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Chapter 1

Introduction

The main goal of this document is to provide an algorithm whose input is a closed, orientable 3–manifold triangulation M and whose output is a 4–manifold triangulation whose boundary is M. This algorithm mirrors the constructive proof that a smooth, closed, orientable 3–manifold bounds some 4–manifold presented in Chapter 3.

Chapter 2

How do you obtain a 4–manifold with a specific boundary?

Given a smooth, closed 3–manifold M, there are infinitely many 4–manifolds with boundary M. We do not ensure that the constructed 4–manifold has any properties other than a specified boundary, so our construction ensures easy verification that the constructed 4–manifold's boundary is exactly M. This is done by setting $W = M \times [0,1]$, so W has boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

We then attach stratified handles to the boundary of W away from M_0 until only M_0 remains.

The concept of a stratified handle attachment needs some explanation. First, we define attachment of topological spaces, and use that language to define handle attachment.

Definition 2.0.1 (Attachment)**.** Let X and Y be topological spaces, $A \subset X$ a subspace, and $f : A \to Y$ a continuous map. We define a relation $\sim$ by putting $f(x) \sim x$ for every x in A. Denote the quotient space $X \sqcup Y/\sim$ by $X \cup_f Y$. We call the map f the *attaching map*. We say that X is *attached* or *glued* to Y over A. A space obtained through attachment is called an *adjunction space* or *attachment space*.

Alternatively, we let A be a topological space and let $i_X : A \to X$, $i_Y : A \to Y$ be inclusions. Here, the adjunction is formed by taking $i_X(a) \sim i_Y(a)$ for every $a \in A$ and we denote the adjunction space by $X \cup_A Y$.

Definition 2.0.2 (Handle)**.** Take $n = \lambda + \mu$ and M a smooth n–manifold with nonempty boundary ∂M. Let D^λ be the closed λ–disk and put $H^\lambda = D^\lambda \times D^\mu$. Let $\varphi : \partial D^\lambda \times D^\mu \to \partial M$ be an embedding and an attaching map between M and H^λ. The attached space H^λ is an *n–dimensional λ–handle*, and $M \cup_\varphi H^\lambda$ is the result of an n–dimensional *λ–handle attachment*.

Handle attachment is defined for smooth manifolds, but the resulting attachment space is not a smooth manifold. Rather, the result is a stratified manifold. We use the definition from [3].

Definition 2.0.3 (Stratification)**.** X is a *filtered space* on a finite partially ordered indexing set S if

1. there is a closed subset X_s for each $s \in S$,
2. $s \leq s'$ implies that $X_s \subset X_{s'}$, and
3. the inclusions $X_s \hookrightarrow X_{s'}$ satisfy the homotopy lifting property.

The X_s are the *closed strata* of X, and the differences

$$X^s = X_s \setminus \bigcup_{r<s} X_r$$

are *pure strata*.

A *filtered map* between spaces filtered over the same indexing set is a continuous function $f : X \to Y$ such that $f(X_s) \subset Y_s$, and such a map is *stratified* if $f(X^s) \subset Y^s$. This leads to definitions of stratified homotopy, therefore stratified homotopy equivalence.

Immediate examples of stratified manifolds are manifolds with boundary and manifolds with corners. Many handles (e.g. $D^1 \times D^1$) are manifolds with corners, and the result of a smooth handle attachment is a manifold with corners at $\varphi(\partial D^\lambda \times \partial D^\mu)$. Hence both are stratified manifolds.

A *stratified handle attachment* is a handle attachment where the handle, the manifold to which we attach the handle, and the attaching map are each stratified. The main distinctions between stratified handle attachment and handle attachment are:

1. the strata of the handle include, but are not exclusive to, the naturally occurring corners from the formation of the handle as the Cartesian product of a pair of disks,

2. the manifold to which we attach the handle is necessarily stratified, and
3. the attaching map ensures that there is a coherent identification between the strata of the handle and the strata of the manifold (i.e. the stratification of the resulting attachment space is well-defined).

Chapter 3

Constructive proof that a smooth, closed, orientable 3–manifold is the boundary of some 4–manifold

We prove that every smooth, closed, orientable 3–manifold is the boundary of some 4–manifold. We do so by explicitly constructing such a 4–manifold from a given 3–manifold. This construction is mirrored in Chapter 4 where we prove the same for a given closed, orientable 3–manifold triangulation and provide an algorithm.

Let M be a smooth, closed, orientable 3–manifold and take $W = M \times [0, 1]$. Then W is a 4–manifold with boundary

$$\partial W = (M \times \{0\}) \cup (M \times \{1\}) = M_0 \cup M_1.$$

A 4–manifold with only one boundary component, M_0, is obtained from W by iteratively attaching 4–dimensional 2–, then 3–, then 4–handles to W over the M_1 boundary component until that component has been "filled in."

Instructions for handle attachment come from defining a projection $f : M_1 \to \mathbb{R}^2$ that induces a stratification of $M_1 \subset \partial W$. We call a closed 3–dimensional stratum of M_1 a *block*, and we impose conditions on f to ensure that every block can be classified as one of the following:

face block: An attachment neighbourhood for a stratified 2–handle. Each face block is diffeomorphic to $S^1 \times G_n$, the product of the circle with an n-gon for some n.

edge block: A partial attachment neighbourhood for a stratified 3–handle. Each edge block is diffeomorphic to one of $D^2 \times [0,1]$, $A \times [0,1]$, or $P \times [0,1]$, where A is the annulus $S^1 \times [0,1]$ and P is a pair-of-pants surface. Attachment of stratified 2–handles over our face blocks "fill in" the annular boundary components of edge blocks, forming full attachment neighbourhoods for stratified 3–handles.

vertex block: A partial attachment neighbourhood for a stratified 4–handle. Each vertex block is homeomorphic to a (3,1)-handlebody of genus at most 3. When stratified 2– and 3–handles are attached to W, the genus of these handlebodies are reduced until the remaining boundary of W consists of M_0 union a collection of stratified 3–spheres. The 3–spheres are then coned away.

The remainder of this chapter is spent ensuring that such a stratification can be achieved for any smooth, orientable, closed 3–manifold, detailing how the stratification is induced, proving that the attachment of stratified 2– and 3–handles has the previously stated effects, and discussing the resulting 4–manifold.

3.1 Projections from 3–manifolds to $\mathbb{R}^2$

Our stratification of M is induced by a decomposition of the plane, itself induced by the singular values of a smooth map $M \to \mathbb{R}^2$. To prove that a stratification suitable for our construction exists for any smooth orientable 3–manifold, we show first that an inducing decomposition of $\mathbb{R}^2$ exists. To prove that such a decomposition of $\mathbb{R}^2$ exists, we present the properties of $f : M \to \mathbb{R}^2$ required to induce the decomposition, and argue why a map possessing such properties exists for any smooth, orientable 3–manifold.

Let $f : M \to \mathbb{R}^2$ be a smooth map, let df be the differential of f, and let $S_r(f)$ be the set of points in M such that df has rank r. Then we require that the following be true of f:

1. $S_0(f)$ is empty.

2. $S_1(f)$ consists of smooth non-intersecting curves. We call these the *fold curves* of f.

3. The set of points where $f|_{S_1(f)}$ has zero differential (these can appear in the image of $S_1(f)$ as cusps in the plane) is empty.

4. Let $\gamma_i, \gamma_j, \gamma_k \in S_1(f)$ be fold curves. Then $f(\gamma_{i,j,k})$ are submanifolds of $\mathbb{R}^2$ such that
   (a) $f(\gamma_i)$ and $f(\gamma_j)$ intersect transversely,
   (b) $f(\gamma_i) \cap f(\gamma_j) \cap f(\gamma_k)$ is empty (i.e. there are no triple-intersections), and
   (c) self-intersections of $f(\gamma_i)$ are transverse
5. If $p \in S_1(f)$ then there exist coordinates (u, z_1, z_2) centred at p and (x, y) centred at $f(p)$ such that f takes the form of either
   (a) $(x, y) = (u, \pm(z_1^2 + z_2^2))$, or
   (b) $(x, y) = (u, \pm(z_1^2 - z_2^2))$

   in a neighbourhood of p. If f takes the form of 5a then we further classify p as a *definite fold*, and if f takes the form of 5b then p is an *indefinite fold*.
6. The set of singular values of f in the plane is connected
7. $f(M)$ is bounded.

We call these the *stratification conditions* on f.

3.2 Decomposing $\mathbb{R}^2$

Let $f : M \to \mathbb{R}^2$ be a smooth map possessing the stratification conditions of Section 3.1 and let $X_f = f(S(f))$, the set of singular values of f. X_f is a connected collection of arcs in the plane that intersect only transversely.

We fit closed neighbourhoods (*sleeves*) around the singular values of f and classify these sleeves by the maximum codimension (with respect to $\mathbb{R}^2$) of singular values they contain. Because X_f consists of codimension 1 and codimension 2 singular values (i.e. arcs and arc-crossings respectively), we decompose $\mathbb{R}^2$ into face regions that contain no singular values, edge regions that contain only codimension 1 singular values, and vertex regions, each of which contain exactly 1 codimension 2 singular value. Figures 3.1-3.3 are used to illustrate the decomposition resulting from sleeve-fitting.

Figure 3.1: **Forming vertex regions.** Octagonal sleeves are fit around codimension 2 singular values to form vertex regions.

We begin by fitting octagonal sleeves around codimension 2 singular values as in Figure 3.1. Octagons are used here solely to simplify descriptions further down the line of proof.

Let x be a codimension 2 singular value. x is the result of an arc crossing, and a small neighbourhood around an arc crossing is divided into four regions of regular values. The octagon around x is fit so its edges alternate between being fully contained in a region of regular values and orthogonally intersecting one of the arcs of singular values that creates x. See Figure 3.1 for a model fitting.

The interiors of the octagons along with the octagonal boundaries form the vertex regions of this decomposition. The octagons are chosen to be small enough that no two vertex regions overlap and such that the octagonal edges that intersect arcs of codimension 1 singular values are all the same length.

Figure 3.2: **Forming edge regions.** Vertex region corners are connected to fit sleeves around arcs of codimension 1 singular values to form edge regions.

Let γ be an arc of codimension 1 singular values with endpoints a pair codimension 2 singular values. γ orthogonally intersects one edge from each of the octagonal vertex regions fit around its endpoints, and we use these edges to form the edge region associated to γ by connecting the endpoints of these edges to one another using a pair of arcs parallel to γ. See Figure 3.2 for a model fitting.

The closures of the interiors of the shapes formed by the arcs and octagon edges form the edge regions of this decomposition. The octagonal edge endpoints are also vertices of the octagons, and the formation of edge regions uses every octagonal vertex as the endpoint of exactly one arc.

Figure 3.3: **Forming face regions.** All remaining regions contain no singular values, and we take these to be the face regions.

Removing from $f(M)$ all vertex and edge regions, we are left with a collection of simply connected regions in the plane, each of which consists entirely of regular values. We take the closures of these to be the face regions of this decomposition. The boundary of each face region is an alternating collection of arcs from edge regions and octagonal edges from vertex regions. See Figure 3.3 for a model fitting.

3.3 Stratifying M

Decomposing $\mathbb{R}^2$ via the singular values of f also induces a stratification of M by considering the fibers of f above the of the decomposing regions. The interiors of face regions have preimage through f a disjoint collection of face blocks, the interiors of edge regions have preimage of edge blocks, and of vertex regions, vertex blocks.

To understand the structure of face, edge, and vertex blocks we investigate the preimages of regular and singular values of f in the plane.

Definition 3.3.1. Because M is closed, f is proper. Thus, for any point q in $f(M)$, a fiber of f above q (i.e. a connected component of $f^{-1}(q)$) is either a closed 1–manifold (i.e. S^1) or contains a critical point of f.

We define a *singular fiber* to be a fiber that contains a critical point of f, and a *regular fiber* to be a fiber consisting entirely of regular points.

The subsets used to stratify M are the fibers of f above the corners of the octagonal vertex regions, the edges of the vertex and edge regions, and the regions themselves. Because the corners of the vertex regions are regular values, their fibers are regular, hence a finite collection of disjointly embedded circles in M. We take these circles as the first collection of subsets that filter M, and assign to them the indices $(1, i)$ for $i = 1 \ldots N_1$, where N_1 is the number of circles. These circles are disjoint, so $(1, i) \nleq (1, j)$ for any i, j.

The arcs of the decomposition connect the vertices and either consist entirely of regular values or contain exactly one singular value. When an arc contains exactly one singular value, exactly one fiber above that value is a singular fiber, with the rest regular fibers. A decomposing arc is diffeomorphic to the unit interval and f is a smooth submersion between smooth manifolds, so a fiber above a decomposing arc is a surface with boundary the fibers above the arc endpoints. When the fiber is regular, the surface is diffeomorphic to an annulus $S^1 \times [0, 1]$. When the fiber is singular, the surface classification depends on the type of singularity. The following theorem and illustration (Theorem 3.15 from [2]) shows that the fiber containing the singularity either has the structure of a figure-of-eight (when the singularity is part of an indefinite fold) or is a single point (when the singularity is part of a definite fold), hence the singular fiber above the arc is diffeomorphic to either a 2–disc or a pair-of-pants.

Theorem 3.3.2 (Adapted Theorem 3.15 in Saeki [2])**.** Let $f : M \to N$ be a proper

C^∞ stable map of an orientable 3–manifold M into a surface N. Then, every singular fiber of f is equivalent to the disjoint union of one of the fibers as in Figure 3.4 and a finite number of copies of a fiber of the trivial unit circle bundle. Furthermore, no two fibers in the list are equivalent to each other even after taking the union with regular circle components.

Figure 3.4: **Singular fibers.** List of singular fibers of proper C^∞ stable maps of orientable 3–manifolds into surfaces. κ is the codimension of the singularity in the surface.

The surface fibers above the decomposing arcs are the second collection of subsets that filter M, and they are assigned the indices $(2, j)$ for $j = 1 \dots N_2$, where N_2 is the number of surfaces. The boundary circles of the surfaces are each subsets of the filtration, indexed by the $(1, i)$ indices, so $(1, i) \leq (2, j)$ if and only if $M_{(1,i)}$ is one of the boundary components of $M_{(2,j)}$. These surfaces intersect one another only when they share a boundary circle, so $(2, j) \not\leq (2, k)$ for any j, k.

There are three types of region in the decomposition: face, edge, and vertex. Regardless of the type of region, they are indexed in our filtration similarly to the edges. A fiber above a region is a 3–manifold with corners formed by the $(1, i)$- and $(2, j)$-level strata, and fibers are disjoint away from their boundaries. We therefore index fibers above regions with the indices $(3, k)$ for $k = 1 \dots N_3$ where N_3 is the total number of fibers above regions, put $(n, i) \leq (3, k)$ if and only if $M_{(n,i)}$ is contained in the boundary of $M_{(3,k)}$.

Recall that we call a closed 3–dimensional stratum of M a *block*. We now prove that the stratification conditions impose the desired structure on the blocks of M, as discussed in the introduction to this chapter.

Theorem 3.3.3. Let M be a closed, smooth, orientable 3–manifold, let $f : M \to \mathbb{R}^2$ be a map satisfying the *stratification conditions* of Section 3.1, suppose $\mathbb{R}^2$ has been decomposed as in Section 3.2 and M has been stratified as in this section. Then each closed strata $M_{(3,k)}$ is classified as either a *face*, *edge*, or *vertex block* depending on whether it is a fiber above a face, edge, or vertex region respectively, and a block has one of the following structures:

face block: Let B be a face block and let F be the face region that B is a fiber over. Then B is homeomorphic to $S^1 \times F$.

edge block: Let B be an edge block and let E be the edge region that B is a fiber over. Let A be the annulus $S^1 \times [0,1]$ and P the pair-of pants surface (i.e. D^2 minus a pair of disjoint open balls). If B is a regular fiber over E then B is homeomorphic to $S^1 \times E$, hence also homeomorphic to $A \times [0,1]$. Otherwise, B is a singular fiber over E and contains part of definite or indefinite fold. In this case we call B a *definite* or *indefinite edge block*. A definite edge block is homeomorphic to $D^2 \times [0,1]$ and an indefinite edge block is homeomorphic to $P \times [0,1]$.

vertex block: Let B be a vertex block and let V be the vertex region that B is a fiber over. If B is a regular fiber then it is homeomorphic to $S^1 \times V$, therefore homeomorphic to a $(3,1)$-handlebody of genus 1. Otherwise, we see from Figure 3.4 that the singular fiber above the codimension 2 singularity contained in V is either connected or disconnected. If the singular fiber is disconnected then there are a pair of disjoint vertex blocks that each contain one of the singular fibers, hence part of a definite or indefinite fold. We therefore classify these blocks as *definite* or *indefinite vertex blocks*. If the singular fiber is connected, then the block containing it is an *interactive vertex block*. A definite(resp. indefinite) vertex block extends and connects definite(resp. indefinite) edge blocks, and is homeomorphic to a $(3,1)$-handlebody of genus 0(resp. 2). An interactive vertex block is homeomorphic to a $(3,1)$-handlebody of genus 3.

Remark 3.3.4. The structures of the blocks described in Theorem 3.3.3 are roughly disc bundles over a representative fiber for the given region. For a block that is a regular fiber, the representative is a circle. For a definite or indefinite block, the representative is the singular fiber containing a definite or indefinite fold, and for an interactive block the representative fiber is the singular fiber above the codimension 2 singular value.

Proof of Theorem 3.3.3. We split the proof into three parts. The first part proves that if a block B is a regular fiber over the region R then B is homeomorphic to $S^1 \times R$. In the second part, we prove that definite and indefinite blocks are homeomorphic to $D^2 \times [0,1]$ or $P \times [0,1]$ respectively. In the final part we discuss interactive vertex blocks, and show that they are homeomorphic to $(3,1)$-handlebodies of genus 3. Figures illustrate the block structures.

Part 1: Let B be a block over the region R, and suppose B consists entirely of regular fibers over R. Then $(B, R, f|_B, S^1)$ has the structure of a circle bundle over R. A fiber bundle over a contractible space is trivial, so B is homeomorphic to

$S^1 \times R$. Furthermore, this homeomorphism is stratified by ensuring the strata of B are mapped to the strata of $S^1 \times R$, where the stratification of $S^1 \times R$ is defined by the manifold with corners structure induced by the product topology. See Figure 3.5.

Figure 3.5: **Regular blocks.** Three types of regular blocks. These are found as regular fibers over face, edge, and vertex regions.

Part 2: Let B be a definite or indefinite block over the region R. R is a subset of the plane homeomorphic to D^2 with an arc $\gamma_s \subset X_f$ of singular values running from one of its edges to another. Let γ_t be a second simple arc that crosses γ_s transversely, and consider the cross-sectional surface obtained by $f^{-1}(\gamma_t)$. Figure 3.6 illustrates the possible surfaces containing the singular fiber over $x = \gamma_s \cap \gamma_t$.

Figure 3.6: **Surfaces over codimension 1 singularities.** γ_s is an arc of singular values and γ_t is an arc with endpoints $\partial\gamma_t = \{p, q\}$ that intersects γ_s transversely at $x = \gamma_s \cap \gamma_t$. The three surfaces shown are the three possible cross-sectional surfaces that can project through f over γ_t.

This cross section is general, so we fit a tubular neighbourhood $\nu(\gamma_s)$ about γ_s in R to obtain a bundle structure for $f^{-1}(\nu(\gamma_s))$ whose fiber is one of the cross-sectional surfaces (a disc or a pair-of-pants) and whose base is the arc γ_s, i.e. an interval. The interval is contractible, so $f^{-1}(\nu(\gamma_s))$ is homeomorphic to $\Sigma \times [0,1]$ for Σ a disc or a pair-of-pants surface. Away from $\nu(\gamma_s)$, R consists entirely of regular values so we obtain solid tori (cf. Part 1) that extend the $\Sigma \times [0,1]$ structure as seen in Figure 3.7.

Figure 3.7: **Definite and indefinite blocks.** The blocks containing sections of definite and indefinite folds that project over codimension 1 singular values. These are found as singular fibers over edge and vertex regions.

As with Part 1, the homeomorphism described is stratified by ensuring the strata of B are mapped to the strata of $\Sigma \times [0,1]$, where the stratification of $\Sigma \times [0,1]$ is defined by the manifold with corners structure induced by the product topology.

Part 3:

Let B be an interactive block over the region R. Interactive blocks occur over octagonal vertex regions where the singular fiber above the region's codimension 2 singularity is connected, so we investigate these fibers. The codimension 2 singular value lies at the intersection of a pair of arcs of codimension 1 singular values. Call the arcs γ_1 and γ_2, let $x = \gamma_1 \cap \gamma_2$, and denote the interactive singular fiber over x by $B_x = B \cap f^{-1}(x) = \{b \in B \mid f(b) = x\}$. Our method of investigation begins by examining the possible resolutions of B_x and combining those resolutions to form a genus 3 surface.

Figure 3.8 demonstrates resolutions of the singular points of B_x when B_x has the first interactive singular fiber form presented in Figure 3.4. We first note that all of the displayed fibers have inherited an orientation from M. This forces fiber resolution to be unambiguous, and allows us to identify fibers when forming the surface shown in Figure 3.9.

Figure 3.8: **Resolutions of the singular points in the first interactive fiber.** B_x and its possible resolutions over nearby codimension 1 singular values and regular values. The fibers inherit orientation from M, and this illustration is presented without loss of generality.

We form the surface shown in Figure 3.9 by gluing together the surfaces that project over simple arcs transversing over the codimension 1 singular values. Gluing is performed over the boundary circles of these surfaces, and is prescribed by the resolutions in Figure 3.8. In the first case, we obtain four pair-of-pants surfaces and two cylinders, and these glue together to form a genus 3 surface.

Figure 3.9: **Surface Σ near the first interactive fiber that projects over $\partial\nu(x)$.** The surface and B_x are presented as embedded in S^3, where $H(B_x)$ is the genus-3 (3,1)–handlebody on the 'outside' of Σ in S^3.

The surface Σ in Figure 3.9 is the boundary of $H(B_x)$, a regular neighbourhood of B_x in M, i.e. a genus-3 (3,1)–handlebody inside of M. $H(B_x)$ projects through f over $\bar{\nu}(x)$, a closed tubular neighbourhood of x, and Σ projects over $\partial(\nu(x))$.

Figure 3.9 presents Σ and B_x as objects embedded in S^3, where Σ bounds genus-3 (3-1)–handlebodies on both sides. We take the 'outside' component of $S^3 \setminus \Sigma$ (i.e. the component containing B_x) to be our model $H(B_x)$.

Outside of $\nu(x)$ we use the investigations from Parts 1 and 2 of this proof. The rest of R, $f^{-1}(R \setminus \nu(x))$, has the structure of a Σ-bundle over the interval, and the bundle extends $H(B_x)$ to the boundary of R, preserving the structure as a genus-3 (3,1)–handlebody. We conclude that B is homeomorphic to a genus-3 (3,1)–handlebody.

An identical argument is made when B_x has the second interactive singular fiber form, using Figures 3.10 and 3.11 in place of Figures 3.8 and 3.9 respectively.

Figure 3.10: **Resolutions of the singular points in the second interactive fiber.** B_x and its possible resolutions over nearby codimension 1 singular values and regular values. The fibers inherit orientation from M, and this illustration is presented without loss of generality.

Figure 3.11: **Surface Σ near the second interactive fiber that projects over $\partial\nu(x)$.** The surface and B_x are presented as embedded in S^3, where $H(B_x)$ is the genus-3 (3,1)–handlebody on the 'outside' of Σ in S^3.

Figure 3.12 displays both possible interactive block structures, highlighting their boundaries. The figure explains that the blocks are the handlebodies on the 'outside' of the illustrated surfaces. This is specifically to aid visualization of the effects of 2– and 3–handle attachment in the next two sections, as the result of these attachments will fill the genus-3 (3,1)–handlebody on the 'inside' of the illustrated surface.

Figure 3.12: **Possible interactive block structures embedded in S^3.** Interactive blocks with indicated boundary stratification induced by $f^{-1}\partial R$. Blocks are embedded in S^3, outside of the illustrated stratified boundary surfaces.

$\square$

3.4 Attach 2–handles

We now investigate the consequences of stratified 2–handle attachment over the face blocks of the stratified M_1 boundary component of $W = M \times [0, 1]$. This investigation is performed by comparing ∂W to $\partial W'$, where

$$W' = W \cup \{H^2_\alpha\}_{\alpha\in A}/\sim,$$

the 4–manifold obtained by attaching stratified 2-handles to W.

Attaching these stratified 2–handles to W alters the boundary of W via surgery on M_1 that is equivalent to replacing the interiors of each face block with a solid torus whose meridians are longitudes of the replaced face block.

3.5 Attach 3–handles

The annular boundary components of each edge block are "filled-in" by cylinders found between meridians of these newly introduced solid tori, in each case forming a stratified $S^2 \times [0,1]$. These stratified $S^2 \times [0,1]$ are taken as attachment neighbourhoods for stratified 3–handles, forming W''.

Finally, we compare the boundaries of W' and W'' to show that $\partial W''$ is the disjoint union of M_0 with a collection of stratified 3–spheres. The 3–sphere boundary components are then coned away.

Chapter 4

Algorithm for constructing a triangulated 4–manifold with prescribed 3–manifold boundary

The steps to construct a triangulated 4–manifold with prescribed 3–manifold boundary broadly follow the steps to construct a 4–manifold with prescribed smooth, orientable 3–manifold boundary. Let N be a 3–manifold triangulation. Then the steps of construction are:

Step 1: Define a projection $f : N \to \mathbb{R}^2$.

Step 2: Induce a subdivision of N from f. The result is a 3–manifold triangulation M that is equivalent to N in the sense of triangulations.

Step 3: Let $W = M \times [0, 1]$ be a 4–manifold with boundary components $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$.

Step 4: Attach 4–dimensional 2–handles to W over its M_1 boundary as prescribed by the subdivision of M from f. Call the result W' and call the boundary of W' different from M_0 by M_1'.

Step 5: Attach 4–dimensional 3–handles to W' over M_1' as prescribed by the subdivision induced by f and the surgery induced by 2–handle attachment. Call the result W''.

Step 6: The boundary of W'' consists of M_0 and a collection of copies of S^3 that we now cone off. The result is a 4–manifold whose boundary is exactly M_0.

Each of these steps is made algorithmic, and these algorithms are chained in series to form a single algorithm. The result has input a closed, orientable 3–manifold triangulation M and output a 4–manifold triangulation W whose sole boundary component is a triangulated 3–manifold that is equivalent to M in the sense of triangulations. In this case, we find that ∂W is a subdivision of M, and this subdivision is the subdivision induced by the projection f in Step 1.

Throughout this chapter N refers to the initial input 3–manifold triangulation, f to the projection defined in Section 4.1, M is the subdivision of N induced by f, W is the 4–manifold $M \times [0,1]$, W' is the result of attaching 2–handles to W, and W'' is the result of attaching 3–handles to W'.

4.1 Define projection

The projection's utility is in defining a subdivision of the initial closed, orientable 3–manifold triangulation N such that attaching regions for triangulated 2– and 3–handles can be found. This is done before forming the base 4–manifold so that the subdivided triangulation is used in the algorithm that provides W.

Our subdivision is obtained by imposing four conditions on $f : N \to \mathbb{R}^2$:

1. f maps vertices to the circle, i.e. for each vertex $v \in N^0$, $f(v)$ lies on the unit circle in $\mathbb{R}^2$.

2. The images of vertices are distinct, i.e. for every pair of vertices $u, v \in N^0$, $f(u) \neq f(v)$.

3. f is linear on each simplex of N and piecewise-linear on N, i.e. if $x \in \sigma$ is a point in the simplex σ with vertices v_i, then $x = \sum_i a_i v_i$ with $\sum_i a_i = 1$ and $f(x) = \sum_i a_i f(v_i)$.

4. Edge intersections are distinct, i.e. for every triple of edges $e_1, e_2, e_3 \in N^1$ that share no vertices, $f(e_1) \cap f(e_2) \neq f(e_2) \cap f(e_3)$.

Conditions 1 and 2 ensure that every simplex of N is mapped to the plane in standard position (i.e. the images of the vertices in the plane form a convex set). This, along with conditions 3 and 4, allows us to use concepts and language from normal surface theory to describe the subdivision of N in the next section. We call these four conditions the *subdivision conditions* on f.

All conditions are satisfied by fixing an odd integer k greater than or equal to the number of vertices in N, injecting the vertices of N to the k^{th} complex roots of unity in the plane, then extending linearly over the skeletons of N. The first three conditions are clearly satisfied by this procedure, and the last is satisfied by applying the results in [1].

The algorithm presented in this section takes as input the triangulated 3–manifold N and produces a projection $f : N \to \mathbb{R}^2$ satisfying the subdivision conditions.

4.2 Induce subdivision

The goal of subdividing N is to create and identify analogues to the face, edge, and vertex blocks of Chapter 3 where we may iteratively attach 2–, 3–, then 4– handles. We use a similar technique to that found in Chapter 3, first decomposing $\mathbb{R}^2$ with the projection, then examining preimages to define our subdivision.

Decomposition of $\mathbb{R}^2$ is done through $f(N_1)$. A point in $f(N_1)$ is the image of either a vertex, exactly one edge, or exactly two edges (i.e. is an edge crossing), so we refer to these as the *vertices*, *edges*, and *crossings* of the decomposition. Because $f(N) \setminus f(N_1)$ is a disjoint collection of simply connected regions, we call the connected components of $f(N) \setminus f(N_1)$ the *faces* of the decomposition.

We construct our subdivision of N using the decomposition component preimages. The preimage of a face component defines substructures analogous to face blocks, of edge components to edge blocks, and vertices and crossings to vertex blocks. Inside of an individual tetrahedron of N, edge and crossing preimages are well-defined, supplying a natural subdivision of N into a cell complex. Then, the subdivision of a cell complex into a triangulation is well-defined.

The algorithm presented in this section takes as input a closed, orientable 3–manifold triangulation N and a projection $f : N \to \mathbb{R}^2$ and produces as output a closed, orientable 3–manifold triangulation M that is a subdivision of N. Furthermore,

1. the 3–cells of M are partitioned into subsets that serve the same purpose as the face blocks of Chapter 3: attaching regions for 2–handles,

2. the 2–cells of M are partitioned into subsets that are either interior to face blocks, or subsets that serve the same purpose as the edge blocks of Chapter 3, i.e. buffers between face blocks.

4.3 Form base 4–manifold

The algorithm presented in this chapter takes as input a closed, orientable 3–manifold triangulation M and produces as output a 4–manifold triangulation W whose boundary is the disjoint union of two copies of M. We do this by explicitly triangulating the cell complex $W \times [0, 1]$.

4.4 Attach 2–handles

At this point in our procedure we have a 4–manifold W with triangulated boundary components M_0 and M_1. We aim to attach handles to W over the boundary of W away from M_0 until the boundary of W is exactly M_0. The first step is to attach 2–handles to W over the closed solid torus triangulations that partition M_1.

The algorithm of this section takes as input a closed solid torus triangulation T with a collection of triangulated parallel longitudes Γ in its boundary (each γ_i in Γ is an explicit 0-framing for the 2–handle attachment) and produces as output a D^4 triangulation whose S^3 boundary triangulation has genus 1 Heegard splitting over ∂T, and γ_i bounds a triangulated disk in $\partial D^4 \setminus T$ for each i. Such a D^4 triangulation is taken as a 4–dimensional 2–handle and attached to W over T. We then attach such a 2–handle over each closed solid torus in our partition of M_0.

4.5 Attach 3–handles

We now have a triangulated 4–manifold with boundary components M_0 and the result of surgery on M_1 induced by 2–handle attachment. This surgery had the following effect on M_1:

1. Each solid torus in the partition of M_1 induced by the projection from Section 4.1 is associated with a triangulated D^4 in Section 4.4.

2. For each (torus, 2–handle) pair (T, H^2), the boundary of H^2 contains T and $T^* = H^2 \setminus \texttt{int}(T)$ is another solid torus whose boundary triangulation is identical to that of T.

3. Thus the effect of surgery on M_1 is of replacing each T with T^* over their shared boundary.

Chapter 5

Conclusion

Bibliography

[1] Bjorn Poonen and Michael Rubinstein. The number of intersection points made by the diagonals of a regular polygon. *SIAM Journal on Discrete Mathematics*, (1) 11:135–156, 1998.

[2] Osamu Saeki. *Topology of Singular Fibers of Differentiable Maps.* Lecture Notes in Mathematics 1854. Springer, 2004.

[3] Shmuel Weinberger. *The topological classification of stratified spaces.* Chicago, 1994.