Theorem 3.15 in Saeki [9]). Let $f : M \rightarrow N$ be a proper C^∞ stable map of an orientable 3-manifold M into a surface N . Then, every singular fiber of f is equivalent to the disjoint union of:

1. one of the fibers in Figure 3.5, and
2. the disjoint union of a finite number of copies of S^1 .

Furthermore, no two fibers in the list are equivalent to each other even after taking the union with regular circle components.

The surface fibers above the $(1, \cdot)$ -strata are the second collection of subsets that filter M , and they are assigned the indices $(2, j)$ for $j = 1 \dots N_2$, where N_2 is the number of surfaces. The boundary circles of the surfaces are each subsets of the filtration, indexed by the $(1, i)$ indices, so $(1, i) \leq (2, j)$ if and only if $M_{(1, i)}$ is one of the boundary components of $M_{(2, j)}$. These surfaces intersect one another only when they share a boundary circle, so $(2, j) \not\leq (2, k)$ for any j, k .

There are three types of region in the decomposition: face, edge, and vertex. Regardless of the type of region, they are indexed in our filtration similarly to the edges. A fiber above a region is a 3-manifold with corners formed by the $(1, i)$ - and $(2, j)$ -level strata, and fibers are disjoint away from their boundaries. We therefore index fibers above regions with the indices $(3, k)$ for $k = 1 \dots N_3$ where N_3 is the total number of fibers above regions, put $(n, i) \leq (3, k)$ if and only if $M_{(n, i)}$ is contained in the boundary of $M_{(3, k)}$.

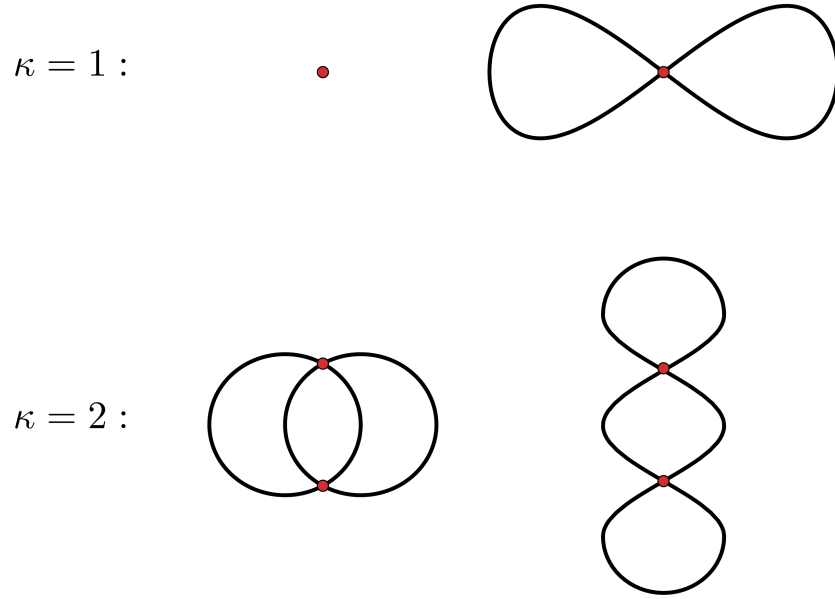


Figure 3.5: **Connected singular fibers.** List of connected singular fibers of proper C^∞ stable maps of orientable 3-manifolds into surfaces. κ is the codimension of the singularity in the surface. The singular fiber above a codimension 2 singular value may be disconnected, in which case the fiber is the disjoint union of a pair of singular fibers from $\kappa = 1$.

Recall that we call a closed 3-dimensional stratum of M a *block*. We now restate the desired block structure laid out at the beginning of this chapter and prove that the stratification conditions impose this structure on the blocks of M .

Theorem 3.3.3. Let M be a smooth, closed, orientable 3-manifold, let $f : M \rightarrow \mathbb{R}^2$ be a stratifying map, suppose \mathbb{R}^2 has been decomposed as in Section 3.2 and M has been stratified as in this section. Then each closed strata $M_{(3,k)}$ is classified as either a *face*, *edge*, or *vertex block* depending on whether it is a fiber above a face, edge, or vertex region respectively, and a block has one of the following structures:

Face block: Let B be a face block that fibers over the face region F . Then B is stratified-homeomorphic to $S^1 \times F$.

Edge block: Let B be an edge block that fibers over the edge region E . Let A be the annulus $S^1 \times [0, 1]$ and P the pair-of pants surface (i.e. D^2 minus a pair of disjoint open balls). If B is a regular fiber over E then B is stratified-homeomorphic to $S^1 \times E$, hence also stratified-homeomorphic to $A \times [0, 1]$. Otherwise, B is a singular fiber over E and contains part of definite or indefinite

fold. In this case we call B a *definite* or *indefinite edge block*. A definite edge block is stratified-homeomorphic to $D^2 \times [0, 1]$ and an indefinite edge block is stratified-homeomorphic to $P \times [0, 1]$.

Vertex block: Let B be a vertex block that fibers over the vertex region V . If B is a regular fiber then it is homeomorphic to $S^1 \times V$, therefore homeomorphic to a (3,1)-handlebody of genus 1. Otherwise, we see from Figure 3.5 that the singular fiber above the codimension 2 singularity contained in V is either connected or disconnected. If the singular fiber is disconnected then there are a pair of disjoint vertex blocks that each contain one of the singular fibers, hence part of a definite or indefinite fold. We therefore classify these blocks as *definite* or *indefinite vertex blocks*. If the singular fiber is connected, then the block containing it is an *interactive vertex block*. A definite (resp. indefinite) vertex block extends and connects definite (resp. indefinite) edge blocks, and is homeomorphic to a (3,1)-handlebody of genus 0 (resp. 2). An interactive vertex block is homeomorphic to a (3,1)-handlebody of genus 3.

Remark 3.3.4. The structures of the blocks described in Theorem 3.3.3 are roughly disk bundles over a representative fiber for the given region or, equivalently, regular neighbourhoods of that fiber. For a block that is a regular fiber, the representative is a circle. For a definite or indefinite block, the representative is the singular fiber containing a definite or indefinite fold, and for an interactive block the representative fiber is the singular fiber above the codimension 2 singular value.

Proof of Theorem 3.3.3. We split the proof into three parts. The first part proves that if a block B is a regular fiber over the region R then B is stratified-homeomorphic to $S^1 \times R$. In the second part, we prove that definite and indefinite blocks are stratified-homeomorphic to $D^2 \times [0, 1]$ or $P \times [0, 1]$ respectively. In the final part we discuss interactive vertex blocks, and show that they are homeomorphic to (3,1)-handlebodies of genus 3. Figures illustrate the block structures.

Part 1: Let B be a block over the region R , and suppose B consists entirely of regular fibers over R . Then $(B, R, f|_B, S^1)$ has the structure of a circle bundle over R . A fiber bundle over a contractible space is trivial, so B is homeomorphic to $S^1 \times R$. Furthermore, this homeomorphism is stratified by ensuring the strata of B are mapped to the strata of $S^1 \times R$, where the stratification of $S^1 \times R$ is defined by the manifold with corners structure induced by the product topology.

Part 2: Let B be a definite or indefinite block over the region R . R is a subset of the plane homeomorphic to D^2 with an arc $\gamma_s \subset X_f$ of singular values running from one of its edges to another. Let γ_t be a second simple arc that crosses γ_s transversely, and consider the cross-sectional surface obtained by $f^{-1}(\gamma_t)$. Figure 3.6 illustrates the possible surfaces containing the singular fiber over $x = \gamma_s \cap \gamma_t$.

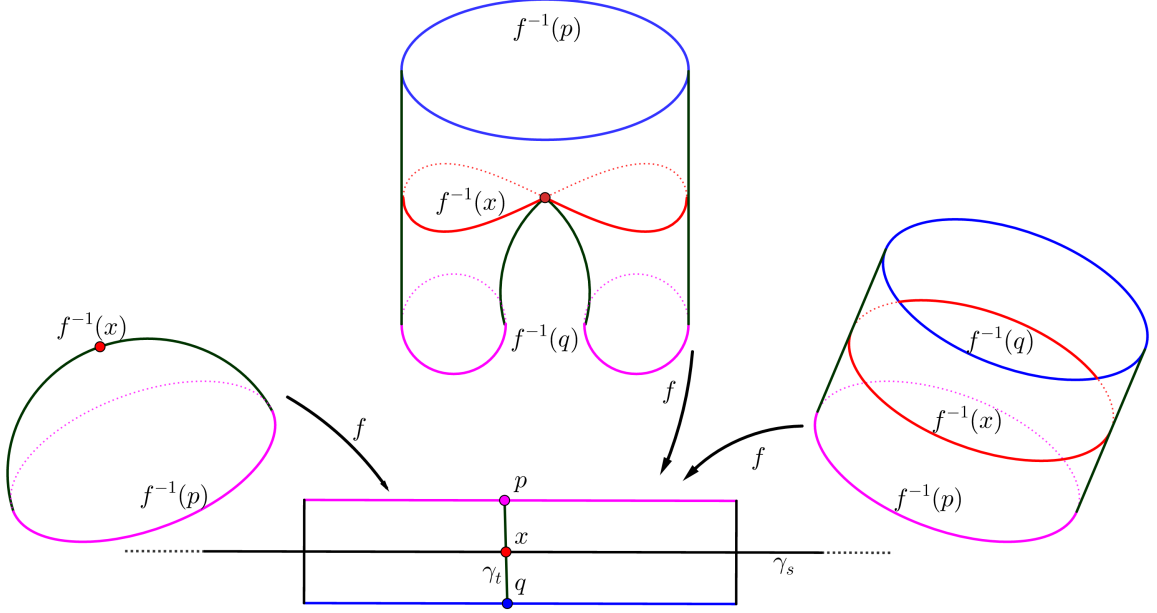


Figure 3.6: **Surfaces over codimension 1 singularities.** γ_s is an arc of singular values and γ_t is an arc with endpoints $\partial\gamma_t = \{p, q\}$ that intersects γ_s transversely at $x = \gamma_s \cap \gamma_t$. The three surfaces shown are the three possible cross-sectional surfaces that can project through f over γ_t .

This cross section is general, so we fit a tubular neighbourhood $\nu(\gamma_s)$ about γ_s in R to obtain a bundle structure for $f^{-1}(\nu(\gamma_s))$ whose fiber is one of the cross-sectional surfaces (a disk or a pair-of-pants) and whose base is the arc γ_s , i.e. an interval. The interval is contractible, so $f^{-1}(\nu(\gamma_s))$ is homeomorphic to $\Sigma \times [0, 1]$ for Σ a disk or a pair-of-pants surface. Away from $\nu(\gamma_s)$, R consists entirely of regular values so we obtain solid tori (cf. Part 1) that extend the $\Sigma \times [0, 1]$ structure as seen in Figure 3.7.

As with Part 1, the homeomorphism described is stratified by ensuring the strata of B are mapped to the strata of $\Sigma \times [0, 1]$, where the stratification of $\Sigma \times [0, 1]$ is defined by the manifold with corners structure induced by the product topology.

Part 3: Let B be an interactive block over the region R . Interactive blocks occur over octagonal vertex regions where the singular fiber above the region's codimension 2 singularity is connected, so we investigate these fibers. The codimension 2 singular

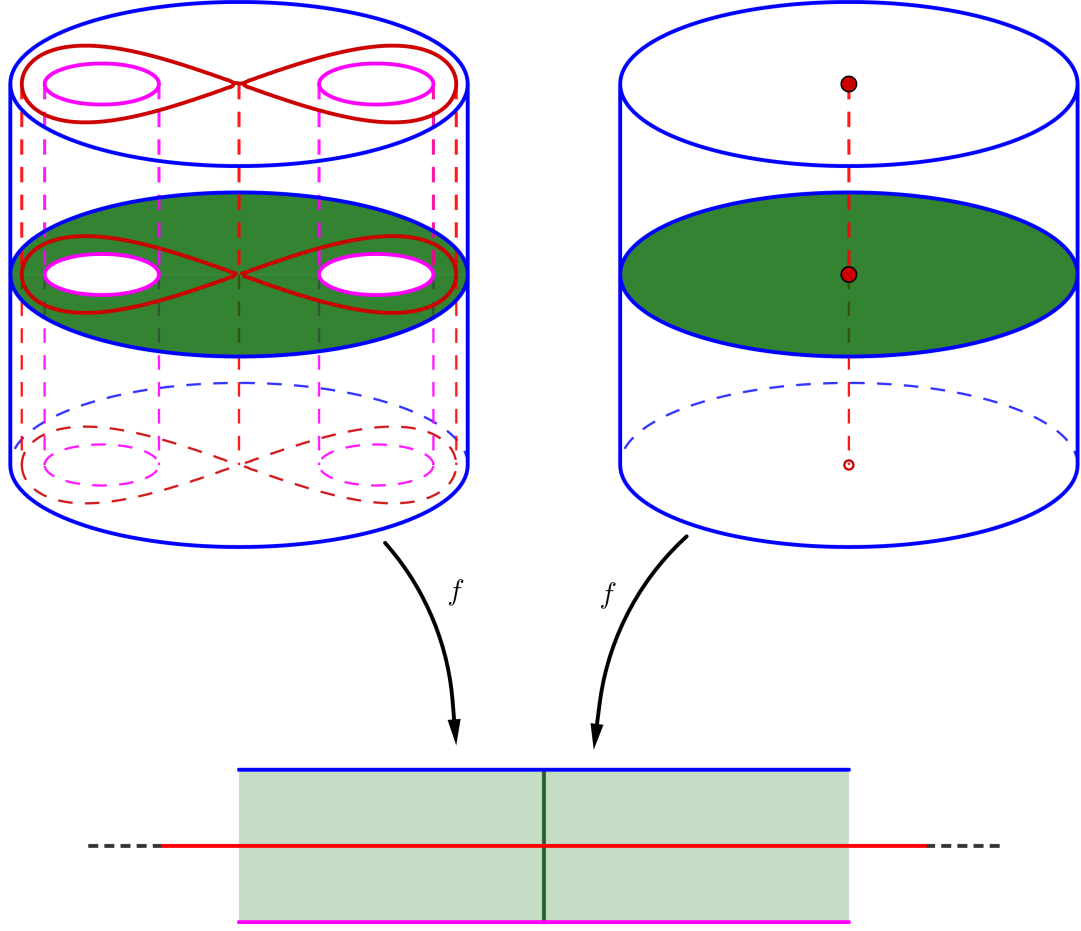


Figure 3.7: **Definite and indefinite blocks.** The blocks containing sections of definite and indefinite folds that project over codimension 1 singular values. These are found as singular fibers over edge and vertex regions.

value lies at the intersection of a pair of arcs of codimension 1 singular values. Call the arcs γ_1 and γ_2 , let $x = \gamma_1 \cap \gamma_2$, and denote the interactive singular fiber over x by $B_x = B \cap f^{-1}(x) = \{b \in B \mid f(b) = x\}$. Our method of investigation begins by examining the possible resolutions of B_x and combining those resolutions to form a genus 3 surface.

Figure 3.8 demonstrates resolutions of the singular points of B_x when B_x has the first interactive singular fiber form presented in Figure 3.5. We first note that all of the displayed fibers have inherited an orientation from M . This forces fiber resolution to be unambiguous, and allows us to identify fibers when forming the surface shown in Figure 3.9.

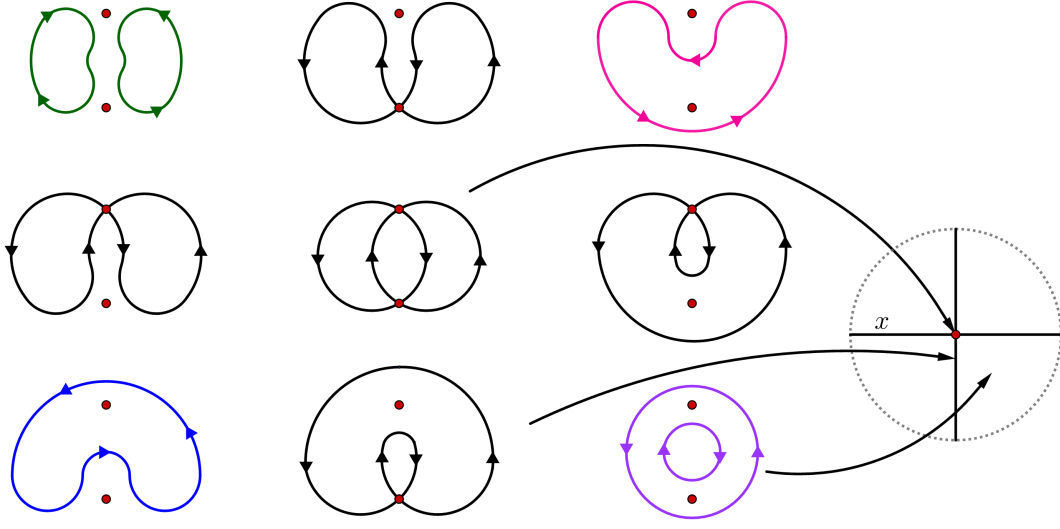


Figure 3.8: **Resolutions of the singular points in the first interactive fiber.** The singular fiber inside of B_x and its possible resolutions over nearby codimension 1 singular values and regular values. The fibers inherit orientation from M , and this illustration is presented without loss of generality. This figure is modeled after Figure 18 from [3].

We form the surface shown in Figure 3.9 by gluing together surfaces that project over simple arcs transversing the codimension 1 singular values. Gluing is performed over the boundary circles of these surfaces, and is prescribed by the resolutions in Figure 3.8. The transverse preimage containing the left column of fibers in Figure 3.8 is a pair of pants with two green ‘cuffs’ (top left) and a blue ‘waist’ (bottom left). The preimage containing the top row is a pair of pants with two green cuffs (top left) and a pink waist (top right). The first gluing that helps realize the surface in Figure 3.9 is of the ‘left’ transverse preimage surface with the ‘top’ transverse preimage surface over their shared green cuffs. The surfaces recovered as transversal preimages are then:

Left: a pair of pants with blue waist and green cuffs

Top: a pair of pants with pink waist and green cuffs

Right: a pair of pants with pink waist and purple cuffs

Bottom: a pair of pants with blue waist and purple cuffs

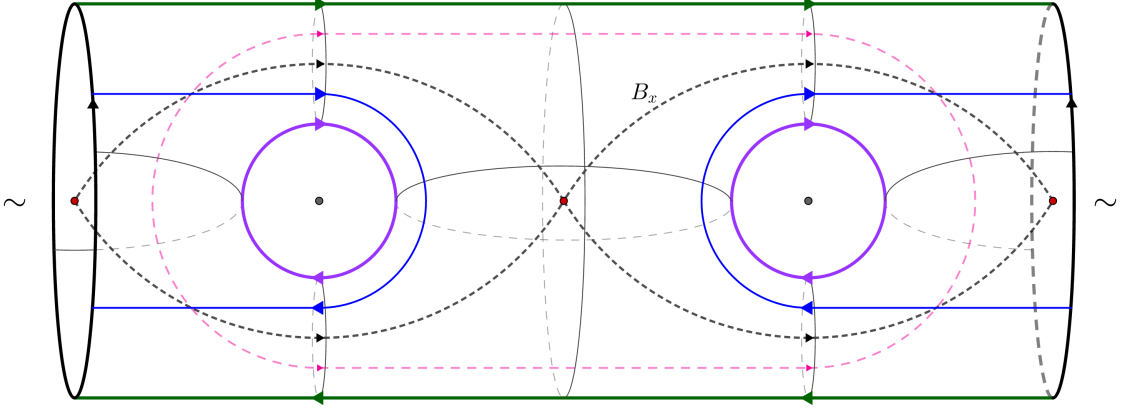


Figure 3.9: **Surface Σ near the first interactive fiber that projects over $\partial\bar{\nu}(x)$.** The surface and B_x are presented as embedded in S^3 , where $H(B_x)$ is the genus-3 (3,1)–handlebody on the ‘inside’ of Σ in S^3 .

The surface Σ in Figure 3.9 is the boundary of $H(B_x)$, a regular neighbourhood of B_x in M , i.e. a genus-3 (3,1)–handlebody inside of M . $H(B_x)$ projects through f over $\bar{\nu}(x)$, a closed tubular neighbourhood of x , and Σ projects over $\partial(\bar{\nu}(x))$.

Figure 3.9 presents Σ and B_x as objects embedded in S^3 , where Σ bounds genus-3 (3,1)–handlebodies on both sides. We take the ‘inside’ component of $S^3 \setminus \Sigma$ (i.e. the component containing B_x) to be $H(B_x)$.

Outside of $\bar{\nu}(x)$ we use the investigations from Parts 1 and 2 of this proof. The rest of R , $f^{-1}(R \setminus \bar{\nu}(x))$, has the structure of a Σ -bundle over the interval, and the bundle extends $H(B_x)$ to the boundary of R , preserving the structure as a genus-3 (3,1)–handlebody. We conclude that B is homeomorphic to a genus-3 (3,1)–handlebody.

An identical argument is made when B_x has the second interactive singular fiber form, using Figures 3.10 and 3.11 in place of Figures 3.8 and 3.9 respectively. In this case the surfaces recovered as transversal preimages are:

Left: a pair of pants with blue waist and green cuffs

Top: the disjoint union of a pair of pants with green waist and pink cuffs with an annulus with one green boundary circle and one pink boundary component

Right: the disjoint union of a pair of pants with purple waist and pink cuffs with an annulus with one purple boundary circle and one pink boundary component

Bottom: a pair of pants with blue waist and purple cuffs

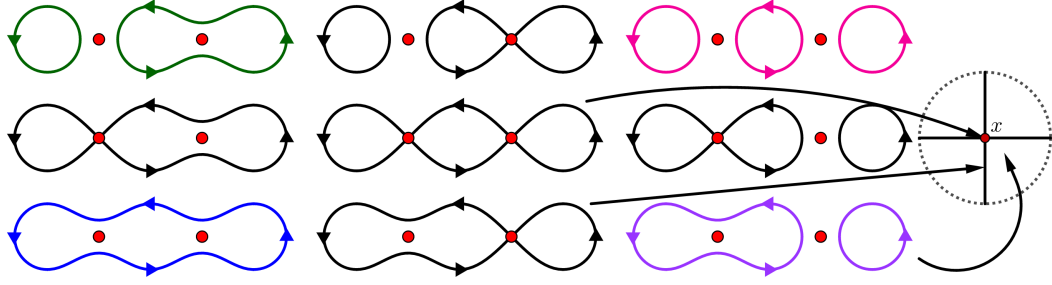


Figure 3.10: **Resolutions of the singular points in the second interactive fiber.** The singular fiber inside of B_x and its possible resolutions over nearby codimension 1 singular values and regular values. The fibers inherit orientation from M , and this illustration is presented without loss of generality. This figure is modeled after Figure 16 from [3].

□

A smooth map f satisfying the stratification conditions of Section 3.1 induces a decomposition on \mathbb{R}^2 and a stratification of M . It is important to note here that the restrictions on f can induce a wide variety of possible stratifications of M , highlighting the variability of the resulting 4-manifold. We end this section with a lemma that guarantees the stratified 2-handle attachments of the next section can be made over our face blocks in any order, hence we can assume all attachments occur simultaneously.

Lemma 3.3.5. Let M be a 3-manifold with stratification induced as in Theorem 3.3.3. Then blocks of the same type (i.e. face, edge, vertex) are disjoint.

Proof. The fibers above a given region are disjoint, so the blocks above that region are disjoint. Regions of the same type are disjoint, hence blocks that are fibers above differing regions are also disjoint. □

3.4 Attach Handles

Stratifying M allows the definition of attachment neighbourhoods for stratified 2-, 3-, and 4-handles in $W = M \times [0, 1]$. A 4-dimensional 2-handle is attached over a closed solid torus embedded in the boundary of a 4-manifold, so attachment neighbourhoods for our stratified 2-handles are straightforward: they are the face blocks of M . We

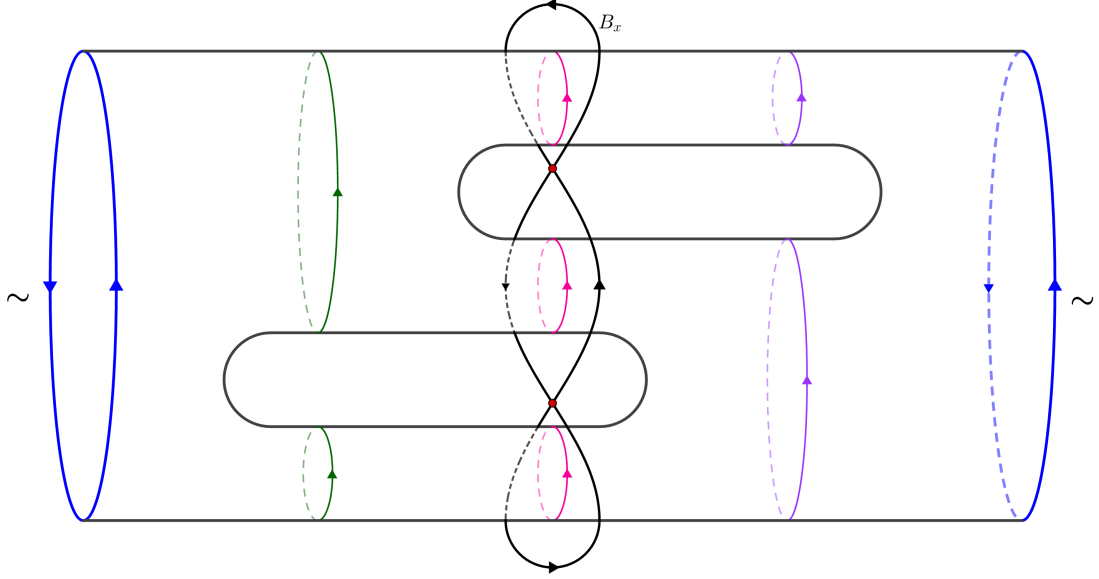


Figure 3.11: **Surface Σ near the second interactive fiber that projects over $\partial\bar{v}(x)$.** The surface and B_x are presented as embedded in S^3 , where $H(B_x)$ is the genus-3 (3,1)-handlebody on the ‘outside’ of Σ in S^3 .

alter the boundary of W by attaching handles, so the attachment neighbourhoods for 3-handles (4-handles, resp.) must be found after 2-handles (3-handles, resp.) are attached. For a 3-handle, an attachment neighbourhood consists of the union of an edge block with some strata introduced by 2-handle attachment. For a 4-handle, an attachment neighbourhood consists of the union of an edge block with some strata introduced by 2- and 3-handle attachment. We investigate the consequences of handle attachment by comparing the boundary of the initial manifold with the boundary of the manifold resulting from handle attachment, and use a precisely defined handle structure to focus the investigation. Our first step is to precisely define the structure of the stratified 2-handles that we are attaching.

3.4.1 2-handles

Let B be a face block of M_1 . By Theorem 3.3.3, B is a closed solid torus that is stratified-homeomorphic to $S^1 \times G_n$, where G_n is an n -gon for some n . Consider an unknotted embedding of B in S^3 . The complement of the interior of B is another closed solid torus B' . The tori B and B' are depicted as cylinders with top and bottom identified in Figure 3.12.

We stratify B' as follows. First include the shared stratified boundary of B . Next, introduce meridinal disks of B' that are bounded by the $(1, i)$ -indexed strata in B , i.e. the boundary curves in B corresponding to $S^1 \times c_m$ where c_m is a corner of G_n , $m = 1 \dots n$. Finally, add the homeomorphic 3-disks of B' whose boundaries consist of one longitudinal annulus of B along with the two meridinal disks in B' that are bounded by the annulus's circular boundary components. The filtration of B' is created identically to that of the original face blocks, using inclusion as a partial ordering and indexing a stratum by its dimension as a submanifold of B' . Figure 3.12 illustrates a face block B and its complement inside of S^3 .

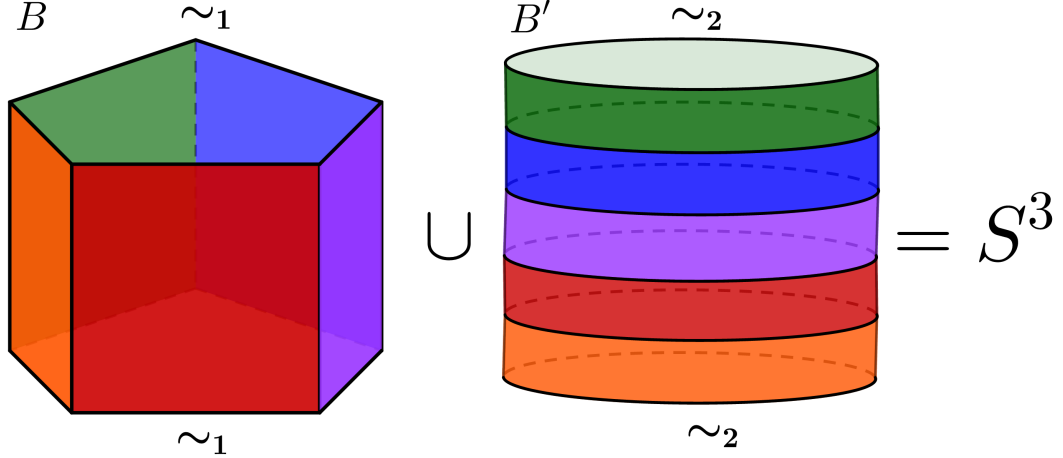


Figure 3.12: **A face block and its complement form S^3 .** A face block B is a stratified closed solid torus that is stratified-homeomorphic to $S^1 \times G_n$ for some n -gon G_n . The complement of its unknotted interior in S^3 is another stratified closed solid torus B' . B and B' are depicted as cylinders with top and bottom identified.

Taking S_B^3 to be the stratified S^3 formed as the union of B and B' , we craft a stratified 2-handle structure as $C(S_B^3)$, the cone of S_B^3 . We call $C(S_B^3)$ the *stratified 2-handle induced by B* . Attaching $C(S_B^3)$ to W over $B \subset M_1$ alters the boundary of W by replacing $B \subset M_1$ with B' . The full extent of this surgery can be detected by examining the edge and vertex blocks of the stratification that are incident to B .

Theorem 3.4.1. Let M be a smooth, closed, stratified, orientable 3-manifold with stratification induced by a stratifying map f , let \mathfrak{B} be the set of face blocks of M_1 , and let $W = M \times [0, 1]$. Consider the 4-manifold W' constructed from W as

$$W' = W \cup \{C(S_B^3)\}_{B \in \mathfrak{B}} / \sim,$$

where \sim is defined by $b \sim \iota(b)$, ι the identity map $C(S_B^3) \supset B \xrightarrow{\iota} B \subset M_1$. Then $M'_1 = \partial W' \setminus M_0$ is a stratified 3-manifold with a decomposition into *primed edge blocks* and *primed vertex blocks* such that the decomposition is well-defined and the primed blocks have the following structure:

Primed edge block: A primed edge block E' is identical to an edge block E from Theorem 3.3.3 with $(3,2)$ -handles (i.e. cylinders: $D^2 \times [0, 1]$) attached over all annular boundary strata of E (i.e. each closed strata A of E such that $A = E \cap B$ for some face block $B \in \mathfrak{B}$). Thus E' is a stratified 3-manifold homeomorphic to $S^2 \times [0, 1]$

Primed vertex block: A primed vertex block V' is identical to a vertex block V from Theorem 3.3.3 with $(3,2)$ -handles attached over each annular boundary stratum A of V such that $A = V \cap B$ for some face block $B \in \mathfrak{B}$. Furthermore, we show that V' is a stratified $(3,2)$ -handlebody.

Proof. We split the proof of this theorem into three parts. In the first two parts, we prove that we can find the prescribed primed edge and vertex block structures in M'_1 . In the third part, we show that these structures exhaust M'_1 .

Part 1: Edge blocks occur in three possible forms: regular edge blocks as $A \times [0, 1]$, definite edge blocks as $D^2 \times [0, 1]$, and indefinite edge blocks as $P \times [0, 1]$. Figure 3.13 displays the possible edge block forms and indicates the boundary strata of an edge block that are shared by face blocks.

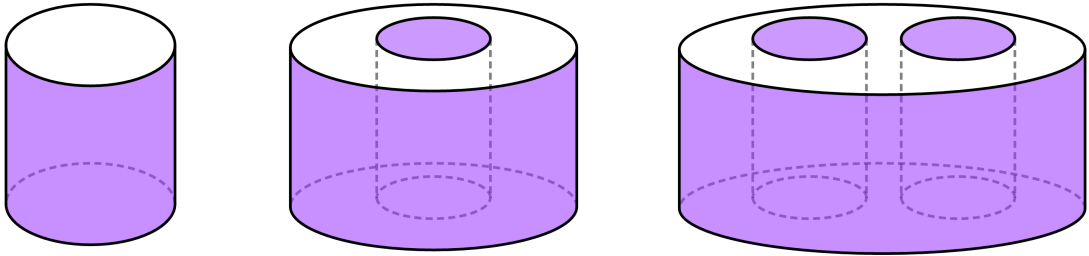


Figure 3.13: **Edge blocks.** The possible edge blocks of M_1 . Annular boundary strata that are incident with face blocks of M_1 are indicated in purple. For each block, attaching $(3,2)$ -handles over the indicated annuli results in a stratified 3-manifold homeomorphic to $S^2 \times [0, 1]$.

Figure 3.14 illustrates how 2-handle attachment affects edge blocks. The effect on ∂W near E of attaching the $(4,2)$ -handle $C(S_B^3)$ over B is equivalent to attaching a $(3,2)$ -handle from the stratification of B' to E over the annular boundary stratum

$E \cap B$. Attachment of all (4,2)–handles to W applies this (3,2)–handle attachment to all annular strata of E , and attaching these (3,2)–handles to all annular strata of E results in a 3–manifold homeomorphic to $S^2 \times [0, 1]$.

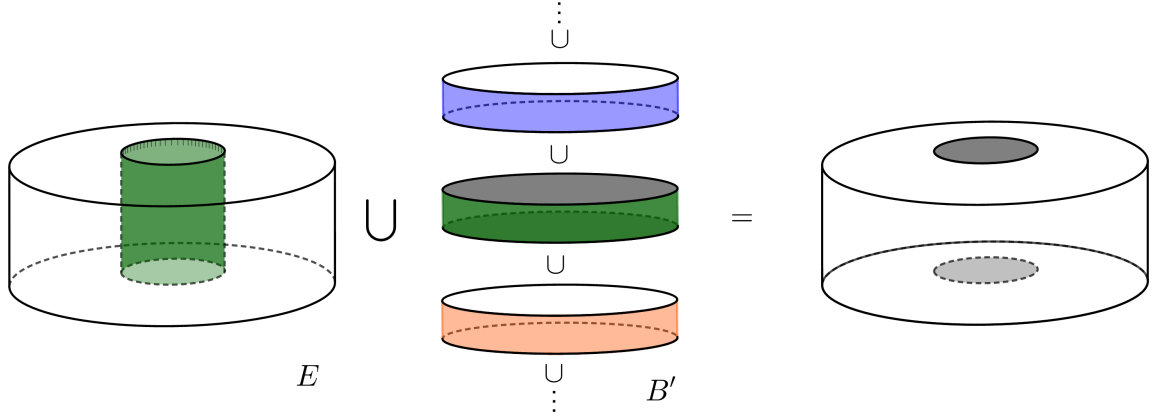


Figure 3.14: **The effect of stratified (4,2)–handle attachment on an example edge block.** An edge block E , a face block complement B' , and the result of attaching $C(S_B^3)$ to W over B on E . The boundary stratum shared by E and B in M_1 is indicated in green, as is the corresponding boundary stratum of B' in S_B^3 .

Part 2: Let V be a vertex block. All vertex blocks are stratified (3,1)–handlebodies of some genus, and V and V' are related via (3,2)–handle attachments to V induced by (4,2)–handle attachments. The attaching regions are disjoint, so the resulting manifold is independent of the order in which we attach handles.

Let g be the genus of V . Proving that V' is a (3,2)–handlebody when $g = 0$ is trivial, because V is then a 3–ball. This means V' is a 3–ball with a collection of (3,2)–handles attached to it, i.e. a (3,2)–handlebody. The case of $g > 0$ is nontrivial.

Suppose V is a (3,1)–handlebody of genus $g > 0$. To prove that V' is a (3,2)–handlebody, we show that each (3,2)–handle attachment either cancels a (3,1)–handle of V or adds a new S^2 boundary component away from the (3,1)–handles of V . When $g > 0$, V is a regular neighbourhood of either a circle or one of the nontrivial singular fibers in Figure 3.5. Each case provides a (3,1)–handlebody structure for V . Because we are interested in handle cancellation, the salient point is the identification of (3,1)–handle belt spheres. When V is a solid torus, i.e. a regular vertex block, we set any meridional circle of V as a belt sphere for the (3,1)–handle. Otherwise, give the fiber the graph structure implied by Figure 3.5. Then V has a (3,1)–handlebody structure with one or two (3,0)–handles, depending on the number of singular points in the fiber (i.e. vertices in the graph structure), and a (3,1)–handle for each edge.

With a (3,1)–handlebody structure in place, let α_i be the belt spheres of each (3,1)–handle. This implies that the α_i are pairwise non-parallel. Let β_i be the attaching spheres of the (3,2)–handles that we attach to V in the construction of V' . For each i, j , $|\alpha_i \cap \beta_j| = 0$ or 1 by construction. Moreover, for every i , $\alpha_i \cap \{\beta_j\}_j > 0$ and $\beta_i \cap \{\alpha_j\}_j > 0$. In words, every (3,1)–handle belt sphere transversely intersects at least one of the (3,2)–handle attaching spheres, every (3,2)–handle attaching sphere intersects at least one of the (3,1)–handle belt spheres, and all intersection numbers are at most 1. Thus every (3,2)–handle attached to V either cancels out an existing (3,1)–handle or adds a new 2–sphere boundary component. Because $\partial V'$ is a disjoint collection of 2–spheres we conclude that every (3,1)–handle of V has been cancelled by a (3,2)–handle, thus V' is a (3,2)–handlebody. We present an example of how these handle attachment sites are arranged in Figure 3.15.

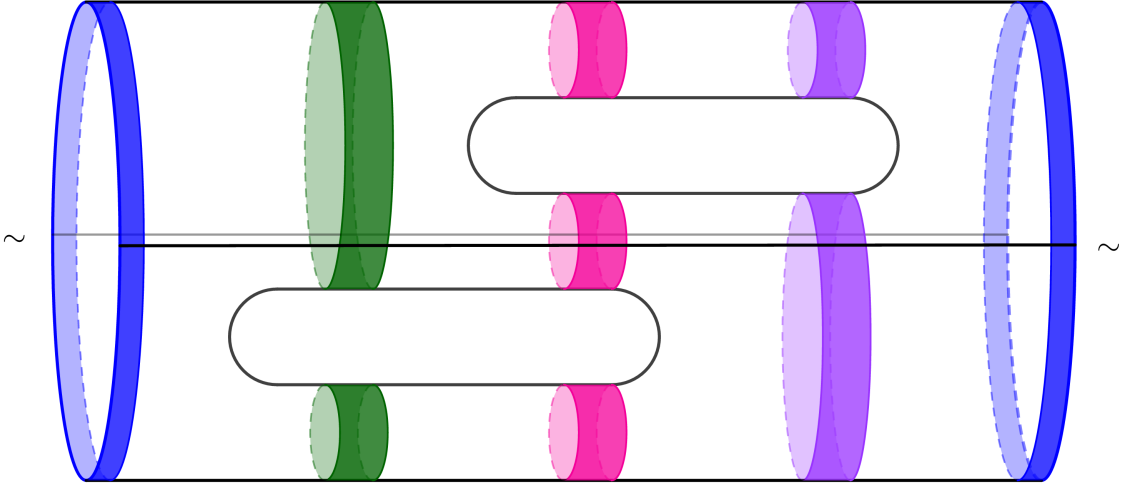


Figure 3.15: **Vertex block with (3,2)–handle attachment sites indicated.** We display an example vertex block of M_1 . Recall that this is the second type of interactive vertex block and, as in Figure 3.11, the surface is presented as embedded in S^3 and the block is the stratified (3,1)–handlebody on the ‘outside’ of the surface in S^3 . Annular boundary strata that are incident with face blocks of M_1 are indicated in color and correspond to the shared vertex-face region boundary edge. The 1–handle belt spheres are also indicated as black horizontal arcs across the surface. Attaching (3,2)–handles over the indicated annuli as prescribed results in a (3,2)–handlebody.

Part 3: To show that primed edge and vertex blocks exhaust M'_1 , we show that the entirety of B' has been apportioned among the primed blocks for each B' . The $(3, k)$ strata of B' are each cylinders whose annular boundary strata corresponds directly to a longitudinal boundary annulus of B . Face regions do not intersect even on their

boundary, so all of the annular boundary strata of B , hence B' , are shared only by edge and vertex blocks. Thus each cylinder of B' has been assigned as a (3,2)–handle and attached to a primed edge or vertex block. Because (4,2)–handle attachment altered M_1 to M'_1 only by replacing each face block B with its complementary B' , the primed blocks must exhaust M'_1 . \square

3.4.2 3–handles

Decomposing M'_1 into primed edge and vertex blocks provides us with stratified 3–handle attachment neighbourhoods. A 4–dimensional 3–handle is attached over an $S^2 \times [0, 1]$ embedded in the boundary of a 4–manifold, so attachment neighbourhoods for our stratified 3–handles are the primed edge blocks of M'_1 . We begin 3–handle attachment by precisely defining the structure of a stratified 3–handle so that it may be attached over a primed edge block.

Let E' be a primed edge block of M'_1 . By Theorem 3.4.1, E' is homeomorphic to $S^2 \times [0, 1]$. In particular, $\partial E'$ is a disjoint pair of stratified 2–spheres. We form a 4–disk containing E' in its boundary by a double coning method on E' : we first cone the spherical boundary components of E' to form a pair of 3–disks, glue these 3–disks to E' to form a 3–sphere, then cone the 3–sphere to obtain a 4–disk.

For each stratified boundary sphere $S_i^2 \in \partial E'$ we form the stratified 3–disk $C(S_i^2)$ and glue $C(S_i^2)$ to E' over S_i^2 . The result is a stratified 3–sphere, and we further cone that 3–sphere to form a stratified 4–disk. We denote the 4–disk by $C^2(E')$ and call the resulting stratified 3–handle structure the *stratified 3–handle induced by E'* . Attaching $C^2(E')$ to W' over $E' \subset M'_1$ alters the boundary of W' by replacing E' with the 3–disks $C(S_i^2)$ glued over the corresponding stratified boundary spheres in M'_1 . The full extent of this surgery can be detected by examining the primed vertex blocks of M'_1 in Corollary 3.4.2.

Corollary 3.4.2. Let W' be the 4–manifold resulting from the construction described in Theorem 3.4.1 and let \mathfrak{E}' be the set of primed edge blocks of $M'_1 \subset \partial W'$. Consider the 4–manifold W'' constructed from W' as

$$W'' = W' \cup \{C^2(E')\}_{E' \in \mathfrak{E}'} / \sim,$$

where \sim is defined by $e \sim \iota(e)$, ι the identity map $C^2(E') \supset E' \xrightarrow{\iota} E' \subset M'_1$. Then $M''_1 = \partial W'' \setminus M_0$ is a stratified 3–manifold with a decomposition into *double*

primed vertex blocks V'' such that the decomposition is well-defined and each V'' is homeomorphic to S^3 .

Proof. We need only prove that each connected component of M_1'' is homeomorphic to S^3 . Let V' be a primed vertex block. We know that V' is a $(3,2)$ -handlebody and $\partial V'$ is a disjoint collection of 2-spheres. Thus the attachment of $(3,3)$ -handles to V' over each 2-sphere boundary component results in a 3-sphere, proving the theorem. \square

After attaching 3-handles over primed edge blocks, W'' is a 4-manifold with boundary consisting of M_0 and a collection of stratified 3-spheres. We attach stratified 4-handles over these 3-spheres, so we begin by precisely defining the structure of these handles.

3.4.3 4-handles

Let V'' be a double primed vertex block of M_1'' . By Corollary 3.4.2, V'' is homeomorphic to S^3 . We form a 4-disk whose boundary is V'' by taking the cone of V'' . We denote this 4-disk by $C(V'')$ and call the resulting 4-handle structure the *stratified 4-handle induced by V''* . Attaching $C(V'')$ to W'' over V'' alters the boundary of W'' by replacing V'' with \emptyset .

Corollary 3.4.3. Let W'' be the 4-manifold resulting from the construction described in Corollary 3.4.2 and let \mathfrak{V}'' be the set of double primed vertex blocks of $M_1'' \subset \partial W''$. Consider the 4-manifold W''' constructed from W'' as

$$W''' = W'' \cup \{C(V'')\}_{V'' \in \mathfrak{V}''} / \sim,$$

where \sim is defined by $v \sim \iota(v)$, ι the identity map $C(V'') \supset V'' \xrightarrow{\iota} V'' \subset M_1''$. Then $M_1''' = \partial W''' \setminus M_0 = \emptyset$, hence W''' is a 4-manifold whose boundary is exactly M .

Chapter 4

Algorithm for triangulated manifolds

Algorithmic construction of a triangulated 4-manifold with a prescribed closed, orientable 3-manifold boundary broadly follows the steps we used in the construction of a 4-manifold with prescribed smooth, orientable 3-manifold boundary. Let N be a closed, orientable 3-manifold triangulation. Then the steps of construction are:

1. Define a projection $f : N \rightarrow \mathbb{R}^2$.
2. Induce a subdivision of N from f . The result is a 3-dimensional cell complex M that is equivalent to N . $M \times [0, 1]$ is a 4-dimensional cell complex with boundary components $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$. Let W be the triangulation obtained by inductively triangulating each cell of M .
3. Attach 4-dimensional 2-handles to W over its M_1 boundary as prescribed by the subdivision of M from f . Call the result W' and let $M'_1 = \partial W' \setminus M_0$.
4. Form W'' by attaching 4-dimensional 3-handles to W' over M'_1 as prescribed by the subdivision induced by f and the surgery induced by 2-handle attachment.
5. The boundary of W'' consists of M_0 and a collection of copies of S^3 . Attaching a 4-handle to each S^3 boundary component results in a triangulated 4-manifold whose boundary is exactly M_0 .

This construction has input a closed, orientable 3-manifold triangulation N and output a 4-manifold triangulation W''' whose sole boundary component is a triangulated 3-manifold that is equivalent to N . In this case, we find that $\partial W'''$ is a

subdivision of N , and this subdivision is the subdivision induced by the projection f in Step 1.

It is necessary that the input 3-manifold triangulation is *edge-distinct*, i.e. if u, v are vertices of N then $\{u, v\}$ is the boundary of at most one edge. If this condition is not satisfied by a given N , then it is satisfied by the barycentric subdivision of N . We assume that N is edge-distinct for the remainder of the chapter.

Throughout this chapter N refers to the input closed, orientable 3-manifold triangulation, f refers to the subdividing map defined in Section 4.1, M is the subdivision of N induced by f , W is the 4-manifold triangulation obtained from $M \times [0, 1]$, W' is the result of attaching 2-handles to W , and W'' is the result of attaching 3-handles to W' .

4.1 Define projection

In this algorithm, we use a projection of $f : N \rightarrow \mathbb{R}^2$ in the same way that we used a stratifying map in Chapter 3: preimages of sleeves around important features of the projection in the plane define handle attachment sites. The important features are the images of the 1-skeleton of N , and we want a high amount of control over how these features present. We first lay out conditions required of the projection, describe a method for ensuring these conditions are met, and then turn the method into an algorithm. We do all of this before forming the base 4-manifold so that the boundary components of W contain the desired handle attachment sites.

Our subdivision of N is obtained by imposing four conditions on $f : N \rightarrow \mathbb{R}^2$:

1. f maps vertices to the circle, i.e. for each vertex $v \in N^0$, $f(v)$ lies on the unit circle in \mathbb{R}^2 .
2. The images of vertices are distinct, i.e. for every pair of vertices $u, v \in N^0$, $f(u) \neq f(v)$.
3. f is linear on each simplex of N and piecewise-linear on N , i.e. if $x \in \sigma$ is a point in the simplex σ with vertices v_i , then $x = \sum_i a_i v_i$ with $\sum_i a_i = 1$ and $f(x) = \sum_i a_i f(v_i)$.
4. Edge intersections are distinct, i.e. for every triple of edges $e_1, e_2, e_3 \in N^1$ that share no vertices, $f(e_1) \cap f(e_2) \neq f(e_2) \cap f(e_3)$.

We call these the *subdivision conditions* on f , and we call a map satisfying the subdivision conditions a *subdividing map*. Conditions 1 and 2 ensure that every simplex of N is mapped to the plane in standard position, where a simplex σ of N is mapped to the plane in *standard position* if every point in $f(\sigma^0)$ is essential in forming the convex hull of $f(\sigma^0)$ as shown in Figure 4.1. This, along with conditions 3 and 4, allows us to use concepts and language from normal surface theory to describe the subdivision of N in the next section.

All conditions are satisfied by fixing an odd integer k greater than or equal to the number of vertices in N , injecting the vertices of N to the k^{th} complex roots of unity in the plane, then extending linearly over the skeletons of N . The first three conditions are clearly satisfied by this procedure, and the last is satisfied by the results in [8].

Algorithm 1 takes as input the closed, orientable 3-manifold triangulation N and produces a subdividing map $f : N \rightarrow \mathbb{R}^2$. The subdividing map projects each tetrahedron to the plane in *standard position*, as shown in Figure 4.1.

	Data: A closed, orientable 3-manifold triangulation N
	Result: A subdividing map $f : N \rightarrow \mathbb{R}^2$
1	begin
2	$k =$ the smallest odd number greater than or equal to $ N^0 $
3	foreach vertex v_i in N^0 , $i = 1, \dots, N^0 $ do
4	$f(v_i) = (\cos(\frac{2\pi i}{k}), \sin(\frac{2\pi i}{k}))$
5	end
6	foreach n in $\{1, 2, 3\}$ do
7	foreach simplex σ in N^n do
8	By definition, σ is the set of convex combinations of σ^0 :
9	$\sigma = \{x \mid x = \sum_{i=0}^n a_i v_i, \sum a_i = 1, a_i \geq 0, v_i \in \sigma^0\}$
10	Define f on $x \in \sigma$ by requiring linearity over simplices:
11	$f(x) = f(\sum_{i=0}^n a_i v_i) = \sum_{i=0}^n a_i f(v_i)$
12	end
13	end
14	end

Algorithm 1: Constructing a subdividing map $f : N \rightarrow \mathbb{R}^2$

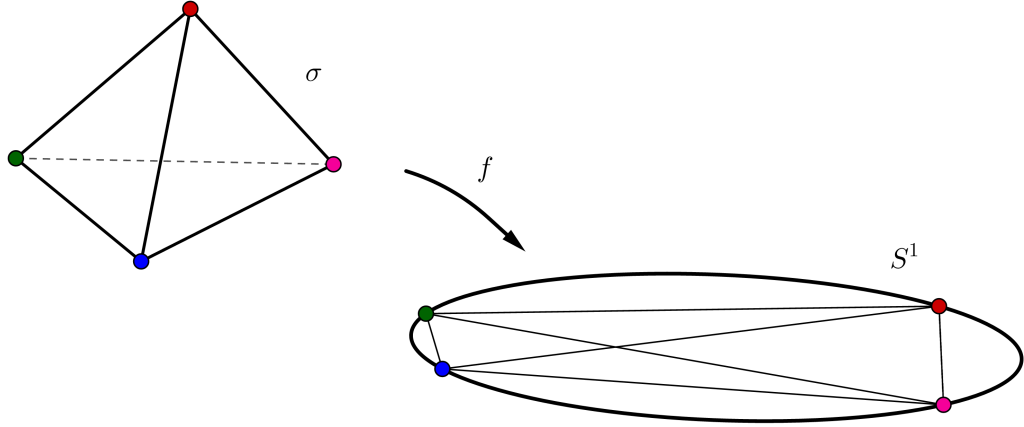


Figure 4.1: A tetrahedron σ projected to the plane in standard position through a subdividing map f . The four vertices of σ^0 map to the unit circle in the plane, thus form a convex arrangement. Four of the six edge of σ^1 map to the boundary of $f(\sigma)$, connecting $f(\sigma^0)$. The last two edges map across $f(\sigma)$, forming an intersection interior to $f(\sigma)$. Each vertex of the arrangement is essential in forming the convex hull of $f(\sigma^0)$.

4.2 Induce subdivision

The goal of subdividing N is to create and identify handle attachment sites analogous to the face, edge, and vertex blocks of Chapter 3. We use a similar technique to that found in Chapter 3, iteratively subdividing the tetrahedra of N by certain preimages of f . Tetrahedron subdivisions are compatible, i.e. tetrahedron subdivisions fit together exactly as the undivided tetrahedra do inside of N .

Let σ be a tetrahedron of N and let s be a line segment in the plane such that $s \cap f(\sigma)$ is nonempty, the endpoints of s are outside of $f(\sigma)$, and $s \cap f(\sigma)$ is a line segment in $f(\sigma)$ disjoint from any vertices or crossings of $f(\sigma)$. Figure 4.2 demonstrates the possible configurations of these line segments, and shows that their preimages inside of σ are triangles and quads. We refer to these preimages as *exterior triangles* and *exterior quads*. When a pair of these preimages intersect, the intersection is a line segment with endpoints interior to a pair of triangles of σ^2 .

For each $\sigma \in N^3$ and each edge e of N^1 such that $e \notin \sigma^1$, $f(e)$ is a line segment in the plane that is either disjoint from $f(\sigma)$ or induces an exterior triangle or quad in σ . There are two edges of σ whose preimages in σ form *interior triangles* and these are the *interior edges* of σ . The image of σ is a quadrilateral, and four of σ 's edges map to the boundary of that quadrilateral. The remaining two edges are the interior

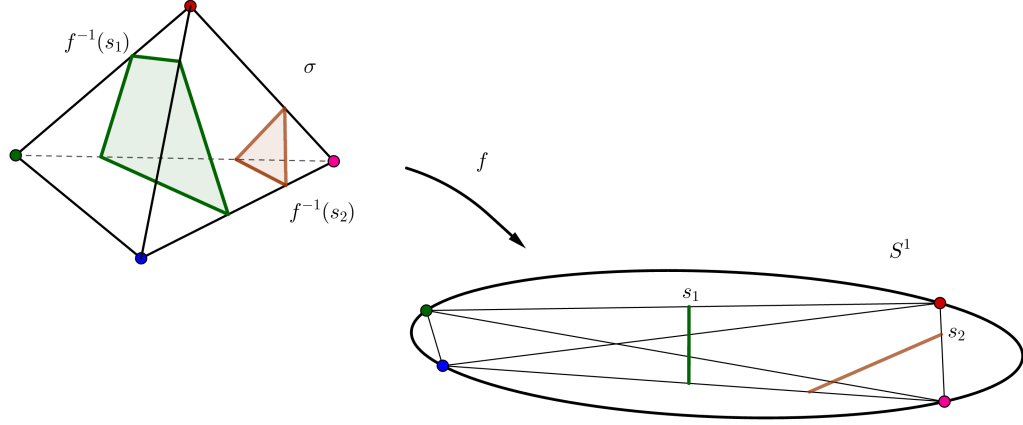


Figure 4.2: **A tetrahedron σ in standard position, intersecting edges, and preimage triangles and quads.** An intersecting edge separates the vertices of σ . If one is separated from the other three, its preimage is a triangle. If the vertices are separated into two pairs of two, the preimage is a quad.

edges. Figure 4.3 shows one interior triangle along with its possible intersections with exterior triangles and quads. The interior triangles of σ always intersect in a line segment with endpoints inside of the interior edges of σ , shown in Figure 4.4

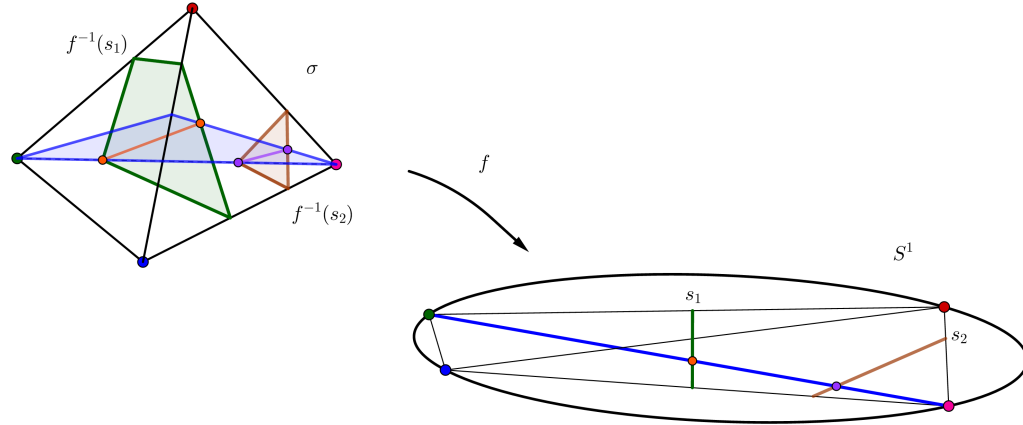


Figure 4.3: **A tetrahedron σ in standard position, one interior triangle, one exterior triangle, and one exterior quad.** There are two special preimage triangles in σ , called *interior triangles*, that occur as the preimages of the edges of σ that map through f across $f(\sigma)$ as in Figure 4.1.

Recall that the goal of this subdivision is to identify combinatorial analogues for the face, edge, and vertex blocks of Chapter 3. If we were to subdivide N using the quads and triangles induced by the edges of N , such blocks are ill-defined. We amend

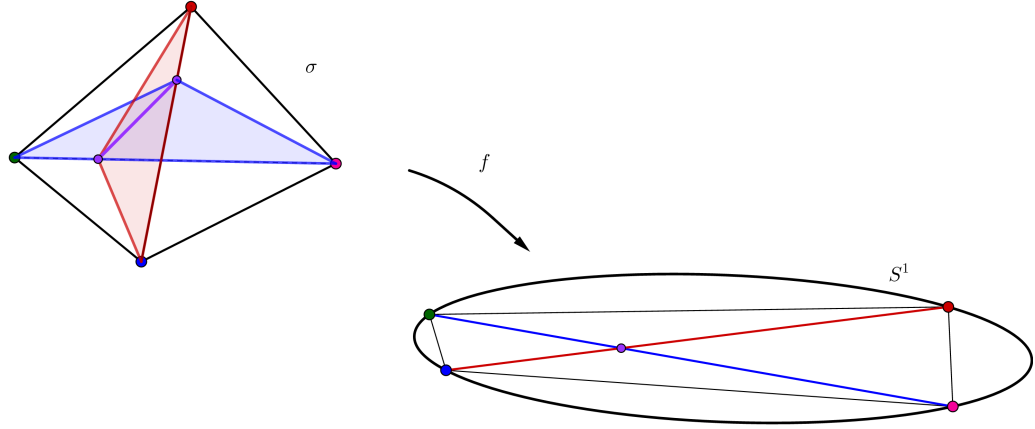


Figure 4.4: **A tetrahedron σ in standard position with both interior triangles.** As in Figure 4.3, but displaying both interior triangles.

this by introducing a set of line segments analogous to the sleeves of Section 3.2.

The first set of line segments separates the images of the vertices of N from the rest of the image. Because the vertices of N are mapped to the k^{th} roots of unity

For each edge e of N^1 , consider $e_\varepsilon^+ = s(e + \varepsilon_e e_\perp)$ and $e_\varepsilon^- = s(e - \varepsilon_e e_\perp)$, secants in the circle that are parallel to $f(e)$ and located a small orthogonal distance away from $f(e)$. Here we are using $s(\cdot)$ as a function that extends and trims line segments in the plane into secants in the circle. We require that, for each $e \in N^1$, ε_e is small enough that the rectangle R_e in the plane defined by e_ε^+ and e_ε^- does not fully contain g_ε^\pm for any $g \in N^1$. This requirement also ensures that the only crossings of $f(N^1)$ contained in R_e are crossings involving e . The segments e_ε^\pm for each $e \in N^1$ are called *sleeve segments*, and their preimages form exterior triangles and quads inside of the tetrahedra of N , as shown in Figure 4.5.

Take G to be the set of line segments in the plane that consists of $f(e)$ for each $e \in N^1$ along with the sleeve segments e_ε^\pm . For each tetrahedron σ of N , the triangles, quads, and intersections of $f^{-1}G \cap \sigma$ define a subdivision of σ into a cell complex. We implement this subdivision in Algorithm 2 by constructing a new cell complex that is equivalent to N . The subdivision iterates over all of the triangles and quads inside of σ , and the subdivision induced by a polygon is fully realized before moving onto the next.

For an example of subdivision, refer to Figure 4.3, where a tetrahedron is being subdivided by an exterior triangle, an interior triangle, and an exterior quad. We construct the subdivision of σ as a 3-complex τ , starting by setting $\tau = \sigma$. Starting at the loop on line 6 of Algorithm 2, we set g to be the brown segment that separates

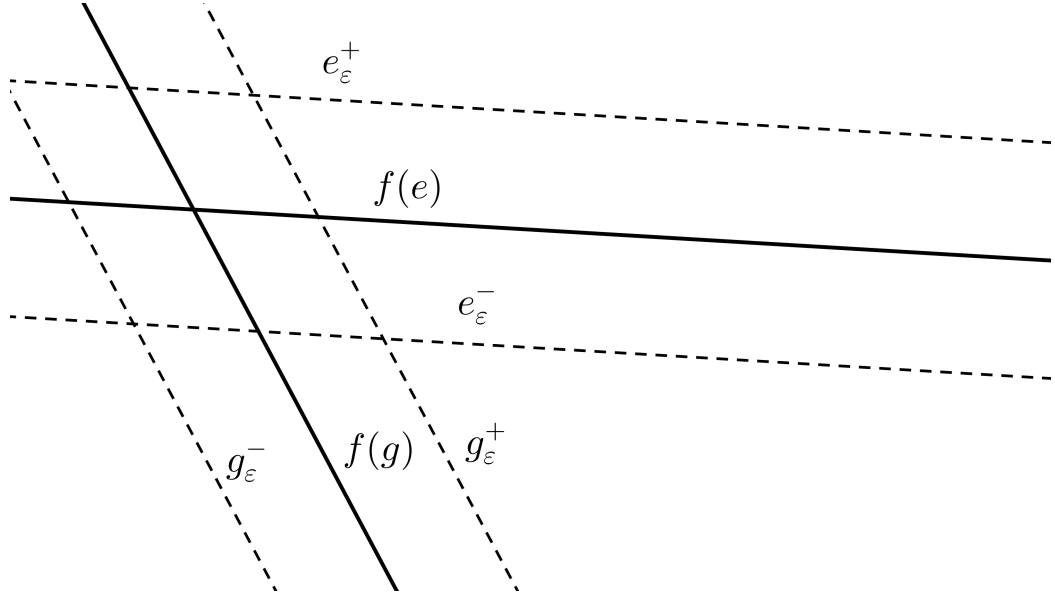


Figure 4.5: **A pair of sleeve segments in the plane.** We show a pair of edges projections $f(e)$ and $f(g)$ in the plane, along with their associated sleeve segments. Sleeve segments are necessary to ensure well-defined face, edge, and vertex block analogues in the subdivision of N .

the pink vertex from the rest of $f(\sigma)$, making δ the brown exterior triangle in σ . Then the loop on line 8 loops over all 1-cells of τ that intersect the exterior triangle δ which are the three 1-cells of τ incident with the pink vertex. We subdivide these 1-cells by their intersection with δ , which amounts to including the corners of the δ exterior triangle into the complex, splitting the relevant 1-cells into a pair of 1-cells on either side of the corner. We move to the loop on line 12, where we subdivide the three boundary pentagons (formerly triangles) of τ that contain the pink vertex by the three triangular edges of δ . Each pentagon is subdivided into a triangle and a quadrilateral. Finally, the loop at line 16 we subdivide the only 3-cell of τ into a tetrahedron containing the pink vertex and a triangular cylinder away from the pink vertex.

Heading back to the top of the loop at line 6, we set g to be the blue interior edge of Figure 4.3 hence δ to be the blue interior triangle. Moving through the loop at line 8, we find

1. the intersection of δ with the 1-cell of τ containing the blue and red vertices and
2. the intersection of δ with the brown exterior triangle located on the shared

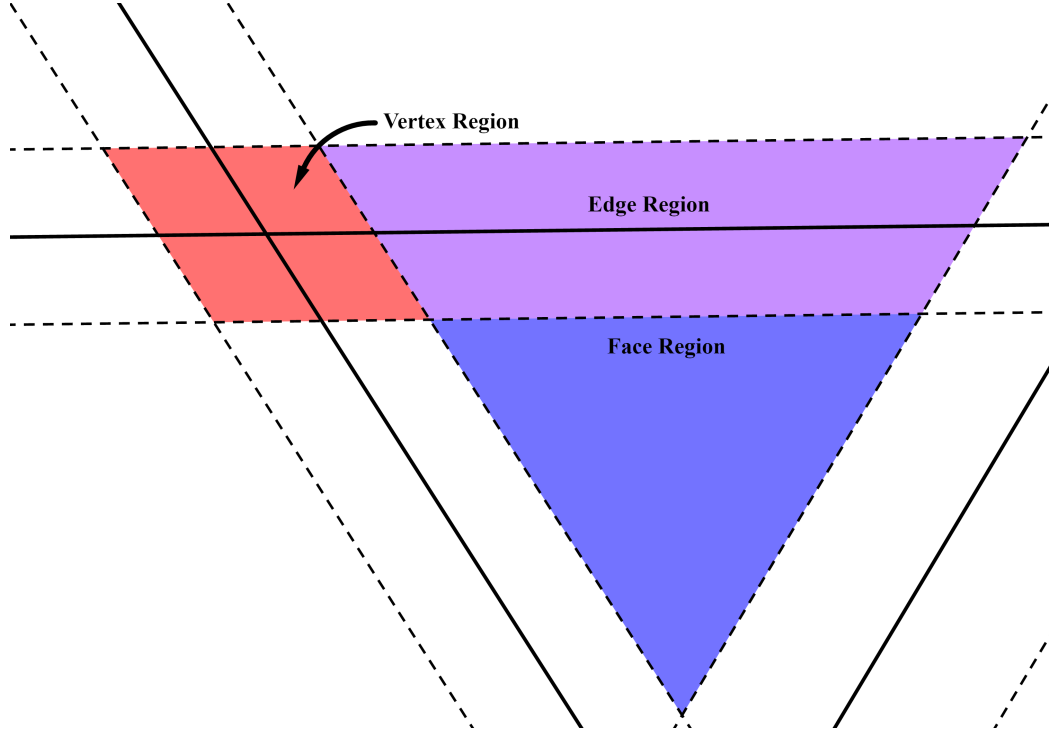


Figure 4.6: **Regions in the plane that correspond to combinatorial vertex, edge, and face block.** The sleeve segments provide the same functionality as the sleeves in Chapter 3, in that the preimages of the regions they form behave similarly to the vertex, edge, and face blocks defined in Chapter 3.

boundary of the pair of 2-cells in $\partial\tau$ induced by the subdivision of the 2-cell in $\partial\sigma$ opposite the green vertex.

We then insert 0-cells into τ at these intersections, subdividing 1-cells as necessary. The loop at line 12 now runs over the intersections of δ with the 2-cells of τ , and these intersections occur with the triangle in $\partial\tau$ opposite the pink vertex and with the 2-cells in $\partial\tau$ induced by the subdivision of the 2-cell in $\partial\sigma$ opposite the green vertex. We add 1-cells to τ that divide the quadrilateral into a pair of quadrilaterals and the triangles into pairs of triangles. Finally, the loop at line 16 subdivides both 3-cells of τ , splitting them each in half as polyhedrons. It is left to the reader to continue the subdivision of τ in Figure 4.3, where the remaining exterior quad subdivides five 1-cells, five 2-cells, and two 3-cells.

Observe also that the gluing defined on the tetrahedra of N is used to produce M as well. To see this, let σ and σ' be tetrahedra of N that are glued together over a triangular face, $\Delta \in \sigma_2$ and $\Delta' \in \sigma'$. Then Algorithm 2 produces a pair of 3-complexes τ and τ' , and Δ, Δ' are subdivided equivalently inside of $\partial\tau$ and $\partial\tau'$, so

the gluing that defines M is constructed by subdividing the gluing that defined N .

	<p>Data: A closed, orientable 3-manifold triangulation N with subdividing map $f : N \rightarrow \mathbb{R}^2$</p> <p>Result: A closed, orientable 3-dimensional cell complex M that is a subdivision of N</p> <pre> 1 begin 2 $G = \{f(e) \mid e \in N^1\} \cup \{e_\varepsilon^\pm \mid e \in N^1\}$ 3 foreach <i>tetrahedron</i> σ of N^3 do 4 We construct a 3-dimensional cell complex τ that is the subdivision of σ prescribed by f 5 $\tau = \sigma$ 6 foreach <i>segment</i> g in G such that $g \cap f(\sigma) \neq \emptyset$ and $g \not\subseteq \partial f(\sigma)$ do 7 $\delta =$ the intersection of σ with $f^{-1}g$ 8 foreach <i>intersection</i> $d_0 \in \delta \cap \tau^1$ such that $d_0 \notin \tau^1$ do 9 $d_0 = \delta \cap t_1$ for some 1-cell t_1 of τ 10 Subdivide t_1 by d_0 into a pair of 1-cells td_1 and dt_1 11 end 12 foreach <i>intersection</i> $d_1 \in \delta \cap \tau^2$ do 13 $d_1 = \delta \cap t_2$ for some 2-cell t_2 of τ 14 Subdivide t_2 by d_1 into a pair of 2-cells td_2 and dt_2 15 end 16 foreach <i>intersection</i> $d_2 \in \delta \cap \tau^3$ do 17 $d_2 = \delta \cap t_3$ for some 3-cell t_3 of τ 18 Subdivide t_3 by d_2 into a pair of 3-cells td_3 and dt_3 19 end 20 end 21 end 22 $M =$ the set of 3-complexes τ constructed above, glued together as prescribed by the gluing maps of N and the identification of the τ as subdivisions of the tetrahedra of N 23 end </pre>
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Algorithm 2: Subdividing N

We subdivided N into M in order to identify analogues to the face, edge, and vertex blocks of Chapter 3. The analogous blocks are defined exactly as they were in Chapter 3 — the sleeve segments subdivide the plane into regions homeomorphic to disks, those disks are classified by whether they contain a crossing of $f(N^1)$, intersect $f(N^1)$ but do not contain a crossing, or do not intersect $f(N^1)$ at all. The preimage of a region that contains a crossing is a *combinatorial vertex block*, the preimage of a region that intersects $f(N^1)$ but does not contain a crossing is an *combinatorial*

edge block, and the preimage of a region that is disjoint from $f(N^1)$ is a *combinatorial face block*. Symmetric names are used for the regions that these blocks project onto — combinatorial face (resp. edge, vertex) blocks map through f to *combinatorial face* (resp. *edge*, *vertex*) *regions*. Figure 4.6 illustrates the regions in the plane that produce such blocks. These preimages exhaust the cells of M , providing a decomposition into subcomplexes. This decomposition is not a partition — some cells are assigned to more than one block. Such an assignment happens precisely when a cell is mapped through f to the shared boundary of different combinatorial regions.

The Cartesian product of a pair of cell complexes is again a cell complex, so we form the base 4-dimensional cell-complex to which we attach handles as $M \times [0, 1]$. Every cell of $M \times [0, 1]$ is homeomorphic to a disk, so we form a 4-manifold triangulation W as a subdivision of $M \times [0, 1]$. This triangulation is obtained by inductively subdividing each cell of $M \times [0, 1]$ that is not a simplex.

4.3 Attach Handles

At this point in the chapter we have constructed a 4-manifold triangulation W with two boundary components, each equivalent via subdivision to the 3-dimensional cell complex M obtained in Section 4.2. Furthermore, the simplices of M are partitioned into subtriangulations analogous to the face, edge, and vertex blocks of Chapter 3. Explicit attachment sites are available for (4,2)-handles, so our first step is construction of a (4,2)-handle.

4.3.1 2-handles

The algorithm of this subsection takes as input a pair (T, Γ) where T is a closed solid torus triangulation and Γ is a collection disjoint parallel longitudes in the boundary of T , given as subtriangulations of ∂T . The output is a 4-disk triangulation H satisfying the following:

1. ∂H is a triangulated S^3 with a genus 1 Heegard splitting over ∂T , i.e. T is a subtriangulation of ∂H such that $\partial H \setminus \text{int}(T)$ is a solid torus.
2. For each γ_i in Γ , γ_i bounds a triangulated disk in ∂H , i.e. each γ_i is an explicit 0-framing for the (4,2)-handle attachment.

We take H to be a (4,2)–handle and attach H to W over T . Constructing the handle involves building a solid torus T' that complements T , combining T and T' over their shared boundary to build a triangulated 3–sphere, then coning that 3–sphere into the 4–disk H .

We find pairs (T, Γ) inside $M_1 \subset \partial W$, the subdivision of N induced by the subdividing map $f : N \rightarrow \mathbb{R}^2$. Let B be a combinatorial face block of M_1 . B is a 3–dimensional subtriangulation of M_1 that forms a closed solid torus, and B projects through f over an n –gon for some n . Call the n corners of $f(B)$ by c_i , $i = 1 \dots n$, and take $C = \{c_1 \dots c_n\}$. Then the $f^{-1}c_i$ are disjoint parallel triangulated longitudes of B . We take $(B, f^{-1}C)$ as our pair (T, Γ) for each combinatorial face block B of M .

For a closed solid torus T , we find a complementary torus T' , i.e. $\partial T = \partial T'$ as triangulations. We begin by investigating the boundary of T , setting $T'_0 = \partial T$. Inside T'_0 is the set of disjoint parallel longitudes Γ that are required to bound disks inside T' , so for each γ_i we attach a disk D_i with $\partial D_i = \gamma_i$ to T'_0 over γ_i . Call the result of these attachments T'_1 . Adjacent longitudes γ_i and γ_j bound an annulus A_{ij} in T'_1 , and $D_i \cup A_{ij} \cup D_j$ is a triangulated 2–sphere that we cone into a 3–disk D_{ij} . For each ij , we attach the 3–disk D_{ij} to T'_1 over $D_i \cup A_{ij} \cup D_j$ to form the closed solid torus T' .

By construction $\partial T' = \partial T$, hence the tori are complementary. The longitudinal curves of T , the γ_i , bound triangulated disks in T' thus are meridians of T' . Hence, gluing T and T' over their shared boundary forms a 3–sphere. We then cone that 3–sphere to produce a 4–disk with an explicit (4,2)–handle structure. This is the basic structure of Algorithm 3, which we iterate over the combinatorial face blocks of $M_1 \subset \partial W$. Finally, we attach these handles to W , forming W' .

Let \mathfrak{B} be the collection of combinatorial face blocks of $M_1 \subset \partial W$. Iterating Algorithm 3 over \mathfrak{B} yields a collection of (4,2)–handles $\{H_B^2\}_{B \in \mathfrak{B}}$. The combinatorial face blocks in \mathfrak{B} are disjoint by construction, so we attach our handles in any order. We form the 4–manifold triangulation W' from W as

$$W' = W \cup \{H_B^2\}_{B \in \mathfrak{B}} / \sim,$$

where \sim is defined by $b \sim \iota(b)$, ι the identity map $H_B^2 \supset B \xrightarrow{\iota} B \subset M_1$.

The manifold W' is formed by attaching triangulated (4,2)–handles to W over the combinatorial face blocks of M_1 . The boundary of W' consists of $M_0 \cup M'_1$, where $M'_1 = \partial W' \setminus M_0$ and M'_1 is related to M_1 via triangulated (3,2)–handle attachments to M_1 induced by the triangulated (4,2)–handle attachments to W . We decompose M'_1

	<p>Data: A pair (T, Γ) where T is a closed solid torus triangulation and Γ is a collection of disjoint parallel triangulated longitudes in ∂T</p> <p>Result: A 4-disk triangulation H_T^2 such that ∂H_T^2 is a triangulated 3-sphere with genus 1 Heegard splitting over ∂T and such that $\gamma_i \in \Gamma$ bounds a triangulated disk in $\partial H_T^2 \setminus \text{int}(T)$ for each i</p> <pre> 1 begin 2 First, we construct a complementary solid torus T' with $\partial T = \partial T'$ 3 $T'_0 = \partial T$ 4 foreach $\gamma_i \in \Gamma$ do 5 $D_i^2 = C(\gamma_i)$, the cone of γ_i 6 end 7 $T'_1 = T'_0 \cup \{D_i^2\} / \sim$, where \sim is induced by $\partial D_i^2 = \gamma_i \xrightarrow{\iota} \gamma_i \subset T'_0$ 8 foreach <i>adjacent pair of longitudes</i> γ_i, γ_j <i>from</i> Γ do 9 A_{ij} = the annulus in T'_1 bounded by γ_i and γ_j 10 S_{ij}^2 = the triangulated 2-sphere in T'_1 that is precisely $D_i^2 \cup A_{ij} \cup D_j^2$ 11 $D_{ij}^3 = C(S_{ij}^2)$, the cone of S_{ij}^2 12 end 13 $T' = T'_1 \cup \{D_{ij}^3\} / \sim$, where \sim is induced by $\partial D_{ij}^3 = S_{ij}^2 \xrightarrow{\iota} S_{ij}^2 \subset T'_1$ 14 $S_T^3 = T \cup T' / \sim$, where \sim is induced by $\partial T \xrightarrow{\iota} \partial T'$ 15 $H_T^2 = C(S_T^3)$, the cone of S_T^3 16 end </pre>
--	---

Algorithm 3: Constructing a (4,2)-handle

into primed combinatorial edge or vertex block subtriangulations in a manner similar to that found in Chapter 3.

Primed combinatorial edge blocks are the result of attaching triangulated (3,2)-handles over annular boundary components of combinatorial edge blocks. These (3,2)-handles are the triangulated 3-disks D_{ij}^3 formed on Line 11 of Algorithm 3, and their attaching regions are the annuli of combinatorial edge blocks that project through f over edge-face region boundaries.

Let E be a combinatorial edge block. Then E is the product of an orientable surface with an interval, i.e. $E = \Sigma \times [0, 1]$, where Σ is S^2 minus some number of disjoint open balls. The annuli $\partial \Sigma \times [0, 1]$ map through f to edge-face region boundaries, hence these are the attaching sites of (3,2)-handles. Primed combinatorial edge blocks are constructed by performing all such (3,2)-handle attachments, thus each primed combinatorial edge block is a triangulated $S^2 \times [0, 1]$.

Differing slightly from Chapter 3, this exhausts the new cells of M'_1 introduced by 2-handle attachment. This is due to the different shape of combinatorial vertex

regions as compared to the vertex regions of Chapter 3. In Chapter 3 vertex regions were octagonal and shared edge boundaries with face regions. Here, combinatorial vertex regions are quadrilateral and share point boundaries with combinatorial face regions. This point boundary is also shared by a pair of combinatorial edge regions.

Let x be a vertex-face-edge-edge boundary. Then $f^{-1}(x) = \{\gamma^i\}$ is a collection of triangulated circles, and each γ^i is a longitude of a toroidal combinatorial face block in M_1 that projects over the face region whose boundary contains x . Thus, every γ^i has been filled by a 2-disc constructed on Line 5 of Algorithm 3. Each of these new disks has been assigned to two primed combinatorial edge blocks, but we will also assign them to the primed vertex block that contains γ^i .

This means that each primed combinatorial vertex block is now equivalent to a triangulated (3,2)-handlebody. Let V be a combinatorial vertex block and V' be V primed. Considering each new disc in V' to be a flattened (3,2)-handle, we conclude that V' is a (3,2)-handlebody using the argument found in Theorem 3.4.1. The attaching spheres of (3,2)-handles transversely intersect the belt spheres of the (3,1)-handles at most once, each belt sphere intersects at least one attaching sphere and vice versa, and the boundary of V' is a disjoint collection of 2-spheres, thus V' is a (3,2)-handlebody.

4.3.2 3-handles

We now construct (4,3)-handles to attach over our primed combinatorial edge blocks. Let E' be such a block. Then $\partial E'$ is a disjoint pair of triangulated 2-spheres. We form a 4-disk containing E' in its boundary by a double coning method on E' : we first cone the spherical boundary components of E' to form a pair of triangulated 3-disks, glue these 3-disks to E' to form a 3-sphere, then cone the 3-sphere to obtain a 4-disk. This idea is identical to that found in Section 3.4.2 and is formalized in Algorithm 4. Finally, we attach (4,3)-handles built using Algorithm 4 to W' over primed combinatorial edge blocks.

Let \mathfrak{E}' be the collection of primed combinatorial edge blocks of $M'_1 \subset \partial W'$. Iterating Algorithm 4 over \mathfrak{E}' yields a collection of (4,3)-handles $\{H_{E'}^3\}_{E' \in \mathfrak{E}'}$. The primed combinatorial edge blocks in \mathfrak{E}' intersect only in discs that are also shared by primed combinatorial vertex blocks and these disks are invariant under the (4,3)-handle attachment so we attach our handles in any order. We form the 4-manifold

triangulation W'' from W' as

$$W'' = W' \cup \{H_{E'}^3\}_{E' \in \mathfrak{E}'} / \sim,$$

where \sim is defined by $e \sim \iota(e)$, ι the identity map $H_{E'}^3 \supset E' \xrightarrow{\iota} E' \subset M_1'$.

Data: E' , a triangulated $S^2 \times [0, 1]$
Result: A 4-disk triangulation $H_{E'}^3$ such that $\partial H_{E'}^3$ is a triangulated 3-sphere containing E' as a subtriangulation

```

1 begin
2   First, we attach 3-disks to  $E'$  over its two 2-sphere boundary components
3   foreach boundary sphere  $S_i^2 \in \partial E'$  do
4      $C(S_i^2)$  = the cone of  $S_i^2$ 
5     Attach  $C(S_i^2)$  to  $\partial E'$  over  $S_i^2 \in \partial E'$ 
6   end
7   The result of the above attachments is the 3-sphere  $S_{E'}^3$ 
8   Then  $H_{E'}^3 = C(S_{E'}^3)$ 
9 end
```

Algorithm 4: Constructing a (4,3)-handle

The manifold W'' is formed by attaching triangulated (4,3)-handles to W' over the primed combinatorial edge blocks of M_1' . The boundary of W'' consists of $M_0 \cup M_1''$, where $M_1'' = \partial W'' \setminus M_0$ and M_1'' is related to M_1' via triangulated (3,3)-handle attachments to M_1' induced by the triangulated (4,3)-handle attachments to W' . These (3,3)-handle attachments turn primed combinatorial vertex blocks into double primed combinatorial vertex blocks. A primed combinatorial vertex block is a (3,2)-handlebody, so a double primed combinatorial vertex block is a (3,2)-handlebody with each S^2 boundary component filled in with a (3,3)-handle, thus each double primed combinatorial vertex block is a triangulated S^3 .

4.3.3 4-handles

We now construct (4,4)-handles to attach over our double primed combinatorial vertex blocks. Let V'' be such a block. Then V'' is a triangulated S^3 , so a (4,4)-handle structure that can be attached over V'' is constructed by coning V'' . This is made explicit in Algorithm 5, which serves double duty as a general algorithm for coning spheres into discs. These handles are attached to W'' to form a triangulated 4-manifold whose only boundary component is M_0 .

Let \mathfrak{V}'' be the collection of double primed combinatorial vertex blocks of $M_1'' \subset \partial W''$. Coning each element of \mathfrak{V}'' yields a collection of (4,4)–handles $\{H_{V''}^4\}_{V'' \in \mathfrak{V}''}$. The double primed combinatorial vertex blocks in \mathfrak{V}'' are disjoint, so we attach our handles in any order. We form the 4–manifold triangulation W''' from W'' as

$$W''' = W'' \cup \{H_{V''}^4\}_{V'' \in \mathfrak{V}''} / \sim,$$

where \sim is defined by $v \sim \iota(v)$, ι the identity map $H_{V''}^4 \supset V'' \xrightarrow{\iota} V'' \subset M_1''$.

Data: V'' , a triangulated S^3
Result: A 4–disk triangulation $H_{V''}^4$ such that $\partial H_{V''}^4 = V''$

```

1 begin
2    $H_{V''}^4 = V''$ 
3   Add a 0–cell  $x$  to  $H_{V''}^4$ 
4   for  $i = 0 \dots 3$  do
5     foreach  $i$ –cell  $y$  of  $V'' \subset H$  do
6        $Z(y) = \{z \in H_{V''}^4 \mid z \notin V'' \text{ and } \partial z \cap \partial y \neq \emptyset\}$ 
7       Attach an  $(i + 1)$ –cell to  $H_{V''}^4$  over the triangulated  $i$ –sphere in
          $H_{V''}^4$  that is precisely  $y \cup Z(y) \cup x$ 
8     end
9   end
10 end
```

Algorithm 5: Constructing a (4,4)–handle; equivalently, coning a sphere

The triangulated 4–manifold W''' has only one boundary component: M_0 . By construction, M_0 is equivalent to the original input 3–manifold triangulation N by subdivision of N , hence W''' is a triangulated 4–manifold with boundary N , as desired. Each algorithm in the construction is now chained together in Algorithm 6.

	Data: N , a 3-manifold triangulation
	Result: A 4-manifold triangulation whose boundary is equivalent to N through subdivision
1	begin
2	Obtain a subdividing map f for N from Algorithm 1
3	Construct M , a 3-dimensional cell complex equivalent to N from Algorithm 2 applied to (N, f)
4	Construct W , the triangulation obtained by inductively subdividing the cells of $M \times [0, 1]$ that are not simplices
5	\mathfrak{B} = the collection of combinatorial face blocks of $M_1 \subset \partial W$
6	foreach $B \in \mathfrak{B}$ do
7	H_B^2 = the (4,2)-handle produced by Algorithm 3 applied to B
8	end
9	W' = the 4-manifold triangulation obtained by attaching H_B^2 to W over B for each $B \in \mathfrak{B}$
10	$M'_1 = \partial W' \setminus M_0$
11	Apportion the cells of $\{H_B^2\}_{B \in \mathfrak{B}}$ to the combinatorial edge and vertex blocks of W' to form primed combinatorial edge and vertex blocks
12	\mathfrak{E}' = the collection of primed combinatorial edge blocks of $M'_1 \subset \partial W'$
13	foreach $E' \in \mathfrak{E}'$ do
14	$H_{E'}^3$ = the (4,3)-handle produced by Algorithm 4 applied to E'
15	end
16	W'' = the 4-manifold triangulation obtained by attaching $H_{E'}^3$ to W' over E' for each $E' \in \mathfrak{E}'$
17	$M''_1 = \partial W'' \setminus M_0$
18	Apportion the cells of $\{H_{E'}^3\}_{E' \in \mathfrak{E}'}$ to the primed combinatorial vertex blocks of W'' to form double primed combinatorial vertex blocks
19	\mathfrak{V}'' = the collection of double primed combinatorial vertex blocks of $M''_1 \subset \partial W''$
20	foreach $V'' \in \mathfrak{V}''$ do
21	$H_{V''}^4$ = the (4,4)-handle produced by Algorithm 5 applied to V''
22	end
23	W''' = the 4-manifold triangulation obtained by attaching $H_{V''}^4$ to W'' over V'' for each $V'' \in \mathfrak{V}''$
24	$\partial W''' = M_0$ and $M_0 \equiv N$ through subdivision
25	return W'''
26	end

Algorithm 6: Full construction of a triangulated 4-manifold with prescribed 3-manifold boundary

Chapter 5

Discussion

The main chapters of this thesis serve different roles in the greater mathematical ecosystem. It is evident from the discussion in Section 2.2 of [3] that the ideas behind the constructive proof in Chapter 3 is not itself novel, but the details had yet to be published. Chapter 3 fills a hole in the current literature, whereas the contents of Chapter 4 are entirely new, serving as a solid foundation for future research in computational topology.

The immediate next step from creation of the algorithm is an implementation for a topology software project such as Regina [1]. Such an implementation furthers progress toward algorithmic computation of the Rokhlin invariant for 3-manifolds and serves as another tool in generating 4-manifold censuses. In its current form, the construction algorithm guarantees no topological features of the constructed 4-manifold save that its boundary is the given input 3-manifold. Lucrative avenues of exploration would therefore include an investigation into invariants (e.g. the fundamental group) or constructs (e.g. spin structures) on the constructed manifold.

This work was initially an attempt to adapt the Turaev Reconstruction Theorem as an algorithm on 3-manifolds triangulations. The theorem was introduced by Turaev [12] and is also covered by Costantino [2]. The algorithm would build on the process outlined in Chapter 4 of Costantino and Thurston [3], where a quadratic bound is achieved for the number of simplices needed to triangulate a 4-manifold constructed using the reconstruction theorem.

Such an adaptation necessarily passes through the shadow realm. The shadows explored in [12] are 2-complexes with strict structural restrictions imposed upon them. In the reconstruction theorem, a shadow serves as a set of instructions for a handle decomposition of the desired 4-manifold. Each 0-cell prescribes a $(4,0)$ -handle, each

1-cell a $(4,1)$ -handle, and each 2-cell a $(4,2)$ -handle. The resulting 4-manifold W has boundary exactly the original 3-manifold M . Two major obstructions eventually caused this adaptation attempt to be scrapped:

1. Precisely defining $(4,2)$ -handle attachment sites in this process is difficult.
2. The fact that $\partial W = M$ at the end of this process is not obvious and requires a deep examination of shadows (briefly, $\partial W = M$ because M and W have the same shadow).

Instead, the results of this work were obtained by turning the reconstruction theorem on its head: we begin by making the precise definition of $(4,2)$ -handle attachment sites as simple as possible. Once $(4,2)$ -handles have been attached, attachment of $(4,3)$ - and $(4,4)$ -handles is straightforward. This method is dual to the Turaev Reconstruction Theorem in its approach and handle indexing ($((n, \lambda)$ - and $(n, n - \lambda)$ -handles are identical, just attached over complementary portions of the handle's boundary), but the goals are different. Turaev's work is focused on studying the topology of the shadows themselves, where our ultimate goal is algorithmic construction of 4-manifolds with prescribed boundary.

Beyond its raw results, this research is a demonstration of adapting an existence proof into a constructive proof. More adaptations of existence proofs should not be surprising as the bounds of scientific computability are pushed.

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