The Optimal Elastic Net: Finding Solutions to the Travelling Salesman Problem

J.V. Stone

Department of Computer Science, The University of Wales, Aberwystwyth, Dyfed, Wales, SY23 3DB.
In Proceedings of the International Conference on Artificial Neural Networks, Brighton, England, pp 170-174, Sept. 1992.

Abstract

We introduce the *optimal elastic net* (OEN) method for finding solutions to the travelling salesman problem (TSP). The OEN method is related to the elastic net (EN) method[1]. The EN method encourages each city to become associated with at least one unit, and requires a large number of units in order to ensure that each city is matched to a unit at convergence. In contrast, the OEN method encourages each city to become associated with at least one unit, *and* each unit to become associated with at least one city. One consequence of this is that the OEN method requires fewer units than the EN method.

The OEN method has applications for problems in which there are two sets of variable (movable) components (e.g. VLSI placement), and an arrangement of components is required which minimises the total interconnect length between the two sets.

1 Introduction

In 1987 Durbin and Willshaw introduced the elastic net (EN) method for finding short routes for the TSP. This method is one of a class of algorithms which ensures that solutions are legal (e.g. generates a route consisting of a complete set of cities), and optimal (e.g. generates short routes), by applying 'soft' constraints. Such methods perform gradient descent on an energy function, where only a sub-set of the minima of the function correspond to legal solutions. Unlike conventional methods, which search the discrete space of legal solutions, these methods search a continuous space in which a relatively small total volume corresponds to legal solutions. Legal solutions are obtained by including terms in the energy-function which penalise the emergence of illegal solutions during minimsation. Thus the conditions of legality and optimality are implicit in the cost-function, and either can be violated by inappropriate minimisation techniques.

The EN method is analogous to laying a circular loop of elastic on a plane containing a number of points (cities), and allowing each city to exert an attractive force on the loop. (In practice the EN method models this loop as a finite set of points). Each city is associated with a Gaussian envelope which (using our physical analogy) effectively means that each city sits in a Gaussian depression on the plane. The standard deviation of the Gaussians associated with all cities is identical. Each city attracts a region on the elastic loop according to the proximity of that region to a given city and the standard deviation of the Gaussian. Initially the standard deviation is large, so that all regions on the loop are attracted to every city to the same extent (approximately) as every other region. As the standard deviation is decreased a city differentially attracts one or more nearby regions on the loop, at the expense of more distant loop regions. These loop-city interactions are modulated by the elasticity of the loop, which tends to keep the loop length short; the elasticity also tends to ensure that each city becomes associated with only one loop region. For a given standard deviation the integral of loop motion associated with each city is constant, and is the same for all cities, so that each city induces the same amount of motion of the loop. By slowly decreasing the standard deviation each city becomes associated with a single point on the loop. Thus the loop ultimately defines a route of the cities it passes through. If the standard deviation is reduced sufficiently slowly then the loop passes through every city, and therefore defines a complete route through all cities.

1.1 Establishing Unit-City Correspondences

In evaluating the force between a given city and a unit the EN method utilises the the relative proximity of all units to that city. This is done in order to ensure that each city is allocated at least one unit at convergence.

However, it does not ensure that each unit is allocated at least one city, so that many more units than cities are required in order to obtain legal solutions. In contrast, by evaluating the attractive force between a city and a unit, the OEN method utilises both the relative distances of all units from the city, *and* the relative distances of all cities from the unit. Initially a city which is closest to a given unit is not necessarily the closest city to that unit, so that (for a given unit-city pair) the city→unit force and the unit→city force are not colinear. As the standard deviation is reduced a city's nearest unit is that unit's nearest city, so that the unit→city and city→unit forces tend to be colinear.

These considerations have led to a new energy function. As with the EN energy function, minimising this energy function encourages each city to become associated with at least one unit. However, the new energy function also encourages each unit to become associated with at least one city. The resultant method requires a smaller number of units than the EN method.

1.2 The Optimal Elastic Net Method

Let the coordinates of a city, i, be denoted by the vector c_i, and let the coordinates of a route point (or unit), u, be t_u. After [1] we define the TSP in terms of a mapping from a circle to a plane. The TSP consists of finding a mapping from a circle, h, to a finite set of points, $C = c_i..c_N$, in the plane, which minimises the distance:

$$D = \oint_h f(s)\ ds$$

Where the circle, h, is parameterised by arc-length, s, and $f(s)$ provides a unique mapping from each point on h to a point on the plane. The locus of points defined by $f(s)$ forms a loop, l, in the plane which passes through each city exactly once.

The loop, l, is modelled as a finite set, $t_1..t_M$, of points, or *units*. Due to the difficulty in obtaining a one-to-one mapping of units to plane points (cities) it is customary to set $M > N$ in order that each city can become associated with at least one unit.

The energy function to be minimised is:

$$E = -\alpha \sum_{i=1}^{N} \ln \sum_{u=1}^{M} \phi(|c_i - t_u|, K) - \kappa \sum_{u=1}^{M} \ln \sum_{i=1}^{N} \phi(|c_i - t_u|, K) + \beta \sum_{u=1}^{M} (|t_u - t_{u+1}|)^2 \qquad (1)$$

Where $\phi(d, K) = \exp(-(d^2/2K^2))$, and K is the standard deviation of the function, ϕ. Note that the original Durbin-Willshaw energy function can be recovered by setting $\kappa = 0$.

Differentiating E with respect to t_u, and multiplying through by K, gives the change $(= -K\partial E/\partial t_u)$, Δt_u, in the position of a unit, u:

$$\Delta t_u = (\alpha \sum_{i=1}^{N} f_{ui} + \kappa \sum_{u=1}^{M} b_{ui}\ )(c_i - t_u) + \beta K(t_{u+1} - 2t_u + t_{u-1}) \qquad (2)$$

Where f_{ui} is identical to the Durbin-Willshaw term (w_{ij}):

$$f_{ui} = \frac{\phi(|c_i - t_u|, K)}{\sum_{v=1}^{M} \phi(|c_i - t_v|, K)} \qquad (3)$$

This term ensures that each city becomes associated with at least one unit. It is the Gaussian weighted distance of t_u from c_i expressed as a proportion of the distances to all other units from c_i. In contrast, the term b_{ui} ensures that each unit becomes associated with at least one unit. It is the Gaussian weighted distance of c_i from t_u, expressed as a proportion of the distances to all other cities from t_u:

$$b_{ui} = \frac{\phi(|c_i - t_u|, K)}{\sum_{i=1}^{N} \phi(|c_i - t_v|, K)} \qquad (4)$$

Whereas the term f_{ui} measures the relative forces that all units exert on a single city, and the term b_{ui} measures the relative forces that all cities exert on a single unit. In both cases it is the units which move, but (with the

OEN) the forces which influence a unit are no longer only determined by the relative distance of that unit from each city. Instead, (with the OEN) the motion of a unit also depends upon the relative proximity of each city to that unit.

2 Results

Preliminary results have been obtained with the five sets of 50 randomly placed cities used in [1]. The EN method was used as described in [1], the value of K being reduced by one percent every $n = 5$ iterations. Other parameter values were also taken from [1]: $\alpha = 0.2$, $\beta = 2.0$, $N = 50$. With these parameter values the EN requires approximately 1200 iterations to converge.

For comparison we used the same number of units, $M = 125$, as in [1], and obtained good agreement with the results obtained therein (except for city-set 5 for which a value of 6.49 is given in [1]). With $M = 80$ the EN produced illegal solutions (marked as IS in Table 1) for three of the five city sets. An illegal solution occurs because two or more cities claim the same unit as their closest unit, making the final configuration uninterpretable as a complete route.

City Set	EN(M=125)	EN(M=80)	OEN(M=80)
1	5.99	IS	6.02
2	6.02	IS	6.05
3	5.75	5.70	5.71
4	5.76	6.52	6.49
5	5.89	IS	5.94

Table 1: Results for the EN and the OEN methods. IS=illegal solution

Using a value of $\kappa = 0.4$ and $M = 80$ the OEN converged to a legal route for all five city sets in about 1200 iterations. The route lengths are comparable to those using the conventional EN method with $M = 125$, but results were obtained in approximately half the time required for the EN method. This is because the most expensive part of the both methods is calculating the Gaussian weighted unit-city distances; once these have been obtained to compute values for f_{ui} then it is relatively inexpensive to compute the values for the b_{ui} defined in (4), for the OEN method.

Our choice of parameter values was guided partly by those in [1], and partly by trial and error. One unsatisfactory aspect of this class of methods is that there does not appear to be a principled way to choose parameter values.

3 Conclusion

The objective of the current work is to investigate the effects of modifying the Durbin-Willshaw energy function. We have reported preliminary results which suggest that an approximation to a one-to-one mapping between units and cities can be obtained by the addition of one term to the Durbin-Willshaw function. The OEN method requires 50% fewer units, and approximately 50% less computation than the EN method.

Ideally we would like to use only as many units as there are cities, and then to construct a one-to-one mapping between units and cities. Unfortunately, setting $M = N$ leads to illegal solutions for both the EN and OEN methods. Future work will investigate the minimisation of other energy functions designed to work with only as many units as cities(e.g. [2]).

References

[1] Durbin R, Willshaw D, "An Anlaogue Approach to the Travelling Salesman Problem Using an Elastic Net Approach", Nature, 326, 6114, pp 689-691, 1987.

[2] Simic P, "Statistical Mechanics as the underlying theory of "elastic" and "neural" optimization", NETWORK: Comp. Neural Systems, 1(1), pp 89-103, 1990.