Contents

Chapter 1

Introduction

The main goal of this document is to provide an algorithm whose input is a closed, orientable 3–manifold triangulation M and whose output is a 4–manifold triangulation whose boundary is M. This algorithm mirrors the constructive proof that a smooth, closed, orientable 3–manifold bounds some 4–manifold presented in Chapter 3.

Chapter 2

How do you obtain a 4–manifold with a specific boundary?

Given a smooth, closed 3–manifold M, there are infinitely many 4–manifolds with boundary M. We do not ensure that the constructed 4–manifold has any properties other than a specified boundary, so our construction ensures easy verification that the constructed 4–manifold's boundary is exactly M. This is done by setting $W = M \times [0,1]$, so W has boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

We then attach handles to the boundary of W away from M_0 until only M_0 remains.

The concept of a stratified handle attachment needs some explanation. First, we define attachment of topological spaces, and use that language to define handle attachment.

Definition 2.0.1 (Attachment)**.** Let X and Y be topological spaces, $A \subset X$ a subspace, and $f : A \to Y$ a continuous map. We define a relation $\sim$ by putting $f(x) \sim x$ for every x in A. Denote the quotient space $X \sqcup Y/\sim$ by $X \cup_f Y$. We call the map f the *attaching map.* We say that X is *attached* or *glued* to Y over A. A space obtained through attachment is called an *adjunction space* or *attachment space.*

Alternatively, we let A be a topological space and let $i_X : A \to X$, $i_Y : A \to Y$ be inclusions. Here, the adjunction is formed by taking $i_X(a) \sim i_Y(a)$ for every $a \in A$ and we denote the adjunction space by $X \cup_A Y$.

Definition 2.0.2 (Handle)**.** Take $n = \lambda + \mu$ and M a smooth n–manifold with nonempty boundary ∂M. Let D^λ be the closed λ–disk and put $H^\lambda = D^\lambda \times D^\mu$. Let $\varphi : \partial D^\lambda \times D^\mu \to \partial M$ be an embedding and an attaching map between M and H^λ. The attached space H^λ is an *n–dimensional λ–handle*, and $M \cup_\varphi H^\lambda$ is the result of an n–dimensional *λ–handle attachment*.

Handle attachment is defined for smooth manifolds, but the resulting attachment space is not a smooth manifold. Rather, the result is a stratified manifold. We use the definition from [3].

Definition 2.0.3 (Stratification)**.** X is a *filtered space* on a finite partially ordered indexing set S if

1. there is a closed subset X_s for each $s \in S$,
2. $s \leq s'$ implies that $X_s \subset X_{s'}$, and
3. the inclusions $X_s \hookrightarrow X_{s'}$ satisfy the homotopy lifting property.

The X_s are the *closed strata* of X, and the differences

$$X_s \setminus \bigcup_{r<s} X_r$$

are *pure strata*. The pure strata are denoted X^s. The singular declension of the word strata is *stratum*.

A *filtered map* between spaces filtered over the same indexing set is a continuous function $f : X \to Y$ such that $f(X_s) \subset Y_s$, and such a map is *stratified* if $f(X^s) \subset Y^s$. This leads to definitions of stratified homotopy, therefore stratified homotopy equivalence.

Immediate examples of stratified manifolds are manifolds with boundary and manifolds with corners. The result of a smooth handle attachment is a manifold with corners at $\varphi(\partial D^\lambda \times \partial D^\mu)$, hence a stratified manifold.

Chapter 3

Constructive proof that a smooth, closed, orientable 3–manifold is the boundary of some 4–manifold

We prove that every smooth, closed, orientable 3–manifold is the boundary of some 4–manifold. We do so by explicitly constructing such a 4–manifold from a given 3–manifold. This construction is mirrored in Chapter 4 where we prove the same for a given closed, orientable 3–manifold triangulation and provide an algorithm.

Let M be a smooth, closed, orientable 3–manifold and take $W = M \times [0,1]$. Then W is a 4–manifold with boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

A 4–manifold with only one boundary component, M_0, is obtained from W by iteratively attaching 4–dimensional 2–, then 3–, then 4–handles to W over the M_1 boundary component until that component has been "filled in."

Instructions for handle attachment come from defining a projection $f : M_1 \to \mathbb{R}^2$ that induces a stratification of $M_1 \subset \partial W$, indexed by dimension of the pure strata as a submanifold of M_1, such that the pure stratum M_1^3 is a disjoint collection of open submanifolds of M_1. We call the closure of a connected component of M_1^3 a *block*, and we impose conditions on f to ensure that every block can be classified as one of the following:

face-block: An attachment neighbourhood for a stratified 2–handle. Each face-block is diffeomorphic to $S^1 \times G_n$, the product of the circle with an n-gon for some n.

edge-block: A partial attachment neighbourhood for a stratified 3–handle. Each edge-block is diffeomorphic to one of $D^2 \times [0,1]$, $A \times [0,1]$, or $P \times [0,1]$, where A is the annulus $S^1 \times [0,1]$ and P is a pair-of-pants surface. Attachment of stratified 2–handles over our face-blocks "fill in" the annular boundary components of edge-blocks, forming full attachment neighbourhoods for stratified 3–handles.

vertex-block: A partial attachment neighbourhood for a stratified 4–handle. Each vertex-block is homeomorphic to a genus 2 or 3 (3,1)-handlebody. When stratified 2– and 3–handles are attached to W, the remaining boundary is M_0 union a collection of stratified 3–spheres which are then coned away.

The remainder of this chapter is spent ensuring that such a stratification can be achieved for any smooth, orientable, closed 3–manifold, detailing how the stratification is induced, proving that the attachment of stratified 2– and 3–handles has the previously stated effects, and discussing the resulting 4–manifold.

3.1 Projections from 3–manifolds to $\mathbb{R}^2$

We begin by stating the conditions required of the projection $f : M \to \mathbb{R}^2$ for the desired decomposition of $\mathbb{R}^2$, hence stratification of M, to exist. For proof that an f satisfying these conditions exists and that such functions are generic, see [1].

3.2 Decomposing $\mathbb{R}^2$

We use a projection f satisfying the conditions in Section 3.1 to induce a particular decomposition on $\mathbb{R}^2$. We fit closed neighbourhoods around the singular values of f and classify these sleeves by the maximum codimension (with respect to $\mathbb{R}^2$) of singular values they contain. Because S_f, the singular values of f, consists of codimension 1 and codimension 2 singular values (i.e. arcs and arc-crossings respectively), we decompose $\mathbb{R}^2$ into face-regions that contain no singular values, edge-regions that contain only codimension 1 singular values, and vertex-regions, each of which contain exactly 1 codimension 2 singular value.

3.3 Stratifying M

Decomposing $\mathbb{R}^2$ via the singular values of f also induces a stratification of M by considering the preimage through f of the decomposing regions. The interiors of a face-regions have preimage through f a disjoint collection of face-blocks, the interiors of edge-regions have preimage of edge-blocks, and of vertex-regions, vertex-blocks.

3.4 Handle attachment

We now investigate the consequences of stratified handle attachment over the face-blocks of the stratified M_1 boundary component of $W = M \times [0, 1]$. This investigation is performed by comparing ∂W to $\partial W'$, where

$$W' = W \cup \{H^2_\alpha\}_{\alpha \in A} / \sim,$$

the 4–manifold obtained by attaching stratified 2-handles to W.

Attaching these stratified 2–handles to W alters the boundary of W via surgery on M_1 that is equivalent to replacing the interiors each face-block with a solid torus whose meridians are longitudes of the replaced face-block. The annular boundary components of each edge-block are "filled-in" by cylinders found between meridians of these newly introduced solid tori, in each case forming a stratified $S^2 \times [0, 1]$. These stratified $S^2 \times [0, 1]$ are taken as attachment neighbourhoods for stratified 3–handles, forming W''.

Finally, we compare the boundaries of W' and W'' to show that $\partial W''$ is the disjoint union of M_0 with a collection of stratified 3–spheres. The 3–sphere boundary components are then coned away.

3.5 The constructed 4–manifold

Chapter 4

Algorithm for constructing a triangulated 4–manifold with prescribed 3–manifold boundary

The steps to construct a triangulated 4–manifold with prescribed 3–manifold boundary broadly follow the steps to construct a 4–manifold with prescribed smooth, orientable 3–manifold boundary. Let N be a 3–manifold triangulation. Then the steps of construction are:

Step 1: Define a projection $f : N \to \mathbb{R}^2$.

Step 2: Induce a subdivision of N from f. The result is a 3–manifold triangulation M that is equivalent to N in the sense of triangulations.

Step 3: Let $W = M \times [0, 1]$ be a 4–manifold with boundary components $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$.

Step 4: Attach 4–dimensional 2–handles to W over its M_1 boundary as prescribed by the subdivision of M from f. Call the result W' and call the boundary of W' different from M_0 by M_1'.

Step 5: Attach 4–dimensional 3–handles to W' over M_1' as prescribed by the subdivision induced by f and the surgery induced by 2–handle attachment. Call the result W''.

Step 6: The boundary of W'' consists of M_0 and a collection of copies of S^3 that we now cone off. The result is a 4–manifold whose boundary is exactly M_0.

Each of these steps is made algorithmic, and these algorithms are chained in series to form a single algorithm. The result has input a closed, orientable 3–manifold triangulation M and output a 4–manifold triangulation W whose sole boundary component is a triangulated 3–manifold that is equivalent to M in the sense of triangulations. In this case, we find that ∂W is a subdivision of M, and this subdivision is the subdivision induced by the projection f in Step 1.

Throughout this chapter N refers to the initial input 3–manifold triangulation, f to the projection defined in Section 4.1, M is the subdivision of N induced by f, W is the 4–manifold $M \times [0, 1]$, W' is the result of attaching 2–handles to W, and W'' is the result of attaching 3–handles to W'.

4.1 Define projection

The projection's utility is in defining a subdivision of the initial closed, orientable 3–manifold triangulation N such that attaching regions for triangulated 2– and 3–handles can be found. This is done before forming the base 4–manifold so that the subdivided triangulation is used in the algorithm that provides W.

Our subdivision is obtained by imposing four conditions on the $f : N \to \mathbb{R}^2$:

1. f maps vertices to the circle. i.e. for each vertex $v \in N^0$, $f(v)$ lies on the unit circle in $\mathbb{R}^2$.

2. The images of vertices are distinct. i.e. for every pair of vertices $u, v \in N^0$, $f(u) \neq f(v)$.

3. f is linear on each simplex of N and piecewise-linear on N. i.e. if $x \in \sigma$ is a point in the simplex σ with vertices v_i, then $x = \sum_i a_i v_i$ with $\sum_i a_i = 1$ and $f(x) = \sum_i a_i f(v_i)$.

4. Edge intersections are distinct. i.e. for every triple of edges $e_1, e_2, e_3 \in N^1$ that share no vertices, $f(e_1) \cap f(e_2) \neq f(e_2) \cap f(e_3)$.

Conditions 1 and 2 ensure that every simplex of N is mapped to the plane in standard position (i.e. the images of the vertices in the plane form a convex set). This, along with conditions 3 and 4, allows us to use concepts and language from normal surface theory to describe the subdivision of N in the next section. We call these four conditions the *subdivision conditions* on f.

All conditions are satisfied by fixing an odd integer k greater than or equal to the number of vertices in N, injecting the vertices of N to the k^{th} complex roots of unity in the plane, then extending linearly over the skeletons of N. The first three conditions are clearly satisfied by this procedure, and the last is satisfied by applying the results in [2].

The algorithm presented in this section takes as input the triangulated 3–manifold N and produces a projection $f : N \to \mathbb{R}^2$ satisfying the subdivision conditions.

4.2 Induce subdivision

The goal of subdividing N is to create and identify analogues to the face- edge- and vertex-blocks of Chapter 3 where we may iteratively attach 2–, 3–, then 4– handles. We use a similar technique to that found in Chapter 3, first decomposing $\mathbb{R}^2$ with the projection, then examining preimages to define our subdivision.

Decomposition of $\mathbb{R}^2$ is done through $f(N_1)$. A point in $f(N_1)$ is the image of either a vertex, exactly one edge, or exactly two edges (i.e. is an edge crossing), so we refer to these as the *vertices*, *edges*, and *crossings* of the decomposition. Because $f(N) \setminus f(N_1)$ is a disjoint collection of simply connected regions, we call the connected components of $f(N) \setminus f(N_1)$ the *faces* of the decomposition.

We construct our subdivision of N using the decomposition component preimages. The preimage of a face component defines substructures analogous to face-blocks, of edge components to edge blocks, and vertices and crossings to vertex-blocks. Inside of an individual tetrahedron of N, edge and crossing preimages are well-defined, supplying a natural subdivision of N into a cell complex. Then, the subdivision of a cell complex into a triangulation is well-defined.

The algorithm presented in this section takes as input a closed, orientable 3–manifold triangulation N and a projection $f : N \to \mathbb{R}^2$ and produces as output a closed, orientable 3–manifold triangulation M that is a subdivision of N. Furthermore, the 3–cells of M are partitioned into subsets that serve the same purpose as the face-blocks of Chapter 3: attaching regions for 2–handles.

4.3 Form base 4–manifold

The algorithm presented in this chapter takes as input a closed, orientable 3–manifold triangulation M and produces as output a 4–manifold triangulation W whose boundary is the disjoint union of two copies of M. We do this by explicitly triangulating the cell complex $W \times [0, 1]$.

4.4 Attach 2–handles

At this point in our procedure we have a 4–manifold W with triangulated boundary components M_0 and M_1. We aim to attach handles to W over the boundary of W away from M_1 until the boundary of W is exactly M_1. The first step is to attach 2–handles to W over the closed solid torus triangulations that partition M_0.

The algorithm of this section takes as input a closed solid torus triangulation T with a triangulated curve γ in its boundary (γ is an explicit 0-framing for the 2–handle attachment) and produces as output a D^4 triangulation whose S^3 boundary triangulation has genus 1 Heegard splitting over ∂T, and γ bounds a disk in $\partial D^4 \setminus T$. Such a D^4 triangulation is taken as a 4–dimensional 2–handle and attached to W over T. We then attach such a 2–handle over each closed solid torus in our partition of M_0.

4.5 Attach 3–handles

4.6 Cone away final boundary components

Chapter 5

Conclusion

Bibliography

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