Homework no. 5

Let $p \in \mathbf{N}^*$ and $n \in \mathbf{N}^*$ be the size of matrix $A \in \mathbf{R}^{p\times n}$, ϵ - computation error, $k_{\max}$ the maximum number of iteretations in the power method,vector $b \in \mathbf{R}^p$.

- For $p = n > 500$ generate randomly a square, sparse, symmetric matrix ($A = A^T$), with positive non-zero elements, using the storing method described in *Homework 3*. Generate also the sparse structures for the sparse matrix stored in the file posted here.

- For $p = n$ and A symmetric ($A = A^T$) and sparse, implement the power method in order to approximate the largest eigenvalue of matrix A and an associated eigenvector. After reading the matrix from the file, verify that the matrix is symmetric. Display the eigenvalue computed with the power method both for the randomly generated matrix and the posted ones. Display the norm:

$$\|Au^{\max} - \lambda_{\max} u^{\max}\|.$$

  where $u^{\max}$ and $\lambda_{\max}$ are the eigenvector and respectively the eigenvalue computed with the power method.

- Case $p > n$ (non-sparse, classical matrices): using the **S**ingular **V**alue **D**ecomposition) implemented in the library from *Homework no. 2*, compute and display:

  - the singular values of matrix A,
  - the rank of matrix A,
  - the conditioning number of matrix A,
  - Moore-Penrose pseudo-inverse of matrix A, $A^I \in \mathbf{R}^{n\times p}$,

  $$A^I = VSU^T$$

  - the vector $x^I \in \mathbf{R}^n$, $x^I = A^I b$ the solution of system $Ax = b$ and the norm:

  $$\|b - Ax\|_2.$$

  For the rank and the conditioning number of the matrix implement the formulae described in this document and also use the functions from the library that compute these values.

Eigenvectors and eigenvalues - definitions

Consider $A \in \mathbb{R}^{n\times n}$ a square, real matrix of size n. An *eigenvalue* for matrix A, is a complex number $\lambda \in \mathbb{C}$, for which there exists a non-zero vector $u \neq 0$, $u \in \mathbb{C}^n$, called şi *eigenvector* associated to eigenvalue λ such that:

$$Au = \lambda\, u$$

The eigenvalues of matrix A can also be defined as the roots of the characteristic polynomial for matrix A, $p_A(\lambda)$:

$$p_A(\lambda) \;=\; \det(\lambda I - A) = 0$$

The characteristic polynomial has degree n and real coefficients, thus any matrix of size n has n eigenvalues (real and/or complex conjugated).

The symmetric matrices have only real eigenvalues.

Power Method

Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. It can be proved that, for a vector $x \in \mathbb{R}^n, x \neq 0$, the sequence of vectors:

$$\frac{x}{||x||_2},\ \frac{Ax}{||Ax||_2},\ \frac{A^2x}{||A^2x||_2},\ \frac{A^3x}{||A^3x||_2},\ \ldots \tag{1}$$

converges to the eigenvector associated to the greatest (in module) eigenvalue.

For a vector $x \in \mathbb{R}^n$, the *Rayleigh coefficient* is the real number:

$$r(x) = \frac{x^TAx}{x^Tx} = \frac{(Ax, x)_{\mathbb{R}^n}}{||x||_2^2}$$

In the above relations we denoted by $||\cdot||_2$, $(\cdot,\cdot)_{\mathbb{R}^n}$ the Euclidean norm for a vector and the inner product of two vectors, respectively.

The Rayleigh coefficient has the property that if vector x is an eigenvector for matrix A associated to eigenvalue λ then $r(x) = \lambda$. If, along with the computations of the sequence (1), one also computes Rayleigh coefficients of the vectors, you obtain a method for approximating the largest in module eigenvalue.

Power Method

chose vector $v^{(0)} \in \mathbb{R}^n$ randomly, but s.t. $||v^{(0)}||_2 \;=\; 1$;

$w = Av^{(0)}\,;$

$\lambda_0 = (w, v^{(0)})_{\mathbb{R}^n}\,;$

$k \;=\; 0;$

do

$$v^{(k+1)} = \frac{1}{||w||_2}\, w\,;$$

$w \;=\; Av^{(k+1)}\,;$

$\lambda_{k+1} = (w, v^{(k+1)})_{\mathbb{R}^n}\;\;;$

$k{+}{+}\quad;$

while $(||w - \lambda_k v^{(k)}||_2 > n\epsilon$ and $\;k \leq k_{\max})$;

($\epsilon \leq 10^{-9}$ and $k_{\max} = 1000000$ are introduced when reading the data)

If the exit from loop *while* is for $k > k_{\max}$ the algorithm did not produce the sought eigenvalue and eigenvector . In this situation we can try to increase the value of ϵ and restart the computations in order to obtain the results.

If the other relation is true,i.e. $||Av^{(k)} - \lambda_k v^{(k)}||_2 \leq n\epsilon$, the value of λ_{k+1} is an approximation of the largest in module eigenvalue for matrix A, and the vector în $v^{(k+1)}$ is an approximation of an eigenvector associated to this eigenvalue.

In this algorithm, you do not need to store all the elements of the sequences λ_k and $v^{(k)}$, you can use one element for each sequence: $\lambda \in \mathbb{R}$ for storing the value of λ_k and $v \in \mathbb{R}^n$ for $v^{(k)}$.

The initial vector $v^{(0)}$ with $||v^{(0)}||_2 \;=\; 1$ can be chosen starting from a non-zero, randomly generated vector, $x \in \mathbb{R}^n$, $x \neq 0$, and setting:

$$v^{(0)} = \frac{1}{||x||_2} x\,.$$

Power Method

randomly choose vector $v \in \mathbb{R}^n$, with $||v||_2 = 1$;

$(v = \frac{1}{||x||_2} x \ , \ x \in \mathbb{R}^n, x \neq 0)$

$w = Av\,;$

$\lambda = (w, v)_{\mathbb{R}^n}\,;$

$k = 0;$

do

$\quad v = \frac{1}{||w||_2}\, w\,;$

$\quad w = Av\,;$

$\quad \lambda = (w, v)_{\mathbb{R}^n}\,;$

$\quad k{+}{+}\quad;$

while $(||w - \lambda v||_2 > n\epsilon$ and $k \leq k_{\max});$

Singular Value Decomposition

Let $A \in \mathbb{R}^{p\times n}$. The Singular Value Decomposition for matrix A is given by the following relation:

$$A = USV^T \quad , \quad U \in \mathbb{R}^{p\times p} \ , \ S \in \mathbb{R}^{p\times n} \ , \ V \in \mathbb{R}^{n\times n}$$

with $U = [u_1 \ u_2 \ ... \ u_p]$ (the u_i vectors are the columns of matrix U) and $V = [v_1 \ v_2 \ ... \ v_n]$ orthogonal matrices and S a diagonal matrix:

$$\text{for } p \leq n \qquad S = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & \sigma_p & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{p\times n}$$

$$\text{for } p > n \qquad S = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \sigma_n \\ \vdots & & & \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{p\times n}$$

where the nonnegative numbers $\sigma_i \geq 0, \forall i$ are the singular values of matrix A.

The rank of matrix A can be computed as the number of strictly positive singular values:

$$\text{rank } (A) = \text{ number of singular values } \sigma_i > 0 \ \ (\sigma_i > \epsilon, \text{in your program}).$$

The conditioning number of matrix A is the ratio between the largest singular value and the smallest strictly positive singular value: strict pozitivă.

$$k_2(A) = \frac{\sigma_{\max}}{\sigma_{\min}} \quad ,$$

$$\sigma_{\max} = \max\{\sigma_i; \sigma_i \text{ singular value }\} \ ,$$

$$\begin{aligned} \sigma_{\min} &= \min\{\sigma_i; \sigma_i > 0 \text{ singular value}\} \\ &\approx \min\{\sigma_i; \sigma_i > \epsilon \text{ singular value}\} \end{aligned}$$

The Moore-Penrose pseudo-inverse of matrix A is computed using the formula:

$$A^I = VS^IU^T.$$

The matrix S^I is computed using the formula displayed below. Assume that we computed for matrix $A \in \mathbb{R}^{p\times n}$ the singular value decomposition. Let $\sigma_1, \sigma_2, \ldots, \sigma_r > \epsilon$ be the (numerically) strictly positive singular values for matrix A, $r = \text{rang}(A)$.

Matrix $S^I \in \mathbb{R}^{n\times p}$ is computed using the relation:

$$S^I = \begin{pmatrix} \frac{1}{\sigma_1} & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma_2} & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & \frac{1}{\sigma_r} & \cdots & 0 \\ & & & & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{n\times p}.$$

The vector $x^I = VS^IU^Tb$ can be considered the solution of the linear system $Ax = b$ even when $p \neq n$ and the system does not have a classical solution. When $p = n$ and the matrix A is non-singular the vector x^I coincides with the classical solution of system $Ax = b$.