3–manifolds Algorithmically Bound 4–manifolds

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Abstract

The question "which closed 3–manifolds bound 4–manifolds?" has been answered since the early 1950's. The answer is "all of them." There are many methods of proof of this fact, and the 2008 paper "3–manifolds efficiently bound 4–manifolds" by Dylan Thurston and Francesco Costantino gives an overview for a process that produces a bound on the number of 4–simplices needed to triangulate a 4–manifold for a given input triangulated 3-manifold. We take the groundwork laid by Costantino and Thurston and provide algorithms to generate an explicit triangulation of a 4–manifold that bounds the given input triangulated 3–manifold.

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Chapter 1

Introduction

Every 3–manifold is the boundary of some 4–manifold. The earliest proofs of this fact are due to René Thom in the early 1950's [19]. Since then the problem has been approached from multiple avenues. A series of papers in the 1960's by Hirsch [5], Rokhlin [13], and Wall [21] built up the theory of immersing and then embedding 3–manifolds in $\mathbb{R}^5$. The embedded 3–manifold then bounds a "Seifert 4–manifold" as with knots bounding Seifert surfaces in $\mathbb{R}^3$. Also in the 60's, Lickorish showed that all orientable 3–manifolds may be represented as surgery on a 3–sphere, and thus bound 4–manifolds. A later result by Rourke [15] demonstrates something similar, this time by investigating Heegaard diagrams of the given 3–manifolds.

Historical arguments prove 3–manifolds bound 4–manifolds, but we normally describe our 3– and 4–manifolds using triangulations. We would like a process that takes us from a triangulated 3–manifold to a triangulated 4–manifold. The work done by Turaev in [20] paved the way for an argument in [2] that includes a process used to estimate a bound on the number of 4–dimensional simplices needed to triangulate a 4–manifold whose boundary is a given 3–manifold. This process involved the definition of structures called *shadows* that feature prominently in the work of Thurston and Costantino at the time. A shadow is, essentially, a 2–dimensional CW–complex with some extra structure. Most importantly, a shadow puts restrictions on how many 2–cells may be incident to a single vertex or attached over a single edge. The extra structure of the shadow is not needed in the scope of this work, so we define only the concept of the 2–dimensional CW–complex.

What follows is a brief overview of what to expect, only slightly more descriptive than our table of contents.

- Chapter 2 consists entirely of geometric topology fundamentals. Most of the material is used only to support the main topics of chapter 2, which are handle attachment and the definition of triangulations. A reader that is well read in those topics is encouraged to skim Chapter 2 in order to familiarize themselves with the notation used.

- Chapter 3 is a build up to an argument that 3–manifolds bound 4–manifolds in the smooth case. We make this argument in a constructive way so that it can be turned into an algorithm in the piecewise–linear case. The main focus is Section 3.3, and the first two sections cover definitions and the techniques of proof in the case of orientable 2–manifolds bounding 3–manifolds that we draw from in Section 3.3.

- Chapter 4 is our collection of algorithms. We initialize some key definitions and present some minor results before diving directly into the algorithms.

Chapter 2

Manifolds

Our first task is to build the machinery necessary to describe manifolds. We begin with a quick, formal definition, and build some basic properties. Next, we talk about a method of building and augmenting manifolds using handles. We end by defining triangulations: a combinatorial description of a manifold that allows us to cleanly describe our algorithms.

This chapter is intended as a review of common tools in geometric topology so, in most places, full proof will be omitted. A knowledge of the fundamental properties and definitions of smooth maps between real vector spaces is assumed. For full details on the fundamentals of smooth manifold theory, refer to the texts [4], [6], [8], or [10]. For details on manifolds with corners, refer to Chapter 3 of [18].

2.1 Fundamentals

At its simplest, most colloquial definition, an n–dimensional manifold is a space that locally looks like an n–dimensional real, half, or cornered space. We use *charts* and *atlases* to make explicit what is meant by "looks like."

Definition 2.1.1 (Real, Half, and Cornered Spaces)**.** Let $n > 0$ and $0 \leq k \leq n$ be integers. The *n–dimensional real space* is denoted $\mathbb{R}^n$ and is defined by

$$\mathbb{R}^n = \{(x_0, x_1, \dots x_{n-1}) : x_i \in \mathbb{R} \text{ for each } i\},$$

and the *closed half space*, denoted $\mathbb{H}$, is defined by

$$\mathbb{H} = \{x \in \mathbb{R} : x \geq 0\},$$

and is a topological subspace of $\mathbb{R}$ endowed with the subspace topology. We use the notations $\texttt{int}(\mathbb{H})$ and $\partial\mathbb{H}$ to denote the topological interior and boundary of $\mathbb{H}$ as subsets of $\mathbb{R}$ which are $\{0\}$ and $\{x \in \mathbb{R} : x > 0\}$ respectively.

We take products of $\mathbb{H}^k$ with $\mathbb{R}^{n-k}$ to form *n–dimensional real space with a k–corner.* Explicitly, this space is

$$\mathbb{H}^k \times \mathbb{R}^{n-k} = \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_i \geq \text{ for } i = 0, \ldots, k-1\},$$

with the origin $\mathbf{0} = (0, \ldots, 0)$ called the *model k-corner point.*

We use the notations $\texttt{int}(\mathbb{H}^k \times \mathbb{R}^{n-k})$ and $\partial(\mathbb{H}^k \times \mathbb{R}^{n-k})$ to denote the topological interior and boundary of $\mathbb{H}^k \times \mathbb{R}^{n-k}$ as subsets of $\mathbb{R}^n$ which, when $n > 0$, are

$$\begin{aligned}
\texttt{int}(\mathbb{H}^k \times \mathbb{R}^{n-1}) &= \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_i > 0 \text{ for } i = 0, \ldots, k-1\}, \\
\partial(\mathbb{H}^k \times \mathbb{R}^{n-1}) &= \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n : x_i = 0 \text{ for some } i \in \{0, \ldots, k-1\}\}.
\end{aligned}$$

When we allow $n = 0$, $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\texttt{int}(\mathbb{H}^0) = \mathbb{R}^0$ and $\partial\mathbb{H}^0 = \emptyset$. The *dimension* of $\mathbb{H}^k \times \mathbb{R}^{n-k}$ is n.

Definition 2.1.2 (Coordinates)**.** Let M be a second-countable Hausdorff space. The pair (U, φ) where U is an open subset of M and φ is a homeomorphism from U onto an open set of $\mathbb{H}^k \times \mathbb{R}^{n-k}$ is called a *chart.* If p is a point in M and (U, φ) is a chart such that $p \in U$, then (U, φ) is a *chart about p.* The map φ is a *coordinate system* on U and its inverse φ^{-1} is a *parameterization* of U. Writing φ as

$$\varphi(u) = (\xi_0(u), \ldots, \xi_{n-1}(u)),$$

the functions ξ_i are *coordinate functions*

Definition 2.1.3 (Atlas)**.** Fix a nonnegative integer n. Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in A\}$ be a collection of charts of M such that the codomain for each chart is of the fixed dimension n. If $\bigcup_A U_\alpha$ contains M, then $\mathcal{A}$ is an *atlas* for M.

First , recall that a map $f : \mathbb{H}^{k_\alpha} \times \mathbb{R}^{n-k_\alpha} \to \mathbb{H}^{k_\beta} \times \mathbb{R}^{n-k_\beta}$ is *smooth* if it can be extended to a smooth map $\tilde{f} : \mathbb{R}^n \to \mathbb{R}^n$, i.e. if it can be extended to a map $\tilde{f}$ that has continuous partial derivatives of all orders.

The homeomorphisms $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ are *transition maps* of $\mathcal{A}$. We say that $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are *smoothly compatible* if either

$U_\alpha \cap U_\beta$ is empty or the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ are smooth as maps $\mathbb{H}^{k_\beta} \times \mathbb{R}^{n-k_\beta} \to \mathbb{H}^{k_\alpha} \times \mathbb{R}^{n-k_\alpha}$ and $\mathbb{H}^{k_\alpha} \times \mathbb{R}^{n-k_\alpha} \to \mathbb{H}^{k_\beta} \times \mathbb{R}^{n-k_\beta}$ respectively.

If every pair of charts in an atlas is smoothly compatible then that atlas is a *smooth atlas*. Two smooth atlases are equivalent if their union is a smooth atlas.

Definition 2.1.4 (Manifold)**.** Let M be a second-countable Hausdorff topological space, $\mathcal{A}$ an atlas for M, and n the dimension of the codomain of each chart in $\mathcal{A}$. If $\mathcal{A}$ is a smooth atlas, then the pair $(M, \mathcal{A})$ is a *smooth n–manifold*, *n–manifold*, or just *manifold*. If $\mathcal{A}$ is a maximal smooth atlas then we call it a *smooth structure* on M.

It eases the notational burden to assume that our manifolds are always equipped with a smooth structure. This assumption allows us to omit writing an atlas when talking about a manifold. We also assume that our manifolds are connected.

We classify the points of a manifold by the charts that exist around them.

Definition 2.1.5. Let M be a smooth n–manifold, $p \in M$, and (U, φ) a chart about p. Consider $\varphi(p) = (x_0, \ldots, x_{n-1}) \in \mathbb{H}^k \times \mathbb{R}^{n-k}$. Then p is classified as a *j–corner point*, where j is the number of entries $x_0, \ldots, x_{k-1}$ that are 0. For p a j–corner, we define the *depth* of p by $\operatorname{depth}(p) = j$. By smooth compatibility of our charts, examining the image of p through exactly one chart is sufficient this classification.

To coincide with the topological notion of interior and boundary, we call a 0–corner an *interior point*, and a j–corner is a *boundary point* for $j > 0$. For $j > 0$, we use $\partial_j M$ to denote the set of points $\{p \in M : p \text{ is a } j\text{–corner}\}$, and call $\partial_j M$ the *j–boundary* of M. The union

$$\bigcup_{j>1} \partial_j M$$

is denoted ∂M and is called the *boundary* of M.

This classification let's us define the subclass of smooth manifolds with corners that is most useful to us.

Definition 2.1.6. Let M be a smooth n–manifold, and $\partial_1(M)$ the 1–boundary of M. A *face* of M is the closure of a connected component of $\partial_1 M$, and we denote the set of faces of M by $\partial_f M$. We say that M is a *manifold with faces* if, for every j–corner point $p \in M$, p belongs to exactly j different faces of M, There are several advantages to considering the subclass of manifolds with faces. First, if M is a manifold with faces then any connected face or disjoint union of connected faces of M is, itself, a manifold with faces. Second, if M is a manifold with faces then so is $M \times \mathbb{H}$.

Definition 2.1.7. If M is a compact smooth manifold and every point of M is interior, then M is *closed* as a manifold.

There is a potential conflict of this definition with the concept of topological closure. In the rare case that we want to say that a manifold is topologically closed, we will specify that the type of closure is topological. Otherwise, a "closed manifold" will refer to a manifold with empty boundary.

Example 2.1.8. The n–sphere, denoted S^n and defined as a subset of $\mathbb{R}^{n+1}$, is

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}.$$

The $(n+1)$–dimensional topologically open ball, or $(n+1)$–ball, defined as a subset of $\mathbb{R}^{n+1}$, is

$$B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| < 1\}.$$

We use $\|\cdot\|$ to denote the Euclidean norm defined for $x \in \mathbb{R}^{n+1}$ by

$$\|x\| = \left(\sum x_i^2\right)^{1/2}.$$

The space D^{n+1}, the topological closure of B^{n+1}, is the closed $(n+1)$–disc. Both B^{n+1} and D^{n+1} are $(n+1)$–manifolds. The disc D^{n+1} is a manifold whose boundary is the n–sphere, a closed n–manifold. Note that the closed 1–disc is the unit interval, which we denote by $\mathbb{I} = [\,0, 1\,]$.

It is often more efficient, in terms of notation, to consider S^1, D^2, S^3, and D^4 as submanifolds of the complex plane $\mathbb{C}$ or $\mathbb{C}^2$. We do exactly this in Theorem 2.4.24 while defining maps $S^1 \times D^2 \to S^1 \times D^2$.

Example 2.1.9. The unit square $\mathbb{I} \times \mathbb{I} = \mathbb{I}^2$ is a 2–manifold with faces that is homeomorphic to D^2. The 1–boundary is

$$\partial_1(\mathbb{I} \times \mathbb{I}) = \Big((0,1) \times \{0\}\Big) \cup \Big((0,1) \times \{1\}\Big) \cup \Big(\{0\} \times (0,1)\Big) \cup \Big(\{1\} \times (0,1)\Big),$$

which yields connected faces of

$$\mathbb{I} \times \{0\}, \quad \mathbb{I} \times \{1\}, \quad \{0\} \times \mathbb{I}, \text{ and } \quad \{1\} \times \mathbb{I}.$$

The 2–corners of $\mathbb{I} \times \mathbb{I}$ are the points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$.

The product $\mathbb{I} \times \mathbb{I} \times S^1$ is also a manifold with faces. It's faces are exactly the products of the faces of $\mathbb{I} \times \mathbb{I}$ with S^1. We also obtain the 2–corners of $\mathbb{I} \times \mathbb{I} \times S^1$ by taking the product of the 2–corners of $\mathbb{I} \times \mathbb{I}$ with S^1.

These results are due to the fact that S^1 is a closed manifold. For contrast, consider $\mathbb{I} \times \mathbb{I} \times \mathbb{I} = \mathbb{I}^3$, the unit cube:

1. The set of 1–corners of $\mathbb{I}^3$ is the union of $\partial_0(\mathbb{I} \times \mathbb{I}) \times \partial_1\mathbb{I}$ with $\partial_1(\mathbb{I} \times \mathbb{I}) \times \partial_0\mathbb{I}$.

2. The set of 2–corners of $\mathbb{I}^3$ is the union of $\partial_0(\mathbb{I} \times \mathbb{I}) \times \partial_2\mathbb{I}$ with $\partial_1(\mathbb{I} \times \mathbb{I}) \times \partial_1\mathbb{I}$ and $\partial_2(\mathbb{I} \times \mathbb{I}) \times \partial_0\mathbb{I}$.

3. The set of 3–corners of $\mathbb{I}^3$ is the union of $\partial_1(\mathbb{I} \times \mathbb{I}) \times \partial_2\mathbb{I}$ with $\partial_2(\mathbb{I} \times \mathbb{I}) \times \partial_1\mathbb{I}$.

This pattern generalizes. If X and M are manifolds with faces then the k–corners of $X \times M$ are found as

$$\partial_k(X \times M) = \bigcup_{\substack{i+j=k \\ i,j \geq 0}} \partial_i X \times \partial_j M,$$

and the set of faces of $X \times M$ is the union $(\partial_0 X \times \partial_f M) \cup (\partial_f X \times \partial_0 M)$.

At times it is necessary to form a *partition of unity* on a manifold. To do this, we first require a suitable atlas.

Definition 2.1.10. Let M be a manifold and $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ an atlas on M. Then $\mathcal{A}$ is *adequate* if it is locally finite, the image of U_α through φ_α is the whole of $H^k \times \mathbb{R}^{n-k}$ for the appropriate value of k, and the union

$$\bigcup_\alpha \varphi_\alpha^{-1}(B^n)$$

covers M.

Theorem 2.1.11 (Theorem I.2.2 in [8])**.** Let $\mathcal{V} = \{V_\beta\}$ be a covering of the manifold M. Then there is an adequate atlas $\{U_\alpha, \varphi_\alpha\}$ such that $\{U_\alpha\}$ is a refinement of $\mathcal{V}$

Definition 2.1.12. Let M be an n–manifold, let $\{U_\alpha, \varphi_\alpha\}$ be an adequate atlas on M, and let λ be a smooth nonnegative function on $\mathbb{R}^n$ that takes the value of 1 on the disc of radius 1 and is zero outside of the disc of radius 2. Define the smooth functions λ_α on M by $\lambda_\alpha = \lambda \circ \varphi_\alpha$ on U_α and $\lambda_\alpha = 0$ outside of U_α. Then $\{\lambda_\alpha\}$ is *the family of bump functions* associated to $\{U_\alpha\}$, and the family $\{\mu_\alpha\}$ of functions defined by

$$\mu_\alpha(p) = \frac{\lambda_\alpha(p)}{\sum_\alpha \lambda_\alpha(p)}$$

is the *partition of unity* associated to $\{U_\alpha, \varphi_\alpha\}$.

We conclude this section by abbreviating our terminology. An *n–manifold* refers to a *smooth n–dimensional manifold with faces*. In other words, "manifold" refers to the broadest definition of the term. We may get more specific by saying *M is a closed manifold*, and in this case every point of M is a 0–corner, and M is compact.

2.2 Smooth Maps

Definition 2.2.1 (Smooth Map)**.** Let $(X, \{U_\alpha, \varphi_\alpha\})$ and $(M, \{V_\beta, \psi_\beta\})$ be smooth n– and k–manifolds, with $\{U_\alpha, \varphi_\alpha\}$ and $\{V_\beta, \psi_\beta\}$ maximal smooth atlases. Let $f : X \to M$ be a map. If, for any α and β, the composition $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is smooth as a map

$$\psi_\beta \circ f \circ \varphi_\alpha^{-1} : \mathbb{R}^n \supset \varphi_\alpha(U_\alpha) \to \varphi_\beta(V_\beta) \subset \mathbb{R}^k,$$

then we say f is *smooth* as a map between manifolds. The space of smooth maps $X \to M$ is denoted $C^\infty(X, M)$.

If f is smooth and a well-defined f^{-1} exists and is smooth, then f is called a *diffeomorphism* between manifolds. We say that manifolds are *diffeomorphic* if there exists a diffeomorphism between them.

Proposition 2.2.2. Diffeomorphism is an equivalence relation on the space of smooth manifolds.

Manifolds are defined by their local homogeneity, and a tangent space is a precise description of that homogeneity near a point. There are many ways to define tangent spaces, and we do so using derivations as in [10]. This approach is used for two reasons. First, it causes the vector space structure of the tangent space to follow directly from the definition. Second, the definition is applicable to manifolds with faces without any fiddling around with details, and the tangent spaces obtained are still vector spaces of an appropriate dimension.

Definition 2.2.3 (Derivations on $\mathbb{R}^n$)**.** Let x be a point in $\mathbb{R}^n$, and denote the space of smooth functions $\mathbb{R}^n \to \mathbb{R}$ be $C^\infty(\mathbb{R}^n)$. A linear map $A : C^\infty(\mathbb{R}^n) \to \mathbb{R}$ is called a *derivation at* x if it satisfies the product rule

$$A(\varphi\,\psi) = \varphi(x)A\psi + \psi(x)A\varphi$$

for any $\varphi, \psi \in C^\infty(\mathbb{R}^n)$. Denote the set of all derivations at x of $C^\infty(\mathbb{R}^n)$ by $T_x\mathbb{R}^n$. The derivations $T_x\mathbb{R}^n$ form a vector space under the operations

$$\begin{aligned}(A+B)\varphi &= A\varphi + B\varphi,\\ (cA)\varphi &= c(A\varphi).\end{aligned}$$

Proposition 2.2.4 (Corollary 3.3 in [10])**.** For any x in $\mathbb{R}^n$, the n derivations

$$\left.\frac{\partial}{\partial x_1}\right|_x, \ldots, \left.\frac{\partial}{\partial x_n}\right|_x$$

defined by

$$\left.\frac{\partial}{\partial x_i}\right|_x \varphi = \frac{\partial \varphi}{\partial x_i}(x)$$

form a basis for $T_x\mathbb{R}^n$, and $T_x\mathbb{R}^n$ is therefore an n–dimensional vector space.

Our definition of derivation extends to the spaces $\mathbb{H}^k \times \mathbb{R}^{n-k}$, and Proposition 2.2.4 extends as well after we establish the following lemma.

Lemma 2.2.5 (Extension Lemma, 2.7 in [10])**.** Let M be a smooth manifold, let $V \subset M$ be a closed subset, and let $f : V \to \mathbb{R}^d$ be a smooth function. Let U be any open set containing V. Then there exists a smooth extension $\tilde{f} : M \to \mathbb{R}^d$ such that $\left.\tilde{f}\right|_V = f$ and $\operatorname{supp}\tilde{f} \subset U$.

Definition 2.2.6 (Derivations on $\mathbb{H}^n$)**.** Let x be a point in $\mathbb{H}^n$. A linear map $A : C^\infty(\mathbb{H}^n) \to \mathbb{R}$ is called a *derivation at* x if it satisfies the product rule

$$A(\varphi\,\psi) = \varphi(x)A\psi + \psi(x)A\varphi$$

for any $\varphi, \psi \in C^\infty(\mathbb{H}^n)$. Denote the set of all derivations at x of $C^\infty(\mathbb{H}^n)$ by $T_x\mathbb{H}^n$.

The derivations $T_x\mathbb{H}^n$ form a vector space under the operations

$$\begin{aligned}(A+B)\varphi &= A\varphi + B\varphi,\\ (cA)\varphi &= c(A\varphi).\end{aligned}$$

Proposition 2.2.7 (Lemma 3.10 in [10])**.** For any x in $\mathbb{H}^n$, the n derivations

$$\left.\frac{\partial}{\partial x_1}\right|_x, \ldots, \left.\frac{\partial}{\partial x_n}\right|_x$$

defined by

$$\left.\frac{\partial}{\partial x_i}\right|_x \varphi = \frac{\partial \varphi}{\partial x_i}(x)$$

form a basis for $T_x\mathbb{H}^n$, and $T_x\mathbb{H}^n$ is therefore an n–dimensional vector space.

Proof. The case where x is in the interior of $\mathbb{H}^n$ is exactly the same as in Proposition 2.2.4. We concern ourselves only with the case when x is a boundary point of $\mathbb{H}^n$.

Let $i : \mathbb{H}^n \to \mathbb{R}^n$ be inclusion. We will show that $i_* : T_x\mathbb{H}^n \to T_x\mathbb{R}^n$ is an isomorphism. In the case where $i_*A = 0$, let φ be a smooth, real–valued function defined on a neighbourhood of x in $\mathbb{H}^n$ and let $\tilde{\varphi}$ extend φ to a smooth function on an open subset of $\mathbb{R}^n$. From this, we have $\tilde{\varphi} \circ i = \varphi$, so

$$A\varphi = A(\tilde{\varphi} \circ i) = (i_*A)(\tilde{\varphi}) = 0.$$

This means that $i_*A = 0$ implies $A = 0$, so i_* is injective. For surjectivity, let $B \in T_x\mathbb{R}^n$ and let $\varphi \in C^\infty(\mathbb{H}^n)$. Define $A \in T_x\mathbb{H}^n$ by

$$A\varphi = B\tilde{\varphi},$$

where $\tilde{\varphi}$ is any extension of φ. Expressing B in terms of the basis for $T_x\mathbb{R}^n$ on the right hand side of this expression gives us

$$A\varphi = \sum_{i=1}^{n} B_i \frac{\partial \tilde{\varphi}}{\partial x_i}(x).$$

The derivatives $\partial\tilde{\varphi}/\partial x_i$ are determined by the derivatives of φ in $\mathbb{H}^n$, and this definition of A is shown to be a derivation by evaluating $A(\varphi\psi)$ as $B(\tilde{\varphi}\tilde{\psi})$. Confirming that $i_*A = B$ uses $\varphi = \tilde{\varphi} \circ i$ as above, and i_* is therefore surjective. We conclude that i_* is an isomorphism. $\square$

Note that $\mathbb{H}^n$ contains model corner points for every type of j–corner, $0 \leq j \leq n$, along the intersections of its coordinate planes. Investigating the tangent spaces in $\mathbb{H}^k \times \mathbb{R}^{n-k}$ allows us to fully realize tangent spaces to manifolds through the use of charts.

Definition 2.2.8. Let M be an n–manifold and p a point in M. A linear map $A : C^\infty(M) \to \mathbb{R}$ is a *derivation at p* if it satisfies

$$A(\varphi\,\psi) = \varphi(p)A\psi + \psi(p)A\varphi$$

for any $\varphi, \psi \in C^\infty(M)$. Denote the set of all derivations at p of $C^\infty(M)$ by T_pM. As in the case with $\mathbb{R}^n$, T_pM is a vector space under

$$\begin{aligned} (A+B)\varphi &= A\varphi + B\varphi, \\ (cA)\varphi &= c(A\varphi). \end{aligned}$$

We call T_pM the *tangent space to M at p* and call the elements of T_pM *tangent vectors*.

To find a basis, and therefore dimension, of a tangent space we first define how smooth maps act on tangent vectors.

Definition 2.2.9 (Pushforward)**.** Let $f : X \to M$ be a smooth map between manifolds. For every $p \in X$, we define the map

$$f_* : T_pX \to T_{f(p)}M$$

by

$$(f_*A)(\varphi) = A(\varphi \circ f).$$

If $\varphi \in C^\infty(X)$ then $\varphi \circ f \in C^\infty(M)$, so $X(\varphi \circ f)$ is, at least, defined. The operator is clearly linear and it is a derivation at $f(p)$ by

$$\begin{aligned} (f_*A)(\varphi\,\psi) &= A((\varphi\,\psi) \circ f) \\ &= X((\varphi \circ f)(\psi \circ f)) \\ &= (\varphi \circ f(p))A(\psi \circ f) + (\psi \circ f(p))A(\varphi \circ f) \\ &= \varphi(f(p))(f_*A)(\psi) + \psi(f(p))(f_*A)(\varphi). \end{aligned}$$

The map f_* is called the *pushforward* of f, and it relates how the tangent vectors in T_pX are transformed into tangent vectors of $T_{f(p)}M$ when p is passed through f.

Lemma 2.2.10 (Lemma 3.5 in [10])**.** Let $f : X \to M$ and $g : M \to N$ be smooth maps. Let $p \in X$.

1. $f_* : T_pX \to T_{f(p)}M$ is linear.
2. $(g \circ f)_* = g_* \circ f_* : T_pX \to T_{g \circ f(p)}N$
3. $(\mathrm{id}_X)_* = \mathrm{id}_{T_pX} : T_pX \to T_pX$
4. If f is a diffeomorphism then $f_* : T_pX \to T_{f(p)}M$ is an isomorphism.

Proposition 2.2.11 (Proposition 3.7 in [10])**.** Let M be a smooth manifold and $U \subset M$ an open submanifold. Let $i : U \to M$ be inclusion. For any $p \in U$, the pushforward $i_* : T_pU \to T_pM$ is an isomorphism.

We know that $T_x\mathbb{H}^n$ is n–dimensional for any x in $\mathbb{H}^n$. Combining Lemma 2.2.10 and Proposition 2.2.11 with a smooth chart proves the corresponding dimension theorem for tangent spaces to smooth manifolds.

Lemma 2.2.12. Let M be a smooth n–manifold and p a point in M. Then T_pM is an n–dimensional vector space with basis given by the coordinate vectors

$$\left.\frac{\partial}{\partial x_1}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p$$

in any smooth chart.

Proof. Let (U, φ) be a chart about p. Then φ is a diffeomorphism from U to an open subset $\tilde{U}$ of $\mathbb{H}^k \times \mathbb{R}^{n-k}$ for some value of k. By Lemma 2.2.10 and Proposition 2.2.11, the pushforward φ_* is an isomorphism $T_pM \to T_{\varphi(p)}\mathbb{H}^k \times \mathbb{R}^{n-k}$. By Proposition 2.2.7 $T_x\mathbb{H}^k \times \mathbb{R}^{n-k}$ is an n–dimensional vector space with basis the derivations

$$\left.\frac{\partial}{\partial x_1}\right|_x, \dots, \left.\frac{\partial}{\partial x_n}\right|_x,$$

so T_pM is as well. $\square$

We also classify smooth maps between manifolds. This classification helps us define a submanifold and provides some of the groundwork for defining handles in the next section.

Definition 2.2.13 (Embedding)**.** Let $f : X \to M$ be a smooth map between manifolds. The *rank* of f at the point $p \in X$ is the rank of the pushforward f_* as a linear map. This is computed as the rank of the matrix of partial derivatives of f in any smooth chart or as the dimension of $f_*(T_pX) \subset T_{f(p)}M$. If f has rank k for every p in X, then f is of *constant rank* and we say $\mathtt{rank}(f) = k$.

If f_* is injective at p, then f is *immersive* at p. If f is an everywhere immersive, then it is an *immersion.* This is equivalent to $\mathtt{rank}(f) = \mathtt{dim}(X)$. If f_* is surjective at p, then f is *submersive* at p, and p is a *regular point* of f. If f is everywhere submersive, then it is a *submersion.* This is equivalent to $\mathtt{rank}(f) = \mathtt{dim}(M)$. If $\mathtt{rank}(f_*)$ is less than maximal, then p is a *critical point* of f. In this case, $f(p) = q$ is a *critical value.* The set of critical points of f in M is called the *singular set* of M, and is denoted s_f. If p is a critical point and the Hessian of f, i.e. the matrix of second partial derivatives of f in any smooth chart, is of less than full rank, then p is a *degenerate critical point.* For $q \in M$, the set of points x in X such that $f(x) = q$ is called the *preimage* of q through f or the *fiber* of f over q and is denoted $f^{-1}(q)$. A point $q \in M$ whose fiber consist entirely of regular points is a *regular value* of f.

If f is an injective immersion that is a homeomorphism onto its image $f(X) \subset M$ under the subspace topology, then it is a *smooth embedding* or just *embedding.*

Let $X \subset M$ where M is a manifold, and let $i : X \hookrightarrow M$ be inclusion. If i is an embedding, then we call X an *embedded submanifold* or *submanifold* of M, and X is, by itself, a manifold with smooth structure induced by the embedding.

Proposition 2.2.14. Let M be an $(n+1)$–manifold. Any connected face in $\partial_f M$ is an n–dimensional submanifold of M.

Theorem 2.2.15 (Regular Value Theorem)**.** Let $f : X \to M$ be a smooth map between manifolds. Let $q \in M$ be a regular value. The preimage $f^{-1}(q)$ is a $(\dim X - \dim M)$–submanifold of X.

Maps that are "close enough" to one another via perturbation tend to be considered equivalent. This concept is made precise through homotopy.

Definition 2.2.16 (Homotopy)**.** Let $f, g : X \to M$ be smooth maps between smooth manifolds. Denote the closed unit interval $[0,1]$ by $\mathbb{I}$. Let H be a continuous function

$$\begin{aligned} H : \quad X \times \mathbb{I} &\to \quad M \\ (x,t) &\mapsto \quad H_t(x) \end{aligned}$$

with H_t a continuous function $X \to M$ for all t, $H_0(x) = f(x)$, and $H_1(x) = g(x)$. Then H is called a *homotopy* between f and g. If a homotopy exists, then f and g are *homotopic*. Less formally, f and g being homotopic means that one can be continuously deformed into the other. The topological spaces X and M are *homotopy equivalent* if there exist continuous maps $f : X \to M$ and $g : M \to X$ for which $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_M.

Because we are primarily interested in smooth functions to build our machinery, we extend our definition of homotopy to a smooth version. With the notation above, a smooth map $H : X \times \mathbb{I} \to M$ with $H_0(x) = f(x)$ and $H_1(x) = g(x)$ is a *smooth homotopy* between f and g. If a smooth homotopy exists, then f and g are *smoothly homotopic*.

Let M be a manifold and $X \subset M$ a subspace that is not necessarily a submanifold. A continuous map $F : M \times \mathbb{I} \to M$ such that, for every m in M and x in X, $F(m, 0) = m$, $F(m, 1) \in X$, and $F(x, 1) = x$, is called a *deformation retraction*, and the subspace X is called a *deformation retract* of M. Because a deformation retraction is a homotopy equivalence between M and X, a deformation retraction from M onto X exists if and only if M and X are homotopy equivalent.

Proposition 2.2.17. For a fixed pair (X, M) of spaces, homotopy and smooth homotopy are equivalence relations on the space of maps $X \to M$.

Definition 2.2.18. Let X be a topological space, let P be a property that an element of X can possess, and let X_P be the subset of elements of X that posses property P. A property P is *stable under perturbation* if, for any element x in X_P, there is an open set U of X such that $x \in U$ and $U \subset X_P$. A property is *generic* if X_P is open and dense in X

We are mainly concerned with the property of having no degenerate critical points. Consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. The critical point at $x = 0$ is degenerate, but the maps $H_\varepsilon(x) = x^3 + \varepsilon x$ have either a pair of nondegenerate critical points (for $\varepsilon < 0$) or no critical points (for $\varepsilon > 0$), and in either case $H_\varepsilon(x)$ has no degenerate critical points and that property is stable. Degenerate critical points are easily "fixed," so our model generic functions are those without degenerate critical points.

Proposition 2.2.19. A generic smooth map between manifolds has no degenerate critical points.

Intersections between embedded submanifolds also fit in the language of genericity. We call this property transversality.

Definition 2.2.20. Let M be an n–manifold with submanifolds X and Y. We say that X and Y *intersect transversely* as submanifolds if, at every intersection point, the sum of their tangent spaces is the tangent space of M at that point. Symbolically, transversality is denoted by $\pitchfork$ and the condition is stated by $X \pitchfork Y$ if and only if for every point $p \in X \cap Y$, $T_pX + T_pY = T_pM$. We may also say that an intersection point is, itself, *transverse*, or that X and Y *intersect transversally at a point.*

Transversality of a pair of submanifolds X, Y is stable under small perturbations of the embeddings $X \hookrightarrow M$ and $Y \hookrightarrow M$. A point of $X \cap Y$ that is not a transverse intersection may be perturbed into stable transversality or non-intersection, so we say that intersections of submanifolds with complementary codimension are generically transverse.

Proposition 2.2.21. If X, Y are generic embedded submanifolds of M, then they intersect transversely.

The main result of this work is applicable to orientable 3–manifolds, so we will review what is meant by a space being orientable. Orientability is the guarantee that when you go for a walk through a manifold you don't return home to find that your right and left sides have switched places.

We first define orientations on real vector spaces. Orientations of real vector spaces let us define orientations of manifolds. Orientations of manifolds let us define a smooth map as orientation preserving or orientation reversing.

Definition 2.2.22. Let V^n be an n–dimensional real vector space and $\mathcal{B}(V^n)$ the set of ordered bases for V^n. Then $\mathcal{B}(V^n)$ has exactly two path–connected components. An *orientation* of V^n is a choice of one of these components.

Let $A : V \to W$ be a linear map between n–dimensional oriented real vector spaces. If A maps the orientation of V to the orientation of W, A is *orientation preserving.* Otherwise, A maps the the orientation of V to the orientation opposite that of W, and in this case A is *orientation reversing.*

Example 2.2.23. Consider $V = \mathbb{H}^k \times \mathbb{R}^{n-k}$. At $x \in V$, the basis vectors of T_xV

are the derivations

$$\left.\frac{\partial}{\partial x_1}\right|_x, \ldots, \left.\frac{\partial}{\partial x_n}\right|_x.$$

This means that there is a canonical orientation of T_xV induced by the orientation of V for every $x \in V$. Now, let $y \in V$ different from x. Because all of the tangent spaces of V have the same basis, an orientation of T_xV can be meaningfully compared to an orientation of T_yV. In particular, we can determine whether T_xV and T_yV have the same orientation.

The fact that orientations of tangent spaces in $\mathbb{H}^k \times \mathbb{R}^{n-k}$ can be compared leads to our definition of orientation for manifolds. We define an orientation of a manifold M to be a continuous orientation of the tangent spaces to M. We use continuous here to mean that the orientation is constant on any chart.

Definition 2.2.24. Let M be an n–manifold with maximal smooth atlas $\mathcal{A}$, let $(\varphi, U) \in \mathcal{A}$. A *continuous choice of orientation* on U is an orientation of every tangent space to U such that, for every $p, q \in U$, the linear isomorphisms

$$\begin{aligned} \varphi_*^p : T_pU &\rightarrow T_{\varphi(p)}\varphi(U) \\ \varphi_*^q : T_qU &\rightarrow T_{\varphi(q)}\varphi(U) \end{aligned}$$

take the orientations of T_pU, T_qU to the same orientation of $T_{\varphi(p)}\varphi(U) \cong T_{\varphi(q)}\varphi(U)$. An *orientation* of M is an orientation of every tangent space to M that induces a continuous choice of orientation for every chart of $\mathcal{A}$.

Definition 2.2.25. Let $f : X \rightarrow M$ be a smooth map between oriented manifolds. If f_* is orientation preserving (reversing) for every $p \in X$, f is an *orientation preserving (reversing)* smooth map.

2.3 Bundles

Important tools in both constructing and describing manifolds are *bundles*. The ultimate construction in this section is the tubular neighbourhood of a submanifold, an object that is unique up to a fiber-preserving isotopy.

Definition 2.3.1 (Bundle)**.** A real *vector bundle* is a tuple $\beta = (E, B, \pi : E \rightarrow B)$ where B is called the *base space*, E the *total space*, and π the *projection*. The projection is a continuous map for which the subspaces $\pi^{-1}(b) = V_b$ all have the structure of a k–dimensional vector space. The space V_b is *fiber* over b.

A bundle is *locally trivial.* That is, for every point $b \in B$ there exists an open neighbourhood $U \subset B$ containing b and a homeomorphism

$$\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$$

such that the map $v \mapsto \varphi(b, v)$ is a vector space isomorphism $\mathbb{R}^n \mapsto V_b$ for every $b \in U$. Such a homeomorphism is called a *local trivialization*, and such a pair is a *local coordinate system.* Any pair of local trivializations

$$\begin{array}{rrcl} \varphi_U : & U \times \mathbb{R}^n & \to & \pi^{-1}(U), \\ \varphi_V : & V \times \mathbb{R}^n & \to & \pi^{-1}(V) \end{array}$$

must be compatible in the sense that the composition

$$\varphi_V^{-1} \circ \varphi_U : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n,$$

is well defined on $U \cap V$ and satisfies

$$\varphi_V^{-1} \circ \varphi_U(b, v) = (b, [A_{UV}(b)](v))$$

for every b in $U \cap V$, where A_{UV} is a function

$$A_{UV} : U \cap V \to \mathrm{GL}_n(\mathbb{R})$$

that assigns a linear transformation from $\mathrm{GL}_n(\mathbb{R})$ to every point b of $U \cap V$. If it is possible to take U to be all of B, then β is a *trivial bundle* and the map $U \times \mathbb{R}^n \to E$ is a *trivialization* of the bundle.

When E and B are smooth manifolds, π is smooth, and the local trivializations are diffeomorphisms, β is a *smooth vector bundle.* We assume that the vector bundle structures we encounter over smooth manifolds are always smooth.

A continuous map $s : B \hookrightarrow E$ that is a right inverse of π is called a *section.* The *zero section* of β is used to refer to both the map

$$\begin{array}{rccc} z: & B & \to & E \\ & b & \mapsto & (b,0), \end{array}$$

and its image $z(B)$ in E. Note that the compatibility condition on local trivialization guarantees that the zero section is well defined.

For β a k–vector bundle, we can form new objects called *k–disc bundles* by restricting the fibers of β to vectors with length at most 1. The machinery we build for vector bundles restricts to this construction.

Other spaces, such as spheres, may be taken as fibers to form the general *fiber bundle.* The definition follows that for vector bundles, except we lose the structure that guarantees the existence of a zero section.

Definition 2.3.2 (Bundle Isomorphism)**.** Let $\Phi : \beta_0 \to \beta_1$ be a map between vector bundles. We call Φ a *fiber map* if $\Phi : E_0 \to E_1$ covers a map $\varphi : B_0 \to B_1$. For Φ to cover φ, that means the following diagram commutes:

$$\begin{tikzcd} E_0 \arrow[r, "\Phi"] \arrow[d, "\pi_0"'] & E_1 \arrow[d, "\pi_1"] \\ B_0 \arrow[r, "\varphi"] & B_1 \end{tikzcd}$$

This means that if $b \in B_0$ and $\varphi(b) = c$, then Φ maps V_b to V_c by a map we will denote Φ_b.

If Φ_b is a linear map for every $b \in B_0$, then we call Φ a *bundle morphism.* If Φ is a bundle morphism, $B_0 = B_1 = B$, Φ_b is bijective for each $b \in B$, and φ is id_B, then Φ is a *bundle isomorphism.* A bundle isomorphism $\Phi : E \to E$ is a *bundle automorphism.*

Definition 2.3.3 (Tangent Bundle)**.** Let M be an n–manifold. We define the total space TM of a smooth vector bundle with base space M and fibers $\mathbb{R}^n \approx T_pM$ at any $p \in M$ by taking the disjoint union of all tangent spaces:

$$TM = \bigsqcup_{p \in M} T_pM.$$

The projection $\pi : TM \to M$ is defined by $\pi(q) = p$ for every $q \in T_pM$. We call TM the *tangent bundle* over M. A section $s : M \to TM$ is a *vector field* on M.

Definition 2.3.4 (Normal Bundle)**.** Let X be a k–dimensional submanifold of the $(n+k)$–manifold M. At a point $p \in X \subset M$, the tangent space T_pX is a subspace of the tangent space T_pM. Denote the orthogonal complement to T_pX in T_pM by N_pX and call it the normal space at p in X. That is, $T_pX \oplus N_pX = T_pM$. From linear algebra, N_pX is an n–dimensional vector space. We define the total space N_MX of a vector bundle with base space X and fibers $\mathbb{R}^n \approx N_pX$ at any $p \in X$ by taking the disjoint union of all normal spaces:

$$N_MX = \bigsqcup_{p \in X} N_pX.$$

The projection $\pi : N_MX \to X$ is defined by $\pi(q) = p$ for every $q \in N_pX$. We call N_MX the *normal bundle* over X in M.

Definition 2.3.5 (Tubular Neighbourhood)**.** Let X be a closed submanifold of the closed manifold M. A smooth embedding $f : N_MX \to M$ with $f(x,0) = x$ and $f(N_MX)$ an open neighbourhood of X in M is called a *tubular neighbourhood* of X in M. We often denote the pair (X, f) by ν_MX.

Similar to the tubular neighbourhood is the *closed tubular neighbourhood*, which has instead the structure of a normal disc bundle over X. We use $D_MX \subset N_MX$ to denote the disc bundle over X, and $\overline{\nu}_MX$ to denote a closed tubular neighbourhood. This naming convention refers to topological closure.

Theorem 2.3.6 (Existence of Tubular Neighbourhoods, Theorem 4.5.2 in [6])**.** Let X be a closed submanifold of the closed manifold M. Then X has a tubular neighbourhood in M.

Definition 2.3.7. Let M be an n–manifold and $X \subset \partial M$. A *collar* of X is a diffeomorphism $f : X \times \mathbb{H} \to X_U$ such that U_X is an open neighbourhood of X in M, $f(x, 0) = (x)$ for every $x \in X$, and U_X is an n–submanifold of M. The set U_X is a *collar neighbourhood* of X. As with tubular neighbourhoods, we denote a collar by (X, f). If a collar exists for X, we say that X *admits a collar.*

Proposition 2.3.8. Let M be a manifold with nonempty $\partial_1 M$ such that $\partial_j M$ is empty for all $j > 0$. Then any open subset of $\partial_1 M$ admits a collar.

Collars are a standard object in the study of manifolds without corners, and Proposition 2.3.8 is the standard result. The proof found in Section I.7 of [8] relies on the existence of solutions to inward pointing vector fields defined on ∂M, and the vector field defined in [8] can be adapted to manifolds with faces.

Definition 2.3.9. Let M be a manifold, let $p \in \partial M$, and let $v \in T_pM$. Suppose $\varphi : U \to \mathbb{H}^k \times \mathbb{R}^{n-k}$ is a chart about p such that $\varphi(p) = \mathbf{0}$. Then

$$v = \sum_i v_i \left.\frac{\partial}{\partial x_i}\right|_p$$

is *inward pointing* if $v_i > 0$ for $i = 0, \ldots, k-1$, and *outward pointing* if $v_i \leq 0$ for any $i = 0, \ldots, k-1$.

Proposition 2.3.10. If M is a manifold with faces then there is a vector field s on M that is inward pointing on ∂M.

Proof. Let $\{(U_\alpha, \varphi)\alpha)\}$ be an *adequate* atlas (Definition 2.1.10) for M and let $\{\mu_\alpha\}$ be an associated partition of unity. The field s defined by

$$s(p) = \sum_\alpha \left(\mu_\alpha \sum_{i=1}^n \left.\frac{\partial}{\partial x_i}\right|_p \right)$$

is inward pointing on ∂M. □

Proposition 2.3.11. Let M be a manifold with nonempty ∂M. Then ∂M admits a collar.

Proof. Replacing I.7.1 of [8] with Proposition 2.3.10, the proof in Section I.7 of [8] applies to manifolds with faces. □

Proposition 2.3.12 (Lemma 2.1.6 of [9])**.** If M is a manifold with faces and $F_1, \ldots, F_k$ are faces of M such that $F_i \cap F_j$ is a face of F_i and F_j for $i \neq j$, then $\bigcup_i F_i$ admits a collar.

A collar of ∂M can be thought of as a half–tubular neighbourhood of the boundary. In this sense the collar is a fiber bundle with base ∂M, fiber $\mathbb{H}$, and trivial structure. This allows us to relate our future results on tubular neighbourhoods to include collars. In particular, tubular neighbourhoods are unique up to a fiber-preserving isotopy, which we define now.

Definition 2.3.13 (Isotopy)**.** Let X be a smoothly embedded submanifold of M. An *isotopy* of X in M is a smooth homotopy

$$\begin{array}{rrcl} F: & X \times \mathbb{I} & \to & M \\ & F(x,t) & = & F_t(x) \end{array}$$

such that the related map

$$\begin{array}{rrcl} \hat{F}: & X \times \mathbb{I} & \to & M \times \mathbb{I} \\ & (x,t) & \mapsto & (F_t(x), t) \end{array}$$

is an embedding. The submanifolds $F_0(X)$ and $F_1(X)$ are *isotopic*. When $X = M$ and F_t is a diffeomorphism for each t, F is a *diffeotopy* of M.

Let X be a smoothly embedded closed submanifold of M and consider a pair of tubular neighbourhoods $f, g : N_M X \to M$ of X in M. An isotopy

$$\begin{array}{rrcl} F: & N_M X \times \mathbb{I} & \to & M \\ & F(x,t) & = & F_t(x) \end{array}$$

satisfying the following properties:

1. $F_0 = f$ and $F_1 = g$,
2. $F_0(N_M X) = F_1(N_M X)$,
3. $F_1^{-1} \circ F_0$ is a vector bundle isomorphism $N_M X \to N_M X$,

is an *isotopy of tubular neighbourhoods*, and the tubular neighbourhoods (X, f) and (X, g) are *isotopic*.

Proposition 2.3.14. For M a manifold, isotopy and diffeotopy are equivalence relations on the space of submanifolds of M.

With these definitions at hand, we can precisely state the uniqueness result for tubular neighbourhoods and the similar result for collars.

Theorem 2.3.15 (Uniqueness of Tubular Neighbourhoods, Theorem III.3.1 of [8])**.** Let X be a closed submanifold of M. Any pair of tubular neighbourhoods of X in M are isotopic.

Theorem 2.3.16 (Uniqueness of Collars)**.** Let M be a manifold and let X be a codimension 1 submanifold of M with $X \subset \partial M$. Let (X, f) and (X, g) be collars of X. Then $f(X \times \mathbb{H})$ and $g(X \times \mathbb{H})$ are isotopic through an isotopy $F : (X \times \mathbb{H}) \times \mathbb{I} \to M$ with $F_0 = f$, $F_1 = g$, and $F_t(x, 0) = x$ for every $x \in X$.

There is a stronger uniqueness theorem for tubular neighbourhoods that uses a tighter definition. It essentially says that two of these neighbourhoods are isotopic through a diffeotopy that is stationary outside of a small neighbourhood of the tube.

Theorem 2.3.17 (Isotopy Extension)**.** Let X be a smooth compact submanifold of the smooth closed manifold M, and let

$$\begin{array}{rrcl} F: & X \times \mathbb{I} & \to & M \\ & F(x,t) & = & F_t(x) \end{array}$$

be an isotopy of X in M. Let L be the subset of M equal to the union of the images of F_t for each t. More precisely,

$$L = \bigcup_{t \in [0,1]} F_t(X).$$

Then there exists a diffeotopy

$$\begin{array}{rrcl} G: & M \times \mathbb{I} & \to & M \\ & G(y,t) & = & G_t(y) \end{array}$$

with $G_0 = \mathrm{id}_M$, G_1 equal to F_1 on $X \subset M$, and F_t the identity on M outside of an arbitrarily small neighbourhood of L for all t.

Definition 2.3.18 (Ambient Isotopy)**.** Let X, M, F be as in Theorem 2.3.17. The isotopy $G : M \times \mathbb{I} \to M$ guaranteed by Theorem 2.3.17 is called an *ambient isotopy*. The images $F_0(X)$ and $F_1(X)$ are *ambiently isotopic* as submanifolds of M.

When the neighbourhood is closed, isotopy extension can strengthen the uniqueness of tubular neighbourhoods theorem to one that is unique through an ambient isotopy. We can perform a similar strengthening on open tubular neighbourhoods as long as we restrict the definition slightly.

Definition 2.3.19. A *proper tubular neighbourhood* is a tubular neighbourhood that is the interior of a closed tubular neighbourhood.

Examples of improper tubular neighbourhoods would be $\mathbb{R}^n$ as a neighbourhood of the origin in $\mathbb{R}^n$, or the strip $\{(x,y) \in \mathbb{R}^2 : |x| < \frac{\pi}{2}\}$ as a neighbourhood of the line $x = 0$ in $\mathbb{R}^2$ whose fibers are the curves $y = \tan x + C$ for each $C \in \mathbb{R}$. An example of a proper tubular neighbourhood would be the same strip, but with fibers $y = C$. It should be clear that the interior of a closed tubular neighbourhood is proper, and every proper tubular neighbourhood is the interior of a closed tubular neighbourhood.

Theorem 2.3.20 (Uniqueness of Proper Tubular Neighbourhoods, Theorem III.3.5 of [8])**.** Let X be a closed submanifold of M. Any two proper tubular neighbourhoods of X are isotopic through an isotopy that can be extended to an ambient isotopy.

2.4 Operations on Manifolds

We construct manifolds in later chapters mainly by gluing simpler manifolds together. These operations include handle attachment and connected sum, special cases of the general construction of joining two manifolds along a submanifold. Special attention is given to the definitions and results needed to meaningfully attach 1– and 2–handles to a 4–manifold.

There is a common tool in topology called the *attaching map* that we use in this section and the next to build our machinery. We take this opportunity to lay out the definition and notation.

Definition 2.4.1. Let X and Y be topological spaces, $A \subset X$ a subspace, and $f : A \to Y$ a continuous map. We define a relation $\sim$ by putting $f(x) \sim x$ for every x in A. Denote the quotient space $X \sqcup Y/\sim$ by $X \cup_f Y$. We call the map f the *attaching map*. We say that X is *attached* or *glued* to Y over A. A space obtained through attachment is called an *adjunction space* or *attachment space*.

Alternatively, we let A be a topological space and let $i_X : A \to X$, $i_Y : A \to Y$ be inclusions. Here, the adjunction is formed by taking $i_X(a) \sim i_Y(a)$ for every $a \in A$ and we denote the adjunction space by $X \cup_A Y$.

Our first application of adjunction is in the connected sum operation.

Definition 2.4.2. Let M and X be closed, oriented n–manifolds. The manifold obtained by identifying the boundary $(n-1)$–spheres of $M \setminus B^n$ and $X \setminus B^n$ in an orientation reversing way is called the *connected sum* of M and X and is denoted $M\#X$. Symbolically,

$$M\#X = (M \setminus B^n) \cup_f (X \setminus B^n)$$

where f is any diffeomorphism $S^n \to S^n$.

Proposition 2.4.3 (VI.1.1 of [8])**.** Connected sum is a well defined operation up to diffeomorphism on the space of closed, oriented n–manifolds with identity element S^n.

We present handle attachment in its standard form before delving into the cases involving manifolds with faces. When discussing handles we have $n = \lambda + \mu$, M a smooth n–manifold with nonempty $\partial_1 M$ and $\partial_j M = \emptyset$ for $j > 1$, and $H^\lambda = D^\lambda \times D^\mu$. Handle attachment is the process of joining M and H^λ along an embedding of $\partial D^\lambda \times D^\mu$ into ∂M.

Definition 2.4.4 (Handle)**.** Take $n = \lambda + \mu$ and M a smooth n–manifold with nonempty $\partial_1 M$. Let $\varphi : \partial D^\lambda \times D^\mu \to \partial_1 M$ be an embedding and an attaching map between M and H^λ. The attached space H^λ is an *n–dimensional λ–handle*, and $M \cup_\varphi H^\lambda$ is the result of an n–dimensional *λ–handle attachment*.

Proposition 2.4.5. Let $M' = M \cup_\varphi H^\lambda$ be the result of an n–dimensional λ–handle attachment. Then M' is a smooth n–manifold with faces, the 1–boundary of M' consists of $\partial M \setminus \varphi(\partial D^\lambda \times D^\mu)$ and $D^\lambda \times \partial D^\mu$, the 2–boundary consists of $\varphi(\partial D^\lambda \times \partial D^\mu)$, and ∂_j is empty for $j > 2$.

Our results regarding attaching manifolds over faces are found in [18]. They tell us that manifolds can be attached over faces, the attachment space is a manifold with faces, and the faces of the attachment space are well defined.

Proposition 2.4.6. Let X and X' be smooth n–manifolds with faces, let F be a manifold with faces, and let $\iota : F \to X$ and $\iota' : F \to X'$ be inclusion maps such that $\iota(F)$ is a face of X and $\iota'(F)$ is a face of X'. Then the adjunction $X \cup_F X'$ is a smooth n–manifold with faces.

For each j, the j–boundary of $X \cup_F X'$ is decomposed into the j–boundary of X minus the image of F in X, the j–boundary of X' minus the image of F in X', and the j–boundary of F seen as a subset of $X \cup_F X'$. Symbolically, this is

$$\partial_j(X \cup_F X') = (\partial_j X \setminus \iota(F)) \cup (\partial_j X' \setminus \iota'(F)) \cup (\partial_j F).$$

{ASIDE FOR EDITING: The following proof is based on some informal notes by Jim Davis found here. Is this even citable?}

Proof. To prove this proposition we put a smooth structure on $X \cup_F X'$ so that if $x \in F$ is a j–corner then $\iota(x) \sim \iota'(F) \in X \cup_F X'$ is a j–corner as well. We first decompose X into the triad $X = (X; K, M)$ where

1. $K = \iota(F)$,
2. $M \cup K = \partial X$, and
3. $M \cap K = \partial K = \partial M = \iota(\partial F)$.

We do the same, respectively, to $X' = (X'; K', M')$.

Let $\psi : K \times \mathbb{H} \to X$ and $\psi' : K' \times \mathbb{H} \to X'$ be collars of K and K' in X and X'. Also take

$$\begin{aligned} \varphi_K : \quad \partial F \times \mathbb{H} &\to K \\ \varphi_M : \quad \partial F \times \mathbb{H} &\to M \\ \varphi'_K : \quad \partial F \times \mathbb{H} &\to K' \\ \varphi'_M : \quad \partial F \times \mathbb{H} &\to M' \end{aligned}$$

to be collars of ∂K and $\partial K'$ in their respective codomains, ensuring that

$$\iota^{-1} \circ \varphi_K = (\iota')^{-1} \circ \varphi'_K.$$

Defining

$$\begin{aligned} X \setminus K &\hookrightarrow X \cup_F X', \text{ and} \\ X' \setminus K' &\hookrightarrow X \cup_F X' \end{aligned}$$

by inclusion, and

$$\begin{aligned} \Psi : \quad (F \setminus \partial F) \times \mathbb{R} &\hookrightarrow X \cup_F X' \\ \Phi : \quad (\partial F) \times \mathbb{R} \times \mathbb{H} &\hookrightarrow X \cup_F X' \end{aligned}$$

by

$$\Psi(x,t) = \begin{cases} \psi(\iota(x), t), & \text{if } t \geq 0 \\ \psi'(\iota'(x), -t), & \text{if } t \leq 0 \end{cases}$$

and

$$\Phi(x, r\cos\theta, r\sin\theta) = \begin{cases} \psi(\varphi_M(x, r\cos(2\theta)), r\sin(2\theta)) & \text{if } \theta \in [\,0, \pi/4\,] \\ \psi(\varphi_K(x, -r\cos(2\theta)), r\sin(2\theta)) & \text{if } \theta \in [\,\pi/4, 2\pi/4\,] \\ \psi'(\varphi'_K(x, -r\cos(2\theta)), -r\sin(2\theta)) & \text{if } \theta \in [\,2\pi/4, 3\pi/4\,] \\ \psi'(\varphi'_M(x, r\cos(2\theta)), -r\sin(2\theta)) & \text{if } \theta \in [\,3\pi/4, 4\pi/4\,] \end{cases}$$

gives $X \cup_F X'$ a smooth structure induced by the open cover

$$X \cup_F X' = (X \setminus K) \cup (X' \setminus K') \cup \Psi((F \setminus \partial F) \times \mathbb{R}) \cup \Phi((\partial F) \times \mathbb{R} \times \mathbb{H}).$$

In this open cover, the j–corners of $\iota(F) = \iota'(F)$, $j \geq 1$, appear only as j–corners of $\Phi((\partial F) \times \mathbb{R} \times \mathbb{H})$, hence they are j–corners in $X \cup_F X'$, thus the proposition is proved. $\square$

Corollary 2.4.7. Let X, X' be manifolds with faces and let $F_1, \ldots, F_k$ be faces of X such that $F_i \cap F_j$ is a face of F_i and F_j for every $i \neq j$. Let $\mathbf{F} = \cup_i F_i$ and let $\varphi : \mathbf{F} \to \partial X'$ be an embedding such that $\varphi(F_i)$ is a face of X' for each i. Then the adjunction $X \cup_{\mathbf{F}} X'$ is a smooth manifold with faces.

If p is a j–corner of X, $j \geq 0$, and k is the number of faces in $\mathbf{F}$ that contain p, then p is a $(j-k)$–corner of $X \cup_{\mathbf{F}} X'$. Similarly, if p is a j–corner of X', $j \geq 0$, and k is the number of faces in $\varphi(\mathbf{F})$ that contain p, then p is a $(j-k)$–corner of $X \cup_{\mathbf{F}} X'$.

Example 2.4.8. We first see an example of Proposition 2.4.5. Let V be a 3–dimensional solid torus (i.e. $V = D^2 \times S^1$) and recall that a 3–dimensional 2–handle has the form $H^2 = D^2 \times D^1$. Let φ be 2–handle attachment map defined by

$$\begin{array}{rccc} \varphi : & \partial D^2 \times D^1 & \to & \partial D^2 \times S^1 = \partial V \\ & (e^{i\theta}, t) & \mapsto & (e^{i\theta}, e^{\pi i t/2}). \end{array}$$

The 3-manifold $V \cup_\varphi H^1$ that results from this handle attachments has one boundary component. It consists of the faces $D^2 \times \{e^{i\pi/2}\}$, $D^2 \times \{e^{-i\pi/2}\}$, and $\partial D^2 \times [e^{i\pi/2}, e^{-i\pi/2}]$, where we use $[e^{i\pi/2}, e^{-i\pi/2}]$ to denote the oriented arc in S^1, orientation agreeing with that of S^1, from the angle $\theta = \pi/2$ to $\theta = -\pi/2$.

Next, we use Corollary 2.4.7 and add a copy of $D^2 \times D^1$ to $V \cup_\varphi H^1$. We define an attachment map ψ on $\partial(D^2 \times D^1)$ piecewise by

$$\begin{array}{rccl} \psi|_{S^1\times D^1}: & S^1 \times D^1 & \to & \partial(V \cup_\varphi H^1) \\ & (e^{i\theta}, t) & \mapsto & (e^{i\theta}, e^{\pi i(t+2)/2}), \\ \psi|_{D^2\times\{-1\}}: & D^2 \times \{-1\} & \to & \partial(V \cup_\varphi H^1) \\ & (re^{i\theta}, -1) & \mapsto & (re^{i\theta}, e^{\pi i/2}), \\ \psi|_{D^2\times\{1\}}: & D^2 \times \{1\} & \to & \partial(V \cup_\varphi H^1) \\ & (re^{i\theta}, 1) & \mapsto & (re^{i\theta}, e^{-\pi i/2}). \end{array}$$

The resulting 3–manifold $(V \cup_\varphi H^1) \cup_\psi (D^2 \times D^1)$ is smooth and closed.

The 3–manifold found in Example 2.4.8 is S^3. We spend the remainder of this section developing tools that help classify the manifolds that arise from our operations.

Put $n = \mu + \lambda$, recall that $H^\lambda = D^\lambda \times D^\mu$, let M be an n–manifold and let $M' = M \cup_\varphi H^\lambda$ be the result of an n–dimensional λ–handle attachment. Note that the domain of φ has the structure of a trivial μ–disc bundle over $S^{\lambda-1}$ and, using the language of vector bundles, the image of φ has the structure of a closed tubular neighbourhood of $f_0(S^{\lambda-1}) = \varphi \circ z(S^{\lambda-1})$. Our uniqueness theorems for tubular neighbourhoods tell us that the closed tubular neighbourhood

$$f : D_{\partial M} f_0(S^{\lambda-1}) \to \overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$$

of $f_0(S^{\lambda-1})$ is unique up to ambient isotopy of the embedding f_0. Thus, if we have an embedding $f_0 : S^{\lambda-1} \to \partial M$ with trivial disc bundle, then a handle attachment can be defined using f_0 and an explicit embedding

$$f : S^{\lambda-1} \times D^\mu \to D_{\partial M}\, f_0(S^{\lambda-1})$$

for a closed tubular neighbourhood $\overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$.

In other words, the characteristics of handle attachment that fully describe the smooth n–manifold $M \cup_\varphi H^\lambda$ up to diffeomorphism are:

1. the isotopy class of an embedding $f_0 : S^{\lambda-1} \to \partial M$ with trivial normal disc bundle, and

2. the isotopy class of an identification of $S^{\lambda-1} \times D^\mu$ with $\overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$.

There is some specific language that is useful to the description of handle attachment.

Definition 2.4.9. Let $M \cup_\varphi H^\lambda$ be an n–manifold with a λ–handle attached. The embedding $f_0 : S^{\lambda-1} \to \partial M$ is called the *attaching map*, its image $f_0(S^{\lambda-1})$ is the *attaching sphere*, and its tubular neighbourhood is the *attaching neighbourhood.* The embedding $f : S^{\lambda-1} \times D^\mu \to \overline{\nu}_{\partial M}\, f_0(S^{\lambda-1})$ is called a *normal framing* or just *framing.* Inside of the λ–handle $H^\lambda = D^\lambda \times D^\mu$, the disc $D^\lambda \times \{\vec{0}\}$ is the *core*, and the disc $\{\vec{0}\} \times D^\mu$ is the *cocore.* The boundary circle $\{\vec{0}\} \times S^{\mu-1}$ of the cocore is the *belt sphere.* The integer λ is the *index* of the handle.

It is desirable to describe a manifold entirely via handle attachments. For this we define the general notion of a decomposition.

Definition 2.4.10. Let W be a compact n–manifold with boundary $M_0 \sqcup M_1$, where M_0 and M_1 are smooth closed $(n-1)$–manifolds. Describe W as a chain of inclusions

$$M_0 \times \mathbb{I} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{k-1} \subset W_k = W$$

where W_i is obtained from W_{i-1} by attaching a handle and M_0 is identified with $M_0 \times \{0\}$. This is called a *handle decomposition* of the pair (W, M_0).

Definition 2.4.11. A connected manifold M that has a handle decomposition consisting of exactly one 0–handle and g 1–handles is called an $(m,1)$*–handlebody* of *genus* g. Let V be a $(m,1)$–handlebody of genus g. A simple closed curve in ∂V is called *essential* if it is not homotopic to a point. A simple closed curve J in ∂V that is essential in ∂V and that bounds a 2–disc in V is called a *meridian.* The properly embedded disc in V that has boundary J is called a *meridinal disc.*

The special case of the oriented genus 1 $(m,1)$–handlebody is called a *solid torus.* More generally, any space that is homeomorphic to $S^1 \times D^{n-1}$ is called a *solid n–torus.* In our most common case of $n = 3$, we just say that $S^1 \times D^2$ is a *solid torus.* A simple closed curve J in the boundary of a solid torus that intersects a meridian at a single point is called a *longitude.* A longitude is essential in a solid torus and there are infinitely many isotopy classes of longitudes, but there is exactly one isotopy class of meridians.

Definition 2.4.12. Let U and V be 3–dimensional handlebodies of genus g and let $f : \partial U \to \partial V$ be an orientation preserving diffeomorphism. The adjunction

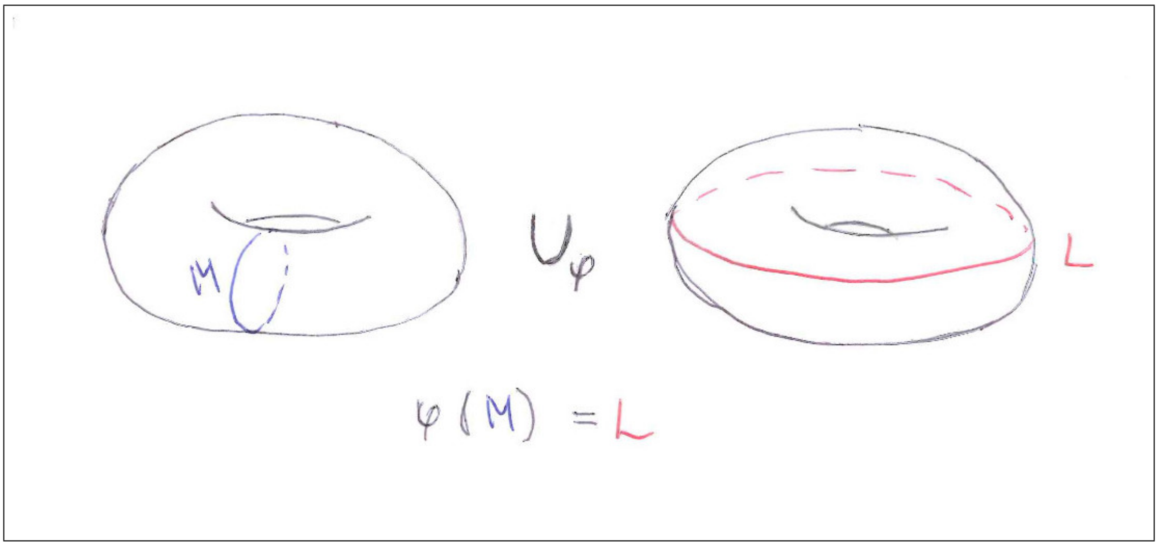

Figure 2.1: Genus 1 Heegaard splitting of S^3

$M = U \cup_f V$ is a *Heegaard splitting* of M, the shared boundary $H = \partial U = \partial V$ is the *Heegaard surface* of the splitting, and the shared genus of U and V is the *genus* of the splitting as well. A Heegaard splitting is also denoted by the pair (M, H).

We use the notion of equivalence between splittings from [17]. A pair of splittings (M, H) and (M, H') are *equivalent* if there is a homeomorphism $h : M \to M$ such that h is isotopic to id_M and $h|_H$ is an orientation preserving homeomorphism $H \to H'$.

Example 2.4.13. The 3–sphere S^3 has two standard Heegaard splittings. The first is the genus 0 splitting, usually realized by considering S^3 as the set of unit vectors in $\mathbb{R}^4$. Take the Heegaard surface to be the intersection of S^3 with the xyz–hyperplane in $\mathbb{R}^4$. This is a copy of S^2 that separates S^3 into two connected components. This splitting is written (S^3, S^2)

The second is the genus 1 splitting, which is visualized using the realization of S^3 as the one–point compactification of $\mathbb{R}^3$. Take solid tori U and V and identify ∂U with ∂V by the homeomorphism that swaps a meridian with a longitude. The adjunction is S^3, and the pair of solid tori are displayed in Figure 2.1. This splitting is written (S^3, T^2). This description of S^3 may also be obtained by examining the boundary of a 4–dimensional 2–handle $\partial(D^2 \times D^2)$.

Definition 2.4.14. Let $(M, H) = U \cup_f V$ be a Heegaard splitting of M. The connected sum $(M, H)\#(S^3, T^2)$ is called an *elementary stabilization* of M, and itself is a splitting $(M, H\#T^2)$. A Heegaard splitting (M, H) is called a *stabilization* of another splitting (M, H') if it obtained from (M, H') via a finite number of elementary stabilizations.

Consider the meridians of the solid tori in the standard genus 1 splitting of S^3. They each bound a disc in their respective handlebody, and they intersect in exactly one point. We would expect to be able to find such curves in any 3–manifold obtained as a stabilization.

Definition 2.4.15. Let $(M,H) = U \cup_f V$ be a Heegaard splitting of genus g and α, β a pair of simple, closed, essential curves in H. Let α be a meridian of U and β be a meridian of V with associated meridinal discs D_α, D_β. If α and β intersect exactly once, then the pair (α, β) is a *meridinal pair* or *destabilizing pair* of the splitting.

To see why (α, β) would be called a destabilizing pair, remove a tubular neighbourhood of D_α from U and add it to V as a 2–handle along a tubular neighbourhood of α in H. In the proof of Waldhausen's Theorem it is shown that these altered spaces U' and V' are handlebodies of genus $g-1$ with $H' = \partial U' = \partial V'$, and (M,H) is a stabilization $(M,H')\#(S^3,T^2)$.

Theorem 2.4.16 (Waldhausen's theorem, [17])**.** If (S^3,H) and (S^3,H') are Heegaard splittings of the same genus, then they are equivalent.

In the proof of Waldhausen's theorem, destabilizing pairs are identified and reduced. We don't need the full power of Waldhausen's theorem, just a weaker version to use in Chapter 3.

Theorem 2.4.17. Let M be a closed, orientable 3–manifold with a Heegaard splitting (M,H) of genus g. If there is a collection of g disjoint destabilizing pairs in H, then M is homeomorphic to S^3.

Heegaard splittings are a specific application of a specific type of handle decomposition. General decompositions do not arise in this way. They usually occur naturally by studying Morse functions as discussed in Chapter 3. More structure can be demanded out of a decomposition with handles of varying indices by isotoping attaching maps.

Proposition 2.4.18 (Proposition 4.2.7 of [4])**.** Any handle decomposition of the compact pair (W, M_0) can be modified so that handles are attached in order of increasing index. Handles of the same index can be attached in any order.

Because a λ–handle is also a μ–handle of the same dimension, any handle decomposition actually defines a pair of dual decompositions.

Definition 2.4.19. Let W be a compact manifold with $\partial W = M_0 \sqcup M_1$ and handle decomposition

$$M_0 \times \mathbb{I} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_{k-1} \subset W_k = W.$$

By attaching a copy of $M_1 \times \mathbb{I}$ to M_1 and removing the collar $M_0 \times \mathbb{I}$, we obtain a handle decomposition of the pair (W, M_1). Each λ–handle in the decomposition of (W, M_0) gives us a μ–handle in the decomposition of (W, M_1) that can be seen by reversing the roles of the core and cocore within the handle. The decomposition obtained is called the *dual handle decomposition* of the pair (W, M_1).

Notice that if the handles of a decomposition are attached in order of increasing index, then the handles of the dual decomposition are as well.

There are two special cases of handle attachment to mention. The first is 1–handle attachment, which reduces to investigating the orientations of the attaching maps. The second is 4–dimensional 2–handle attachment, which reduces to identifying the homology class of a single oriented curve.

Remark 2.4.20. Let M be orientable and path–connected with ∂M compact, connected, and nonempty. The attaching sphere of a 1–handle is $S^0 = \partial D^1$, which is a pair of points. There is a unique isotopy class of embeddings $f_0 : S^0 \to \partial M$. This means that $M \cup_\varphi H^1$ is determined entirely by the framing f. The normal disc bundle of $f_0(S^0)$ is a bundle over S^0, so it is vacuously trivial. Using the vector bundle structure of the tubular neighbourhood, we write an embedding of $S^0 \times D^{n-1} \to S^0 \times D^{n-1}$ as a pair of length–preserving linear transformations $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, i.e. elements of the orthogonal group $\mathcal{O}_{n-1}(\mathbb{R})$, each restricted to act on one of the connected components of $S^0 \times D^{n-1}$. The determinant of an element of $\mathcal{O}_{n-1}(\mathbb{R})$ is either 1 or -1, and $\mathcal{O}_{n-1}(\mathbb{R})$ has two path–connected components corresponding to these two cases. Every element of $\mathcal{O}_{n-1}(\mathbb{R})$ is an embedding $\mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, so any path in $\mathcal{O}_{n-1}(\mathbb{R})$ is an isotopy of its endpoints. It is then easy to see that are four isotopy classes of our trivialization, and those fall into two types. Either both transformations are are orientation preserving (reversing), or one is orientation preserving (reversing) and the other is orientation reversing (preserving). Under the first type of automorphism,

$M \cup_\varphi H^1$ is a non-orientable manifold. Under the second, $M \cup_\varphi H^1$ is orientable.

Let W be a 4–manifold with orientable boundary M. To attach a 2–handle to W, we need an embedding $f_0 : S^1 \to M$ and a framing $S^1 \times D^2 \to \overline{\nu}_{\partial M} f_0(S^1)$.

Once an embedding $f_0 : S^1 \to M$ is chosen, $W \cup_\varphi H^2$ is determined by the framing $f : S^1 \times D^2 \to \overline{\nu}_{\partial M}\, f(S^1)$. To investigate possible framings, we consider the mapping class group of the solid torus.

Definition 2.4.21. Let X be a manifold and let $\mathrm{Aut}(X)$ be the group of automorphisms of X (isomorphisms $X \to X$ under composition). Define an equivalence relation $\sim$ on $\mathrm{Aut}X$ by $f \sim g$ if f is isotopic to g. The group $\mathrm{Aut}(X)/\sim$ is the *mapping class group* of X and is denoted $mpg(X)$.

For V a solid torus, $mpg(V)$ is very closely linked to the mapping class group of the torus $T^2 = S^1 \times S^1 = \partial V$, which is isomorphic to the special linear group $\mathrm{SL}_2(\mathbb{Z})$, the group of 2×2 matrices with integer entries and determinant ± 1.

Let $\partial V = S^1 \times S^1 = T^2$. There is a well–defined group action of $mpg(T^2)$ on $H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. We represent an element of $H_1(T^2)$ by (μ, λ) using the basis where $(1,0)$ represents the meridinal curve $S^1 \to T^2$ defined by

$$e^{i\theta} \mapsto (e^{i\theta}, 1)$$

and $(0,1)$ represents the longitudinal curve $S^1 \to T^2$ defined by

$$e^{i\theta} \mapsto (1, e^{i\theta}).$$

Then an element of $mpg(T^2) \cong \mathrm{SL}_2(\mathbb{Z})$ acts on an element of $H_1(T^2)$ by matrix multiplication on the right.

Lemma 2.4.22. Let V be a solid torus and let f be an automorphism of the torus ∂V. Then f extends to an automorphism of V if and only if f maps a meridian to a meridian.

Lemma 2.4.23. A pair of automorphisms f, g of V that agree on ∂V and map a meridian to a meridian are isotopic.

Theorem 2.4.24. The mapping class group of a solid torus V is isomorphic to the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ whose elements are of the form

$$\begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & k \\ 0 & \pm 1 \end{pmatrix}.$$

Proof. An element F of $mpg(V)$ must restrict to an element f of $mpg(\partial V)$. Because f is an automorphism of ∂V that extends to an automorphism of V, it sends a meridian to a meridian. In particular, f is of the form

$$\begin{pmatrix} 1 & k \\ 0 & n \end{pmatrix} \text{ or } \begin{pmatrix} -1 & k \\ 0 & n \end{pmatrix}.$$

Because $f \in \mathrm{SL}_2(\mathbb{Z})$, n must be ± 1.

Conversely, every element f in $mpg(\partial V)$ of the form

$$\begin{pmatrix} \pm 1 & k \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & k \\ 0 & \pm 1 \end{pmatrix}$$

maps a meridian to a meridian, so f extends to an automorphism F of V. $\square$

Remark 2.4.25. Let W be a 4–manifold with $\partial W = M$, let $f : S^1 \times D^2 \to M$ be a handle attachment map, and let $V = f(S^1 \times D^2)$.

Let J be an oriented longitude in V that we will call the *preferred framing* of V, and let $[J]$ be the isotopy class of J. There is a class $[F]$ of orientation preserving automorphisms of V in $mpg(V)$ mapping $[J]$ to $[f(S^1 \times \{1\})]$. We represent $[F]$ by

$$[F] = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},$$

which is possible because J is oriented and $[F]$ is orientation preserving. To sum up, any handle attachment is entirely determined by the embedding $f_0 : S^1 \to M$ and an integer k, provided the attachment is orientation preserving and we can define a preferred framing. We call the integer k the *framing coefficient* of the handle attachment.

Sometimes, there is a canonical class of preferred framings. One important case is when M is S^3. In this case, let $f_0 : S^1 \to S^3$ be an embedding and let V be a solid torus that is a closed tubular neighbourhood of the knot $f_0(S^1)$ in S^3. There is exactly one isotopy class $[J]$ of longitudes of V, unique up to ambient isotopy of f_0, such that any curve in $[J]$ bounds a disc in $S^3 \setminus V$. Any representative of $[J]$ can be taken as a preferred framing of V.

2.5 Triangulations

The *CW–complex* is our first foray into combinatorial descriptions of manifolds.

Definition 2.5.1. For $n > 0$, a topological space that is homeomorphic to the n–ball is called an *open n–cell*, and a topological space that is homeomorphic to the n–disc is called a *closed n–cell.* For $n = 0$, we say that a singleton set is both a closed and open 0–cell.

Let M be a Hausdorff space. A *cell decomposition* of M is a collection $T = \{t_\alpha\}$ such that every t_α is an open cell, every pair of cells of T is disjoint, and the union of the cells of T is exactly M. The *n–skeleton* of T is the disjoint union of the cells of T with dimension at most n, and it is denoted T^n.

Let T be a cell decomposition of the Hausdorff space M. Then T is a *closure–finite weak–topology complex* if the following are satisfied:

1. For any n–cell t_α in T, there exists a continuous map $f : D^n \to M$ such that $f|_{B^n}$ is a homeomorphism onto t_α and $f(S^{n-1})$ is contained in T^{n-1}.

2. The closure of any cell of T intersects only a finite number of other cells in T (i.e. T is closure finite).

3. A subset X of M is closed if and only if $X \cap \overline{t_\alpha}$ is closed for every t_α in T (i.e. the topology of M is weak).

It is standard to abbreviate such a cell decomposition as a *CW–complex.* The *dimension* of a CW–complex T is the highest dimension among the cells of T.

The 1–skeleton of a CW–complex is a graph with vertex set T^0 and edge set the collection of pairs (u, v) where $u \cup v$ is the boundary of the closure of a 1–cell. A CW–complex is *edge distinct* if no pair of edges e, f in T^1 have $\partial e = (u, v) = \partial f$, and in this case T^1 is a simple graph.

A map $\phi : T \to X$ where X is an m–manifold and T is a n–complex is an *embedding* if ϕ restricted to any cell of T is an embedding. That embedding is *flat* if for every point $\phi(p)$ of any top dimensional embedded n–cell $\phi(c)$ of T, there is a chart (U, f) of X about $\phi(p)$ for which $f(U \cap X)$ sits inside of $\mathbb{R}^n \subset \mathbb{R}^m$.

We concern ourselves mostly with a specific type of cell decomposition called a *triangulation.* The cells of a triangulation are the interiors of *n–simplices*, which can be thought of as generalizations of the most basic 2–dimensional building block, the triangle, to arbitrary dimensions.

Definition 2.5.2. Let $E = \{e_0, \ldots, e_n\}$ be a set of $n+1$ points in some n–dimensional affine space F such that

$$e_i \neq \sum_{j \neq i} t_j e_j$$

for each i and any collection of nonnegative real t_j with $\sum t_j = 1$. The *standard n–simplex* σ is defined as

$$\sigma = \{p \in F : p = \sum_{i=1}^{n} t_i e_i,\ e_i \in E,\ t_i \geq 0 \text{ for each } i, \text{ and } \sum_{i=1}^{n} t_i = 1\}.$$

The above construction is called taking the *convex hull* over E.

Let E' be a $k+1$ element subset of E. The convex hull over E' is called a *facet* or *k–facet* of σ and is itself a k–simplex. A zero–dimensional facet is a *vertex*, one–dimensional an *edge*, two–dimensional a *triangle*, three–dimensional a *tetrahedron* and four–dimensional a *pentachoron.* These names are also applied to the 0–, 1–, 2–, 3–, and 4–simplices. A facet with dimension $n-1$ is called a *face* of the simplex that contains it. We number the vertices of the n–simplex with the numbers $0, \ldots, n$. Every face of σ contains all but one vertex of σ, and this gives a numbering to the faces of σ. The i^{th} face of σ is found via the *face map* $F^i(\sigma)$. Removing all of the proper faces of a simplex leaves us with the *interior* of σ.

Let's start building some cell structures out of simplices. First, notice that a simplex has a canonical cell decomposition into the interiors of its facets. Next, let σ, τ be n–simplices, $n > 0$. A linear attaching map $F^i(\sigma) \to F^j(\tau)$ can be defined by mapping the vertices of $F^i(\sigma)$ to the vertices of $F^j(\tau)$ and then extending linearly over the facets of $F^i(\sigma)$. Let $g : F^i(\sigma) \to F^j(\tau)$ be such an attaching map, and form the space $\sigma \cup_g \tau$. There is also a canonical cell decomposition of $\sigma \cup_g \tau$ built from the interiors of the facets of σ and τ. Extending to a collection of n–simplices and gluing maps, a cell decomposition continues to be simple to define, and we may say the adjunction formed from simplices in a collection $\{\sigma_i\}$ over gluing maps $\{g_j\}$ has a canonical cell decomposition.

Definition 2.5.3. A CW–complex matching the description above is called an n–*dimensional simplicial gluing complex* or just *gluing complex.* The faces of a gluing complex T can be partitioned into two subsets based on whether the face has been glued to another face. If a face $F^i(\sigma)$ has been glued to another face $F^j(\tau)$, both faces are *glued.* Otherwise, $F^i(\sigma)$ is *unglued.* The set of unglued faces and all facets contained in those faces form the *boundary* of a gluing complex, and is denoted ∂T. The remaining facets are *internal.* A gluing complex with empty boundary is *closed.*

Put $n = k+m$ and let σ be an m–simplex in an n–gluing complex T and consider an n–simplex τ containing σ. There is a k–facet τ_k that is "opposite" σ in the sense that $\sigma \cap \tau_k = \emptyset$ and τ is the convex hull of $\sigma \cup \tau_k$. The subset of C built as the union of all τ_k corresponding to τ containing σ is the *link* of σ and is denoted $\mathtt{lk}(\sigma)$. Note that $\mathtt{lk}(\sigma)$ is an m–gluing complex in its own right.

If M is an n–manifold and T is an n–simplicial gluing complex that is homeomorphic to M, then we say that T is a *triangulation* of M. A triangulation T with the property that any pair of distinct edges share at most one vertex is called *edge distinct.*

There is a very slick way of checking whether a gluing complex is a triangulation of some n–manifold using only the links of vertices.

Theorem 2.5.4. An n–gluing complex T is the triangulation of some n–manifold M if and only if the link of every internal vertex v is homeomorphic to S^{n-1} and the link of every boundary vertex w is homeomorphic to D^{n-1} with $\partial\mathtt{lk}(w)$ contained in ∂T.

Definition 2.5.5. When we have a triangulation T for a closed, oriented n–manifold M, we can also define a structure dual to that triangulation that is useful for computation. This structure, denoted T^*, is a cell decomposition dual to T in the sense that each $(n-k)$–cell of T^* is associated with a k–cell of T. For a k–simplex σ of T, denote the dual cell in T^* to σ by σ^*. When T^* is fully defined, this forms a bijection between the k–cells of T and the $(n-k)$–cells of T^*. Now, σ is contained in a finite number of n–simplices τ_i, and we can fully describe σ^* by describing how σ^* meets these τ_i.

For a given τ containing σ, there are 2^{n-k} subsets of the vertices of τ, i.e. the 0–skeleton of τ, i.e. τ^0, that contain the vertices of σ. Each subset $s_j \subset \tau^0$ has a barycentre $b(s_j)$, so we form a collection $B_\tau(\sigma) = \{b(s_j)\}$ of these barycentres. Then the intersection of σ^* with τ is defined as the convex hull over $B_\tau(\sigma)$ inside of τ. The union of these hulls over all τ_i containing σ is then a closed $(n-k)$–cell whose interior is then defined to be the dual cell σ^*. The union of all σ^* forms a CW–complex T^* for M that we call the *dual decomposition* to T.

When T is a triangulated 3–manifold, the top dimensional dual cells of T^* are all solid polyhedra. We call a 3–dimensional CW–complex whose top–dimensional cells are all polyhedra a *solid polyhedral gluing.* Note that a 3–manifold triangulation is also a solid polyhedral gluing.

A description of a manifold through a cell decomposition offers us a finite, combinatorial object that we are able to tinker with and investigate using methods that are computable, and the algorithms found in future chapters all use cell decompositions as their main data type. The next logical question is whether cell decompositions accurately represent smooth manifolds. Fortunately, we are working in dimension at most 4. This means that a triangulation has an essentially unique smoothing, and triangulations of smooth manifolds exist and are well defined. The standard references for these facts are [7] with respect to unique smoothings, and [22] with respect to the existence of triangulations.

Chapter 3

Cobordism

We now delve into some arguments for smooth manifolds. Once the general idea is in place, the following chapters make explicit the corresponding construction for triangulations of manifolds. The problems of this chapter lie in the realm of cobordism.

Definition 3.0.1. Let W be a manifold with boundary $\partial W = M_1 \sqcup M_2$. We say that M_1 and M_2 are *cobordant* and W is a *cobordism* of the pair (M_1, M_2).

As discussed in the introduction, there are many ways to prove that a closed, oriented 3–manifold is cobordant to $\emptyset$. Our goal is to take a 3–manifold triangulation M and produce an explicit triangulated cobordism W of the pair $(M, \emptyset)$.

3.1 Morse Theory

Our primary use of Morse theory is to define a pair of related constructions. The first is the handle decomposition of a manifold that is induced by a Morse function, and this decomposition comes with a dual decomposition. The second is the *Stein complex* of a Morse function on a manifold. The main results from this chapter connect a Stein complex with a handle decomposition, and this construction is extended to triangulations in the remaining chapter.

Definition 3.1.1. A generic smooth function $f : M \to \mathbb{R}$ with M an n–manifold is called a *Morse function*. For a critical point $p \in M$ of f, the *index* of p is the dimension of the largest subspace of T_pM such that $d^2 f_p$ is negative definite. Less formally, this corresponds to the number of independent directions in M along which

f decreases. We denote by M^a the subspace $f^{-1}(-\infty, a\,]$ where f is a fixed Morse function on M.

We can expand the interpretation of the index of a critical point via the Morse lemma.

Lemma 3.1.2 (Lemma 2.2 of [11]). Let M be an n–manifold, let $f : M \to \mathbb{R}$ be a Morse function, and let p be an index i critical point of f. Then there exists a chart $(U, g = (\xi_0, \ldots, \xi_{n-1}))$ around p such that $\xi_j(p) = p_j = 0$ for each j and

$$g(q) = g(p) - q_0^2 - \cdots - q_{i-1}^2 + q_i^2 + \cdots + q_{n-1}^2$$

for each q in U.

A key question of Morse theory is how the topology of M^a changes as a passes the critical values of f. This question is answered by the following two theorems.

Theorem 3.1.3 (Theorem 3.1 of [11]). Let $f : M \to \mathbb{R}$ be a Morse function with no critical values in $(a, b\,]$, $a < b$. If $f^{-1}[\,a, b\,]$ is compact, then M^a and M^b are diffeomorphic, and M^b deformation retracts onto M^a.

Theorem 3.1.4 (Theorem 3.2 of [11]). Let $f : M \to \mathbb{R}$ be a Morse function. Let p be a critical point of f of index λ with associated critical value $f(p) = q$. If $f^{-1}[\,q - \varepsilon, q + \varepsilon\,]$ is compact and contains no critical points other than p, then $M^{q+\varepsilon}$ is diffeomorphic to $M^{q-\varepsilon} \cup_\varphi H^\lambda$ for some attaching map $\varphi : S^{\lambda-1} \times D^\mu \to \partial(M^{q-\varepsilon})$.

Let $f : M \to \mathbb{R}$ be a Morse function on a closed n–manifold M. If f has critical points $\{p_1, \ldots, p_k\}$ of indices $\{\lambda_1, \ldots, \lambda_k\}$ such that

$$0 \leq t_0 < f(p_1) < t_1 < f(p_2) < \cdots < t_{k-1} < f(p_k) < t_k \leq 1$$

and

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k,$$

then Theorem 3.1.4 guarantees that, for each $i \geq 1$, $f^{-1}[\,t_{i-1}, t_i\,]$ is diffeomorphic to $(f^{-1}(t_{i-1}) \times \mathbb{I}) \cup H^{\lambda_i}$. Such a Morse function then defines a handle decomposition

$$\emptyset = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{k-1} \subset M_k = M$$

where M_i is obtained from M_{i-1} by attaching a λ_i–handle, and the Morse function $1 - f$ defines the dual decomposition.

Similar definitions can be made for manifolds with boundary. Let W be an n–manifold with boundary $\partial W = M_0 \sqcup M_1$, where both M_0 and M_1 are compact. The Morse functions on W that extend Theorems 3.1.3 and 3.1.4 are those that fix $f(M_0) = 0$, $f(M_1) = 1$. Now, the handle decomposition of W is built on top of the component $M_1 \times \{1\}$ of $M_0 \times \mathbb{I}$, with $M_0 \times \{0\}$ corresponding to the boundary component M_0 in the finished product.

We use Stein factorizations and complexes to get our hands on explicit handle decompositions.

Definition 3.1.5. A function $f : X \to Y$ is *proper* if, for any compact subset $C \subset Y$, the preimage $f^{-1}(C)$ is compact in X. A *Stein factorization* of a proper function f is a decomposition of f into $g \circ h$ with

$$X \xrightarrow{h} S \xrightarrow{g} Y$$

such that g is a finite–to–one map and h has connected fibers. The intermediate space S is called the *Stein complex* of the factorization.

The Stein complex of a proper function $f : X \to Y$ is the quotient space of X by a particular relation $\sim$. For a pair of points x_1, x_2 in X with $f(x_1) = f(x_2) = y$, we define $x_1 \sim x_2$ if x_1 and x_2 are in the same connected component of $f^{-1}(y)$.

The particulars of a Stein complex are explored in the next two sections, where our ultimate goal is to use the Stein complex as a set of instructions for constructing explicit cobordisms.

3.2 2–Manifolds Bound 3–Manifolds

We now have the necessary framework to demonstrate that a closed, oriented 2–manifold bounds a 3–manifold through an explicit construction. This is useful as a jumping off point for the case one dimension higher which draws on the same framework and methods.

Let Σ be a closed, oriented 2–manifold. Our construction of a 3–manifold M for which $\partial M = \Sigma$ begins by letting $f : \Sigma \to [-1, 1]$ be a Morse function. We take as our base for a 3–manifold $M = \Sigma \times [\,-1, 1\,]$. Then M has two boundary components Σ^- and Σ^+ corresponding to $\Sigma \times \{-1\}$ and $\Sigma \times \{+1\}$ respectively. Our goal is to attach handles to M over Σ^+ until that boundary component is reduced to $\emptyset$. The construction follows the general structure of

1. Decompose Σ^+ via our Morse function.

2. Some of the pieces into which Σ^+ is decomposed serve as attaching regions in M for 1–handles.

3. The attachment of 1–handles alters Σ^+ in a controlled way, and yields attaching regions for 2–handles. Attaching 2–handles over these regions empties Σ^+.

Before detailing our decomposition, we define the pieces into which our surface will decomposed.

Definition 3.2.1. An *annulus* is any surface homeomorphic to $S^1 \times \mathbb{I}$. An annulus's boundary consists of two components that are homeomorphic to S^1. Another description of an annulus is as a 2–disc minus an open ball.

Definition 3.2.2. A *pair of pants* is a surface that is homeomorphic to a 2–disc minus two disjoint open balls. The boundary of a pair of pants consists of three components that are homeomorphic to S^1.

Lemma 3.2.3. Let Σ be a closed, oriented 2–manifold and let $f : \Sigma \to [\,-1, 1\,]$ be a Morse function. Then f induces a decomposition of Σ as

$$\Sigma = A \cup D \cup P$$

where A is a collection of annuli, D is a collection of 2–discs, and P is a collection of pairs of pants. The intersection between any pair of elements in this decomposition is either empty or a boundary component of both. Further, discs and pants only intersect annuli.

Theorem 3.2.4. Let Σ be a closed orientable surface and $f : \Sigma \to [\,0, 1\,]$ a proper Morse function with distinct critical values. Then there exists a cobordism of the pair $(\Sigma, \emptyset)$ with an explicit handle decomposition described fully by a Stein complex of f.

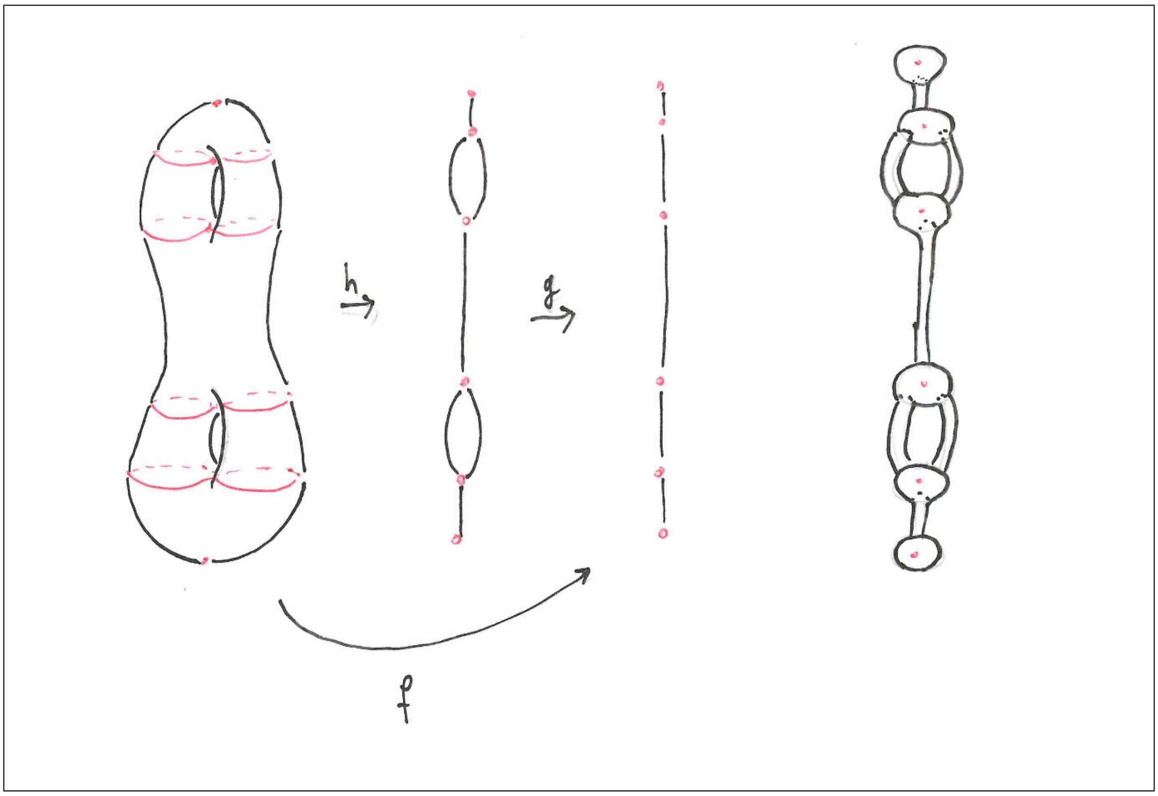

Figure 3.1: A typical Morse function $T^2 \# T^2 \to \mathbb{R}$, its Stein factorization, and corresponding dual handle decomposition.

Proof. Figure 3.1 depicts a typical Morse function acting on the closed oriented surface of genus 2. We begin by examining the preimages of points $x \in \mathbb{R}$. Denote the space $f^{-1}(x)$ by Σ_x, a small interval around x by $\varepsilon(x) = [\, x - \varepsilon, x + \varepsilon \,])$, and the neighbourhood in Σ around Σ_x by $f^{-1}(\varepsilon(x)) = \varepsilon(\Sigma_x)$. The spaces Σ_x and $\varepsilon(\Sigma_x)$ may not be connected, so we index the connected components by superscript. Let p be a critical point of f with critical value $f(p) = x$. By our assumption that critical values are distinct, p is the only critical point of f in Σ_x. The connected component of Σ_x containing p is called the *singular fiber* at p and is denoted Σ_x^p. The connected component of $\varepsilon(\Sigma_x)$ containing p is called the *critical neighbourhood* at p and is denoted $\varepsilon^p(\Sigma_x)$.

By the regular value theorem, any regular value pulls back through f^{-1} to a disjoint collection of circles in Σ called *regular fibers*. Likewise, a neighbourhood $\varepsilon(x)$ that contains no critical values of f pulls back to a disjoint collection of annuli in Σ. The restriction of f to the components of Σ_x that do not contain a critical point has x as a regular value, so the remaining connected components of Σ_x are all copies of S^1 that we index by Σ_x^i for $i = 1, \ldots, k$, and their associated neighbourhoods are the annuli $\varepsilon^i(\Sigma_x)$. When referring to an arbitrary connected subspace of Σ_x that could be the singular fiber Σ_x^p or a regular fiber Σ_x^i, we will use the notation Σ_x^*.

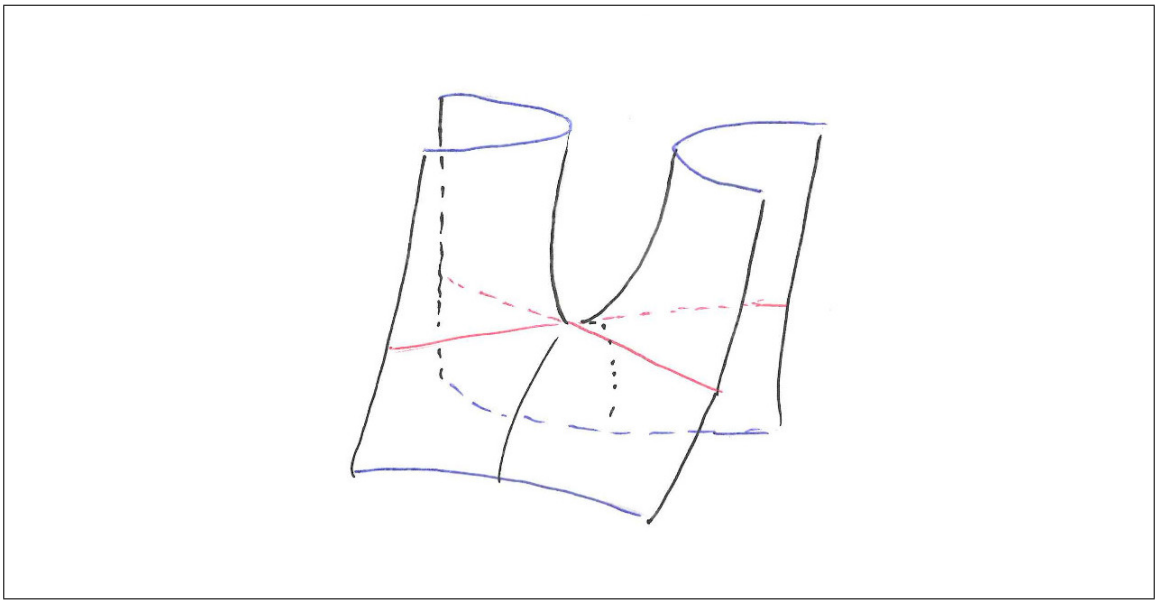

Figure 3.2: Smoothing a cross into a saddle

The shape of a singular fiber Σ_x^p is determined by the index of p. Because f is a Morse function whose domain is a surface, its critical points are easily classified by the Morse lemma (Lemma 3.1.2). Locally in Σ, a critical point of index 0 is a minimum, of index 1 a saddle, and of index 2 a maximum.

Let x be a critical value for the critical point p and take ε to be small enough that $\varepsilon(x)$ contains no critical values other than x. When p is of index 0 or 2, we can immediately deduce that $\varepsilon^p(\Sigma_x)$ is diffeomorphic to a disc. When p is of index 1, the Morse lemma tells us that Σ looks like a standard saddle near p. The intersection of Σ_x^p with this saddle is a cross whose centre is p. For $y \in \varepsilon(x)$, y is a regular value whose preimage is a disjoint union of circles. The circles above and below the saddle singularity are the result of smoothing out the cross into a pair of oriented arcs, done in two possible ways. The orientations of these circles orient the cross, which has two incoming arms and two outgoing arms which appear in alternating order. A Morse function has distinct singular fibers, so the cross we know about in Σ_x^p must have its arms connected in Σ_x^p through nonsingular orientation–preserving arcs. We can then see that Σ_x^p is a figure 8, and $\varepsilon^p(\Sigma_x)$ is a pair of pants in Σ.

This analysis is sufficient to form a handle decomposition for a cobordism of $(\Sigma, \emptyset)$. We begin with the 3–manifold $\Sigma \times \mathbb{I}$ whose boundary components $\Sigma_0 = \Sigma \times \{0\}$ and $\Sigma_1 = \Sigma \times \{1\}$ are copies of Σ, and let $f : \Sigma_0 \to \mathbb{I}$ be a Morse function with distinct critical values x_i. The general idea is that we attach handles to Σ_0, altering that boundary component until it is empty. The x_i partition $\mathbb{I}$ into open intervals containing only regular values. We take the associated regular annuli to be attaching regions for 3–dimensional 2–handles, and then fill in what remains with 3–dimensional 3–handles.

For an interval (x_i, x_{i+1}), consider the subinterval $\varepsilon(t_i)$ where $x_i < t_i < x_{i+1}$ and ε is small enough that $\varepsilon(t_i)$ contains neither x_i nor x_{i+1}. Take the associated regular annuli $\varepsilon(\Sigma_{t_i})$ to be the attaching regions for 3–dimensional 2–handles. The boundary component that was once Σ_0 is now the result of removing the interiors of the annuli $\varepsilon(\Sigma_{t_i})$ from Σ_0 for each t_i, and then introducting discs parallel to the cores of the attached 2–handles. These discs are seen as $D^2 \times \{0\}$ and $D^2 \times \{1\}$ inside of a 2–handle $D^2 \times D^1$. The boundary circles of the discs are $\Sigma_{t_i-\varepsilon}$ and $\Sigma_{t_i+\varepsilon}$ for each i.

There are now intervals about each critical point that we know, from our analysis above, pull back to discs, annuli, and pairs of pants. Each of these subsurfaces have circle boundaries that correspond to the regular fibers $\Sigma_{t_i-\varepsilon}$ and $\Sigma_{t_i+\varepsilon}$ which were capped with discs in the previous step. The altered boundary described before is therefore a collection of copies of S^2, which we take to be the attaching regions of 3–dimensional 3–handles.

With these handles attached, we obtain $(\Sigma \times \mathbb{I}) \cup \{2\text{–handles}\} \cup \{3\text{–handles}\}$, which is a 3–manifold whose only boundary component is Σ_1, i.e. is a cobordism of the pair $(\Sigma, \emptyset)$.

The corresponding dual handle decomposition is realized by turning the process upside down. Where we previously had 3–handle attachments with cores $D^3 \times \{\vec{0}\}$ and cocores $\{\vec{0}\} \times \{\vec{0}\}$, we now have 0–handles with cores $\{\vec{0}\} \times \{\vec{0}\}$ and cocores $D^3 \times \{\vec{0}\}$. In other words, the dual construction begins by taking the disjoint union of a collection of 3–discs that correspond to the space obtained by capping the boundary circles of the components $\varepsilon(\Sigma_{x_i})$ with 2–discs and then filling the spheres with 3–balls. Where we previously had 2–handle attachments with cores $D^2 \times \{\vec{0}\}$ and cocores $\{\vec{0}\} \times D^1$, we now have 1–handle attachments with cores $\{\vec{0}\} \times D^1$ and cocores $D^2 \times \{\vec{0}\}$. In other words, we connect the 0–handles together using 1–handles. The old belt 0–spheres of the 2–handles in the previous construction correspond here to new attaching 0–spheres, and the attaching maps are chosen to preserve orientability cf. Remark 2.4.20. The new attaching 0–spheres bound the new cores, the old core 2–discs are now cocores. This construction yields a 3–manifold whose boundary is exactly Σ, and is indeed another cobordism of the pair $(\Sigma, \emptyset)$.

A Stein factorization $f = g \circ h$ is simple to describe. Define the equivalence relation $\sim$ on the points in Σ by putting $p \sim q$ if and only if $f(p) = f(q) = x$ and p and q are in the same subspace Σ_x^*. Then h is the quotient map $\Sigma \to \Sigma/\sim$ where the points of $\Sigma/\sim$ are the subspaces Σ_x^*, and g is the map $\Sigma/\sim \to \mathbb{R}$ defined by $g(\Sigma_x^*) = x$.

The Stein complex $S = \Sigma/\sim$ can be viewed as a graph G. A critical value $x = f(p)$ has as its preimage one associated singular fiber Σ_x^p in S plus a number of copies of S^1 (possibly 0) given by Σ_x^i. We take the associated points $h(\Sigma_x)$ in S for each critical value to be the vertex set $v(G)$. For a pair of adjacent critical values x_j, x_{j+1}, an appropriate choice of ε and $x \in (x_j, x_{j+1})$ yields a collection of regular annuli $\varepsilon(\Sigma_x)$ in Σ that has boundary inside of $\Sigma_{x_j} \cup \Sigma_{x_{j+1}}$. In S, we find $h(\varepsilon(\Sigma_x))$ to consist of 1–dimensional strands that connect components of Σ_{x_j} and $\Sigma_{x_{j+1}}$. The set of pairs (v, w) of connected subspaces with $v \in \Sigma_{x_j}$ and $w \in \Sigma_{x_{j+1}}$ such that v and w are connected by a strand in S forms the edge set $e(G)$.

At this point, G corresponds exactly to the dual handle decomposition given earlier. For each vertex of G we get a 0–handle. For each edge, a 1–handle. We attach 1–handles with only one demand: that the resulting space continues to be orientable. $\square$

The Stein complex obtained in the proof of Theorem 3.2.4 may have superfluous vertices. In particular, a vertex v of G that is adjacent to exactly two verticex u and w via the edges (u, v) and (v, w) may be replaced, along with its adjacent edges, by a single edge (u, w). To see that the handle decomposition described by this Stein complex graph is equivalent, examine the attaching region of the 1–handle corresponding to (u, v) as it sits inside the boundary of the 0–handle corresponding to v. This region may be isotoped over the 1–handle corresponding to (v, w), where it ends up in the boundary of the 0–handle corresponding to w. We are left with a 0–handle v and 1–handle (v, w) that are contractible into w.

3.3 3–Manifolds Bound 4–Manifolds

To extend the results of the previous section to the case of 3–manifolds bounding 4–manifolds, we consider generic proper smooth maps from closed orientable 3–manifolds into the 2–ball B^2. Such a map is Morse–like in the sense that its singular

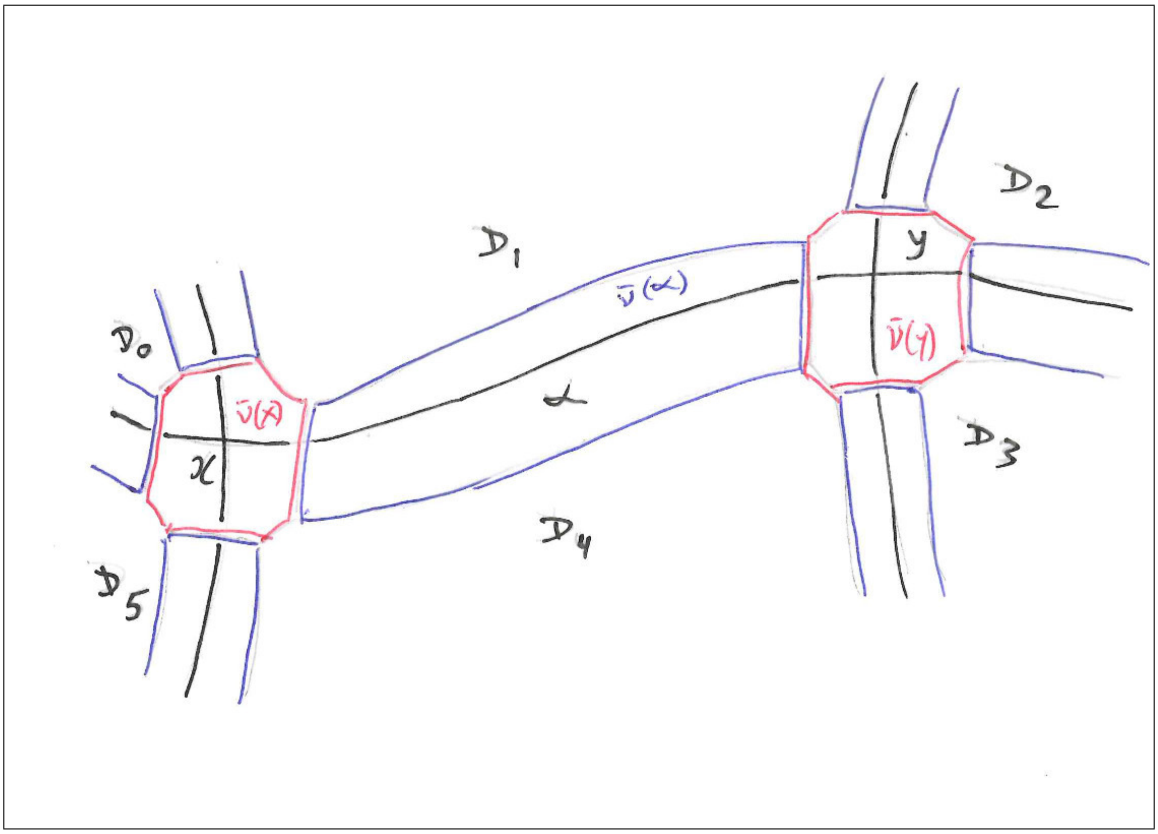

Figure 3.3: Closed sleeve around an arc of codimension 1 critical values and the two neighbouring codimension 2 critical values

set is well behaved, it can be studied via the same techniques as in Section 3.2, and we may recover a Stein factorization and Stein complex that define a handle decomposition for a bounding 4–manifold.

Throughout this section M is a closed orientable 3–manifold and f is a generic proper smooth map $M \to B^2$. The image $X = f(s_f)$ forms the space of critical values for f in the plane. By Sard's theorem, X is a set of closed arcs in the plane. By genericity of f, we can take the set of arcs to intersect only transversely and in such a way that $B^2 \setminus X$ is a set of regions $R_0, \ldots, R_k, R_\infty$ such that each R_i, $i < \infty$ is homeomorphic to B^2, R_∞ is homeomorphic to a open annulus, and one of the boundary components of $\overline{R_\infty}$ inside of $\overline{B^2} = D^2$ is exactly ∂D^2.

We decompose X into arc crossings and arcs that connect crossings. The arcs are *codimension 1 critical values* and the crossings are *codimension 2 critical values.* This induces a structure on X as a 4–valent graph whose vertex set consists of the set of codimension 2 critical values. The edge set of X then consists of pairs p, q such that p and q corresponds to codimension 2 critical values connected by an arc of codimension 1 critical values in X. For this reason we also call an arc of critical values of X an *edge*, and a codimension 2 critical value of X a *vertex* of X.

Lemma 3.3.1. There is a decomposition of B^2 induced by f into a collection of 2–manifolds with corners. All but one of these cornered surfaces are homeomorphic to D^2, these cornered discs intersect one another along intervals embedded in their boundaries, and boundary points of these intervals are exactly the disc corners.

Proof. Refer to Figure 3.3 for the general idea. Around each vertex x of X we fit an octagon $\overline{\nu}(x)$ such that the edges of the octagon alternate between transverse intersection and nonintersection with an arc of codimension 1 critical values.

Consider $B^2_* = \overline{B^2 \setminus \cup_x \overline{\nu}(x)}$. The topological boundary of B^2_* consists of the octagonal boundaries of the $\overline{\nu}(x)$. The edges of X connect the boundary components of B^2_*, so we may fit neat closed tubular neighbourhoods around the edges in such a way that the corners of this neighbourhood agree with the corners of the $\overline{\nu}(x)$. For α an edge of X, $\overline{\nu}(\alpha)$ is the sleeve around α.

Now, $B^2_{**} = \overline{B^2_* \setminus \cup_\alpha \overline{\nu}(\alpha)}$. Each connected component is either a subset of R_∞ or an open octagonal subset of a region $\mathbb{R}_i$. We take the topologinal closure of these octagons in B^2 and call these spaces by D_i.

The union of the topological boundaries of the decomposing blocks form a subspace Y of B^2 that intersects X only along edges of X. By construction, Y has the structure of a 3–valent graph whose vertices are its corners and whose edges are the edges of the decomposing blocks for B. □

Lemma 3.3.2. There is a decomposition of M induced by f and Lemma 3.3.1 into distinct subspaces, each of which is a manifold with corners that is homeomorphic to a $(3,1)$–handlebody of genus 0, 1, 2, or 3. These cornered handlebodies intersect one another along $(2,1)$–handlebodies embedded in their boundaries and the boundary circles of these embedded $(2,1)$–handlebodies are exactly the corners of the $(3,1)$–handlebodies.

Proof. Take Y to be the graph obtained in Lemma 3.3.1 that is embedded in B^2 and decomposes B^2. We examine how $f^{-1}(Y)$ sits in M. In particular, $M \setminus f^{-1}(Y)$ is a collection of open $(3,1)$–handlebodies, and their topological closure decomposes M exactly as desired.

First consider the vertices of Y. Some of these vertices are outside the image of f, so we disregard them entirely. The rest are regular points of f, so their preimages are disjoint collections of circles in M. For a vertex v of Y, we call the preimage by M_v and index the individual circles by $M_{v,k}$.

Now, the edges of Y. Again, some edges of Y are outside of $f(M)$, so we disregard them. Of the rest, we first consider those that do not intersect X. These are strands of regular points of f, so their preimages are disjoint collections of annuli in M as in Lemma **??**. An edge e of this type is one of the edges bounding an octagon D_i interior to R_i, and it does so as the shared boundary of D_i with either an edge sleeve $\overline{\nu}(\alpha)$ of X or a vertex sleeve $\overline{\nu}(x)$ of X. We call the preimage annuli by $M_{e,1}$ and index the connected components by $M_{e,1}^{k_u,k_v}$, where u and v are the vertices of Y that bound e and M_{v,k_v} and M_{u,k_u} are the boundary circles of the component. The term "1" in the subscript denotes the genus of the $(2,1)$–handlebody that projects over the edge e.

If an edge e of Y intersects an edge α of X, then $f^{-1}(e)$ is the disjoint union of a collection of some number of annuli (possibly zero) with a pair of pants or a disc, as in Lemma **??**. We call the preimage annuli in this collection by $M_{e,1}^{k_u,k_v}$ as before. If there is a disc projecting over e, it is denoted $M_{e,0}^{k_u}$, and if there is a pair of pants it is $M_{e,2}^{k_u,k_v,k_w}$, where the indexing follows the established pattern.

Pulling back Y to $f^{-1}(Y) = M_\Sigma$ yields a collection of $(2,1)$–handlebodies of genus 0, 1, and 2 that intersect other handlebodies only along their boundary circles. These circles are the union

$$M_\Sigma^c = \bigcup_{v \in v(Y)} \Big(\bigcup_k M_{v,k} \Big),$$

and $M_\Sigma \setminus M_\Sigma^c$ is just the collection of open $(2,1)$–handlebodies that project over the interiors of the edges of Y.

We now show that $M \setminus M_\Sigma$ is a disjoint collection of open sets, each of which is homeomorphic to the interior of a $(3,1)$–handlebody of genus 0, 1, or 2.

Let D be a disc in the decomposition of B^2. For p a point interior to D, p is regular. We invoke the same arguments made in Lemmas **??** and **??**, finding a small tubular neighbourhood about p, then taking the union of tubular neighbourhoods to form an open neighbourhood that projects surjectively over D. This implies that the preimage of $\texttt{int}(D)$ is a collection of open solid tori whose topological boundary in M sits inside of M_Σ as a collection of disjoint cornered surfaces homeomorphic to tori. We know that a pair of tubular neighbourhoods to disjoint circles over the same point reside in two disjoint solid tori over D because a pair of fibers join into a single circle only at a critical point, and every value in D is regular. This also means that the annuli over the edges of.

For an arc sleeve $\overline{\nu}(\alpha)$,

$\square$

Lemma 3.3.3. Let M have the decomposition of Lemma 3.3.2. If M is the boundary of some 4–manifold W, then 4–dimensional 2–handles can be attached to W over the regular genus 1 handlebodies in the decomposition of M in such a way that the induced surgery on M turns every genus 0, 1, and 2 codimension 1 handlebody into a copy of $S^2 \times D^1$.

Lemma 3.3.4. Let M^2 have the decomposition of Lemma 3.3.2 surgered by the 2–handle attachments of Lemma 3.3.3, and take M^2 to be the boundary of some 4–manifold W^2. Then 4–dimensional 3–handles can be attached to W^2 over the copies of $S^2 \times D^1$ in M^2 formed in Lemma 3.3.3, and the surgery on M^2 induced by the attachments of this lemma and Lemma 3.3.3 turns every genus 0, 1, 2, and 3 codimension 2 handlebody into a copy of S^3.

Theorem 3.3.5. The map f determines entirely a handle decomposition for a cobordism of the pair $(M, \emptyset)$.

Proof. The preceding lemmas provide a

$\square$

Theorem 3.3.6. Let M be a closed orientable 3–manifold. Then a proper generic smooth map $f : M \to B^2$ determines a handle decomposition for a cobordism of the pair $(M, \emptyset)$.

Proof. By Sard's theorem, the image of the singular set $f(s_f)$ in B^2 has Lebesgue measure 0. By genericity, and because the maps constructed in Chapter 4 satisfy this requirement, we take s_f to consist of a set of arcs in the plane that intersect each other only pairwise and transversally, in such a way that $B^2 \setminus s_f$ consists of a collection of regions R_i homeomorphic to copies of B^2 along with a single annular region R_∞ that does not intersect $f(M)$, and in such a way that $f(s_f)$ has a natural structure as a simple undirected 4–valent planar graph. We classify the critical values of f as codimension 1 inside of arcs away from crossings, and codimension 2 at arc crossings. This classification makes the simple undirected planar graph structure of $f(s_f)$ explicit. The vertex set is given by the set of codimension 2 critical values and the edge set is the collection of codimension 1 critical values.

We obtain a 4–manifold with boundary M by attaching 2–handles corresponding to the regular regions R_i, 3–handles corresponding to codimension 1 singularities, and 4–handles corresponding to codimension 2 singularities.

We will first focus on a region R in B^2 that is not R_∞. Begin by "shrinking" R away from $f(s_f)$ into a space D that is homeomorphic to D^2 Every point of D has preimage a disjoint collection of circles by the regular value theorem, so we centre our analysis on an arbitrary regular value x and its regular fibers, where $M_{x,k}$ denotes the k^{th} regular fiber mapped over x by f. An appropriate closed tubular neighbourhood of $M_{x,k}$ will consist entirely of regular fibers that map into D and has the structure of a closed 2–disc bundle over the circle, hence is a solid torus. The neighbourhood can be extended to enclose the entirety of $f^{-1}(D)$.

In summary, an arbitrary point x_i is chosen from a closed 2–disc D_i which itself is taken as a shrinking of an open region R_i. For each k, the k^{th} regular fiber $M_{i,k}$ over x_i is taken as the zero section of a 2–disc bundle that maps over D_i. Such a bundle is an open solid torus, and the k^{th} bundle is denoted $V_{i,k}$. Of particular importance in $V_{i,k}$ are the zero section and a certain isotopy class of longitudes. The zero section $z(V_{i,k})$ is a regular fiber that maps over a point interior to D_i, and the longitudinal isotopy class is the one that contains regular fibers that map to single boundary points of D_i. This pair determines an attaching map for 4–dimensional 2–handle attached over $V_{i,k}$.

Consider an arc α of codimension 1 critical values that separates a pair of regions R_i and R_j where either i or j may be ∞. As when we considered regions, α has been shrunk away from the codimension 2 critical values it connects. Let $\overline{\nu}(\alpha)$ be a closed "sleeve" around α whose boundary intersects the boundaries of D_i and D_j as in Figure 3.3. The restriction of f to the preimage of a linear transversal to α that connects D_i and D_j is, itself, a Morse function. The intersection with α is a critical point of this restriction, and the same techniques from the proof of Theorem 3.2.4 may be applied to $f^{-1}(\overline{\nu}(\alpha))$. A strand of singular fibers over α consists of an interval in the case of an extremum for the associated Morse function or an interval crossed with the figure 8 when the associated Morse function yields a saddle singularity. The critical neighbourhood $f^{-1}(\overline{\nu}(\alpha))$ contains a connected component that is either $D^2 \times \mathbb{I}$ when we have cross sectional extremum, or $P \times \mathbb{I}$, where P is the pair of pants surface, i.e. the 2–sphere with three interior 2–balls removed, when the cross section yields a saddle. All remaining connected components are intervals

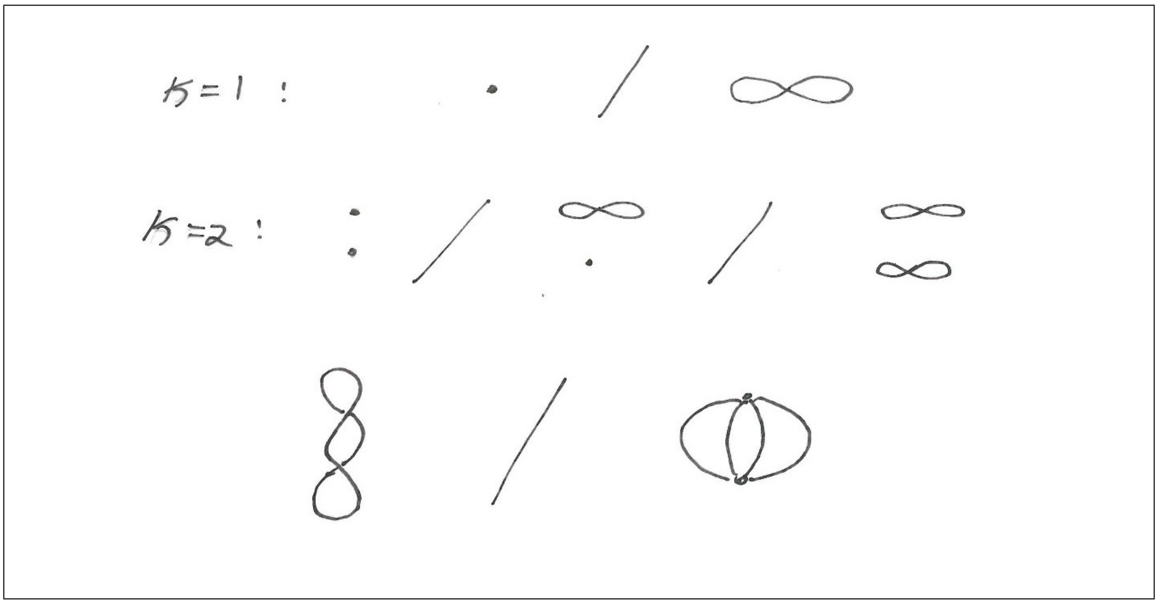

Figure 3.4: Possible singular fibers of a proper generic smooth map from an orientable 3–manifold to a surface

crossed with regular annuli. We use $A = S^1 \times D^1$, alternatively the 2–sphere with two interior 2–balls removed, to denote an annulus, and find $A \times \mathbb{I}$ to be the form of all remaining connected components.

Whenever one of the regions that α separates is R_∞, the cross section necessarily gives us an extremum, and the critical neighbourhood over $\overline{\nu}(\alpha)$ is $D^2 \times \mathbb{I}$. We call α a *definite fold* when the critical neighbourhood is $D^2 \times \mathbb{I}$, and an *indefinite fold* when the critical neighbourhood is $P \times \mathbb{I}$.

Let x be a codimension 2 critical value and $\overline{\nu}(x)$ the closed neighbourhoods whose boundary agrees with the shrunken regions and arc sleeves it is near as in Figure 3.3. The possible singular fibers over x are cataloged in [16], and Figure 3.4 displays them. The singular fibers over x may be disconnected. When that is the case, the fibers have the form seen in our codimension 1 analysis. Otherwise, the singular fiber has the shape of a 4–valent directed graph. The regular fibers are still copies of S^1. In each case $f^{-1}(\overline{\nu}(x))$ is a 3–dimensional thickening of the corresponding fibers, hence is a disjoint collection of handlebodies of genus 0,1,2, or 3. The neighbourhood of a regular fiber is still a solid torus, so all but at most two of the connected components of $f^{-1}(\overline{\nu}(x))$ will be genus 1 handlebodies.

We may now begin describing a handle decomposition for a cobordism W of the pair $(M, \emptyset)$. Begin with $M \times \mathbb{I}$, whose boundary is $M_0 \sqcup M_1$ with $M_0 = M \times \{0\}$ and $M_1 = M \times \{1\}$, and a proper generic smooth map $f : M_0 \to B^2$.

For each $i < \infty$ indexing the connected regions of $B^2 \setminus f(s_f)$, we index over k

the connected components of $f^{-1}(R_i)$. Consider $D_i \subset R_i$, a 2–disc of regular values, and $V_{i,k}$, the k^{th} regular solid torus that maps over D_i. The solid torus $V_{i,k}$ has a trivial disc bundle structure over the regular fiber $M_{i,k} = z(V_{i,k})$, and $f(M_{i,k}) = x$ in the interior of D_i. Because we consider a single solid torus for the rest of this argument, we abbreviate $V_{i,k}$ to V, $M_{i,k}$ to $z(V)$ and D_i to D.

Let's remember what we're doing here — we're attaching handles in such a way that, once all handles have been attached, the boundary component M_0 of $M \times \mathbb{I}$ has been filled in. In this step we attach 2–handles over V. Our goal is to do so in such a way that we remove the "bad" solid torus V and replace it with a solid torus that is "nice," where "nice" means that the newly introduced solid torus can be filled in by attaching 3– and 4–handles. This happens by deleting V and gluing 2–discs to the longitudes in ∂V that are regular fibers of points in the boundary of D. To attach a 2–handle over V, we need

1. The isotopy class of an embedding $g : S^1 \to M_0$, and
2. the isotopy class of an identification $G : S^1 \times D^2 \to \overline{\nu}(g(S^1))$.

The embedding $S^1 \to M_0$ is easy enough to define as we will just be identifying S^1 with $z(V)$. This S^1 lives as the attaching sphere $S^1 \times \{0\}$ inside of the 2–handle $D^2 \times D^2$ that we plan to attach. Taking $\partial(D^2 \times D^2) = (S^1 \times D^2) \cup (D^2 \times S^1)$ to be the genus 1 Heegard splitting of S^3, we will be defining a map $G : V \to S^1 \times D^2$, where $S^1 \times D^2$ is the first torus component. We call the other torus V^*. In order to satisfy our desired criteria, we need to have G be in the isotopy class of a map that takes a regular fiber J over a point in the boundary of D to $S^1 \times \{1\}$. When this is true, we may form the adjunction $(M \times \mathbb{I}) \cup_{G^{-1}} (D^2 \times D^2)$ and J bounds a disc in V^*. Every fiber in the interior of V now bounds a disc, and the new boundary is exactly V^*, whose meridians are the fibers that project over the boundary of D. This is the property we wanted — we've replaced the "bad" solid torus V with the "nice" solid torus V^*.

A class $[G]$ satisfying the above is unique and it exists. To see why, let $\varphi : V \to S^1 \times D^2$ be any trivialization of V, take $K = \varphi^{-1}(\{1\} \times S^1)$ to represent the meridinal isotopy class of V, and take L be the longitude in ∂V defined by $L = \varphi^{-1}(S^1 \times \{1\})$. Let $J = f^{-1}(q)$ in ∂V represent the isotopy class of regular fibers over boundary points of D, where q is an arbitrary point in the boundary of D. Figure 3.5 shows the curves J, K, and L with orientations sitting on $\varphi(V)$. Thinking of $S^1 \times D^2$ as

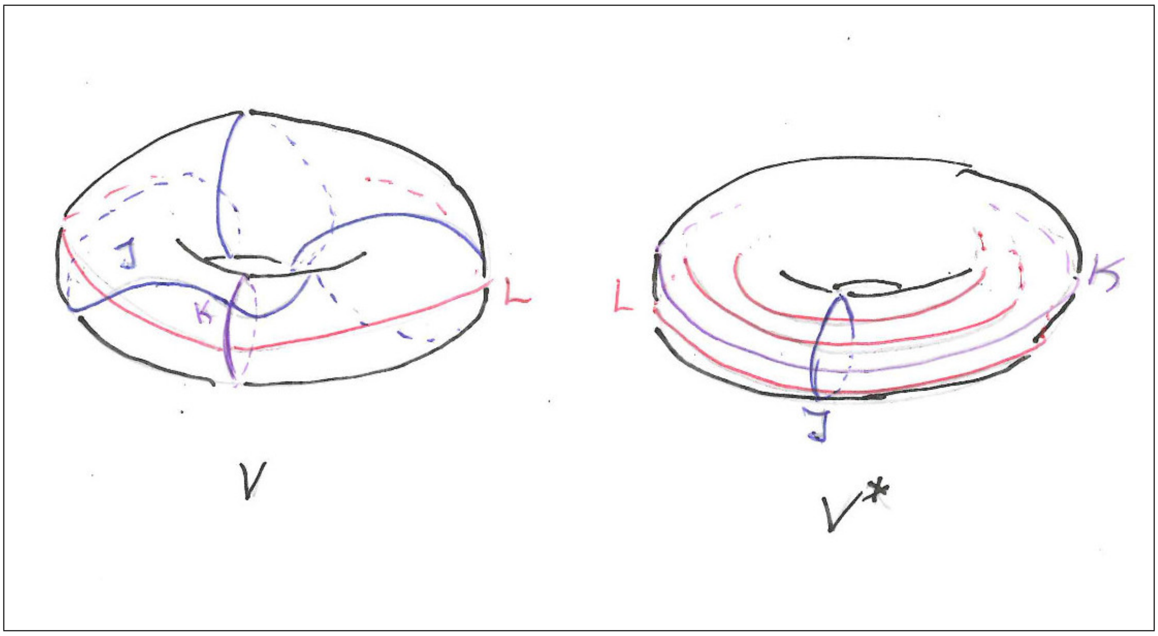

Figure 3.5: The solid tori V and V^* with boundary curves J, K, and L.

a subset of $\mathbb{C}^2$ gives it a natural orientation, which we have pulled back through φ onto K and L. We give J a compatible orientation so that its intersections with K and L can be counted. Put κ to be the oriented intersection number of J and L. Then $\varphi(J) \in \partial\varphi(V)$ is in the isotopy class of $h_M^\kappa(\varphi(L))$, where h_M is the meridinal twist defined in Theorem 2.4.24. Take $G = H_M^{-\kappa} \circ \varphi$, where H_M is the extension of h_M to the solid torus defined in Remark 2.4.25. In the boundary of $G(V)$, $G(J)$ is in the isotopy class of $S^1 \times \{1\}$, $G(K)$ is in the isotopy class of $\{1\} \times S^1$, and $G(L) = h_M^\kappa(S^1 \times \{1\})$. Existence of G is shown, and uniqueness up to isotopy comes from the same uniqueness in trivializations of V and of H_M.

With 2–handles attached, we move onto the preimages of arc sleeves. Let $\overline{\nu}(\alpha)$ be an arc sleeve, and consider the connected components of $f^{-1}(\overline{\nu}(\alpha))$. Figure 3.6 displays the possible connected components of arc sleeve preimages, each of which has the form of a surface crossed with the interval. The boundary circles of these surfaces at a cross section $\Sigma \times \{t\}$ project through f over the boundaries of shrunken regions, so they are filled with discs that sit inside of attached 2–handles from the previous step. In each case, we obtain a copy of $S^2 \times D^1$ over which we attach a 3–handle. The further modification to M_0 that takes place when we attach 3–handles can be thought of as the deletion of a copy of $S^2 \times D^1$ followed by gluing 3–discs over the newly created 2–sphere boundary components.

Finally, let x be a codimension 2 critical value and let $\overline{\nu}(x)$ be its sleeve. We find x at the crossing of a pair of strands of codimension 1 critical values, and those

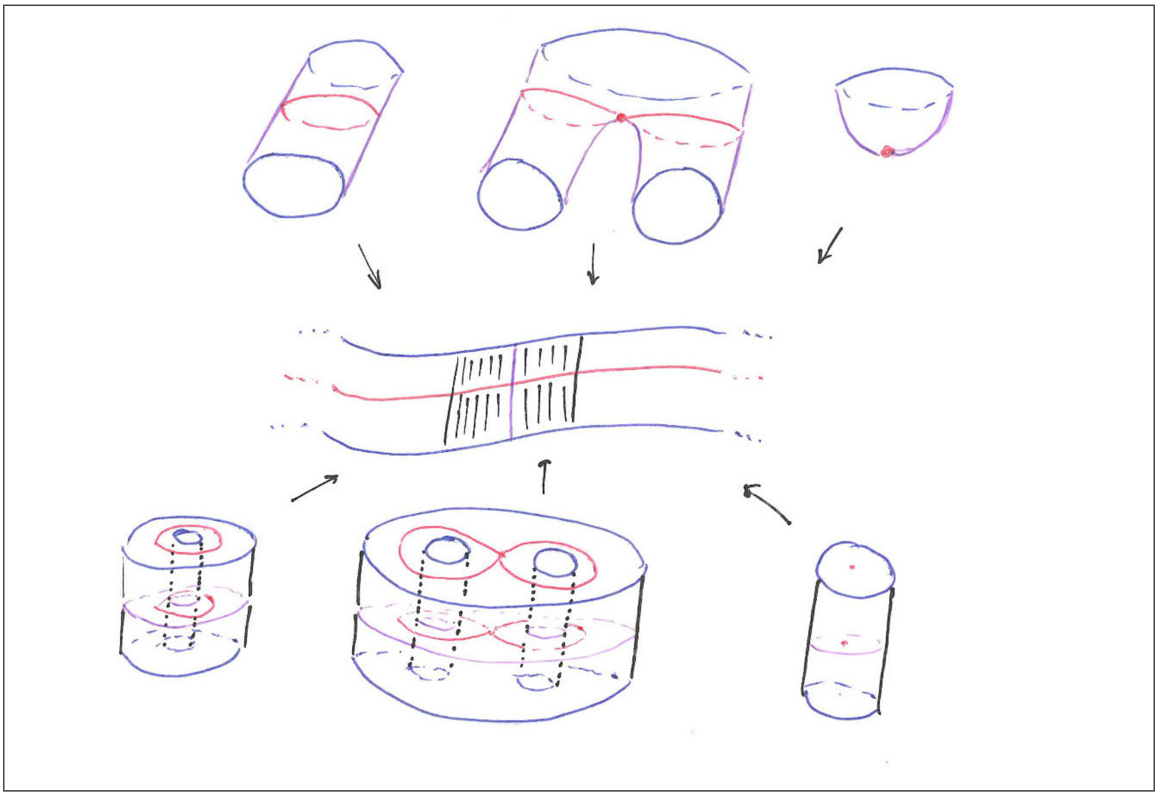

Figure 3.6: Possible connected components of arc sleeve preimages

strands are classified as definite or indefinite folds. The analysis of the codimension 2 critical value is then broken down into five cases. In four cases, we are able to extend the codimension 1 situation. This happens when the singular fiber is disconnected, called an *uninteractive* singular fiber, or when one of crossing folds is definite. When both folds are indefinite and the singular fiber is connected, then the singular fiber is called *interactive* and we defer to the analysis in [2].

First we look at the connected components of $f^{-1}(\overline{\nu}(x))$ that do not contain a singular fiber over x. These connected components are made from regular fibers, i.e. circles, that project over $\overline{\nu}(x)$, i.e. a 2–disc. As in the case of regions of regular values, they are solid tori $V_{x,k}$, using the same naming convention that has been established for solid tori that map over the regions D_i. Seeing $V_{x,k}$ as a 2–disc bundle over $f^{-1}(x)$, we can find an isotopy class of longitudes that project over single points in the boundary $\partial\overline{\nu}(x)$. These longitudes bound discs introduced during 2– and 3–handle attachment. The union of all such discs yields a solid torus $V^*_{x,k}$ in the boundary. The boundary component consisting of $V_{x,k}$ and $V^*_{x,k}$ is described by the Heegaard splitting $V_{x,k} \cup_\varphi V^*_{x,k}$ where φ takes a longitude of $V_{x,k}$ to a meridian of $V^*_{x,k}$. This is the standard genus 1 Heegaard splitting of S^3, so we have a copy of $S^3 \times D^0$ over which we may attach a 4–handle.

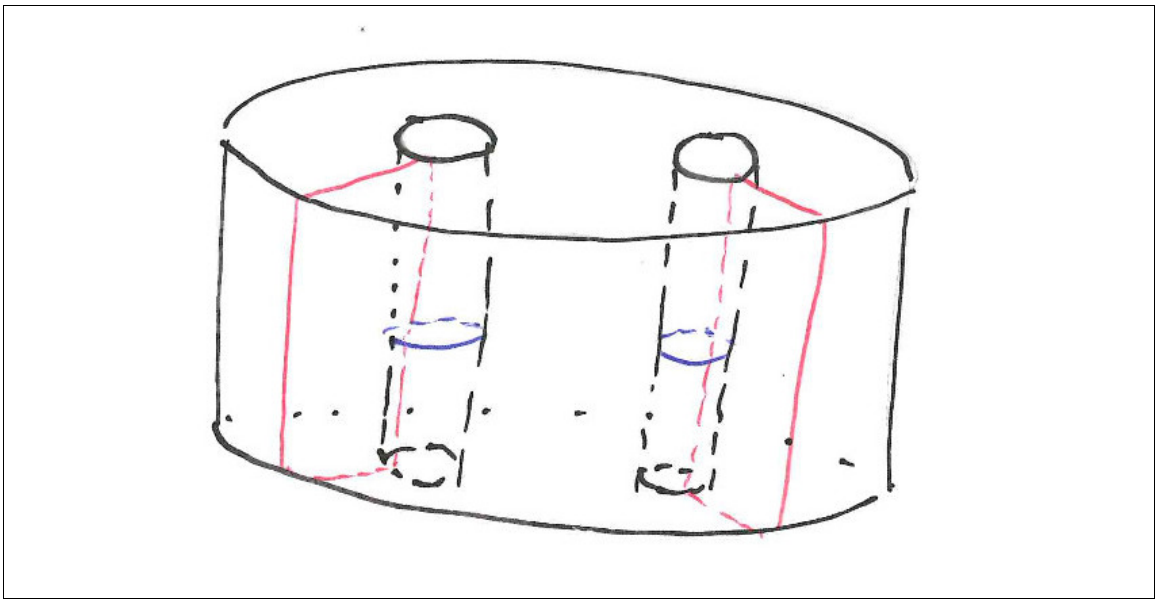

Figure 3.7: Destabilizing pairs for the pair of pants

The component that comes from an uninteractive definite fold is of the form $D^2 \times \mathbb{I}$ as in the case of codimension 1 critical values. The 2–sphere boundary of this shape is filled with 2–discs from our 2– and 3–handles, so this shape is the adjunction of a pair of 3–discs glued over their boundary. This is the genus 0 Heegaard splitting of S^3, so we have a copy of $S^3 \times D^0$ over which we may attach a 4–handle.

An uninteractive singular fiber that comes from an indefinite fold is the figure 8, and the preimage $f^{-1}(\overline{\nu}(x))$ is a closed tubular neighbourhood of the figure 8 graph, which is homeomorphic to the pair of pants crossed with an interval. Here, we find a Heegaard splitting of genus 2 with two destabilizing pairs, showing that the splitting is a splitting of S^3, and we can attach a 4–handle to it by Theorem 2.4.17. An appropriate choice of meridians in $f^{-1}(\overline{\nu}(x))$ would be a pair of curves that run from cuff to waist along the inside of each of the legs in the pair of pants, crest the waist, then down the outside of the legs back to the cuff. Meridians for the solid filled by our 2– and 3–handles would run around the cuffs of each leg. Figure 3.7 illustrates the idea.

When an indefinite and definite fold interact, we get the same shapes found in the case of an uninteractive definite fold. From a 1–dimensional Morse function viewpoint, we witness a homotopy of a pair of peaks separated by a saddle come together into a single peak, eliminating the saddle.

This leaves us with the case of interacting indefinite folds. The full analysis is covered in Section 4.4 of [2]. They show that filling in $f^{-1}(\partial\,\overline{\nu}(x))$ with 2–discs and gluing the two genus 3 handlebodies together over $f^{-1}(\partial\,\overline{\nu}(x))$ results in a Heegard splitting of S^3 by explicitly finding 3 destabilizing pairs. We attach 4–handles over these 3–spheres. Attaching a 4–handle over a 3–sphere fills in the 3–sphere with a 4–disc, so these final modifications completely eliminate what is left of M_0. We are left with a handle decomposition for a cobordism of the pair $(M, \emptyset)$ built on top of M_0 in $M \times \mathbb{I}$. $\square$

Let $f : M :\to B^2$ be a proper generic smooth function and W a the cobordism of the pair $(M, \emptyset)$, both as per the construction in Theorem 3.3.5. Because the handle decomposition produced for W consists of handles of index 2,3, and 4, the associated dual decomposition contains handles of degree 0,1, and 2. Our goal is to obtain a combinatorial description of this decomposition, and we find this through the Stein complex.

Recall that the Stein complex for a Morse function on a surface was given by a 1–complex. In the case of a proper generic smooth map $M \to B^2$, the Stein complex is given by a 2–complex S. The vertices of S correspond to 0–handles, the edges to 1–handles and the 2–cells to 2–handles.

We define the Stein factorization and complex of f as usual. Let $\sim$ be an equivalence relation on M defined by $p \sim q$ if and only if $f(p) = f(q) = x$ and p, q are in the same connected component of $f^{-1}(x)$. Then $f = g \circ h$ where h is the quotient map $M \to M/{\sim}$ and g takes a point in $f^{-1}(x)$ to x.

Theorem 3.3.7. Let f, M, S, and W be as we defined above. Then S is a 2–dimensional CW–complex that embeds flatly in W.

Proof. There are only a few locations where S could fail to be a CW–complex, and those are found over the critical values of f. We construct S as we would any other CW–complex, by iteratively attaching cells of increasing dimension, and define a map $\varphi : S \hookrightarrow W$ that is almost a flat embedding in a similar fashion. Let S be empty and begin by adding 0–cells to S.

For a codimension 2 singular value x of f, the fibers of over x were discussed in Theorem 3.3.5. The Stein factorization $f = g \circ h$ has h crush each fiber to a point, which we take as a new 0–cell in S. Let $\overline{\nu}(x)$ be a sleeve of x as in Theorem 3.3.5. In the construction of W, we attached 4–handles over 3–spheres with Heegaard decompositions over $f^{-1}(\overline{\nu}(x))$. The cocore of a 4–handle is a single point, so we associate the 0–cells of S with the cocores of 4–handles in W. Let x_i be a fiber over

x, let $H_i^4 \subset W$ be the 4–handle associated to x_i, let $c_i \in H_i^4$ be the cocore of H_i^4, and let v_i be the vertex of S associated to x_i. Define $\varphi(v_i) = c_i$.

We can now add edges to S. Let α be an open strand of codimension 1 critical points of f with $\partial\overline{\alpha} = \{x, y\}$, a pair of codimension 2 critical points of f. Pulling back α to M yields an open interval crossed with the fibers over any point of α and pulling back $\overline{\alpha}$ connects the fibers over x and y. For a fiber α_i over α with endpoints α_i^x sitting inside of a fiber over x and α_i^y sitting in a fiber over y, we add an edge attached over the vertices v_i^x and v_i^y associated to the fibers over x and y. This corresponds to the action of h on α_i, which crushes the interval of fibers to an interval.

Let $\overline{\nu}(\alpha)$ be a sleeve of α as in Theorem 3.3.5. To construct W we attached 3–handles over copies of $S^2 \times D^1$, those $S^2 \times D^1$ contained copies of $\Sigma \times D^1$ with Σ the 2–sphere with one, two, or three open 2–balls removed, and those $\Sigma \times D^1$ projected over $\overline{\nu}(\alpha)$. The cocore of a 3–handle is a copy of D^1, which corresponds to an edge in S. Let α_i be a fiber over α, H_i^3 the 3–handle associated with α_i, c_i the interval cocore of H_i^3, and e_i the edge of S corresponding to α_i with $\partial e_i = \{u, v\}$. Because e_i is a 1–cell, it is homeomorphic to an interval $[\,0, 1\,]$, so we take a slightly smaller closed subinterval e_i' in e_i (just as we can take $[\,\varepsilon, 1-\varepsilon\,]$ inside of $[\,0, 1\,]$) and define $\varphi(e_i') = c_i$. The endpoints of c_i are the belt sphere of H_i^3, and they intersect the 4–handles H_u^4 and H_v^4 corresponding to u and v. The intersection points are in the boundary 3–spheres of these 4–handles, so there are straight lines inside of the 4–handle that connect the cocore to these boundary points. Define φ on $e_i \setminus e_i'$ to be those straight line segments in the appropriate continuous way.

And finally, the 2–cells. Let R be an open ball of regular values in the plane, and let D be the 2–disc inside of R that pulls back through f to a disjoint collection of solid tori over which we attached 2–handles in our construction of W. Let H be the 2–handle attached over a solid torus V projecting over D. The boundary of H^2 is a 3–sphere with genus 1 Heegaard splitting consisting of V and a dual torus V^*. We attached H^2 because we wanted to fill the boundary of V with 2–discs that were easy to attach 4– and 3–handles over, and V^* is also contained in the union of these 4– and 3–handles. The vertices and edges of S corresponding to this collection 4– and 3–handles forms a cycle C in the 1—skeleton of the Stein complex already built. We know that R contains only regular values, so it pulls back to a disjoint collection of open solid tori. Let U be the open solid torus in this collection with

$f(U) = R$ and $V \subset U$. In particular, $\partial V = \partial V^* \subset U$. The fibers of S are fibers of f that project over the regular values in R, so $h(U)$ is a 2–ball. The closure of U intersects the singular fibers corresponding to the 4– and 3–handles that contain V^*. We take $h(U)$ in S to be a 2–cell attached over C.

To define φ on the 2–cell σ_U corresponding to U, we start by defining it on σ_V, a subdisc of σ_U (cf. the case of defining φ on the edges of S, where $\sigma_V \subset \sigma_U$ just like how $\{z \in \mathbb{C} : |z|^2 \leq 1 - \varepsilon\} \subset D^2$). Recall that H^2 is the 2–handle attached over V. It is useful to think of H^2 as a D^2 bundle over D^2, where the base space and zero section are the cocore C_V of H^2. We define $\varphi(\sigma_V) = C_V$, and then examine the intersection of C_V with the 4– and 3–handles of W. The collection of higher index handles that intersect H^2 is exactly the collection that contains V^*, and the intersection is exactly V^*. The cocore C_V intersects V^* in exactly the belt sphere of H^2. We then extend φ to $\sigma_U \setminus \sigma_V$ exactly as we did with the edges of S, by connecting the belt sphere of H^2 with the cocores of the 4– and 3–handles by line segments.

This demonstrates first that the Stein complex $S = h(M)$ has the structure of a 2–complex. Secondly, the map $\varphi : S \hookrightarrow W$ constructed here is piecewise–linear where it is not smooth, so can be smoothed into a flat embedding of a CW–complex. ☐

In Section 4.1 it is convenient to remove open balls from a given 3–manifold M and build a projection of M' instead. We want the results of this section to still apply, so we state this in Theorem 3.3.8

Lemma 3.3.8. Let M be a closed orientable 3–manifold, and let M' be M with a finite number of disjoint 3–balls removed. Call the disjoint 2–sphere boundary components of M' by $\partial_i M'$. Let $f : M' \to D^2$ be a proper generic smooth map such that $f(\partial M')$ is a disjoint collection of intervals in the boundary of D^2 and the image of the singular set of f away from $\partial M'$ is as described in the proof of Theorem 3.3.5. Then the Stein complex recovered for f in the manner of Theorem 3.3.7 is exactly the Stein complex recovered for an extension of f to M.

Proof. The singular fibers near $f(\partial M')$ are all definite folds, and extending f to 3-balls attached over the boundary components of M' yields an extension of that definite fold over the image of the 3–ball. ☐

All that's left is to realize the Stein complex as a set of instructions to build a handle decomposition of W, dual to the one obtained in Theorem 3.3.5, that relies only on f and S. We state this more precisely in Theorem 3.3.9.

Theorem 3.3.9. Let f, M, S, W be as defined above. Then there exists a 4–manifold W^* that is a cobordism of the pair $(M, \emptyset)$ in which S embeds flatly as in Theorem 3.3.7, and this manifold may be reconstructed from the combinatorics of S and the map f.

Proof. Let G be the 1–skeleton of S. We consider first the 4–handlebody obtained by attaching to $\emptyset$ a 4–dimensional 0–handle H^0_v for each vertex v of G, embedding v as the core of H^0_v, attaching a 1–handle H^1_e with orientation preserving attaching maps into ∂H^0_u and ∂H^0_v for each edge $e = (u, v)$ of G, embedding e as the core of H^1_e plus a pair of line segments to the cores ∂H^0_u and ∂H^0_v. The result is a 4–dimensional handlebody called the *4–thickening* of G, and is denoted by $H_4(G)$.

Consider now $U_S(G)$, a closed regular neighbourhood of G in S. This is equivalent to S minus an open ball inside of each 2–cell (as in Theorem 3.3.7 when we considered $\sigma_U \setminus \sigma_V$), or can also be seen as G plus an annulus attached by one of its boundary components over each cycle that a 2–cell was attached over in Theorem 3.3.7. In either case, $U_S(G)$ collapses onto G and the 4–thickening of $U_S(G)$ is homeomorphic to the 4–thickening of G. To build W, we make a 3–thickening of $U_S(G)$, and then a 4–thickening of the 3–handlebody obtained. This time, however, we pay attention to how the 2–cells of S interact with the thickening.

We built S to have a 1–skeleton G whose interior vertices are all of degree 4 and whose boundary vertices are all of degree 3. The 3–thickened regular neighbourhoods of the five possible vertex types and the three possible edge types in S are depicted in Figure 3.8. We use these to build the 3–thickening $H_3(U_S(G))$ of $U_S(G)$ where $U_S(G)$ is embedded in $H_3(U_S(G))$ and intersects the boundary of $H_3(U_S(G))$ in the curves depicted on the blocks in Figure 3.8. Thinking of $H_3(U_S(G))$ as generically a bundle of $[-1, 1]$ over $U_S(G)$, the curves would be the zero section of the bundle over the boundary of $U_S(G)$. Once we have $H_3(U_S(G))$, there is a unique orientable $[-1, 1]$ bundle over $H_3(U_S(G))$, and this is the 4–thickening of G. This bundle can be obtained either by specifying local trivializations that reverse the orientation of the $[-1, 1]$ factor along the curves in $H_3(U_S(G))$ along which the space becomes nonorientable, or by first building the 3–thickening of $T \subset G$, a maximal spanning

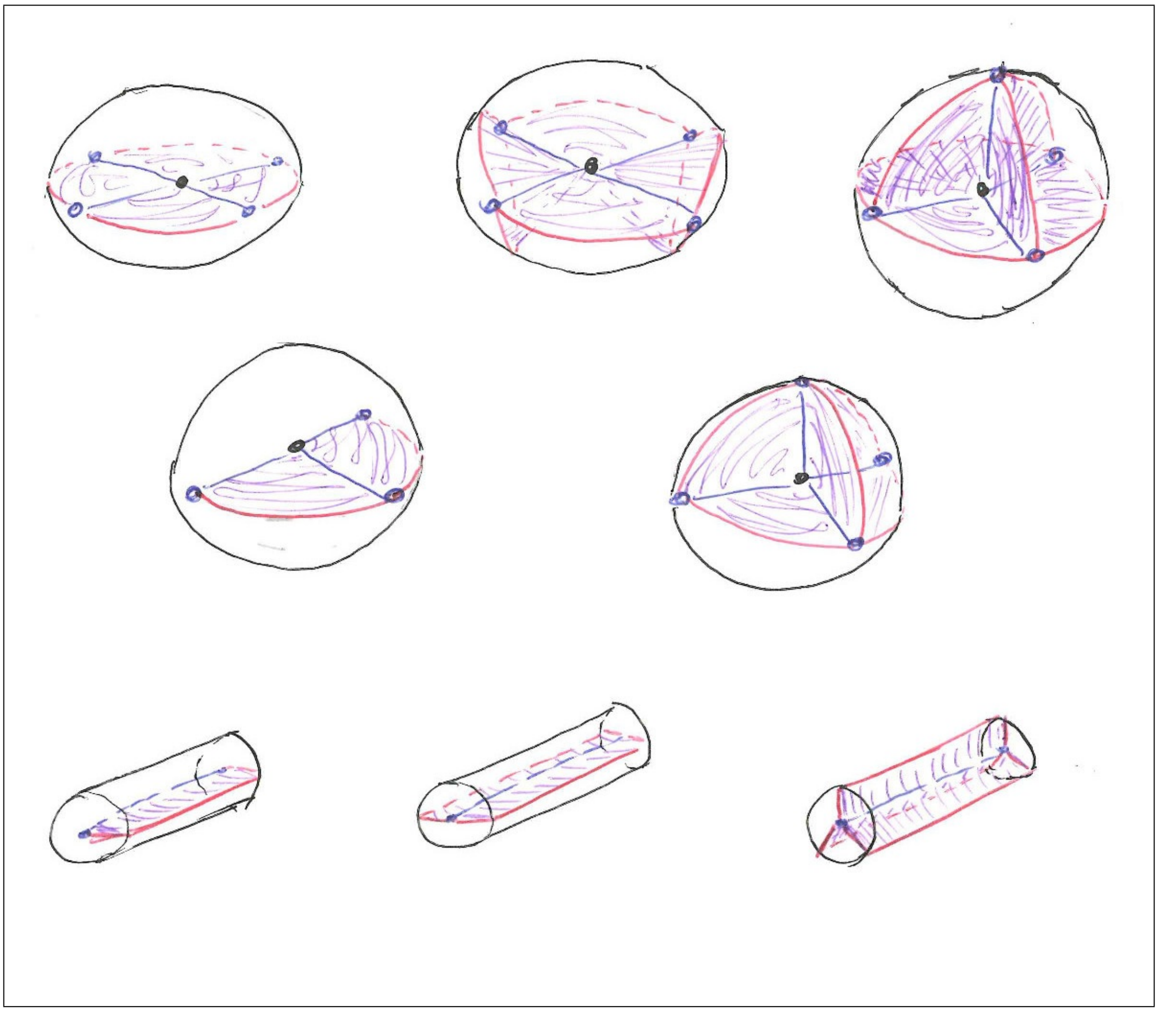

Figure 3.8: 3–thickening blocks for the 1–skeleton of the Stein complex

tree of G, 4–thickening that space, and then attaching all remaining 1–handles in the unique orientation preserving way. We choose the second construction.

Building $H_3(U_S(T))$ by adding an appropriate vertex block to $H_3(U_S(T))$ for each vertex of T, then connect them together with edge blocks in the unique way that is forced by the combinatorics of S for each edge of T. Take the product of this space with $[-1, 1]$. The result is a $[-1, 1]$–bundle over $H_3(U_S(T))$ and generically a $[-1, 1]^2$–bundle over $U_S(T)$. To extend this to G, we add 1–handles for each edge in G that is not in T. Every such edge has at least one 2-handle in S attached over it, and every 2–cell of S attaches over at least one edge added in this step. We use the product of our edge blocks with $[-1, 1]$ as the 4–disc structure for our 1–handles. The combinatorics of S and the requirement that $H_4(U_S(G))$ is orientable make these handle attachments unique. Our 4–thickening $H_4(U_S(G))$ is once again a $[-1, 1]$–bundle over $H_3(U_S(G))$ and generically a $[-1, 1]^2$–bundle over $U_S(G)$. The boundary circles of $U_S(G)$, corresponding to 2–cells of S and thus 2–handles of W, are now thickened to solid tori in $H_4(U_S(G))$. The last step to this construction is the attachment of 2–handles over these solid tori.

Let v^* be a boundary circle of $U_S(G)$ corresponding to a 2–cell c_v of S with thickening V^* in the boundary of $H_4(U_S(G))$. The first thing we must do is specify a canonical 0–framing on V^*, which we do by investigating the diagonal of the square fibers $[\,-1,1\,]$. As a subspace of V^*, the union of the fibers have the structure of an annulus or a Möbius strip. To see which we get, we turn to the completion of $H_4(U_S(T))$ into $H_4(U_S(G))$ accomplished by attaching 4–dimensional 1–handles. At least one such 1–handle corresponded to an edge of ∂c_v, and n of them were attached with an orientation reversal in both $[\,-1,1\,]$ factors. If n is even then the diagonal is an annulus and we put the canonical 0–framing as the trivialization that takes one boundary component of that annulus to $S^1 \times \{1\}$. If n is odd then the diagonal is a Möbius strip. In this case we take a curve that follows one half of the boundary component of the Möbius strip (i.e. once around V^* in the longitudinal direction along the boundary), then connects to itself via one half positive twist in the orientation of V^* (which is canonically oriented by the canonical orientations of the double intervals $[\,-1,1\,]$).

With a 0–framing specified, we just need framing coefficients. To recover those, we ask how our canonical framing sits inside of M. This question is well defined because the solid tori we are attaching 2–handles over are the V^* we examined in the proof of Theorem 3.3.5, and the boundaries of these tori are contained in M.

The complex S embeds flatly into W, and the restriction of that embedding to the 0– and 1–handles of W is exactly how $U_S(G)$ embeds in $H_4(U_S(G))$. Specifically, the zero section of the torus V^* above is the belt sphere of the 2–handle in W corresponding to the 2–cell c_v in S. This belt sphere is then the attaching sphere of the dual 2–handle in the handle decomposition of W^* and the framing coefficient is then the number of meridinal twists from the 0–framing to the longitude in V^* that is a meridian of V. Attaching a 2–handles over every such V^*'s yields a handle decomposition of a 4–manifold W^* that is dual to the decomposition of W acquired in Theorem 3.3.5. We conclude that W^* is the desired cobordism of $(M, \emptyset)$. □

There are a couple of points left to address in this chapter. The first is that of explicitly recovering the 0–framing curve used in the attachment of 2–handles in Theorem 3.3.9, a necessity for turning this process into an algorithm. The second is an acknowledgment of the origin of the ideas in this Section.

The 0–framing L is a curve in the shared boundary of V and V^* that is a number of meridinal twists of V^* away a meridian of V. This number is the framing coefficient we equip to the 2–cell of S representing 2–handle attachment over V^* in the construction of W^*.

Because L is a longitude of V^*, it wraps exactly once around the meridinal direction of V, and can thus be realized as a section over ∂D where $f(V) = D$. Let D_0 be the zero section of V as a disc bundle over an interior regular fiber of V. Then the oriented intersection number of J with D yields the framing coefficient. We compute the framing coefficient for a 2–cell of S in Section 4.3.3 using this method.

This work is based on that of Costantino and Thurston in [2] and of Turaev in [20]. Theorem 3.3.9 in particular is a version of the Turaev Reconstruction Theorem. Three distinct versions can be found as Theorem 4.1 of [3], 3.8 of [2], and 19.1 of [20]. The proof presented is closest in form to that found in [3]. These selections offer a decent introduction to the theory of shadows of 4–manifolds. A shadow is a 2–complex with extra structure, and the Stein complex found in this section is almost a shadow.

Chapter 4

Algorithms

This final chapter develops and details a composite of algorithms that takes as input an oriented edge–distinct triangulation T of a 3–manifold M, and produces a triangulation of a 4–manifold W whose boundary is M. Section 3.3 outlines the general flow we follow.

We begin with a triangulation T of a 3–manifold M. First we find suitable map from T to the 2–disc. This map yields a Stein complex. The Stein complex provides instructions for building a triangulation for 4–manifold W whose boundary is M.

4.1 The Generic Piecewise–linear Map

We focus on obtaining the piecewise linear analogue to the generic proper smooth map of Section 3.3. The properties this map must satisfy are laid out in Section 4.1.6, and the algorithms used to ensure these conditions are laid out in the intermediate sections.

From T a closed, oriented, edge–distinct 3–manifold triangulation, we obtain:

1. A solid polyhedral gluing T' that is homeomorphic to T minus some 3–balls.
2. A piecewise–linear map $\pi' : T' \to \mathbb{C}$ analogous to a generic proper smooth map.
3. A 2–complex T'^* dual to T'.
4. A 2–complex X' generated by π'.

These are the data carried forward to Section 4.2 that store the information used to construct a Stein complex for π'.

4.1.1 Piecewise–linear Maps and 2–complexes

We establish some properties of piecewise–linear maps from polyhedra to $\mathbb{C}$, and develop an algorithm to produce an associated 2–dimensional CW–complex. This CW–complex is used as a data type to keep track of the piecewise linear version of the generic proper smooth maps of Section 3.3. In Section 4.2, we use the 2–complex produced to form a Stein complex.

Definition 4.1.1. Let σ be a tetrahedron with the four vertices u, v, w, x, six edges uv, uw, ux, vw, vx, wx, and faces $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{x}$, where a face is named by the vertex of σ it does not contain. Define a projection $\pi : \sigma \to \mathbb{C}$ by first choosing a map from the vertices of σ to distinct points in S^1. Each point p of σ is described by the convex combination

$$p = t_u u + t_v v + t_w w + t_x x$$

with the t_* nonnegative and summing to 1. We can define π at p by

$$\begin{aligned} \pi(p) &= \pi(t_u u + t_v v + t_w w + t_x x) \\ &= t_u\pi(u) + t_v\pi(v) + t_w\pi(w) + t_x\pi(x) \end{aligned} \tag{4.1}$$

Without loss of generality, we assume that the points $\pi(u), \pi(v), \pi(w), \pi(x)$ are ordered in a clockwise orientation about S^1. We call $\pi : \sigma \to \mathbb{C}$ a *linear tetrahedral projection.*

Definition 4.1.2. A point of $\mathbb{C}$ in the image of π is one of five types:

1. The four points of type 1 are the images of the vertices of σ under π.
2. The single point of type 2 is the intersection $\pi(uw) \cap \pi(vx)$.
3. All points in $\pi(uv) \cup \pi(vw) \cup \pi(wx) \cup \pi(xu)$, excluding the points of type 1, are of type 3.
4. All points in $\pi(uw) \cup \pi(vx)$ excluding the points of type 1 or 2 are of type 4.
5. The points of type 5 are the points outside of the image of any vertex or edge of σ.

By definition, the preimage of a point of type 1 is a vertex of σ. The preimage of the single point of type 2 is a line segment between the edges uw and vx interior to σ. A point of type 3 is the image of exactly one point in an edge of σ. A point of type 4 is in the image of exactly one edge and one face, so the preimage of one of these points is a line segment interior to σ between those two facets. Finally, points of type 5 are in the image of exactly two faces of σ, and pull back to line segments interior to σ between those two faces. Our classification of points in the image of π naturally lends itself to the structure of a 2–complex. We detail this construction in Algorithm 1.

Data: a simplex σ, a map $\pi : \sigma \to \mathbb{C}$
Result: a 2–complex X

```
begin
    X^1 ⟵ σ^1;
    while there is a pair of edges (e, f) = ((u, v), (w, x)) in X that cross at
      the point y ∈ R^2 do
        delete e, f;
        add a vertex y′ to X;
        define π(y′) = y;
    end
    foreach chordless cycle c = (v_1, v_2, ..., v_k, v_1) in X do
        attach a 2–cell over c;
    end
end
```

Algorithm 1: Generating a 2–complex from a linear tetrahedral projection

For T a 3–dimensional edge–distinct connected simplicial complex, the above algorithm breaks down when a triple of edges cross at the same point. A sufficient condition to guarantee that a crossing point between a pair of edges is the crossing point of at most two edges (by the results in [12]) is if $\pi(T^1)$ is a subgraph of an odd dimensional complete graph K_n in its standard position (i.e. with the n vertices each at a distinct n^{th} complex root.)

Definition 4.1.3. Let T be a 3–dimensional edge–distinct connected simplicial complex and $\pi : T \to \mathbb{C}$ a map that is linear on each simplex of T. If, for some odd positive integer n, $\pi(T^0)$ is injective into the n^{th} roots of unity, then we say π is a *crossing distinct* projection of T^1 into $\mathbb{C}$.

The 2–complex construction can now be extended to simplicial complexes. In Algorithm 2 we continue to let T be a 3–dimensional edge–distinct connected simplicial complex with crossing distinct projection $\pi : T \to \mathbb{C}$.

Data: T, $\pi : T \to \mathbb{C}$
Result: a 2–complex X
begin
  $X^1 \longleftarrow T^1$;
  while *there is a pair of edges* $(e, f) = ((u, v), (w, x))$ *in* X *that cross at the point* $y \in \mathbb{R}^2$ **do**
    delete e, f;
    add a vertex y' to X;
    define $\pi(y') = y$;
  end
  foreach *chordless cycle* $c = (v_1, v_2, \ldots, v_k, v_1)$ *in* X **do**
    attach a 2–cell over c;
  end
end

Algorithm 2: Generating a 2–complex from a crossing distinct projection

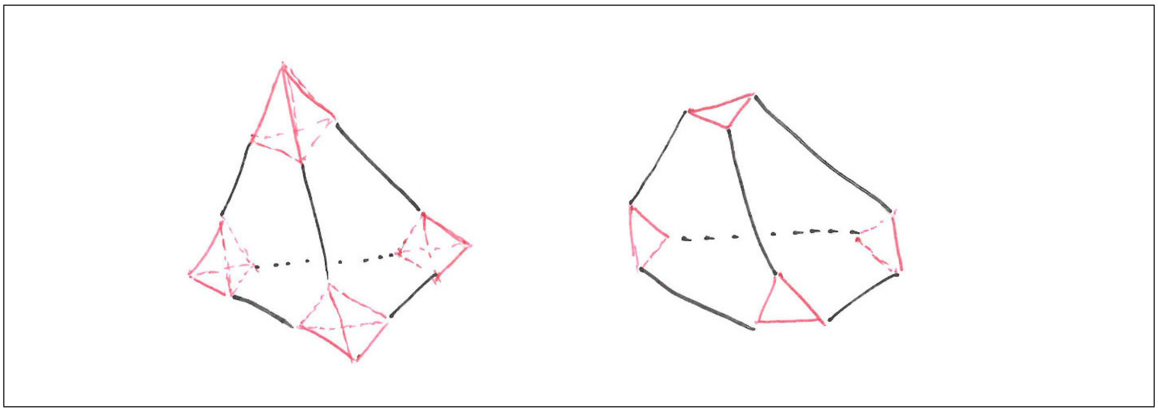

Figure 4.1: A truncated tetrahedron

4.1.2 Tetrahedral Truncation

We are interested in what happens to the 2–complex obtained in Section 4.1.1 when we begin truncating tetrahedra. Truncation must be done in a controlled way, to keep gluing maps between faces well–defined. A model truncated tetrahedron can be seen in Figure 4.1

Let T be a 3–dimensional edge–distinct connected simplicial complex and $\pi : T \to \mathbb{C}$ a crossing distinct projection of T^1 into $\mathbb{C}$. One way to turn each tetrahedron of T into a truncated tetrahedron is to remove a neighbourhood from each vertex of T. Each ball removed leaves a D^2 boundary in each truncated tetrahedron that contained the removed vertex. Once all removals are complete, we have a polyhedral gluing complex whose polyhedral 3–cells are truncated tetrahedra. We choose the removed balls so that if B is the ball removed about the vertex v in T^0, then $B = \pi^{-1}(B_\varepsilon(\pi(v)))$ where B_ε is a ball of radius ε about $\pi(v)$. Figure 4.2 depicts this process.

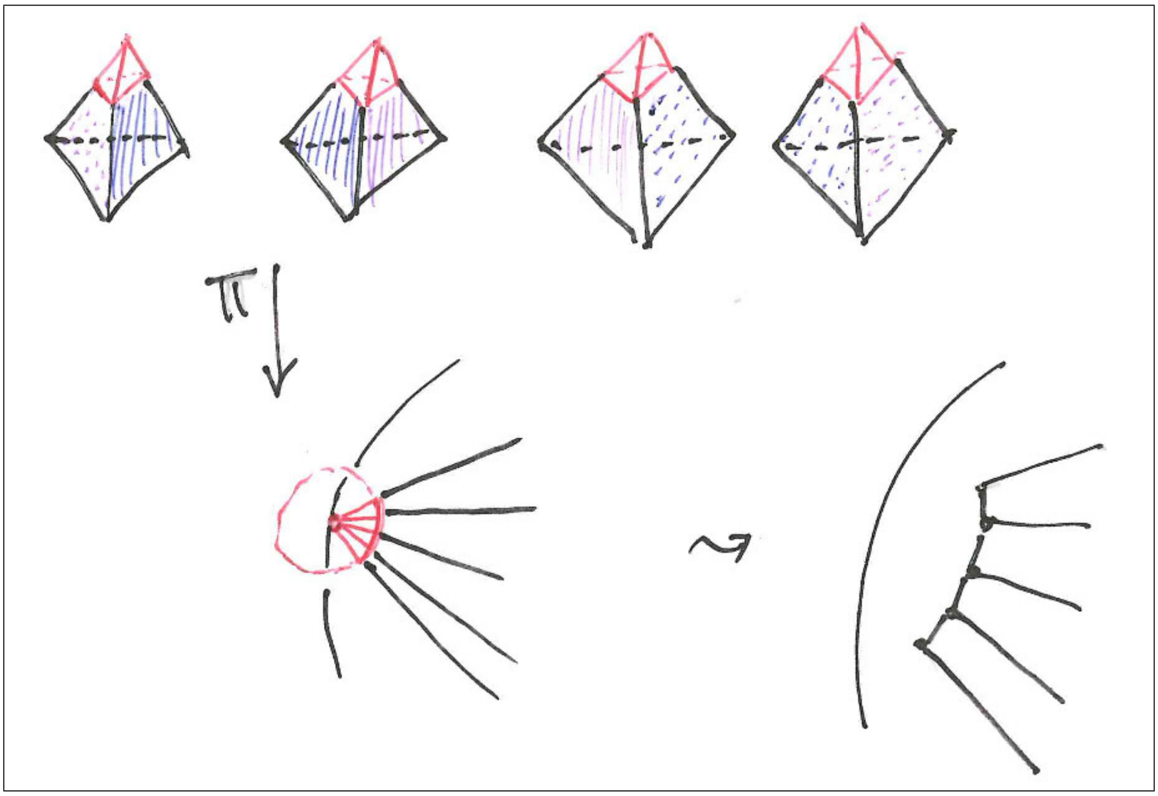

Figure 4.2: Balls to remove and the effect of Algorithm 3 on the 2–complex

We also make explicit the effect this has on the 2–complex we recover from $\pi' : T' \to \mathbb{C}$, where π' is just π restricted to $T' \subset T$. This is done in Algorithm 3, taking T and π to be as in the input to Algorithm 2. Figure 4.2 depicts the result of an application around a specific vertex.

4.1.3 Edge Blowups

We truncate simplicial complexes because we want to be able to perform edge blowups. Let T' be a truncated simplicial complex as defined in Section 4.1.2, let E be an internal edge of the polyhedral gluing with $\partial E = (u, v)$ where each of u, v are boundary vertices of T', and let σ, τ be 3–cells of T' with $\sigma \cap \tau = E$. A blowup of E across σ and τ is the process of replacing E, σ and τ by $\hat{E}$, $\hat{\sigma}$ and $\hat{\tau}$ where $\hat{E}$ is a rectangular 2–cell and $\hat{\sigma}$, $\hat{\tau}$ are 3–cells with $\hat{\sigma} \cap \hat{\tau} = \hat{E}$. This does not modify any of the other adjacencies of σ or τ. A model blowup is depicted in Figure 4.3. The explicit modification of T' involves replacing a small cylinder containing E with a small cylinder that contains $\hat{E}$ in its place. We then need to define π' on this cylinder. There is already definition of π' on the boundary of the cylinder, so we define π' on the edges of $\hat{E}$ internal to T' to be a pair of lines that are parallel to $\pi'(E)$, and extend linearly. Note that there are exactly two ways to define π' on

Data: T, $\pi : T \to \mathbb{C}$

Result: a 2–complex X' corresponding to the truncated complex T'

1 **begin**
2     $X \longleftarrow$ result of Algorithm 2 applied to T ;
3   **foreach** *vertex* v *of* X *with* $v \in \pi(T^0)$ **do**
4   delete all 2–cells containing v;
5   **foreach** *edge* $e = (v, u)$ *of* X **do**
6   **foreach** *edge* $f = (v, w)$, $f \neq e$ **do**
7   temporarily label w by $\theta_{u,w}$, the positive clockwise rotation in radians about v that takes e onto f;
8   **end**
9   delete e;
10   add a vertex u_v to X;
11   add an edge (u_v, u) to X;
12   **end**
13   **foreach** *vertex* u_v *added at Line 10* **do**
14             $w_v \longleftarrow$ the vertex with smallest $\theta_{u,w}$ between 0 and π;
15   add an edge (u_v, w_v) to X;
16   **end**
17   delete v;
18   **end**
19 **end**

Algorithm 3: Generating a 2–complex from a truncated linear tetrahedral projection

$\hat{E}$ that correspond to flipping $\pi'(\hat{E})$ over a central axis.

The effect an edge blowup has on the 2–complex associated to T' is depicted in Figure 4.4. The level of bookkeeping involved in writing an explicit algorithm only serves to obfuscate understanding so, because Figure 4.4 depicts a representative case of modification, we take the figure to be a full demonstration.

4.1.4 Triangulations and Dual Complexes

An edge distinct triangulation T of a closed, connected, orientable 3–manifold is a simplicial complex that can be run through the algorithms of the previous sections. We also consider the dual polyhedral complex to T^* and observe how it changes as T is changed.

To begin, let $\pi : T \to \mathbb{C}$ be a crossing distinct projection, and let X be the 2–complex produced by Algorithm 2. We want π to be the piecewise–linear analogue to the generic, proper, smooth map of Section 3.3. Because π is linear on each tetrahedron of T, the only analogues to critical values would occur on the images of edges of T. This leaves the 2–cells of X as our analogue to regions of regular

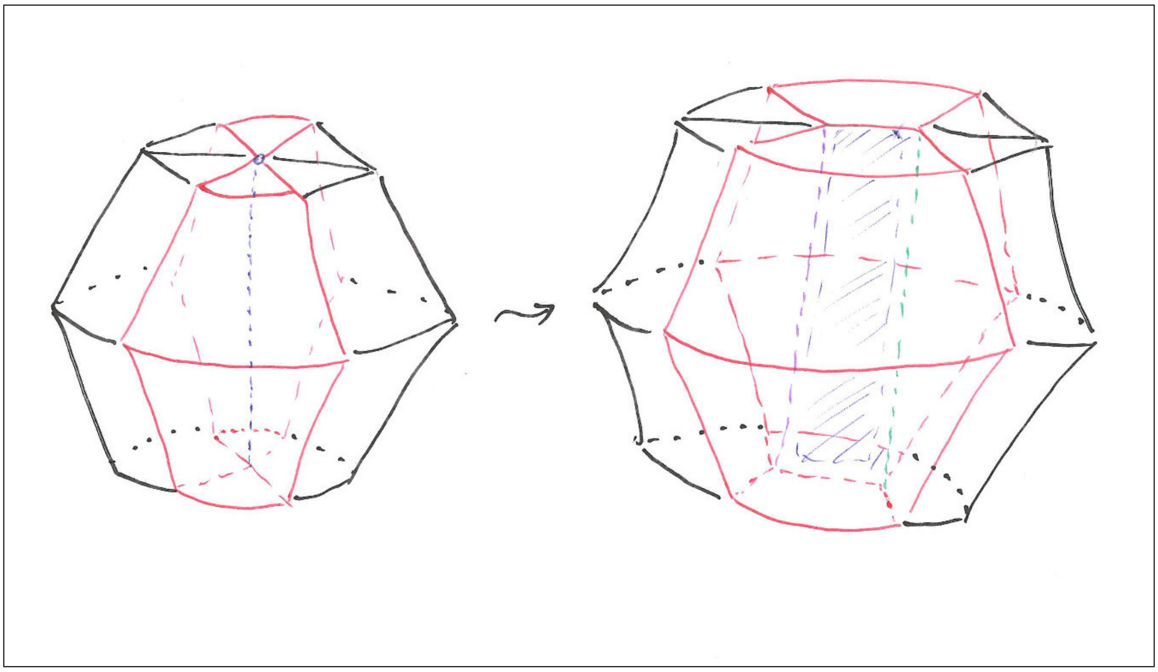

Figure 4.3: Blowing up an edge in a polyhedral gluing

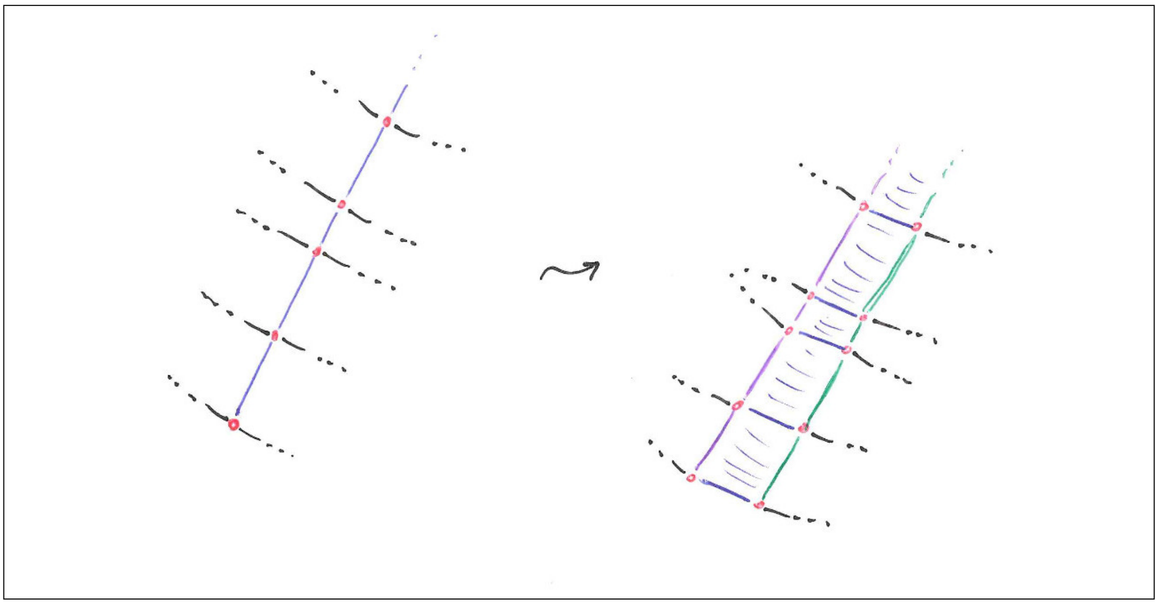

Figure 4.4: Blowing up the associated path in the 2–complex

values. Under the classification in from Definition 4.1.2, these are points of type 5. Restricting our focus to a single tetrahedron σ, a point of type 5 would pull back through π to a line segment s in σ that intersects the faces $F^i(\sigma)$ and $F^j(\sigma)$. Because p is continuous and T is closed, there are tetrahedra σ_i and σ_j that are incident to σ over $F^i(\sigma)$ and $F^j(\sigma)$. Then s meets the endpoints of two more line segments that pass through σ_i and σ_j. This continues until the segments join up into a simplicial circle interior to T, just as in the smooth regular value theorem.

Data: a 2–cell c of X, T, T^*, $\pi : T \to \mathbb{C}$
Result: a collection $C = \{C_i\}$ of oriented simplicial circles in T^* representing the circles in T that project through π over the region in the plane that correspond to c

begin
  $c_\pi \longleftarrow$ the region of type 5 points in the plane corresponding to c;
  $C \longleftarrow \emptyset$;
  foreach *edge* o^* *in* T^* **do**
    if *the dual 2–cell* o *in* T *has* $c_\pi \subset \pi(o)$ **then**
      add o^*, including its boundary vertices, to C;
    end
  end
  orient the edges of C by the orientation of T;
end

Algorithm 4: Building a collection of circles that map over a region

The circles we find this way unambiguously define cycles in the 1–skeleton of T^*, the dual complex to T, so we also take the result Algorithm 4 to be the corresponding dual cycles. We use the dual complex as a central data structure alongside the 2–complex X to store information about π. The first example of this is in Algorithm 4, where we find all of the simplicial circles in T^* that map over a region of X. We produce C, a collection of disjoint simplicial circles in T^* that project over c_π, the region of type 5 points in the plane corresponding to c, and we decorate the associated 2–cell c of X by a positive integer: the number of connected components of C.

In Section 4.1.2, we deleted balls about each vertex of T to find T'. The best we can do to reflect this modification in T^* is to delete the interior of each 3–cell. This modification yields a 2–complex T'^* and preserves the simplicial circles found during Algorithm 4.

In Section 4.1.3 we blew up edges into 2–cells. By duality of dimension, the appropriate action to consider would be to split a 2–cell by an edge. An edge o^* of

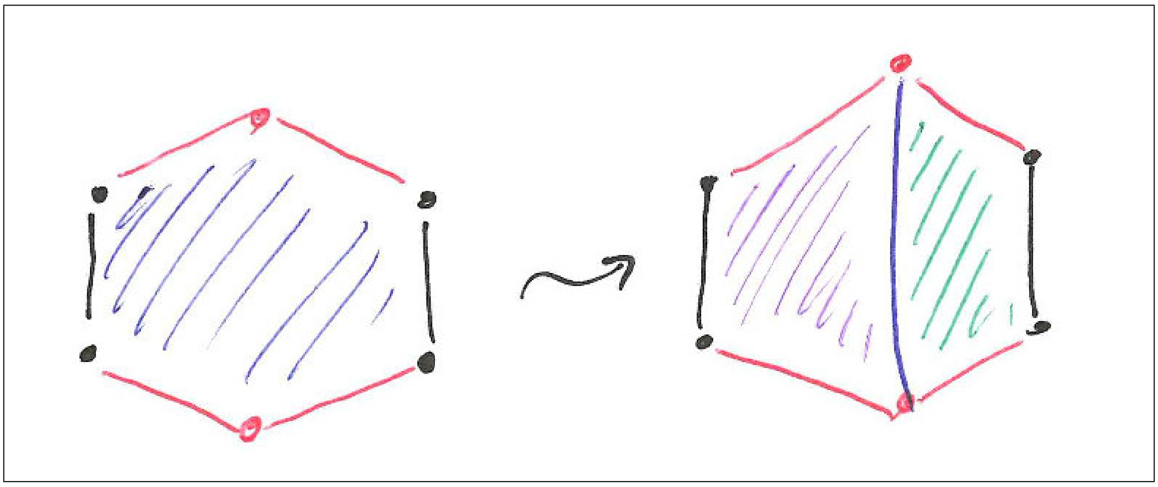

Figure 4.5: Dual edge blowup

T'^* connects vertices that are dual to 3–cells. These 3–cells are adjacent over the 2–cell o that is dual to o^*. We can therefore consider an edge in T'^* to indicate adjacency between a pair of 3–cells.

Let E be the edge and σ, τ be the 3–cells that we blow up. The modification of T'^* appropriate to a blowup over E then begins by introducing a 1–cell $\hat{E}^*$ into T^* over the vertices $\hat{\sigma}^*$ and $\hat{\tau}^*$. We then remove E^* and introduce a pair of 2–cells corresponding to the new edges of $\hat{E}$ over the newly created chordless cycles containing $\hat{E}^*$. We can see the modified structure in Figure 4.5.

It is worth noting here that we need only build directly the collection of circles that map over a single region. Once that is built, the circles mapping over neighbouring regions are determined from our first collection. This is especially useful once edges are blown up, as tracking the specifics of a projection π through such modifications is tedious.

Let X be a 2–complex generated as described by any of our algorithms from a polyhedral gluing T and let T^* be the appropriate dual complex to T. Let c be a 2–cell of X, let c' be a neighbouring 2–cell over the edge e in X, let C be the collection of circles mapping over c, and C' the collection over c'. There is an edge E of T and dual 2–cell E^* that correspond to e. By Algorithm 4 the only set of edges in T^* on which C and C' differ is ∂E^*. We can then obtain C' from C by taking the symmetric difference of C with ∂E^*, which we denote by $C' = C \triangle \partial E^*$.

4.1.5 Genericity Via Blowups

In Section 4.1.4 we mentioned that we would like π to be the piecewise linear analogue to the generic proper smooth map of Section 3.3. The important condition that we do not necessarily have is genericity. An investigation of the analogue to critical points of π demonstrates why an appropriate choice of edge blowups will introduce the genericity we are looking for.

Generic singular fibers in the smooth case have the form of either a point or a figure 8 graph. Taking T, $\pi : T \to \mathbb{C}$, and X be as in Section 4.1.4. A singular fiber of a π is found by checking how simplicial circles change when we pass over the images of edges in T. Let $X(c_i)$, $X(c_j)$ be 2–cells in X that intersect at a 1–cell $X(e)$, let c_i, c_j be the regions of points of type 5 in the plane corresponding to $X(c_i)$, $X(c_j)$, and let e be the strand of points of type 4 that corresponds to $X(e)$. Let E be the edge of T with $e \subset \pi(E)$, let $C_i = \{C_{i,k}\}$ be the collection of simplicial circles in T^* that correspond to the circles of T that map through π over c_i, and let $C_j = \{C_{j,l}\}$ be the similar collection for c_j. Considering C_i and C_j as sets of edges in T^*, the method of building C_i and C_j found in Algorithm 4 tells us that the symmetric difference of C_i and C_j is either the boundary of the dual 2–cell $E*$, or is empty.

The form of the singular fiber over a point in e is then seen by pulling the centres of the edges of ∂E^* towards the centre of E^*. This is justified by considering the centre of E^* as the sole intersection of E and E^*, and recognizing that the preimage of a point in e through π must intersect the edge E. We illustrate this situation in Figure 4.6.

The form of the singular fiber is then the wedge of n simplicial circles. We define n to be the *wedge number* of E, and this can be computed by investigating the link of E in T through π, which is precisely what we do in Algorithm 5.

Using the fact that T is closed, we get that $\texttt{lk}(E)$ is a simplicial circle in the 1–skeleton of T. Because π maps the vertices of these circles to $S^1 \subset \mathbb{C}$, we classify the edges in $\texttt{lk}(E)$ into those that pass over $e = \pi(E)$ and those that do not. There is an even number of edges that cross e because $\texttt{lk}(E)$ is closed. An edge F of $\texttt{lk}(E)$ with $f = \pi(F)$ crossing e determines uniquely a tetrahedron σ with $F, E \in \sigma^1$. For $x_i \in c_i$ and $x_j \in c_j$, take γ to be a simple path from x_i to x_j that crosses e transversely exactly once. Then $\pi^{-1}(x_i)$ and $\pi^{-1}(x_j)$ each intersect σ, and

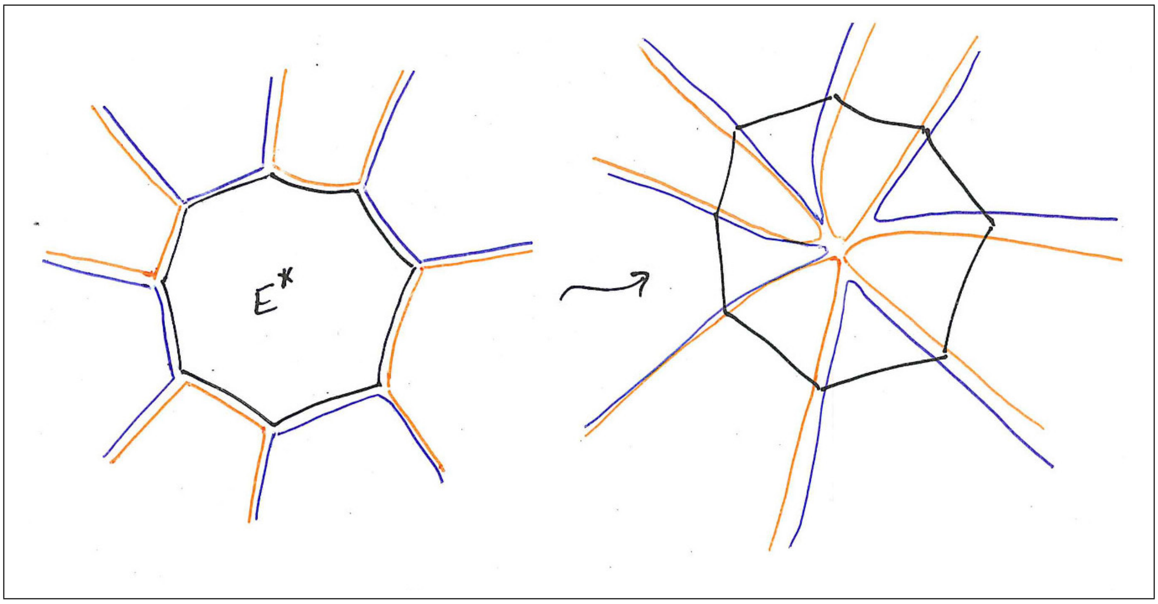

Figure 4.6: Simplicial circles near an edge

Data: T, $\pi : T \to \mathbb{C}$, and edge E of T
Result: the wedge number of E, denoted w_E
begin
  $C \longleftarrow \mathtt{lk}(E) = \{e_0, e_1, \ldots, e_k, e_0\}$ as a cycle of the graph T^1;
  $w_E \longleftarrow 0$;
  foreach *edge e in C* **do**
    if $\pi(e)$ *crosses* $\pi(E)$ **then**
      $w_E \longleftarrow w_E + 1/2$;
    end
  end
end

Algorithm 5: Wedge number computations

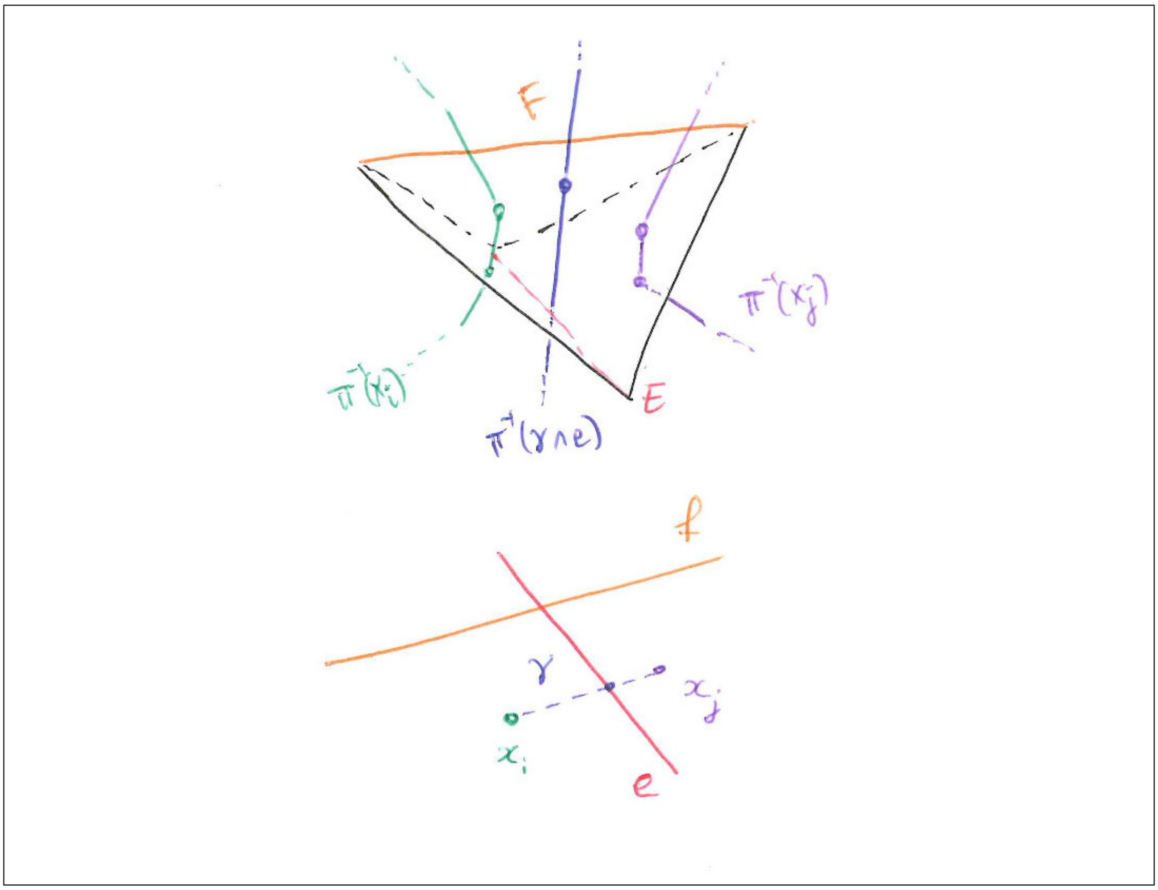

Figure 4.7: Preimages of points on either side of e as they sit in σ

the preimage of every point in γ does so as well. A representative of this situation is seen in Figure 4.7. To turn $\pi^{-1}(x_i)$ into $\pi^{-1}(x_j)$ along γ, the simplicial circle passes through σ and over E.

The edges of $\mathtt{lk}(E)$ that do not cross e determine tetrahedra that do not have this property. Each edge of $\mathtt{lk}(E)$ that crosses e then contributes one half of one circle in the bouquet of circles that map over e, hence contributes $+1/2$ to the computation of the wedge number.

We strive for the piecewise analogue of genericity and this happens when the wedge number of each edge in T is 0, 1, or 2. One way to achieve this is through the use of edge blowups. In order to perform edge blowups, we modify our triangulation to a truncated triangulation. Truncating in this way and then performing edge blowups makes tracking data through edges more trouble than its worth, and leads to confusion. Instead, we turn to the dual decomposition for a better way.

We keep T as our 3–manifold triangulation and T^* as its dual. Let E be an edge of T with dual 2–cell of E^*. The computation of a wedge number for E assigned each edge in $\mathtt{lk}(E)$ a contributing coefficient of 0 or $+1/2$. In the justification of this assignment, it was demonstrated that the contribution comes from the tetrahedrons uniquely determined by the edges of $\mathtt{lk}(E)$ rather than the edges themselves. For each edge E of T (2–cell E^* of T^*), the tetrahedra containing E (the vertices in the boundary of E^*) are assigned a wedge coefficient of 0 or $+1/2$. The wedge number for E (E^*) is then computed as the sum of the wedge coefficients surrounding it.

Recall that the modification to T^* equivalent to an edge blowup was the splitting of a 2–cell by a new edge. In the view from T, the 3–cells over which we blowup still contain and map over each of the new edges, while the remaining 2–cells contain only one of the edges. The 3–cells over which we split then contribute to the wedge numbers of both edges, while the rest contribute to only one.

In T^*, we have split a 2–cell by a 1–cell connecting a pair of vertices. The vertices over which we connect contribute their wedge coefficient to both of the new 2–cells because the corresponding 3–cells project over both of the new edges. The wedge numbers of the new 2–cells are then easily recomputed as displayed in Figure 4.8. The details to reduce a single edge number are found in Algorithm 6

Data: a dual 2–cell E^* of a truncated dual complex T'^* with wedge number $w_E > 2$

Result: a pair of dual vertices (σ^*, τ^*) of ∂E^* over which we may split E^*, so that the two new 2–cells have wedge numbers of 2 and $w_E - 1$

begin
- $C \longleftarrow \partial E^* = \{\sigma_0^*, \ldots, \sigma_k^*\}$ as a cycle in the 1–skeleton of T'^*;
- $\sigma^* \longleftarrow \sigma_i^*$ where σ_i^* is the first dual vertex in C with nonzero wedge coefficient;
- $w, j \longleftarrow 0$;
- **while** $w < 2$ **do**
  - $j \longleftarrow j + 1$;
  - $c_j \longleftarrow$ the wedge coefficient of the dual vertex σ_{i+j}^*;
  - $w \longleftarrow w + c_j$;
- **end**
- $\tau^* \longleftarrow \sigma_{i+j}^*$;

end

Algorithm 6: Wedge number reduction

4.1.6 Building the Generic Piecewise–linear Map

This section comes to fruition with the definition of the piecewise–linear analog to a generic proper smooth map. Let T be a closed, orientable, edge–distinct 3–manifold triangulation. We build a map $\pi' : T' \to \mathbb{C}$ satisfying the following criteria.

First, the space T' is a solid polyhedral gluing whose boundary components are

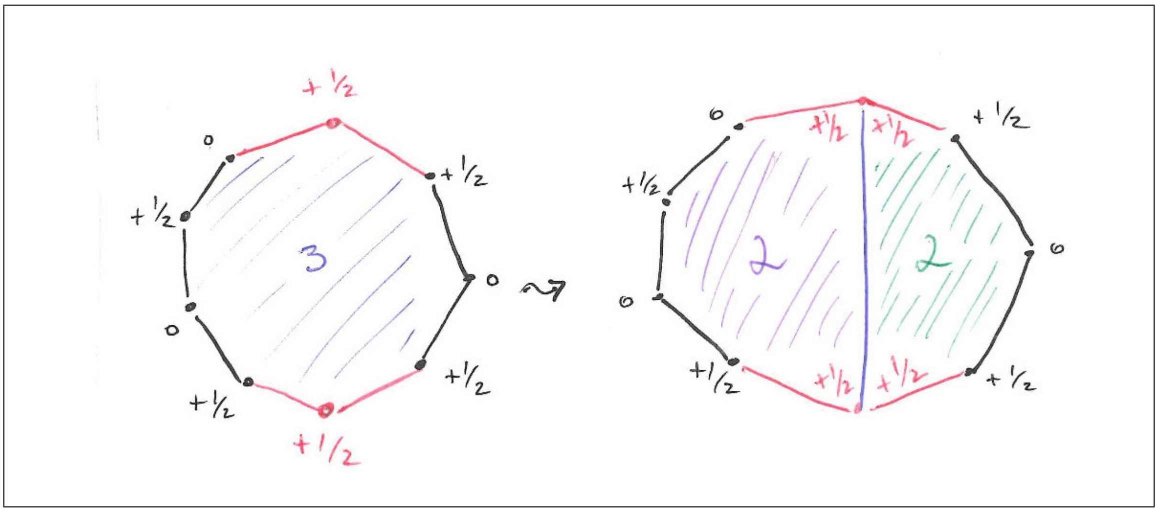

Figure 4.8: Wedge numbers after a blowup

homeomorphic to S^2. Second, attaching 3–balls over the boundary components of T' results in a complex homeomorphic to T. Third, for any pair of edges E, F in T', if $\pi'(E) \cap \pi(F) = z$ is nonempty, then z is not the intersection of any other pair of edges in T'. Fourth, the wedge number of every edge of T' is 0, 1, or 2.

The first two conditions are met by modifying T only through truncation and the blowing up of edges. Condition three we obtain by requiring π to be crossing distinct on T, and then modifying T only in the controlled ways discussed in this section. For condition four, we blow up the appropriate edges. The details are laid out in Algorithm 7, where we also obtain the 2–complex associated to π' along the way. We pass the results of Algorithm 7 into the algorithms of Section 4.2

4.2 The Stein Complex

We enter this section with a piecewise linear analogue to the generic proper smooth map of Section 3.3, and focus on leaving it with a Stein complex to the map. A generic proper smooth map already has a Stein complex that is described as a 2–complex, so this section is fairly straightforward and relatively light. Only in Section 4.3.3 do we delve into complicated subject matter. As we are running fully parallel with Section 3.3, the arguments can be easily compared.

Section 4.1 ended by providing us with some explicit data structures. These structures are taken as given in the following sections. What we have to work with:

1. A solid polyhedral gluing T' that is homeomorphic to T minus some 3–balls.
2. A piecewise–linear map $\pi' : T' \to \mathbb{C}$ analogous to a generic proper smooth

Data: the closed, orientable, edge–distinct 3–manifold triangulation T
Result: a map $\pi' : T' \to \mathbb{C}$ satisying the above criteria, and the associated 2–complex X'

begin
  $m \longleftarrow |T^0|$;
  put an arbitrary order on T^0;
  foreach *integer* k *in* $[1, m]$ **do**
    define $\pi(x_k) = e^{2\pi i \frac{k}{m}}$
  end
  extend π linearly as in Section 4.1.1;
  foreach *edge* E *of* T **do**
    $w_E \longleftarrow$ result of Algorithm 5;
    distribute w_E across the tetrahedra containing E;
  end
  truncate T to T' and π to π';
  $X \longleftarrow$ result of Algorithm 3 on π;
  foreach *internal edge* E' *of* T' **do**
    while $w_E > 2$ **do**
      $(\sigma^*, \tau^*) \longleftarrow$ result of Algorithm 6;
      $\pi', T', T'^*, X' \longleftarrow$ result of blowing up E' over σ and τ;
      $w_E \longleftarrow w_E - 1$;
    end
  end
end

Algorithm 7: Building the piecewise–linear analogue to a generic proper smooth map

map.

3. A 2–complex T'^* dual to T'.

4. A 2–complex X' generated by π'.

We build the Stein complex for π' in the order of increasing cells. Our 2–complex X' corresponds completely to the image of T' through π', so the factorization of π' should yield a 2–complex very similar to X'. The 2–complex S built in this section is the only data structure passed to the algorithms of Section 4.3.

4.2.1 Stein as a CW–complex

Our first task is to determine how to generate the vertices of the Stein complex S. Each vertex v of X' yields a number of vertices in the Stein complex of π', each of which map over v it when we factor π'. Here we have referred to v as both the vertex of X and the type 2 point in the image of π' that the vertex corresponds to, and we do the same with the edges and 2–cells of X'. According to the analysis in Section 3.3, a given vertex of X' has at most two vertices in S that will be difficult to find. For v a vertex of X', introducing a vertex of S over v is based entirely on the behaviour of the circles that map over the regions incident to v.

In our first case, v is an internal vertex. There are four 2–cells $c_0, \dots, c_3$ of X' that contain v in their boundaries. Put C_i to be the collection of simplicial circles in T'^* that project over c_i as in the result of Algorithm 4. There are also four 1–cells $e_0, \dots, e_3$ where e_i is the intersection of c_i and c_{i+1}, indices computed modulo 4. See Figure 4.9.

An edge E_0 of T' maps through π' over e_0 and e_3, and E_1 maps over e_1 and e_3. From Section 4.1.4 we know:

1. $C_0 \triangle \partial E_0^* = C_1$.

2. $C_1 \triangle \partial E_1^* = C_2$.

3. $C_2 \triangle \partial E_0^* = C_3$.

4. $C_3 \triangle \partial E_1^* = C_0$.

In the smooth case this is equivalent to finding cobordisms between sets of circles C_i and C_{i+1}. We say that an element $C_{i,k}$ of C_i *interacts* with an element $C_{i+1,j}$

of C_{i+1} if $C_{i,k} \triangle C_{i+1,j}$ is either empty or exactly $\partial E^*_{\lfloor i/2 \rfloor}$. Take the union of the C_i's and partition it into subsets that such that any element of a subset interacts with at least one other element of that subset. Call the set of subsets of $\bigcup_i C_i$ by $N(v)$ and call it the set of *interactive elements* over v. Every element of $N(v)$ adds a vertex to S that is linked to the subset of $N(v)$ that spawned it.

In our second case, v is a boundary vertex. The analysis is exactly the same, except that there are only two 2–cells of X' that are incident with v. In $\mathbb{C}$, the last region incident to v is R_∞, so our analysis is simplified somewhat. Otherwise, we perform the same steps as the internal case.

With the 0–cells of S determined, we find 1–cells. As with the vertices, the 1–cells of the Stein complex project over the 1–cells of X'. Take u, v vertices of S adjacent over an edge e of X'. To find the associated edges of S, examine the sets of interactive elements over u and v. The shared circles within the elements of $N(u)$ and $N(v)$ tell us whether a pair of vertices are adjacent in S as detailed in Algorithm 9.

Attaching the 2–cells is very similar to attaching the 1–cells. A 2–cell of S projects over a 2–cell c of X', and each 2–cell corresponds to a simplicial circle projecting through π' over c. The attachment is determined entirely by a cycle in the 1-skeleton of S, found in Algorithm 10.

Data: a vertex v of X'
Result: the set $N(v)$ of sets of interacting circles near v
begin
  $c_i \longleftarrow$ the 2–cells of X' incident to v;
  $C_i \longleftarrow$ the result of Algorithm 4 with input c_i;
  $N(v) \longleftarrow \emptyset$;
  foreach *circle* $C_{0,k}$ *in* C_0 **do**
    $y \longleftarrow \{C_{0,k}\}$;
    add each circle incident to $C_{0,k}$ to y;
    add each circle incident to a circle incident to $C_{0,k}$ to y;
    add y to $N(v)$;
  end
end

Algorithm 8: Finding vertices for the Stein complex

4.2.2 Building the Stein Complex

The preceding algorithms are now combined into Algorithm 11 to build our integer decorated Stein complex.

```
Data: an edge e = (u, v) of X′
Result: the set N(e) of pairs (U, V) with U ∈ N(u) and V ∈ N(v) such that
        U and V are adjacent over an edge of S
begin
    N(u), N(v) ⟵ the results of Algorithm 8 with inputs u, v;
    N(e) ⟵ ∅;
    foreach element U of N(u) do
        foreach element V of N(v) do
            if U ∩ V ≠ ∅ then
                add (U, V) to N(e);
            end
        end
    end
end
```

Algorithm 9: Finding edges for the Stein complex

```
Data: a 2–cell c of X′ with {v_0, ..., v_k} = ∂c
Result: the set N(c) of cycles {V_0, ..., V_k} where V_i is in N(v_i) such that a
        2–cell is attached over {V_0, ..., V_k} in S
begin
    C ⟵ the result of Algorithm 4 with input c;
    N(v_i) ⟵ the result of Algorithm 8 with input v_i;
    N(c) ⟵ ∅;
    foreach simplicial circle C_j of C do
        N_j ⟵ ∅;
        foreach integer i in [0, k] do
            foreach element V_i of N(v_i) do
                if C_j ∈ V_i then
                    add V_i to N_j;
                end
            end
        end
        add N_j to N(c);
    end
end
```

Algorithm 10: Finding 2–cells for the Stein complex

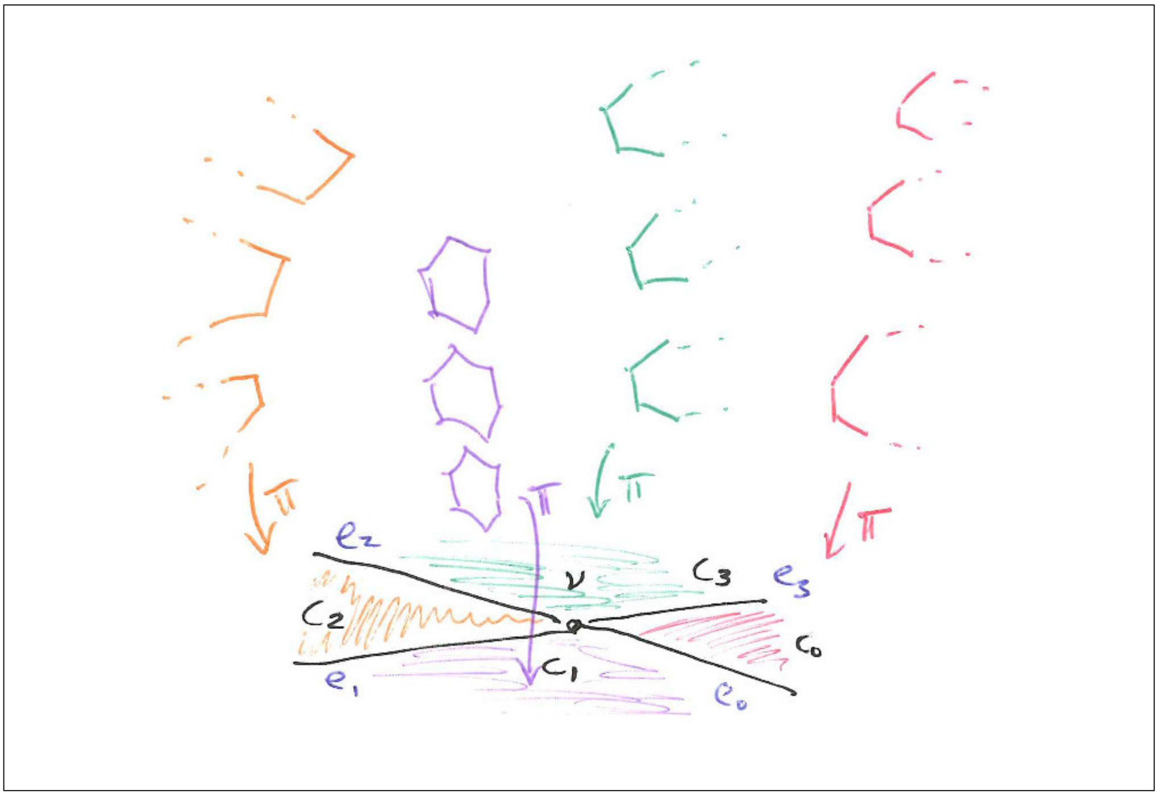

Figure 4.9: A vertex in X' and the circles that map over incident 2–cells

Data: the closed, orientable, edge–distinct 3–manifold triangulation T
Result: an integer decorated Stein complex S

begin
    $X' \longleftarrow$ the result of Algorithm 7 with input T;
    $S \longleftarrow \emptyset$;
    foreach *vertex v of X'* **do**
        foreach *element V_i of $N(v)$* **do**
            add a vertex v_i to S;
        end
    end
    foreach *edge $e = (u, v)$ of X'* **do**
        foreach *element (U_i, V_j) of $N(e)$* **do**
            add an edge (u_i, v_j) to S;
        end
    end
    foreach *2–cell c of X'* **do**
        foreach *element N_j of $N(c)$* **do**
            attach a 2–cell to S over the cycle N_j;
        end
    end
end

Algorithm 11: Building the Stein complex

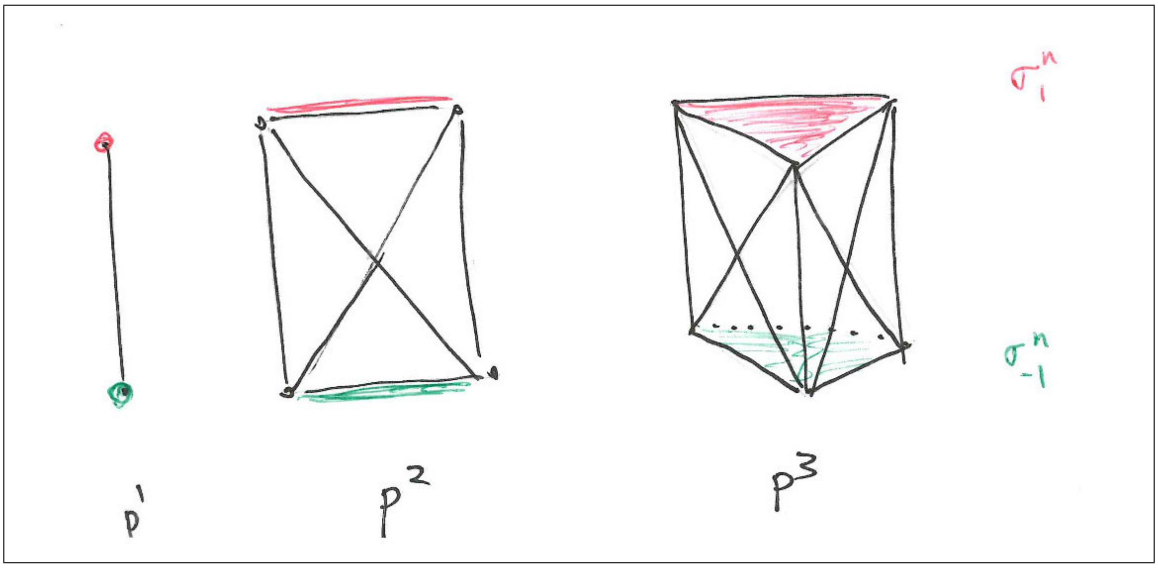

Figure 4.10: Sample 1–, 2–, and 3–prisms

4.3 The 4–manifold

We leave Section 4.2 with an integer decorated 2–complex that is taken to be the Stein complex for a map constructed in Section 4.1 analogous to the generic proper smooth map of Section 3.3. This complex is a set of instructions for building a cobordism of the pair $(M, \emptyset)$, shown in Section 3.3. First, build a 3–thickening of a regular neighbourhood of a maximal spanning tree of the 1–skeleton of the Stein complex. Second, perform a 4–thickening. Third, complete the thickening into a 4–handlebody. Finally, attach the appropriate 2–handles. Before embarking on this construction, we define some standard building blocks that we can use to keep track of essential information in the defining of a 0–framing for 2–handle attachment.

4.3.1 The triangulated prism

The basic building block that we used in our 4–manifold reconstruction is the triangulated n–prism with identical walls. Take σ^n to be the standard n–simplex. By n–prism, we mean the n–disc seen as $\sigma^{n-1} \times [-1, 1]$ in Figure 4.10. The prism has canonical "top" and "bottom" faces which are $\sigma^{n-1} \times \{1\}$ and $\sigma^{n-1} \times \{-1\}$ respectively. It also has n identical "walls" which are $F^i(\sigma^{n-1}) \times [-1, 1]$ for each $i = 0, \ldots, n-1$. The triangulation we construct for the n–prism is designed so that the walls have identical triangulations. We do this so that we are able to perform an n–thickening of an $(n-1)$–dimensional triangulation.

If a pair of σ^{n-1} are glued over a face, then we take those simplices to be the bottom faces of a pair of prisms. We glue these prisms together over their corresponding walls. This provides the two σ^{n-1} with an n–dimensional thickening. If we can do this to any pair of simplices, then we can do this to an entire triangulation, as in Algorithm 12

Data: an n–dimensional triangulation T
Result: an $(n+1)$–dimensional triangulation $H(T)$ that is the $(n+1)$–thickening of T
begin
    $H(T) \longleftarrow \emptyset$;
    foreach *n–simplex σ of T* **do**
        add a triangulated $(n+1)$–prism $H(\sigma)$ to T';
    end
    foreach *face $F^i(\sigma) = F^j(\tau)$ over which a pair σ, τ of n–simplices are glued in T* **do**
        glue the i^{th} wall of $H(\sigma)$ to the j^{th} wall of $H(\tau)$;
    end
end

Algorithm 12: Thickening a triangulation

Considering $F^i(\sigma^{n-1})$ is an $(n-2)$–simplex, it is reasonable to build our prisms recursively. The intuition is that we build an $(n-1)$–sphere whose walls are $(n-2)$–prisms, themselves with identical walls. Then, we cone the $(n-1)$–sphere to make a triangulated n–prism. The details are in Algorithm 13, assuming the base case of the 1–prism being an edge.

4.3.2 Building a Handlebody

The first construction in the proof of Theorem 3.3.9 is a 3–handlebody regular neighbourhood of the 1–skeleton of our Stein complex S. Put G to be the 1–skeleton of S and $A \subset G$ a maximal spanning tree. We know that G is a graph where, for any vertex v of G, $\deg_G v$ is 3 if v is boundary and 4 if it is interior.

We first triangulate the building blocks of Figure 3.8, where each block is depicted as a 3–disc whose boundary is decorated by a graph with blue vertices and red edges. Our initial triangulations appear in Figure 4.11, where we see triangulated 2–spheres from one perspective. There are two important properties of these triangulations. The first is that the red edges appear as triangulated 1–submanifolds (in these basic block they are single edges). The second is that there are triangulated 2–disc submanifolds about each blue vertex (appearing as single triangles in the basic blocks), and these are regular in the sense that the triangulated edge blocks can be

Data: a positive integer n
Result: P^n, a triangulated n–prism with identical walls

begin
  if $n = 1$ **then**
    $P^1 = [-1, 1]$;
  end
  else
    $\sigma^n_{-1}, \sigma^n_1 \longleftarrow$ standard n–simplices;
    $P^n \longleftarrow \sigma^n_{-1} \sqcup \sigma^n_1$;
    $P^{n-1}_0, P^{n-1}_1, \ldots, P^{n-1}_n \longleftarrow$ result of Algorithm 13 with input $n-1$;
    foreach $(n-1)$ *prism* P^{n-1}_i **do**
      attach σ^{n-1}_{-1} of P^{n-1}_i to $F^i(\sigma^n_{-1})$ of σ^n_{-1} in P^n;
      attach σ^{n-1}_1 of P^{n-1}_i to $F^i(\sigma^n_1)$ of σ^n_1 in P^n;
    end
    foreach $n-2$ *simplex* $F^k(F^i(\sigma^n_1)) = F^l(F^j(\sigma^n_1))$ *of* σ^n_1 **do**
      attach the wall of P^{n-1}_i containing $F^k(F^i(\sigma^n_1))$ to the wall of P^{n-1}_j containing $F^l(F^j(\sigma^n_1))$;
    end
    $P^n \longleftarrow CP^n$, the cone on P^n;
  end
end

Algorithm 13: Building a standard n–prism

glued to the vertex blocks in such a way that the red edges line up. The cone on each of these 2–spheres yields initial 3–disc blocks. Two barycentric subdivisions of these blocks maintain the above properties and deliver a third desirable property: that the triangulated half–tubular neighbourhoods of distinct curves are disjoint. This means that upon 4–thickening any 3–prism used in the thickening that intersects one thickened curve intersects no other 3–prisms that intersect a different curve. We take these subdivided triangulations as our *vertex* and *edge blocks*, and the special triangulated red boundary curves as *framing pieces*.

The proof of Theorem 3.3.9 is procedural in its creation of handlebodies, and the details for the triangulated case are contained in Algorithm 14. In this section we use the notation laid out in the proof of Theorem 3.3.9, and we take $A \subset G$ to be fixed throughout.

The result of Algorithm 14 is a 4–handlebody that is a 4–thickening of $U_S(G)$ along with a set of curves in the boundary of that handlebody. In Algorithm 15 we find a set of solid torus regions for 2–handle attachment and their associated 0–framing strips as discussed in Theorem 3.3.9. The curves of Algorithm 14 are the boundaries of the strips of Algorithm 15, and we find the 0–framing curve of the attaching regions based on the type of strip. When the strip is an annulus, the curve is found as one of the boundary components of the annulus. When it is a Möbius strip, there is one boundary component 2γ. We take 2γ halfway around its length to one full longitude of V, and perform one last positive half twist in order to get an actual 0–framing curve γ. To be precise, the half twist is a curve γ in the meridian direction of the boundary of V connecting the endpoints of $\frac{1}{2}(2\gamma)$, and the positive direction is the anticlockwise direction determined by the $[-1,1]^2$ fibers over the boundary component of $U_S(G)$. We perform this half twist on the boundary of a part of the solid torus contributed by a vertex block so that we can make more well structured arguments in Section 4.3.3.

Data: a Stein complex S
Result: a triangulated 4–manifold A_4 that is the 4–thickening of $U_S(G)$ and a set of embedded framing curves in the boundary of $U_S(G)$

begin
  $A_3, C \longleftarrow \emptyset$;
  foreach *vertex* v *of* A **do**
    add the triangulated vertex block v_B associated to the type of vertex v is in S to A;
    add the framing pieces in the boundary of v_B to C_A;
  end
  foreach *edge* $e = (u, v)$ *of* A **do**
    attach the triangulated edge block e_B associated to the type of edge e is in S over the blocks u_B and v_B;
    add framing pieces in the boundary of e_B to C_A;
    combine the pieces in C_A over identical boundaries;
  end
  $A_4 \longleftarrow$ result of Algorithm 12 with input A_3;
  $C_A \longleftarrow$ the inclusions of C_A into the copies of A_3 at $A_3 \times \{\pm 1\}$ in A_4;
  foreach *edge* $e = (u, v)$ *of* $e(G) \setminus e(A)$ **do**
    $e_B \longleftarrow$ the triangulated edge block associated to the type of edge e is in S;
    $E_B \longleftarrow$ the result of Algorithm 12 with e_B;
    attach E_B over the thickenings U_B and V_B of u_B and v_B in A_4 in the unique orientation preserving way;
    add the framing pieces included into the copies of e_B at $e_B \times \{\pm 1\}$ in E_B to C_A;
    combine the curves in C_A over identical boundaries;
  end
end

Algorithm 14: Building the 4–handlebody of the 1–skeleton of the Stein complex

Data: a Stein complex S

Result: a triangulated 4–manifold A_4 that is the 4–thickening of $U_S(G)$, a set of trianglated solid tori $\mathcal{V}$ in the boundary over which we will attach 2–handles, and a set of strips $\mathcal{C}$ in the boundary tori of $\mathcal{V}$ used to determine 0–framings as in the proof of Theorem 3.3.9

begin
- $A_4, C_A \longleftarrow$ result of Algorithm 14 with input S;
- $\mathcal{V}, \mathcal{C} \longleftarrow \emptyset$;
- **foreach** *curve c of C_A* **do**
  - $V \longleftarrow \emptyset$, the solid torus associated with c;
  - $C \longleftarrow \emptyset$, the strip in V that determines 0–framing;
  - **foreach** *4–prism P^4 in the 4–thickening structure of A_4 intersecting c nontrivially* **do**
    - **foreach** *wall P^3 of P^4* **do**
      - **if** *$P^3 \cap c$ is nonempty* **then**
        - **foreach** *wall P^2 of P^3* **do**
          - **if** *$P^2 \cap c$ contains edges of c & P^2 is not yet in C* **then**
            - add P^2 to C;
          - **end**
        - **end**
        - add P^3 to V;
      - **end**
    - **end**
  - **end**
- **end**
- combine the elements of V over identical boundaries;
- add V to $\mathcal{V}$;
- combine the elements of C over identical boundaries;
- add C to $\mathcal{C}$;

end

Algorithm 15: Finding attaching regions and 0–framing strips for 2–handle attachment

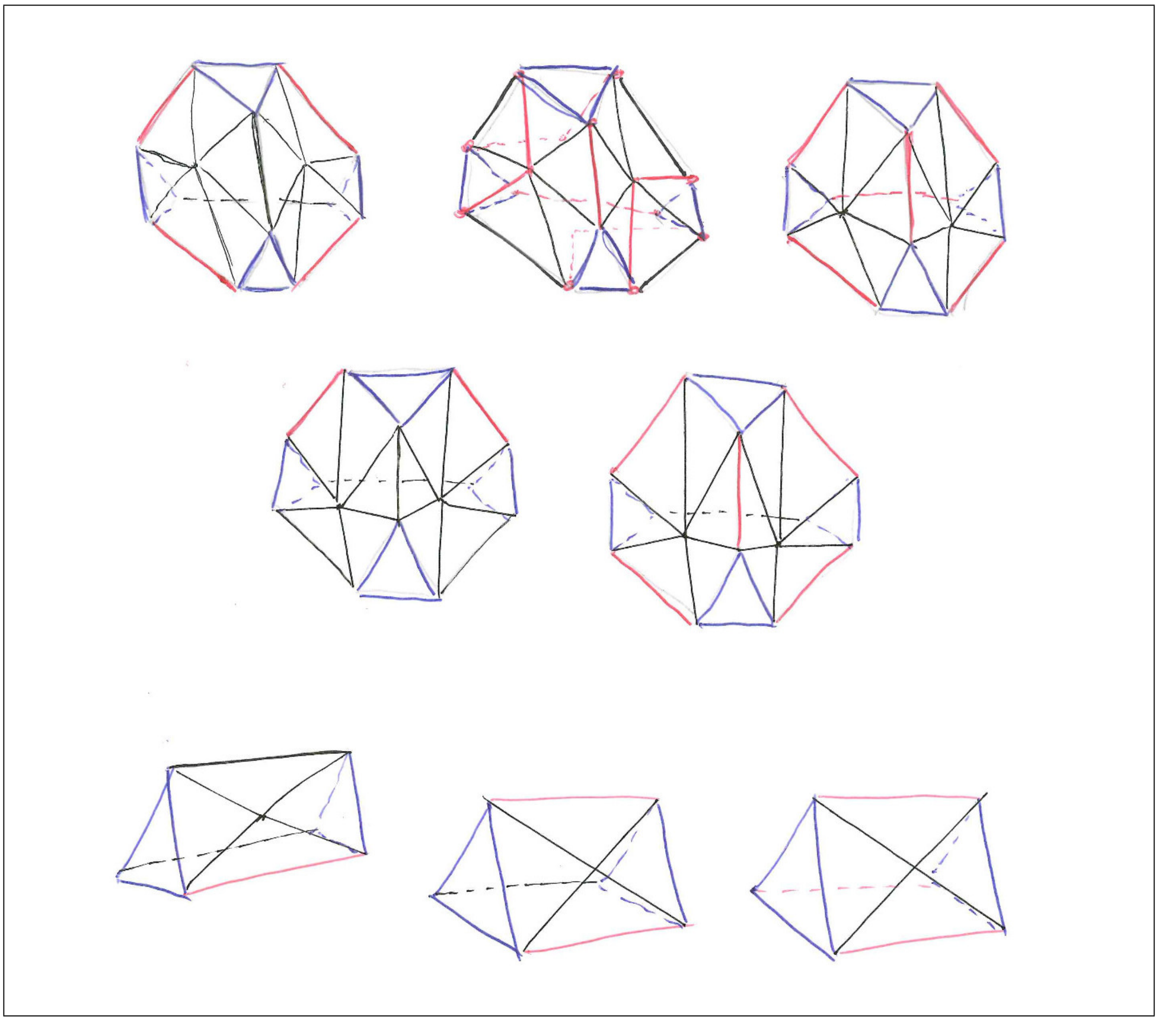

Figure 4.11: The triangulated blocks used in the 3–thickening of the 1–skeleton of the Stein complex

4.3.3 Framing Coefficient Computation

Section 3.3 ends with a discussion of framing coefficients with respect to a canonical 0–framing, and it does so by identifying the boundaries of tori V and V^* with $\partial V = \partial V^*$. As in the case of the effect of edge blowups on the 2–complex recovered in Section 4.1.3, the details of this algorithm are not easily digested in the form of pseudocode, so Algorithm 16 deals only in broad strokes.

Let c be a 2–cell of S and V^* the triangulated solid torus associated to c in the 4–handlebody A_4 found by Algorithm 15. The solid torus V^* also comes equipped with a triangulated boundary curve γ that determines its 0–framing, and γ has a natural partitioning into the framing pieces from which it was built in Algorithm 14.

We acquire a triangulation of an open solid torus V projecting through π over the disc of regular values D associated to c. First equip ∂D with a cell decomposition parallel to ∂c, which we write as

$$\partial c = \{e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_k = (v_k, v_0), e_0\},$$

the vertices and edges being those contained in S. Triangulate ∂D with an edge for each vertex and each edge of ∂c. Take f_i to be the edge corresponding to the edge e_i and y_i the edge corresponding to the vertex v_i. We triangulate D' as the cone on this decomposition. If a 2–cell of T' intersects V it does so in a meridinal disc, so we take a copy of D' for each such 2–cell. We connect these discs together with triangulated 3–prisms to form V. Now V has a collection of meridinal discs m_i that are copies of the triangulated discs D'_i and whose boundaries ∂m_i are meridians of V. We may also equip V with a longitude corresponding to a meridian of V^*. Indeed, we obtain a set of parallel triangulated longitudes over each vertex of $\partial D'$, each corresponding to a meridinal disc of V^*.

Now, we turn to the blocks of V^* containing the framing pieces of γ. Such a framing piece is a curve in the boundary of a 4–thickened vertex or edge block, vertices and edges coming from ∂c. We call the portion of V^* corresponding to a vertex block over v_i by Y_i and the portion corresponding to an edge block over e_i by F_i. These blocks share triangulated boundaries that are parallel meridinal discs of V^*.

Cut ∂V along its parallel triangulated boundary longitudes into the longitudinal strips λ_j and ∂V^* along its parallel triangulated boundary meridians into meridinal strips μ_j. Each λ_j contains a set of parallel triangulated curves between its boundary components corresponding to its intersection with the m_i, and each μ_j contains a triangulated curve between its boundary components corresponding to its intersection with the framing piece of γ over the corresponding edge or vertex block. The longitudinal strip λ_i is equivalent to the meridinal strip μ_i inside of M by our analysis of the smooth case, so we identify these annuli by building a common triangulation. First, we justify that the framing pieces in edge blocks run parallel to the meridinal pieces in the associated annuli in the boundary of V.

Let e be an edge of ∂c with parallel f in $\partial D'$. The 0–framing curve γ is a section over $\partial D'$, as per Theorem 3.3.9. First examine a section over e, which is a portion of an edge E in T'. An appropriate section over f, denoted $s(f)$, sits entirely inside of a single 3–cell σ of T' that contains E. If this is true, then $s(f)$ intersects no 2–cells of T', hence is parallel to the set of meridians of V we have found.

It is appropriate to take $s(f)$ completely inside of σ because of the piecewise–linear nature of π. The section we construct will have the same oriented intersection number with any of the m_i of V. If $s(f)$ leaves σ, it leaves through a boundary 2–cell of σ. If it does not travel at least once around E, then it is identical in our computations to the version of $s(f)$ that sits entirely inside of σ. Suppose that $s(f)$ does travel around E, and recall that we have a collection of meridians m_i corresponding to the 2–cells mapping over D. Here, if a 2–cell incident to E projects over D then all of them must project over D and we must have that no other 2–cell of T' may project over D in order to keep the oriented intersection number invariant. In this case, E forms a definite fold singularity so $s(f)$ can be isotoped through the 3–ball in M associated to this type of singularity back to a position in which it sits entirely inside of σ.

We conclude that every intersection of $s(f)$ with an m_i occurs near a vertex of S. This means that if μ is the annulus corresponding to a framing piece $s(f)$ over an edge block, then the associated annulus λ may be subdivided to contain a triangulated curve between its boundaries that is parallel to the meridinal curves in λ and sits between the two meridinal curves that are closest to the 3–cell σ in which $s(f)$ resides. At this point, we connect λ to μ over a triangulated 3–manifold. Subdivide the boundary curves of λ to be identical to the boundary curves of μ but with opposite orientation, glue λ to μ over their boundary curves so that the boundaries of $s(f)$ in each annulus agree and so that the space is orientable, and attach a triangulated disc over the copies of $s(f)$ in each annulus. Cutting this space along the disc with boundary $2s(f)$ yields a 2–sphere, so it's cone is a 2–disc glued to itself in an orientation preserving way over a boundary disc, hence is a solid torus. We build a solid torus like this for each pair μ_i, λ_i that arise from edge blocks, then move onto the framing pieces near vertices.

Let μ_{i-1}, μ_i, and μ_{i+1} be annuli corresponding to an edge block, a vertex block, and an edge block respectively. Take Λ_* be the solid tori constructed above that identify the meridinal curves of V inside of λ_* with curves in μ_* and the framing pieces inside of μ_* with curves in λ_*. Of the pair of curves in $\partial\Lambda_*$ over which we attached μ_* and λ_*, one corresponds to a boundary curve of μ_i, and we will call that curve by l_*. Attach each of the Λ_* to μ_i by identifying the curves l_* with their corresponding boundary curve of μ_i. There are now points in the boundary of μ_i corresponding exactly to how a meridian of V lies in the boundary of V^* near a vertex. We subdivide the boundary curves of λ_i as before, but this times we glue λ_i and μ_i together to that the meridians of V agree. The result is a torus t_i. There is an ordered pair of points (a, b) realized as the boundary of the oriented framing piece $s(y_i)$ in Y_i, and (a, b) sit in λ_i is a way that corresponds to how the 0–framing of V^* lies in the boundary of V near a vertex of S. The points (a, b) bound the curve $s(y_i)$ in μ_i, and we form another oriented curve $s(y_i)'$ in λ_i connecting a and b by first traveling through λ_i parallel to the meridians of V from b to the shared boundary component A of μ_i and λ_i containing a. Once inside of A, we examine the orientation of A as it sits inside of μ_i, and follow that direction to a. In keeping with the construction of the Λ_*'s, we demand that the curve $s(y_i) + s(y_i)'$ as well as the meridians of V as they appear in t_i all bound discs in the completion of t_i into a solid torus. This forces the meridians to complete in a unique way inside of μ_i. Glue a disc over $s(y_i) + s(y_i)'$ and cone the complex as before to obtain a solid torus.

The collection $\{\Lambda_i\}$ of solid tori can be glued together over the parallel meridians of V^* (also parallel longitudes of V), which we denoted by $ł_i$. This is a CW–complex that is not a simplicial gluing, but we don't really need it to be. All we need to do is recognize that the space obtained relates the boundaries of V and V^*, showing us how to get from one to the other. More important are the annuli μ_i and λ_i, which now each contain triangulated curves representing the 0–framing of V^* and meridians of V.

Take ∂V^* to be the 2–complex obtained by gluing together the μ_i over the curves l_i. Then ∂V^* contains a triangulated curve γ that determines the 0–framing of V^* and a set of triangulated curves $\{m_i\}$ that represent meridians of V. The oriented intersection number of γ with any m_i determines the framing coefficient for the 2–handle attached over V^*.

Data: a 2–cell c of the Stein complex S
Result: the framing coefficient n for c

begin
  $V^*, \gamma \longleftarrow$ the attaching region corresponding to c that results from Algorithm 15 with input S;
  $V, \{m_i\} \longleftarrow$ triangulation of V with ordered meridians;
  $C_V \longleftarrow$ the dual simplicial cycle in T'^* that represents regular fibers of π' in V;
  $\lambda_i \longleftarrow$ the triangulated strips from ∂V cut by parallel longitudes of V;
  $\mu_i \longleftarrow$ the triangulated strips from ∂V^* cut by parallel meridians of V^*;
  foreach *edge* e_i *in* $\partial c = \{e_0, e_1, \ldots, e_k, e_0\}$ **do**
    determine a lift $s(f_i)$ into a 3–cell τ for f_i parallel to e_i;
    $w_j^*, w_{j+1}^* \longleftarrow$ the dual edges in C_V incident to τ^*;
    subdivide λ_i so that its boundary components match μ_i;
    subdivide λ_i so that there is room for $s(f_i)$ between ∂w_j and ∂w_{j+1};
    add the meridinal curves of λ_i to μ_i;
    add $s(f)$ to λ_i between $\partial w_j = m_j$ and $\partial w_{j+1} = m_{j+1}$;
  end
  $n \longleftarrow 0$;
  foreach *vertex* v_i *in* $\partial c = \{e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_k = (v_k, v_0), e_0\}$ **do**
    $\Lambda \longleftarrow$ the torus formed by gluing λ_i to μ_i;
    $\hat{s}(y_i) \longleftarrow$ the completion of $s(y_i)$ over λ_i in Λ;
    $\hat{m}_0 \longleftarrow$ the completion of m_0 over μ_i in Λ;
    $\kappa \longleftarrow$ the oriented intersection number of $\hat{s}(y_i)$ with $\hat{m}_0$;
    $n \longleftarrow n + \kappa$;
  end
end

Algorithm 16: Computation of Framing Coefficients

4.3.4 Building 2–handles

We now concern ourselves with 2–handle attachment, and the necessary data has already been computed. Every 2–cell of S is equipped with an integer framing coefficient, a triangulated solid torus that serves as an attaching region, and a curve in the boundary torus that represent a 0–framing.

Let c be the 2–cell, V the solid torus with boundary $\partial V = v$, λ the curve representing the 0–framing, and n_c the framing coefficient. The general idea is that we build a triangulated 3–sphere Z containing V as a subtriangulation such that a curve λ^n at n_c meridinal twists away from $\lambda \subset v$ bounds a disc in $Z \setminus V$. We assume that V has been subdivided enough to admit a curve satisfying the conditions of λ^n. With Z built, the cone CZ is a 4–disc with the desired 2–handle structure to attach over V.

Data: a triangulated solid torus V, a curve λ in ∂V that determines 0–framing, an integral framing coefficient n

Result: a triangulated 4–disc H^4_V that contains V as a subtriangulation of the boundary and λ^n bounds a disc in the boundary

begin
- $v \longleftarrow \partial V$;
- $\lambda^n \longleftarrow$ a curve in v that is n_c meridinal twists away from λ;
- $v|\lambda^n \longleftarrow$ the annulus obtained by splitting v along λ^n;
- $C \longleftarrow$ the triangulated cone on $v|\lambda^n$;
- $D \longleftarrow$ the triangulated cone on $\partial(v|\lambda^n)$;
- **foreach** *pair of boundary edges e_i, e'_i in D with e_i, e'_i corresponding to the same edge of λ^n* **do**
  - $D \longleftarrow$ the result of gluing e_i to e'_i;
- **end**
- $B \longleftarrow$ the triangulated cone on D;
- $Z \longleftarrow$ the result of gluing C to B along their shared boundary;
- $H^4_V \longleftarrow CZ$;

end

Algorithm 17: Building a 2–handle

4.3.5 Building the 4–manifold

We tie the algorithms of Section 4.3 together into Algorithm 18.

Data: the closed, orientable, edge–distinct 3–manifold triangulation T
Result: a triangulated 4–manifold W with boundary triangulation homeomorphic to the closed orientable 3–manifold M whose triangulation was taken as input to Algorithm 7

begin
  $S \longleftarrow$ the result of Algorithm 11 with input T;
  $W, \mathcal{V}, \mathcal{C} \longleftarrow$ the result of Algorithm 14 with input S;
  foreach *2–cell* c *of* S **do**
    decorate c with the result of Algorithm 16 with input c;
  end
  $\Lambda \longleftarrow$ the set of pairs (V, λ) with V from $\mathcal{V}$ and λ the 0–framing of V;
  foreach *pair* (V, λ) *of* Λ **do**
    $n \longleftarrow$ the framing coefficient of the 2–cell of S corresponding to V;
    $W \longleftarrow$ the result of subdividing V to a triangulated solid torus V' for which a curve λ^n as described in Section 4.3.4 exists;
    $\lambda' \longleftarrow$ the subdivision of λ from the previous line;
    $H^4 \longleftarrow$ the result of Algorithm 17 with input V',λ',n;
    $W \longleftarrow$ the result of gluing H^4 to W over V';
  end
end

Algorithm 18: Building a 4–manifold

Chapter 5

Conclusion

We now have an algorithmic method to obtain a 4–manifold with a given boundary 3–manifold, provided that 3–manifold is orientable and given as a triangulation. This work fits into some active areas of low dimensional topology, and there are some clear directions that we could take in the future.

The first consideration is an actual software implementation of the algorithm into a framework such as Regina: software for low dimensional topology [1]. This is especially useful if we want to get our hands on explicit triangulations of 4–manifolds for the purposes of building a census or because we would like to compute cobordism invariants such as the Rohklin invariant.

In that same vein, the work can be refined somewhat. For example, the handlebody built in Section 4.3.2 is the gluing of several copies of the 3–disc, all of which have undergone two barycentric subdivisions and are thus triangulated by $21 \cdot 24^2$ tetrahedra. We then 4–thicken each tetrahedron of that handlebody with a 4–prism. The number P_n of n–simplices used to triangulated the n–prism designed in Section 4.3.1 is $nP_{n-1} + 2$, with the first few numbers being $P_1 = 1$, $P_2 = 4$, $P_3 = 14$, and $P_4 = 58$. We quickly approach numbers of pentachora in the thousands, making some computations unfeasible without suitable simplification of triangulation.

It is also worth making modifications to the algorithm to ensure that the 4–manifold produced is simply connected, as in Theorem 5.4 of [2], or to acquire a spin structure for the 4–manifold as in Theorem 6.1 of [2]. We said before that the Stein complex was ‘almost’ a shadow, and one stop on the route to each of these would be to explicitly turn the Stein complex acquired in Section 4.2 into a gleamed shadow.

The final and most general avenue is the conversion of theorem proofs in smooth low–dimensional topology to algorithms on triangulations. Much of the groundwork that went into triangulating the smooth case in this document was explored in [2], but the history of the problem has yielded a large number of smooth proofs. The first step in our algorithm was to develop a piecewise–linear analogue to the generic proper smooth map, and this is entirely because our proof of smooth 3–manifolds bounding 4–manifolds did this as well. Investigating other proofs that smooth 3–manifolds bound 4–manifolds may allow explicit constructions to arise, but these constructions need to be directly obtainable from a triangulation.

There are standard arguments in the literature that any closed, orientable 3–manifold has a Heegaard decomposition, and this is shown by explicitly constructing the decomposition from a triangulation of the given 3–manifold. Combining the construction of Heegaard diagrams with proof in the smooth case of 3–manifolds bounding 4–manifolds is a promising direction for future work.

Bibliography

[1] Benjamin Burton, Ryan Budney, William Pettersson, et al. Regina: Software for low-dimensional topology. `http://regina.sourceforge.net/`, 1999–2016.

[2] Francesco Costantino and Dylan Thurston. 3-manifolds efficiently bound 4-manifolds. *Journal of Topology*, (3) 1, 2008.

[3] Franceso Costantino. A short introduction to shadows of 4–manifolds, 2005. Available on arxiv.

[4] Robert Gompf and Andràs Stipsicz. *4–Manifolds and Kirby Calculus.* American Mathematical Society, 1999.

[5] Morris Hirsch. Immersions of almost parallelizable manifolds. *Proc. Amer. Math. Soc.*, 12:845–846, 1961.

[6] Morris Hirsch. *Differential Topology.* Springer, 1967.

[7] Morris Hirsch and Barry Mazur. *Smoothing of Piecewise Linear Manifolds.* Princeton University Press.

[8] Antoni Kosinski. *Differential Manifolds.* Academic Press, Inc., 1993.

[9] Gerd Laures. On cobordisms of manifolds with corners. *Transactions of the American Mathematical Society*, (12) 352:5667–5688, 2000.

[10] John M. Lee. *Introduction to Smooth Manifolds.* Springer, 2000.

[11] John Milnor. *Morse Theory.* Princeton University Press, 1963.

[12] Bjorn Poonen and Michael Rubinstein. The number of intersection points made by the diagonals of a regular polygon. *SIAM Journal on Discrete Mathematics*, (1) 11:135–156, 1998.

[13] Vladimir Rokhlin. The embedding of non-orientable three-manifolds into five-dimensional euclidean space. *Dokl. Akad. Nauk SSSR*, 160:549–551, 1965.

[14] Dale Rolfsen. *Knots and Links.* AMS Chelsea Publishing.

[15] Colin Rourke. A new proof that Ω_3 is zero. *J. London Math. Soc.*, (2) 31:373–376, 1985.

[16] Osamu Saeki. *Topology of Singular Fibers of Differentiable Maps.* Lecture Notes in Mathematics 1854. Springer, 2004.

[17] Saul Schleimer. Waldhausen's theorem. *Geometry & Topology Monographs*, 12:299–317, 2007.

[18] Christopher J. Schommer-Pries. The classification of two-dimensional extended topological field theories, 2014. .

[19] René Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, 38:17–86, 1954.

[20] Vladimir Turaev. Topology of shadows. Preprint, 1991.

[21] Charles Terence Clegg Wall. All 3-manifolds imbed in 5-space. *Bull. Amer. Math. Soc.*, 71:564–567, 1965.

[22] John Henry Constantine Whitehead. On C^1 complexes. *Annals of Mathematics*, Second Series, (41) 4:809–824, 1940.