Homework no. 6

Given $(n+1)$ distinct points, x_0, x_1, ..., x_n ($x_i \in \mathbb{R}\ \forall i$, $x_i \neq x_j$, $i \neq j$) and the $(n+1)$ values of an unknown function$f$ at these points, $y_0 = f(x_0)$, $y_1 = f(x_1)$, ..., $y_n = f(x_n)$:

x	x_0	x_1	$\cdots$	x_n
f	y_0	y_1	$\cdots$	y_n

approximate the value of function f in $\bar{x}$, $f(\bar{x})$, for a given $\bar{x}$, a value which is not in the above table, $\bar{x} \neq x_i$, $i = 0, \ldots, n$ using:

- polynomial approximation computed with the least squares method. For computing the value of least squares polynomial in $\bar{x}$, use Horner's algorithm; display $P_m(\bar{x})$, $|P_m(\bar{x}) - f(\bar{x})|$ and $\displaystyle\sum_{i=0}^{n} |P_m(x_i) - y_i|$. For m introduce values smaller than 6.
- trigonometric interpolation. In this case one considers that the function$f$ is periodic with period 2π and the number of interpolation nodes is odd, $n = 2m$, $0 \leq x_0 < x_1 < \cdots < x_{2m} < 2\pi$. Display $T_n(\bar{x})$ şi $|T_n(\bar{x}) - f(\bar{x})|$.

For solving the linear systems involved in solving the above interpolation problems use the library employed in *Homework 2*.

Generate the interpolation nodes $\{x_i, i = 0, ..., n\}$ in the following way: read x_0 şi x_n from keyboard or from a file such that $x_0 < x_n$. The interpolation points x_i are randomly generated such that $x_i \in (x_0, x_n)$ and $x_{i-1} < x_i$; the values $\{y_i, i = 0, ..., n\}$ are computed using a function f hard-coded in your program (examples of initialization for nodes x_0, x_n and function $f(x)$ are displayed at the end of this document), $y_i = f(x_i), i = 0, ..., n$;

Bonus (10 pt): Draw the graphs of function f and of the approximative computed functions P_m and T_n.

Least Squares Interpolation

Let $a = x_0 < x_1 < \cdots < x_n = b$. Given $\bar{x} \in [a, b]$ approximate $f(\bar{x})$ knowing that the $n+1$ values y_i of function f in the interpolation nodes.

One computes a polynomial of degree m:

$$S_m(x) = S_m(x; a_0, a_1, \ldots, a_m) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0 = \sum_{k=0}^{m} a_k\, x^k$$

the coefficients $\{a_i; i = \overline{0, m}\}$ being the solution to the optimization problem:

$$\min \left\{ \ \sum_{r=0}^{n} \left| S_m(x_r; a_0, a_1, \ldots, a_m) - y_r \right|^2 \ ; \ a_0, a_1, \ldots, a_m \in \mathbb{R} \ \right\}$$

Solving this problem leads to solving the linear system:

$$Ba = f$$

$$B = (b_{ij})_{i,j=0,\ldots,m} \in \mathbb{R}^{(m+1)\times(m+1)} \quad f = (f_i)_{i=0,\ldots,m} \in \mathbb{R}^{m+1}$$

$$\sum_{j=0}^{m} \Big(\sum_{k=0}^{n} x_k^{i+j} \Big) a_j \ = \ \sum_{k=0}^{n} y_k x_k^i \quad , \quad i = 0, \ldots, m$$

This linear system can be solved with the same numerical library that was used for *Homework 2*.

The value of function f in $\bar{x}$ is approximated by the value of the polynomial S_m in $\bar{x}$:

$$f(\bar{x}) \approx S_m(\bar{x}; a_0, a_1, \ldots, a_m)$$

For computing the value of polynomial $S_m(\bar{x})$ use Horner's method described below.

Horner's method for computing $P(v)$

Let P be a polynomial of degree p:

$$P(x) = c_0 x^p + c_1 x^{p-1} + \cdots + c_{p-1} x + c_p \ , \quad (c_0 \neq 0)$$

We can write polynomial P also as:

$$P(x) = ((\cdots(((c_0 x + c_1)x + c_2)x + c_3)x + \cdots)x + c_{p-1})x + c_p$$

Taking into account this grouping of the coefficients, one obtains an efficient way to compute the value of polynomial P in any point $v \in \mathbb{R}$, this procedure is knowns as *Horner's method*:

$$\begin{array}{rcl} d_0 & = & c_0 \, , \\ d_i & = & c_i + d_{i-1} v \ , \quad i = \overline{1,p} \end{array} \tag{1}$$

In the above sequence:

$$P(v) := d_p$$

the rest of the computed elements of the sequence $d_i, i = 1, ..., p-1$, are the coefficients of the quotient polynomial Q, obtained in the division:

$$\begin{array}{rcl} P(x) & = & (x - v)Q(x) + r \ , \\ Q(x) & = & d_0 x^{p-1} + d_1 x^{p-2} \cdots + d_{p-2} x + d_{p-1} \ , \\ r & = & d_p = P(v). \end{array}$$

Computing $P(v)$ (d_p) with formula (1) can be performed using only one real variable $d \in \mathbb{R}$ instead of using a vector $d \in \mathbb{R}^p$.

Trigonometric Interpolation

The trigonometric interpolation is used for approximating periodic functions of period T:

$$f(x+T) = f(x) \quad , \quad \forall x.$$

We consider the case when $T = 2\pi$. Assume we have an odd number of interpolation node $n = 2m$ placed in the interval $[0, 2\pi)$:

$$0 \leq x_0 < x_1 < \cdots < x_{n-1} < x_n < 2\pi.$$

The function f is approximaed by a linear combination of sin and cos function in the following way:

$$f(x) \approx T_n(x) = a_0 + \sum_{k=1}^{m} a_k \cos(k\,x) + \sum_{k=1}^{m} b_k \sin(k\,x).$$

Denote by:

$$\phi_0(x) = 1 \quad , \quad \phi_{2k-1}(x) = \sin(k\,x) \quad , \quad \phi_{2k}(x) = \cos(k\,x) \quad , \quad k = 1, \ldots, m.$$

The coefficients $\{a_k; k = 0, \ldots m\}$ şi $\{b_k; k = 1, \ldots m\}$ are computed by solving the linear system:

$$TX = Y \quad , \quad X = (a_0, b_1, a_1, b_2, a_2, \ldots, b_m, a_m)^T$$

$$T = \begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \phi_2(x_0) & \ldots & \phi_{n-1}(x_0) & \phi_n(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \ldots & \phi_{n-1}(x_1) & \phi_n(x_1) \\ \phi_0(x_2) & \phi_1(x_2) & \phi_2(x_2) & \ldots & \phi_{n-1}(x_2) & \phi_n(x_2) \\ \vdots & & & & & \\ \phi_0(x_{n-1}) & \phi_1(x_{n-1}) & \phi_2(x_{n-1}) & \ldots & \phi_{n-1}(x_{n-1}) & \phi_n(x_{n-1}) \\ \phi_0(x_n) & \phi_1(x_n) & \phi_2(x_n) & \ldots & \phi_{n-1}(x_n) & \phi_n(x_n) \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}$$

$$Y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} \in \mathbb{R}^{n+1} \quad , \quad X = \begin{pmatrix} a_0 \\ b_1 \\ a_1 \\ \vdots \\ b_m \\ a_m \end{pmatrix} \in \mathbb{R}^{2m+1}.$$

For solving this linear system use the library employed in *Homework 2*.

$$f(\bar{x}) \approx T_n(\bar{x}) = a_0\phi_0(\bar{x}) + b_1\phi_1(\bar{x}) + a_1\phi_2(\bar{x}) + \cdots + b_m\phi_{2m-1}(\bar{x}) + a_m\phi_{2m}(\bar{x}).$$

Input data - examples

$$x_0 = a = 1 \quad , \quad x_n = b = 5 \quad , \quad f(x) = x^4 - 12x^3 + 30x^2 + 12$$

$$x_0 = 0 \quad , \quad x_n = \frac{31\pi}{16} \quad , \quad f(x) = \sin(x) - \cos(x)$$

$$x_0 = 0 \quad , \quad x_n = \frac{31\pi}{16} \quad , \quad f(x) = \sin(2x) + \sin(x) + \cos(3x)$$

$$x_0 = 0 \quad , \quad x_n = \frac{63\pi}{32} \quad , \quad f(x) = \sin^2(x) - \cos^2(x)$$