National Polytechnic School

Mechanical Engineering Department

Finite Elements Method

HomeWork #1

SEMMAD Mohamed
01/02/2021

Problem Specification:

APPLICATION 5:

Part 1:

Finding the displacement at three nodes of a truncated cone shaped bar using two models:

1. Variable section area.
2. Constant section area for each node.

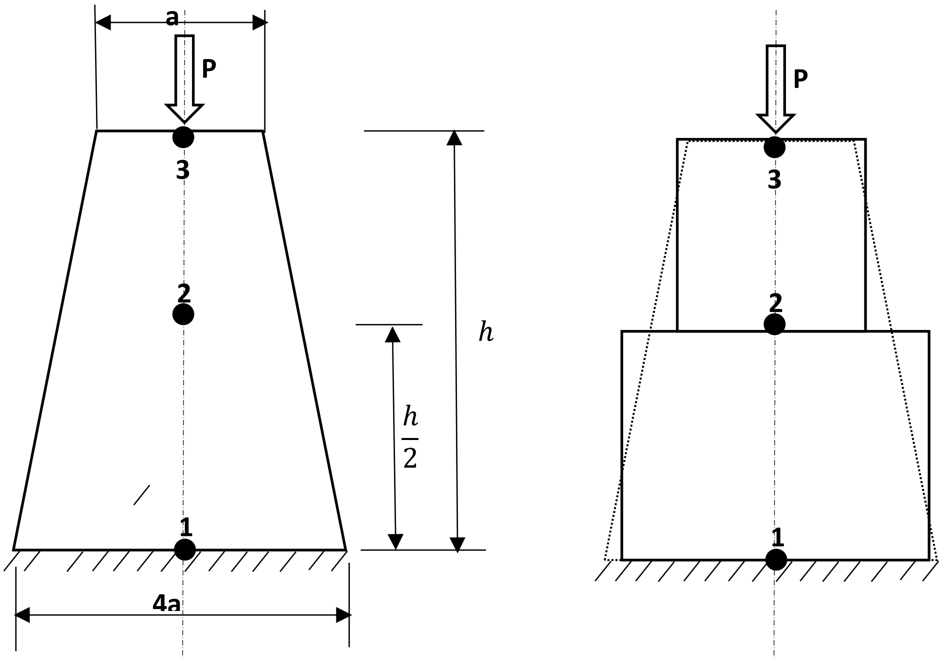


Figure 1 : Model 1 (in the left): The section area is taken as a function of the postion-Model2 (Right): The section area is the average of extreme node's section area for the respective element

Part 2:

For the bar presented above, increase the number of element until reaching the convergence of the two models (an error of 5% is proposed).

APPLICATION 6:

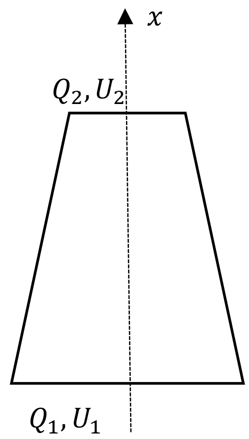
Reconsider the problem presented in APPLICATION 5 with replacing the rigid support at node 1 with an elastic support with an elasticity constant k.

SOLUTION

Application 5: Part 1:

MODEL-1:

Elementary matrix for a truncated cone shaped bar:



The Deformation Energy is given by:

$$U_D = \frac{1}{2}\int_l [ES(x)]\left(\frac{dU}{dx}\right)^2 dx \qquad (1)$$

With :

$$U(x) = \left(\frac{x_2 - x}{l}\right)U_1 + \left(\frac{x - x_1}{l}\right)U_2$$

Where $U_1$and $U_2$are the displacements at nodes 1 and 2 respectively.

To compute U_D, S must be formulated first, So we know that:

$$S(x) = \pi r^2(x)$$

r is given by a linear function of x, then:

$$r(x) = ax + b$$

We replace a and b by the respective values we will be able to write r as:

$$r(x) = \left(\frac{x_2 - x}{l}\right)\frac{d_1}{2} + \left(\frac{x - x_1}{l}\right)\frac{d_2}{2} \qquad (2)$$

Where d_1, d_2 are the diameters of the cone at node 1 and 2 respectively.

By replacing (2) in (1), and computing the integral we find:

$$U_D = \frac{\pi E}{24l}(d_2^2 + d_1 d_2 + d_1^2)[-U_1 + U_2]^2$$

Castigliano Theorem states that:

$$Q_i = \frac{\partial U_D}{\partial U_i}$$

So Q_1, Q_2 are given by:

$$Q_1 = \frac{\pi E}{12l}(d_2^2 + d_1 d_2 + d_1^2)[U_1 - U_2]$$

$$Q_1 = \frac{\pi E}{12l}(d_2^2 + d_1 d_2 + d_1^2)[U_1 - U_2]$$

Finally, the elementary matrix for a truncated cone shaped bar is given as follows:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{\pi E}{12l}(d_2^2 + d_1 d_2 + d_1^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

Application to a bar with two elements (our problem's case):

For element 1:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{\pi E}{6h}(d_2^2 + d_1 d_2 + d_1^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \quad (I)$$

For element 2 :

$$\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \frac{\pi E}{6h}(d_3^2 + d_2 d_3 + d_2^2)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} \quad (II)$$

NOTE that for the present case (two elements) $l = \frac{h}{2}$

We compute d_2 and from (2), by taking:

$$\begin{cases} d_1 = 4a \\ d_3 = a \end{cases}$$

We find:

$$d_2 = 2 * r\left(x_2 = \frac{h}{2}\right) = \frac{5}{2}a$$

So, (I) and (II) becomes respectively:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{43\pi a^2 E}{8h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} \quad (I')$$

And :

$$\begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = \frac{13\pi a^2 E}{8h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} \quad (II')$$

For generality sake, we define the following quantity:

$U^* = \frac{a^2 E}{hP}U$: adimensionalized displacement.

$Q^* = \frac{Q}{P}$: adimensionalized force.

The adimensionalized form of (I') and (II') is:

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \end{Bmatrix} = \frac{43\pi}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^* \\ U_2^* \end{Bmatrix} \quad (I'')$$

And :

$$\begin{Bmatrix} Q_2^* \\ Q_3^* \end{Bmatrix} = \frac{13\pi}{8}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix} \quad (II'')$$

For the assembly of the two systems, we rewrite them in the following form:

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{bmatrix} \frac{43\pi}{8} & -\frac{43\pi}{8} & 0 \\ -\frac{43\pi}{8} & \frac{43\pi}{8} & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{13\pi}{8} & -\frac{13\pi}{8} \\ 0 & -\frac{13\pi}{8} & \frac{13\pi}{8} \end{bmatrix} \begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

<u>By adding the two systems, we obtain:</u>

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{bmatrix} \frac{43\pi}{8} & -\frac{43\pi}{8} & 0 \\ -\frac{43\pi}{8} & 7\pi & -\frac{13\pi}{8} \\ 0 & -\frac{13\pi}{8} & \frac{13\pi}{8} \end{bmatrix} \begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

<u>Boundary Conditions imply that:</u>

$$\begin{cases} U_1^* = 0 \\ Q_2^* = 0 \\ Q_3^* = -1 \end{cases}$$

The reduced form is obtained by elimination the row and column correspondent to the null displacement, the system to solve reduces to:

$$\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{bmatrix} 7\pi & -\frac{13\pi}{8} \\ -\frac{13\pi}{8} & \frac{13\pi}{8} \end{bmatrix} \begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix}$$

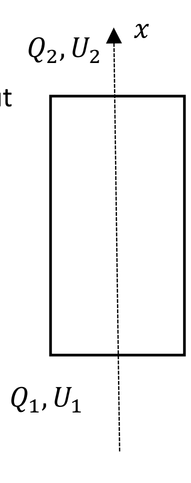
$$\Rightarrow \begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix} = \begin{bmatrix} 7\pi & -\frac{13\pi}{8} \\ -\frac{13\pi}{8} & \frac{13\pi}{8} \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix} = \frac{64}{559\,\pi^2} \begin{bmatrix} \frac{13\pi}{8} & \frac{13\pi}{8} \\ \frac{13\pi}{8} & 7\pi \end{bmatrix} \begin{Bmatrix} 0 \\ -1 \end{Bmatrix}$$

$$\Rightarrow \begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix} = \begin{Bmatrix} -0{,}0592204 \\ -0{,}2551035 \end{Bmatrix}$$

MODEL-2:

<u>Elementary matrix for a cylindrical bar:</u>



The linear system id already established in the previous lectures, so we present it without step-by-step calculations:

$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{ES}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}$$

Application to a bar with two elements (our problem's case):

We define the main section area of each element as the average of the section areas on its side nodes, based on that, it comes that:

The main section area for element (1) is:

$$S_1 = \frac{S(x_1) + S(x_2\,)}{2} = 4\pi a^2 + \frac{25}{16}\pi a^2 = \frac{89}{16}\pi a^2$$

The main section area for element (2) is:

$$S_2 = \frac{S(x_2) + S(x_3\,)}{2} = \frac{\pi a^2}{4} + \frac{25}{16}\pi a^2 = \frac{29}{16}\pi a^2$$

So we obtain the following adimensionalized form equations for elements(1) and (2) respectively:

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \end{Bmatrix} = \frac{89\pi}{16}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_1^* \\ U_2^* \end{Bmatrix}$$

And :

$$\begin{Bmatrix} Q_2^* \\ Q_3^* \end{Bmatrix} = \frac{29\pi}{16}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix}$$

By assembling the previous system as we did for MODEL-1, we obtain:

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{bmatrix} \frac{89\pi}{16} & -\frac{89\pi}{16} & 0 \\ -\frac{89\pi}{16} & \frac{59\pi}{8} & -\frac{29\pi}{16} \\ 0 & -\frac{29\pi}{16} & \frac{29\pi}{16} \end{bmatrix}\begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

We apply the same boundary conditions as in Model-1, which leads to the following reduced system:

$$\begin{Bmatrix} 0 \\ -1 \end{Bmatrix} = \begin{bmatrix} \frac{59\pi}{8} & -\frac{29\pi}{16} \\ -\frac{29\pi}{16} & \frac{29\pi}{16} \end{bmatrix}\begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix}$$

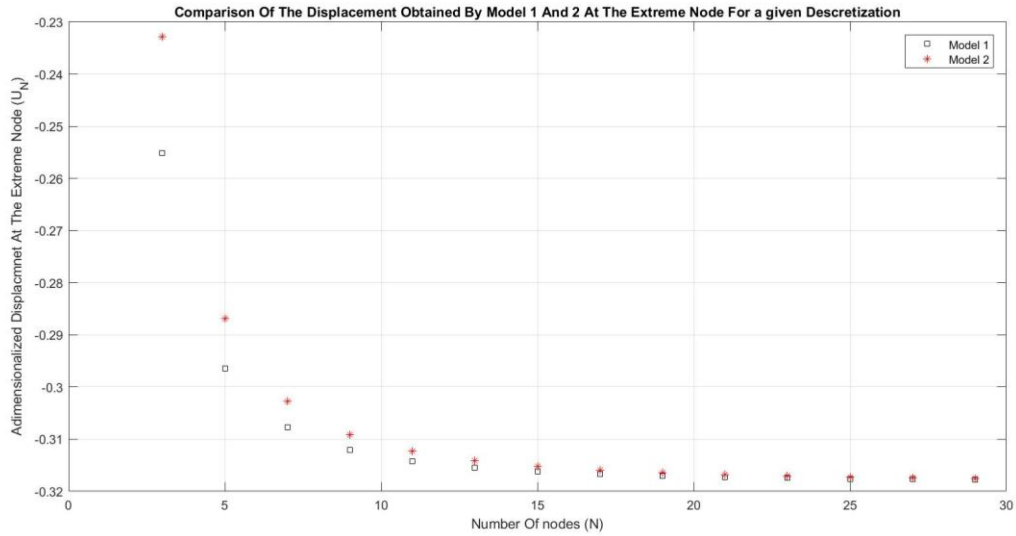
By inverting the rigidity matrix we find:

$$\begin{Bmatrix} U_2^* \\ U_3^* \end{Bmatrix} = \begin{Bmatrix} -0.0572242 \\ -0.2328435 \end{Bmatrix}$$

Part 2:

By generalizing the systems of equations obtained previously for two elements, and writing a MATLAB script based on these results, we could performs computation for a sufficient number of elements in order to show the convergence of the models.

The MATLAB script are provided in a separate file.



<u>**NOTE:**</u> ***We note that greater discard between the two models is obtained at the last node, so we propose to visualize this difference in the following plots (for both applications).***

<u>Results discussion:</u>

-The two model give a very close results, even for a small number of elements (an error less that 9% at the extreme node is achieved only by using 2 elements).

<u>Application 6: Part 1:</u>

For Application 6 we reconsider the same linear system obtained for each model in application 5. The only thing that needs to be changed is the boundary conditions, since now we have an elastic liaison at node 1, instead on a rigid one.

<u>MODEL-1:</u>

As previously, the system to solve is given by:

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{bmatrix} \frac{43\pi}{8} & -\frac{43\pi}{8} & 0 \\ -\frac{43\pi}{8} & 7\pi & -\frac{13\pi}{8} \\ 0 & -\frac{13\pi}{8} & \frac{13\pi}{8} \end{bmatrix} \begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

In that case, boundary Conditions imply that:

$$\begin{cases} Q_1^* = -\dfrac{kh}{a^2E} U_1^* \\ Q_2^* = 0 \\ Q_3^* = -1 \end{cases} \quad (B2)$$

Where k is the constant of elasticity, of the elastic support.

So, the system to solve is given by:

$$\begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{bmatrix} (\frac{43\pi}{8} + \frac{kh}{a^2E}) & -\frac{43\pi}{8} & 0 \\ -\frac{43\pi}{8} & 7\pi & -\frac{13\pi}{8} \\ 0 & -\frac{13\pi}{8} & \frac{13\pi}{8} \end{bmatrix} \begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

NOTE: ***We have moved the force at node 1 to the other side of the equation 1 (displacements side) this form is more convenient for computer resolution.***

By solving the system we find:

$$\begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{C} \\ -\frac{8C + 43\pi}{43C\pi} \\ -\frac{448C + 559\pi}{559C\pi} \end{Bmatrix}$$

Where C is defined as:

$$C = \frac{kh}{a^2E}$$

For $C = 1$, the solution is :

$$\begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1.05922 \\ -1.255103 \end{Bmatrix}$$

<u>MODEL-2 :</u>

As in application 5, the system to solve is given by:

$$\begin{Bmatrix} Q_1^* \\ Q_2^* \\ Q_3^* \end{Bmatrix} = \begin{bmatrix} \frac{89\pi}{16} & -\frac{89\pi}{16} & 0 \\ -\frac{89\pi}{16} & \frac{59\pi}{8} & -\frac{29\pi}{16} \\ 0 & -\frac{29\pi}{16} & \frac{29\pi}{16} \end{bmatrix} \begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

By applying the Boundary conditions, given by (B2), the following set of equation is obtained:

$$\begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{bmatrix} (\frac{89\pi}{16} + \frac{kh}{a^2E}) & -\frac{89\pi}{16} & 0 \\ -\frac{89\pi}{16} & \frac{59\pi}{8} & -\frac{29\pi}{16} \\ 0 & -\frac{29\pi}{16} & \frac{29\pi}{16} \end{bmatrix} \begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix}$$

By solving the system we find:

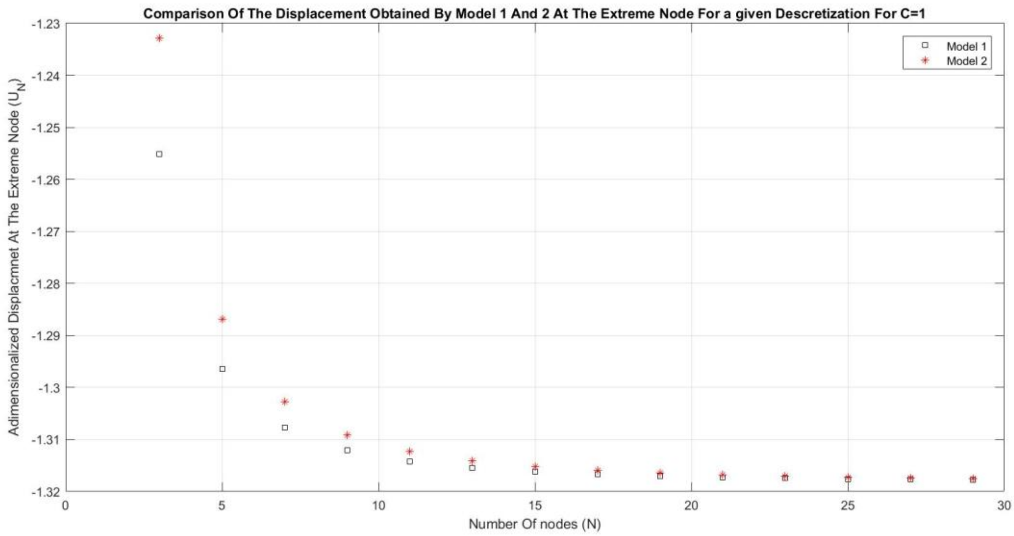
$$\begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix} = \begin{Bmatrix} -\dfrac{1}{C} \\ -\dfrac{16C + 89\pi}{89C\pi} \\ -\dfrac{1888C + 2581\pi}{2581C\pi} \end{Bmatrix}$$

For $C = 1$, the solution is :

$$\begin{Bmatrix} U_1^* \\ U_2^* \\ U_3^* \end{Bmatrix} = \begin{Bmatrix} -1 \\ -1.057224 \\ -1.232843 \end{Bmatrix}$$

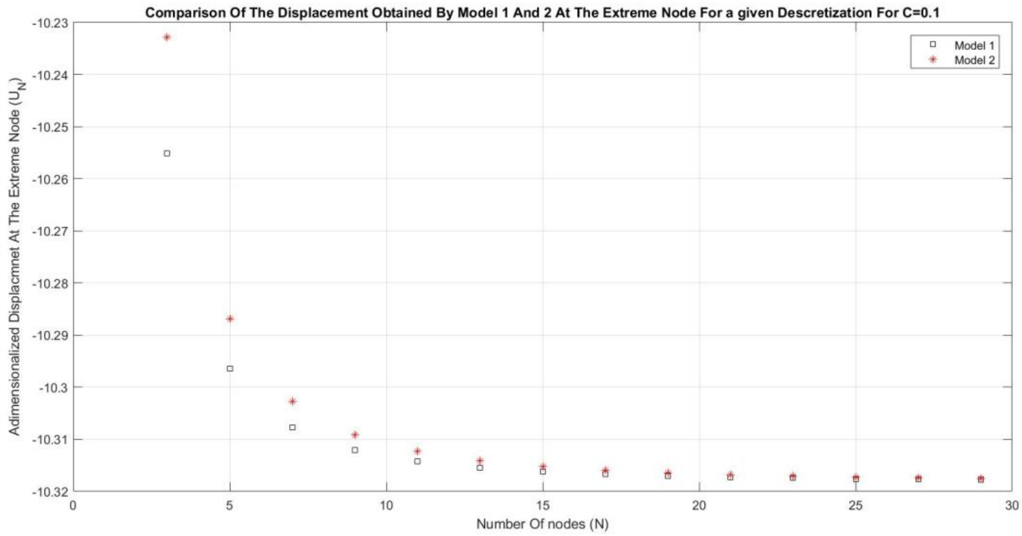
Part 2:

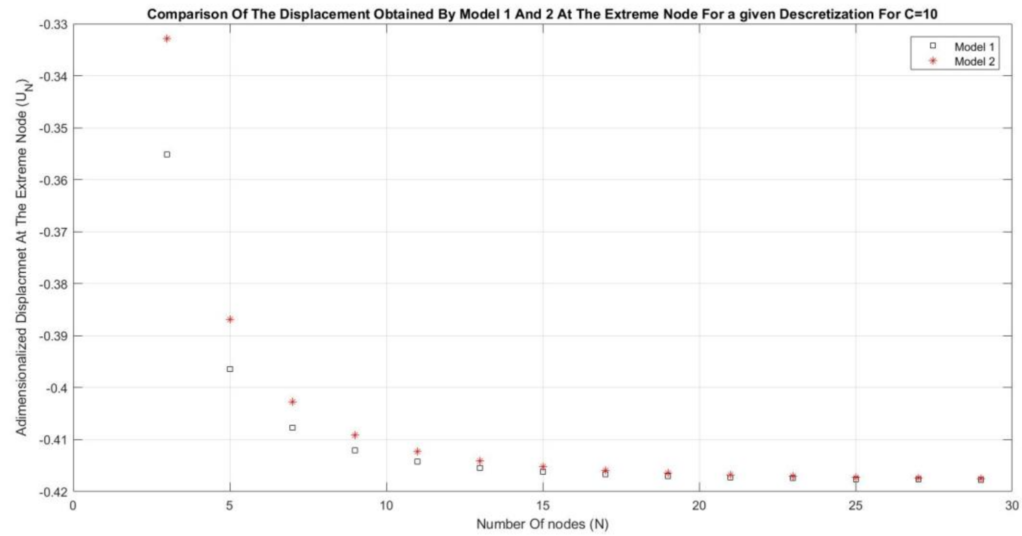
A MATLAB script is used to study the convergence of the two models proposed for Application 6. As we did for Application 5, the displacements of the last node obtained by each model are plotted in function of the number of nodes taken in that model.

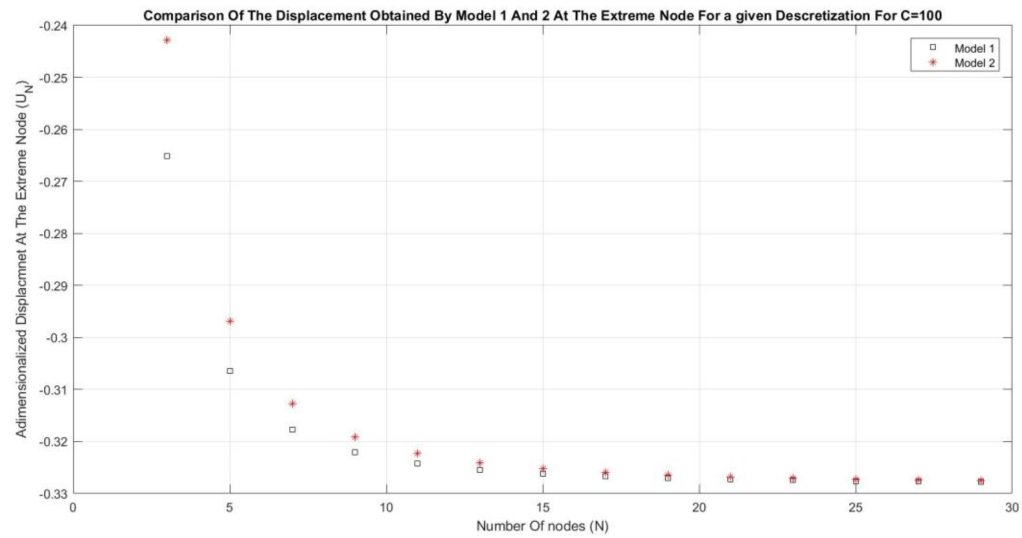


Study of the Influence of the parameter C:

Since the parameter C is arbitrary fixed, we propose to study its effect on the convergence, to do that we did the same calculation for three other different values of C which are $C = 0.1, 10, 100$. The obtained plots are presented below:







Results discussion:

-The Models gives a very close results, even for a small number of elements (an error less that 21% at the extreme node is achieved only by using 2 elements).

-The parameter C has a negligebale influace on the convergence of the models, as shown clearly by the plots corresponding to a values of $C = 0.1,10, 100$.

Conclusion:

The result obtained are predictable; as number of element increases the solution based on the approximated section area (MODEL-2) converge to the one taking the real section area (MODEL-1), we can resume our comment in the following point:

For Application 5:

-with 3 element we can achieve an error less than 4% , so the required error between the two models which is 5% is satisfied with only 3 elements.

-Even with an approximated area Model-2 gives a very accurate result (even with a reduced number of elements), so its result could be trusted in a wide range of application.

For Application 6:

-With 5 element we achieved an error of 4.82% between the models, so the required error between the two models which is 5% is satisfied with only 5 elements

- Also for the case of application 6, Model-2 gives a very accurate result (even with a reduced number of elements), indeed its result could be trusted in a wide range of application

-The factor $C = \frac{kh}{a^2E}$ does not have a considerable effect on the convergence of the models, as can be shown by the different plots for $C = 0.1, 10, 100$.

In conclusion, the level of accuracy is primarily defined by the type of application, which would in its turn defines the number of elements taken in the model to achieve the requirement, So model-2 can be a very good candidate for its appreciated precision and simplicity.