2020 SIME Solutions and Results

Summer Mathematics Competitions

July 2020

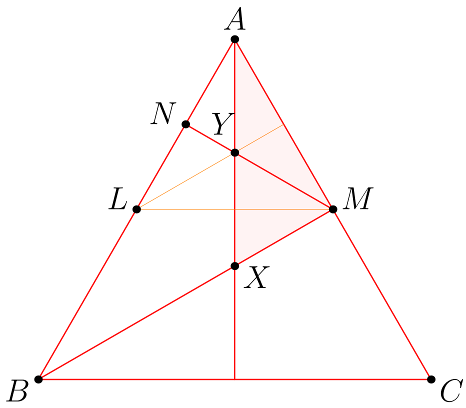
Contents

§1 Problems and Solutions

Problem 1

Let $\triangle ABC$ be an equilateral triangle with side length 1 and M be the midpoint of side AC. Let N be the foot of the perpendicular from M to side AB. The angle bisector of angle $\angle BAC$ intersects MB and MN at X and Y, respectively. If the area of triangle $\triangle MXY$ is $\mathcal{A}$, and $\mathcal{A}^2$ can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Solution:



Let L be the midpoint of AB. Notice that Y is the center of equilateral triangle ALM and X is the center of equilateral triangle ABC. Now since a homothety (dilation) of a factor of 2 from A sends the former to the latter, Y is the midpoint of X. Hence by same base same height we have that the area of $\triangle MXY$ is equal to the area of $\triangle AYM$.

But clearly $\triangle AYM$ is just a third of $\triangle ALM$ which is a fourth of $\triangle ABC$. It follows that

$$\mathcal{A} = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{\sqrt{3}}{4} = \frac{1}{16\sqrt{3}},$$

so $\mathcal{A}^2 = \frac{1}{768}$, and $m + n = \boxed{769}$. ■

Problem 2

Andrew rolls two fair six sided die each numbered from 1 to 6, and Brian rolls one fair 12 sided die numbered from 1 to 12. The probability that the sum of the numbers obtained from Andrew's two rolls is less than the number obtained from Brian's roll can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

Solution: If Brian rolls 1, we do not get the desired condition, so lets ignore that and multiply by $\frac{11}{12}$ at the end.

It is easy to see that now both Andrew's roll and Brian's roll are both symmetric with respect to 6. Hence the probability that Andrew's roll is less than Brian's is the same as the probability that Andrew's roll is greater than Brian's.

It suffices to find the complement that is not in either of these and divide by two, i.e. the probability Brian's roll is exactly equal to Andrew's roll.

But clearly no matter what Andrew rolls (any number from 2 to 12), Brian matches that number with $\frac{1}{11}$ probability. Hence the answer is

$$\frac{11}{12} \cdot \frac{1 - \frac{1}{11}}{2} = \frac{5}{12},$$

so $m + n = \boxed{017}$. ■

Problem 3

Real numbers $x, y > 1$ are chosen such that the three numbers

$$\log_4 x,\ 2\log_x y,\ \log_y 2$$

form a geometric progression in that order. If $x + y = 90$, then find the value of xy.

Solution: Note that if real values r, s, t form a geometric sequence, then we must have $s^2 = rt$. Using this, we get that

$$4(\log_x y)^2 = \log_y 2 \cdot \frac{1}{2}\log_2 x \implies 8(\log_x y)^2 = \log_y x$$

and since $\log_y x = \frac{1}{\log_x y}$, we get

$$8 = (\log_y x)^3 \implies \log_y x = 2 \implies x = y^2.$$

Substituting this into $x + y = 90$ gives $y^2 + y = 90 \implies (y + 10)(y - 9) = 0$. Since y is positive, we must have $y = 9$, so $x = 81$. The desired product is $xy = 81 \cdot 9 = \boxed{729}$.

■

Problem 4

Suppose that $(\underline{AB}, \underline{CD})$ is a pair of two digit positive integers (digits A and C must be nonzero) such that the product $\underline{AB} \cdot \underline{CD}$ divides the four digit number $\underline{ABCD}$. Find the sum of all possible values of the three digit number $\underline{ABC}$.

Solution: Let $\underline{AB} = x$ and $\underline{CD} = y$ where x and y are values. We have that $xy \mid 100x + y$. Suppose that we let $\frac{100x+y}{xy} = k$ for some positive integer k. Cross multiplying the fractions yields the equation $kxy = 100x + y$. Multiply both sides by k and using a well-known factoring gives $(kx - 1)(ky - 100) = 100$.

If $k = 1$ then by size, $(kx - 1)(ky - 100)$ must be nonpositive for $x, y \in [1, 100]$, which is not possible.

If $k = 2$ then things get a little more interesting. $2x-1$ is odd and $2y-100$ is even. Furthermore, since x is a two-digit number, $2x - 1 \geq 19$ so in fact the only possible factor of 100 that fits this criteria is 25. Thus, $2x - 1 = 25 \implies x = 13$ so $2y - 100 = 4$ must hold so $y = 52$. We get the solution $(13, 52)$.

If $k = 3$ then we actually have very few possibilities by size. $3x - 1 \geq 29$ so the only possible value it can take on is 50 or 100. In the former case, we get $(x, y) = (17, 34)$ and in the latter case is not possible.

If $k \geq 4$ we only need to check $kx - 1 = 50, 100$ for size reasons. However clearly 101 is prime so $kx - 1 = 100$ will never work. Furthermore, $kx - 1 = 50 \implies kx = 51$ is also not possible since x is two digits.

Thus our only solutions are $(13, 52)$ and $(17, 34)$. We sum $135 + 173 = \boxed{308}$. ■

Problem 5

Let $ABCD$ be a rectangle with side lengths $\overline{AB} = \overline{CD} = 6$ and $\overline{BC} = \overline{AD} = 3$. A circle ω with center O and radius 1 is drawn inside rectangle $ABCD$ such that ω is tangent to $\overline{AB}$ and $\overline{AD}$. Suppose X and Y are points on ω that are not on the perimeter of $ABCD$ such that BX and DY are tangent to ω. If the value of XY^2 can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

Solution: Let ω be tangent to AB and AD at X' and Y', respectively. Clearly (X, X') and (Y, Y') are reflections of each other over BO and CO, respectively, by tangency properties. Hence, we can let $\theta_1 = \angle BOX = \angle BOX'$ and $\theta_2 = \angle DOY = \angle DOY'$.

Notice that $2\theta_1 + 2\theta_2 + 90^\circ = 360^\circ + \angle XOY$. Hence, $\angle XOY = 2\theta_1 + 2\theta_2 - 270^\circ$. We can just find $\cos \angle XOY$ and the rest of the problem is finished by a simple length chase.

By $\cos x = \cos(x + 360^\circ)$ and $\cos x + 90^\circ = \sin x$, we get $\cos(2\theta_1 + 2\theta_2 - 270^\circ) = \sin(2\theta_1 + 2\theta_2)$ and this becomes $2\sin(\theta_1 + \theta_2)\cos(\theta_1 + \theta_2)$. We actually have all the information we need to compute this. With the computed lengths $BX' = 5, OX' = 1, OY' = 1, DY' = 2$, we may use Pythagorean to find the hypotenuses and get that $\sin\theta_1 = \frac{5}{\sqrt{26}}, \cos\theta_1 = \frac{1}{\sqrt{26}}, \sin\theta_2 = \frac{2}{\sqrt{5}}$, and $\cos\theta_2 = \frac{1}{\sqrt{5}}$.

This is enough to compute our desired terms. Using sine and cosine addition, we may compute $\sin(\theta_1 + \theta_2) = \frac{7}{\sqrt{130}}$ and $\cos(\theta_1 + \theta_2) = \frac{9}{\sqrt{130}}$ hence $\cos(\angle XOY) = \frac{63}{65}$.

We finish the problem with Law of Cosines:

$$XY^2 = 2 - 2\cos(\angle XOY) = \frac{4}{65},$$

hence our final answer of $4 + 65 = \boxed{069}$ as desired. ■

Problem 6

Let $P(x) = x^2 + ax + b$ be a quadratic polynomial. For how many pairs (a, b) of positive integers where $a, b < 1000$ do the quadratics $P(x+1)$ and $P(x)+1$ have at least one root in common?

Solution: If the quadratics $P(x+1), P(x)+1$ are to share a common root, that must be the root of the linear function $P(x+1) - (P(x)+1)$ since common roots are preserved in addition and subtraction. Thus, after computing,

$$P(x+1) - P(x) - 1 = 2x + a$$

which has the root $-\frac{a}{2}$. Now this must also be a root of $P(x)+1$ (which also then implies that it is a root of $P(x+1)$ by the preservation of common roots through addition) so plugging it in must give 0 as follows:

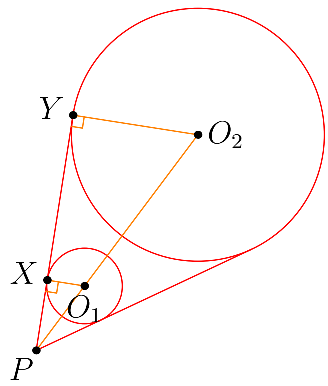
$$\frac{a^2}{4} - \frac{a^2}{2} + b + 1 = 0 \implies 4(b+1) = a^2.$$

So a must be even and has a upper bound of $\sqrt{4000} \approx 63.25$. To make sure b is positive, we must restrict $a > 2$. So a can just be any even positive integer less than 63 but greater than 2, and with every such a comes a unique positive integer value of b, hence a total of 30 possible value of a and therefore $\boxed{030}$ such pairs (a, b). ■

Problem 7

Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers $(1,1)$ and $(4,5)$ and radii $r_1 < r_2$, respectively, are drawn on the coordinate plane. The product of the slopes of the two common external tangents of $\mathcal{C}_1$ and $\mathcal{C}_2$ is 3. If the value of $(r_2 - r_1)^2$ can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

Solution:



Let O_1 and O_2 denote the centers of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. Let X and Y be the feet of the perpendiculars from O_1 and O_2 to the top tangent as shown, and let P be the the instersection of the common external tangents. Let ϕ be the angle formed between PX and O_1O_2, and let θ be the angle formed between O_1O_2 and the x axis.

It's easy to see that

$$r_2 - r_1 = PO_2 \sin\phi - PO_1 \sin\phi = O_1O_2 \sin\phi = 5\sin\phi.$$

Now the slope of line O_1O_2 is $\frac{4}{3}$, so $\tan\theta = \frac{4}{3}$. Similarly, we deduce that the condition is equivalent to $\tan(\theta+\phi)\tan(\theta-\phi) = 3$. Two quick applications of tangent addition/subtract later, we arrive at

$$3 = \frac{\frac{4}{3} + \tan\phi}{1 - \frac{4}{3}\tan\phi} \cdot \frac{\frac{4}{3} - \tan\phi}{1 + \frac{4}{3}\tan\phi} = \frac{16 - 9\tan^2\phi}{9 - 16\tan^2\phi}.$$

Cross multiplying gives $27 - 48\tan^2\phi = 16 - 9\tan^2\phi$, so $\tan^2\phi = \frac{11}{39}$ and $\tan\phi = \frac{\sqrt{11}}{\sqrt{39}}$. It follows that $\sin\phi = \frac{\sqrt{11}}{\sqrt{11+39}} = \frac{\sqrt{11}}{5\sqrt{2}}$. Now

$$(r_2 - r_1)^2 = \left(5 \cdot \frac{\sqrt{11}}{5\sqrt{2}}\right)^2 = \frac{11}{2},$$

so $m + n = \boxed{013}$.

■

Problem 8

Find the number of positive integers n between 1 and 1000, inclusive, satisfying

$$\lfloor\sqrt{n-1}\rfloor + 1 = \left\lfloor\sqrt{n+\sqrt{n}+1}\right\rfloor$$

where $\lfloor n\rfloor$ denotes the greatest integer not exceeding n.

Solution: We can write every positive integer n in the form k^2+m where k^2 is the largest perfect square smaller than n and m is a positive integer in the range $[1, 2k+1]$. Plugging this into the problem statement yields

$$k+1 = \left\lfloor\sqrt{k^2+\sqrt{k^2+m}+m+1}\right\rfloor.$$

Since $k^2 < k^2+m \leq (k+1)^2$ we write $\sqrt{k^2+m}$ as $k+\epsilon$ where $\epsilon \in \{0,1\}$. We need

$$k^2+k+1+m+\epsilon \geq k^2+2k+1$$

which simplifies to $m+\epsilon \geq k$. Since $\epsilon = 0$ unless $m = 2k+1$, this is saying that $m \geq k$. Therefore our intervals of solutions are of the form $[k^2+k, (k+1)^2]$ for positive integers k. Each such interval contains $k+2$ numbers. Note that since $31^2 < 1000 < 32^2$, the large interval $[1, 1000]$ contains full such intervals for $k = 1, 2, \ldots, 30$. However, note that $31^2+31 = 992 \leq 100$ so in fact 9 numbers from the 31th interval should also be included. Finally, we cannot forget that 1 is a solution to our original problem.

Hence, our final answer is

$$1+\left(\sum_{k=1}^{30}(k+2)\right)+9 = 10+(3+4+\ldots+32) = 10+15\cdot 35 = \boxed{535}$$

as desired. ■

Problem 9

William writes the number 1 on a blackboard. Every turn, he erases the number N currently on the blackboard and replaces it with either $4N+1$ or $8N+1$ until it exceeds 1000, after which no more moves are made. If the minimum possible value of the final number on the blackboard is M, find the remainder when M is divided by 1000.

Solution: Convert the number on the board to binary. Note that in binary, multiplying a number by 4 and then adding 1 is equivalent to concatenating 01 right after that number. Similarly, multiplying by 8 and then adding 1 is equivalent to concatenating 001.

Since 1 is just 1 in binary, what William is really doing is appending either 01 or 001 right after the blackboard number in binary until the number exceeds 1000.

Note that it is impossible to use only 10 digits because the 2^8 spot must be 0. This is not large enough to exceed 1000.

Hence, the final number must have 11 digits. The 2^{10} spot must be 1 since the original number on the board is 1. We want to fill the rest of the 10 spots with 001 and 01 to minimize the value of the number. Note that in the binary string, 1's must either be spaced one zero apart or two zeroes apart.

Suppose that after the 1 in the leftmost digit, we have a 01 strings and b 001 strings. Clearly, the positive integers a and b must satisfy $2a+3b=10$, which is only possible when $a=b=2$. To minimize value, we must push the ones to as right as possible, so we put the 001's closer to the left.

This yields the answer of $M = 10010010101_2$ in binary, which converts to

$$2^{10}+2^7+2^4+2^2+2^0 = 1173 \equiv \boxed{173} \pmod{1000}$$

in base 10.

■

Problem 10

Consider all 2^{20} paths of length 20 units on the coordinate plane starting from point $(0,0)$ going only up or right, each one unit at a time. Each such path has a unique *bubble space*, which is the region of points on the coordinate plane at most one unit away from some point on the path. The average area enclosed by the bubble space of each path, over all 2^{20} paths, can be written as $\frac{m+n\pi}{p}$ where m,n,p are positive integers and $\gcd(m,n,p)=1$. Find $m+n+p$.

Solution: It definitely helps on this problem to draw a couple of examples of what a bubble space might look like for shorter paths of say, length 5 units.

Define the *boxey space* of a path to be the set of all unit grid squares touching the path at some point. Notice the similarity between a path's boxey space and bubble space; the bubble space is formed by rounding off $90°$ corners of the boxey space.

Claim: The number of unit squares in the boxey space of a path is $2l+4$, where l is the length of the path.

Proof: This is actually quite surprising, because the area of the boxey space of some path only depends on its length, and not the shape of the path itself. Nevertheless, we will prove this with induction (which turns out to be surprisingly easy).

Our base case is trivial. For a path of length 1, its boxey space is just a $2\cdot 3$ grid of squares and does indeed consist of $2\cdot 1+4=6$ squares.

The inductive step also follows quite immediately; Suppose the boxy space of an arbitrarily chosen k-length path consists of $2k+4$ squares. It suffices to show that adding one more unit to the length of the path adds exactly two squares to the path's boxey space.

Consider the last point, say P, of the k-length path. All four unit squares touching P must be part of the original boxey space. If we add one upward unit to the path, the two squares directly above the previously mentioned four unit squares will be added into the boxey space of the new path. If we add one rightward unit to the path, the two squares to direct right of the previously mentioned four unit squares will be added into the boxey space of the new path.

Hence, our inductive step is finished and so is our induction. ■

It remains to find the expected number of corners that are rounded off over all paths of length 20. Note that the two "bottom-most/leftmost" and "top-most/rightmost" squares bordering each of the first and last point must be rounded off, which gives us 4 rounded-off corners already. Furthermore, we see that each time the path changes direction, it creates a $270°$ and $90°$ angle in the boxey space of the path. Hence, aside from the already counted 4 corners, each time there is a "turn" in the path, we round off one more corner.

We just need to find the expected number of turns in a 20-unit long path. Clearly the path cannot "change direction" in the first move, but in each of the 19 subsequent moves, there is a $\frac{1}{2}$ chance of changing direction. Therefore, by Linearity of Expectation, the expected number of turns is $\frac{19}{2}$, so the expected number of rounded corners is $\frac{19}{2}+4=\frac{27}{4}$.

Each rounded corner has area $\frac{4-\pi}{4}$, and the over all 20-unit long paths, the boxey space always has area 44, so the expected value of the area of a path's bubble space is therefore

$$44-\frac{27}{2}\cdot\left(\frac{4-\pi}{4}\right)=\frac{244-27\pi}{8}$$

so our desired answer is $244+27+8=\boxed{279}$. ■

Problem 11

Let $d_1, d_2, \ldots, d_k$ be the distinct positive integer divisors of 6^8. Find the number of ordered pairs (i, j) such that $d_i - d_j$ is divisible by 11.

Solution: Let $d_i = 2^a 3^b$ and $d_j = 2^c 3^d$. The key idea is that 2 is a primitive root modulo 11 since 11 divides neither 4 nor 32. Now we have

$$2^a 3^b \equiv 2^c 3^d \pmod{11} \iff 2^{a-c} \cdot (4^{-1})^{b-d} \equiv 1 \pmod{11} \iff 2^{a-c-2(b-d)} \equiv 1 \pmod{11},$$

so the condition is equivalent to $10 \mid a - c - 2(b - d)$.

We do the remaining counting with PIE.

Since a, b, c, d are from 0 to 8, there is always exactly a that works for any selection of b, c, d with the exception of when $c + 2(b - d) \equiv 9 \pmod{10}$.

Now similarly to above, in this exception case, there is always one c that is valid, unless

$$2(b - d) \equiv 0 \pmod{10} \iff b \equiv d \pmod{5}.$$

If both b and d are 4 mod 5, then both must be 4. Otherwise, any other residue modulo 5 has two choices for each, so there are 17 such b, d. The answer is $9^3 - 9^2 + 17 = \boxed{665}$. ■

Problem 12

Two sets S_1 and S_2, which are not necessarily distinct, are each selected randomly and independently from each other among the 512 subsets of $S = \{1, 2, \ldots, 9\}$. Let $\sigma(X)$ denote the sum of the elements of set X. Note that $\sigma(\emptyset) = 0$ where $\emptyset$ denotes the empty set. If $S_1 \cup S_2$ stands for the union of S_1 and S_2, the probability that $\sigma(S_1 \cup S_2)$ is divisible by 3 can be expressed as a common fraction of the form $\frac{m}{2^n}$ where m is odd and n is a positive integer. Find $m + n$.

Solution: Notice that in determining two sets A and B, we can equivalently just choose four disjoint sets that are to be assigned to $A \setminus B, B \setminus A, A \cap B$, and everything else.

Notice that the three disjoint sets $A \setminus B, B \setminus A$, and $A \cap B$ pretty much make up $A \cup B$. So in our original splitting into four sets, we consider the sum of all numbers that end up in 3 out of 4 of them.

We use generating functions. Since each element has a $\frac{3}{4}$ chance of ending up in one of the three sets we are considering, we consider the function

$$f(x) = \prod_{i=1}^{9} \left(\frac{1 + 3x^i}{4} \right) = \left(\frac{1 + 3x^1}{4} \right) \left(\frac{1 + 3x^2}{4} \right) \cdots \left(\frac{1 + 3x^9}{4} \right)$$

where in the expansion of this polynomial, the fraction before an x_k term denotes the probability that $\sigma(S_1 \cup S_2) = k$, where S_1 and S_2 denote the chosen sets in the problem. We want to find the sum of the coefficients of all x^{3k} terms.

This is just a classic roots of unity filter. Let $\omega = e^{\frac{2\pi i}{3}}$. Using the well known cube roots of unity identities $1 + \omega + \omega^2 = 0$ and $\omega^3 = 1$, we see that our desired expression is

$$\frac{f(1) + f(\omega) + f(\omega^2)}{3}.$$

Clearly $f(1) = 1$. Notice that in evaluating $f(\omega)$ and $f(\omega^2)$, we only need to evaluate the first three terms in the expression then cube, since $\omega^3 = 1$. So in fact, both

$$f(\omega), f(\omega^2) = \left(\frac{(1 + 3\omega)(1 + 3\omega^2)}{16} \right)^3.$$

Evaluating this should not be hard. Again, expanding and using $\omega^2 + \omega = -1$ and $\omega^3 = 1$, we see that $f(\omega), f(\omega^2) = \frac{7^3}{2^{12}}$. Plugging these into our expression for our final desired expression, we get

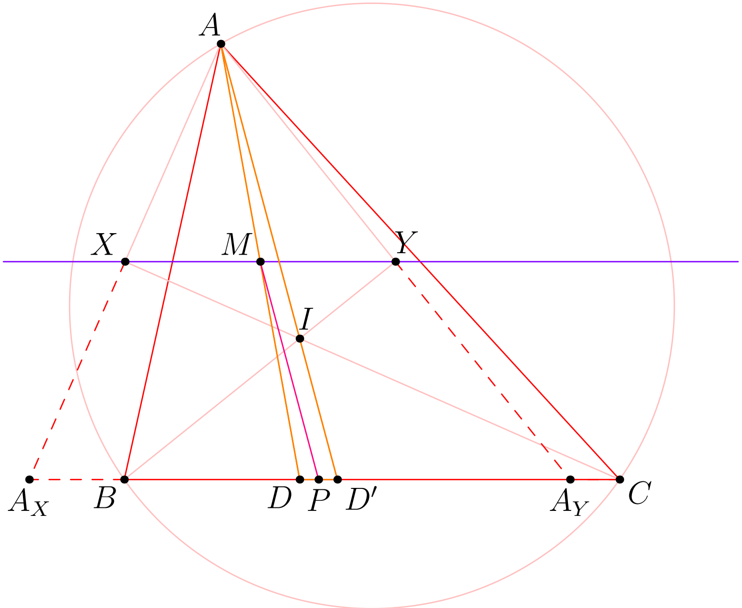
$$\frac{1 + \frac{343}{2^{11}}}{3} = \frac{2391}{3 \cdot 2^{11}} = \frac{797}{2^{11}}$$

hence our final answer of $797 + 11 = \boxed{808}$. ■

Problem 13

In acute triangle $\triangle ABC$, $\overline{AB} = 20$ and $\overline{AC} = 21$. Let the feet of the perpendiculars from A to the angle bisectors of $\angle ACB$ and $\angle ABC$ be X and Y, respectively. Let M be the midpoint of $\overline{XY}$. Suppose P is the point on side BC such that MP is parallel to the angle bisector of $\angle BAC$. If given that $\overline{BP} = 11$, then the length of side BC can be expressed as a common fraction in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

Solution:



Let A_X and A_Y be the reflections of A over X and Y, respectively. By the definition of angle bisector and perpendicularity, we know that points A_X and A_Y both lie on BC. We now see that clearly XY is the A-midline.

Note that since the incenter I is on both the perpendicular bisectors of AA_X and AA_Y, it must be the circumcenter of $\triangle AA_XA_Y$. Hence, the foot from I to BC, which we denote D, is the midpoint of A_XA_Y. By homothety, points A, M, D must be collinear, with $AM = MD$.

Now let AI hit BC at point D'. Since $MP \parallel AD'$ and passes throuch the midpoint of AD, line MP must be the D-midline of triangle $\triangle DAD'$. Therefore, P is the midpoint of DD'.

The rest is a length chase. Let $BC = a$. Since D is also the A-intouch point, we have

$$BD = \frac{a + 20 - 21}{2} = \frac{a-1}{2}.$$

By the Angle Bisector Theorem, we have $\frac{BD'}{CD'} = \frac{20}{21}$ so $BD' = \frac{20a}{41}$. Since P is the midpoint of DD', we have

$$BD + BD' = 2BP = 22$$

which upon the following equation

$$\frac{a-1}{2} + \frac{20a}{41} = 22$$

yields $BC = a = \frac{205}{9}$ so our answer is $205 + 9 = \boxed{214}$ as desired. ■

Problem 14

Let $P(x) = x^3 - 3x^2 + 3$. For how many positive integers $n < 1000$ does there not exist a pair (a, b) of positive integers such that the equation

$$\underbrace{P(P(\dots P}_{a \text{ times}}(x)\dots)) = \underbrace{P(P(\dots P}_{b \text{ times}}(x)\dots))$$

has exactly n distinct real solutions?

Solution: The key is to notice that the coefficients of P roughly follow the coefficients of $(x-1)^3$. Indeed we can "move" the constant coefficient of P to the x term by adding $3x - 3$. It follows that $P(x) = (x-1)^3 - 3(x-1) + 1$.

This reminds us of the identity $t^3 + \frac{1}{t^3} = \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right)$. Hence setting $x = t + \frac{1}{t} + 1$ where t is a complex number either on the unit circle or on the real line (so that x is real), gives us $P(x) = t^3 + \frac{1}{t^3} + 1$, hence

$$\underbrace{P(P(\dots P}_{k \text{ times}}(x)\dots)) = t^{3^k} + \frac{1}{t^{3^k}} + 1.$$

It follows that the equation reduces to $t^{3^a} + \frac{1}{t^{3^a}} = t^{3^b} + \frac{1}{t^{3^b}}$.

WLOG assume $a > b$ (equality gives infinitely many solutions) and clear denominators. We obtain

$$0 = t^{2\cdot 3^a} - t^{3^a+3^b} - t^{3^a-3^b} + 1 = \left(t^{3^a+3^b} - 1\right)\left(t^{3^a-3^b} - 1\right)$$

Now t must be $3^a + 3^b$ or $3^a - 3^b$ root of unity (which lie on the unit circle, so x is real). The gcd of these numbers is $2 \cdot 3^b$, hence there are a total of $(3^a + 3^b) + (3^a - 3^b) - 2 \cdot 3^b = 3^{a+1} - 2 \cdot 3^b$ solutions in t due to overlap.

Now since x is twice the real part of t plus one, and the solutions are symmetric over the real axis (with the exception of $t = 1$), there are only $\frac{3^{a+1} - 2\cdot 3^b + 1}{2}$ unique values of x.

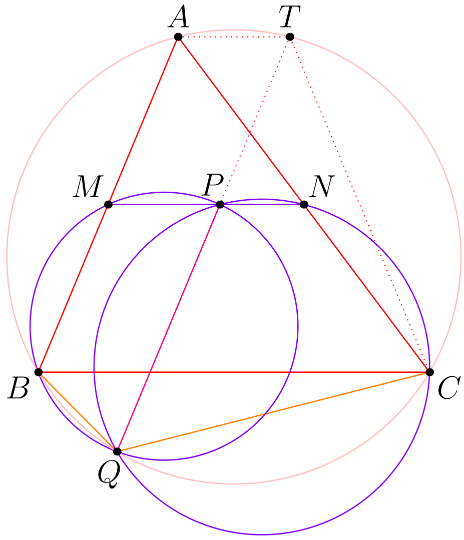
It follows that n cannot be of the form $\frac{3^a+1}{2} - 3^b$. Proceed with complementary counting. Notice that since $b < a$, we must have $a \leq 7$, otherwise things are too large.

Notice that for $a \leq 6$, we can have $b = 1, 2, \dots, a-1$, so this gives $1 + 2 + 3 + 4 + 5 = 15$ cases. But if $a = 7$, $\frac{3^a+1}{2} = 1094$, hence we can only have $b = 5, 6$, so we have a total of 17 bad cases. It follows that the answer is $999 - 17 = \boxed{982}$. ■

Problem 15

Triangle $\triangle ABC$ has side lengths $\overline{AB} = 13, \overline{BC} = 14$, and $\overline{AC} = 15$. Suppose M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let P be a point on $\overline{MN}$, such that if the circumcircles of triangles $\triangle BMP$ and $\triangle CNP$ intersect at a second point Q distinct from P, then PQ is parallel to AB. The value of AP^2 can be expressed as a common fraction of the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution:



First, we prove that Q lies on (ABC). This should not be hard:

$$\angle BQC = \angle BQP + \angle CQP = \angle AMP + \angle ANP = \angle ABC + \angle ACB = 180^\circ - \angle BAC$$

as desired.

Next, we prove a not so obvious result.

Claim: Let $\overrightarrow{QP}$ intersect (ABC) again at point T. Then, $AT \parallel BC \implies ABCT$ is an isosceles trapezoid.

Proof: This is actually also an angle chase. Note that

$$\angle QPM = 180^\circ - \angle ABQ = \angle ACQ = \angle ATQ$$

so $\angle ATQ = \angle MPQ$ and thus $AT \parallel MP$. Clearly $MP \parallel BC$ so $AT \parallel BC$, as desired. $\square$

By symmetry of isosceles trapezoid $ABCT$, we get $BT = 15$ and $CT = 13$. Using Ptolemy's Theorem, we length chase to get

$$AT \cdot 14 + 13^2 = 15^2 \implies AT = 4.$$

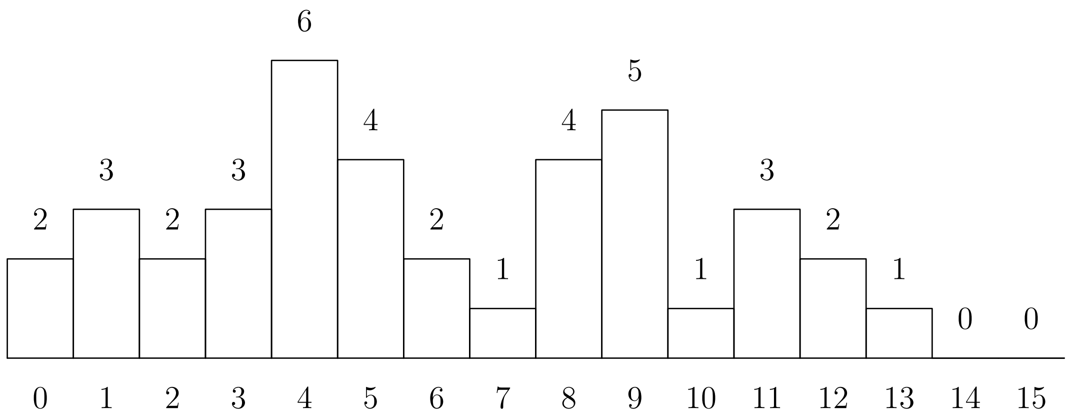
Since $TQ \parallel AB$ and $AT \parallel MN$, quadrilateral $ATPM$ is a parallelogram so $MP = AT = 4$.

The rest is just a length chase using properties of the $13-14-15$ triangle. Let D be the foot from A to BC and D' be the foot from A to MN. By properties, $AD = 12 \implies AD' = 6$, and $BD = 5 \implies MD' = \frac{5}{2}$. Furthermore, $PD' = MP - MD' = 4 - \frac{5}{2} = \frac{3}{2}$.

It remains to try Pythagorean Theorem on $\angle AD'P = 90^\circ$. We get $AP^2 = 36 + \frac{9}{4} = \frac{153}{4}$ so our final answer is $153 + 4 = \boxed{157}$ as desired. ■

§2 Score Summary

N	39	1st Q	3.5	Max	13
μ	6.08	Median	5	Top 3	12
σ	3.64	3rd Q	9	Top 10	9



§3 Problem Statistics

Problem	# Solves	% Solves	Off-by-Ones
P1	29	74.36%	2
P2	24	61.54%	0
P3	31	79.49%	0
P4	19	48.72%	0
P5	24	61.54%	0
P6	20	51.28%	7
P7	12	30.77%	0
P8	16	41.03%	3
P9	22	56.41%	2
P10	10	25.64%	0
P11	9	23.08%	0
P12	10	25.64%	1
P13	6	15.38%	0
P14	0	0.00%	0
P15	5	12.82%	0

§4 Full Rankings

Rank	Username																Score
1.	vvluo	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	13
2.	Mathscienceclass	1	1	1	1	1	1	1	0	1	0	1	1	1	0	1	12
2.	kvedula2004	1	1	1	1	0	1	1	1	1	1	0	1	1	0	1	12
4.	v4913	1	1	1	0	1	0	1	1	1	1	1	1	0	0	1	11
4.	Puddles_Penguin	1	1	1	1	1	0	1	1	1	1	1	1	0	0	0	11
4.	ppanther	1	1	1	1	1	1	1	1	1	0	1	1	0	0	0	11
7.		1	1	1	1	1	1	1	0	1	1	0	1	0	0	0	10
8.		1	1	1	1	0	1	0	1	1	1	0	0	1	0	0	9
8.	youyanli	1	1	1	1	1	1	0	1	1	0	1	0	0	0	0	9
8.	mathtiger6	1	1	1	1	1	1	0	1	1	1	0	0	0	0	0	9
8.	RadiantCheddar	1	1	1	1	1	1	1	0	1	1	0	0	0	0	0	9
8.	hwu32	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	9
13.	fidgetboss_4000	1	1	1	0	1	0	0	1	0	1	1	1	0	0	0	8
13.	zhao_andrew	1	1	1	0	1	0	1	1	1	0	0	1	0	0	0	8
13.		1	1	1	1	1	0	0	1	1	0	0	1	0	0	0	8
13.	tauros	1	1	1	1	1	0	1	1	1	0	0	0	0	0	0	8
17.	billybillybobjoejr.	1	0	1	0	1	1	0	1	1	0	1	0	0	0	0	7
18.		0	1	1	1	1	1	0	0	1	0	0	0	0	0	0	6
18.	pandax2007	0	1	1	1	1	1	0	0	1	0	0	0	0	0	0	6
20.	Blossomstream	1	0	0	1	1	0	0	0	0	1	0	0	1	0	0	5
20.	kred9	0	0	1	1	0	1	0	1	1	0	0	0	0	0	0	5
20.	vsamc	1	1	1	0	0	1	0	1	0	0	0	0	0	0	0	5
20.	Anonymous	1	1	1	0	1	1	0	0	0	0	0	0	0	0	0	5
24.	maththinkingman2000	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	4
24.	Hayasaka best girl	1	1	1	0	0	0	0	0	0	0	1	0	0	0	0	4
24.		0	1	1	0	0	1	0	0	1	0	0	0	0	0	0	4
24.	lrjr24	1	0	1	1	0	0	0	0	1	0	0	0	0	0	0	4
24.		1	0	1	1	1	0	0	0	0	0	0	0	0	0	0	4
24.		1	1	1	0	1	0	0	0	0	0	0	0	0	0	0	4
30.	usernameyourself	1	0	1	0	0	0	0	0	1	0	0	0	0	0	0	3
30.	Random_Person_921	1	0	1	0	0	1	0	0	0	0	0	0	0	0	0	3
30.		1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	3
33.	Jatmoz	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	2
33.		1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	2
35.	franchester	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
35.	intellegence30	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	1
35.	proguamkid	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1
38.	mathicorn	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
38.		0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0