Group Structure on an Elliptic Curve

Sam Mergendahl

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  (ie) $y^2 = f(x) = x^3 + ax^2 + bx + c$ where $a, b, c \in \mathbb{Z}$
- Let $C(\mathbb{Q}) = \{P = (x, y) \in C \mid x, y \in \mathbb{Q}\}$ be the set of points on C with rational cordinates

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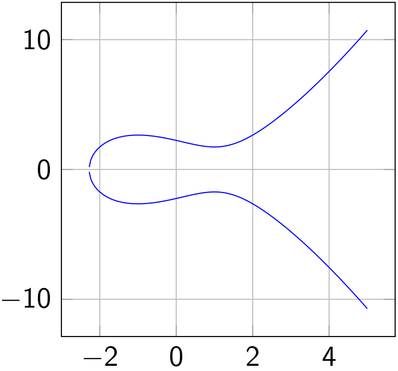

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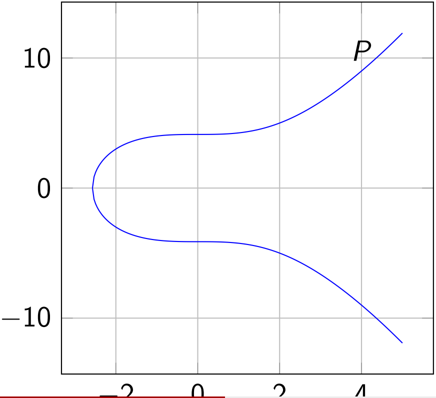

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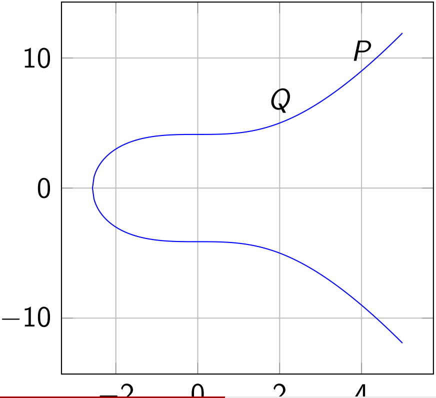

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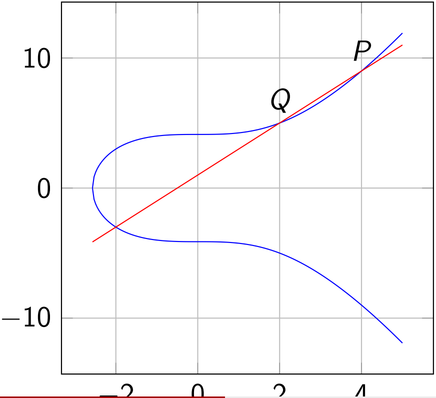

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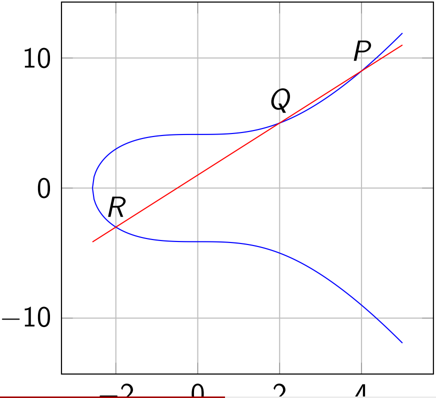

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- Unless P is an inflection point, then it has multiplicity three and will not intersect anywhere else

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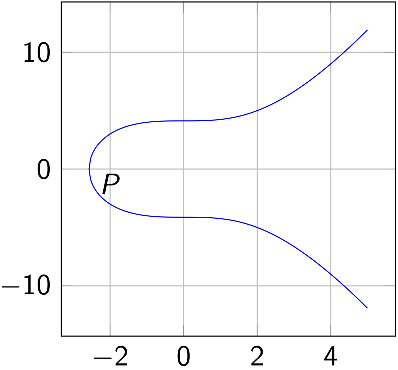

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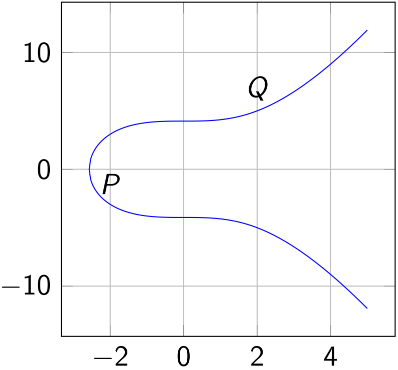

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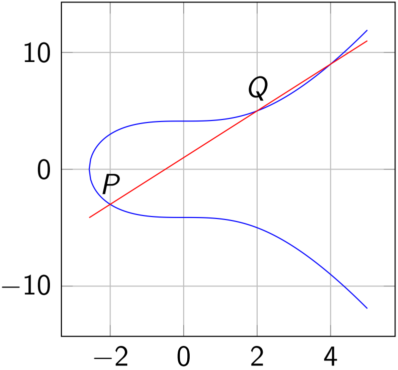

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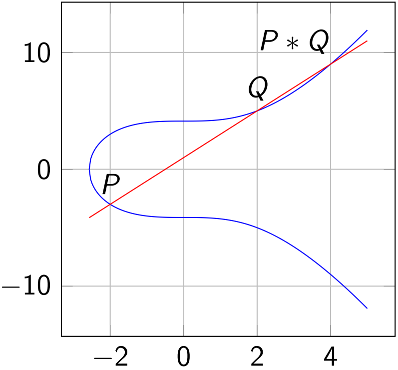

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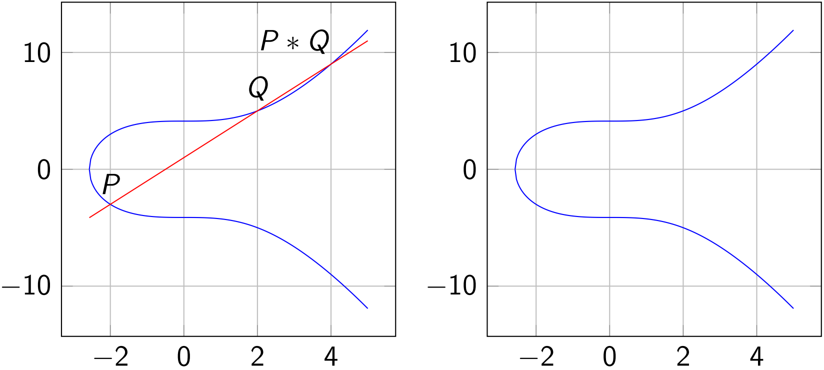

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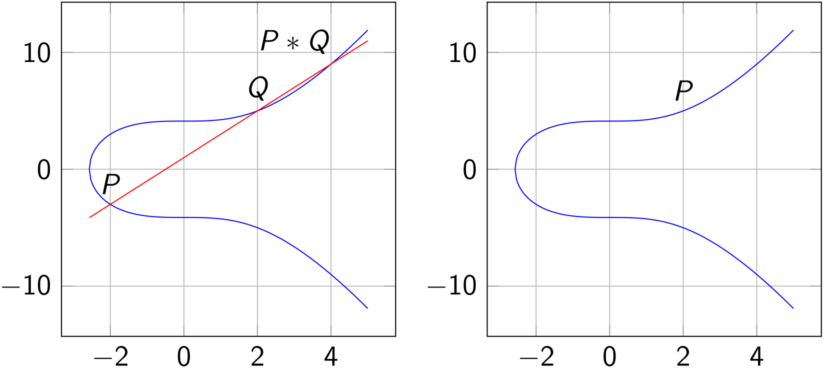

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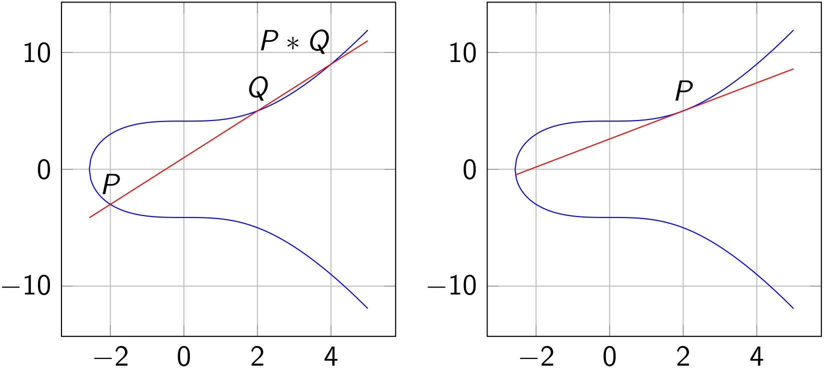

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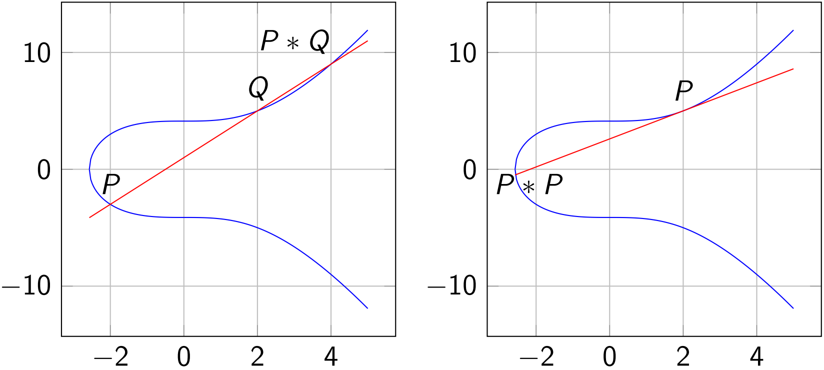

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- Instead let $\mathcal{O}$ be a fixed point in $C(\mathbb{Q})$ and let $P + Q = (P * Q) * \mathcal{O}$

Group Law (*cont.*)

- (eg) if $P = (-2, 3)$, $Q = (-1, 4)$, $\mathcal{O} = (-2, -3)$, then $P + Q = (2, 5)$ on $y^2 = x^3 + 17$

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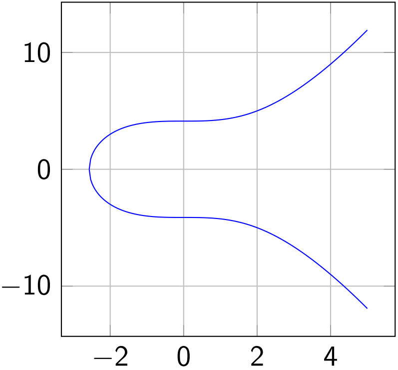

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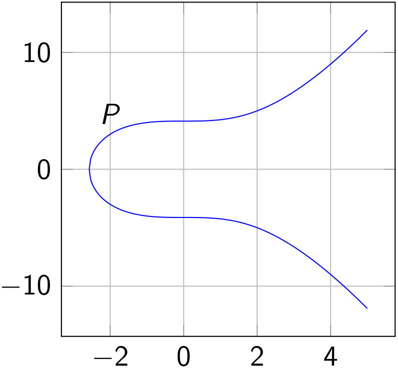

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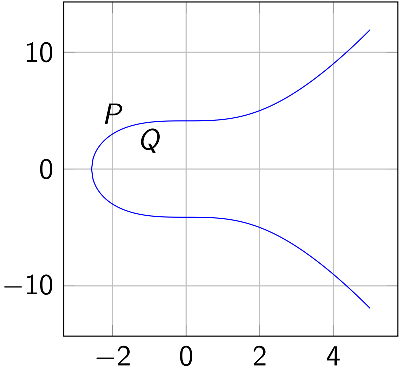

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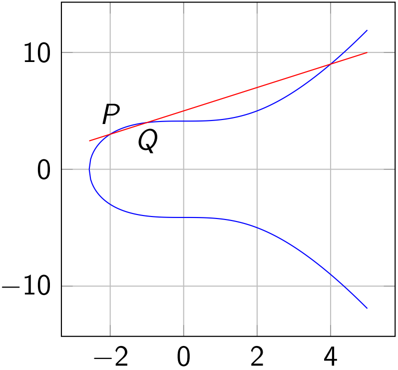

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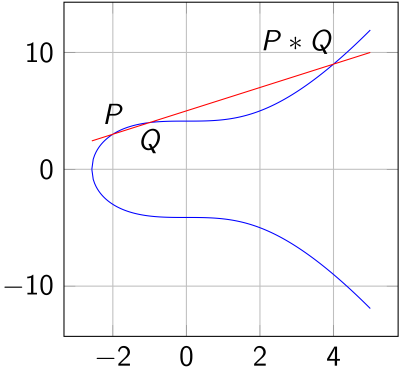

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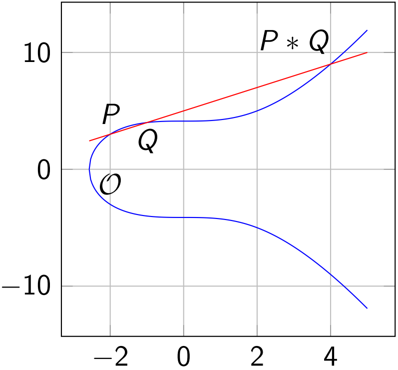

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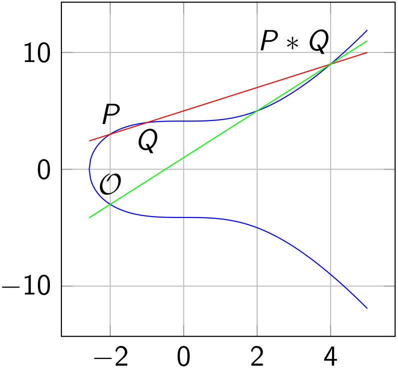

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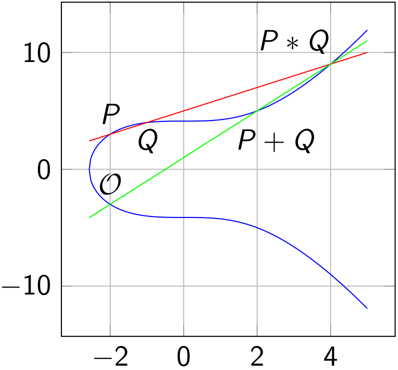

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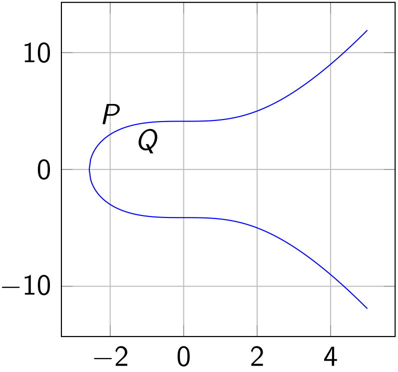

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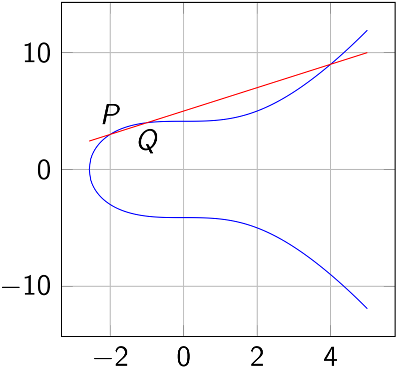

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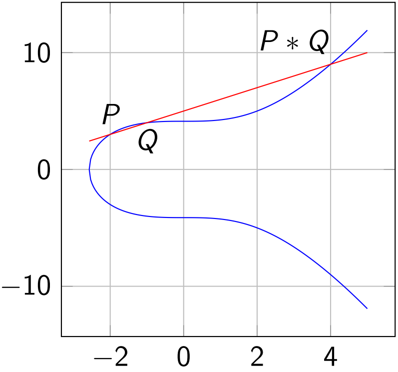

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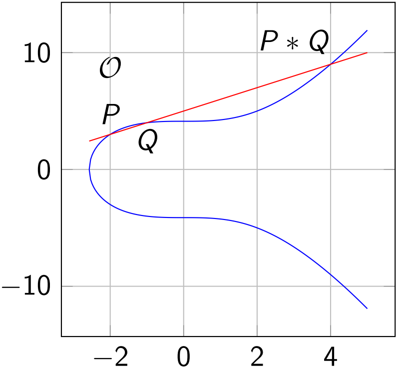

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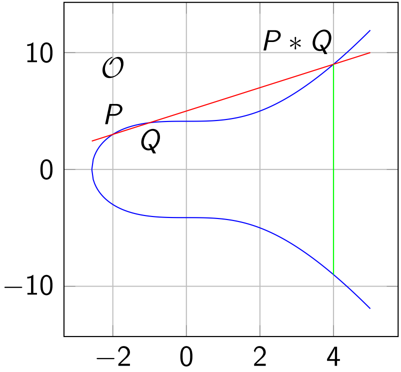

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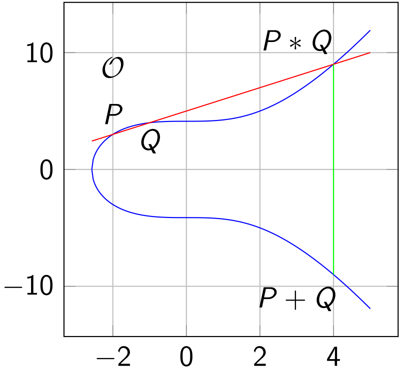

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- Because of a surprising theorem by Nagell-Lutz (both independently discovered), we know that if $P = (x, y) \in \Phi$, then $x, y \in \mathbb{Z}$

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    (ie) $\hat{C}$ is non-singular as long as $p \nmid D$ (and $p \neq 2$)

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  (ie) $\Phi \to \hat{C}(\mathbb{F}_p)$ st $P = (x, y) \mapsto \hat{P} = (\hat{x}, \hat{y})$ (note: $\mathcal{O} \mapsto \mathcal{O}$)

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- $\#\Phi$ divides $\#\hat{C}(\mathbb{F}_p)$

Example

- Let $C : y^2 = x^3 + 3$

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- Let's look at $\hat{C}(\mathbb{F}_5)$ and $\hat{C}(\mathbb{F}_7)$

Example (*cont.*)

- $\hat{C}(\mathbb{F}_5)$:

x	$x^2 \pmod 5$	$x^3 + 3 \pmod 5$
0	0	3
1	1	4
2	4	1
3	4	0
4	1	2

Example (*cont.*)

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- $\hat{C}(\mathbb{F}_5) = \{(1,2),(1,3),(2,1),(2,4),(3,0),\mathcal{O}\}$

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  $\Rightarrow \#\hat{C}(\mathbb{F}_5) = 6$

Example (*cont.*)

- $\hat{C}(\mathbb{F}_7)$:

x	$x^2 \pmod 7$	$x^3 + 3 \pmod 7$
0	0	3
1	1	4
2	4	4
3	2	2
4	2	4
5	4	2
6	1	2

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- $\hat{C}(\mathbb{F}_7) =$
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  $(3,3),(3,4),(4,2),(4,5),$
  $(5,3),(5,4),(6,3),(6,4),\mathcal{O}\}$

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$\Rightarrow \#\hat{C}(\mathbb{F}_7) = 13$

Example (*cont.*)

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x	$x^2 \pmod 7$	$x^3 + 3 \pmod 7$
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  $\Rightarrow \#\hat{C}(\mathbb{F}_7) = 13$
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$\Rightarrow \#\Phi$ divides both 6 and 13
$\Rightarrow \#\Phi = 1$

Example (*cont.*)

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x	$x^2 \pmod 7$	$x^3 + 3 \pmod 7$
0	0	3
1	1	4
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4	2	4
5	4	2
6	1	2

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  $\Rightarrow \#\hat{C}(\mathbb{F}_7) = 13$
  $\Rightarrow \#\Phi$ divides both 6 and 13
  $\Rightarrow \#\Phi = 1 \Rightarrow \Phi = \{\mathcal{O}\}$